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Bounded Geometry and Property A for Nonmetrizable Coarse Spaces

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To the Graduate Council:

I am submitting herewith a dissertation written by Jared R Bunn entitled "Bounded Geometry and Property A for Nonmetrizable Coarse Spaces." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Jerzy Dydak, Major Professor

We have read this dissertation and recommend its acceptance:

Nikolay Brodskiy, Stefan Richter, Mark Hector

Accepted for the Council:

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Bounded Geometry and Property A for Nonmetrizable Coarse Spaces

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Abstract

We begin by recalling the notion of a coarse space as defined by John Roe. We show that metrizability of coarse spaces is a coarse invariant. The concepts of bounded geometry, asymptotic dimension, and Guoliang Yu's Property A are investigated in the setting of coarse spaces. In particular, we show that bounded geometry is a coarse invariant, and we give a proof that finite asymptotic dimension implies Property A in this general setting. The notion of a metric approximation is introduced, and a characterization theorem is proved regarding bounded geometry.

Chapter 7 presents a discussion of coarse structures on the minimal uncountable ordinal. We show that it is a nonmetrizable coarse space not of bounded geometry. Moreover, we show that this space has asymptotic dimension 0; hence, it has Property A.

Finally, Chapter 8 regards coarse structures on products of coarse spaces. All of the previous concepts above are considered with regard to 3 different coarse structures analogous to the 3 different topologies on products in topology. In particular, we see that an arbitrary product of spaces with any of the 3 coarse structures with asymptotic dimension 0 has asymptotic dimension 0.

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Notation

$\mathbb{N}$	positive integers
$\mathbb{R}$	real numbers
$\mathbb{R}_+$	positive real numbers
$\mathbb{Z}$	integers
$\mathbb{C}$	complex numbers
X^n	n-tuples formed from a set X
$\mathcal{P}(X)$	the power set of X
$\setminus$	set minus
$\subset$	set containment (possibly equality)
$\lfloor x \rfloor$	the greatest integer less than or equal to x
$\lceil x \rceil$	the least integer greater than or equal to x
Δ_X	points of the form (x, x) in X^2 (subscript may be omitted when clear)
$A \Delta B$	the symmetric difference of the sets A and B: $(A \setminus B) \cup (B \setminus A)$
$c(E)$	the smallest coarse structure containing E
S_Ω	the minimal uncountable ordinal
$a \vee b$	the maximum of the numbers a and b
$a \wedge b$	the minimum of the numbers a and b

Chapter 1

Introduction

1.1 Background and Purpose

The general goal of coarse geometry is to obtain large-scale perspective of a set X endowed with some mathematical structure. Typically, X is a metric space; however, the tools of coarse geometry generalize to apply for other (nonmetrizable) structures as well. In [Roe03], John Roe sets forth a general theory of coarse structures motivated by uniformly bounded families in the metric space setting. One of the motivations he mentions for this approach comes from its analogy to topology. There is much benefit from the abstract viewpoint of topological spaces; one can avoid some of the details with epsilons and deltas and focus on the overall notion of continuity without any extra complications.

In coarse geometry, we can mimic the abstraction in topology and evolve from metric spaces to coarse spaces. Much of the basic theory is detailed in Chapter 2, providing a backbone for the later chapters. Since a metric is a function on the square of a set X, a coarse structure is defined on X^2. The most fundamental example is the *bounded coarse structure*, which is the coarse structure induced by a metric. In this case, one can think of every controlled set being paired with a number describing a uniformly bounded family. As with every area of mathematics, we will not only be concerned with coarse spaces, but we will also need to discuss the morphisms between them, called *coarse maps*, which leads to the notion of a *coarse equivalence*, the isomorphism in the coarse category. With the theory intact, we will need a collection of coarse invariants in order to distinguish between examples of coarse spaces, up to coarse equivalence.

The first such coarse invariant discussed is *metrizability*. There is a relatively simple criterion for metrizability: a coarse space is metrizable if and only if its coarse structure is countably generated [Roe03, p. 34]. One can think of a generating set for a coarse structure $\mathcal{C}$ as a collection of sets $\mathcal{D} := \{C_\alpha\}_\alpha$ for which the smallest coarse structure containing $\mathcal{D}$ equals $\mathcal{C}$. We carefully show that metrizability is a coarse invariant by exploiting this criterion.

Next, Chapter 3 contains background on the coarse invariant of *bounded geometry*. Roughly speaking, a discrete metric space is of bounded geometry if each ball of a prescribed size has the same (finite) maximum number of points. In more general settings, we say that a metric (or coarse) space is of bounded geometry if there is a *gauge* E, which acts as a minimal distance, so that for each ball of a prescribed size, there is a maximum number of points that, pairwise, are not in E. In the metric case, saying a pair of points is not in E amounts to saying that they are r-separated. The main goal of Chapter 3 outside of introducing bounded geometry is to carefully verify that it is a coarse invariant in both the metric and coarse settings.

In Chapter 4, we consider Property A, which is also a coarse invariant. Property A, first conceived by Guoliang Yu in [Yu00], gives a sufficient condition for a discrete metric space to coarsely embed into a Hilbert space. This has proven to be very significant in proving results regarding the Baum-Connes conjecture and the Novikov conjecture, which are outside the scope of this dissertation. We will instead focus on the definition, equivalent definitions, and definitions in the more general setting of coarse spaces with hopes of finding interesting examples, with particular interest in those that are nonmetrizable coarse spaces. These more general ideas are considered in the later chapters.

As alluded to in the last paragraph, we seek to study the coarse invariants bounded geometry and Property A in the setting of nonmetrizable coarse spaces. Much research has been done in the metric setting, see [CDV08], [Now07], and [Wil09]. In particular, the latter paper has an excellent bibliography and a brief set of notes describing results regarding Property A from those references. Our main focus will be on developing some theory in the nonmetrizable setting and trying to construct examples.

1.2 Results

Our first attempt to understand nonmetrizable spaces utilizes *metric approximations*, as discussed in Chapter 5. These are metrizable coarse structures contained in a given nonmetrizable coarse structure that approximate the larger structure, with a suitable notion of approximation. The goal of such an approach is to extend results for metrizable spaces to nonmetrizable spaces via approximations. The single result obtained is

Theorem 1.1. *A coarse space $(X, \mathcal{C})$ is of bounded geometry with gauge E if and only if there exists a metric approximation $(X, \mathcal{C}_1)$ of bounded geometry with gauge E and any approximation $(X, \mathcal{C}_\alpha)$ with $C_1 \subset C_\alpha$ is of bounded geometry with gauge E.*

While considering nonmetrizable structures to approximate, infinite sets with the *discrete* coarse structure were considered since they are nonmetrizable. This is discussed in Chapter 2. This suggested studying S_Ω, the minimal uncountable ordinal. In Chapter 6, we begin by introducing a nonmetrizable coarse structure on S_Ω called the *ordered coarse structure*. We show that this structure is not of bounded geometry but has Property A. This structure does not appear to be very useful since it uses only the well-order on S_Ω and not the addition operation; J. Dydak suggested a different approach that incorporates ordinal addition.

We take a brief detour to discuss *asymptotic dimension*. Although not a main focus of the dissertation, the asymptotic dimension of a coarse space is important due to its connection with Property A. For example, in [CDV08], the authors present a proof that finite asymptotic dimension implies Property A in the metric setting. This is the main result of Chapter 6, except that we generalize the proof to the coarse setting.

In Chapter 7, we give a brief review of ordinal addition, and then we formulate two coarse structures: the *pivotal coarse structure* $\mathcal{C}_p$ and the *translational coarse structure* $\mathcal{C}_t$. Both structures are nonmetrizable, although the controlled sets mimic uniformly bounded families from a metric space. The former is shown to be the same as the ordered coarse structure, while the latter is shown to properly contain the ordered coarse structure. We proceed to show

Theorem 1.2. *The coarse space* $(S_\Omega, \mathcal{C}_t)$ *is not of bounded geometry.*

Theorem 1.3. *The coarse space* $(S_\Omega, \mathcal{C}_t)$ *has asymptotic dimension 0.*

Thus, $(S_\Omega, \mathcal{C}_t)$ is not of bounded geometry and has Property A, using the result from Chapter 6.

We conclude the dissertation with Chapter 8, which deals with product structures. Coarse structures on finite products are defined in [Roe03]. Some results regarding asymptotic dimension on finite products are investigated in [BD08a] and [Gra06]. We define three coarse structures on an arbitrary product of coarse spaces. These three structures coincide for finite products, but the structures are different in general, similar to the situation with the box, uniform, and product topologies [Mun00]. One of the main results gives a way to construct nonmetrizable spaces using an infinite product. Combining this with the following theorem, we can construct nonmetrizable spaces with Property A using the *product coarse structure.*

Theorem 1.4. *Suppose* $(X_\alpha, \mathcal{C}_\alpha)$ *are coarse spaces for each* $\alpha \in A$*. Create a product coarse space by setting* $Y = \prod_\alpha X_\alpha$ *and equip* Y *with any of the three coarse structures* $\mathcal{D}_*$*. If* $asdim(X_\alpha) = 0$ *for all* α*, then* $asdim(Y) = 0$*.*

Chapter 2

Coarse Spaces

2.1 The Coarse Category

We begin with the basic definitions needed to understand the coarse category.

Definition 2.1. Let X be a set. The **product** of two sets $C, D \subset X^2$, denoted $C \circ D$ is given by

$$C \circ D = \{(x, y) \in X^2 \,|\, \exists z \in X \ni (x, z) \in C, (z, y) \in D\}.$$

For multiple products, we will use the notation C^n. This should not be confused with set-theoretic powers.

Definition 2.2. Let X be a set. A **coarse structure** on X is a collection of subsets of $\mathcal{C} \subset \mathcal{P}(X^2)$ satisfying $\Delta \in \mathcal{C}$ in addition to the following four closure properties:

1. $C \in \mathcal{C} \Rightarrow D \in \mathcal{C}$ for any $D \subset C$ (closed under subsets)
2. $C \in \mathcal{C} \Rightarrow C^t \in \mathcal{C}$ (closed under transpositions)
3. $C, D \in \mathcal{C} \Rightarrow C \cup D \in \mathcal{C}$ (closed under finite unions)
4. $C, D \in \mathcal{C} \Rightarrow C \circ D \in \mathcal{C}$ (closed under finite products)

A **coarse space** is a set X endowed with a coarse structure $\mathcal{C}$. The sets in $\mathcal{C}$ are called **controlled sets**. Any subset B of X for which B^2 is controlled is called **bounded**. A coarse space is **connected** if every point $(x, y) \in X^2$ lies in some controlled set. Note that if X is a coarse space and $Y \subset X$, then Y has a coarse structure comprised of the sets $C \cap Y^2$ where C is controlled in X.

Definition 2.3. Let X, Y be coarse spaces.

1. A map $f : X \to Y$ is said to be **bornologous** if f^2 maps controlled sets in X to controlled sets in Y.
2. A map $f : X \to Y$ is said to be **proper** if $f^{-1}(B)$ is bounded for any bounded set of Y.
3. A map that is both bornologous and proper is said to be a **coarse map**.

Most of the time, we will use f in place of f^2 when the context is clear.

We can now describe the coarse category as the class whose objects are coarse spaces and morphisms are coarse maps. Of course, we also want to understand what an isomorphism is between coarse spaces. Since coarse geometry focuses on large-scale properties of a space, we will not need a composition to be equal to the identity, only to be *close* to the identity, as defined below.

Definition 2.4. Let X and Y be coarse spaces.

1. Two maps $f, g : K \to Y$ are **close** if the set

$$\{(f(x), g(x)) \mid x \in K\}$$

is controlled in Y.

2. Two coarse spaces X and Y are **coarsely equivalent** if there exists coarse maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are close to their respective identity maps. We say that g is the **coarse inverse** of f.

Next we generalize the concept of balls into the coarse setting, then we build the idea of a *coarse embedding*.

Definition 2.5. Let X, Y be coarse spaces.

1. Given a controlled set $E \subset X^2$, an **E-ball** about a point $x \in X$ is described by either of the following two sets

$$E_x = \{z \mid (z, x) \in E\} \qquad \text{or} \qquad E^x = \{z \mid (x, z) \in E\}.$$

Note that if E is **symmetric** ($E = E^t$), then $E_x = E^x$. Also, $B \subset X$ is bounded if and only if $B = E_x$ for some controlled set E in X.

2. Given a controlled set $E \subset X^2$ and a set $K \subset X$, an **E-neighborhood** of K by E is given by the set

$$E[K] = \{z \in X \mid (z, x) \in E \text{ for some } x \in K\}.$$

We will usually be interested in the case where E is symmetric so that $E[K] = E^t[K]$.

Definition 2.6. Let X, Y be coarse spaces.

1. A map $f : X \to Y$ is said to be **effectively proper** if for every controlled set E in Y, there exists a controlled set F in X such that for every $x \in X$,

$$f^{-1}(E_{f(x)}) \subset F_x.$$

2. A map $f : X \to Y$ is a **coarse embedding** if f is a coarse equivalence onto its image in Y.

Recall that a proper map pulled back bounded sets to bounded sets, without any uniform control on the "size" of the bounded sets. An effectively proper map forces such a uniform control. It is not hard to see that an effectively proper map is also proper. We now see how to characterize coarse embeddings.

Proposition 2.7. *A map $f : X \to Y$ is a **coarse embedding** if and only if f is bornologous and effectively proper.*

Proof. Let $\mathcal{C}$, $\mathcal{D}$ be the coarse structures for X and Y, respectively. Moreover, let $\mathcal{D}_r$ be the coarse structure on $f(X)$ given by the sets $D \cap f(X)^2$ with $D \in \mathcal{D}$.

Suppose that f is a coarse embedding. Then f is trivially bornologous onto $(Y, \mathcal{D})$ since $\mathcal{D}_r \subset \mathcal{D}$. Let $g : f(X) \to X$ be the coarse inverse of f (with the range restricted to $f(X)$), and let $W \in \mathcal{C}$ be the controlled set $W = \{(x, g(f(x))) \,|\, x \in X\}$. Let $D \in \mathcal{D}$, and define $E \in \mathcal{C}$ by $E = g(D \cap f(X)^2)$. We will show that for $C = W \circ E \circ W^t$

$$f^{-1}(D_{f(x)}) \subset C_x$$

for all $x \in X$. Let $z \in f^{-1}(D_{f(x)})$. Then $(f(z), f(x)) \in D \cap f(X)^2$. Thus, $(z, x) \in W \circ E \circ W^t = C$; hence, $z \in C_x$. Therefore, f is effectively proper.

Conversely, suppose that f is bornologous and effectively proper. We want to show that f is a coarse equivalence onto $f(X)$. We will continue to use the symbol f in this context. As before, f is immediately bornologous and proper (being effectively proper). To define a map $g : f(X) \to X$, we use the axiom of choice. Given $f(x) \in f(X)$, define $g(f(x))$ as any point in $f^{-1}(\{f(x)\})$. Then, $f(g[f(x)]) = f(x)$, that is, $f \circ g$ equals the identity on $f(X)$. Using that f is effectively proper, there exists a controlled set $C \in \mathcal{C}$ such that

$$f^{-1}((\Delta_Y)_{f(x)}) \subset C_x$$

for all $x \in X$. By the definition of g, $f(g[f(x)]) = f(x)$. Thus, $g(f(x)) \in C_x$ for all $x \in X$. Therefore, $g \circ f$ is close to the identity on X.

To conclude the proof, we need to show that g is bornologous and proper. Let $D \in \mathcal{D}_r$. Since f is effectively proper, there exists $C \in \mathcal{C}$ such that

$$f^{-1}(D_{f(x)}) \subset C_x$$

for all $x \in X$. Then $g(D) \subset C$. For if $(g(y), g(w)) \in g(D)$ with $(y, w) \in D$, then $(f(g(y)), f(g(w))) = (y, w) \in D$; in other words, $g(y) \in f^{-1}(D_{f(g(w))})$. So $g(y) \in C_{g(w)}$, yielding the desired result.

To see that g is proper, start with a bounded set $B \subset X$. There exists $C \in \mathcal{C}$ such that $B = C_x$ for some $x \in X$. Let $D = f(C) \in \mathcal{D}_r$. Then $g^{-1}(B) \subset D_{f(x)}$. For if $g(y) \in B = C_x$, then $(y, f(x)) = (f(g(y), f(x)) \in D$. Since $D^2_{f(x)}$ is controlled, so is $g^{-1}(B)^2$. Hence, $g^{-1}(B)$ is bounded. □

2.1.1 Examples

There are few basic examples to know. Let X be a set. The first (trivial) example of a coarse structure on X is the power set of X^2, called the **maximal coarse structure**. Another is the collection $\mathcal{C}_{\text{dis}}$ consisting of all sets containing only finite many points off the diagonal Δ; this is called the **discrete coarse structure** on X. The discrete coarse structure is the smallest, connected coarse structure on X. See [Roe03] for many other examples.

Perhaps the most fundamental nontrivial example of a coarse space is a metric space (X, d) endowed with the **bounded coarse structure**. This is the structure consisting of all sets C such that

$$\sup\{d(x, y) \,|\, (x, y) \in C\} < \infty.$$

Another interesting example is the following, which is based on a large-scale structure in [DH08]. Let G be a finitely generated group. Define a coarse structure $\mathcal{C}$ on G by declaring $C \in \mathcal{C}$ if there is a finite subset F of G such that for all $(x, y) \in C$, $x = f_1 + z$ and $y = f_2 + z$ for some $f_1, f_2 \in F$ and $z \in G$. It is easy to check that $\Delta_G \in \mathcal{C}$ and that $\mathcal{C}$ is closed under subsets and transpositions. If $C, D \in \mathcal{C}$ with respect to finite sets F_1 and F_2, then $C \cup D$ is controlled with respect to $F := F_1 \cup F_2$. To see that $C \circ D \in \mathcal{C}$, start by enlarging F_1 and F_2 by including inverses and the identity of G. Then let $F = (F_1 + F_1 + F_2) \cup F_2$, which is still finite. Then if $(x, y) \in C \circ D$, there exists $w \in G$ such that $(x, w) \in C$ and $(w, y) \in D$. So there exists $f_1, f_2 \in F_1$, $f_3, f_4 \in F_2$, and $z, z_1 \in G$ such that $x = f_1 + z$, $y = f_4 + z_1$, and $w = f_2 + z = f_3 + z_1$. We can express $z = -f_2 + f_3 + z_1$. Thus, $x = f_1 - f_2 + f_3 + z_1$ where $f_1 - f_2 + f_3 \in F_1 + F_1 + F_2 \subset F$. Therefore, $C \circ D$ is controlled.

2.2 Subbases and Bases

Our goal is to set up a system of thinking about coarse structures in a similar way to topological structures. That is, just as a topology can be created by moving from a subbasis to a basis and then to a topology, a coarse structure should be able to be created in a similar fashion. This is directly related to the notion of metrizability.

Definition 2.8. A coarse space X is **metrizable** if its coarse structure is the bounded coarse structure induced by some metric on X.

The idea will come from John Roe's proof that a countably generated coarse structure is metrizable. In that proof, one takes the countable generating set for the coarse structure and creates a new generating set that satisfies some "nicer" properties. It is exactly this process that we will think of as moving from a subbasis to a basis.

Definition 2.9. A **subbasis** for a coarse structure on a set X is any collection of subsets $\mathcal{S}$ of X^2.

We can consider the smallest coarse structure containing a subbasis $\mathcal{S}$ to obtain a coarse structure on X, which is given by the intersection of all coarse structures on X containing the collection $\mathcal{S}$. We will denote this coarse structure by $c(\mathcal{S})$. The downside to creating coarse structures this way is that it is not very clear how the controlled sets are built from the sets in $\mathcal{S}$. The next definition clarifies the situation a bit.

Definition 2.10. A **basis** for a coarse structure on a set X is any collection of subsets $\mathcal{F}$ such that

$$\{C \mid C \subset F, F \in \mathcal{F}\}$$

is a coarse structure on X. We call this structure the coarse structure **induced** by $\mathcal{F}$

In other words, a basis $\mathcal{F}$ on X generates a coarse structure on X where each controlled set $C \subset F$ for some $F \in \mathcal{F}$. This makes $c(\mathcal{F})$ is easy to describe.

2.2.1 Countable Subbases

Before we discuss countable bases, meaning *infinite* countable subbases, let us consider finite bases. Note first that $c(S_1, \ldots, S_n) = c(S_1 \cup \cdots \cup S_n)$ for any finite collection of subsets of X^2. This means that any coarse structure generated by a finite subbasis is actually generated by one set. Such structures are called **monogenic**. In [Roe03, p. 34], it is shown that a coarse space X is monogenic if and only if X is coarsely equivalent to a geodesic metric space.

We now present Roe's aforementioned proof that we can upgrade to a basis from a subbasis whenever $\mathcal{S}$ is countable. When $\mathcal{S}$ is infinite, the idea is to enumerate $\mathcal{S}$ as $S_1, S_2, \ldots$ and to inductively define the (countable) basis $\mathcal{F}$ by $F_0 = \Delta$ and

$$F_n = F_{n-1} \circ F_{n-1} \cup S_n \cup S_n^t.$$

When $\mathcal{S}$ is finite, we define $S = S_1 \cup \cdots \cup S_n$, and let

$$F_n = F_{n-1} \circ F_{n-1} \cup S \cup S^t.$$

The proof that $\mathcal{F}$ is a basis involves showing the special chain relation:

$$\Delta \subset F_{n-1} \subset F_{n-1} \circ F_{n-1} \subset F_n.$$

To show this relation, one uses induction on the truth of the whole chain to verify the first inclusion. Then the other inclusions follow in a straightforward manner. Note that the definition of F_n above implies that $F_n = F_n^t$ (by induction).

Once we have this chain relation, we can easily prove that $\mathcal{F}$ is a basis and that the induced coarse structure equals $c(\mathcal{F}) = c(\mathcal{S})$. In particular, use induction to show $c(\mathcal{F}) \subset c(\mathcal{S})$.

The benefit of starting with a countable basis is that one can define a metric d on X so that the bounded coarse structure induced by d coincides with $c(\mathcal{S})$. This metric is given by

$$d(x, y) = \min\{n \mid (x, y) \in F_n\},$$

where the presence of infinite values simply means that the coarse structure is not connected. So, as Roe showed, a coarse space is metrizable if and only if it is countably generated.

The purpose of this section is clarify that if one wants to work with metrizable coarse structures by considering a countable subbasis, it is convenient to upgrade to a basis in order to adequately describe the controlled sets in the coarse structure.

It seems that one should just use the metric to describe the controlled sets. However, the metric is awkward to work with since we aren't explicitly given it. To give the explicit definition of the metric, one must perform the process of upgrading to a basis anyway. Therefore, we might as well work with the coarse structure itself, interpreting it as being induced by the basis. This allows us to use the more general coarse setting, rather than working in the metric setting. Moreover, the applications are facilitated by using the basis sets rather than the subbasis sets, since we can easily describe the controlled sets as subsets of the basis sets.

2.3 Metrizability as a Coarse Invariant

Our current goal is to prove that metrizability is a coarse invariant. Roe implies that this is true through his argument that the topological coarse structure of a locally compact topological space with a second countable compactification is *nonmetrizable* [Roe03, p. 33]. We would eventually like to produce a precise proof of this being an invariant. The following definition and lemma will help us to get started. Note that in what follows, we commonly abuse notation and write f when we mean f^2. Context should make the meaning clear.

Definition 2.11. Given set X, a coarse space $(Y, \mathcal{D})$, and a function $f : X \to Y$, the **pull-back** coarse structure on X is induced by the basis

$$\mathcal{F} = f^{-1}(\mathcal{D}) = \{f^{-1}(D) \mid D \in \mathcal{D}\}.$$

We omit the proof that $\mathcal{F}$ is a basis, since the argument is similar to the one given in the proof of the lemma below.

Lemma 2.12. *Suppose Y is a coarse space with metrizable coarse structure $\mathcal{D}$ and $f : X \to Y$ is a map. If $\mathcal{B}$ is the basis upgraded from the subbasis of the countable set of generators for $\mathcal{D}$, then $f^{-1}(\mathcal{B})$ is a basis for the pull-back coarse structure $\mathcal{E}$.*

Proof. Recall that $\mathcal{E}$ has the basis $f^{-1}(\mathcal{D})$. We want to describe $\mathcal{E}$ as subsets of

$$\{f^{-1}(B_i) \mid B_i \in \mathcal{B}\}.$$

We apply the construction process detailed above to describe $\mathcal{D}$ in terms of $\mathcal{B}$ so that

$$\mathcal{D} = c(B_i)$$

and each $D \in \mathcal{D}$ is a subset of B_i for some i.

First we show that

$$\{f^{-1}(B_i) \mid B_i \in \mathcal{B}\}$$

is indeed a basis. The diagonal of X is in this collection since $\Delta_X \subset f^{-1}(\Delta_Y)$. Closure under subsets is trivial. By construction, $B_i = B_i^t$ for all i; thus if $C \subset f^{-1}(B_i)$ for some i, then $C^t \subset f^{-1}(B_i)$ as well.

If $C_j \subset f^{-1}(B_{i_j})$ for $j = 1, 2$, then

$$C_1 \cup C_2 \subset f^{-1}(B_{i_1} \cup B_{i_2}) \subset f^{-1}(B_{i_1 \vee i_2})$$

and

$$C_1 \circ C_2 \subset f^{-1}(B_{i_1} \circ B_{i_2}) \subset f^{-1}(B_{i_1 \vee i_2} \circ B_{i_1 \vee i_2}) \subset f^{-1}(B_{i_1 \vee i_2 + 1}).$$

This finishes verifying that $f^{-1}(\mathcal{B})$ is a basis.

To finish, we show that the coarse structure induced by $f^{-1}(\mathcal{B})$ coincides with $\mathcal{E}$. Suppose $E \subset f^{-1}(B_i)$ for some i. Since each $B_i \in \mathcal{D}$, $E \subset f^{-1}(B_i) \in f^{-1}(\mathcal{D})$ implies that $E \in \mathcal{E}$. Conversely, let $C \subset f^{-1}(D)$ for some $D \in \mathcal{D}$. By hypothesis, $D \subset B_i$ for some i. Hence, $C \subset f^{-1}(D) \subset f^{-1}(B_i)$. Therefore, $f^{-1}(\mathcal{B})$ is a basis for the coarse space $(X, \mathcal{E})$. □

The next lemma completes the machinery needed to prove that metrizability is a coarse invariant.

Lemma 2.13. *Let X be a coarse space with coarse structure $\mathcal{C}$, and let Y be a set. Suppose there exists maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is close the the identity on X. Then there exists a unique coarse structure on Y such that f and g are coarse equivalences with $f = g^{-1}$ in the coarse sense.*

Proof. We start by showing uniqueness. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two coarse structures on Y such that f and g are coarse equivalences. Let

$$F = \{(y, fg(y)) \mid y \in Y\}.$$

Furthermore, enlarge F to be symmetric.* Then by hypothesis, F is controlled with respect to either coarse structure on Y. Suppose that $D \in \mathcal{D}_1$. Then $D \subset F \circ fg(D) \circ F$, which is controlled with respect to either coarse structure. Thus, $\mathcal{D}_1 \subset \mathcal{D}_2$. The other inclusion follows by a similar argument.

For existence, we define a coarse structure $\mathcal{D}$ on Y using the pull-back structure via g. First we prove that $f \circ g$ is close to the identity on Y. Consider F as above; we want to show that $F \subset g^{-1}(E)$ where

$$E = \{(x, gf(x)), x \in X\}^\dagger$$

is controlled in X since $g \circ f$ is close to the identity on X. This is simple; for if $(y, fg(y)) \in F$, then

$$g(y, fg(y)) = (g(y), g[fg(y)]) = (g(y), gf[g(y)]) \in E.$$

Next we show that f and g are bornologous. If $D \in \mathcal{D}$, then by definition $D \subset g^{-1}(C)$ for some $C \in \mathcal{C}$. Hence, $g(D) \subset gg^{-1}(C) \subset C$, showing the $g(D)$ is controlled in X. Thus, g is bornologous. If $C \in \mathcal{C}$, then $f(C) \subset g^{-1}(E \circ C \circ E) \in \mathcal{D}$. For if $(c_1, c_2) \in C$, then $(gf(c_1), c_1) \in E$ and $(c_2, gf(c_2)) \in E$, meaning that $(f(c_1), f(c_2)) \in g^{-1}(E \circ C \circ E) \in \mathcal{D}$. Hence, f is bornologous.

Lastly, we show that f and g are proper. If $B \subset Y$ is bounded, then $B^2 \subset g^{-1}(C)$ for $C \in \mathcal{C}$, and since g is bornologous

$$f^{-1}(B) \times f^{-1}(B) \subset E \circ g(B^2) \circ E \in \mathcal{C}$$

by an argument similar to the one at the end of the previous paragraph. Hence, f is proper since the preimage of a bounded set is bounded. Similarly, if $A \subset X$ is bounded, then

$$g^{-1}(A) \times g^{-1}(A) \subset F \circ f(A^2) \circ F \in \mathcal{D},$$

where F is controlled in Y^2 since we have already shown that f is bornologous. This concludes the proof. $\square$

We now reach our goal through the following theorem.

Theorem 2.14. *Suppose $f : X \to Y$ is a coarse equivalence between coarse spaces where X is metrizable. Then Y is metrizable.*

Proof. Let $g : Y \to X$ be the coarse inverse of f. Then by the first lemma, we obtain a countable basis for the pull-back coarse structure defined on Y^2 via g. By the uniqueness part of the previous lemma, we must have that the initial coarse structure on Y is precisely this structure. Therefore, Y is metrizable. $\square$

*This is done by redefining F as $F \cup F^t$.

†Enlarge E to be symmetric.

2.3.1 Examples

Any countable set X equipped with the discrete coarse structure is metrizable. An easy example is $X = \mathbb{N}$. The easiest basis to work with consists of $F_0 = \Delta_{\mathbb{N}}$ and

$$F_n = \{(x, y) \mid x \vee y \leq n\}.$$

Thus, an explicit metric inducing $\mathcal{C}_{\text{dis}}$ on $\mathbb{N}$ is given by

$$d(x, y) = \begin{cases} x \vee y & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

Note that any uncountable set X equipped with the discrete coarse structure is nonmetrizable. This is because any countable basis $\mathcal{C}$ for the *connected* coarse structure $\mathcal{C}_{\text{dis}}$ satisfies

$$\bigcup_{C \in \mathcal{C}} C \setminus \Delta = X^2 \setminus \Delta;$$

the left hand hand side is countable, while the right hand side is uncountable, a contradiction. This will benefit us later when we need examples of nonmetrizable coarse spaces in order to have nontrivial metric approximations.

Chapter 3

Bounded Geometry

3.1 Metric Space Setting

This chapter will detail the notion of bounded geometry, beginning with the metric setting, and moving toward the coarse setting. First we start with our standard definition for discrete metric spaces.

Definition 3.1. A discrete metric space is said to be of **bounded geometry** if there exists a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that every ball of radius r contains at most $f(r)$ points.

Example 3.2. Any finitely generated group with the word metric is of bounded geometry [dlH00, p. 104].

We have proposed that the definition for a (nondiscrete) metric space is the following.

Definition 3.3. A metric space is said to be of **bounded geometry** if every ε-net is of bounded geometry for all $\varepsilon > 0$.

An ε-net is a subset which is both ε-separated and covers the space when ε-balls are taken. In [BS07], Buyalo and Schroeder give the following definition.

Definition 3.4. A metric space is said to be of **BS-bounded geometry** if there exists a constant $r \geq 0$ and a function $M : [1, \infty) \to \mathbb{N}$ such that for every $\rho \geq 1$, each ball of radius $r\rho$ can be covered by $M(\rho)$ balls of radius r.

It follows that if a space is of BS-bounded geometry with the constant r, then it does with any larger constant as well. Note that discrete spaces of (discrete) bounded geometry are of BS-bounded geometry with $r = 0$. There are discrete spaces of (discrete) bounded geometry that are not of BS-bounded geometry. Consider the sequence $\{1/n\}$ viewed as a metric space X. The space X is discrete but *not* uniformly discrete as $1/n$ converges to 0 in $\mathbb{R}$, and X is clearly not of (discrete) bounded geometry. However, X is of BS-bounded geometry since the space is bounded, that is, we can cover X with a single ball of radius 1 centered at any point of X.

It isn't too hard to see that our definition of bounded geometry is stronger than Buyalo and Schroeders' definition. Here is the proof.

Proposition 3.5. *If (X, d) is of bounded geometry, then it is of BS-bounded geometry.*

Proof. Let Y be any 1-net of X. Let $r = 1$ and let $M : [1, \infty) \to \mathbb{N}$ be defined by $M(\rho) = f(\rho + 2)$. Given a ball $B(x, \rho)$ in X, we show that it can be covered by $M(\rho)$ balls of radius 1. We can cover $B(x, \rho)$ with the balls $B(y, 1)$, where each y lies in $B(x, \rho+1) \cap Y$. This is true since for each $x_1 \in B(x, \rho)$, there exists $y_1 \in Y$ such that $d(x_1, y_1) < 1$; hence, $d(x, y_1) < \rho + 1$.

We finish the proof by showing that there are only $M(\rho)$ elements of Y in $B(x, \rho + 1)$, thereby placing an upper bound on the number of balls $B(y, 1)$. Choose $y_0 \in Y$ such that $d(x, y_0) < 1$. Then if $y \in B(x, \rho + 1) \cap Y$, we have

$$d(y, y_0) \leq d(y, x) + d(x, y_0) < \rho + 1 + 1 = \rho + 2.$$

Therefore, $B(x, \rho + 1) \cap Y \subset B(y_0, \rho + 2) \cap Y$. Since $|B(y_0, \rho + 2) \cap Y| \leq f(\rho + 2) = M(\rho)$, we know that $B(x, \rho)$ can be covered by $M(\rho)$ balls of the type $B(y, 1)$ where $y \in Y$. □

The converse is not entirely true. We only get a partial result. This isn't too surprising since our definition doesn't have a large-scale feel.

Proposition 3.6. *If (X, d) is of BS-bounded geometry with constant $r > 0$, then every ε-net, $\varepsilon \geq 2r$, is of bounded geometry.*

Proof. Let Y be an ε-net with $\varepsilon \geq 2r$. Also let $f(R) = M(R/r)$ for $R \geq r$ and $f(R) = M(1)$ for $R < r$. Consider the ball $B(y, R) \subset Y$. If $R \geq r$, then by hypothesis we can cover $B(y, R)$ with $M(R/r)$ balls of radius r, considering everything as a subset of X. Each of these balls can contain at most one point from Y. For if y_1, y_2 lie in any such ball, then $d(y_1, y_2) < 2r$; however, $d(y_1, y_2) \geq \varepsilon \geq 2r$ since Y is ε-separated. We conclude that $B(y, R)$ can have no more than $M(R/r) = f(R)$ points of Y.

If $R < r$, then $B(y, R) \subset B(y, r)$. We then use the argument above for the case of $R = r$ to show here that $B(y, R)$ contains at most $f(R) = M(1)$ points of Y. □

From this point on, bounded geometry means in the sense of Buyalo and Schroeder unless specifically mentioned otherwise. At this point, we will present an equivalent form of bounded geometry which is formulated in terms of the maximal number of r-separated points in a ball, that is, points whose distance is *strictly* greater than r. This form of the definition will be generalized later to coarse spaces; thus, we will use this definition from now on in the metric space setting.

Proposition 3.7. *A metric space X is of bounded geometry if and only if there exists a constant $r' \geq 0$ and a function $M' : [1, \infty) \to \mathbb{N}$ such that for every $\rho' \geq 1$, each ball of radius $r'\rho'$ contains at most $M'(\rho')$ points that are r'-separated.*

Proof. Suppose that this property holds. Let $r = r'$ and $M = M'$. For any $B(x, \rho r)$, consider a maximal r-separated set S_ρ. Then the collection of r-balls centered at points of S_ρ forms a cover of $B(x, \rho r)$ using only $M(\rho)$-balls. Thus, X is of bounded geometry.

Suppose that X is of bounded geometry with constant r and function M. Then the above property holds with $r' = 2r$ and $M'(\rho') = M(2\rho')$. This is because any $B(x, \rho' r') = B(x, 2\rho' r)$ can be covered by $M(2\rho')$ balls of radius r, which corresponds to $M'(\rho')$ balls of radius $r'/2$. Each ball in this cover can contain at most one point from any r'-separated set. □

In the case of *discrete* metric spaces, we have the following results. The proofs are straightforward.

Proposition 3.8. *If X is of bounded geometry, then so is any subset $Y \subset X$.*

Proposition 3.9. *Suppose $f : X \to Y$ is an effectively proper map and X is of bounded geometry. Then $f(X)$ is of bounded geometry.*

Proposition 3.10. *Suppose $f : X \to Y$ is an injective bornologous map and Y is of bounded geometry. Then X is of bounded geometry.*

Proposition 3.11. *Suppose $f : X \to Y$ is a coarse equivalence and that Y is of bounded geometry. Then there exists $Z \subset X$ such that Z is of bounded geometry with inclusion being a coarse equivalence.*

The definitions for bornologous, proper, and effectively proper in the metric setting are given via the bounded coarse structure induced by the metric. More specifically,

Definition 3.12. Let (X, d_X) and (Y, d_Y) be metric spaces.

1. A map $f : X \to Y$ is said to be **bornologous** if there exists a function $b : \mathbb{R}_+ \to \mathbb{R}_+$ such that for every $R > 0$, $d_X(x_1, x_2) \le R$ implies $d_Y(f(x_1), f(x_2)) \le b(R)$ for all $x_1, x_2 \in X$. We call the function b the **bornologous control function** for f.

2. A map $f : X \to Y$ is said to be **proper** if for each $x \in X$, there exists a function $p_x : \mathbb{R}_+ \to \mathbb{R}_+$ such that for every $S > 0$, $d_Y(f(x_1), f(x)) \le S$ implies $d_X(x_1, x) \le p(S)$ for all $x_1 \in X$.

3. A map $f : X \to Y$ is said to be **effectively proper** if there exists a function $p : \mathbb{R}_+ \to \mathbb{R}_+$ such that for every $S > 0$,

$$f^{-1}(B(f(x), S)) \subset B(x, p(S))$$

for all $x \in X$. We call the function p the **effective control function** for f.

The next step was to try and prove these four propositions in the general setting of nondiscrete metric spaces. It turns out to work for the B-S definition of bounded geometry, with some additional hypotheses. Proposition 3.13 is straightforward, and the Propositions 3.15 and 3.16 follow from Proposition 3.14.

Proposition 3.13. *If X is of bounded geometry, then so is any subset $Y \subset X$.*

Proposition 3.14. *Suppose $f : X \to Y$ is a coarse embedding and X is of bounded geometry. Then $f(X)$ is of bounded geometry.*

Proof. Suppose that X is of bounded geometry with constant $r \ge 0$ and function M. Let b be the bornologous control function for f, and let p be the effective control function for f. Let $r' = b(r)$, and let $M'(\rho) = M(p(\rho r')/r)$ for $\rho \in [1, \infty)$.

Consider the ball $B := B(f(x), \rho r') \cap f(X)$. We want to show that B contains at most $M'(\rho)$ points that are r'-separated. Let A be an r'-separated set in B. For each $a \in A$, chose some $s \in X$ such that $f(s) = a$. This forms an r-separated set S in $f^{-1}(B)$. For if $d(s_1, s_2) \le r$, then $d(f(s_1), f(s_2)) \le b(r) = r'$, which is a contradiction since $f(s_1), f(s_2) \in A$ and A is r'-separated.

Next note that since f is effectively proper,

$$S \subset f^{-1}(B) \subset B(x, p(\rho r')).$$

Since X is of bounded geometry, $B(x, p(\rho r'))$ has at most $M(p(\rho r')/r) = M'(\rho)$ points that are r-separated. Hence, $|S|$ is at most $M'(\rho)$. Since S is r-separated, S is in one-to-one correspondence with A, so that we must have that $|A|$ is at most $M'(\rho)$, as desired. □

Proposition 3.15. *Suppose $f : X \to Y$ is a coarse embedding and Y is of bounded geometry. Then X is of bounded geometry.*

Proposition 3.16. *Suppose $f : X \to Y$ is a coarse equivalence and that Y is of bounded geometry. Then there exists $Z \subset X$ such that Z is of bounded geometry with inclusion being a coarse equivalence.*

The previous proposition can actually be strengthened to show that X is of bounded geometry. Thus, bounded geometry is in fact a coarse invariant. We will need Proposition 3.17. It follows that bounded geometry is a coarse invariant by combining Propositions 3.16 and 3.17. The theorem that follows shows the relationship between the discrete and nondiscrete cases.

Proposition 3.17. *Suppose a subset $Z \subset X$ is of bounded geometry and is coarsely equivalent to X via the inclusion map. Then X is of bounded geometry.*

Proof. Suppose Z is of bounded geometry with respect to the constant $r \geq 0$ and function M. Let $i : Z \to X$ be the inclusion map; there is a map $g : X \to Z$ such that $i \circ g$ is close to the identity of X. So there exists a number $S \geq 0$ so that every x is within S of a point in Z. Let $r' = 2S + r$ and let $M'(\rho) = M(\frac{\rho r' + 2S}{r})$. We will show that X is of bounded geometry with respect to the constant r' and the function M'.

For $\rho \geq 1$, let $B := B(x, \rho r')$ be a ball in X. Let $z \in Z$ be a point that such that $d(x, z) \leq S$. Choose $\rho_1 = \frac{\rho r' + 2S}{r}$. Then if $x_1 \in B(x, \rho r')$,

$$d(x_1, z) \leq d(x_1, x) + d(x, z) < \rho r' + 2S = \rho_1 r.$$

Hence, $B \subset B(z, \rho_1 r)$.

Now let A be an r'-separated set in B. Consider the function g restricted to A; its image is an r-separated set in $B(z, \rho_1 r)$. To see this, recall that $d(x, g(x)) \leq S$ for all $x \in X$. So if two points in the image z_1, z_2 satisfy $d(z_1, z_2) \leq r$, we must have that

$$d(x_1, x_2) \leq d(x_1, z_1) + d(z_1, z_2) + d(z_2, x_2) \leq 2S + r = r',$$

a contradiction. Thus, the image of g on A must be an r-separated set. Moreover, this argument shows that g restricted to A is injective.

Using that Z is of bounded geometry, we know that in $B(z, \rho_1 r)$, there are at most $M(\rho_1)$ points in an r-separated set. Thus, A contains at most $M(\rho_1) = M(\frac{\rho r' + 2S}{r}) = M'(\rho)$ points. Therefore, X is of bounded geometry. □

Theorem 3.18. *A metric space X is of bounded geometry if and only if it is coarsely equivalent to a discrete space of bounded geometry (in the discrete sense).*

Proof. Suppose that X is of bounded geometry with constant r and function M. We can apply Zorn's Lemma to find a maximal r-separated subset Y of X, which is coarsely equivalent to X via inclusion. Then Y is immediately of bounded geometry, being a subset of X. Moreover, it is of (discrete) bounded geometry since any ball consists only of r-separated points, the number of which is bounded due to the bounded geometry of X.

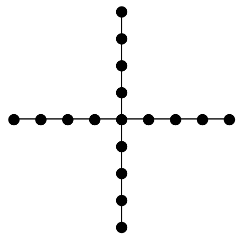

Figure 3.1: C_4

Conversely, suppose that X is coarsely equivalent to a discrete space Y that is of (discrete) bounded geometry. The space Y is of bounded geometry with $r = 0$; any ball of radius s in Y has finitely many points, which implies that the number of 0-separated points in any ball of radius s is bounded by this same finite number. Finally, we use that bounded geometry is a coarse invariant to obtain that X is of bounded geometry. $\square$

3.1.1 Another Definition

In [BD08b], the authors proposed following definition for bounded geometry of a metric space.

Definition 3.19. A metric space X is said to be of **B-D bounded geometry** if for every $M > 0$, there exists a uniformly bounded cover of finite multiplicity and Lebesgue number at least M.

For our purposes, the Lebesgue number of a cover $\mathcal{U}$ of X is

$$\inf_x \sup\{d(x, X \setminus U) \,|\, U \in \mathcal{U}\}.$$

We would like to ask some questions regarding this new definition. In particular, is it equivalent to Buyalo and Schroeders' definition.

I was unable to prove any implications relating this new definition to the old one. I have since discovered a relatively simple example of a metric space of (B-D) bounded geometry that is not of bounded geometry. The example is a disjoint graph with infinite distances allowed between points in different components. Our space will be the set of vertices of the graph along with the graph metric.

We can describe the graph G in $\mathbb{C} \times \mathbb{R}$. It is defined by its components. The component C_n lies in $\mathbb{C} \times \{n\}$ and is defined as follows. We let C_0 be a point, C_1 is two points, one at 0 and one at i, connected by an edge, and C_2 is a graph of 5 points with points at 0, i, $2i$, $-i$, and $-2i$. We create each successive component by adding a point to each "leg," and then we add an additional leg copying the previous ones. A picture of C_4 is shown above.

So it should be apparent that C_j consists of a central point x_j called the *node* along with j legs of length j. It is easy to see that G *is not* of bounded geometry since given any $r > 0$, the balls $B(x_j, \lceil r \rceil + 1)$, $j \geq r$, all have at least $j + 1$ points that are r-separated.

These $j + 1$ points are the node along with the points that are $\lceil r \rceil$ steps away from the node on each of the j legs. Since j approaches ∞, the r-capacity of these balls approaches infinity. Thus, G is not of bounded geometry.

However, the space G is of (B-D) bounded geometry. Given $M > 0$, we define a uniformly bounded cover of G as follows. First we cover the nodes using $U_j = B(x_j, 2\lceil M \rceil)$. Next we cover the remaining space using $B(x, \lceil M \rceil)$ for each $x \in C_j$ with $d(x_j, x) \geq \lceil M \rceil$. Let $\mathcal{U} = \{U_j\} \cup \{U_x\}$. Clearly this is a uniformly bounded cover, so we claim that it is also of finite multiplicity with Lebesgue number at least M. Any point which is at least M away from the node is only in as many U_y sets as the number of points in U_x, with a slight exception for points of distance close to M from the node. It is clear then that the local multiplicity of these points are finite. The only question regards the local multiplicity of the node and nearby points.

The node x_j is only in the ball U_j since all of the other balls are centered at points x with $d(x, x_j) \geq \lceil M \rceil$. So for all j, the local multiplicity of x_j is 1. If a point x lies on a leg and $d(x, x_j) < \lceil M \rceil$, then x can only lie in U_y for which y lies in the same leg. For if $d(x, y) < M$ for y in another leg, we would necessarily have that $d(x_j, y) < \lceil M \rceil$, a contradiction. Therefore, every point has finite local multiplicity, which is uniformly bounded above.

Finally we need to verify that the Lebesgue number of the cover is at least M. It suffices to show that for each $x \in G$,

$$B(x, M) \subset U,$$

for some $U \in \mathcal{U}$. If $d(x, x_j) \geq \lceil M \rceil$ for some j, then just let $U = U_x$. If $d(x, x_j) < \lceil M \rceil$, let $U = U_j$. Then if $y \in B(x, M)$, we see that

$$d(y, x_j) \leq d(y, x) + d(x, x_j) < M + \lceil M \rceil \leq 2\lceil M \rceil,$$

so that $y \in U$. Since x was arbitrary, taking an infimum over all x shows that the Lebesgue number is at least M. Therefore, G is of (B-D) bounded geometry.

3.2 Coarse Space Setting

The next goal is to define and understand bounded geometry in the more general setting of coarse spaces. A definition is given in Chapter 3 of [Roe03]. It is a bit cryptic at first, but it emerges to be a natural generalization of our definition of bounded geometry for metric spaces.

A different definition can be formulated in terms of *metric approximations*. An intuitive guess would state that a coarse space is of bounded geometry if every metric approximation is of bounded geometry in the B-S sense. We will see this definition at the end of Chapter 5, where we will investigate the relationship between this new definition and Roe's definition.

For a preview of this new definition, we think of a *metric approximation* $\mathcal{X}$ of a coarse space X as a collection consisting of coarse structures X_M on the same set X endowed with a metrizable coarse structure for which the identity map from X_M to X is bornologous, along with a few other properties that reflect the notion of approximation. Then our guess would say that a space X is said to be of bounded geometry if every approximation in $\mathcal{X}$ is of bounded geometry. When we revisit this definition later, we will see that a slightly different result is true.

Now onto Roe's definition. He uses the notion of capacity to generalize the notion of r-separated subsets. If $A \subset X$ and $E \subset X^2$, then

$$\mathrm{cap}_E A$$

is defined to be the maximum cardinality of a set $S \subset A$ such that $(s_1, s_2) \notin E$ for all distinct points $s_1, s_2 \in S$. We say that the points of S are *not E-related* or *E-separated.* So here E plays the role of r in the B-S definition of bounded geometry in the sense that saying "s_1 and s_2 are E-separated" is analogous to saying "s_1 and s_2 are r-separated."

Since bounded geometry says something about the capacity of balls which are arbitrarily larger expansions of a given ball, we need an analog of how to expand the radius of a ball in the coarse setting. This is accomplished by considering the set

$$F[E_x] := \{z \mid \exists z' \ni \ (z, z') \in F \text{ and } (z', x) \in E\},$$

where E, F are controlled sets. One can think of the section E_x as a ball centered at x and the set $F[E_x]$ as all points which are F-related to points that are E-related to x. Refer to the definition of E-balls in Chapter 2.

Now it is a natural transition to get a definition for coarse spaces. We define the capacity of a controlled set F by writing

$$\mathrm{cap}_E F = \sup_x \{\mathrm{cap}_E F[E_x], \mathrm{cap}_E F^t[E_x]\}.$$

We call the sets F satisfying $\mathrm{cap}_E F < \infty$ **uniform** with respect to the gauge E. We define a **gauge** for a given coarse structure to be a symmetric controlled set containing the diagonal which is uniform with respect to itself, that is, $\mathrm{cap}_E E < \infty$. This concept is similar to doubling metric spaces; see Chapter 8 of [BS07].

Definition 3.20. We say that $(X, \mathcal{C})$ is of **bounded geometry** if there exists a *gauge* E such that every controlled set F is uniform with respect to the gauge E.

The uniform bounds are what correspond to the function M in the metric setting, since there is a different number associated to each controlled set F. Note, however, that F is not a direct analog of the number ρ used in the metric case, because using ρr enlarges a ball of radius r multiplicatively, whereas the ball $F[E_x]$ enlarges the ball E_x additively. That is, if the controlled space is metrizable and the sets F and E correspond to distances s and r, respectively, then the "radius" of $F[E_x]$ is $s + r$.

3.2.1 Categorical Properties

It is also natural to ask whether or not the propositions above hold in the category of coarse spaces and coarse maps. We see that bounded geometry is hereditary for Roe's definition. We will restate the propositions with the new proofs.

Proposition 3.21. *If X is of bounded geometry, then so is any subset $Y \subset X$.*

Proof. Suppose X is of bounded geometry with respect to a gauge F. Let $H = F \cap Y^2$. We claim that H is a gauge for Y. First, H contains the diagonal of Y since (y, y) is in Y^2 and the diagonal of X, hence is in F. Now we need to see that H is doubling, that is,

$$\mathrm{cap}_H H < \infty.$$

First note that if $(y_1, y_2) \notin H$, then $(y_1, y_2) \notin F$. It follows that

$$\text{cap}_H H[H_y] = \text{cap}_F H[H_y] = \text{cap}_F H[F_y] < M, \qquad \text{(for all } y \in Y)$$

since H is controlled in X.

We conclude the proof by showing that $\text{cap}_H E < \infty$ for all controlled sets E of Y. This is exactly the same as above since the same equation holds:

$$\text{cap}_H E[H_y] = \text{cap}_F E[H_y] = \text{cap}_F E[F_y] < M. \qquad \text{(for all } y \in Y)$$

□

Proposition 3.22. *Suppose $f : X \to Y$ is a coarse embedding and X is of bounded geometry. Then $f(X)$ is of bounded geometry.*

Proof. To simplify the argument, suppose without loss of generality that f is surjective. Suppose X is of bounded geometry with respect to a gauge E. Let $F = f^2(E)$. We claim that F is a gauge for Y. We get that F is controlled since f is bornologous. To show that F contains the diagonal of Y, we observe that for each $y \in Y$, there exists an $x \in X$ such that $f(x) = y$. Hence, $f^2(x, x) = (y, y)$, so that $(y, y) \in F$.

As with the last proof, the bulk of the work comes from proving that F is doubling, that is,

$$\text{cap}_F F < \infty.$$

Given $y_0 \in Y$, let A be an F-separated set in $F[F_{y_0}]$. Clearly, $f^{-1}(A)$ is an E-separated set in $f^{-1}(F[F_{y_0}])$ since if two points in X are E-related, their images must be F-related.

Next, since f is effectively proper and $E \supset \Delta_X$, there exists a set E' controlled in X such that

$$f^{-1}(F[F_{f(x_0)}]) = f^{-1}((F \circ F)_{f(x_0)}) \subset E'_{x_0} \subset (E' \circ E)_{x_0} = E'[E_{x_0}]$$

where $x_0 \in f^{-1}(\{y_0\})$. Hence,

$$f^{-1}(A) \subset f^{-1}(F[F_{y_0}]) \subset E'[E_{x_0}].$$

Since X is of bounded geometry, we see that $f^{-1}(A)$ can only have as many points as a maximal E-separated set in $E'[E_{x_0}]$, which is bounded above by $\text{cap}_E E' < \infty$. Therefore, $|A| = |f^{-1}(A)|$ is bounded above by $\text{cap}_E E'$, which does not depend on y_0.

To conclude that Y is of bounded geometry, we must extend the last argument by proving that

$$\text{cap}_F F' < \infty$$

for all controlled sets F' in Y. However, the proof will be nearly verbatim upon inspection. □

Proposition 3.23. *Suppose $f : X \to Y$ is a coarse embedding and Y is of bounded geometry. Then there exists $Z \subset X$ of bounded geometry with inclusion being a coarse equivalence.*

Proposition 3.24. *Suppose $i : Z \to X$ is a coarse equivalence with i being the inclusion map. If Z is of bounded geometry, then so is X.*

Proof. Since Z is of bounded geometry, there exists a gauge E for which $\mathrm{cap}_E D < \infty$ for all sets D controlled in Z. Let $g : X \to Z$ be the coarse inverse to i. Then since $i \circ g$ is close to the identity map on X, the set $V := \{(x, g(x)) \mid x \in X\}$ is controlled in X. Enlarge and redefine V to be symmetric.

Define a gauge F on X by

$$F = V \circ E \circ V.$$

Since E and V are symmetric, so is F. Moreover, F contains the diagonal of X. For if $x \in X$, then $(x, g(x)) \in V$, $(g(x), g(x)) \in E$, and $(g(x), x) \in V$. Next we need to show that F is doubling, that is, $\mathrm{cap}_F F < \infty$. Given $x \in X$, let A be an F-separated set in $F[F_x]$. Then $g(A)$ is an E-separated set; otherwise, there would exist $(g(a_1), g(a_2)) \in g(A)^2 \cap E$, which implies that $(a_1, a_2) \in F$. To finish showing that F is doubling, we need to find a controlled set D in Z such that $g(A) \subset D[E_{g(x)}]$. Given $a \in A$, there exists $w \in X$ such that $(a, w) \in F$ and $(w, x) \in F$. Hence, $(g(a), g(x)) \in g(F) \circ g(F)$, which is controlled in Z since g is bornologous. Let $D = g(F)^2$. Note that $D \subset D \circ E$ since $E \supset \Delta_Z$; thus, $g(A) \subset D[E_{g(x)}]$ is an E-separated set, whence

$$|g(A)| \leq \mathrm{cap}_E D[E_{g(x)}] \leq \mathrm{cap}_E D.$$

Note that g restricted to A is injective since $E \supset \Delta_Z$; it follows that $|A| = |g(A)|$, which allows us to conclude that

$$\mathrm{cap}_F F \leq \mathrm{cap}_E D < \infty.$$

To show that $\mathrm{cap}_F C < \infty$ for any controlled set C in X, just repeat the argument above with $D = g(C) \circ g(F)$. □

Theorem 3.25. *Bounded geometry is a coarse invariant.*

Proof. If $f : X \to Y$ is a coarse equivalence with coarse inverse g, then by Proposition 3.22, $f(X)$ is of bounded geometry. Moreover, $i : f(X) \to Y$ is a coarse equivalence with coarse inverse $f \circ g$; therefore, we use Proposition 3.24 to conclude that Y is of bounded geometry. □

Chapter 4

Property A

4.1 What is Property A?

In [Yu00], Guoliang Yu first defined Property A as a sufficient condition for a metric space to coarsely embed into a Hilbert space. It can be thought of as a weak amenability condition. The paper [BCV95] inspired his findings. We begin with the original definition.

Definition 4.1. Let (X, d) be a discrete metric space. We say that X has **Property A** if for every $R, \varepsilon > 0$, there exists a collection $\{A_x\}$ of nonempty, finite subsets of $X \times \mathbb{N}$ such that

1. $d(x, y) < R \Longrightarrow \dfrac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \varepsilon$
2. There exists $S > 0$ such that $A_x \subset B(x, S) \times \mathbb{N}$

The interpretation of the definition is that for any "small" number ε and for any "large" number R, one can construct the *Yu* sets A_x so that any pair of points within a large set of diameter R have nearly identical A_x sets. Also, each Yu set lives inside of a cylinder of uniform diameter.

The reason for the "height" factor $\mathbb{N}$ is to insure that the property is invariant under quasi-isometries. In fact, Property A is a coarse invariant.

Example 4.2.

1. Any bounded discrete metric space has Property A. Just let $A_x = X$.

2. The integers have Property A. Given any $R, \varepsilon > 0$, let $A_x = B(x, r)$ where $r > 0$ is chosen so that $r > 2R/\varepsilon$.

3. For $n \geq 2$, the free group on n generators, F_n, has Property A. We utilize the Cayley graph of F_n since it is a tree. First, fix a geodesic ray γ starting at the identity. Given any $R, \varepsilon > 0$, let $n > R(\varepsilon + 2)/\varepsilon$ (in particular, $n > R$) and define A_x to be the unique geodesic γ_x of length n starting at x and traveling in the direction γ. That is, γ_n travels along the geodesic from x to γ, and then coincides with γ for the duration of its length. Clearly, each A_x is nonempty and finite. Given $x, y \in F_n$ with $d(x, y) < R$, it isn't too hard to see that $|A_x \Delta A_y| \leq 2R$ and $|A_x \cap A_y| \geq n - R$. Hence,
$$\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} \leq \frac{2R}{n - R} < \frac{2R}{R(\varepsilon + 2)/\varepsilon - R} = \varepsilon.$$

For a short exposition on the basics of Property A, see [NY08]. These examples as well as others are discussed there.

It is a nontrivial matter to construct a space that does not have Property A. One way is to use Yu's result and construct a space that does not coarsely embed into a Hilbert space. This can be done using expander graphs; see [Wil09] for example. In [Now07], Piotr Nowak constructs spaces without Property A that do coarsely embed into a Hilbert space. The easiest example is the infinite cube complex described as a disjoint union of powers of $\mathbb{Z}_2$, the two element group, where the disjoint union is given a particular metric so that each component of the union has Property A "arbitrarily badly."

4.2 Equivalent Definitions

We will briefly consider some alternate formulations of Property A. For certain applications, other definitions are more suitable. Many of the results mentioned will have proofs similar to those in Willett's notes [Wil09].

In Willett's notes, he presents proofs that 8 other definitions of Property A are all equivalent to this definition. We will investigate the first 3. The second and third definitions replace the sets A_x with functions $\xi_x \in l^p(X)$, the first condition with $||\xi_x - \xi_y||_p < \varepsilon$, and the second condition with a condition that requires a uniform bound on $|\mathrm{supp}(\xi_x)|$. The proofs here are adapted from his notes. For more background on the proofs, see [HR00] and [Dra06].

The proof given for the equivalence requires that we restrict to spaces of bounded geometry. The only proof that requires this restriction is $(iii) \Rightarrow (i)$.

Theorem 4.3. *Suppose (X, d) is a discrete metric space of bounded geometry. The following statements are equivalent.*

i. X has Property A.

ii. There exists $1 \leq p < \infty$ such that for every $\varepsilon, R > 0$, there exists $\xi : X \to l^p(X)$ given by $x \mapsto \xi_x$ such that each ξ_x is a unit vector and

  (a) $d(x, y) \leq R \Longrightarrow ||\xi_x - \xi_y||_p < \varepsilon$;

  (b) There exists an $S > 0$ such that $\mathrm{supp}(\xi_x) \subset B(x, S)$.

iii. For all $1 \leq p < \infty$ and for every $\varepsilon, R > 0$, there exists $\xi : X \to l^p(X)$ given by $x \mapsto \xi_x$ such that each ξ_x is a unit vector and

  (a) $d(x, y) \leq R \Longrightarrow ||\xi_x - \xi_y||_p < \varepsilon$;

  (b) There exists an $S > 0$ such that $\mathrm{supp}(\xi_x) \subset B(x, S)$.

Proof. $(i) \Rightarrow (ii)$: Given $R, \varepsilon > 0$, define $\xi : X \to l^1(X)$ by

$$\xi_x(y) = \frac{|(y \times \mathbb{N}) \cap A_x|}{|A_x|}.$$

It is immediate that $\|\xi_x\|_1 = 1$. Also, if $\xi_x(y) \neq 0$, then there exists an n such that $(y, n) \in A_x$. By hypothesis, $d(x, y) \leq S$. Thus, $\mathrm{supp}(\xi_x) \subset B(x, S + 1)$.

Finally, suppose $d(x_1, x_2) \leq R$. Then the following norm counts a subset of $|A_{x_1} \Delta A_{x_2}|$:

$$\||A_{x_1}|\xi_{x_1} - |A_{x_2}|\xi_{x_2}\|_1 = \sum_{y \in X} ||(y \times \mathbb{N}) \cap A_{x_1}| - |(y \times \mathbb{N}) \cap A_{x_2}|| \leq |A_{x_1} \Delta A_{x_2}|.$$

Next we use our hypothesis to bound the ratio $|A_{x_2}|/|A_{x_1}|$. First

$$\varepsilon > \frac{|A_{x_1} \Delta A_{x_2}|}{|A_{x_1} \cap A_{x_2}|} = \frac{|A_{x_1}| + |A_{x_2}| - 2|A_{x_1} \cap A_{x_2}|}{|A_{x_1} \cap A_{x_2}|} \geq 1 + \frac{|A_{x_2}|}{|A_{x_1}|} - 2.$$

We could repeat the same estimate by dividing by $|A_{x_2}|$ to obtain

$$\frac{1}{\varepsilon + 1} < \frac{|A_{x_2}|}{|A_{x_1}|} < \varepsilon + 1.$$

This implies the estimate we need

$$-\varepsilon < \frac{|A_{x_2}|}{|A_{x_1}|} - 1 < \varepsilon.$$

Therefore,

$$\begin{aligned}
\|\xi_{x_1} - \xi_{x_2}\|_1 &\leq \left\|\xi_{x_1} - \frac{|A_{x_2}|}{|A_{x_1}|}\xi_{x_2}\right\|_1 + \left\|\frac{|A_{x_2}|}{|A_{x_1}|}\xi_{x_2} - \xi_{x_2}\right\|_1 \\
&< \frac{|A_{x_1} \Delta A_{x_2}|}{|A_{x_1}|} + \varepsilon \\
&\leq \frac{|A_{x_1} \Delta A_{x_2}|}{|A_{x_2} \cap A_{x_2}|} + \varepsilon \\
&\leq 2\varepsilon.
\end{aligned}$$

(ii) $\Rightarrow$ *(iii)*: Suppose for some $p \in [1, \infty)$, there exists $\xi : X \to l^p(X)$ with the given properties above. By the reverse triangle inequality, we can assume that ξ_x takes only nonnegative values (use $|\xi_x|$). Given $q \in [p, \infty)$, define $\eta : X \to l^q(x)$ by $\eta_x(y) = (\xi_x(y))^{\frac{p}{q}}$. Then

$$\|\eta_x\|_q^q = \sum_{y \in X} |\eta_x(y)|^q = \sum_{y \in X} ((\xi_x(y))^{\frac{p}{q}})^q = 1.$$

Thus, each η_x is a unit vector in $l^q(X)$. Since η_x and ξ_x have the same support, the (b) part of the definition holds. If $d(x_1, x_2) \leq R$, then

$$\|\eta_{x_1} - \eta_{x_2}\|_q = \|\xi_{x_1}^{\frac{p}{q}} - \xi_{x_2}^{\frac{p}{q}}\|_q \leq \|\xi_{x_1} - \xi_{x_2}\|_q^{\frac{p}{q}} < \varepsilon^{\frac{p}{q}} \to 0,$$

where the middle inequality is a general result that can be proved with elementary calculus: $|a^q - b^q| \leq |a - b|^q$ where $a, b \geq 0$ and $q \leq 1$.

For the case of $q \in [1, p)$, we define $\eta : X \to l^q(x)$ by $\eta_x(y) = (\xi_x(y))^p$. We will prove that η satisfies (ii) with $p = 1$, and then we apply the argument of the last paragraph to obtain (ii) for our $q \in [1, p)$. As before the (b) part of (ii) will follow easily; we are concerned with the (a) part. Suppose $d(x_1, x_2) \leq R$. Then by factoring we can obtain the following:

$$\|\eta_{x_1} - \eta_{x_2}\|_1 = \|\xi_{x_1}^p - \xi_{x_2}^p\|_1 = \sum_{y\in X} |\xi_{x_1} - \xi_{x_2}| \left|\sum_{k=0}^{p-1} \xi_{x_1}^k \xi_{x_2}^{p-1-k}\right|.$$

Now we apply Hölder's Inequality to obtain the estimate

$$\|\eta_{x_1} - \eta_{x_2}\|_1 \leq \|\xi_{x_1} - \xi_{x_2}\|_p \left\|\sum_{k=0}^{p-1} \xi_{x_1}^k \xi_{x_2}^{p-1-k}\right\|_r,$$

where $pr = p + r$. We have control over the first factor, so we need to estimate the second factor. We start with two similar calculations:

$$\xi_{x_1}^{kr}\xi_{x_2}^{pr-r-kr} = \left(\frac{\xi_{x_1}}{\xi_{x_2}}\right)^{kr} \xi_{x_2}^{pr-r} = \left(\frac{\xi_{x_1}}{\xi_{x_2}}\right)^{kr} \xi_{x_2}^{p}, \qquad \text{and}$$

$$\xi_{x_1}^{kr}\xi_{x_2}^{pr-r-kr} = \xi_{x_1}^{(kr+r-pr)-r+pr}\xi_{x_2}^{pr-r-kr} = \xi_{x_1}^{p}\left(\frac{\xi_{x_2}}{\xi_{x_1}}\right)^{pr-r-kr}.$$

Then, after analyzing the two cases $\xi_{x_1}(y) \leq \xi_{x_2}(y)$ and $\xi_{x_2}(y) \leq \xi_{x_1}(y)$, we can conclude from these equations that

$$\xi_{x_1}^{kr}\xi_{x_2}^{pr-r-kr} \leq \xi_{x_1}^p + \xi_{x_2}^p.$$

We now refine our earlier estimate to

$$\begin{aligned}\|\eta_{x_1} - \eta_{x_2}\|_1 \leq \|\xi_{x_1} - \xi_{x_2}\|_p \sum_{k=0}^{p-1} \left\|\xi_{x_1}^k \xi_{x_2}^{p-1-k}\right\|_r &\leq p\|\xi_{x_1} - \xi_{x_2}\|_p \left(\sum_{y\in X} \xi_{x_1}^p + \xi_{x_2}^p\right)^{\frac{1}{r}} \\ &= 2^{\frac{1}{r}}p\|\xi_{x_1} - \xi_{x_2}\|_p < 2^{\frac{1}{r}}p\varepsilon.\end{aligned}$$

That is, we get a Lipschitz condition that guarantees that $\|\eta_{x_1} - \eta_{x_2}\|_1 \to 0$ as $\varepsilon \to 0$.

$(iii) \Rightarrow (i)$: Given $R, \varepsilon > 0$, there exists $\xi : X \to l^1(X)$ satisfying the given properties in (iii). As before, we can assume that each ξ_x takes only nonnegative values. Since X is of bounded geometry, there exists an $N > 0$ such that $|\text{supp } \xi_x| \leq |B(x, S)| \leq N$. We define

$$A_x := \{(y, n) \mid 0 < n \leq \theta_x(y)\},$$

where $\theta_x(y) = \lceil M\xi_x(y)\rceil$ with $M > N/\varepsilon$. Since each ξ_x is a unit vector with finite support, the sets A_x are nonempty and finite. Suppose $d(x_1, x_2) \leq R$. Before calculating $|A_{x_1}\Delta A_{x_2}|$, we require the following estimate

$$\frac{1}{M}|\theta_{x_1} - \theta_{x_2}| \leq |\frac{\theta_{x_1}}{M} - \xi_{x_1}| + |\xi_{x_1} - \xi_{x_2}| + |\xi_{x_2} - \frac{\theta_{x_2}}{M}|,$$

which we can refine by noting that for $i = 1, 2$

$$0 = \frac{M\xi_{x_i}}{M} - \xi_{x_i} \leq \frac{\theta_{x_i}}{M} - \xi_{x_i} \leq \frac{M\xi_{x_i} + 1}{M} - \xi_{x_i} = \frac{1}{M}$$

to obtain

$$\frac{1}{M}\|\theta_{x_1} - \theta_{x_2}\|_1 \leq \frac{2N}{M} + \varepsilon < 2\varepsilon + \varepsilon = 3\varepsilon.$$

Now we see that
$$|A_{x_1}\Delta A_{x_2}| = M \cdot \frac{1}{M}\|\theta_{x_1} - \theta_{x_2}\|_1 < M \cdot 3\varepsilon.$$
Upon observing $|A_{x_i}| = \sum_{y\in X}\theta_{x_i}(y) \geq \sum_{y\in X} M\xi_x(y) = M$ for $i = 1, 2$, we have
$$\frac{|A_{x_1}\Delta A_{x_2}|}{|A_{x_i}|} \leq \frac{|A_{x_1}\Delta A_{x_2}|}{M} < 3\varepsilon.$$
It follows that X has Property A. $\square$

Chapter 5

Metric Approximations

5.1 Definition and Examples

Definition 5.1. Let $(X, \mathcal{C})$ be a coarse space. A **metric approximation** $\mathcal{X}$ of the given coarse space is a collection $\{\mathcal{C}_\alpha\}_{\alpha \in I}$ of coarse structures on X such that

1. $\mathcal{C}_\alpha$ is metrizable for all $\alpha \in I$
2. $\mathcal{C}_\alpha \subset \mathcal{C}$ for all $\alpha \in I$
3. For all $C \in \mathcal{C}$, there exists $\alpha \in I$ such that $C \in \mathcal{C}_\alpha$
4. For all $\alpha_1, \alpha_2 \in I$, there exists $\alpha_3 \in I$ such that $\mathcal{C}_{\alpha_1}, \mathcal{C}_{\alpha_2} \subset \mathcal{C}_{\alpha_3}$

We also refer to the approximating structures themselves as metric approximations, or more briefly as approximations.

One goal of introducing a metric approximation is to gain information about a nonmetrizable coarse space. Thus, we need some examples of nonmetrizable coarse spaces to get us started. As we saw in Chapter 2, the discrete coarse structure on an uncountable set is always nonmetrizable.

Example 5.2. One way of constructing a metric approximation for an uncountable discrete coarse space X is by considering the collection of all countable subsets of X. We then have a metric approximation $\mathcal{X}$ consisting of collections $\mathcal{C}_C$ that contain all sets with only finitely many points off the diagonal having coordinates in C. The crucial property to check is that any controlled set in X is controlled in some $\mathcal{C}_C$. This is easy; since every such controlled set contains only finitely many points off the diagonal, we can let C be the set of those coordinates. Then $\mathcal{C}_C$ contains this controlled set. Note that any approximation of the discrete coarse structure is necessarily disconnected since the discrete coarse structure is the smallest connected coarse structure on a set X.

All of these examples regard approximating discrete coarse spaces. What if the space is a generic nonmetrizable coarse space? If $(X, \mathcal{C})$ is a coarse space, then we can mimic the construction above by considering the metrizable coarse structures $\mathcal{C}_C$, where C is a countable subset of X. The definition of $\mathcal{C}_C$ in this setting is given by considering all controlled sets in X whose coordinates all lie in C. However, this collection may not be a metric approximation; there may be controlled sets whose set of coordinates is uncountable.

In light of the previous paragraph, we would like to investigate the possibilities of coarse structures on S_Ω, the minimal uncountable well-ordered set. For example, one could use the discrete coarse structure.

So the metric approximation of the previous example should work here. However, we are more interested in whether an ordered set like S_Ω has a noteworthy coarse structure related to its order. Another coarse structure, called the **ordered coarse structure**, denoted $\mathcal{C}_o$, is given by the following. Let X be an ordered set without a largest element. We say $E \subset X^2$ is controlled if there exists $M \in X$ such that points with *distinct* coordinates have coordinates bounded above by M. One can verify that this is a coarse structure that is connected and larger than the discrete coarse structure. It is not the maximal coarse structure since horizontal lines in X^2 are not controlled (the x-coordinates have no upper bound). Can we approximate this structure? We return to this question in a bit.

We will show that the ordered coarse structure is indeed a coarse structure. The diagonal is controlled vacuously since all points have identical coordinates. If C is controlled with **control element** M, that is, the smallest such M satisfying the definition above, then so is C^t and any subset $D \subset C$. If C_1 and C_2 are controlled with control elements M_1 and M_2, respectively, then $C_1 \cup C_2$ is controlled with control element $M := M_1 \vee M_2$. For if $(x, y) \in C_1 \cup C_2$ with $x \neq y$, then $x, y \leq M_i \leq M$ whenever $(x, y) \in C_i$, $i \in \{1, 2\}$. In addition, $C_1 \circ C_2$ is controlled with control element M. For if $(x, y) \in C_1 \circ C_2$ with $x \neq y$, then there exists $z \in X$ such that $(x, z) \in C_1$ and $(z, y) \in C_2$. Suppose for the moment, without loss of generality, that $x > y$. If $z \geq x$, then $z > y$ yielding $z, y \leq M_2 \leq M$, so that $x, y \leq M$. If $x > z$, then $x, z \leq M_1 \leq M$, so that $x, y \leq M$.

Now we address whether the ordered coarse structure is metrizable. Well it is when $X = \mathbb{R}$. This is because we can create a countable basis by considering controlled sets C_k with control element k, where k is a positive integer. Now, any controlled set C of X has a control element M; so choose $k > M$ to get that C is contained in C_k. In the general case, we need to be able to choose elements $M_1 < M_2 < \cdots$ of X so that $F_{M_k} = \{(x, y) \mid x, y < M_k\} \cup \Delta$ will work as our countable basis. Can we always find such an unbounded increasing chain? We will see that for $X = S_\Omega$, this is impossible since every countable sequence is bounded above in S_Ω.

We could try the earlier construction where we restricted the controlled sets to those whose coordinates come from a given countable set. These again should be metrizable substructures of the starting coarse structure. Let E be controlled with control element M. The coordinates represented in E may form an uncountable set. So how can we find an approximation in which it is metrizable? Well, this fails again in this case. We need a more clever way to construct approximations.

Here is another idea. If $(X, \mathcal{C})$ is a coarse structure, we can form a metric approximation $\mathcal{X}$ by considering monogenic coarse structures on X generated by controlled sets in $\mathcal{C}$. Clearly, these are substructures that are metrizable. Moreover, if $F \in \mathcal{C}$, then $(X, c(F)) \in \mathcal{X}$ is a structure for which F is controlled. Finally, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are coarse structures that are monogenic and generated by E and F, respectively, then $\mathcal{C}_3 := c(E \cup F)$ is a monogenic coarse structure that contains both $\mathcal{C}_1$ and $\mathcal{C}_2$.

5.2 Bounded Geometry

Note that an uncountable discrete coarse space $(X, \mathcal{C})$ is of bounded geometry with respect to the gauge Δ. This is because any controlled set F consists of at most finitely many points off the diagonal Δ, so $|F_x| < |F \setminus \Delta|$ for all $x \in X$. Hence, $\text{cap}_\Delta F < \infty$.

Thus, the canonical metric approximation $\mathcal{X}$ of X has approximations of bounded geometry by the same argument. This leads one to believe that a good definition for bounded geometry in terms of metric approximations is that a space is of bounded geometry if there exists a metric approximation whose coarse structures are all of bounded geometry (as metric spaces or as coarse spaces). This definition is too restrictive though. Instead, we can say that a space is of bounded geometry if there exists a metric approximation and a gauge E such that some space in the metric approximation is of bounded geometry with respect to the gauge E. This is nearly the right concept, but we need a bit more for this definition to be equivalent to the original definition. This is discussed in the next section. For now we consider an example of a coarse structure that is not of bounded geometry.

Example 5.3. Consider $X = S_\Omega$ with the ordered coarse structure $\mathcal{C}_o$. This coarse structure is nonmetrizable. To see this, let $\{C_i\}$ be a countable basis for this structure. Then there are control elements M_i corresponding to each C_i. Since $\{M_i\}$ is countable, it has an upper bound M in S_Ω. Take any element $M' > M$. Then the controlled set of points $\{(x, y) \,|\, x \neq y \text{ and } x, y \leq M'\}$ is not in the coarse structure generated by the basis: just consider the point (M, M') for example. This contradiction shows that the structure is nonmetrizable.

We can approximate $(S_\Omega, \mathcal{C}_o)$ by considering the coarse spaces $(S_\Omega, \mathcal{C}_M)$, where the coarse structure $\mathcal{C}_M$ consists only of controlled sets with control elements *strictly** less than M. This family of coarse structures, indexed by the elements of S_Ω, forms a metric approximation of the ordered coarse structure on S_Ω.

An interesting series of questions can be asked regarding this coarse space. Does it have Property A? Is it of bounded geometry? What other (generalized) metric space properties does it have? For now we just consider bounder geometry.

The space $(S_\Omega, \mathcal{C}_o)$ is not of bounded geometry. Suppose that $E \in \mathcal{C}$ is a gauge for which S_Ω is of bounded geometry. We seek to find a controlled set F that is not uniform with respect to E. Let N be the control element for E. Consider the sequence of points starting with N constructed by taking immediate successors. This sequence is bounded above, since every countable set in S_Ω is bounded above. Choose any upper bound $M \in S_\Omega$ for this sequence. Then the interval $(N, M]$ is infinite.

Now let F be the controlled set $[0, M]^2$, where 0 is the least element of S_Ω. We claim that

$$\text{cap}_E F[E_x] = \infty$$

for all $x < N$. We will show that for $x < N$, the set $F[E_x]$ has infinitely many points that are E-separated. The E-separated set is $(N, M]$ since any point $(y_1, y_2) \notin E$ if $y_1, y_2 > N$. Moreover, $(N, M] \subset F[E_x]$. For if $y \in (N, M]$, then $(y, x) \in F$ since $y, x \leq M$ and $(x, x) \in E$ since E contains the diagonal (being a gauge). Therefore, we conclude that $\text{cap}_E F = \infty$ for all $F \in \mathcal{C}$ so that $(S_\Omega, \mathcal{C}_o)$ is not of bounded geometry.

The previous example utilized a coarse structure on X that is somewhat trivial. The metric inducing the approximating coarse structures isn't too interesting. In Chapter 7,

*This requirement guarantees that the structure is not the maximal coarse structure.

we will investigate two other coarse structures on S_Ω, one which turns out to coincide with the ordered coarse structure, while the other turns out to be larger and more interesting.

5.3 Characterization of Bounded Geometry

To define bounded geometry for general coarse spaces in terms of metric approximations in such a way that we obtain a definition equivalent to Roe's definition in Chapter 3 of [Roe03], we can use the following theorem.

Theorem 5.4. *A coarse space $(X, \mathcal{C})$ is of bounded geometry with gauge E if and only if there exists a metric approximation $(X, \mathcal{C}_1)$ of bounded geometry with gauge E and any approximation $(X, \mathcal{C}_\alpha)$ with $C_1 \subset C_\alpha$ is of bounded geometry with gauge E.*

Proof. Suppose $(X, \mathcal{C})$ is of bounded geometry with gauge E. Take the metric approximation $\mathcal{X}$ to be the approximation with coarse structures that are monogenic. Then the coarse structure $c(E)$ contains E and is of bounded geometry with gauge E since $\mathrm{cap}_E F < \infty$ for all $F \in c(E)$ since $c(E) \subset \mathcal{C}$. Moreover, if $\mathcal{C}_\alpha \supset c(E)$ is another finitely generated coarse structure on X, then $\mathrm{cap}_E F < \infty$ for all $F \in \mathcal{C}_\alpha$ since $F \in \mathcal{C}$.

Conversely, suppose there exists a metric approximation $(X, \mathcal{C}_1)$ of bounded geometry with gauge E and any approximation $(X, \mathcal{C}_\alpha)$ with $C_1 \subset C_\alpha$ is of bounded geometry with gauge E. We want to show $\mathrm{cap}_E F < \infty$ for all $F \in \mathcal{C}$. By the definition of metric approximation, we can find a structure $\mathcal{C}_2 \in \mathcal{X}$ for which F is controlled. Furthermore, there exists an approximation $\mathcal{C}_\alpha$ containing both $\mathcal{C}_1$ and $\mathcal{C}_2$ which is of bounded geometry with gauge E since $\mathcal{C}_1 \subset \mathcal{C}_\alpha$. Since $F \in \mathcal{C}_\alpha$, we have $\mathrm{cap}_E F < \infty$, as desired. □

5.4 Property A

When S_Ω is equipped with the ordered coarse structure, the approximations of S_Ω have Property A. The argument is rather trivial. First let 0 be the smallest element of S_Ω. Consider the approximation $(S_\Omega, \mathcal{C}_M)$. Let $M_1, M_2, \ldots$ be the control elements corresponding to a countable basis for the coarse space. Suppose $R, \varepsilon > 0$. Let $i = \lfloor R \rfloor$. We define $A_x = \{0\}$ if $x \leq M_i$ and $A_x = \{x\}$ otherwise. Now if $d(x, y) \leq R$, then $d(x, y) \leq i$ since the metric is discrete. By definition of the metric, we know that $x, y \leq M_i$, so that $A_x = A_y = \{0\}$. Thus,

$$\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} = \frac{0}{1} < \varepsilon.$$

To verify the second requirement of Property A, we let $S = i$ with the goal of showing $A_x \subset B(x, S)$ for all $x \in S_\Omega$. We have two cases. If $x \leq M_i$, then $A_x = \{0\}$ and $B(x, S) = [0, M_i]$. So clearly, $A_x \subset B(x, S)$. If $x > M_i$, then $A_x = \{x\} \subset B(x, S)$.

It can be shown that S_Ω has Property A using the coarse definition of Property A below. The argument will be nearly identical to the one given for the approximations. This gives us an example of a nonmetrizable coarse space with Property A that is not of bounded geometry.

Of further interest is whether Property A can be characterized in terms of metric approximations, as was done with bounded geometry. Thus far, I have no found such characterization, although the current example involving S_Ω gives us hope.

For spaces of bounded geometry, there are equivalent formulations of Property A, just as in the metric case. See Chapter 11 in [Roe03], for example. Our definition is in fact valid for any arbitrary coarse space.

Definition 5.5. A coarse space $(X, \mathcal{C})$ has **Property A** if for every $C \in \mathcal{C}$ and for every $\varepsilon > 0$, there exists finite, nonempty sets $A_x \subset X \times \mathbb{N}$ such that the following two conditions hold:

1. $(x, y) \in C \Longrightarrow \dfrac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \varepsilon$;
2. There exists $D \in \mathcal{C}$ such that $A_x \subset D_x \times \mathbb{N}$ for all $x \in X$.

For example, let $X = S_\Omega$ and $\mathcal{C} = \mathcal{C}_o$, and let $C \in \mathcal{C}$ and $\varepsilon > 0$ be given. We can take $A_x = \{0\}$ when $x \leq M$, where M is the control element for C, and take $A_x = \{x\}$ when $x > M$. Then if $(x, y) \in C$, $A_x = A_y$, and the first condition is trivially true. To verify the second condition, let $D = [0, M]^2 \cup \Delta$. Then $A_x \subset D_x$ for all $x \in X$.

Chapter 6

Asymptotic Dimension

First we start with the generalized definition of asymptotic dimension as found in Chapter 9 of [Roe03]. We introduce a slight bit of terminology to facilitate the definition of asymptotic dimension.

Definition 6.1. Let X be a coarse space.

1. We say a collection of bounded sets $\{B_\alpha\}$ is **uniformly bounded** if $\bigcup B_\alpha^2$ is controlled.
2. Given a controlled set C, we say a collection of sets $\{K_\alpha\}$ is **C-disjoint** if $(x, y) \notin C$ whenever $x \in K_{\alpha_1}$ and $y \in K_{\alpha_2}$ with $\alpha_1 \neq \alpha_2$.

Clearly, any finite collection of bounded sets is uniformly bounded; this definition deals with infinite collections. The notion of C-disjointness measures how separated a collection of sets is.

Definition 6.2. A coarse space $(X, \mathcal{C})$ has **asymptotic dimension** at most n (denoted $\operatorname{asdim}(X) \leq n$) if for every $C \in \mathcal{C}$ we can cover X with subsets $X_1, \dots, X_{n+1}$ where each X_i can be partitioned into a uniformly bounded family that is C-disjoint. Moreover, we say $\operatorname{asdim}(X) = n$ if $\operatorname{asdim}(X) \leq n$ and $\operatorname{asdim}(X) \nleq n - 1$.

This definition is the natural generalization of Theorem 19 (2) in [BD08a]. For metric spaces, a common definition of asymptotic dimension involves the Lebesgue number and multiplicity of a cover. In [Gra06], the author proves the equivalence of Roe's definition above with a generalization of the Lebesgue number definition. We will see a proof that finite asymptotic dimension implies Property A using Definition 5.5. Proofs of this fact in the metric setting can be found in [Wil09] or [Roe03], but a more direct proof can be found in [CDV08]. The upcoming proof mimics the proof in [CDV08]. First we state a simple, yet necessary, lemma.

Lemma 6.3. *If $\{K_\alpha\}$ is a uniformly bounded family and C is controlled and symmetric, then $\{C[K_\alpha]\}$ is a uniformly bounded family.*

Proof. Just notice that

$$\bigcup (C[K_\alpha])^2 \subset C \circ \bigcup K_\alpha^2 \circ C$$

□

Theorem 6.4. *Let $(X, \mathcal{C})$ be a coarse space. If asdim$(X) \leq n$ for some positive integer n, then X has Property A.*

Proof. Let $C \in \mathcal{C}$ and $\varepsilon > 0$ be given. First enlarge C to be symmetric and contain Δ and diminish $\varepsilon < .001$. Choose an integer $M > 2n/\varepsilon + 1$. Since $\operatorname{asdim}(X) \leq n$, we can write $X = X_1 \cup \cdots \cup X_{n+1}$ where each X_i can be partitioned as

$$X_i = \bigsqcup_{\alpha \in J_i} K_\alpha^i$$

where each $\{K_\alpha^i\}$ is a uniformly bounded family that is C^{2M}-disjoint.

Now choose a representative x_α^i from each set $C^M[K_\alpha^i]$. These will be the points that we will use to define the sets A_x. For a given point x, we define $A_x \subset X \times \mathbb{N}$ to be the union of the "stacked" representatives $\{x_\alpha^i \times 1, \ldots, x_\alpha^i \times h_\alpha(x)\}$ where $x \in C^M[K_\alpha^i]$ and $h_\alpha(x)$ is the length of a minimal C-chain connecting x to a point $z \notin C^M[K_\alpha^i]$. If no chain exists, let $h_\alpha(x) = M$.

Before proceeding, we should note that the cover $\{C^M[K_\alpha^i]\}$ is of multiplicity at most $n+1$. For if $x \in C^M[K_{\alpha_0}^i]$ for some $i \in \{1, \ldots, n+1\}$ and $\alpha_0 \in J_i$, then $x \notin C^M[K_\alpha^i]$ for any $\alpha \neq \alpha_0$ since each ith partition is C^{2M}-disjoint. Hence, any given x can only belong to at most $n+1$ sets in $\{C^M[K_\alpha^i]\}$ with two never coming from the same i. This allows us to say that $0 < |A_x| < \infty$.

Suppose $(x, y) \in C$. If $x, y \in C^M[K_\alpha^i]$, then $|h_\alpha(x) - h_\alpha(y)| \leq 1$. If $x \in C^M[K_\alpha^i]$ and $y \notin C^M[K_\alpha^i]$ for some α, then $h_\alpha(x) = 1$. Since $(x, y) \in C$, there exists $C^M[K_\alpha^i]$ for some i and α such that $x, y \in C^M[K_\alpha^i]$ (just choose the partition elements K_α^i containing x). Thus,

$$|A_x \cap A_y| > M - 1.$$

Furthermore, there can be at most n sets $C^M[K_\alpha^i]$ for which $x \in C^M[K_\alpha^i]$ and $y \notin C^M[K_\alpha^i]$, or vice-versa; this is due to the partitions being C^{2M}-disjoint. Hence, in total we have at most $2n$ of such sets; therefore, since $(x, y) \in C$

$$|A_x \Delta A_y| \leq 2n.$$

Combining our results, we obtain

$$\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \frac{2n}{M-1} \leq \frac{2n}{2n/\varepsilon} = \varepsilon.$$

Finally, define a controlled set D by

$$D = \bigcup_{i=1}^{n+1} \bigcup_{\alpha \in J_i} (C^M[K_\alpha^i])^2.$$

We claim that $A_x \subset D_x \times \mathbb{N}$ for all $x \in X$. This is easy; for if $z \times m \in A_x$, then z is a representative of any $C^M[K_\alpha^i]$ in which x belongs, and one such set must exist. Thus, $(z, x) \in D$. $\square$

Chapter 7

More on S_Ω

In the following section, many of the results are basic results from ordinal arithmetic. A good source of study is the book [Jec03].

We use 0 to denote the first element of S_Ω. J. Dydak proposed a new coarse structure on S_Ω. To define this structure, we need a definition.

Definition 7.1. An element $\chi \in S_\Omega$ is called a **pivot** when we consider an order-preserving map f on $[0, \Omega)$ such that $0 \mapsto \chi$.

When we want to make the pivot clear, we use the notation f_χ. A pivot allows one to shift elements "to the right" in S_Ω. It has the effect of shifting an element x to $f_\chi(x)$ so that the "distance" from 0 to x is in some sense the same as the "distance" χ to $f_\chi(x)$. Notice that $f_\chi(x) = \chi + x$, so the pivot map f_χ with pivot χ is just ordinal addition on the left by χ. For an ordinal x, we will let $x + 1$ denote the immediate successor of x.

Any element of S_Ω is a pivot: just define the map f using transfinite induction. If x has an immediate predecessor y, let $f(x) = f(y) + 1$. If x does not have an immediate predecessor, let $f(x)$ be the least upper bound of the set $S := \{f(y) \,|\, y < x\}$, which exists since the set S is countable and thus bounded above.

One way to define a coarse structure $\mathcal{C}_p$ via pivots is as follows. We say that C is controlled if there exists a pivot $\chi \in S_\Omega$ such that for every $(x, y) \in C$, we have $x \leq \chi + y$ and $y \leq \chi + x$. Essentially this means that the larger coordinate is smaller than the shift of the smaller coordinate.

There are a number of other basic facts regarding ordinal addition that need to be mentioned.

Proposition 7.2. *Let $\phi, x \in S_\Omega$. Then*

i. The map $\chi \mapsto \chi + x$ is nondecreasing.

ii. The map $y \mapsto \phi + y$ is strictly increasing, takes successor ordinals to successor ordinals, and takes limit ordinals to limit ordinals.

iii. $\phi < \phi + x \qquad (x > 0)$

iv. $x \leq \phi + x$

Proof. To prove (i.) we use transfinite induction on x. If $x = 0$, the result is trivial. If x is a successor ordinal with immediate predecessor $w \geq 0$, suppose that the map $\chi \mapsto \chi + z$

is nondecreasing for all $z \leq w$. Now suppose $\chi < \psi$. Then

$$\chi + x = (\chi + w) + 1 \leq (\psi + w) + 1 = \psi + x.$$

If x is a limit ordinal, suppose that the map $\chi \mapsto \chi + w$ is nondecreasing for all $w < x$. Then if $\chi < \psi$,

$$\chi + x = \sup\{\chi + w \,|\, w < x\} \leq \sup\{\psi + w \,|\, w < x\} = \psi + x.$$

By transfinite induction, $\chi \mapsto \chi + x$ is nondecreasing on S_Ω.

The proof of (ii.) proceeds a bit differently, although transfinite induction is still used. First, note that $\psi + 0 < \psi + 1$ for all $\psi \in S_\Omega$; this obvious result drives the proof. Now suppose that $y \mapsto \phi + y$ is strictly increasing on $[0, z)$, where z is a successor ordinal with immediate predecessor w. Then if $y < z$, $\phi + y \leq \phi + w < (\phi + w) + 1 = \phi + z$, where the equality follows by definition. Thus, $y \mapsto \phi + y$ is strictly increasing on $[0, z]$. Finally, suppose that $y \mapsto \phi + y$ is strictly increasing on $[0, z)$ where z is a limit ordinal. Then if $y < z$, we also know that $y + 1 < z$. Hence, $\phi + y < \phi + (y + 1) \leq \phi + z$ since $\phi + z = \sup\{\phi + w \,|\, w < z\}$. Thus, $y \mapsto \phi + y$ is strictly increasing on $[0, z]$. By transfinite induction, $y \mapsto \phi + y$ is strictly increasing on S_Ω. The second remark in (ii.) follows straight from the definition of addition.

The inequalities in (iii.) and (iv.) follow from (ii.) and (i.), respectively. □

Before we show that this is indeed a coarse structure on S_Ω, we state the following lemmas.

Lemma 7.3. *Given two pivots χ_1 and χ_2 in S_Ω and $y \in S_\Omega$, we have the following equality.*

$$(\chi_1 + \chi_2) + y = \chi_1 + (\chi_2 + y) \tag{7.1}$$

Proof. We proceed by transfinite induction. Note that for $y = 0$, the result is trivial. First suppose that y is a successor ordinal with immediate predecessor w and that w satisfies (7.1) (in place of y). We observe that $\chi_2 + y$ is a successor ordinal since it is by definition the immediate successor of $\chi_2 + w$. Thus, $\chi_1 + (\chi_2 + y)$ is by definition the immediate successor of $\chi_1 + (\chi_2 + w)$. Also, by definition, we know that $(\chi_1 + \chi_2) + y$ is the immediate successor of $(\chi_1 + \chi_2) + w$. Since immediate successors are unique, we use the induction hypothesis involving w to obtain (7.1).

Now suppose y is a limit ordinal. Assume the induction hypothesis that w satisfies (7.1) (in place of y) for all $w < y$. Let

$$\begin{aligned} S_1 &= \{(\chi_1 + \chi_2) + w \,|\, w < y\} \\ S_2 &= \{\chi_1 + w \,|\, w < \chi_2 + y\} \\ a &= \sup S_1 \\ b &= \sup S_2 \end{aligned}$$

Observe that the equation $a = b$ is equivalent to (7.1) since $\chi_2 + y$ is a limit ordinal.

First we show that $a \leq b$. Let $(\chi_1 + \chi_2) + w \in S_1$. Then $w < y$ and the induction hypothesis holds. Moreover, Proposition 7.2 gives $\chi_2 + w < \chi_2 + y$ so that $\chi_1 + (\chi_2 + w) \in S_2$. Owing to the induction hypothesis, $(\chi_1 + \chi_2) + w \in S_2$. Thus, b is an upper bound for S_1. Therefore, $a \leq b$.

Finally we show that $b \leq a$. Let $\chi_1 + w \in S_2$. Then $w < \chi_2 + y$. Since $\chi_2 + y$ is the supremum of the set $\{\chi_2 + z \,|\, z < y\}$, there exists an element $z < y$ such that $w < \chi_2 + z$. Hence, $\chi_1 + w < \chi_1 + (\chi_2 + z) = (\chi_1 + \chi_2) + z \leq a$. Thus, a is an upper bound for S_2. Therefore, $b \leq a$. $\square$

Lemma 7.4. *Define* $n\phi = \underbrace{\phi + \cdots + \phi}_{\text{n times}}$. *Then* $x \leq \phi + x$ *with equality if and only if* $x \geq M$, *where* $M = \lim_{n \to \infty} n\phi$.

Proof. The inequality was already shown in Proposition 7.2. We focus on the aspect of equality. The logical equivalence is shown as follows. Suppose $x \geq M$ where $M = \lim_{n\to\infty} n\phi$. It suffices to restrict to the case that $x = M$; for if $x < \phi + x$ and $x > M$, then

$$M < x < \phi + x < \phi + M = M,$$

a contradiction provided we verify the $x = M$ case. So now suppose $x = M$ and $M < \phi + M$. Since M is a limit ordinal, so is $\phi + M$. Choose $v < M$ such that $M < \phi + v < \phi + M$. Since $v < M$, there exists an n such that $v < n\phi$. Hence, $\phi + v < \phi + n\phi = (n+1)\phi < M$, a contradiction since $\phi + v > M$. Hence, $M = \phi + M$.

Conversely, suppose $x = \phi + x$. Assume $x < M$, and choose n_0 such that $x < n_0\phi$. Then for all n, $n\phi < n\phi + x = x$ by associativity. Hence, $x < n_0\phi < x$, a contradiction. Therefore, $x \geq M$. $\square$

With Proposition 7.2 and Lemma 7.3, it is not difficult to verify that $\mathcal{C}_p$ is a coarse structure, which we'll call the **pivotal coarse structure on** S_Ω. Of course, $\Delta \in \mathcal{C}_p$ since $x \leq \chi + x$ for any $\chi \in S_\Omega$. Subsets of a controlled set C are controlled with respect to the same pivot that works for C. If C is controlled, then so is C^t since the slot of a coordinate is independent of the order of the elements in S_Ω. If $C, D \in \mathcal{C}_p$, then $C \cup D \in \mathcal{C}_p$ by letting $\chi = \chi_1 \vee \chi_2$, where χ_1, χ_2 are the pivots of C, D, respectively. To check, we let $(x, y) \in C \cup D$. Suppose without loss of generality that $x > y$. If $(x, y) \in C$, then $x \leq \chi_1 + y \leq \chi + y$ by Proposition 7.2. If $(x, y) \in D$, then $x \leq \chi_2 + y \leq \chi + y$. Finally, to check that $C \circ D$ is controlled, where C, D have pivots χ_1, χ_2, respectively, let $\chi = (\chi_2 + \chi_1) \vee (\chi_1 + \chi_2)$. Then if $(x, y) \in C \circ D$, with $x > y$, then there exists $z \in S_\Omega$ such that $(x, z) \in C$ and $(z, y) \in D$. Hence, $x \leq \chi_1 + z \leq \chi_1 + (\chi_2 + y) = (\chi_1 + \chi_2) + y \leq \chi + y$ by associativity lemma. The case $y < x$ follows similarly.

Unfortunately, the pivotal coarse structure is nothing new. It is the same as the ordered coarse structure $\mathcal{C}_o$. This follows from Lemma 7.4. For if $C \in \mathcal{C}_p$ is controlled with pivot χ, then $x = \chi + x$ for all $x \geq M$ where $n\chi \to M$ as $n \to \infty$. Then $C \subset [0, M]^2 \cup \Delta$, which is controlled in $\mathcal{C}_o$. For if $x < y$ with $y \leq \chi + x$ and $y > M$, we obtain $y = \chi + y \leq \chi + x$ while $\chi + x < \chi + y$, a contradiction. The reverse containment is trivial: if $C \in \mathcal{C}_o$, then its control element is a pivot.

Since ordinal addition is not commutative, we can reverse the order of addition to create a different coarse structure.

Definition 7.5. The **translational coarse structure** on S_Ω, denoted $\mathcal{C}_t$, is comprised of controlled sets C for which there exists an $x \in S_\Omega$ such that

$$\chi \leq \phi + x \qquad \text{and} \qquad \phi \leq \chi + x \qquad\qquad \text{for all } (\chi, \phi) \in C.$$

Of course, every controlled set C has a smallest x satisfying the inequalities above. We call this x the **shifting element** for C.

The proof that this is a coarse structure is similar to the proof for the pivotal coarse structure. The diagonal is controlled with respect to the shifting element 0. Suppose $C, D \in \mathcal{C}_t$ with respect to shifting elements x and y, respectively. Then C^t and any subset of C is controlled with respect to the shifting element of C. The set $C \cup D$ is controlled with respect to the shifting element $x \vee y$. To see that $C \circ D \in \mathcal{C}_t$, let $\check{x} = (x+y) \vee (y+x)$. Then if $(\chi, \phi) \in C \circ D$, there exists $\psi \in S_\Omega$ such that $(\chi, \psi) \in C$ and $(\psi, \phi) \in D$. Then $\chi \leq \psi + x \leq (\phi + y) + x = \phi + (y + x) \leq \phi + \check{x}$ and $\phi \leq \psi + y \leq (\chi + x) + y = \chi + (x + y) \leq \chi + \check{x}$.

The translational coarse structure is actually larger than the previous two identical coarse structures. To see this, consider the controlled set

$$C_1 := \{(\chi, \phi) \,|\, \chi \leq \phi + 1, \phi \leq \chi + 1\}.$$

This set is not controlled in the ordered coarse structure; for any $M \in S_\Omega$, $(M, M+1) \in C_1$ trivially. Finally, if $C \in \mathcal{C}_o$ with control element M, then $C \in \mathcal{C}_t$ with shifting element M. For if $\chi, \phi \leq M$, then $\chi \leq M \leq \phi + M$ and $\phi \leq M \leq \chi + M$.

We can now investigate whether this larger coarse structure is metrizable, of bounded geometry, has finite asymptotic dimension, or has Property A. It is again a simple matter to show that this structure is nonmetrizable. If the structure has a countable basis, then each controlled set is associated to its shifting element. Since the collection of such shifting elements is countable, it is bounded above by say $\bar{x}$. Then any controlled set with its shifting element greater than $\bar{x}$ will not be contained in any of the basis sets; just consider the set $\{(0, \bar{x} + 1)\}$ for example. This contradicts the existence of a countable basis. Therefore, $(S_\Omega, \mathcal{C}_t)$ is nonmetrizable.

Theorem 7.6. *The coarse space $(S_\Omega, \mathcal{C}_t)$ is not of bounded geometry.*

Proof. Suppose that $E \in \mathcal{C}_t$ is symmetric and contains the diagonal of S_Ω. Let x be the shifting element for E. Suppose $nx \to y$ as $n \to \infty$ and let

$$C = \{(\chi, \phi) \,|\, \chi \leq \phi + y, \phi \leq \chi + y\}.$$

Consider the set $A = \{0, 2x, 4x, \dots (2n)x, \dots\}$. This set is clearly E-separated: if $m < n$, then $2mx + x < 2mx + 2x = 2(m+1)x \leq 2nx$. Moreover, $A \subset C_0$ since $2mx = 0 + 2mx < 0 + y$ and $0 < 2mx + y$. Since E contains the diagonal, $A \subset C[E_0]$. Therefore,

$$\mathrm{cap}_E C[E_0] = \infty,$$

and S_Ω cannot be of bounded geometry with respect to any gauge. □

We can show that $(S_\Omega, \mathcal{C}_t)$ has asymptotic dimension 0. Then by the main result of Chapter 6, S_Ω must have Property A.

Theorem 7.7. *The coarse space $(S_\Omega, \mathcal{C}_t)$ has asymptotic dimension 0.*

Proof. Given $C \in \mathcal{C}_t$, there exists $x \in S_\Omega$ such that $\chi \leq \phi + x$ and $\phi \leq \chi + x$ for all $(\chi, \phi) \in C$. Partition S_Ω by considering $\chi \sim \phi$ if $\chi + nx$ and $\phi + nx$ both converge to $M \in S_\Omega$. This is indeed a partition since any such chain $\chi + nx$ is bounded and if $\chi + nx \to M$ and $\phi + nx \to N$ with $M < N$, then clearly $\chi \not\sim \phi$.

Let each component be denoted D_M. We have to show two things: first that the components are uniformly bounded, and second, that the components are C-disjoint. Suppose $nx \to L$ as $n \to \infty$. If D_M is a given component, then for any $\chi, \phi \in D_M$ we can find n so that $\chi < \phi + nx$ since $\chi < M$ and $\phi + nx \to M$ as $n \to \infty$. Since $nx < L$, we conclude that $\chi < \phi + L$. Similarly, $\phi < \chi + L$. Thus, the collection is uniformly bounded since $\bigcup D_M^2$ has shifting element L.

If $\chi \in D_M$ and $\phi \in D_N$ with $M < N$, then for large n, we have $\chi < \chi + (n+1)x < M < \phi + nx$. Therefore, it is impossible to have $\phi \leq \chi + x$, for this would imply that $\phi + nx \leq \chi + (n+1)x$. Therefore, such a point $(\chi, \phi) \notin C$, showing that distinct components are C-disjoint. $\square$

Corollary 7.8. *The coarse space $(S_\Omega, \mathcal{C}_t)$ has Property A.*

Chapter 8

Products of Coarse Spaces

8.1 Definitions

In [Roe03], Roe gives a definition for the product coarse space $X \times Y$ of two coarse spaces on X and Y by calling a set controlled in $(X \times Y)^2$ if its projections to $X \times X$ and $Y \times Y$ are controlled. This notion of product naturally generalizes to finite products, as used in [Gra06], for example. We will revisit this definition in a moment for *arbitrary* products of coarse spaces, and we will compare it to 2 other coarse structures.

Let's start with a simple lemma.

Lemma 8.1. *Let X and Y be a coarse spaces, and let C, D be controlled sets in the coarse structure on X. Then if $f : X \to Y$, we have*

$$f^2(C \circ D) \subset f^2(C) \circ f^2(D)$$

Proof. Suppose $(y_1, y_2) \in f^2(C \circ D)$. Then there exists $(x_1, x_2) \in C \circ D$ that maps to (y_1, y_2). So there exists $z \in X$ such that $(x_1, z) \in C$ and $(z, x_2) \in D$. Hence, $(y_1, f(z)) \in f^2(C)$ and $(f(z), y_2) \in f^2(D)$. Therefore, $(y_1, y_2) \in f^2(C) \circ f^2(D)$. $\square$

Let $(X_\alpha, \mathcal{C}_\alpha)$ be coarse spaces with $\alpha \in J$. Let $Y = \prod_{\alpha \in J} X_\alpha$. We will define three coarse structures on Y, starting with the generalization of Roe's definition. The setup should be reminiscent to similar constructions in topology.

Definition 8.2. The map $\pi_\alpha^2 : Y^2 \to X_\alpha^2$ will be denoted by π_α in the definitions below.

1. Define a coarse structure $\mathcal{D}_{\text{prod}}$ on $Y = \prod_{\alpha \in J} X_\alpha$ by declaring D controlled if its projections $\pi_\alpha(D)$ are controlled. This is referred to as the **product coarse structure** on Y.

2. Define a coarse structure $\mathcal{D}_{\text{cap}}$ on $Y = \prod_{\alpha \in J} X_\alpha$ by declaring D controlled if its projections $\pi_\alpha(D)$ are controlled with the restriction that for all but finitely many α, $\pi_\alpha(D) \subset \Delta_{X_\alpha}$. This is referred to as the **capped coarse structure** on Y.

3. Suppose the coarse spaces X_α are all identical. Define a coarse structure $\mathcal{D}_{\text{unif}}$ on $Y = \prod_{\alpha \in J} X_\alpha$ by declaring D controlled if there exists a controlled set C in X_α such that $\pi_\alpha(D) \subset C$ for all α. This is referred to as the **uniform coarse structure** on Y.

Of course for this definition to be valid, we must verify that these are all coarse structures. We will sketch a proof that $\mathcal{D}_*$ is a coarse structure simultaneously. Let $D, E \in \mathcal{D}_*$; the following inclusions are all clear, except the fifth, which is the lemma above:

$$\begin{array}{rcll} \pi_\alpha(\Delta_Y) & \subset & \Delta_{X_\alpha} & \\ \pi_\alpha(D^t) & \subset & \pi_\alpha(D)^t & \\ \pi_\alpha(D) & \subset & \pi_\alpha(E) & \qquad\qquad (\text{if } D \subset E) \\ \pi_\alpha(D \cup E) & \subset & \pi_\alpha(D) \cup \pi_\alpha(E) & \\ \pi_\alpha(D \circ E) & \subset & \pi_\alpha(D) \circ \pi_\alpha(E). & \end{array}$$

First, we see that $\Delta_Y \in \mathcal{D}_*$ since ΔX_α is controlled, contained in itself for all α, and all diagonals are identical when the X_α spaces are identical. For closure under transpositions, subsets, finite unions and products, just note that the given controlled set has properties that place it in the right of one of the inclusions above, and the set under question will be contained in it to the left. For example, suppose $D \subset E$ and E is controlled. Then if all $\pi_\alpha(E)$ are controlled, the same holds for D; if all but finitely many $\pi_\alpha(E)$ are contained in Δ_{X_α}, then so are the $\pi_\alpha(D)$; in the uniform case, if there is a controlled set C such that $\pi_\alpha(E) \subset C$ for all α, then $\pi_\alpha(D) \subset C$ as well for all α. Note that in the union and product cases, one may have to take unions or products in the proof, but it is straightforward. For example, if $D, E \in \mathcal{D}_{\text{cap}}$, then $\pi_\alpha(D) \subset \Delta_{X_\alpha}$ and $\pi_\alpha(E) \subset \Delta_{X_\alpha}$ for all but finitely many α. Hence, $\pi_\alpha(D \cup E) \subset \Delta_{X_\alpha}$ and $\pi_\alpha(D \circ E) \subset \Delta_{X_\alpha}$ for all but finitely many α too (Note $\Delta^2_{X_\alpha} = \Delta_{X_\alpha}$).

It should be clear that $\mathcal{D}_{\text{cap}} \subset \mathcal{D}_{\text{prod}}$, and when the X_α are identical

$$\mathcal{D}_{\text{cap}} \subset \mathcal{D}_{\text{unif}} \subset \mathcal{D}_{\text{prod}}.$$

Moreover, when the indexing set for the product is finite, all of the coarse structures coincide. To see that the inclusions are proper, just take $X = \mathbb{R}$ with the bounded coarse structure induced by the euclidean metric, and let $Y = \prod_{i=1}^{\infty} \mathbb{R}$. Let

$$E_n = \{(x, y) \mid d(x, y) \leq n\}. \tag{8.1}$$

Then $\prod_{i=0}^{\infty} E_i \in \mathcal{D}_{\text{prod}} \setminus \mathcal{D}_{\text{unif}}$ and $\prod_{i=0}^{\infty} E_5 \in \mathcal{D}_{\text{unif}} \setminus \mathcal{D}_{\text{cap}}$.

We want to investigate questions related to the coarse concepts of metrizability, bounded geometry, and asymptotic dimension. In particular, we want to determine how these properties behave for the different coarse structures on a product space. One can check that the product coarse structure is a product in the category of coarse spaces. Thus, this coarse structure proves to be the most important one. However, the other two structures are useful in their own regard.

Before we delve into the main concepts, there is one point about connectivity worth mentioning. It is true that given an infinite collection of connected coarse spaces X_α, we obtain a connected product Y in the product coarse structure, but not in the capped coarse structure. This fact is easily seen for the product coarse structure simply by considering the product of all controlled sets containing the projections of a point $(\mathbf{y_1}, \mathbf{y_2})$. However, in the capped coarse structure case, we would need a point $(\mathbf{y_1}, \mathbf{y_2})$ to satisfy $\pi_\alpha(\mathbf{y_1}) = \pi_\alpha(\mathbf{y_2})$ to hold for all but finitely many α.

8.2 Metrizability

First we tackle metrizability. The capped coarse structure does indeed prove to be metrizable for *countable products* of metrizable spaces, while the product coarse structure can, in some cases, be nonmetrizable. We give one such example using a countable product of $\mathbb{R}$, but any unbounded, connected space will do. The uniform coarse structure for a product of metrizable spaces is always metrizable; just use the uniform metric induced by the metrics of each factor (infinite distances are possible).

Proposition 8.3. *Let $(X_\alpha, \mathcal{C}_\alpha)$ be metrizable coarse spaces with $\alpha \in \mathbb{N}$. Let $Y = \prod_\alpha X_\alpha$, and equip Y with the capped coarse structure. Then Y is metrizable.*

Proof. Let $C_1^\alpha, C_2^\alpha, \ldots$ be a countable basis for X_α. We proposed that the family

$$\mathcal{F} := \{\prod_\alpha E_\alpha \,|\, E_\alpha \subset \Delta_{X_\alpha} \text{ for all but finitely many } \alpha \text{ and } E_\alpha = C_i^\alpha \text{ for the remaining } \alpha\}$$

is a countable basis for the capped coarse structure on Y. This family is countable since it can be realized as a countable union of countable sets. It is a basis since given any $D \in \mathcal{D}_{\text{cap}}$ with $\pi_\alpha(D) = C_\alpha$, we know that for some finite set of indices J_f, $C_\alpha \subset C_i^\alpha$ for some i when $\alpha \in J_f$, and $C_\alpha \subset \Delta_{X_\alpha}$ when $\alpha \notin J_f$. So let

$$E_\alpha = \begin{cases} \Delta_{X_\alpha} & \text{if } \alpha \notin J_f \\ C_i^\alpha & \text{if } \alpha \in J_f. \end{cases}$$

Then $D \subset \prod_\alpha E_\alpha$. For if $(\mathbf{x}, \mathbf{y}) \in D$, then for $\alpha \in J_f$, $(x_\alpha, y_\alpha) \in C_i^\alpha = E_\alpha$, and for $\alpha \notin J_f$, $(x_\alpha, y_\alpha) \in C_\alpha \subset \Delta_{X_\alpha} = E_\alpha$; hence, $(\mathbf{x}, \mathbf{y}) \in \prod_\alpha E_\alpha$. □

The following example shows that the previous proposition does not hold true for the product coarse structure. Note that this example generalizes to any collection of coarse spaces X_α that are unbounded and coarsely connected.

Example 8.4. Consider $Y = \prod_{n=1}^\infty \mathbb{R}$ equipped with the product coarse structure. Then Y is nonmetrizable. We will verify this conclusion by contradiction. Suppose Y is metrizable with a countable basis $\{D_i\}$. Define $C_i^n = \pi_n(D_i)$. Since $\mathbb{R}$ is coarsely connected and unbounded, we can choose a point in $\mathbb{R}^2 \setminus C_n^n$ and a controlled set F_n containing (x, y) to obtain a "larger" controlled set $E_n := C_n^n \cup F_n$. We claim that $\prod_{n=1}^\infty E_n$ is not contained in D_i for any i. For if $\prod_{n=1}^\infty E_n \subset D_{i_0}$ for some i_0, then $E_{i_0} \subset C_{i_0}^{i_0}$. However, E_{i_0} was built to *properly* contain $C_{i_0}^{i_0}$, creating a contradiction.

8.3 Bounded Geometry

Bounded geometry does not behave well under products using any of the coarse structures above. The reason for this is that even though we may have control over E-separated sets in the projection of a $D \circ E$ ball in the product, we have no uniform control over them when the indexing set for the product is infinite. Even in the uniform case with $(X, \mathcal{C})$ of bounded geometry, examples of $Y = \prod_\alpha X$ without bounded geometry can be constructed. Do note that since each factor X_α coarsely embeds into the product Y, it is necessary that each X_α be of bounded geometry if the product Y is of bounded geometry.

Example 8.5. Consider $X = \mathbb{R}$, which is of bounded geometry being coarsely equivalent to the discrete space $\mathbb{Z}$ via the map $x \mapsto \lfloor x \rfloor$. Let Y be the countably infinite product of $\mathbb{R}$ with the uniform coarse structure. Suppose that Y is of bounded geometry with respect to the gauge E where $r = \sup\{(\rho(\mathbf{x}, \mathbf{y}) \mid (\mathbf{x}, \mathbf{y}) \in E\}$. Let F be the controlled set $\{(\mathbf{x}, \mathbf{y}) \mid \rho(\mathbf{x}, \mathbf{y}) \leq r + 1\}$. Consider the infinite set of points $A = \{\mathbf{x}_i\}_{i=1}^{\infty}$ where

$$\pi_j(\mathbf{x}_i) = \begin{cases} r+1 & \text{if } j < i \\ 0 & \text{if } j \geq i. \end{cases}$$

Since $A \subset F[E_{\mathbf{0}}]$ and E-separated, $\operatorname{cap}_E F = \infty$, so that Y is not of bounded geometry.

8.4 Asymptotic Dimension

In this section, we will show that for spaces X_α satisfying $\operatorname{asdim}(X_\alpha) = 0$ for all α produce a product $Y = \prod X_\alpha$ with asymptotic dimension 0. This is well known in the finite product case; see Section 7 of [BD08a] for a more general result involving products. Our result will work for all three coarse structures on the product.

Theorem 8.6. *Suppose $(X_\alpha, \mathcal{C}_\alpha)$ are coarse spaces for each $\alpha \in A$. Create a product coarse space by setting $Y = \prod_\alpha X_\alpha$ and equip Y with any of the three coarse structures $\mathcal{D}_*$. If asdim$(X_\alpha) = 0$ for all α, then asdim$(Y) = 0$.*

Proof. The proof is relatively straightforward. Suppose $D \in \mathcal{D}_*$ is given; let $D_\alpha = \pi_\alpha(D)$. Since $\operatorname{asdim}(X_\alpha) = 0$ for all α, we can partition each X_α by writing

$$X_\alpha = \bigsqcup_{\beta_\alpha \in J_\alpha} K_{\beta_\alpha},$$

where $\{K_{\beta_\alpha}\}_{\beta_\alpha \in J_\alpha}$ is a uniformly bounded family that is D_α-disjoint.

In the capped coarse structure case, we modify the decomposition of the X_α where $\alpha \in J_i$ is an index where $D_\alpha \subset \Delta_{X_\alpha}$. In this case, we can and will simply partition X_α into the union of its singleton sets. This is sufficient since the components will be Δ_{X_α}-disjoint trivially and

$$\bigcup_{x_\alpha \in X_\alpha} \{x_\alpha\}^2 = \Delta_{X_\alpha}$$

is controlled. Therefore, in the capped coarse structure case, we will always assume X_α has this decomposition when $\alpha \in J_i$; in the other cases, we will just use the hypothesized decomposition above.

Now we construct a D-disjoint partition of Y. We partition Y as

$$Y = \bigsqcup_{\beta \in J} \left(\prod_\alpha K_{\beta_\alpha} \right)$$

where $J = \prod_\alpha J_\alpha$ and $\beta \in J$ is given by $\beta = (\beta_\alpha)_{\alpha \in A}$. It is not hard to see that the union is indeed disjoint.

First, we show that $\{\prod_\alpha K_{\beta_\alpha}\}_{\beta \in J}$ is D-disjoint. Suppose

$$\mathbf{x} \in \prod_\alpha K_{\beta_\alpha} \qquad \text{and} \qquad \mathbf{y} \in \prod_\alpha K_{\gamma\alpha}$$

with $\beta \neq \gamma$. Then for some α, $\beta_\alpha \neq \gamma_\alpha$. For this α, K_{β_α} and K_{γ_α} are D_α-disjoint, whence $(x_\alpha, y_\alpha) \notin D_\alpha$. Therefore, $(\mathbf{x}, \mathbf{y}) \notin D$. Note that this part of the proof works same for all three structures $\mathcal{D}_*$.

Finally, we seek to show that $\{\prod_\alpha K_{\beta_\alpha}\}_{\beta \in J}$ is uniformly bounded. In the product coarse structure case, this follows easily from the inclusion

$$\bigcup_{\beta \in J} \left(\prod_\alpha K_{\beta_\alpha} \right)^2 \subset \prod_\alpha \bigcup_{\beta_\alpha \in J_\alpha} (K_{\beta_\alpha})^2 , \tag{8.2}$$

since the set on the right is controlled in the product coarse structure.

This is not immediately obvious for the other two structures. For the uniform coarse structure, the decompositions of each projection X_α can of course be chosen to be identical, so that each of the factors in the right-hand side of (8.2) are the same controlled set, yielding a controlled set in Y. For the capped coarse structure, we made sure to decompose certain X_α carefully so that all but finitely many of the factors in the right-hand side of (8.2) are contained in Δ_{X_α}. Again, this insures that the right-hand side of (8.2) is controlled. $\square$

8.5 Further Questions

It would be of interest to see if one can construct a coarse space that does not have Property A using products. Is there a way to relate this process to coarse embeddings into a Hilbert space? Could coarse structures on S_Ω play a role here?

In addition, it would be of interest to research metric approximations more thoroughly. For example, is there a characterization of Property A using metric approximations? Can it be used to construct examples of spaces without Property A more easily? Are there more interesting coarse structures on S_Ω that would require approximations to understand? These questions represent just a handful of further research opportunities.

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Vita

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