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Decay Rates of the Energy for Damped Wave Equations with Critical Potential and Defocusing Nonlinearity

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Decay Rates of the Energy for Damped Wave Equations with Critical Potential and Defocusing Nonlinearity, Part I

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Abstract

We establish new results on the weighted L^2 and L^{p+1} estimates for the nonlinear wave equation with variable damping

$$u_{tt} - \Delta u + a(x) u_t + \lambda |u|^{p+1} u = 0 \ in \ \mathbb{R}^n,$$

and critical potential $a(x) \geq a_1 \left(1 + |x|^2\right)^{\frac{1}{2}}$ with $a_1 > 0$. The presence of the critically decaying potential drastically changes the asymptotic profile of solutions and creates many additional difficulties. We use a modification of the Todorova-Yordanov techniques to a certain extent. But later on, more precisely at the region of small p where the Klein-Gordon effects are really strong, the critical potential does not allow us to rely anymore on their technique and we derive our own approach. Surprisingly, we show that the energy of solutions decays at a polynomial rate $t^{-\min\{a_1, n-1\}}$, where n is the space dimension. We derive these results by using a special version of the multiplier method.

1 Introduction

In this paper, we are concerned with finding an exact energy estimate for the following nonlinear wave equation with damping:

$$u_{tt} - \Delta u + a(x) u_t + f(u) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, n \geq 3 \tag{1.1}$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n \tag{1.2}$$

$$f(u) = \lambda |u|^{p-1} u, \lambda \quad \geq 0, \quad 1 \leq p \leq \tfrac{n+2}{n-2} \tag{1.3}$$

$$a(x) \geq a_1 \left(1 + |x|^2\right)^{-\frac{1}{2}}, \quad a_1 \geq 0 \tag{1.4}$$

We impose several assumptions on the initial data:

$$(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \tag{1.5}$$

where the initial data are compactly supported, such that $u_0(x) = u_1(x) = 0$ for $|x| > R$, where R is some positive constant.

Note also that $a(x)$ is assumed to be a C^0-coefficient, and $f(u)$ is assumed to be a C^1 nonlinearity.

It is worth mentioning that the absorption problem in a much simpler setting, namely with constant potential $a(x) = 1$ has been treated by many authors ([KNO], [K], [HKN], [EK], [NZ], and [INZ]) from 1985 until 2007. Despite their effort, the absorption problem with constant potential $a(x) = 1$ was open until 2007. There was a gap between the regions with different decay rates– the so-called supercritical and subcritical regions. Namely, there was a region where the decay rate was unknown.

Currently, Dr. Todorova and Dr. Yordanov in [TY] have given a complete answer to the absorption problem not only in the above simple case of constant potential, but in the extremely delicate case of the space-dependent potential $a(x)$ by using the newly developed approach in [TY1]. In 2007, they solved the absorption problem with slow decaying potential $0 \leq a(x) \leq a_1(1+|x|)^{-\alpha}$, where $a_1 > 0$ and $\alpha \in [0,1)$, and they found the exact decay of the solutions. Surprisingly, they observed new effects which do not show up in the case of constant potential. As a byproduct of their results, the gap between the supercritical and subcritical regions in the case of a constant coefficient have been closed for any n. Their method works very successfully up until $\alpha < 1$. What we have to deal with is $\alpha = 1$. To solve this, we use a modification of their approach.

Ikehata, Todorova, and Yordanov in [ITY] found the exact decay for the **linear** wave equation with critical potential. Our equation is **nonlinear**, which creates some difficulties. We use their approach, with some modifications, to find the sharp decay of the solution for large exponent p of the nonlinearity. For small p, we have to create our own approach, which will be presented in part II of this paper.

Our results are as follows:

Theorem 1.1. *Assume (1.4) and (1.5), again with the restriction that* $a_1 \geq 1$. *Then there exists a global solution to the problem (1.1)-(1.3) with non-weighted energy satisfying*

$$\int_{\mathbb{R}^n} \left(u_t(t,x)^2 + |\nabla u(t,x)|^2 + \frac{2\lambda}{p+1} |u(t,x)|^{p+1} \right) dx \leq C t^{-m-1}$$

with C *a non-negative constant, and* $m := \min\{a_1 - 1, n-2\} \geq 0$.

Theorem 1.2. *Assume (1.4) and (1.5). Furthermore, suppose that* $0 < a_1 < 1$. *Then there exists a global solution to the problem (1.1)-(1.3) satisfying*

$$\int_{\mathbb{R}^n} \left(u_t(t,x)^2 + |\nabla u(t,x)|^2 + \frac{2\lambda}{p+1} |u(t,x)|^{p+1} \right) dx \leq C t^{-a_1}.$$

The rest of this thesis is organized as follows. In section 2, we prove many useful lemmas and the main theorem, Theorem 1.1. In section 3, we prove more useful lemmas and Theorem 1.2. In section 4, we list some ideas concerning future research.

2 Proof of Theorem 1.1

There is a classical result [S3] which confirms local existence and uniqueness of the nonlinear problem (1.1) with defocusing nonlinearity and any compactly supported data in the energy space $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Following directly from this classical local existence result and the continuation principle, this problem has global existence for any compactly supported data in the energy space.

The main idea in the approach in [TY1] is to find an approximate solution with convenient properties. Here, in our problem, we do not have an approximate solution. Instead, we use $w(x) := \langle x \rangle^{-m}$ as a factor, with $\langle x \rangle := \left(1 + |x|^2\right)^{\frac{1}{2}}$. Then, we define $v := \frac{u}{w}$ and write (1.1) for v.

Lemma 2.1. *Let u be the weak solution to (1.1-1.5). Substitution into (1.1) yields the following equation on v:*

$$v_{tt} - \Delta v + Av_t B \cdot \nabla v + Cv + \lambda D |v|^{p-1} v = 0 \tag{2.1}$$

with new coefficients as follows:

$$\begin{gathered} A = a_1 \\ B = 2mx \langle x \rangle^{-2} \\ C = mn \langle x \rangle^{-2} - m(m+2)|x|^2 \langle x \rangle^{-4} \\ D = \langle x \rangle^{-m(p-1)} \end{gathered} \tag{2.2}$$

Proof. We first substitute $u = wv$ into (1.1), yielding:

$$(wv)_{tt} - \Delta(wv) + a(x)(wv)_t + f(wv) = 0$$

Since w is not time-dependent, this gives:

$$wv_{tt} - \Delta(wv) + a(x) wv_t + f(wv) = 0 \tag{2.3}$$

We must evaluate the expression $\Delta(wv)$:

$$\Delta(wv) = \sum_{i=1}^{n} \frac{\partial^2 (wv)}{\partial x_i^2}$$

$$\frac{\partial (wv)}{\partial x_i} = v \frac{\partial w}{\partial x_i} + w \frac{\partial v}{\partial x_i}$$

$$\frac{\partial^2 (wv)}{\partial x_i^2} = v \frac{\partial^2 w}{\partial x_i^2} + 2 \frac{\partial w}{\partial x_i} \frac{\partial v}{\partial x_i} + w \frac{\partial^2 v}{\partial x_i^2}$$

Summing from 1 to n, we obtain $\Delta(wv) = v\Delta w + 2\nabla v \cdot \nabla w + w\Delta v$, and note also that $f(wv) = \lambda |wv|^{p-1} wv$, which, substituted into (2.3), gives the expression:

$$wv_{tt} - v\Delta w - 2\nabla v \cdot \nabla w - w\Delta v + a(x)\, wv_t + \lambda w^p |v|^{p-1} v = 0$$

Dividing through by w yields:

$$v_{tt} - \Delta v + a(x)\, v_t - 2w^{-1}\nabla w \cdot \nabla v - w^{-1}\Delta w v + \lambda w^{p-1} |v|^{p-1} v = 0 \qquad (2.4)$$

Set $A := a(x)$, $B := -2w^{-1}\nabla w$, $C := -w^{-1}\Delta w$, and $D := w^{p-1}$ to obtain:

$$v_{tt} - \Delta v + Av_t + B \cdot \nabla v + Cv + \lambda D |v|^{p-1} v = 0 \qquad (2.5)$$

Now explicit forms for B, C, and D must be found as functions of x. To do so, w^{-1}, ∇w, and Δw must be found as functions of x:

$$w^{-1} = \left(\langle x\rangle^{-m}\right)^{-1} = \langle x\rangle^{m}$$

$$\frac{\partial w}{\partial x_i} = -\frac{m}{2}\left(1+|x|^2\right)^{-\frac{m}{2}-1} \cdot 2x_i = -m\left(1+|x|^2\right)^{\frac{-m-2}{2}} x_i = -mx_i \langle x\rangle^{-m-2}$$

$$\frac{\partial^2 w}{\partial x_i^2} = -m\left(1+|x|^2\right)^{\frac{-m-2}{2}} + m(m+2)\, x_i^2 \left(1+|x|^2\right)^{\frac{-m-4}{2}}$$

$$\nabla w = \left(\frac{\partial w}{\partial x_1}, \dots, \frac{\partial w}{\partial x_n}\right) = -mx \langle x\rangle^{-m-2}$$

$$\Delta w = -mn \langle x\rangle^{-m-2} + m(m+2)\, |x|^2 \langle x\rangle^{-m-4}$$

Now the explicit forms of the constants may be found:

$$B = -2w^{-1}\nabla w = -2\langle x\rangle^{m}\left(-mx\langle x\rangle^{-m-2}\right) = 2mx\langle x\rangle^{-2}$$

$$C = -w^{-1}\Delta w = -mn\langle x\rangle^{-2} + m(m+2)\,|x|^2\langle x\rangle^{-4}$$

$$D = w^{p-1} = \left(\langle x\rangle^{-m}\right)^{p-1} = \langle x\rangle^{-m(p-1)}$$

□

Now we will use the multiplier method for the modified equation (2.1) with weights P and Q, defined in terms of x as $P(x) := k\langle x\rangle w = k\langle x\rangle^{-m+1}$ and $Q(x) := w = \langle x\rangle^{-m}$, where k is a positive constant.

Lemma 2.2. *Multiplying the modified equation (2.1) by $Pv_t + Qv$ and integrating over $\mathbb{R}^n$ yields the following weighted energy identity:*

$$\frac{\partial}{\partial t}E_w(t) + F(v_t, v) + G(v) + H(h, v_t, v) = 0 \tag{2.6}$$

where E, F, G, and H are defined as follows, with $E_w(t)$ being the weighted energy:

$$\begin{aligned} E_w(t) := \; & \tfrac{1}{2}\int_{\mathbb{R}^n}\left[P\left(v_t^2 + |\nabla v|^2\right) + 2Qv_tv + (CP + AQ)v^2\right]dx \\ & + \tfrac{1}{2}\int_{\mathbb{R}^n}\left[\tfrac{\lambda k}{p+1}\langle x\rangle^{-mp+1}|v|^{p+1}\right]dx \end{aligned}$$

$$F(v_t, v) := \frac{1}{2}\int_{\mathbb{R}^n}(2AP - 2Q)v_t^2dx + \int_{\mathbb{R}^n}(\nabla P + BP)v_t\nabla v dx + \frac{1}{2}\int_{\mathbb{R}^n}2Q|\nabla v|^2dx$$

$$G(v) := \frac{1}{2}\int_{\mathbb{R}^n}(-\Delta Q - \nabla(QB) + 2CQ)v^2dx$$

$$H(v) := \int_{\mathbb{R}^n}\lambda\langle x\rangle^{-mp}|v|^{p-1}v^2dx \tag{2.7}$$

Proof. Multiplying the modified equation (2.1) by Pv_t gives the equation:

$$Pv_tv_{tt} - Pv_t\Delta v + PAv_t^2 + PB\cdot\nabla vv_t + PCvv_t + \lambda w^{p-1}|v|^{p-1}vPv_t = 0 \tag{2.8}$$

Note that P, A, B, and C are all time-independent, so:

$$\frac{d}{dt}\left(\frac{1}{2}Pv_t^2\right) = \frac{1}{2}Pv_tv_{tt} + \frac{1}{2}Pv_{tt}v_t = Pv_tv_{tt}$$

$$\frac{d}{dt}\left(\frac{1}{2}CPv^2\right) = \frac{1}{2}CPvv_t + \frac{1}{2}CPv_tv = CPvv_t$$

$$\frac{1}{2}\frac{d}{dt}\left(P|\nabla v|^2\right) = P\nabla v_t\nabla v$$

$$\nabla(Pv_t\nabla v) = \nabla P\cdot\nabla vv_t + P\nabla v_t\nabla v + Pv_t\Delta v$$

$$-Pv_t\Delta v = \nabla P\cdot\nabla vv_t + \frac{1}{2}\frac{d}{dt}\left(P|\nabla v|^2\right) - \nabla(Pv_t\nabla v)$$

Substituting these equations into (2.8) gives:

$$\frac{1}{2}\frac{d}{dt}\left[Pv_t^2 + CPv^2 + P|\nabla v|^2\right] + APv_t^2 + PB\cdot\nabla vv_t$$

$$+\nabla P\cdot\nabla vv_t - \nabla(Pv_t\nabla v) + \lambda w^{p-1}|v|^{p-1}vPv_t = 0$$

Multiplying the original equation (2.5)by Qv gives the equation:

$$Qvv_{tt} - Qv\Delta v + AQvv_t + QB\cdot\nabla vv + QCv^2 + \lambda Qw^{p-1}|v|^{p-1}v^2 = 0$$

Note that Q, A, B, and C are all time-independent, so:

$$\frac{d}{dt}(Qvv_t) = Qvv_{tt} + Qv_t^2 \Longrightarrow Qvv_{tt} = \frac{d}{dt}(Qvv_t) - Qv_t^2$$

$$\frac{d}{dt}\left(\frac{1}{2}AQv^2\right) = AQvv_t$$

$$\nabla\left(QBv^2\right) = \nabla\left(QB\right)v^2 + 2QBv\nabla v \Rightarrow QB\cdot\nabla vv = -\frac{1}{2}\nabla\left(QB\right)v^2 + \frac{1}{2}\nabla\left(QBv^2\right)$$

$$\nabla\left(Qv^2\right) = \nabla Q\cdot v^2 + 2Qv\nabla v$$

$$\Delta\left(Qv^2\right) = \Delta Qv^2 + 4\nabla Qv\nabla v + 2Q\left|\nabla v\right|^2 + 2Qv\Delta v$$

$$2\nabla Qv\nabla v = \nabla\cdot\left(\nabla Qv^2\right) - \Delta Qv^2$$

$$-Qv\Delta v = Q\left|\nabla v\right|^2 - \frac{1}{2}\Delta Qv^2 + \nabla\cdot\left(\nabla Qv^2\right) + \Delta\left(Qv^2\right)$$

Substituting these equations in gives:

$$\frac{d}{dt}\left(Qv_tv\right) - Qv_t^2 + Q\left|\nabla v\right|^2 - \frac{1}{2}\Delta Qv^2 + \nabla\cdot\left(\nabla Qv^2\right) + \Delta\left(Qv^2\right) +$$

$$\frac{d}{dt}\left(\frac{1}{2}AQv^2\right) + \frac{1}{2}\nabla\left(QBv^2\right) - \frac{1}{2}\nabla\left(QB\right)v^2 + QCv^2 + \lambda w^p\left|v\right|^p v = 0$$

We must also deal with combining terms in the nonlinearity:

$$\lambda w^{p-1}\left|v\right|^{p-1}v\left(Pv_t + Qv\right) = \lambda w^{p-1}P\left|v\right|^{p-1}vv_t + \lambda w^p\left|v\right|^{p-1}v^2$$

$$\frac{d}{dt}\left(\frac{\lambda}{p+1}Pw^{p-1}\left|v\right|^{p+1}\right) = \lambda w^{p-1}P\left|v\right|^{p-1}vv_t$$

$$\Rightarrow \lambda w^{p-1}\left|v\right|^{p-1}v\left(Pv_t + Qv\right) = \frac{d}{dt}\left(\frac{\lambda k}{p+1}\left\langle x\right\rangle^{-mp+1}\left|v\right|^{p+1}\right) + \lambda\left\langle x\right\rangle^{-mp}\left|v\right|^{p-1}v^2$$

Adding these two equations yields:

$$\frac{1}{2}\frac{d}{dt}\left[P\left(v_t^2 + \left|v\right|^2\right) + 2Qv_tv + \left(CP + AQ\right)v^2 + \frac{\lambda k}{p+1}\left\langle x\right\rangle^{-mp+1}\left|v\right|^{p+1}\right] +$$

$$\frac{1}{2}\left(2AP - 2Q\right)v_t^2 + \left(\nabla P + BP\right)v_t\nabla v + \frac{1}{2}\left(2Q\right)\left|\nabla v\right|^2 +$$

$$\frac{1}{2}\left[-\Delta Q - \nabla\left(QB\right) + 2CQ\right]v^2 +$$

$$\lambda\left\langle x\right\rangle^{-mp}\left|v\right|^{p-1}v^2 +$$

$$\nabla\cdot\left(\nabla Qv^2\right) + \Delta\left(Qv^2\right) + \frac{1}{2}\nabla\left(QBv^2\right) - \nabla\left(Pv_t\nabla v\right) = 0$$

where the divergences in the last line are all zero when integrated over $\mathbb{R}^n$ due to the finite speed of propagation and compact support. Integrating over $\mathbb{R}^n$ gives us the equation:

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{R}^n}\left[P\left(v_t^2+|v|^2\right)+2Qv_tv+(CP+AQ)\,v^2+\frac{\lambda k}{p+1}\langle x\rangle^{-mp+1}|v|^{p+1}\right]dx+$$

$$\int_{\mathbb{R}^n}\left[\frac{1}{2}(2AP-2Q)\,v_t^2+(\nabla P+BP)\,v_t\nabla v+\frac{1}{2}(2Q)\,|\nabla v|^2\right]dx+$$

$$\int_{\mathbb{R}^n}\frac{1}{2}\left[-\Delta Q-\nabla(QB)+2CQ\right]v^2dx+$$

$$\int_{\mathbb{R}^n}\lambda\langle x\rangle^{-mp}|v|^{p-1}v^2dx=0$$

□

Turning back to the weighted energy identity (2.6), we will start to write sufficient conditions on the weights P and Q, which will guarantee that the terms F and G in (2.6) are non-negative. From that, we will yield that the weighted energy is a decreasing function.

The term F is a complicated quadratic form with respect to v_t and ∇v. To ensure non-negativity of F, we use the standard properties of quadratic forms, namely $4w\,(a\,(x)\,P-w)\geq|\nabla\ P-2P\nabla lnw|^2$. To ensure that G is non-negative, it is enough to require that $\Delta w\leq 0$.

Lemma 2.3. *The condition that* $4w\,(a\,(x)\,P-w)\geq|\nabla\ P-2P\nabla lnw|^2$ *implies that* $F\geq 0$.

Proof. This follows directly from the quadratic form of F, which gives a sufficient condition for $F\geq 0$ in terms of the coefficients of the quadratic form: $(2AP-2Q)\,(2Q)\geq|\nabla\ P+BP|^2$. Substituting in $Q:=w$, $A:=a\,(x)$, and $B:=-2w^{-1}\nabla w$, we obtain the following equivalent inequality:

$$(2a\,(x)\,P-2w)\,(2w)\geq\left|\nabla P-2w^{-1}\nabla wP\right|^2$$

Pulling constants in front and noting that $w^{-1}\nabla w=\nabla lnw$, we obtain the condition required for F to be non-negative:

$$4w\,(a\,(x)\,P-w)\geq|\nabla\ P-2P\nabla lnw|^2$$

□

Lemma 2.4. *The condition that* $\Delta w\leq 0$ *implies that* $G\geq 0$.

Proof. Substituting the values $Q:=w$, $B:=-2w^{-1}\nabla w$, and $C:=-w^{-1}\Delta w$ into $G\,(v)=\int_{\mathbb{R}^n}\frac{1}{2}\left[-\Delta Q-\nabla(QB)+2CQ\right]v^2dx$ gives the following equation for G:

$$G\,(v)=\frac{1}{2}\int_{\mathbb{R}^n}\left(-\Delta w-\nabla\left(w\left(-2w^{-1}\nabla w\right)\right)+2\left(-w^{-1}\Delta ww\right)\right)v^2dx$$

Cancelling out pairs of w, w^{-1} and noting that $\nabla \cdot (\nabla 2w) = 2\Delta w$ gives us a simplified equation:

$$G(v) = \frac{1}{2} \int_{\mathbb{R}^n} (-\Delta w + 2\Delta w - 2\Delta w) v^2 dx$$

Combining the Δw terms, we obtain a simple expression for G:

$$G(v) = \frac{1}{2} \int_{\mathbb{R}^n} -\Delta w v^2 dx$$

Which is clearly non-negative if $\Delta w \leq 0$. □

In order to proceed, we must find constraints on our multipliers P and Q that will give us the conditions of Lemmas (2.3) and (2.4). Specifically, we find sufficient conditions on our constant m to finally be able to drop the terms F and G out of the equation (2.6) and deal solely with the weighted energy.

Lemma 2.5. *The following two constraints on m will give the conditions of Lemmas 2.3 and 2.4:*

1. $m \leq a_1 - 1$

2. $m \leq n - 2$

Thus, the constraint $m = \min\{a_1 - 1, n - 2\}$ *is sufficient to give* $F, G \geq 0$.

Proof. If $m \leq a_1 - 1$, and m and a_1 are both non-negative, then with some simple algebraic manipulations we obtain the following inequality:

$$4^2 a_1 - 4\left(-(1+m)^2\right)(-4) \geq 0$$

Given this inequality, the following quadratic form is positive:

$$-(1+m)^2 k^2 + 4a_1 k - 4 \geq 0$$

Further algebraic manipulations and substitution of terms gives us the condition on F:

$$4(a_1 k - 1) \geq (1+m)^2 k^2$$

$$4w^2 (a(x) k \langle x \rangle - 1) \geq \left|(1+m) kx \langle x \rangle^{-m-1}\right|^2$$

$$4w(aP - w) \geq \left|k \langle x \rangle (-mx) \langle x \rangle^{-m-1} + kx \langle x \rangle^{-m-1} + 2mkx \langle x \rangle^{-m-1}\right|^2$$

$$4w(aP - w) \geq \left|k \langle x \rangle (-mx) \langle x \rangle^{-m-2} + \frac{kx}{\langle x \rangle} \langle x \rangle^{-m} - 2\left(-mx \langle x \rangle^{-m-2}\right) k \langle x \rangle\right|^2$$

$$4w(aP - w) \geq \left|k \langle x \rangle \nabla w + k \nabla x w - 2w^{-1} \nabla w k \langle x \rangle w\right|^2$$

$$4w(aP - w) \geq \left|\nabla P - 2P \ln w\right|^2$$

This proof also gives us the condition that m must be non-negative, thus $a_1 - 1$ must also be non-negative. So for Theorem 1.1, we require that $a_1 \geq 1$.

Expressing Δw in terms of x, we obtain the following expression: $\Delta w = -mn \langle x \rangle^{-m-2} + m(m+2)|x|^2 \langle x \rangle^{-m-4} \geq 0$. Multiplying through by $\frac{\langle x \rangle^{m+4}}{m}$ and adding $-n \langle x \rangle^2$ to both sides, we get the equivalent inequality $(m+2)|x|^2 \leq n \langle x \rangle^2$. This is satisfied if $m \leq n-2$.

Given the sufficient conditions $m \leq a_1 - 1$ and $m \leq n-2$, setting $m := \min\{a_1 - 1, n-2\}$ will satisfy both conditions. □

Now that we have F, G non-negative, we can drop these terms from (2.6), and start finding properties of $E_w(t)$ that will allow us to arrive at a weighted energy estimate. Specifically, we need to know that the weighted energy is a decreasing, non-negative function. This will signify that the energy is decaying.

Lemma 2.6. *The condition that* $m := \min\{a_1 - 1, n-2\}$ *implies that the weighted energy is bounded above by a constant:*

$$E_w(t) \leq E(t_0) \tag{2.9}$$

where t_0 *is the initial time of (1.1)-(1.3).*

Proof. Lemmas 2.2, 2.3, and 2.4 yield that F and G are non-negative. Note also that $H = \int_{\mathbb{R}^n} \lambda \langle x \rangle^{-mp} |v|^{p-1} v^2 dx$ is clearly non-negative. So we can drop F, G, and H from (2.6), which gives us the following inequality:

$$\frac{\partial}{\partial t} E_w(t) \leq 0$$

Integating from t_0 to t, we get the following inequality:

$$E_w(t) \leq E_w(t_0)$$

□

Lemma 2.7. *The condition that* $m := \min\{a_1 - 1, n-2\}$ *implies that the weighted energy is non-negative.*

Proof. Expressing $E_w(t)$ in terms of x by substituting values in for P, Q, A, B, and C, we obtain the following expression for the weighted energy:

$$\begin{aligned} E_w(t) = \quad & \int_{\mathbb{R}^n} \left[k \langle x \rangle^{1-m} \left(v_t^2 + |\nabla v|^2 \right) + \tfrac{\lambda k}{p+1} \langle x \rangle^{1-mp} |v|^{p+1} \right] dx \\ & + \int_{\mathbb{R}^n} \left[(-k \langle x \rangle \Delta w + aw) v^2 + 2 \langle x \rangle^{-m} v_t v \right] dx \end{aligned}$$

The coefficient of v^2 is clearly non-negative, since we have established that given our conditions on m, $\Delta w \leq 0$. It then remains to deal with the only other unsigned term, $2 \langle x \rangle^{-m} v_t v$. To do this, we use the Cauchy inequality and establish a bound from below:

$$2 \langle x \rangle^{-m} v_t v \geq - \langle x \rangle \langle x \rangle^{-m} v_t^2 - \langle x \rangle^{-1} \langle x \rangle^{-m} v^2$$

Substituting this back into the expression for the weighted energy, we obtain an inequality on the weighted energy:

$$\begin{aligned} E_w(t) \quad &\geq \textstyle\int_{\mathbb{R}^n} \left[(k-1) \langle x \rangle^{1-m} \left(v_t^2 + |\nabla v|^2 \right) + \frac{\lambda k}{p+1} \langle x \rangle^{1-mp} |v|^{p+1} \right] dx \\ &\quad + \textstyle\int_{\mathbb{R}^n} \left[(a_1 - 1) \langle x \rangle^{-m-1} v^2 \right] dx \end{aligned} \tag{2.10}$$

Since all terms in this expression are non-negative, we have that the weighted energy is non-negative. This completes the proof of the Lemma. □

Finally, we arrive at the first major lemma of the paper. Now that we know the energy is decaying, we can go on to find an estimate for the decay of the weighted energy. In doing so, we must turn back to an expression in terms of u instead of the factored variable v.

Lemma 2.8. *Assume (1.4) and (1.5). Furthermore, assume that* $0 \leq a_1 \leq 1$, *and define* $m := \min\{a_1 - 1, n - 2\}$. *Then the solution of (1.1)-(1.3) has weighted energy satisfying*

$$\int_{\mathbb{R}^n} \langle x \rangle^{m+1} \left(u_t^2 + |\nabla u|^2 + \lambda |u|^{p+1} \right) dx \leq C \tag{2.11}$$

where C *is a non-negative constant.*

Proof. We know from Lemma 2.6 that $E_w(t) \leq E_w(t_0)$, where $E_w(t_0)$ is a non-negative constant, since we know from Lemma 2.7 that the weighted energy is non-negative at all points. Then we can pull the constants in (2.10) in front of the integral and combine them with the constant $E_w(t_0)$ to give us the following inequality:

$$\int_{\mathbb{R}^n} \left[\langle x \rangle^{1-m} \left(v_t^2 + |\nabla v|^2 \right) + \lambda \langle x \rangle^{1-mp} |v|^{p+1} + \langle x \rangle^{-m-1} v^2 \right] dx \leq C$$

To find the final inequality on the weighted energy, we must substitute $v := uw$ back into the above inequality. To do so, we need to establish a few identities to convert v to u:

$$v^2 = u^2 \langle x \rangle^{2m}$$

$$v_t^2 = u_t^2 \langle x \rangle^{2m}$$

$$|\nabla v|^2 \geq |\nabla\, u|^2 \langle x \rangle^{2m}$$

$$|v|^{p+1} = |u|^{p+1} \langle x \rangle^{mp+m}$$

Substituting in these expressions give us the following weighted energy inequality:

$$\int_{\mathbb{R}^n} \langle x \rangle^{m+1} \left(u_t^2 + |\nabla u|^2 + \lambda |u|^{p+1} \right) dx \leq C$$

where C is a non-negative constant. □

The inequality (2.11) also gives us the following norms for u:

$$\int_{\mathbb{R}^n} \left(\langle x\rangle^{m-1} u^2\right) dx \le C \tag{2.12}$$

$$\int_{\mathbb{R}^n} \left(\langle x\rangle^{m+1} u^{p+1}\right) dx \le C \tag{2.13}$$

These inequalities will prove useful later for describing the interactions between norms on u when p is small.

Now we have an estimate for the weighted energy, with the weights functions of x, but these weights must be converted to time-dependent functions in order to arrive at our final energy estimate. The finite speed of propagation of the problem will not help us here, so we will have to rely on some other method to convert the weighted energy estimate.

We define the following functions to help us convert the energy estimate from an estimate with weights in x to an estimate that relies solely on t:

$$I(t) := \int_t^T \left(\int_{\mathbb{R}^n} \langle x\rangle^{-1}\left(u_t^2 + |\nabla u|^2 + \lambda |u|^{p+1}\right) dx\right) ds$$

$$Y(t) := \int_{\mathbb{R}^n} \left(u_t^2 + |\nabla u|^2 + uu_tW + \frac{1}{2}aWu^2 + \frac{2\lambda}{p+1}|u|^{p+1}\right) dx$$

where $W(x) := \frac{a_1}{2\langle x\rangle}$, $I(t)$ is a very complicated space-time norm, and $Y(t)$ is an auxiliary function that arises naturally from the calculations. The correlations between $E(t)$, $I(t)$, and $Y(t)$ must be known very well to transfer the weight of the energy from x to t, so we prove a set of inequalities on these three functions.

Lemma 2.9. *The functions $E(t)$, $I(t)$, and $Y(t)$ satisfy the following for $n \ge 3$, where c is a non-negative constant that may vary among inequalities:*

1. $Y(t) \ge 0$
2. $Y(t) \ge cI(t)$
3. $Y(t) \le cE(t)$
4. $I(t) \le cE(t)$
5. $E(t) \le c\left(-I'(t)\right)^{\frac{m+1}{m+2}}$
6. $E(t) \le cI\left(\frac{t}{2}\right)$

Proof of 1. Let $\varepsilon > 0$. By Cauchy's inequality, we obtain the following expression on the sole signless term in $Y(t)$:

$$|uu_tw| \le \frac{\varepsilon}{2}u_t^2 + \frac{1}{2\varepsilon}u^2W^2 \le \frac{\varepsilon}{2}u_t^2 + \frac{1}{2\varepsilon}u^2Wa \tag{2.14}$$

Setting $\varepsilon := \frac{3}{2}$ and bounding the signless term from below, we obtain the inequality $uu_t w \geq -\frac{3}{4}u_t^2 - \frac{1}{3}u^2 Wa$. Substituting this inequality into the original definition for $Y(t)$ and combining like terms, we obtain the following inequality:

$$Y(t) \geq \int_{\mathbb{R}^n} \left(\frac{1}{4}u_t^2 + |\nabla u|^2 + \frac{1}{6}aWu^2 + \frac{2\lambda}{p+1}|u|^{p+1} \right) dx$$

where all terms are positive, so $Y(t) \geq 0$.

Proof of 2. For this inequality, we multiply (1.1) by $u_t + \frac{1}{2}Wu$ to obtain helpful equations for proving the inequality $Y(t) \geq cI(t)$. First, we multiply by u_t to obtain the following:

$$u_t u_{tt} - u_t \Delta u + a(x) u_t^2 + \lambda |u|^{p-1} uu_t = 0 \tag{2.15}$$

Observe that:

- $\frac{1}{2}\frac{\partial}{\partial t}\left(u_t^2\right) = u_t u_{tt}$
- $\frac{\lambda}{p+1}\frac{\partial}{\partial t}\left(|u|^{p+1}\right) = \lambda |u|^{p-1} uu_t$
- $u_t \Delta u + \frac{\partial}{\partial t}\left(\frac{|\nabla u|^2}{2}\right) = u_t \Delta u + \nabla u \cdot \nabla u_t + \nabla \cdot (\nabla u \cdot u_t)$
  $\Rightarrow -u_t \Delta u = \frac{\partial}{\partial t}\left(\frac{|\nabla u|^2}{2}\right) - \nabla \cdot (\nabla u \cdot u_t)$

After substituting the above equalities into (2.15) and integrating over $\mathbb{R}^n$, we obtain the following equation:

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{R}^n} \left(u_t^2 + |\nabla u|^2 + \frac{2\lambda}{p+1}|u|^{p+1} \right) dx + \int_{\mathbb{R}^n} a(x) u_t^2 dx = 0, \tag{2.16}$$

where the divergence term becomes 0 after integration due to the finite speed of propagation. Next, we multiply (1.1) by $\frac{1}{2}Wu$ to obtain the following:

$$\frac{1}{2}Wuu_{tt} - \frac{1}{2}Wu\Delta u + \frac{1}{2}Wa(x) uu_t + \frac{\lambda}{2}W|u|^{p-1}u^2 = 0 \tag{2.17}$$

Observe that:

- $\frac{\partial}{\partial t}\left(\frac{1}{2}Wuu_t\right) = \frac{1}{2}Wu_t^2 + \frac{1}{2}Wuu_{tt}$
  $\Rightarrow \frac{1}{2}Wuu_{tt} = -\frac{1}{2}Wu_t^2 + \frac{\partial}{\partial t}\left(\frac{1}{2}Wuu_t\right)$
- $\frac{\partial}{\partial t}\left(\frac{1}{4}Wa(x) u^2\right) = \frac{1}{2}Wa(x) u_t^2$
- $\nabla\left(Wu^2\right) = \nabla Wu^2 + 2Wu\nabla u$
- $\nabla \cdot \nabla\left(Wu^2\right) = \Delta Wu^2 + 4\nabla W \cdot \nabla u \cdot u + 2W|\nabla u|^2 + 2Wu\triangle u$

- $4\nabla\left(W\nabla uu\right)=4\nabla Wu\nabla u+4Wu\Delta u+4W\left|\nabla u\right|^2$

  $\Rightarrow \nabla\cdot\nabla\ \left(Wu^2\right)=\Delta Wu^2+4\nabla\left(W\nabla uu\right)-2W\left|\nabla u\right|^2-2Wu\triangle u$

  $\Rightarrow -\frac{1}{2}Wu\triangle u=\frac{1}{4}\nabla\cdot\nabla\ \left(Wu^2\right)-\frac{1}{4}\Delta Wu^2-\nabla\left(W\nabla uu\right)+\frac{1}{2}W\left|\nabla u\right|^2$

After substituting the above equalities into (2.17) and integrating over $\mathbb{R}^n$, we obtain the following equation:

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{R}^n}\left(uu_tW+\frac{1}{2}Wa\left(x\right)u^2\right)dx+$$

$$\frac{1}{2}\int_{\mathbb{R}^n}\left(-Wu_t^2+W\lambda\left|u\right|^{p+1}u^2-\frac{1}{2}\Delta Wu^2+W\left|\nabla u\right|^2\right)dx=0 \qquad (2.18)$$

where again, the divergence terms become 0 after integration due to the finite speed of propagation. Now, we add (2.17) and (2.18) together to obtain the following equation:

$$\frac{\partial}{\partial t}Y\left(t\right)+K\left(t\right)=0 \qquad (2.19)$$

$$K(t):=\int_{\mathbb{R}^n}\left(W\left|\nabla u\right|^2-\Delta W\frac{u^2}{2}+\left(2a\left(x\right)-W\right)u_t^2+W\lambda\left|u\right|^{p+1}\right)dx$$

We need to show that $K\left(t\right)\geq 0$ to proceed. The coefficient in front of u_t^2 is easily dealt with, since $2a(x)-W=\frac{2a_1}{\langle x\rangle}-\frac{a_1}{2\langle x\rangle}=\frac{3}{2}\frac{a_1}{\langle x\rangle}\geq 0$. We will ensure non-negativity of the coefficient of u^2 by showing that $\Delta W\leq 0$. To do so, we express W radially. If we define $r:=|x|=\left(x_1^2+\ldots+x_n^2\right)^{\frac{1}{2}}$ and $z\left(r\right):=\frac{a_1}{2\left(1+r^2\right)^{\frac{1}{2}}}=\frac{a_1}{2\left(1+|x|^2\right)^{\frac{1}{2}}}=W$, it will be easier to find derivatives for W. Then:

- $W_{x_i}=z'\left(r\right)\frac{\partial r}{\partial x_i}=z'\left(r\right)\frac{x_i}{r}$
- $W_{x_ix_i}=z''\left(r\right)\left(\frac{x_i}{r}\right)^2+z'\left(r\right)\left(\frac{r-\frac{x_i^2}{r}}{r^2}\right)$

  $\Rightarrow \Delta W=z''\left(r\right)+z'\left(r\right)\left(\frac{n-1}{r}\right)$
- $z'\left(r\right)=-\frac{a_1r}{2}\left(1+r^2\right)^{-\frac{3}{2}}$
- $z''\left(r\right)=-\frac{a_1}{2}\left(1+r^2\right)^{-\frac{3}{2}}+\frac{3a_1r^2}{2}\left(1+r\right)^{-\frac{5}{2}}$

These equations give us the following expression for ΔW:

$$\begin{aligned}\Delta W &= -\frac{a_1}{2\langle x\rangle^3}+\frac{3a_1r^2}{2\langle x\rangle^5}-\frac{a_1(n-1)}{2\langle x\rangle^3}=\frac{3a_1r^2-a_1n\langle x\rangle^2}{\langle x\rangle^5}\\ &\leq \frac{3a_1r^2-a_1nr^2}{\langle x\rangle^5}=\frac{a_1r^2(3-n)}{\langle x\rangle^5}\end{aligned}$$

which is non-negative so long as $n\geq 3$, a condition already imposed on our problem. So all terms in $K\left(t\right)$ with unknown sign have been shown to be non-negative, thus $K\left(t\right)\geq 0$.

To proceed, we integrate (2.19) with respect to t to obtain

$$Y(t) = Y(T) + \int_t^T K(t)$$

We will further evaluate the expression on the right hand side of this equation to obtain the desired results. First, we can drop the u^2 term since it is positive and not necessary for our inequality:

$$\begin{aligned} Y(T) + \int_t^T K(t) \quad & \geq Y(T) + \int_t^T \int_{\mathbb{R}^n} \left(W \left|\nabla u\right|^2 + \left(2a\left(x\right) - W\right) u_t^2 + W\lambda \left|u\right|^{p+1} \right) \\ & \geq Y(T) + c_0 \int_t^T \int_{\mathbb{R}^n} \langle x \rangle^{-1} \left(u_t^2 + \left|\nabla u\right|^2 + \lambda \left|u\right|^{p+1} \right) dx \\ & = Y(T) + + \int_t^T \int_{\mathbb{R}^n} \left(\frac{a_1 \left|\nabla u\right|^2}{2\langle x \rangle} + \frac{c_1 u_t^2}{\langle x \rangle} + \frac{c_2 \lambda \left|u\right|^{p+1}}{\langle x \rangle} \right) dx \\ & \geq Y(T) + c_0 \int_t^T \int_{\mathbb{R}^n} \langle x \rangle^{-1} \left(u_t^2 + \left|\nabla u\right|^2 + \lambda \left|u\right|^{p+1} \right) dx \\ & = Y(T) + c_0 I(t) \\ & \geq c_0 I(t) \end{aligned}$$

Thus, $Y(t) \geq c_0 I(t)$.

Proof of 3. We now use the Cauchy inequality in (2.14) with $\varepsilon = 1$ and bound the unsigned term uu_tW from above, which gives us

$$uu_tW \leq \frac{1}{2}u_t^2 + \frac{1}{2}u^2 W a(x)$$

We can further simplify this term to a desirable form using basic manipulations and the Hardy inequality:

$$\int_{\mathbb{R}^n} u^2 W a(x)\, dx \leq \int_{\mathbb{R}^n} u^2 \frac{c}{\left(1 + \left|x\right|^2\right)} dx \leq c \int_{\mathbb{R}^n} \frac{u^2}{\left|x\right|^2} dx$$

$$c \int_{\mathbb{R}^n} \frac{u^2}{\left|x\right|^2} dx \leq c \frac{4}{(n-2)^2} \int_{\mathbb{R}^n} \left|\nabla u\right|^2 dx$$

Note that the use of the Hardy inequality requires that $n \geq 3$. This is not a problem, since we have already required this.

Substituting these inequalities into the expression for $Y(t)$, we obtain the following inequality:

$$\begin{aligned} Y(t) \quad & \leq \int_{\mathbb{R}^n} \left(u_t^2 + \left|\nabla u\right|^2 + \tfrac{1}{2}u_t^2 + \tfrac{1}{2}aWu^2 + \tfrac{1}{2}aWu^2 + \tfrac{2\lambda}{p+1} \left|u\right|^{p+1} \right) dx \\ & \leq \int_{\mathbb{R}^n} \left(\tfrac{3}{2}u_t^2 + c_0 \left|\nabla u\right|^2 + + \tfrac{2\lambda}{p+1} \left|u\right|^{p+1} \right) dx \\ & \leq c_1 \int_{\mathbb{R}^n} \left(u_t^2 + \left|\nabla u\right|^2 + + \lambda \left|u\right|^{p+1} \right) dx \\ & = c_1 E(t) \end{aligned}$$

Thus, $Y(t) \leq cE(t)$.

Proof of 4. This follows immediately from inequalities 2 and 3.

Proof of 5. From Theorem 2.11 we have the following:

$$\int_{\mathbb{R}^n} \langle x\rangle^{m+1} \left(u_t\left(t,x\right)^2 + \left|\nabla u\left(t,x\right)\right|^2 + \frac{2\lambda}{p+1}\left|u\left(t,x\right)\right|^{p+1}\right) dx \leq C$$

We must first manipulate the expression for $E\left(t\right)$ so we may split the integral using the Holder inequality:

$$\begin{aligned} E\left(t\right) &= \int_{\mathbb{R}^n} \left(u_t^2 + \left|\nabla u\right|^2 + \tfrac{2\lambda}{p+1}\left|u\right|^{p+1}\right) dx \\ &= \int_{\mathbb{R}^n} \left[\langle x\rangle^{\frac{m+1}{m+2}} \left(u_t^2 + \left|\nabla u\right|^2 + \tfrac{2\lambda}{p+1}\left|u\right|^{p+1}\right)^{\frac{1}{m+2}}\right] \times \\ &\quad \left[\langle x\rangle^{-\frac{m+1}{m+2}} \left(u_t^2 + \left|\nabla u\right|^2 + \tfrac{2\lambda}{p+1}\left|u\right|^{p+1}\right)^{\frac{m+1}{m+2}}\right] dx \end{aligned}$$

We now use the Holder inequality for $p = m+2$, $q = \frac{m+2}{m+1}$:

$$\begin{aligned} E\left(t\right) &\leq \left(\int_{\mathbb{R}^n} \langle x\rangle^{m+1} \left(u_t^2 + \left|\nabla u\right|^2 + \tfrac{2\lambda}{p+1}\left|u\right|^{p+1}\right) dx\right)^{\frac{1}{m+2}} \times \\ &\quad \left(\int_{\mathbb{R}^n} \langle x\rangle^{-1} \left(u_t^2 + \left|\nabla u\right|^2 + \tfrac{2\lambda}{p+1}\left|u\right|^{p+1}\right) dx\right)^{\frac{m+1}{m+2}} \\ &\leq C^{\frac{1}{m+2}} \left(-I'\left(t\right)\right)^{\frac{m+1}{m+2}} \end{aligned}$$

Thus, we have that $E\left(t\right) \leq \left(-I'\left(t\right)\right)^{\frac{m+1}{m+2}}$.

Proof of 6. Recall that $I\left(\frac{t}{2}\right) = \int_{\frac{t}{2}}^{2T} \left(\int_{\mathbb{R}^n} \langle x\rangle^{-1} \left(u_t^2 + \left|\nabla u\right|^2 + \lambda\left|u\right|^{p+1}\right) dx\right) ds$. By finite speed of propagation, $|x| \leq R+T$, and for large enough $T \geq R$, $|x| \leq 2T$. Outside of this radius, our functions are 0. Note also that $E\left(t\right)$ is a decreasing function, so it achieves its maximal value at the upper limit of integration. By this reasoning, we can obtain the following inequalities:

$$\begin{aligned} I\left(\tfrac{t}{2}\right) &\geq \int_{\frac{t}{2}}^{2T} \left(\int_{|x|\leq 2T} \langle x\rangle^{-1} \left(u_t^2 + \left|\nabla u\right|^2 + \lambda\left|u\right|^{p+1}\right) dx\right) ds \\ &\geq E\left(2T\right) \tfrac{1}{\sqrt{1+4T^2}} \int_{\frac{t}{2}}^{2T} ds \\ &\geq c_0 E\left(2T\right) \tfrac{T}{\sqrt{1+4T^2}} \\ &\geq cE\left(t\right) \end{aligned}$$

Thus, our final inequality is proved.

□

Given these relations between $E\left(t\right)$, $I\left(t\right)$, and $Y\left(t\right)$, we can proceed to transfer the weight of the energy from x to t and obtain our energy estimate.

From inequalities 4 and 5, we have that $I(t) \leq c(-I'(t))^{\frac{m+1}{m+2}}$, which is a simple ordinary differential equation. Solving for $I(t)$, we obtain the inequality $I(t) \leq ct^{-m-1}$. From inequality 6, we have the following:

$$E(t) \leq cI(t) \leq ct^{-m-1}$$

So our energy estimate for the problem in (1.1)-(1.3) with conditions (1.4) and (1.5) and $a_1 \geq 1$ is $E(t) \leq ct^{-m-1}$, where $m = \min\{a_1 - 1, n-2\}$. So we have the following energy estimates that depend on the value of a_1:

$$E(t) \leq \begin{cases} ct^{-a_1} & a_1 \leq n-1 \\ ct^{-n+1} & a_1 \geq n-1 \end{cases}$$

Using the constraints on m, we obtain the following unweighted norms on u from the weighted norms in (2.12) and (2.13):

$$\|u\|_2^2 \leq \begin{cases} ct^{-a_1+2} & a_1 \leq n-1 \\ ct^{-n+3} & a_1 \geq n-1 \end{cases} \tag{2.20}$$

$$\|u\|_{p+1}^{p+1} \leq \begin{cases} ct^{-a_1} & a_1 \leq n-1 \\ ct^{-n+1} & a_1 \geq n-1 \end{cases} \tag{2.21}$$

This completes our work on Theorem 1.1.

3 Proof of Theorem 1.2

We first multiply both sides of the equation in (1.1) by $f(t)u_t + g(t)u$, where $f(t)$ and $g(t)$ are non-negative functions that will be specified later. We then integrate the equation over $\mathbb{R}^n$, and get the following result.

Lemma 3.1. *Let u be the weak solution to (1.1)-(1.3) on $[t_0, T_{max}]$. Then it is true that*

$$\frac{d}{dt}E_w(t) + F(t) = 0$$

where

$$E_w(t) := \int_{\mathbb{R}^n} \left(f\left(u_t^2 + |\nabla u|^2 + \frac{2\lambda}{p+1}|u|^{p+1} \right) + 2u_t u g + (ga(x) - g_t)u^2 \right) dx$$

$$F(t) = \frac{1}{2}\int_{\mathbb{R}^n} \left(u_t^2 (2af - 2g - f) + |\nabla u|^2 (2g - f_t) + u^2 (g_{tt} - g_t a) \right) dx$$
$$+ \frac{1}{2}\int_{\mathbb{R}^n} \lambda |u|^{p+1} \left(g - \frac{f_t}{p+1} \right) dx.$$

Proof. We simply multiply (1.1) by the weight $f(t)\,u_t + g(t)\,u$ and then integrate by parts over $\mathbb{R}^n$, noting that because of finite speed of propagation, any divergences will be 0. Multiplying by fu_t gives $fu_tu_{tt} - fu_t\Delta u + afu_t^2 + \lambda |u|^{p-1} uu_t = 0$. Integrate by parts over $\mathbb{R}^n$, noting that:

$$\left(\frac{1}{2}u_t^2 f\right)_t = u_tu_{tt}f + \frac{1}{2}u_t^2 f_t$$

$$\nabla \cdot (fu_t\nabla u) = \nabla f \cdot u_t\nabla u + f \cdot (\nabla u_t \nabla u + u_t\Delta u)$$

$$\Rightarrow 0 = 0 + \frac{1}{2}f\frac{\partial}{\partial t}|\nabla u|^2 + fu_t\Delta u$$

$$\left(\frac{1}{2}f|\nabla u|^2\right)_t = \frac{1}{2}f_t|\nabla u|^2 + \frac{1}{2}f\frac{\partial}{\partial t}|\nabla u|^2$$

$$\Rightarrow 0 = \frac{1}{2}\frac{\partial}{\partial t}\left[f|\nabla u|^2\right] - \frac{1}{2}f_t|\nabla u|^2 + fu_t\Delta u$$

$$\left(\frac{\lambda}{p+1}|u|^{p+1}f\right)_t = \lambda |u|^{p-1}uu_tf + \frac{\lambda}{p+1}f_t|u|^{p+1}$$

Making the above substitutions, we obtain the following equation:

$$\int_{\mathbb{R}^n}\frac{\partial}{\partial t}\left[\frac{1}{2}u_t^2 f + \frac{1}{2}f|\nabla u|^2 + \frac{\lambda}{p+1}f|u|^{p+1}\right]dx +$$

$$\int_{\mathbb{R}^n}\left[-\frac{1}{2}u_t^2 f_t + afu_t^2 - \frac{\lambda}{p+1}f_t|u|^{p+1} - \frac{1}{2}f_t|\nabla u|^2\right]dx = 0 \qquad (3.1)$$

Multiplying (1.1) by gu gives: $guu_{tt} - gu\Delta u + guau_t + \lambda g|u|^{p+1} = 0$. We integrate by parts over $\mathbb{R}^n$, noting that:

$$(u_tug)_t = \frac{1}{2}g_t\frac{\partial}{\partial t}\left(u^2\right) + gu_{tt}u + u_t^2 g$$

$$\left(\frac{1}{2}g_tu^2\right)_t = \frac{1}{2}g_{tt}u^2 + \frac{1}{2}g_t\frac{\partial}{\partial t}\left(u^2\right)$$

$$\Rightarrow (u_tug)_t = \left(\frac{1}{2}g_tu^2\right)_t - \frac{1}{2}g_{tt}u^2 + gu_{tt}u + u_t^2 g$$

$$\nabla \cdot (g\nabla uu) = \Delta\ g\nabla uu + g\Delta uu + g|\nabla u|^2$$

$$\Rightarrow 0 = g\Delta uu + g|\nabla u|^2$$

$$\left(\frac{1}{2}gau^2\right)_t = \frac{1}{2}g_tau^2 + gauu_t$$

Making these substitutions, we get the following equation:

$$\int_{\mathbb{R}^n}\frac{\partial}{\partial t}\left[u_tug - \frac{1}{2}g_tu^2 + \frac{1}{2}gau^2\right]dx +$$

$$\int_{\mathbb{R}^n}\left[\frac{1}{2}g_{tt}u^2 + g|\nabla u|^2 - \frac{1}{2}g_tau^2 + \lambda g|u|^{p+1} - gu_t^2\right]dx = 0 \qquad (3.2)$$

Adding (3.1) and (3.2) together, we get the desired result. □

Lemma 3.2. *Assume that the smooth functions $f(t)$, $g(t)$, and $h(t) > 0$ satisfy the following conditions for $t \geq t_0$,*

1. $2af - 2g - f_t \geq 0$
2. $2g - f_t \geq 0$
3. $g_{tt} - g_t a \geq 0$
4. $g - \frac{f_t}{p+1} \geq 0$
5. $ga - g_t - gh^{-1} \geq 0$.

Then, if u is the weak solution to (1)-(3) on $[t_0, T_{max}]$, the following is true:

$$\frac{1}{2}\int_{\mathbb{R}^n} (f - hg)\left(u_t^2 + |\nabla u|^2 + \frac{1}{p+1}|u|^{p+1}\right) dx \leq E_w(t_0). \tag{3.3}$$

Proof. Applying inequalities 1.-4. to the result of Lemma 3.1, we get that $\frac{\partial}{\partial t}E_w(t) \leq 0$. Integrating with respect to t, from t_0 to t, we get $\frac{\partial}{\partial t}\int_{t_0}^t E_w(s)ds \leq 0$, which implies that $E_w(t) \leq E_w(t_0)$, where

$$E_w(t) = \frac{1}{2}\int_{\mathbb{R}^n}\left[f\left(u_t^2 + |\nabla u|^2 + \frac{2\lambda}{p+1}|u|^{p+1}\right) + 2u_t ug + (2g - g_t)u^2\right] dx.$$

We wish to control the terms in E_w that have no definite sign, so we use Cauchy's inequality on $2u_t ug$, introducing a helper function, $h(t) \geq 0$, such that

$$|2u_t ug| = \left|2u_t u h^{\frac{1}{2}} h^{-\frac{1}{2}} g^{\frac{1}{2}} g^{\frac{1}{2}}\right| \leq hgu_t^2 + \frac{1}{h}gu^2.$$

Taking a lower bound, we get that $-hgu_t^2 - gu^2h^{-1} \leq 2u_t ug$. So:

$$E_w(t) \geq \frac{1}{2}\int_{\mathbb{R}^n}\left[f\left(u_t^2 + |\nabla u|^2 + \frac{2\lambda}{p+1}|u|^{p+1}\right)\right] dx$$

$$+\frac{1}{2}\int_{\mathbb{R}^n}\left[-hgu_t^2 - hg|\nabla u|^2 - hg\frac{2\lambda}{p+1}|u|^{p+1} + \left(2g - g_t - gh^{-1}\right)u^2\right] dx$$

Using inequality 5. and combining terms, we get the expression $E_w(t) \geq \frac{1}{2}\int_{\mathbb{R}^n}(f - gh)\left(u_t^2 + |\nabla u|^2 + \frac{2\lambda}{p+1}|u|^{p+1}\right) dx$, which will be non-negative if $f - gh \geq 0$. □

Choose $f(t)$, $g(t)$, and $h(t)$ as follows:

$$f(t) := (1+t)^{1-\delta},\ g(t) := \frac{1-\delta}{2}(1+t)^{-\delta},\ h(t) := t+1 \tag{3.4}$$

where δ is a positive constant such that $0 < \delta < 1$ and $\delta > 1 - a_1$. Then, one reaches the following lemma.

Lemma 3.3. *Let f, g, and h be defined by (3.4). Then conditions 1-5 of Lemma 3.2 are satisfied for $t \geq t_0$.*

Proof. First we will calculate the derivatives of $f(t)$ and $g(t)$:

$$f_t = (1-\delta)(1+t)^{-\delta}$$

$$g_t = (-\delta)\frac{1-\delta}{2}(1+t)^{-\delta-1}$$

$$g_{tt} = \delta(\delta+1)\frac{1-\delta}{2}(1+t)^{-\delta-1}.$$

We note that $g_t < 0$ and $g_{tt} > 0$.
Proof of 1. From (1.4) and

$$\begin{aligned} 2af - f_t - 2g &= 2a(1+t)^{1-\delta} - (1-\delta)(1+t)^{-\delta} - (1-\delta)(1+t)^{-\delta} \\ &\geq 2(1+t)^{-\delta}(a(1+t) - 1 + \delta) \\ &\geq 2(1+t)^{-\delta}\left(\tfrac{a_1(1+t)}{(1+|x|^2)^{\frac{1}{2}}} - 1 + \delta\right) \\ &\geq 2(1+t)^{-\delta}\left(\tfrac{a_1(1+t)}{1+|x|} - 1 + \delta\right) \\ &\geq 2(1+t)^{-\delta}\left(\tfrac{a_1(1+t)}{1+t+R} - 1 + \delta\right) > 0 \end{aligned}$$

Where the last inequality comes from our choice of δ, so that

$$\lim_{t\to\infty}\left(\frac{a_1(1+t)}{1+t+R} - 1 + \delta\right) = a_1 - 1 + \delta > 0.$$

Proof of 2. Notice that because $f_t(t) = 2g(t)$, this is trivially true.

Proof of 3. Since $g_{tt}(t) > 0$ and $g_t(t) < 0$, this is trivial as well.

Proof of 4. Since $p > 1$, this is trivial, since

$$\begin{aligned} g - \tfrac{f_t}{1+p} &= \left(\tfrac{1-\delta}{2}\right)(1+t)^{-\delta} - \tfrac{1}{p+1}(1-\delta)(1+t)^{-\delta} \\ &= (1-\delta)(1+t)^{-\delta}\left(\tfrac{1}{2} - \tfrac{1}{p+1}\right) > 0 \end{aligned}$$

Proof of 5. Similar to 1, one gets the following:

$$\begin{aligned} ga - g_t - h^{-1}g &= \tfrac{1-\delta}{2}(1+t)^{-\delta}a + \delta\tfrac{1-\delta}{2}(1+t)^{-\delta-1} - \tfrac{1-\delta}{2(1+t)}(1+t)^{-\delta} \\ &= \tfrac{1}{2}(1-\delta)(1+t)^{-\delta-1}((1+t)a + \delta - 1) > 0 \end{aligned}$$

□

Proof of Theorem 1.2. Now that we have chosen our functions f, g, and h, we can easily reach our final energy estimate. From Lemma 4.2, we have that the weighted energy is bounded above by a positive constant. Furthermore, since the weight on the energy is time dependent, while we are integrating over space, we simply move the term to the other side of the inequality, reaching our decay estimate.

$$\begin{aligned}\int_{\mathbb{R}^n}\left(u_t^2+|\nabla u|^2+\tfrac{1}{p+1}|u|^{p+1}\right)dx &\le E_w(t_0)(f-gh)^{-1}\\ &\le E_w(t_0)\left((1+t)^{1-\delta}-\tfrac{1}{2}(1+t)^{1-\delta}\right)\\ &= E_w(t_0)\tfrac{1+\delta}{2}(1+t)^{1-\delta}\end{aligned}$$

From this, we quickly get the estimate stated in Theorem 1.2.

4 Future Areas of Research

While our research has provided a fast rate of decay for problem (1.1)-(1.5), it is not likely that this will be the optimal decay for the entire range of the exponent p of the nonlinearity. It is likely that the decay for small p will be much faster due to the influence of the Klein-Gordon equation. In order to find this decay, we will use an iterative method based on interactions between the L^2 and L^{p+1} norms of u. In (2.20) and (2.21), it is noteworthy that the decay of the L^2 norm of u differs from the decay of the L^{p+1} norm of u by a power of 2. But for p close to 1, the L^{p+1} norm is approximately the same as the L^2 norm. So for small p, we will use the interactions between these two norms in an attempt to show that the energy of the problem decays faster than any polynomial. These interactions will also likely yield a threshold for the decay. Upon completing this work, we will have completely described the energy of the problem.

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