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Combinatorial Unification of Binomial-Like Arrays

James Stephen Lindsay
University of Tennessee - Knoxville

To the Graduate Council:

I am submitting herewith a dissertation written by James Stephen Lindsay entitled "Combinatorial Unification of Binomial-Like Arrays." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Carl G. Wagner, Major Professor

We have read this dissertation and recommend its acceptance:

Pavlos Tzermias, Xia Chen, Gina Pighetti

Accepted for the Council:

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Combinatorial Unification of Binomial-Like Arrays

A Dissertation
Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

James Stephen Lindsay
May 2010

Dedication

I dedicate this thesis to Tammy and Steve, who started this ball rolling, to Cynthia and John, who kicked it in the present direction, and to Heather, who has to live with it now.

Acknowledgments

I would like to express my sincerest gratitude to my advisor, Professor Carl G. Wagner, for helping me make this so much more than I could have made it on my own. Thanks are due as well to the remainder of the faculty and many of the graduate students in the Mathematics Department at the University of Tennessee, in particular my committee members: Pavlos Tzermias and Xia Chen. I also deeply appreciate my family for their enduring patience with me during this process.

Abstract

This research endeavors to put a common combinatorial ground under several binomial-like arrays, including the binomial coefficients, q-binomial coefficients, Stirling numbers, q-Stirling numbers, cycle numbers, and Lah numbers, by employing symmetric polynomials and related words with specialized alphabets as well as a balls-and-urns counting approach. Using the method of statistical generating functions, q- and p, q-generalizations of the binomial coefficients, Stirling numbers, cycle numbers, and Lah numbers are all discussed as well, unified under a single general triangular array that is herein referred to as the array of Comtet-Lancaster numbers.

Contents

Chapter 1

Comtet's Theorem

1.0 Notation

The notational conventions herein are as follows:

- $\mathbb{N}$ denotes the set $\{0, 1, 2, \ldots\}$ of nonnegative integers.
- $\mathbb{P}$ denotes the set $\{1, 2, 3, \ldots\}$ of positive integers.
- $\mathbb{Q}$ denotes the field of rational numbers.
- $\mathbb{R}$ denotes the field of real numbers.
- $\mathbb{C}$ denotes the field of complex numbers.
- $\mathbb{F}_q$ and $\mathbb{F}_q^n$, for $n \in \mathbb{P}$, denote the finite field of q elements, when $q = p^d$, for some prime p, and the n-dimensional vector space thereupon, respectively, with $\mathbb{F}_q^0 := \{0\}$.
- For all $n \in \mathbb{P}$, $[n] := \{1, 2, \ldots, n\}$, with $[0] := \emptyset$.
- For all $n \in \mathbb{N}$, $[n]^* := \{0, 1, 2, \ldots, n\}$.

Other more specialized notations appear as well[1]. Some of those are as follows:

- For $n \in \mathbb{P}$, and x any indeterminate or complex number,

$$x^{\underline{n}} := x(x-1)(x-2)\cdots(x-n+1),$$

  with $x^{\underline{0}} := 1$, denotes the *falling factorial function* of degree n.

- For $n \in \mathbb{P}$, and x as above,

$$x^{\overline{n}} := x(x+1)(x+2)\cdots(x+n-1),$$

  with $x^{\overline{0}} := 1$, denotes the *rising factorial function* of degree n.

[1]Some of these will have their formal definitions given later in the text, frequently in another (equivalent) form altogether.

- For all $n \in \mathbb{P}$, and q an indeterminate, complex number, or, particularly, a power of a prime number,

$$n_q := 1 + q + q^2 + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1},$$

with $0_q := 0$, denotes a *q-integer.*

- For all $n \in \mathbb{P}$, and p and q indeterminates or complex numbers, including powers of prime numbers,

$$n_{p,q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + pq^{n-2} + q^{n-1} = \frac{q^n - p^n}{q - p},$$

with $0_{p,q} := 0$, denotes a *p, q-integer.*

- For all $n \in \mathbb{P}$ and q as above,

$$n\underset{q}{!} := n_q(n-1)_q(n-2)_q \cdots 2_q 1_q,$$

with $0\underset{q}{!} := 1$ denotes the *q-factorial function*, with *p, q-factorial* defined analogously by $0\underset{p,q}{!} := 1$ and for all $n \in \mathbb{P}$,

$$n\underset{p,q}{!} := n_{p,q}(n-1)_{p,q} \cdots 1_{p,q}.$$

- For all $n \in \mathbb{P}$, $k \in \mathbb{P}$, and q as above,

$$n_q^{\underline{k}} := n_q(n_q - 1_q)(n_q - 2_q) \cdots (n_q - (k-1)_q)$$

denotes the k^{th} *q-falling factorial*, observing $n_q^{\underline{k}} = 0$ for $k > n$, in particular $0_q^{\underline{k}} = 0$ for all $k \in \mathbb{P}$, and p, q-falling factorials are defined analogously.

1.1 Similarities Between Binomial Coefficients, Stirling Numbers, and Their q-Analogues

1.1.1 The Binomial Coefficients

Given $n, k \in \mathbb{N}$, let $\binom{n}{k} := |\{A : A \subset [n] \text{ and } |A| = k\}|$, with $k \in \mathbb{N}$. These are the *binomial coefficients.* Immediately, it follows that

Theorem 1.1.1. *For all $n, k \in \mathbb{N}$,*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{1.1}$$

whenever $0 \leq k \leq n$.

Proof. Since for any $n, k \in \mathbb{N}$ with $k \leq n$ it is well known (see, for instance, [20]) that $n^{\underline{k}} = \frac{n!}{(n-k)!}$ counts the permutations of k elements from $[n]$, $\binom{n}{k}$ counts the k-element subsets of $[n]$ by considering any permutation of k-elements of $[n]$ and mapping it to the (unordered) set of those k elements, which is a $k!$-to-one surjection. $\square$

and

Theorem 1.1.2. *For all $n, k \in \mathbb{N}$,*

$$\binom{n}{k} = \binom{n}{n-k}. \tag{1.2}$$

Proof. Straightforward by set complementation. $\square$

Furthermore, two more results follow from the definition:
First,

Theorem 1.1.3. *Given $k \in \mathbb{N}$, for all $n \in \mathbb{N}$,*

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k. \tag{1.3}$$

Proof. It more than suffices to establish this polynomial identity[2] for all $r \in \mathbb{P}$. Then (1.3) takes the form

$$(r+1)^n = \sum_{k=0}^{n} \binom{n}{k} r^k. \tag{1.4}$$

Then both sides of (1.4) count the n-letter words in the alphabet $[r]^*$. The left-hand side does this by filling n slots with the $r+1$ letters in $[r]^*$. The right-hand side does this in $k+1$ disjoint, exhaustive classes: those with exactly $n-k$ zeros, for $0 \leq k \leq n$. The term $\binom{n}{k} r^k$ chooses k positions from among the n which will have elements of $[r]$, and then the remaining $n-k$ positions are all filled in with 0's.

$\square$

By substitution of $x-1$ in place of x, (1.3) can be rewritten in the form

$$x^n = \sum_{k=0}^{n} \binom{n}{k} (x-1)^k. \tag{1.5}$$

Of course, (1.3) and (1.5) are special cases of the binomial theorem.

And second, a two-term recurrence,

Theorem 1.1.4. *For all $n, k \in \mathbb{P}$,*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \tag{1.6}$$

subject to the boundary conditions $\binom{0}{k} = \delta_{0,k}$ for all $k \in \mathbb{N}$ and $\binom{n}{0} = 1$ for all $n \in \mathbb{N}$.

[2] A polynomial identity involving a polynomial of degree n requires only $n+1$ verified instantiations to establish the result. This fact is sometimes called the "engineer's dream theorem" and can be found and proved in [20], for instance. Here, the infinite number of instances in which this polynomial identity holds far exceeds the necessary $n+1$.

Proof. The boundary conditions are obvious.

For $n, k \in \mathbb{P}$, among all k-element subsets of $[n]$, $\binom{n-1}{k-1}$ counts those that contain the element n, and $\binom{n-1}{k}$ counts those that do not. $\square$

From the recurrence (1.6), we derive the following formulas:
First, a column generating function,

Theorem 1.1.5. *For all $k \in \mathbb{N}$,*

$$\sum_{n\geq 0} \binom{n}{k} x^n = \frac{x^k}{(1-x)^{k+1}}, \tag{1.7}$$

Proof. For every $k \in \mathbb{N}$, let

$$g_k(x) := \sum_{n\geq 0} \binom{n}{k} x^n.$$

Then $g_0(x) = \sum \binom{n}{0} x^n = \sum x^n = \frac{1}{1-x}$ by the geometric series identity.
Using the recurrence (1.6), a recurrence for $g_k(x)$ arises for all $k \in \mathbb{P}$:

$$\begin{aligned} g_k(x) &= \sum_{n\geq 1} \left[\binom{n-1}{k-1} + \binom{n-1}{k}\right] x^n \\ &= \sum_{n\geq 1} \binom{n-1}{k-1} x \cdot x^{n-1} + \sum_{n\geq 1} \binom{n-1}{k} x \cdot x^{n-1} \\ &= x g_{k-1}(x) + x g_k(x). \end{aligned}$$

Whence, $g_k(x) = \frac{x}{1-x} g_{k-1}(x)$. The result follows by induction. $\square$

And second, a closed-form expression,

Theorem 1.1.6. *For all $n, k \in \mathbb{N}$,*

$$\binom{n}{k} = \sum_{\substack{d_0+d_1+\cdots+d_k=n-k \\ d_i\in\mathbb{N}}} 1, \tag{1.8}$$

Proof. The recurrence and boundary conditions in Theorem 1.1.4 are recovered from the right-hand side of (1.8) by considering separately the cases in which $d_k = 0$ and otherwise. $\square$

Alternatively, if choosing k elements from the sequence $(1, 2, \ldots, n)$ to make a k-element subset of $[n]$, then each d_i counts the number of elements in the i^{th} interval between the k chosen elements of the sequence, taking the zeroth interval to mean the one preceding the first chosen element. $\square$

Of course, all of these results are well known, appearing, for instance, in [20].

1.1.2 The Stirling Numbers of the Second Kind

Given a fixed $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$, let $\Pi_{n,k}$ denote the set of partitions of $[n]$ into k nonempty blocks[3], and let $S(n,k) := |\Pi_{n,k}|$. These are the *Stirling numbers of the second kind*[4]. Two results follow from this definition:
First, recalling (1.5) and with the following proof appearing in [20],

Theorem 1.1.7. *For any $n \in \mathbb{N}$,*

$$x^n = \sum_{k=0}^{n} S(n,k)x^{\underline{k}}. \tag{1.9}$$

Proof. It more than suffices to establish this polynomial identity for all $r \in \mathbb{P}$. Then (1.9) takes the form

$$r^n = \sum_{k=0}^{n} S(n,k)r^{\underline{k}}. \tag{1.10}$$

Each side of (1.10) counts the functions $f : [r] \to [n]$, the left-hand side directly. For the right-hand side, note that for each partition of $[n]$ into k nonempty blocks, there are $k!S(n,k)$ *ordered partitions*[5] of $[n]$ into k nonempty blocks since there are $k!$ possible orders for the k blocks. Furthermore, by mapping any ordered partition of $[n]$ into k nonempty blocks to a function $f : [n] \to [k]$ given by $f(i) = j$ whenever $i \in [n]$ appears in block $j \in [k]$, notice that $k!S(n,k)$ counts the surjective functions $f : [n] \to [k]$.

Thus, consider

$$\sum_{k=0}^{n} S(n,k)r^{\underline{k}} = \sum_{k=0}^{n} k!S(n,k)\frac{r!}{k!(r-k)!} = \sum_{k=0}^{n} k!S(n,k)\binom{r}{k}. \tag{1.11}$$

For each $k \in [n]$, the term $k!S(n,k)\binom{r}{k}$ counts those functions $f : [r] \to [n]$ in which $|range(f)| = k$ since $\binom{r}{k}$ chooses the k values in the range and $k!S(n,k)$ counts all surjective functions from $[r]$ to that k-element set. $\square$

And second, a two-term recurrence similar to (1.6),

Theorem 1.1.8. *For all $n, k \in \mathbb{P}$,*

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \tag{1.12}$$

subject to the boundary conditions $S(0,k) = \delta_{0,k}$, for all $k \in \mathbb{N}$ and $S(n,0) = 0$, for all $n \in \mathbb{P}$.

Proof. The boundary conditions are straightforward, and when $k > n$, $\Pi_{n,k} = \emptyset$. Thus, assume $n, k \in \mathbb{P}$ with $1 \le k \le n$. Then among those partitions of $[n]$ into k nonempty

[3] A very common variant on this is "distributions of n labeled balls into k unlabeled urns so that no urn is left empty."

[4] Unless otherwise indicated, "the Stirling numbers" will be taken to mean henceforth "the Stirling numbers of the second kind" unless explicit inclusion of the epithet is demanded for clarity or comparison, e.g. with the (signless) Stirling numbers of the first kind.

[5] An ordered partition is one in which the order in which the blocks appear will distinguish one ordered partition from another.

blocks, $S(n-1,k-1)$ counts those in which the element n appears as the only element in its block, and $kS(n-1,k)$ counts those in which the element n appears in a block containing at least one other element, for which there are k choices. □

From this recurrence, we derive the following formulas:
First, a column generating function similar to (1.7),

Theorem 1.1.9. *For all $k \in \mathbb{P}$,*

$$\sum_{n\geq 0} S(n,k)x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}. \tag{1.13}$$

Proof. For every $k \in \mathbb{N}$, let

$$g_k(x) := \sum_{n\geq 0} S(n,k)x^n.$$

Then $g_0(x) = \sum S(n,0)x^n = \sum \delta_{n,0}x^n = 1$.
Using the recurrence (1.12), a recurrence for $g_k(x)$ arises for all $k \in \mathbb{P}$:

$$\begin{aligned} g_k(x) &= \sum_{n\geq 1} \left[S(n-1,k-1) + kS(n-1,k)\right] x^n \\ &= \sum_{n\geq 1} S(n-1,k-1)x \cdot x^{n-1} + \sum_{n\geq 1} S(n-1,k)kx \cdot x^{n-1} \\ &= xg_{k-1}(x) + kxg_k(x). \end{aligned}$$

Whence, $g_k(x) = \frac{x}{1-kx} g_{k-1}(x)$. The result follows by induction. □

And second, a closed-form expression similar to (1.8),

Theorem 1.1.10. *For all $n, k \in \mathbb{P}$,*

$$S(n,k) = \sum_{\substack{d_0+\cdots+d_k=n-k \\ d_i\in\mathbb{N}}} 0^{d_0}1^{d_1}\cdots k^{d_k}. \tag{1.14}$$

Proof. The recurrence and boundary conditions in (1.12) can be recovered from the right-hand side of (1.14) by considering separately the cases in which $d_k = 0$ and otherwise. □

Another useful structure counted by $S(n,k)$ is the set of restricted growth functions from $[n]$ to $[k]$, first discovered by Stephen Milne [14].

Definition 1.1.11. *A surjective function $f : [n] \to [k]$ is a* restricted growth function *if in the sequence $(f(1), f(2), \ldots, f(n))$, the first occurrence of j precedes the first occurrence of $j+1$ for each $j \in [k-1]$. The set of restricted growth functions from $[n]$ to $[k]$ will be denoted $\mathcal{RGF}(n,k)$.*

Before connecting restricted growth functions to partitions of a set, it is advantageous to designate a canonical form for writing such partitions. In particular, it is useful to

1. List the elements within the blocks in increasing order by their magnitudes, and

2. List the blocks in increasing order by the magnitudes of their smallest (here: initial) elements.

Unless otherwise specified, henceforth such partitions are written in this canonical form.

Theorem 1.1.12. *There is a bijection between the sets* $\Pi_{n,k}$ *and* $\mathcal{RGF}(n,k)$.

Proof. Given a (canonically written) partition $\pi \in \Pi_{n,k}$, define a function $f_\pi : [n] \to [k]$ by $f_\pi(i) = j$ whenever i appears in the j^{th} block of π. Since the first occurrence of j will be from the smallest element of the j^{th} block, and likewise for the first occurrence of $j+1$, the canonical ordering on π provides that this map gives $f_\pi \in \mathcal{RGF}(n,k)$. Furthermore, given any $f \in \mathcal{RGF}(n,k)$, by placing each i in block j of a partition of $[n]$ with k nonempty blocks whenever $f(i) = j$, it is clear that this map is surjective. Finally, since any two distinct members of $\Pi_{n,k}$ have at least one element of $[n]$ appearing in different blocks, their associated functions will return different values for such elements. □

All of these results are well known, appearing, for instance, in [20].

1.1.3 The q-Binomial Coefficients

Given a fixed $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$ and q any power of a prime number, define the q-binomial coefficient $\binom{n}{k}_q$ to be the number of k-dimensional linear subspaces of the n-dimensional vector space $\mathbb{F}_q^n$. This array is, in fact, a q-analogue of the binomial coefficients, the latter being obtained from the former by choosing $q = 1$ and letting $[n]$ stand in place of $\mathbb{F}_1^n$, with subsets acting as the subspace-like structure. A few results follow directly from this definition:
First, q-analogues of (1.1) and (1.5):

Theorem 1.1.13. *For all* $n \in \mathbb{N}$ *and for* q *a prime power,*

$$\binom{n}{k}_q = \frac{n\underset{q}{!}}{k\underset{q}{!}(n-k)\underset{q}{!}}, \tag{1.15}$$

and

$$x^n = \sum_{k=0}^{n} \binom{n}{k}_q (x-1)(x-q)(x-q^2)\cdots(x-q^{k-1}). \tag{1.16}$$

Proof. Formula (1.15) follows from the fact that

$$\frac{n\underset{q}{!}}{k\underset{q}{!}(n-k)\underset{q}{!}} = \prod_{j=0}^{k-1} \frac{(q^n - q^j)}{(q^k - q^j)} = \frac{n^{\underline{k}}_q}{k^{\underline{k}}_q} \tag{1.17}$$

are algebraic variants of the right-hand side of (1.15). The middle expression in (1.17) counts the k-dimensional linear subspaces of $\mathbb{F}_q^n$ by considering first the set

$$\mathcal{C} := \{(x_1, \ldots, x_k) : \text{ the vectors } x_i \text{ are linearly independent in } \mathbb{F}_q^n\}.$$

The map from $\mathcal{C}$, which contains $(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})$ sequences, to the set of k-dimensional subspaces of $\mathbb{F}_q^n$ given by mapping a sequence of vectors to its linear span in $\mathbb{F}_q^n$ is a $(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})$-to-one surjection since there are $(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})$ ordered bases for each linear span, establishing (1.15).

By performing a top-down summation and replacing $\binom{n}{n-k}_q$ with $\binom{n}{k}_q$, it follows that (1.16) can be proved by showing that for all $r \in \mathbb{P}$,

$$(q^r)^n = \sum_{k=0}^{n} \binom{n}{k}_q \prod_{i=0}^{n-k-1} (q^r - q^i). \tag{1.18}$$

Both sides of equation (1.18) count the linear transformations $T : \mathbb{F}_q^n \to \mathbb{F}_q^r$. The right-hand side does so since among all such transformations T,

$$\binom{n}{k}_q \prod_{i=0}^{n-k-1} (q^r - q^i)$$

counts those with a k-dimensional null space. This can be verified by choosing first a k-dimensional subspace W of $\mathbb{F}_q^n$ and letting $(x_1, \ldots, x_k)$ represent any ordered basis of W. Now, extend that basis to an ordered basis $(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n)$ of $\mathbb{F}_q^n$, noting that a linear transformation $T : \mathbb{F}_q^n \to \mathbb{F}_q^r$ will have W as its null space precisely when $T(x_i) = 0$, for $1 \le i \le k$, and $T(x_j)$, for $k+1 \le j \le n$, any linearly independent sequence of vectors in $\mathbb{F}_q^r$. $\square$

These proofs appear in [21].

Observe that $\binom{n}{1}_q = n_q$ as a special case of (1.15). Furthermore, notice that by interchanging the roles of k and $n-k$ in (1.15), it follows directly that

Theorem 1.1.14. *For all $n, k \in \mathbb{N}$,*

$$\binom{n}{k}_q = \binom{n}{n-k}_q. \tag{1.19}$$

Additionally, there is a two-term recurrence analogous to (1.6) and similar to (1.12),

Theorem 1.1.15. *For q a prime power, with boundary conditions $\binom{0}{k}_q = \delta_{0,k}$ and $\binom{n}{0}_q = 1$ for every $n, k \in \mathbb{N}$,*

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q, \quad \forall n, k \in \mathbb{P}. \tag{1.20}$$

Proof. For $k > n$, the recurrence holds in the form 0=0. Also, $\forall n, k \in \mathbb{N}$, the boundary conditions are obvious.

Thus, let $n, k \in \mathbb{P}$ with $k \le n$, and let W be any one-dimensional subspace of $\mathbb{F}_q^n$. Then among those k-dimensional linear subspaces of $\mathbb{F}_q^n$, (i) $\binom{n-1}{k-1}_q$ counts those that contain W as a subspace, while (ii) $q^k\binom{n-1}{k}_q$ counts those that do not.

To see (i), let $\mathcal{A}$ be the set of linearly independent sequences $(x_1, \ldots, x_k)$ in $\mathbb{F}_q^n$ in which $x_1 \in W$. Note that $|\mathcal{A}| = (q-1)(q^n-q)\cdots(q^n-q^{k-1})$ since there are $(q-1)$ choices for x_1, (q^n-q) choices for a linearly independent x_2, and so on. Now, let $\mathcal{B}$ be the set of k-dimensional linear subspaces of $\mathbb{F}_q^n$ that also contain W as a subspace. Then the map from $(x_1, \ldots, x_k)$ to the linear span of $(x_1, \ldots, x_k)$ is a $(q-1)(q^k-q)\cdots(q^k-q^{k-1})$-to-one surjection from $\mathcal{A}$ to $\mathcal{B}$. Hence,

$$|\mathcal{B}| = \frac{(q-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q-1)(q^k-q)\cdots(q^k-q^{k-1})}.$$

The right-hand side of this can be simplified to

$$\frac{(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^{k-1}-1)\cdots(q-1)} = \frac{(n-1)!_q}{(k-1)!_q(n-k)!_q} = \binom{n-1}{k-1}_q.$$

To see (ii), let $\mathcal{A}$ be the set of linearly independent sequences $(x_1, \ldots, x_k)$ in $\mathbb{F}_q^n$ in which no $x_i \in W$. Note that $|\mathcal{A}| = (q^n-q)\cdots(q^n-q^k)$ since there are (q^n-q) choices for $x_1 \notin W$, (q^n-q^2) choices for a linearly independent $x_2 \notin W$, and so on. Now, let $\mathcal{B}$ be the set of k-dimensional linear subspaces of $\mathbb{F}_q^n$ that do not contain W as a subspace. Then the map from $(x_1, \ldots, x_k)$ to the linear span of $(x_1, \ldots, x_k)$ is a $(q^k-q)\cdots(q^k-q^{k-1})$-to-1 surjection from $\mathcal{A}$ to $\mathcal{B}$. Hence,

$$|\mathcal{B}| = \frac{(q^n-q)\cdots(q^n-q^k)}{(q^k-q)\cdots(q^k-q^{k-1})}.$$

This gives

$$|\mathcal{B}| = q^k \frac{(n-1)!_q}{k!_q(n-k-1)!_q} = q^k\binom{n-1}{k}_q.$$

□

This proof also appears in [21].

From the recurrence (1.20), we derive other identities, each similar to a formula given for the binomial coefficients and Stirling numbers:
First, a column generating function analogous to (1.7) and similar to (1.13),

Theorem 1.1.16. *For every $k \in \mathbb{N}$ and for q a prime power,*

$$\sum_{n\geq 0}\binom{n}{k}_q x^n = \frac{x^k}{(1-x)(1-qx)(1-q^2x)\cdots(1-q^kx)}. \tag{1.21}$$

Proof. For every $k \in \mathbb{N}$, let

$$g_k(x) := \sum_{n\geq 0}\binom{n}{k}_q x^n.$$

Then $g_0(x) = \sum\binom{n}{0}_q x^n = \sum x^n = \frac{1}{1-x}$ by the geometric series identity.
Using the recurrence (1.20), a recurrence for $g_k(x)$ arises for all $k \in \mathbb{P}$:

$$\begin{aligned} g_k(x) &= \sum_{n\geq 1}\left[\binom{n-1}{k-1}_q + q^k\binom{n-1}{k}_q\right]x^n \\ &= \sum_{n\geq 1}\binom{n-1}{k-1}_q x\cdot x^{n-1} + \sum_{n\geq 1}\binom{n-1}{k}_q q^k x\cdot x^{n-1} \\ &= xg_{k-1}(x) + q^k x g_k(x). \end{aligned}$$

Whence, $g_k(x) = \frac{x}{1-q^k x} g_{k-1}(x)$. The result follows by induction. $\square$

And second, a closed-form expression analogous to (1.8) and similar to (1.14),

Theorem 1.1.17. *For every $n, k \in \mathbb{N}$ and for an indeterminate q,*

$$\binom{n}{k}_q = \sum_{\substack{d_0+d_1+\cdots+d_k=n-k \\ d_i \in \mathbb{N}}} q^{0d_0+1d_1+2d_2+\cdots+kd_k}. \tag{1.22}$$

Proof. The proof of this fact is similar to the proofs of Theorems 1.1.6 and 1.1.10. $\square$

All of these results are well known, appearing, for instance, in [21].

1.1.4 The Carlitz q-Stirling Numbers

Following the approaches of Milne [14] and Wagner [19], given a fixed $n \in \mathbb{N}$, let V be a k-dimensional vector space over $\mathbb{F}_q$. Then for each $k \in \mathbb{N}$, consider the set of sequences $(U_1, \ldots, U_n)$ of one-dimensional subspaces of V with $\dim(Sp(U_1, \ldots, U_n)) = k$, where by $Sp(U_1, \ldots, U_n)$ is meant the linear span of the the spaces $U_1, \ldots, U_k$. Associate with each such sequence a subsequence $(U_{t_1}, \ldots, U_{t_k})$ obtained by letting U_{t_i} be the first instance in $(U_1, \ldots, U_n)$ in which $\dim(Sp(U_1, \ldots, U_{t_i})) = i$, for each $i \in [k]$. In [19], Wagner shows that the cardinality of the preimage of this map is the same for any choice of $(U_{t_1}, \ldots, U_{t_k})$. Indeed he shows that, denoting the cardinality of the preimage of any such sequence by $\tilde{S}_q(n, k)$,

Theorem 1.1.18 (Wagner)**.** *For every $n, k \in \mathbb{N}$ and q a power of a prime number,*

$$\tilde{S}_q(n, k) = \sum_{\substack{d_1+\cdots+d_k=n-k \\ d_i \in \mathbb{N}}} (1_q)^{d_1} (2_q)^{d_2} \cdots (k_q)^{d_k}. \tag{1.23}$$

These numbers $\tilde{S}_q(n, k)$ are the Carlitz q-Stirling numbers[6] (of the second kind), and two results follow from this definition:

First, analogous to (1.9),

Theorem 1.1.19. *For every $n \in \mathbb{N}$ and q a power of a prime number,*

$$x^n = \sum_{k=0}^{n} \tilde{S}_q(n, k) x(x - 1_q)(x - 2_q) \cdots (x - (k-1)_q). \tag{1.24}$$

[6]Carlitz first proposed these numbers in [2] in an investigation of a class of Abelian fields and subsequently expanded upon them in [3]. An interpretation similar to this one was discovered by Milne [14] by analyzing restricted growth functions. More generally, Wagner in [19] uses the idea of restricted growth on sequences of atoms in a "modular binomial lattice of characteristic q," resulting in a common treatment of $\tilde{S}_q(n, k)$ for $q = 0$ (chains), $q = 1$ (sets), and q a prime power (vector spaces).

Proof. It more than suffices to establish this polynomial identity for r_q for any $r \in \mathbb{P}$ with $r \geq n$. Then (1.24) takes the form

$$(r_q)^n = \sum_{k=0}^{n} \tilde{S}_q(n,k) r_q (r_q - 1_q)(r_q - 2_q) \cdots (r_q - (k-1)_q). \tag{1.25}$$

Let n be fixed. Both sides of (1.25) enumerate the set A of sequences $(U_1, \ldots, U_n)$ of one-dimensional linear subspaces of $\mathbb{F}_q^r$. That the left-hand-side does this is clear since there are r_q one-dimensional subspaces of $\mathbb{F}_q^r$ and n positions in the sequence. For the right-hand-side of (1.25), let B denote the set of sequences of one-dimensional linear subspaces of $\mathbb{F}_q^r$ with any length from 0 to n for which it holds that if the length of a sequence in B is k, then $\dim(Sp(U_1, \ldots, U_k)) = k$. Now for each $k \in [n]^*$, let B_k denote the subset of B composed of sequences of length k. Observe that $(B_0, \ldots, B_n)$ is an ordered partition of B and that for each $k \in [n]^*$, $|B_k| = r_q(r_q - 1_q) \cdots (r_q - (k-1)_q)$ since there are r_q choices for the first element of a sequence in B_k, $r_q - 1_q$ choices for the second element of a sequence in B_k since it must be chosen outside of the span of the first choice, and so on. Further, by definition, if f maps A to B, then by definition any particular sequence in B_k has $\tilde{S}_q(n,k)$ elements in its preimage under f, for each $k \in [n]^*$. □

And second, analogous to (1.12),

Theorem 1.1.20. *For all $n, k \in \mathbb{P}$ and q a power of a prime number,*

$$\tilde{S}_q(n,k) = \tilde{S}_q(n-1, k-1) + k_q \tilde{S}_q(n-1, k), \tag{1.26}$$

subject to the boundary conditions $\tilde{S}_q(n,0) = \delta_{n,0}$, $\tilde{S}_q(0,k) = \delta_{0,k}$.

Proof. The boundary conditions are clear.

When $n, k \in \mathbb{P}$, consider the cases for which $\dim(Sp(U_1, \ldots, U_{n-1})) = k-1$ and for which $\dim(Sp(U_1, \ldots, U_{n-1})) = k$. In the first case, $U_{t_k} = U_n$. Since $(U_{t_1}, \ldots, U_{t_{k-1}})$ has preimage of cardinality $\tilde{S}_q(n-1, k-1)$ with elements of the form $(U_1, \ldots, U_{n-1})$, the preimage of the sequence $(U_{t_1}, \ldots, U_{t_k})$ has elements of the form $(U_1, \ldots, U_n)$ and cardinality $\tilde{S}_q(n-1, k-1)$.

On the other hand, when $\dim(Sp(U_1, \ldots, U_{n-1})) = k$, $U_{t_k} \neq U_n$. Therefore, we have $U_n \in Sp(U_1, \ldots, U_{n-1})$. Since the elements of the preimage are of the form $(U_1, \ldots, U_{n-1})$, and since that set has cardinality $\tilde{S}_q(n-1, k)$, given a sequence $(U_{t_1}, \ldots, U_{t_k})$ it suffices to show that there are k_q ways to choose U_n. This, however, is precisely the number of one-dimensional subspaces of any k-dimensional subspace over $\mathbb{F}_q$, in particular of $Sp(U_1, \ldots, U_{n-1})$. □

From this recurrence, we derive a column generating function analogous to (1.13):

Theorem 1.1.21. *For every $k \in \mathbb{N}$ and q a power of a prime number,*

$$\sum_{n \geq 0} \tilde{S}_q(n,k) x^n = \frac{x^k}{(1-x)(1-2_q x)(1-3_q x) \cdots (1-k_q x)}. \tag{1.27}$$

Proof. For every $k \in \mathbb{N}$, let

$$g_k(x) := \sum_{n \geq 0} \tilde{S}_q(n,k) x^n.$$

Then $g_0(x) = \sum \tilde{S}_q(n,0)x^n = \sum \delta_{n,0}x^n = 1$.
Using the recurrence (1.26), a recurrence for $g_k(x)$ arises for all $k \in \mathbb{P}$:

$$\begin{aligned} g_k(x) &= \sum_{n\geq 1}\left[\tilde{S}_q(n-1,k-1) + k_q\tilde{S}_q(n-1,k)\right]x^n \\ &= \sum_{n\geq 1}\tilde{S}_q(n-1,k-1)x\cdot x^{n-1} + \sum_{n\geq 1}\tilde{S}_q(n-1,k)k_qx\cdot x^{n-1} \\ &= xg_{k-1}(x) + k_qxg_k(x). \end{aligned}$$

Whence, $g_k(x) = \frac{x}{1-k_qx}g_{k-1}(x)$. The result follows by induction. □

These statements are all well known, appearing in [8] for instance.

Another structure counted by $\tilde{S}_q(n,k)$, valid for any $q \in \mathbb{P}$, is presented in Section 1.4.4 as a q-analogue of $\mathcal{RGF}(n,k)$.

1.2 Comtet's Algebraic Unification

There is strong similarity between the four primary equations provided in each of the four cases of the Section 1.1. In 1972, Louis Comtet drew them together with an algebraic unification in [4].

Theorem 1.2.1 (Comtet's Theorem)**.** *Given a sequence $\langle b_i\rangle_{i\geq 0}$ in an integral domain I, the following four statements are equivalent specifications of a rectangular array $C(n,k;\langle b_i\rangle)$ for all $n,k \in \mathbb{N}$:*

1. *With boundary conditions $C(0,k;\langle b_i\rangle) = \delta_{0,k}$ and $C(n,0;\langle b_i\rangle) = b_0^n$ for every $n,k \in \mathbb{N}$,*

$$C(n,k;\langle b_i\rangle) = C(n-1,k-1;\langle b_i\rangle) + b_kC(n-1,k;\langle b_i\rangle),\ \forall n,k \in \mathbb{P}; \tag{1.28}$$

2. *For every $n \in \mathbb{N}$,*

$$x^n = \sum_{k=0}^{n} C(n,k;\langle b_i\rangle)\varphi_k(x), \tag{1.29}$$

where $\varphi_0(x) \equiv 1$, and $\varphi_k(x) := (x-b_0)(x-b_1)(x-b_2)\cdots(x-b_{k-1})$, $\forall k \in \mathbb{P}$;

3. *For every $k \in \mathbb{N}$,*

$$\sum_{n\geq 0} C(n,k;\langle b_i\rangle)x^n = \frac{x^k}{(1-b_0x)(1-b_1x)(1-b_2x)\cdots(1-b_kx)}; \tag{1.30}$$

and

4. *For every $n,k \in \mathbb{N}$,*

$$C(n,k;\langle b_i\rangle) = \sum_{\substack{d_0+d_1+\cdots+d_k=n-k \\ d_i\in\mathbb{N}}} b_0^{d_0}b_1^{d_1}b_2^{d_2}\cdots b_k^{d_k}. \tag{1.31}$$

Proof. First, observe that each of the statements (1) through (4) uniquely defines a triangular array $(C(n,k;\langle b_i\rangle))_{n,k\in\mathbb{N}}$. Thus, it will suffice to show that those arrays defined by (2), (3), and (4) each satisfy the boundary conditions and recurrence given in (1).

(2)⇒(1): Direct examination of (1.29) provides that the boundary condition $C(0,k;\langle b_i\rangle) = \delta_{0,k}$ is satisfied. Furthermore, $C(n,0;\langle b_i\rangle) = b_0^n$ is clear from the annihilation of the sum on the right-hand side of (1.29) that arises from the choice $x = b_0$.

The recurrence is trivially obtained from (1.29) when $k > n$, as both sides are 0. Thus, we may assume that $1 \le k \le n$. In this case,

$$\begin{aligned}
\sum_{k\geq 0} C(n,k;\langle b_i\rangle)\varphi_k(x) = x^n = x\cdot x^{n-1}& \\
&= x\sum_{k\geq 0} C(n-1,k;\langle b_i\rangle)\varphi_k(x)\\
&= \sum_{k\geq 0} C(n-1,k;\langle b_i\rangle)\varphi_k(x)(x-b_k+b_k)\\
&= \sum_{k\geq 0} C(n-1,k;\langle b_i\rangle)(\varphi_{k+1}(x)+b_k\varphi_k(x))\\
&= \sum_{k\geq 0}(C(n-1,k-1;\langle b_i\rangle)+b_kC(n-1,k;\langle b_i\rangle))\varphi_k(x).
\end{aligned}$$

Since $\{\varphi_k(x)\}_{k\geq 0}$ is a basis of the algebra $I[x]$, the recurrence in (1.28) follows by comparison of coefficients.

(3)⇒(4): Given the column generating function in (1.30), applying the geometric series identity $k+1$ times provides

$$\begin{aligned}
\sum_{n\geq 0} C(n,k;\langle b_i\rangle)x^n = x^k \sum_{d_0\geq 0} b_0^{d_0}x^{d_0}\sum_{d_1\geq 0} b_1^{d_1}x^{d_1}\cdots\sum_{d_k\geq 0} b_k^{d_k}x^{d_k}& \\
&= \sum_{\text{each } d_i\geq 0} b_0^{d_0}b_1^{d_1}\cdots b_k^{d_k}x^{d_0+d_1+\cdots+d_k+k}\\
&= \sum_{n\geq 0} x^n \sum_{\substack{d_0+\cdots+d_k=n-k\\ d_i\in\mathbb{N}}} b_0^{d_0}b_1^{d_1}\cdots b_k^{d_k}.
\end{aligned}$$

Comparison of coefficients of x^n produces (1.31).

(4)⇒(1): The boundary conditions of (1.28) are immediate from (1.31). When $n,k\in\mathbb{P}$, however, the sum in (1.31) is split into two cases: when $d_k = 0$ and when $d_k > 0$. Noting that $n-k = (n-1)-(k-1)$, this gives

$$C(n,k;\langle b_i\rangle) = \sum_{\substack{d_0+\cdots+d_{k-1}=n-k\\ d_i\in\mathbb{N}}} b_0^{d_0}b_1^{d_1}\cdots b_k^{d_k} + \sum_{\substack{d_0+\cdots+d_k=n-k\\ d_i\in\mathbb{N};d_k\in\mathbb{P}}} b_0^{d_0}b_1^{d_1}\cdots b_k^{d_k}$$

$$
\begin{aligned}
&= \sum_{\substack{d_0+\cdots+d_{k-1}=n-k \\ d_i\in\mathbb{N}}} b_0^{d_0}b_1^{d_1}\cdots b_k^{d_k} + b_k \sum_{\substack{d_0+\cdots+d_k=n-k-1 \\ d_i\in\mathbb{N}}} b_0^{d_0}b_1^{d_1}\cdots b_k^{d_k} \\
&= C(n-1,k-1;\langle b_i\rangle) + b_k C(n-1,k;\langle b_i\rangle)
\end{aligned}
$$

□

The values of the array $C(n,k;\langle b_i\rangle)$ are the *Comtet numbers associated with* $\langle b_i\rangle_{i\geq 0}$. Also, the function $\varphi_k(x)$ defined in the second point above is called the k^{th} *falling factorial function in* $\langle b_i\rangle_{i\geq 0}$.

Observe that Theorem 1.2.1 reestablishes most of the results from Section 1.1 since from $C(n,k;\langle b_i\rangle)$,:

1. $b_i \equiv 1$ gives the ordinary binomial coefficients,

2. $b_i = i$, for all $i\in\mathbb{N}$, gives the Stirling numbers,

3. $b_i = q^i$, for all $i\in\mathbb{N}$, gives the q-binomial coefficients, and

4. $b_i = i_q$, for all $i\in\mathbb{N}$, gives the Carlitz q-Stirling numbers.

The statement and proof of Comtet's theorem as given here appear in [21] with little modification.

1.3 A Combinatorial Interpretation of the Comtet Numbers

As suggested by Wagner in [23], let $\langle B_i\rangle_{i\geq 0}$ be a sequence of finite, pairwise disjoint sets, with $|B_i| = b_i$ for each $i\in\mathbb{N}$. Then, for all $n,k\in\mathbb{N}$, let $\mathcal{W}(n,k;\langle b_i\rangle)$ denote the set of words of length $n-k$ in $B_0\cup\cdots\cup B_k$ so that for all $0\leq i\leq k-1$, every letter from the alphabet B_i precedes the letters from the alphabet B_{i+1}. The term *from ascending alphabets* will be used to mean that the letters in the words will be chosen in this manner.

Theorem 1.3.1. *For all* $n,k\in\mathbb{N}$ *and each sequence* $\langle b_i\rangle_{i\geq 0}$ *of nonnegative integers,*

$$|\mathcal{W}(n,k;\langle b_i\rangle)| = C(n,k;\langle b_i\rangle). \tag{1.32}$$

Proof. For any word $w\in\mathcal{W}(n,k;\langle b_i\rangle)$, for each i, if d_i counts the number of letters in w chosen from the set B_i, then for all $n,k\in\mathbb{N}$,

$$|\mathcal{W}(n,k;\langle b_i\rangle)| = \sum_{\substack{d_0+d_1+\cdots+d_k=n-k \\ d_i\in\mathbb{N}}} b_0^{d_0}b_1^{d_1}\cdots b_k^{d_k}. \tag{1.33}$$

Then the result follows by (4) in Comtet's Theorem 1.2.1. □

In particular, $C(n,0;\langle b_i\rangle) = b_0^n$ for all $n\in\mathbb{N}$, and $C(n,k;\langle b_i\rangle) = 0$ if $0\leq n<k$. Formula (1.33) may be rewritten in the form

$$C(n,k;\langle b_i\rangle) = \sum_{0\leq i_1\leq i_2\leq\cdots\leq i_{n-k}\leq k} b_{i_1}b_{i_2}\cdots b_{i_{n-k}}, \tag{1.34}$$

by letting i_j, for $1 \leq j \leq n-k$, be the specific subscripts of the letters in a word $w \in \mathcal{W}(n, k; \langle b_i \rangle)$. So we see that $C(n, k; \langle b_i \rangle)$ is the $(n-k)^{th}$ *complete symmetric function* in b_0, b_1, ..., b_k[7].

Using (1.32), a combinatorial proof of (1.28) can be given when each $b_i \in \mathbb{N}$. Recall Equation (1.28): for all $n, k \in \mathbb{P}$,

$$C(n, k) = C(n-1, k-1) + b_k C(n-1, k),$$

subject to the boundary conditions $C(0, k) = \delta_{0,k}$ and $C(n, 0) = b_0^n$, for all $n, k \in \mathbb{N}$.

Proof. On the RHS of (1.28), $C(n-1, k-1)$ counts those words in $\mathcal{W}(n, k)$ that contain no letter from B_k, and $b_k C(n-1, k)$ counts those that contain at least one letter from B_k. $\square$

Before proceeding note that there is another pair of identities involving the Comtet numbers that do not appear above and yet can be proved via this interpretation, see [8]. These are two variants on analogues of the "Hockey Stick Theorem." The first will be referred to as a diagonal variant while the second is called a vertical variant. Though they can both be proved simply by repeatedly expanding one of the terms in the recurrence, to prove them combinatorially it is helpful to introduce another perspective on the words in $\mathcal{W}(n, k; \langle b_i \rangle)$.

Notice that $\mathcal{W}(n, k; \langle b_i \rangle)$ can also be represented as a union of the following sets: let $\mathcal{B}_i$ denote the set of words in $\mathcal{W}(n, k; \langle b_i \rangle)$ with the property that they contain at least one letter from the alphabet B_i. Then clearly $\mathcal{W}(n, k; \langle b_i \rangle) = \mathcal{B}_0 \cup \cdots \cup \mathcal{B}_k$. A convenient way to express $\mathcal{W}(n, k; \langle b_i \rangle)$ is to reorganize the above in terms of the pairwise disjoint sets $\mathcal{B}_k$, $\mathcal{B}_k^c \mathcal{B}_{k-1}$, $\mathcal{B}_k^c \mathcal{B}_{k-1}^c \mathcal{B}_{k-2}$, ..., $\mathcal{B}_k^c \cdots \mathcal{B}_1^c \mathcal{B}_0$, where concatenation denotes the intersection of sets. Note that $\mathcal{B}_k^c \cdots \mathcal{B}_{j+1}^c \mathcal{B}_j$, comprises those words in $\mathcal{W}(n, k; \langle b_i \rangle)$ containing at least one letter from B_j but no letters from any B_i with $i > j$, for $i, j \in [k-1]^*$, i.e. all of the words in $\mathcal{W}(n, k; \langle b_i \rangle)$ with the property that the alphabet with the largest index among the alphabets B_0, ..., B_k from which a letter appears is j. Thus, we claim that

$$\mathcal{W}(n, k; \langle b_i \rangle) = (\mathcal{B}_k) \dot{\cup} (\mathcal{B}_k^c \mathcal{B}_{k-1}) \dot{\cup} \cdots \dot{\cup} (\mathcal{B}_k^c \cdots \mathcal{B}_1^c \mathcal{B}_0), \tag{1.35}$$

i.e., the sets $\mathcal{B}_k$, $\mathcal{B}_k^c \mathcal{B}_{k-1}$, $\mathcal{B}_k^c \mathcal{B}_{k-1}^c \mathcal{B}_{k-2}$, ..., $\mathcal{B}_k^c \cdots \mathcal{B}_1^c \mathcal{B}_0$ form a pairwise disjoint, exhaustive class of subsets of $\mathcal{W}(n, k; \langle b_i \rangle)$.

Proof. It remains to show that these sets are exhaustive within $\mathcal{W}(n, k; \langle b_i \rangle)$. To do so, let $w \in \mathcal{W}(n, k; \langle b_i \rangle)$. Then suppose that the last letter in w, i.e. the letter from the alphabet with largest index, is from B_j. Then $w \in \mathcal{B}_k^c \cdots \mathcal{B}_{j+1}^c \mathcal{B}_j$. $\square$

Now first among the Comtet Hockey Stick Theorems is the diagonal variant:

Theorem 1.3.2. *Subject to the same boundary conditions given in (1.28),* $\forall n, k \in \mathbb{P}$,

$$C(n, k) = \sum_{j=0}^{k} b_{k-j} C(n-1-j, k-j). \tag{1.36}$$

[7]Henceforth, occasionally the sequence $\langle b_i \rangle_{i \geq 0}$ will be omitted to condense the notation, though it will always be present when needed for clarity.

Proof. The term $b_{k-j}C(n-1-j, k-j)$ counts the words in $\mathcal{B}_k^c \cdots \mathcal{B}_{k-j+1}^c \mathcal{B}_{k-j}$, and so (1.36) follows from (1.35). $\square$

And second is the vertical variant,

Theorem 1.3.3. *Subject to the same boundary conditions given in (1.28),* $\forall n, k \in \mathbb{P}$,

$$C(n,k) = \sum_{j=k}^{n} b_k^{n-j} C(j-1, k-1). \tag{1.37}$$

Proof. Among those words in $\mathcal{W}(n,k;\langle b_i \rangle)$, the term $b_k^{n-j}C(j-1,k-1)$ counts those in which exactly $n-j$ letters, necessarily the last $n-j$ letters, are from the alphabet B_k. To see this, choose a word in $\mathcal{W}(n,k;\langle b_i \rangle)$ end in $n-j$ letters from B_k. Map that word to the set $\mathcal{W}(j-1,k-1;\langle b_i \rangle)$ by deleting those $n-j$ letters from B_k. This map is a (b_k^{n-j})-to-one surjection. $\square$

1.4 Bijections to Familiar Structures and Combinatorial Proofs for the Special Cases

For the purpose of connecting $\mathcal{W}(n,k;\langle b_i \rangle)$ with more familiar structures, we will use a canonical representation of the letters forming the alphabets B_i. Specifically, represent each alphabet set B_i (with $|B_i| = b_i$), by $B_i = \{b_{i,1}, b_{i,2}, \ldots, b_{i,b_i}\}$. We adopt this convention to be able to determine the alphabet for each letter directly by inspection: the first subscript of each letter reveals to which alphabet it belongs.

1.4.1 The Binomial Coefficients

When $b_i \equiv 1$, the recurrence (1.28) takes the form

$$C(n,k;\langle 1 \rangle) = C(n-1,k-1;\langle 1 \rangle) + C(n-1,k;\langle 1 \rangle), \text{ for all } n,k \in \mathbb{P}, \tag{1.38}$$

subject to the boundary conditions $C(0,k;\langle 1 \rangle) = \delta_{0,k}$, for all $k \in \mathbb{N}$, and $C(n,0;\langle 1 \rangle) \equiv 1$. These are the same recurrence and boundary conditions satisfied by the binomial coefficients, given in (1.6), and so $C(n,k;\langle 1 \rangle) = \binom{n}{k}$, so here the binomial coefficients enumerate the words in the set $\mathcal{W}(n,k;\langle 1 \rangle)$, i.e. words of length $n-k$ in the ascending alphabets $B_0 \cup \cdots \cup B_k$, with each $B_i = \{b_{i,1}\}$.

Theorem 1.4.1. *For every* $n, k \in \mathbb{N}$, *there is a bijection between* $\mathcal{W}(n,k;\langle 1 \rangle)$ *and the set of* k*-element subsets of* $[n]$.

Proof. In the cases when $k > n$, the map is $\emptyset \to \emptyset$. When $n = k = 0$, the map is from the empty word to the set containing $\emptyset$.
Now let $n \in \mathbb{P}$ and $0 \le k \le n$. Letting the j^{th} letter of a word $w \in \mathcal{W}(n,k;\langle 1 \rangle)$ be $w_j = b_{i_j,1}$, for $j \in [n-k]$, define a map from $\mathcal{W}(n,k;\langle 1 \rangle)$ to the k-element subsets of $[n]$ by

$$w_1 w_2 \cdots w_{n-k} \mapsto \{i_1 + 1, i_2 + 2, \ldots, i_{n-k} + n - k\}^c. \tag{1.39}$$

That this map is a bijection is clear from construction, though it bears mentioning that since k is the largest value i_{n-k} can take, and i_1 can equal 0, that the image of this set is indeed a k-element subset of $[n]$. $\square$

As an aside, there is a second interpretation available from these words, yielding another structure counted by the binomial coefficients essentially for free. Collecting, in order, the sequence of initial subscripts of a word in $\mathcal{W}(n,k;\langle 1\rangle)$ provides the image of a function $f : [n-k] \to [k]^*$ that is monotonically increasing. Thus, the binomial coefficients count the number of such functions.

Also, let $w \in \mathcal{W}(n,k;\langle 1\rangle)$. Then if one takes account of d_i, the number of letters in w with initial subscript i, as i ranges from 0 to k, then $d_1 + d_2 + \cdots + d_k = n-k$, and there is only one possible manifestation of w. For all $n,k \in \mathbb{N}$, this yields the formula

$$\binom{n}{k} = \sum_{\substack{d_1+d_2+\cdots+d_k=n-k \\ d_i\in\mathbb{N}}} 1. \tag{1.40}$$

Furthermore, if the specific initial subscripts in w are $0 \le i_1 \le i_2 \le \cdots \le i_{n-k} \le k$, then there is still only one possible manifestation of w, providing the similar formula

$$\binom{n}{k} = \sum_{0\le i_1\le i_2\le\cdots\le i_{n-k}\le k} 1, \tag{1.41}$$

also valid for all $n,k \in \mathbb{N}$.

Finally, the two variants on the Hockey Stick Theorem given in Theorems 1.3.2 and 1.3.3 take the forms, for all $n,k \in \mathbb{P}$,

$$\binom{n}{k} = \sum_{j=0}^{k} \binom{n-1-j}{k-j}, \tag{1.42}$$

and

$$\binom{n}{k} = \sum_{j=k}^{n} \binom{j-1}{k-1}, \tag{1.43}$$

both subject to the same boundary conditions given in (1.6).

In light of the proofs of Theorems 1.3.2 and 1.3.3 and the bijection (1.39) above, among all k-element subsets of $[n]$,

- the term $\binom{n-1-j}{k-j}$ in (1.42) counts those in which all of the j elements of $[n]$ larger than $n-j$ are present but $n-j$ is not, i.e. $n-j$ is the largest excluded element, and
- the term $\binom{j-1}{k-1}$ in (1.43) counts those in which the largest element of $[n]$ present is j.

These interpretations are in agreement with the ones available by direction inspection of (1.42) and (1.43) using the provided interpretation of the binomial coefficients.

1.4.2 The Stirling Numbers

When $b_i = i$, for all $i \in \mathbb{N}$, the recurrence (1.28) takes the form

$$C(n,k) = C(n-1,k-1) + kC(n-1,k), \text{ for all } n,k \in \mathbb{P}, \tag{1.44}$$

subject to the boundary conditions $C(0,k) = \delta_{0,k}$, for all $k \in \mathbb{N}$, and $C(n,0) = 0$, for all $n \in \mathbb{P}$. These are the same recurrence and boundary conditions satisfied by the Stirling numbers, given in (1.12), and so $C(n,k;\langle i\rangle) = S(n,k)$. Thus, the Stirling numbers also enumerate the words in $\mathcal{W}(n,k;\langle i\rangle)$, i.e. the words of length $n-k$ in the ascending alphabets $B_i := \{b_{i,1}, b_{i,2}, \ldots, b_{i,i}\}$, for $i \in [k]$, taking note that $B_0 = \emptyset$.

Recall that $\mathcal{RGF}(n,k)$ is the set of restricted growth functions $f : [n] \to [k]$, i.e. surjections f that are surjections with the property that the first occurrence of j precedes the first occurrence of $j+1$, for $j \in [k-1]$, in the sequence $(f(1), \ldots, f(n))$.

Theorem 1.4.2. *There is a bijection between $\mathcal{W}(n,k;\langle i\rangle)$ and $\mathcal{RGF}(n,k)$ for every $n,k \in \mathbb{N}$.*

Proof. When $n = 0$, $k = 0$, or $k > n$ with $n,k \in \mathbb{P}$, the map is $\emptyset \to \emptyset$ in every case except when $n = k = 0$, in which case the empty word maps to the trivial restricted growth function that can be represented by the empty sequence. For $n,k \in \mathbb{P}$, with $1 \le k \le n$, consider the map from $\mathcal{W}(n,k;\langle i\rangle)$ to $\mathcal{RGF}(n,k)$ defined for $w \in \mathcal{W}(n,k;\langle i\rangle)$ by first inserting the k different values of $[k]$ in increasing order into w so that all of the heretofore unplaced elements of $[k]$ up to j together with j are placed immediately before the first occurrence of some letter in w from the alphabet B_j. All remaining elements of $[k]$ are placed at the end of this expanded word, also in increasing order. Then map the resulting n-letter word to a sequence of length n by taking the inserted elements to themselves and each letter b_{j,i_j} of w to i_j, preserving the order in which they appear. This sequence is clearly unique to w and can be understood as the image of a function $f \in \mathcal{RGF}(n,k)$. $\square$

A concrete example of this map is helpful for clarity:

Example 1.4.3. *Let $n = 11$ and $k = 5$. Consider the word*

$$w = b_{1,1}\, b_{3,2}\, b_{3,1}\, b_{3,1}\, b_{4,3}\, b_{4,2} \in \mathcal{W}(11,5;\langle i\rangle). \tag{1.45}$$

Then w maps first to

$$1\, b_{1,1}\, 2\, 3\, b_{3,2}\, b_{3,1}\, b_{3,1} 4\, b_{4,3}\, b_{4,2}\, 5, \tag{1.46}$$

which is identified with the restricted growth function from the set $[11]$ *to the set* $[5]$

$$(1,1,2,3,2,1,1,4,3,2,5). \tag{1.47}$$

Observe also that if for a word $w \in \mathcal{W}(n,k;\langle i\rangle)$, one takes account of d_i, the number of letters in w with initial subscript i, as i ranges from 1 to k, then $d_1 + d_2 + \cdots + d_k = n-k$, and there are $1^{d_1} 2^{d_2} \cdots k^{d_k}$ possible manifestations of w. For all $n,k \in \mathbb{N}$, this yields the formula

$$S(n,k) = \sum_{\substack{d_0+d_1+\cdots+d_k=n-k \\ d_i \in \mathbb{N}}} 0^{d_0} 1^{d_1} \cdots k^{d_k}. \tag{1.48}$$

Furthermore, if the specific initial subscripts in w are $0 \leq i_1 \leq i_2 \leq \cdots \leq i_{n-k} \leq k$, then there are $i_1 i_2 \cdots i_{n-k}$ possible manifestations of w, providing the similar formula

$$S(n,k) = \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_{n-k} \leq k} i_1 i_2 \cdots i_{n-k}, \tag{1.49}$$

also valid for all $n, k \in \mathbb{N}$.

Finally, the two variants on the Hockey Stick Theorem given in Theorems 1.3.2 and 1.3.3 take the forms, for all $n, k \in \mathbb{P}$:

$$S(n,k) = \sum_{j=0}^{k} (k-j) S(n-1-j, k-j), \tag{1.50}$$

and

$$S(n,k) = \sum_{j=k}^{n} k^{n-j} S(j-1, k-1), \tag{1.51}$$

both subject to the same boundary conditions as (1.12).

In light of the proofs of Theorems 1.3.2 and 1.3.3 and the bijection described in Theorem 1.4.2, among all elements of $\mathcal{RGF}(n,k)$,

- the term $(k-j)S(n-1-j,k-j)$ in (1.50) counts those in which the last j values in the image sequence are $f(n-j+1) = k-j+1, \ldots, f(n) = k$, with either $f(n-j) < k-j$ or $f(n-j-1) = f(n-j) = k-j$, and
- the term $k^{n-j}S(j-1,k-1)$ in (1.51) counts those in which the first occurrence of $f(i) = k$ in the image sequence is at $i = j$.

These interpretations are in agreement with the ones available by direction inspection of (1.50) and (1.51) using the provided interpretation of the Stirling numbers in terms of restricted growth functions.

1.4.3 The q-Binomial Coefficients

When $b_i = q^i$, $\forall i \in \mathbb{N}$, the recurrence (1.28) takes the form

$$C(n,k) = C(n-1,k-1) + q^k C(n-1,k), \text{ for all } n,k \in \mathbb{P}, \tag{1.52}$$

subject to the boundary conditions $C(0,k) = \delta_{0,k}$, for all $k \in \mathbb{N}$, and $C(n,0) \equiv 1$. These are the same recurrence and boundary conditions satisfied by the q-binomial coefficients, given in (1.20), and so $C(n,k;\langle q^i \rangle) = \binom{n}{k}_q$. Thus, here the q-binomial coefficients count the words in $\mathcal{W}(n,k;\langle q^i \rangle)$, i.e. words of length $n-k$ in the ascending alphabets $B_0 \cup \cdots \cup B_k$ with each $B_i := \{b_{i,1}, b_{i,2}, \ldots, b_{i,q^i}\}$. This interpretation is valid for every $q \in \mathbb{P}$ and can be extended to every $q \in \mathbb{N}$ by taking $B_i = \emptyset$ when $q = 0$ and $i \neq 0$.

A bijection between these words and the set of k-dimensional linear subspaces of $\mathbb{F}_q^n$ would be desirable. One can be forced, though it is unsatisfactory, by considering the vector spaces in terms of $k \times n$ echelon matrices with entries in $\mathbb{F}_q$ with all rows nonzero and using the lexicographical order.

Note that for $w \in \mathcal{W}(n, k; \langle q^i \rangle)$, if one takes account of d_i, the number of letters in w with initial subscript i, as i ranges from 0 to k, then $d_1+d_2+\cdots+d_k = n-k$, then there are $q^{0d_0+1d_1+2d_2+\cdots+kd_k}$ possible manifestations of w. For all $n, k \in \mathbb{N}$, this yields the formula

$$\binom{n}{k}_q = \sum_{\substack{d_1+d_2+\cdots+d_k=n-k \\ d_i \in \mathbb{N}}} q^{0d_0+1d_1+2d_2+\cdots+kd_k}. \tag{1.53}$$

Furthermore, if the specific initial subscripts in w are $0 \leq i_1 \leq i_2 \leq \cdots \leq i_{n-k} \leq k$, then there are $q^{i_1+i_2+\cdots+i_{n-k}}$ possible manifestations of w, providing the similar formula

$$\binom{n}{k}_q = \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_{n-k} \leq k} q^{i_1+i_2+\cdots+i_{n-k}}, \tag{1.54}$$

also valid for all $n, k \in \mathbb{N}$.

In addition, the two variants on the Hockey Stick Theorem given in (1.3.2) and (1.3.3) take the forms, for all $n, k \in \mathbb{P}$:

$$\binom{n}{k}_q = \sum_{j=0}^{k} q^{k-j} \binom{n-1-j}{k-j}_q, \tag{1.55}$$

and

$$\binom{n}{k}_q = \sum_{j=k}^{n} q^{k(n-j)} \binom{j-1}{k-1}_q, \tag{1.56}$$

both subject to the same boundary conditions as (1.20).

By the symmetry of $\binom{n}{k}_q$, there is another variant of each q-binomial Hockey Stick Theorem:

Theorem 1.4.4. *For all $n, k \in \mathbb{N}$ and an indeterminate q,*

$$\binom{n}{k}_q = \sum_{j=k}^{n} q^{j-k} \binom{j-1}{k-1}_q, \tag{1.57}$$

and

$$\binom{n}{k}_q = \sum_{j=0}^{k} q^{j(n-k)} \binom{n-1-j}{k-j}_q, \tag{1.58}$$

subject to the same boundary conditions as (1.20).

Proof. To see (1.57), consider (1.55) and apply the symmetry of the q-binomial coefficient along with a top-down summation:

$$\begin{aligned}\sum_{j=0}^{k} q^{k-j}\binom{n-1-j}{k-j}_q &= \sum_{j=0}^{k} q^{k-j}\binom{n-1-j}{n-1-k}_q \\ &= \sum_{j=n-k}^{n} q^{j-(n-k)}\binom{j-1}{n-k-1}_q.\end{aligned}$$

Interchanging the roles of k and $n-k$ yields (1.57).

To see (1.58), consider (1.56) and apply the symmetry of the q-binomial coefficient along with a top-down summation:

$$\begin{aligned}\sum_{j=k}^{n} q^{k(n-j)}\binom{j-1}{k-1}_q &= \sum_{j=k}^{n} q^{k(n-j)}\binom{j-1}{j-k}_q \\ &= \sum_{j=0}^{n-k} q^{jk}\binom{n-1-j}{n-k-j}_q.\end{aligned}$$

Interchanging the roles of k and $n-k$ yields (1.58). □

These are not explained by (1.3.2) and (1.3.3). In fact, they are special to $\binom{n}{k}_q$ and do not apply to $C(n,k;\langle b_i\rangle)$ in general since $C(n,k;\langle b_i\rangle)$ is not symmetric in k and $n-k$ for an arbitrary sequence $\langle b_i\rangle_{i\geq 0}$.

1.4.4 The Carlitz q-Stirling Numbers

When $b_n = n_q$, for all $n\in\mathbb{N}$, the recurrence (1.28) takes the form

$$C(n,k) = C(n-1,k-1) + k_q C(n-1,k), \text{ for all } n,k\in\mathbb{P}, \tag{1.59}$$

subject to the boundary conditions $C(0,k)=\delta_{0,k}$, for all $k\in\mathbb{N}$, and $C(n,0)=0$, for all $n\in\mathbb{P}$. These are the same recurrence and boundary conditions satisfied by the Carlitz q-Stirling numbers, and so $C(n,k;\langle i_q\rangle)=\tilde{S}_q(n,k)$. Thus, $\tilde{S}_q(n,k)$ counts the words in $\mathcal{W}(n,k;\langle i_q\rangle)$, i.e. the words of length $n-k$ the ascending alphabets $B_1\cup\cdots\cup B_k$ with each $B_i := \{b_{i,1}, b_{i,2},\ldots,b_{i,i_q}\}$, taking note that $B_0=\emptyset$ need not be included. This is valid for any $q\in\mathbb{P}$.

Since the words in $\mathcal{W}(n,k;\langle i\rangle)$, counted by $S(n,k)$ have a connection to restricted growth functions, it would be nice if the words in the q-analogous $\mathcal{W}(n,k;\langle i_q\rangle)$ do as well. Such a connection can be had in a new way by extending the notion of a restricted growth function in a way that slightly relaxes the usual notion of restricted growth, which will apply whenever $q\in\mathbb{P}$. For notational convenience, the notation $\overline{[j_q]} := [j_q]-[(j-1)_q]$, will be applied.

For $n,k\in\mathbb{N}$ and $q\in\mathbb{P}$, let

$\mathcal{RGF}(n,k;q) := \Big\{ f:[n]\to[k_q]$: the first occurrence of an element in

$\overline{[j_q]}$ precedes the first occurrence of an element in $\overline{[(j+1)_q]}$,

$\forall j \in [k]$, with at least one element from each $\overline{[j_q]}$ present$\Big\}$

be the set of *q-analogized restricted growth functions*, more conveniently referred to as *q restricted growth functions*[8].

Refer to those q restricted growth functions in which the first appearance of an element belonging to each $\overline{[j_q]}$ is the largest possible among those, i.e. j_q, by *canonical q-restricted growth functions*. Denote the set of these by $\mathcal{RGF}^*(n, k; q)$.

Remark 1.4.5. *For all $n, k \in \mathbb{N}$ and any $q \in \mathbb{P}$,*

$$|\mathcal{RGF}(n, k; q)| = q^{\binom{k}{2}} |\mathcal{RGF}^*(n, k; q)|. \tag{1.60}$$

Proof. This follows since there are q^{j-1} possible choices for a first element belonging to $\overline{[j_q]}$ for each $j \in [k]$. □

Hence,

Theorem 1.4.6. *For every $n, k \in \mathbb{P}$,*

$$|\mathcal{RGF}^*(n, k; q)| = \tilde{S}_q(n, k). \tag{1.61}$$

Proof. The method of proof will be to show that there is a bijection from the words in $\mathcal{W}(n, k; \langle i_q \rangle)$ to $\mathcal{RGF}^*(n, k; q)$. Observe first that only the subscripts of the letters of the words matter, as they are all b's. In fact, given such a word, it can be represented instead as a sequence of ordered pairs of the form

$$\begin{aligned}((1, i_{1,1}), (1, i_{1,2}), \ldots, (1, i_{1,d_1}), (2, i_{2,1}), (2, i_{2,2}), \ldots, (2, i_{2,d_2}), \ldots \\ (k, i_{k,1}), (k, i_{k,2}), \ldots, (k, i_{k,d_k})),\end{aligned} \tag{1.62}$$

where each $i_{l,m} \in [l_q]$. Now, into that sequence, insert j_q along with i_q for every $i < j$ not yet inserted immediately before the first instance of an ordered pair with first term j, for each $j \in [k]$. If the largest first term is less than k, then the remaining numbers of the form i_q should be placed in increasing order at the end. Since the sequence (1.62) is of length $n - k$, and k values have been added, the result is a sequence of length n. Finally, map the resulting sequence to a canonical q restricted growth function, here represented by a sequence $(s_1, s_2, \ldots, s_n)$ in which those items that are not ordered pairs are mapped to themselves and those that are get mapped to their second components. □

Corollary 1.4.7. *By Remark 1.4.5 and Theorem 1.4.6, for all $n, k \in \mathbb{N}$ and $q \in \mathbb{P}$,*

$$|\mathcal{RGF}(n, k; q)| = q^{\binom{k}{2}} \tilde{S}_q(n, k). \tag{1.63}$$

The numbers $q^{\binom{k}{2}} \tilde{S}_q(n, k)$ are often denoted by $S_q(n, k)$. In this work they have the interpretation of counting the q restricted growth functions from $[n]$ to $[k_q]$. Further discussion

[8]Note that one must take care with the term "q restriction" and realize it is applied for brevity

of $S_q(n,k)$ is omitted since they are not Comtet numbers[9][10].

Observe that for $w \in \mathcal{W}(n,k,\langle i_q \rangle)$, if one takes account of d_i, the number of letters in w with initial subscript i, as i ranges from 1 to k, then $d_1 + d_2 + \cdots + d_k = n - k$, and there are $(1_q)^{d_1}(2_q)^{d_2} \cdots (k_q)^{d_k}$ possible manifestations of w. For all $n, k \in \mathbb{N}$, this yields the formula

$$\tilde{S}_q(n,k) = \sum_{\substack{d_1+d_2+\cdots+d_k=n-k \\ d_i \in \mathbb{N}}} (1_q)^{d_1}(2_q)^{d_2} \cdots (k_q)^{d_k}. \tag{1.64}$$

Furthermore, if the specific initial subscripts in w are $0 \le i_1 \le i_2 \le \cdots \le i_{n-k} \le k$, then there are $(i_1)_q(i_2)_q \cdots (i_{n-k})_q$ possible manifestations of w, providing the similar formula

$$\tilde{S}_q(n,k) = \sum_{0 \le i_1 \le i_2 \le \cdots \le i_{n-k} \le k} (i_1)_q(i_2)_q \cdots (i_{n-k})_q, \tag{1.65}$$

also valid for all $n, k \in \mathbb{N}$.

Finally, the two variants on the Hockey Stick Theorem given in (1.3.2) and (1.3.3) take the forms, for all $n, k \in \mathbb{P}$:

$$\tilde{S}_q(n,k) = \sum_{j=0}^{k} (k-j)_q \tilde{S}_q(n-1-j, k-j), \tag{1.66}$$

and

$$\tilde{S}_q(n,k) = \sum_{j=k}^{n} (k_q)^{n-j} \tilde{S}_q(j-1, k-1), \tag{1.67}$$

both subject to the same boundary conditions as (1.26).

In light of the proofs of Theorems 1.3.2 and 1.3.3 and the bijection in the proof of Theorem 1.4.6, among all elements of $\mathcal{RGF}(n,k)$,

- the term $(k-j)_q S_q(n-1-j, k-j)$ in (1.66) counts those in which the last j values in the image sequence are $f(n-j+1) = (k-j)_q + 1, \ldots, f(n) = k$, with either $f(n-j) < (k-j)_q$ or $f(n-j-1) = f(n-j) = (k-j)_q$, and
- the term $(k_q)^{n-j} S_q(j-1, k-1)$ in (1.67) counts those in which the first occurrence of $f(i) = (k-1)_q$ in the image sequence is at $i = j$.

Observe that these interpretations could be arrived at by direct inspection of (1.66) and (1.67).

[9]For further reading on the subject, see [14] or [19]

[10]The above construction and proofs apply to a class of restricted growth functions counted by the fully general Comtet numbers by replacing each j_q with b_j, *mutatis mutandis*.

1.5 Additional Examples From Comtet's Note

In Comtet's paper [4], he gives a series of examples of what are now called Comtet numbers on the first page, most with combinatorial interpretations. Two such (related) examples are, in the present notation, $C(n,k;\langle(2i)^2\rangle)$ and $C(n,k;\langle(2i+1)^2\rangle)$, which he states count the partitions of $[2n]$ into $2k$ blocks (respectively of $[2n+1]$ into $2k+1$ blocks) where the block cardinality of each block is odd. Since he does not offer proofs of these statements, the goal of this section will be to give such verification explicitly. Our notations for these sets will be $\Pi^{(1)}_{2n,2k}$ and $\Pi^{(1)}_{2n+1,2k+1}$, respectively.

To see Comtet's claim, what needs to be established is that the given structures satisfy the recurrences and boundary conditions of $C(n,k;\langle(2i)^2\rangle)$ and $C(n,k;\langle(2i+1)^2\rangle)$. Specifically,

$$C(n,k;\langle(2i)^2\rangle) = C(n-1,k-1;\langle(2i)^2\rangle) + (2k)^2 C(n-1,k;\langle(2i)^2\rangle), \tag{1.68}$$

and

$$C(n,k;\langle(2i+1)^2\rangle) = C(n-1,k-1;\langle(2i+1)^2\rangle) + (2k+1)^2 C(n-1,k;\langle(2i+1)^2\rangle), \tag{1.69}$$

both subject to the boundary conditions $C(n,0;\langle\cdot\rangle) = \delta_{n,0}$ and $C(0,k;\langle\cdot\rangle) = \delta_{0,k}$.

Proof. Since the two separate cases given in (1.68) and (1.69) are proved similarly, *mutatis mutandis*, only the case in which $b_i = (2i)^2$ will be proved explicitly. Furthermore, the boundary conditions are obvious, following just as in the case for the Stirling numbers.

Among those partitions $\pi \in \Pi^{(1)}_{2n,2k}$, consider separately the disjoint, exhaustive cases when

1. both of the elements $2n-1$ and $2n$ of $[2n]$ appear in blocks of cardinality 1;
2. the elements $2n-1$ and $2n$ occupy the same block; and
3. the elements $2n-1$ and $2n$ occupy distinct blocks not both of which are singletons.

Given a partition in $\Pi^{(1)}_{2n,2k}$ described in the first of these cases, consider the bijection to $\Pi^{(1)}_{2n-2,2k-2}$ given by deleting from the end of that partition the two singleton blocks containing the elements $2n-1$ and $2n$. Thus, there are $C(n-1,k-1;\langle(2i)^2\rangle)$ partitions of $[2n]$ into $2k$ blocks, each with odd cardinality, in which the elements $2n-1$ and $2n$ both appear in a block with cardinality 1.

Given a partition in $\Pi^{(1)}_{2n,2k}$ described in the second of these cases, consider the map to $\Pi^{(1)}_{2n-2,2k}$ given by deleting the elements $2n-1$ and $2n$ from whichever of the $2k$ blocks that they appear together in, resulting in a block with odd cardinality.

Given a partition in $\Pi^{(1)}_{2n,2k}$ described in the third case, consider the map to $\Pi^{(1)}_{2n-2,2k}$ given by deleting the elements $2n-1$ and $2n$ from whichever distinct two of the $2k$ blocks that they appear in, which results in those two blocks each having even, nonzero cardinalities. In that case, examine the two blocks that contained $2n-1$ and $2n$, and among those two blocks, move the smallest element among them to the other block so that both have odd cardinality. The map that considers both of the second and third cases together is a

$(2k)^2$-to-1 surjection from $\Pi^{(1)}_{2n,2k}$ to $\Pi^{(1)}_{2n-2,2k}$, and thus there are $(2k)^2C(n-1,k;\langle(2i)^2\rangle)$ partitions in this class. $\square$

Comtet offers a related example for when the block cardinalities are even, stating that $(2k-1)!!C(n,k;\langle i^2\rangle)$ counts the partitions of $[2n]$ into k nonempty blocks where the block cardinality of each block is even[11]. Our notation for this set will be $\Pi^{(0)}_{2n,k}$. Note that these numbers are not, themselves, Comtet numbers.

To see this, observe that a two-term recurrence can be derived algebraically from the expression $Ev(n,k) := (2k-1)!!C(n,k;\langle i^2\rangle)$ via the recurrence for $C(n,k;\langle i^2\rangle)$:
For all $n,k \in \mathbb{P}$,

$$Ev(n,k) = (2k-1)Ev(n-1,k-1) + k^2 Ev(n-1,k), \tag{1.70}$$

subject to the boundary conditions $Ev(n,0) = \delta_{n,0}$ and $Ev(0,k) = \delta_{0,k}$.

Then what needs to be established is that the given structure satisfies the recurrence and boundary conditions of (1.70).

Proof. The boundary conditions are clear.
Let $n,k \in \mathbb{P}$, and consider the following four-part partition of $\Pi^{(0)}_{2n,k}$: Let

- E_1 denote the set of those partitions in $\Pi^{(0)}_{2n,k}$ that have both of $2n-1$ and $2n$ appearing together in a single block of cardinality 2;
- E_2 denote the set of those partitions in $\Pi^{(0)}_{2n,k}$ that have $2n-1$ and $2n$ appearing in different blocks, at least one of which has cardinality 2, organized so that when exactly one of those blocks has cardinality larger than 2, that larger block follows the block of cardinality 2 in a left-to-right scan;
- E_3 denote the set of those partitions in $\Pi^{(0)}_{2n,k}$ that have both of $2n-1$ and $2n$ appearing together in a block of cardinality larger than 2; and
- E_4 denote the set of those partitions in $\Pi^{(0)}_{2n,k}$ that have $2n-1$ and $2n$ appearing in different blocks, at most one of which has cardinality 2, organized so that when exactly one of those blocks has cardinality 2, the larger of the two blocks precedes the block of cardinality 2 in a left-to-right scan.

Then,

- the map $E_1 \to \Pi^{(0)}_{2n-2,k-1}$ given by deleting the block containing $2n-1$ and $2n$ from the partition is a bijection;
- the map $E_2 \to \Pi^{(0)}_{2n-2,k-1}$ given by deleting $2n-1$ and $2n$ from the partition and then unifying the resulting two blocks of odd cardinalities is a $(2k-2)$-to-one surjection;
- the map $E_3 \to \Pi^{(0)}_{2n-2,k}$ given by deleting $2n-1$ and $2n$ from the partition is a k-to-one surjection; and

[11] The notation $n!!$ is read "n double-factorial" and means $n(n-2)(n-4)\cdots 2$ when n is even and $n(n-2)(n-4)\cdots 1$ when n is odd.

- the map $E_4 \to \Pi^{(0)}_{2n-2,k}$ given by deleting $2n-1$ and $2n$ from the partition and subsequently moving the smallest element from among those two blocks to the other[12] is a $k(k-1)$-to-one surjection.

Combined, these four maps will establish the recurrence (1.70). Notice that the claims on the maps $E_1 \to \Pi^{(0)}_{2n-2,k-1}$ and $E_3 \to \Pi^{(0)}_{2n-2,k}$ are clear.

To verify the claim on the map $E_2 \to \Pi^{(0)}_{2n-2,k-1}$, note that an arbitrary partition in $\Pi^{(0)}_{2n-2,k-1}$ has the form $\{B_1|\cdots|B_{k-1}\}$ with each $|B_i|$ even. Then a member of E_2 can be obtained from any such partition by choosing any of its $k-1$ blocks and replacing that block's smallest element with one of $2n-1$ or $2n$ and then forming a new block of cardinality 2 composed of that displaced element and the other of $2n-1$ and $2n$. This new block will necessarily appear to the left of the block containing the other of $2n-1$ and $2n$.

To verify the claim on the map $E_4 \to \Pi^{(0)}_{2n-2,k}$, note that an arbitrary partition in $\Pi^{(0)}_{2n-2,k}$ has the form $\{B_1|\cdots|B_k\}$ with each $|B_i|$ even. Then a member of E_4 can be obtained from any such partition by first choosing any of the k blocks to insert $2n-1$ into and then any of the remaining $k-1$ blocks to insert $2n$ into, i.e. there are $k(k-1)$ ways to insert both $2n-1$ and $2n$ into the partition in different blocks. Notice that the resulting partition will have exactly two blocks with odd cardinalities of at least 3. To remedy this, move the smallest element among these two blocks to the other. As a result, should this generate a block containing one of $2n-1$ or $2n$ and having cardinality 2, then it will necessarily be to the right of the (larger) block that contains the other of $2n-1$ and $2n$. $\square$

[12] Note that the designated smallest element originally appears in a block with cardinality 4 or greater due to the construction of E_4.

Chapter 2

Lancaster's Theorem

2.1 Similar Arrays Outside of Comtet's Unification

2.1.1 The Cycle Numbers

Let $n, k \in \mathbb{N}$ and define $\mathcal{P}(n,k)$ to be the set of permutations of $[n]$ with exactly k cycles when written in cycle notation. Then denote the *cycle numbers*[1] by $c(n,k) := |\mathcal{P}(n,k)|$. By convention, assume the elements of $\mathcal{P}(n,k)$ are canonically written, as follows:

- the elements appearing in each cycle are ordered so that the least among them appears first, and
- the cycles themselves are listed in increasing order of their least (here: initial) elements.

It follows that

Theorem 2.1.1. *For all $n, k \in \mathbb{P}$,*

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k), \tag{2.1}$$

subject to the boundary conditions $c(0,k) = \delta_{0,k}$ for all $k \in \mathbb{N}$ and $c(n,0) = 0$ for all $n \in \mathbb{P}$.

Proof. The boundary conditions are clear, noting the empty permutation in the case $n = k = 0$. Also when $k > n$, $\mathcal{P}(n,k) = \emptyset$.

Thus, assume $1 \leq k \leq n$. Among those permutations in $\mathcal{P}(n,k)$, observe $c(n-1,k-1)$ counts those in which the cycle (n) appears (necessarily last), appended to the end of any permutation in $\mathcal{P}(n-1,k-1)$.

On the other hand, $(n-1)c(n-1,k)$ counts those in which the element n appears in a cycle with at least one other element. This is so because given a permutation in $\mathcal{P}(n,k)$, the element n could be located in a position following any of the other $n-1$ elements, inside the same cycle as the element it follows. Deleting n from the permutation is therefore an $(n-1)$-to-one surjection onto $\mathcal{P}(n-1,k)$.

$\square$

[1] These are frequently called the *signless Stirling numbers of the first kind.* The name used here reflects the given combinatorial interpretation.

From the recurrence in Theorem 2.1.1, we derive the following formulas:
First,

Theorem 2.1.2. *For all $n, k \in \mathbb{N}$,*

$$x^{\overline{n}} = \sum_{k=0}^{n} c(n,k)x^k. \tag{2.2}$$

Proof. Induct on n. For $n = 0$, the formula (2.2) is satisfied in the form 1=1.
Let $n \in \mathbb{P}$. Then

$$\begin{aligned}
\sum_{k=0}^{n} c(n,k)x^k &= \sum_{k=0}^{n-1} \left(c(n-1,k-1) + (n-1)c(n-1,k)\right) x^k \\
&= \sum_{k=1}^{n-1} c(n-1,k-1)x^k + \sum_{k=0}^{n-1}(n-1)c(n-1,k)x^k \\
&= \sum_{k=0}^{n-1} c(n-1,k)x^{k+1} + \sum_{k=0}^{n-1}(n-1)c(n-1,k)x^k \\
&= (x+n-1)\sum_{k=0}^{n-1} c(n-1,k)x^k = (x+n-1)\cdot x^{\overline{n-1}} = x^{\overline{n}}.
\end{aligned}$$

□

Second,

Theorem 2.1.3. *For all $n, k \in \mathbb{P}$,*

$$c(n,k) = \sum_{j=0}^{k}(n-1-j)c(n-1-j,k-j), \tag{2.3}$$

subject to the same boundary conditions as (2.1).

Proof. Repeatedly expand the term with the unit coefficient in (2.1). □

And third,

Theorem 2.1.4. *For all $n, k \in \mathbb{N}$,*

$$c(n,k) = \sum_{j=k}^{n} c(j-1,k-1)(n-1)^{\underline{n-j}}, \tag{2.4}$$

subject to the same boundary conditions as (2.1).

Proof. Repeatedly expand the term with the $(n-1)$ as its coefficient in (2.1). □

The formulas (2.1) through (2.4) are similar to the recurrence and connection-constants formulations of Comtet's theorem, (1.28) and (1.29), along with the Hockey Stick theorems for the Comtet numbers, (1.36) and (1.37)[2]. Observe, however, that the cycle numbers are not Comtet numbers.

2.1.2 The Lah Numbers

Perhaps the proper motivation for the Lah numbers is their role in completing the symmetry in the connection-constants formulations for the Stirling and cycle numbers. Recall Equations (1.9) and (2.2):

For all $n, k \in \mathbb{N}$,

$$x^n = \sum_{k=0}^{n} S(n,k) x^{\underline{k}}.$$

$$x^{\overline{n}} = \sum_{k=0}^{n} c(n,k) x^k.$$

Ivo Lah sought to combine these by considering the connection constants between the rising and falling factorial polynomials [13]. Thus, the numbers now bearing his name can be defined for all $n, k \in \mathbb{N}$ by

$$x^{\overline{n}} = \sum_{k=0}^{n} L(n,k) x^{\underline{k}}. \tag{2.5}$$

To maintain the symmetry of these subsections, however, the Lah numbers will be defined herein combinatorially instead, deriving (2.5) as a result below.

Let $n, k \in \mathbb{N}$ and define $\overrightarrow{\Pi}_{n,k}$ to be the set of partitions of $[n]$ into k nonempty blocks where each block is equipped with a linear order, and denote the *Lah numbers* by $L(n,k) := \left|\overrightarrow{\Pi}_{n,k}\right|$. By convention, assume the elements of $\overrightarrow{\Pi}_{n,k}$ are canonically written so that the blocks are listed in increasing order of their smallest elements.

It follows that

Theorem 2.1.5. *For all $n, k \in \mathbb{P}$,*

$$L(n,k) = L(n-1,k-1) + (n+k-1)L(n-1,k), \tag{2.6}$$

subject to the boundary conditions $L(0,k) = \delta_{0,k}$ for all $k \in \mathbb{N}$ and $L(n,0) = 0$ for all $n \in \mathbb{P}$.

Proof. The boundary conditions and case when $k > n$ are clear.

When $1 \le k \le n$, among all partitions in $\overrightarrow{\Pi}_{n,k}$, notice first that $L(n-1,k-1)$ counts those in which the k^{th} block contains only n by the bijection that deletes that k^{th} block and the element n from the partition.

On the other hand, $(n+k-1)L(n-1,k)$ counts those in which n appears in a block with at least one other element. To see this, choose a partition in $\overrightarrow{\Pi}_{n,k}$ and delete n. Note that n could appear following any of the $n-1$ other elements (and appearing in the same block as the element it follows) or as the initial element of any of the k blocks. Thus, this map is a $(n-1+k)$-to-one surjection onto $\overrightarrow{\Pi}_{n-1,k}$. □

[2] The formulas given in this section are all well-known, appearing with similar proofs in [8], for example.

Recovering (2.5) from the recurrence is similar to the proof of Theorem 2.1.2:

Theorem 2.1.6. *For all* $n, k \in \mathbb{N}$,

$$x^{\overline{n}} = \sum_{k=0}^{n} L(n,k) x^{\underline{k}}.$$

Proof. Induct on n. For $n = 0$, the formula (2.5) is satisfied in the form 1=1.
Let $n \in \mathbb{P}$. Then

$$\begin{aligned}
\sum_{k=0}^{n} L(n,k)x^{\underline{k}} &= \sum_{k=0}^{n-1} \left(L(n-1,k-1) + (n+k-1)L(n-1,k)\right) x^{\underline{k}} \\
&= \sum_{k=1}^{n-1} L(n-1,k-1)x^{\underline{k}} + \sum_{k=0}^{n-1} (n+k-1)L(n-1,k)x^{\underline{k}} \\
&= \sum_{k=0}^{n-1} L(n-1,k)x^{\underline{k+1}} + \sum_{k=0}^{n-1} (n+k-1)L(n-1,k)x^{\underline{k}} \\
&= (x-k+n+k-1)\sum_{k=0}^{n-1} L(n-1,k)x^{\underline{k}} \\
&= (x+n-1)\cdot x^{\overline{n-1}} = x^{\overline{n}}.
\end{aligned}$$

□

As with the cycle numbers, the Hockey Stick theorems also hold:
First,

Theorem 2.1.7. *For all* $n, k \in \mathbb{P}$,

$$L(n,k) = \sum_{j=0}^{k} (n+k-1-2j)L(n-1-j, k-j), \tag{2.7}$$

subject to the same boundary conditions as (2.6).

Proof. Repeatedly expand the term with the unit coefficient in (2.6). □

And second,

Theorem 2.1.8. *For all* $n, k \in \mathbb{P}$,

$$L(n,k) = \sum_{j=k}^{n} L(j-1, k-1)(n+k-1)^{\underline{n-j}}, \tag{2.8}$$

subject to the same boundary conditions as Theorem 2.1.5.

Proof. Repeatedly expand the term with the coefficient $(n-1+k)$ in (2.6). □

The formulas (2.5) through (2.8) are also similar to those arising in the recurrence and connection-constants formulations in Comtet's theorem, (1.28) and (1.29), along with the

Hockey Stick theorems for the Comtet numbers, (1.36) and (1.37)[3]. Notice, however, that the Lah numbers are not Comtet numbers.

As a last note about the Lah numbers, consider another interpretation for them analogous to the restricted growth functions counted by $S(n,k)$. Wagner, in [19], puts forth functions he calls *Lah restricted growth functions*, defined by first associating with each partition of $[n]$ into k nonempty blocks, each with a linear order (with no canonical ordering assumed), a sequence

$$((u_1,p_1),\ldots,(u_n,p_n)), \tag{2.9}$$

where u_i denotes the number of the block in which ball i is placed, and p_i its position in that block[4]. Then, he notes that in each sequence (2.9), the pair $(j,1)$ occurs exactly once for every $j \in [k]$. Letting $((u_{t_1},1),\ldots,(u_{t_k},1))$ denote the subsequence of all such pairs, then he shows

$$((u_1,p_1),\ldots,(u_n,p_n)) \mapsto (u_{t_1},\ldots,u_{t_k}) \tag{2.10}$$

maps from such sequences to the set of permutations of $[k]$. Moreover, each permutation arising in this way has $L(n,k)$ preimages in the set of sequences of the form (2.9) with respect to the map (2.10). The preimages of the permutation $(1,2,\ldots,k)$, are what he calls Lah restricted growth functions. The nomenclature is justified since for the choice $(1,2,\ldots,k)$, the preimages in question are those sequences of the form (2.9) in which $(j,1)$ precedes $(j+1,1)$ for all $j \in [k-1]$ (in a left-to-right scan).

2.2 Lancaster's Algebraic Unification

For the purposes of the ensuing discussion, consider two sequences $\langle a_i\rangle_{i\geq 0}$ and $\langle b_i\rangle_{i\geq 0}$ in an integral domain I, and, following Lancaster in [8], from them define the following generalizations of the rising and falling factorial polynomials:
For each sequence $\langle a_i\rangle_{i\geq 0}$ in I, define

$$\rho_n(x) = \prod_{i=0}^{n-1}(x+a_i), \tag{2.11}$$

and, for each sequence $\langle b_i\rangle_{i\geq 0}$ in I, define

$$\varphi_n(x) = \prod_{i=0}^{n-1}(x-b_i). \tag{2.12}$$

In his master's thesis, Lancaster defined and used these polynomials to study the array that acts as their connection constants. In doing so, he introduced a way to unify Comtet, cycle, and Lah numbers, along with the mentioned q-generalizations, from a single recurrence.

[3]The formulas given in this section are all well-known, appearing, for instance, in [8].

[4]Wagner actually uses an equivalent formulation of the structure counted by the Lah numbers given in terms of distributions of n labeled balls in k unlabeled urns. The connection between the two is given by letting the blocks be considered urns and the elements in the blocks be considered balls.

Given any pair of sequences $\langle a_i \rangle_{i\geq 0}$, $\langle b_i \rangle_{i\geq 0}$ in an integral domain I, and for all $n, k \in \mathbb{N}$, define an array $(A(n,k;\langle a_i\rangle,\langle b_i\rangle))_{n,k\geq 0}$ so that

$$\rho_n(x) = \sum_{k\geq 0} A(n,k;\langle a_i\rangle,\langle b_i\rangle)\varphi_k(x), \tag{2.13}$$

and call them the *Comtet-Lancaster numbers*[5] associated with the sequences $\langle a_i \rangle_{i\geq 0}$, $\langle b_i \rangle_{i\geq 0}$.

Remark 2.2.1. *For $n, k \in \mathbb{N}$ with $k > n$, $A(n,k;\langle a_i\rangle,\langle b_i\rangle) = 0$, for every pair of sequences $\langle a_i \rangle_{i\geq 0}$, $\langle b_i \rangle_{i\geq 0}$ in an integral domain I.*

To condense the notation, henceforth the references to the particular sequences will occasionally be omitted unless clarification is necessary, and furthermore, the Comtet-Lancaster numbers will henceforth be referred to as the *C-L numbers*, for brevity.

Then,

Theorem 2.2.2 (Lancaster's Theorem)**.** *Given sequences $\langle a_i \rangle_{i\geq 0}$ and $\langle b_i \rangle_{i\geq 0}$ in an integral domain I, the rectangular array $(A(n,k))_{n,k\geq 0}$ can be specified in the following three*[6] *equivalent ways:*

1. *For all $n, k \in \mathbb{N}$,*
$$\rho_n(x) = \sum_{k\geq 0} A(n,k)\varphi_k(x); \tag{2.14}$$

2. *For all $n, k \in \mathbb{P}$,*
$$A(n,k) = A(n-1,k-1) + (a_{n-1} + b_k)A(n-1,k), \tag{2.15}$$
subject to the boundary conditions $A(0,k) = \delta_{0,k}$ and
$$A(n,0) = \prod_{i=0}^{n-1}(a_i + b_0) \text{ for all } n,k \in \mathbb{N};$$

3. *For all $n, k \in \mathbb{P}$,*
$$A(n,k) = \sum_{j=k}^{n} A(j-1,k-1)\prod_{i=j}^{n-1}(a_i + b_k), \tag{2.16}$$
subject to the same boundary conditions as 2.15.

Lancaster gives the following proof of his theorem in [8]:

Proof. (1)⇒(2): Suppose $n \in \mathbb{P}$. Then

$$\sum_{k\geq 0} A(n,k)\varphi_k(x) = \rho_n(x) = \rho_{n-1}(x)(x + a_{n-1})$$

[5]In [8] Lancaster refers to these numbers as the *SLC Numbers of the second kind* where SLC stands for Stirling-Lah-Comtet.

[6]Lancaster actually presents four equivalent conditions here, the fourth applying a difference operator approach. Further, there is a fifth condition in the special case when the b_i's are all distinct numbers in $\mathbb{C}$.

$$
\begin{aligned}
&= \left(\sum_{k\geq 0} A(n-1,k)\varphi_k(x)\right)(x+a_{n-1}) \\
&= \sum_{k\geq 0} A(n-1,k)\varphi_k(x)(x-b_k+b_k+a_{n-1}) \\
&= \sum_{k\geq 0} A(n-1,k)\varphi_k(x)(x-b_k) + \sum_{k\geq 0} A(n-1,k)\varphi_k(x)(b_k+a_{n-1}) \\
&= \sum_{k\geq 0} A(n-1,k)\varphi_{k+1}(x) + \sum_{k\geq 0} A(n-1,k)\varphi_k(x)(b_k+a_{n-1}) \\
&= \sum_{k\geq 1} A(n-1,k-1)\varphi_k(x) + \sum_{k\geq 0}(a_{n-1}+b_k)A(n-1,k)\varphi_k(x).
\end{aligned}
$$

By equating coefficients of $\varphi_k(x)$,[7] for all $n,k \in \mathbb{P}$, the recurrence in (2.15) is established. As for the boundary conditions, the first is obvious since $\rho_0(x) = 1$ and the second is obtained by iterating, noting that $\varphi_0(x) = 1$ as well.

(2)⇒(1): Induct on n. First, when $n = 0$, $A(0,k) = \delta_{0,k}$ by the boundary conditions, so it follows that

$$\rho_0(x) = 1 = \sum_{k\geq 0} A(0,k)\varphi_k(x).$$

Hence, we may assume that the recurrence in (2.15) holds for row $n-1$. So,

$$
\begin{aligned}
\rho_n(x) &= \rho_{n-1}(x)(x+a_{n-1}) = \sum_{k\geq 0} A(n-1,k)\varphi_k(x)(x-b_k+a_{n-1}+b_k) \\
&= \sum_{k\geq 0} A(n-1,k)\varphi_k(x)(x-b_k) + \sum_{k\geq 0} A(n-1,k)\varphi_k(x)(a_{n-1}+b_k) \\
&= \sum_{k\geq 0} A(n-1,k)\varphi_{k+1}(x) + \sum_{k\geq 0}(a_{n-1}+b_k)A(n-1,k)\varphi_k(x) \\
&= \sum_{k\geq 1} A(n-1,k-1)\varphi_k(x) + \\
&\qquad \sum_{k\geq 1}(a_{n-1}+b_k)A(n-1,k)\varphi_k(x) + A(n-1,0)(a_{n-1}+b_0) \\
&= \sum_{k\geq 1} A(n,k)\varphi_k(x) + A(n,0) = \sum_{k\geq 0} A(n,k)\varphi_k(x).
\end{aligned}
$$

(2)⇒(3): Since the boundary conditions for both recurrences are the same, assume that $n,k \in \mathbb{P}$. Then,

$$
\begin{aligned}
&\sum_{j=k}^{n} A(j,k)\prod_{i=j}^{n-1}(a_i+b_k) \\
&\qquad = \sum_{j=k}^{n}\left(A(j-1,k-1)+(a_{j-1}+b_k)A(j-1,k)\right)\prod_{i=j}^{n-1}(a_i+b_k)
\end{aligned}
$$

[7]The polynomials $\varphi_n(x)$ form a basis for the algebra $I[x]$. This can be seen, for instance in [8].

$$\begin{aligned}
&= \sum_{j=k}^{n} A(j-1,k-1) \prod_{i=j}^{n-1} (a_i+b_k) + \sum_{j=k}^{n} A(j-1,k) \prod_{i=j-1}^{n-1} (a_i+b_k) \\
&= \sum_{j=k}^{n} A(j-1,k-1) \prod_{i=j}^{n-1} (a_i+b_k) + \sum_{j=k-1}^{n-1} A(j,k) \prod_{i=j}^{n-1} (a_i+b_k) \\
&= \sum_{j=k}^{n} A(j-1,k-1) \prod_{i=j}^{n-1} (a_i+b_k) + \sum_{j=k}^{n-1} A(j,k) \prod_{i=j}^{n-1} (a_i+b_k),
\end{aligned}$$

the last equality holding since $A(k-1,k)=0$. Thus,

$$A(n,k) = \sum_{j=k}^{n} A(j-1,k-1) \prod_{i=j}^{n-1} (a_i+b_k).$$

(3)$\Rightarrow$(2): Again, assume $n,k \in \mathbb{P}$. Then

$$\begin{aligned}
A(n,k) &= \sum_{j=k}^{n} A(j-1,k-1) \prod_{i=j}^{n-1} (a_i+b_k) \\
&= A(n-1,k-1) + \sum_{j=k}^{n-1} A(j-1,k-1) \prod_{i=j}^{n-1} (a_i+b_k) \\
&= A(n-1,k-1) + (a_{n-1}+b_k) \sum_{j=k}^{n-1} A(j-1,k-1) \prod_{i=j}^{n-2} (a_i+b_k) \\
&= A(n-1,k-1) + (a_{n-1}+b_k) A(n-1,k) \text{ by (3).}
\end{aligned}$$

□

Though Lancaster doesn't mention it, the diagonal variant of the Hockey Stick theorem also applies and is equivalent to the three conditions given above:

Theorem 2.2.3. *For all $n,k \in \mathbb{P}$ and any pair of sequences $\langle a_i \rangle_{i \geq 0}$ and $\langle b_i \rangle_{i \geq 0}$ in an integral domain I,*

$$A(n,k) = \sum_{j=0}^{k} (a_{n-j-1} + b_{k-j}) A(n-1-j, k-j), \tag{2.17}$$

subject to the same boundary conditions as (2.15).

Proof. Repeatedly expand the term with the unit coefficient in the recurrence (2.15). □

Note that Lancaster's Theorem reestablishes many of the results from Section 2.1 while unifying them with the Comtet numbers, as shown in [8]. For examples,

1. $A(n,k;\langle 0 \rangle, \langle b_i \rangle) = C(n,k;\langle b_i \rangle)$ for all $n,k \in \mathbb{N}$ with the specific manifestations thereof discussed previously in Section 1.2.

2. $A(n,k;\langle i \rangle, \langle 0 \rangle) = c(n,k)$ for all $n,k \in \mathbb{N}$.

3. $A(n,k;\langle i\rangle,\langle i\rangle) = L(n,k)$ for all $n,k \in \mathbb{N}$.

2.3 Combinatorial Interpretations of the Comtet-Lancaster Numbers

2.3.1 Symmetric Polynomials

Let $\mathcal{C}_j(b_0,\ldots,b_m)$ denote the *complete symmetric function* of degree j in the variables $b_0,\ldots,b_m$, and $\mathcal{E}_j(a_0,\ldots,a_m)$ denote the *elementary symmetric function* of degree j in the variables $a_0,\ldots,a_m$, i.e., let

$$\mathcal{C}_j(b_0,\ldots,b_m) = \sum_{0\le i_1\le\ldots\le i_j\le m} b_{i_1}\cdots b_{i_j}, \tag{2.18}$$

and

$$\mathcal{E}_j(a_0,\ldots,a_m) = \sum_{0\le i_1<\ldots<i_j\le m} a_{i_1}\cdots a_{i_j}. \tag{2.19}$$

Hence, following [23],

Theorem 2.3.1. *For all $n,k\in\mathbb{N}$ and each pair of sequences $\langle a_i\rangle_{i\ge 0}$, $\langle b_i\rangle_{i\ge 0}$ in $\mathbb{N}$,*

$$\sum_{j=k}^{n} \mathcal{E}_{n-j}(a_0,\ldots,a_{n-1})\mathcal{C}_{j-k}(b_0,\ldots,b_k) = A(n,k;\langle a_i\rangle,\langle b_i\rangle). \tag{2.20}$$

Proof. Recall $\rho_n(x)$ given in (2.11) and $\phi_n(x)$ given in (2.12), the polynomials for which $A(n,k;\langle a_i\rangle,\langle b_i\rangle)$ act as connection constants by Lancaster's Theorem 2.2.2. We will demonstrate that the left-hand side of (2.20) plays the same role for any $n\in\mathbb{N}$.
It is a well-known result [24] that

$$\rho_n(x) = \sum_{j=0}^{n} \mathcal{E}_{n-j}(a_0,\ldots,a_{n-1})x^j. \tag{2.21}$$

Applying applying (1.29) of Comtet's Theorem, along with (1.34) and (2.18), to (2.21) gives

$$\rho_n(x) = \sum_{j=0}^{n}\left(\mathcal{E}_{n-j}(a_0,\ldots,a_{n-1})\sum_{k=0}^{j}\mathcal{C}_{j-k}(b_0,\ldots,b_k)\phi_k(x)\right). \tag{2.22}$$

Finally, by interchanging the order of summation in (2.22), we arrive at

$$\rho_n(x) = \sum_{k=0}^{n}\left(\sum_{j=k}^{n}\mathcal{E}_{n-j}(a_0,\ldots,a_{n-1})\mathcal{C}_{j-k}(b_0,\ldots,b_k)\right)\phi_k(x), \tag{2.23}$$

which establishes the result by Lancaster's Theorem. $\square$

It is easy from this fact to see that $\mathcal{E}_{n-j}(a_0,\ldots,a_{n-1})$ counts the words of length $n-j$ in the alphabets $A_0\cup\cdots\cup A_{n-1}$ in which a word of length 0 or 1 in A_0 appears first, followed by a word of length 0 or 1 in A_1, and so on, finally followed by a word of length 0 or 1 in A_{n-1}. By the main result in Section 1.3, recall that $\mathcal{C}_{k-j}(b_0, \dots, b_k)$ counts the words of length $k-j$ in the ascending alphabets $B_0 \cup \cdots \cup B_k$.

Therefore, let $\langle A_i \rangle_{i \geq 0}$ and $\langle B_i \rangle_{i \geq 0}$ be two sequences of finite, pairwise disjoint sets, with $|A_i| = a_i$ and $|B_i| = b_i$ for each i. Then, for all $n \in \mathbb{N}$, $\mathcal{CL}(n, k; \langle a_i \rangle, \langle b_i \rangle)$ will denote the set of words of length $n-k$ in $A_0 \cup \cdots \cup A_{n-1} \cup B_0 \cup \cdots \cup B_k$ so that

1. for all $0 \leq i \leq n-1$ no more than one letter from any A_i appears, and each letter from the alphabet A_i precedes the letters from any subsequent A_j, $i < j \leq n-1$, as well as all of the letters from the alphabets B_i, $i \in [k]^*$, if any appear, and
2. for all $0 \leq i \leq k-1$, every letter from the alphabet B_i precedes the letters from the alphabet B_{i+1}, for $i \in [k-1]^*$.

The terms *from strictly ascending alphabets* and *from ascending alphabets* will be used to mean the manner in which the letters can be chosen from the alphabets labeled with A's and B's, respectively.

Then,

Theorem 2.3.2. *For all $n, k \in \mathbb{N}$, and for any pair of sequences $\langle a_i \rangle_{i \geq 0}$, $\langle b_i \rangle_{i \geq 0}$ in $\mathbb{N}$,*

$$|\mathcal{CL}(n, k; \langle a_i \rangle, \langle b_i \rangle)| = A(n, k; \langle a_i \rangle, \langle b_i \rangle). \tag{2.24}$$

Proof. This fact follows by first considering separately the cases in which there are precisely $n-j$ letters appearing from alphabets labeled with A, for each $j \in [k]$, the remaining $j-k$ letters coming from alphabets labeled with B and then applying Theorem 2.3.1. $\square$

Since the set $\mathcal{CL}(n, k; \langle a_i \rangle, \langle b_i \rangle)$ is complicated in structure, it is helpful to consider the following, as in [23]:

If for all $0 \leq i \leq n-1$, $\mathcal{A}_i$ denotes the subset of $\mathcal{CL}(n, k; \langle a_i \rangle, \langle b_i \rangle)$ characterized by containing at least one letter from the alphabet A_i and similarly if for all $0 \leq i \leq k$, $\mathcal{B}_i$ denotes the subset of $\mathcal{CL}(n, k; \langle a_i \rangle, \langle b_i \rangle)$ characterized by containing at least one letter from the alphabet B_i, then for all $n, k \in \mathbb{N}$ and any pair of sequences of nonnegative integers $\langle a_i \rangle_{i \geq 0}$ and $\langle b_i \rangle_{i \geq 0}$,

$$\mathcal{CL}(n, k; \langle a_i \rangle, \langle b_i \rangle) = \mathcal{A}_0 \cup \cdots \cup \mathcal{A}_{n-1} \cup \mathcal{B}_0 \cup \cdots \cup \mathcal{B}_k. \tag{2.25}$$

The sets combined in this union are clearly not disjoint, so consider instead the following expression for $\mathcal{CL}(n, k; \langle a_i \rangle, \langle b_i \rangle)$, letting $\dot\cup$ denote disjoint union and concatenation represent set intersection, as previously:

$$\begin{aligned}(\mathcal{A}_{n-1}) \dot\cup (\mathcal{A}_{n-1}^c \mathcal{A}_{n-2}) \dot\cup \cdots \dot\cup (\mathcal{A}_{n-1}^c \cdots \mathcal{A}_1^c \mathcal{A}_0) \dot\cup (\mathcal{A}_{n-1}^c \cdots \mathcal{A}_0^c \mathcal{B}_k) \dot\cup \\ (\mathcal{A}_{n-1}^c \cdots \mathcal{A}_0^c \mathcal{B}_k^c \mathcal{B}_{k-1}) \dot\cup \cdots \dot\cup (\mathcal{A}_{n-1}^c \cdots \mathcal{A}_0^c \mathcal{B}_k^c \cdots \mathcal{B}_1^c \mathcal{B}_0).\end{aligned} \tag{2.26}$$

To see this claim,

Proof. It is clear that the sets in (2.26) are indeed disjoint. They are also exhaustive in $\mathcal{CL}(n, k; \langle a_i \rangle, \langle b_i \rangle)$, so that, in fact, (2.26) is a representation for that set. To establish this fact, let $w \in \mathcal{CL}(n, k; \langle a_i \rangle, \langle b_i \rangle)$, and consider two cases. First, if w contains a letter from some alphabet A_j, for $j \in [n-1]^*$, then let $A_{\hat{j}}$ denote the alphabet with the largest index among the alphabets A_j for which w contains a letter. Then $w \in \mathcal{A}_{n-1}^c \cdots \mathcal{A}_{\hat{j}+1}^c \mathcal{A}_{\hat{j}}$. Second, if w contains no letters from any of the alphabets A_j, for $j \in [n-1]^*$, then it must be comprised entirely of letters from the alphabets $B_0, \ldots, B_k$. Let $B_{\hat{j}}$ denote the alphabet with the largest index among the alphabets B_j for which w contains a letter. Then $w \in \mathcal{A}_{n-1}^c \cdots \mathcal{A}_0^c \mathcal{B}_k^c \cdots \mathcal{B}_{\hat{j}+1}^c \mathcal{B}_{\hat{j}}$. Hence, w is in the union in (2.26). $\square$

Then the the recurrence (2.15) can be shown combinatorially when $\langle a_i \rangle_{i \geq 0}$ and $\langle b_i \rangle_{i \geq 0}$ are sequences in $\mathbb{N}$. Recall for all $n, k \in \mathbb{P}$,

$$A(n,k;\langle a_i \rangle, \langle b_i \rangle) = A(n-1,k-1;\langle a_i \rangle, \langle b_i \rangle) + (a_{n-1} + b_k) A(n-1,k;\langle a_i \rangle, \langle b_i \rangle),$$

subject to $A(0,k;\langle a_i \rangle, \langle b_i \rangle) = \delta_{0,k}$ and

$$A(n,0;\langle a_i \rangle, \langle b_i \rangle) = \prod_{i=0}^{n-1} (a_i + b_0).$$

Proof. The boundary condition in which $n = 0$ is obvious. The other boundary condition follows from the recurrence for it:

$$A(n,0;\langle a_i \rangle, \langle b_i \rangle) = (a_{n-1} + b_0) A(n-1,0;\langle a_i \rangle, \langle b_i \rangle),$$

which holds for all $n \in \mathbb{P}$. Among all n-letter words in $\mathcal{CL}(n,0;\langle a_i \rangle, \langle b_i \rangle)$, $a_{n-1} A(n-1,0;\langle a_i \rangle, \langle b_i \rangle)$ counts those in $\mathcal{A}_{n-1}$ and $b_0 A(n-1,0;\langle a_i \rangle, \langle b_i \rangle)$ counts those in $\mathcal{A}_{n-1}^c \mathcal{B}_0$. Observe that these two sets are disjoint and exhaustive in $\mathcal{CL}(n,0;\langle a_i \rangle, \langle b_i \rangle)$.

When $n, k \in \mathbb{P}$, the recurrence (2.15) follows from

$$\mathcal{CL}(n,k;\langle a_i \rangle, \langle b_i \rangle) = (\mathcal{A}_{n-1}^c \mathcal{B}_k^c) \dot{\cup} [\mathcal{A}_{n-1} \dot{\cup} (\mathcal{A}_{n-1}^c \mathcal{B}_k)], \tag{2.27}$$

which holds since the complement of the right-hand side of (2.27) is

$$\begin{aligned} \left((\mathcal{A}_{n-1}^c \mathcal{B}_k^c) \dot{\cup} [\mathcal{A}_{n-1} \dot{\cup} (\mathcal{A}_{n-1}^c \mathcal{B}_k)]\right)^c &= (\mathcal{A}_{n-1}^c \mathcal{B}_k^c)^c \cap [\mathcal{A}_{n-1} \dot{\cup} (\mathcal{A}_{n-1}^c \mathcal{B}_k)]^c \\ &= [(\mathcal{A}_{n-1} \mathcal{A}_{n-1}^c) \cup (\mathcal{A}_{n-1}^c \mathcal{B}_k)] \cap (\mathcal{A}_{n-1} \cup \mathcal{B}_k^c) \\ &= (\mathcal{A}_{n-1}^c \mathcal{B}_k) \cap (\mathcal{A}_{n-1} \cup \mathcal{B}_k^c) \\ &= (\mathcal{A}_{n-1}^c \mathcal{B}_k) \cap (\mathcal{A}_{n-1}^c \mathcal{B}_k)^c = \emptyset. \end{aligned}$$

$\square$

Also recall the Hockey Stick theorem (2.17), which states that for all $n, k \in \mathbb{P}$ and any pair of sequences $\langle a_i \rangle_{i \geq 0}$ and $\langle b_i \rangle_{i \geq 0}$ in $\mathbb{N}$, which are suppressed temporarily for compactness of notation:

$$A(n,k) = \sum_{j=0}^{k} (a_{n-j-1} + b_{k-j}) A(n-1-j, k-j),$$

subject to the same boundary conditions as (2.15).

Proof. The boundary conditions are the same as in the previous proof. For all $n, k \in \mathbb{P}$, the result follows from the equation

$$\mathcal{CL}(n,k) = \dot{\bigcup_{j=0}^{k}} (\mathcal{A}_{n-1}^c \cdots \mathcal{A}_{n-j}^c \mathcal{A}_{n-j-1} \mathcal{B}_k^c \cdots \mathcal{B}_{k-j+1}^c \dot{\cup} \mathcal{A}_{n-1}^c \cdots \mathcal{A}_{n-j-1}^c \mathcal{B}_k^c \cdots \mathcal{B}_{k-j+1}^c \mathcal{B}_{k-j}). \tag{2.28}$$

Again, pairwise disjointness is clear by the construction or, if needed, can be seen by recovering this equation iteratively in the same way as the recurrence. To see that these classes are indeed exhaustive within $\mathcal{CL}(n,k)$, let w be a word in that set. Then suppose that the alphabet A_{j_1} provides the letter in w that comes from the highest-indexed alphabet among $A_0, \ldots, A_{n-1}$, taking as a notation $j_1 = \infty$ in the case where there are no letters from any alphabet A_i. Likewise, suppose that the alphabet B_{j_2} provides the letter in w that comes from the highest-indexed alphabet among $B_0, \ldots, B_k$, taking as a notation $j_2 = \infty$ in the case where there are no letters from any alphabet B_i. Then define $\hat{j} = \max\{n-1-j_1, k-j_2\}$. Then

$$w \in (\mathcal{A}_{n-1}^c \cdots \mathcal{A}_{n-\hat{j}}^c \mathcal{B}_k^c \cdots \mathcal{B}_{k-\hat{j}+1}^c)(\mathcal{A}_{n-1-\hat{j}} \dot{\cup} \mathcal{A}_{n-1-\hat{j}}^c \mathcal{B}_{k-\hat{j}}).$$

□

The associated interpretation is that for a fixed $j \in [k]^*$, $(a_{n-1-j} + b_{k-j})A(n-1-j, k-j)$ counts those words in $\mathcal{CL}(n,k)$ either

- containing a letter from A_{n-j-1} but no letters from any A_i for $i \in \{n-j, \ldots, n-1\}$ or any B_i for $i \in \{k-j+1, \ldots, k\}$, or
- containing at least one letter from B_{j-k} but no letters from any A_i for $i \in \{n-j-1, \ldots, n-1\}$ or any B_i for $i \in \{k-j+1, \ldots, k\}$.

2.3.2 Selections of Balls from Urns

As an alternative, let $n, k \in \mathbb{N}$ and let $\langle a_i \rangle_{i \geq 0}$, $\langle b_i \rangle_{i \geq 0}$ be two sequences in $\mathbb{N}$. Then consider n labeled urns with labels $0, 1, \ldots, n-1$ and $n-k$ labeled positions for those urns, labeled with $1, \ldots, n-k$. Choose $n-k$ of the urns to be placed into the positions in increasing order by their labels, and let α_j denote the label of the urn in position j. Associate to each α_j a *sublabel* β_j given by $\beta_j := \alpha_j - j + 1$, for all $j \in [n-k]$.[8] For each $j \in [n-k]$, place a_{α_j} labeled white balls and b_{β_j} labeled red balls into the urn in position j, for convenience letting the labels on the red balls in urn j begin with $a_{\alpha_j} + 1$, for each $j \in [n-k]$. Finally, select one ball from each urn and place them in the same order as the urns they were chosen from to create a sequence of ordered pairs of length $n-k$ written as (label on the urn, label on the ball from that urn). Denote the set of all such sequences by $\mathcal{U}(n,k; \langle a_i \rangle, \langle b_i \rangle)$.

Then,

Theorem 2.3.3. *For all $n, k \in \mathbb{N}$ and any pair of sequences $\langle a_i \rangle_{i \geq 0}$, $\langle b_i \rangle_{i \geq 0}$ in $\mathbb{N}$,*

$$|\mathcal{U}(n,k; \langle a_i \rangle, \langle b_i \rangle)| = A(n,k; \langle a_i \rangle, \langle b_i \rangle). \tag{2.29}$$

[8]Thus β_j gives the difference between the ordinance of the label on the urn among the n urns (i.e. the urn labeled 0 is the first urn) and its position. Note, then, that β_j counts cumulatively, up to the placement of urn j, how many urns have been skipped via the selection of $n-k$ of them from among the $n-1$ urns.

Proof. For the moment, let $U(n,k;\langle a_i\rangle,\langle b_i\rangle) := |\mathcal{U}(n,k;\langle a_i\rangle,\langle b_i\rangle)|$.

Then first, when $n=0$, only the subcase $k=0$ is defined, in which case $\mathcal{U}(n,k;\langle a_i\rangle,\langle b_i\rangle)$ contains only the empty sequence. Hence, $U(0,k;\langle a_i\rangle,\langle b_i\rangle)=\delta_{0,k}$, for all $k\in\mathbb{N}$.

Second, if $k=0$ and $n\in\mathbb{P}$, then necessarily the sequence of labels for the urns is $(\alpha_1,\ldots,\alpha_n)=(0,\ldots,n-1)$. Hence, for $i\in[n]$, each $\alpha_i=i-1$ and each $\beta_i=0$. Therefore, for all $j\in[n-1]^*$, there are (a_j+b_0) balls in the urn labeled j. Thus, for all $n\in\mathbb{P}$,

$$U(n,0;\langle a_i\rangle,\langle b_i\rangle)=\prod_{j=0}^{n-1}(a_j+b_0).$$

Third, if $n,k\in\mathbb{P}$, then the possible arrangements of urns can be split into two disjoint, exhaustive classes:

(i) the urn in position $n-k$ is labeled $n-1$, i.e. $\alpha_{n-k}=n-1$ whence $\beta_{n-k}=k$, and

(ii) the urn in position $n-k$ has some label other than $n-1$, i.e. $\alpha_{n-k}<n-1$ whence $\beta_{n-k}<k$.

Given any sequence in (i) within $\mathcal{U}(n,k;\langle a_i\rangle,\langle b_i\rangle)$, map it to a sequence in $\mathcal{U}(n-1,k;\langle a_i\rangle,\langle b_i\rangle)$ by deleting the element of the sequence in position $n-k$. Since $\alpha_{n-k}=n-1$ provides $\beta_{n-k}=k$ for the urn in position $n-k$, there are a_{n-1} white balls and b_k red balls that could have been chosen. This deletion is a $(a_{n-1}+b_k)$-to-one surjection, and the first class is therefore counted by $(a_{n-1}+b_k)U(n-1,k;\langle a_i\rangle,\langle b_i\rangle)$.

Given any sequence in (ii) within $\mathcal{U}(n,k;\langle a_i\rangle,\langle b_i\rangle)$, note that it can also be considered a unique sequence of length $n-k$ in $\mathcal{U}(n-1,k-1;\langle a_i\rangle,\langle b_i\rangle)$ since in this case, the urn labeled $n-1$ is never chosen. Hence, this class is counted by $U(n-1,k-1;\langle a_i\rangle,\langle b_i\rangle)$.

Therefore, since $|\mathcal{U}(n,k;\langle a_i\rangle,\langle b_i\rangle)|$ satisfies the same boundary conditions and recurrence as (2.15) in Lancaster's Theorem 2.2.2, the result follows. □

Despite being *ad hoc*, approaching the C-L numbers from the perspective of $\mathcal{U}(n,k;\langle a_i\rangle,\langle b_i\rangle)$ has some noteworthy advantages. Consider, for instance, the proof of Theorem 2.2.3, the diagonal Hockey Stick theorem for the C-L numbers arising in this context[9]:

Proof. The boundary conditions are the same as in the proof of Theorem 2.3.3 for all $n,k\in\mathbb{N}$. Hence, let $n,k\in\mathbb{P}$. Then for all $j\in[k]^*$, the term $(a_{n-1-j}+b_{k-j})A(n-1-j,k-j)$ counts those sequences in $\mathcal{U}(n,k;\langle a_i\rangle,\langle b_i\rangle)$ arising from an urn placement in which the urn with the largest label is marked $\alpha_{n-k}=n-1-j$, whence the sublabel $\beta_{n-k}=k-j$. □

In addition, there is a simple proof of the vertical Hockey Stick theorem given in Lancaster's Theorem, i.e. Equation (2.16), which appears very difficult to prove with the words from Section 2.3.1. Recall that for all $n,k\in\mathbb{P}$,

$$A(n,k)=\sum_{j=k}^{n}A(j-1,k-1)\prod_{i=j}^{n-1}(a_i+b_k),$$

subject to the same boundary conditions as (2.15). Then,

[9] Again suppressing the references to the sequences to condense notation.

Proof. The boundary conditions are the same as in the previous theorem and proof for all $n, k \in \mathbb{N}$. Hence, let $n, k \in \mathbb{P}$. Then for all $j \in \{k, \ldots, n\}$, the term

$$A(j-1, k-1) \prod_{i=j}^{n-1} (a_i + b_k)$$

counts the sequences in $\mathcal{U}(n, k; \langle a_i \rangle, \langle b_i \rangle)$ arising from an urn placement in which all of the last $n-j$ urns, but no more, have maximally sequentially increasing labels, i.e. $\alpha_{j-k+1} = j$, $\alpha_{j-k+2} = j+1, \ldots, \alpha_{n-k} = n-1$ with $\alpha_{j-k} < j-1$, whence $\beta_{j-k+1} = \cdots = \beta_{n-k} = k$ with $\beta_{j-k} < k$. □

Furthermore, another result is available, expanding Lancaster's theorem. To derive it, first consider the special case $\mathcal{U}(n, k; \langle 0 \rangle, \langle b_i \rangle)$ as it applies to a facet of Comtet's theorem: Notice that for all $n, k \in \mathbb{N}$,

$$|\mathcal{U}(n, k; \langle 0 \rangle, \langle b_i \rangle)| = C(n, k; \langle b_i \rangle), \tag{2.30}$$

and (1.31):

$$C(n, k; \langle b_i \rangle) = \sum_{\substack{d_0 + \cdots + d_k = n-k \\ d_i \in \mathbb{N}}} b_0^{d_0} \cdots b_k^{d_k}.$$

The result is a closed form for $A(n, k; \langle a_i \rangle, \langle b_i \rangle)$ that generalizes (1.31) and arises by extending a proof of (1.31) that uses $\mathcal{U}(n, k; \langle 0 \rangle, \langle b_i \rangle)$.

Alternate Proof of (1.31). Among all sequences in $\mathcal{U}(n, k; \langle 0 \rangle, \langle b_i \rangle)$, the term $b_0^{d_0} \cdots b_k^{d_k}$ counts those in which for all $j \in [k]^*$ the urn placement results in exactly d_j urns being placed so that their sublabels are j. More specifically, this gives, for each $j \in [k]^*$,

$$\beta_{d_0 + \cdots + d_{j-1} + 1} = \cdots = \beta_{d_0 + \cdots + d_j} = j,$$

starting with β_1 in the case $j = 0$, and discontinuing the process if at any point a placement subscript exceeds $n - k$. Note that, for $j \in [k]^*$, each urn with sublabel d_j contains b_j red balls. □

Although not explicitly needed for the above proof, the values of the labels α_l, $l \in [n-k]$, of the urns counted by the same term are given by

$$\alpha_{d_0 + \cdots + d_{j-1} + i} = d_0 + \cdots + d_{j-1} + i + j - 1, \tag{2.31}$$

as i ranges in $[d_j]$ and for each $j \in [k]^*$, discontinuing the process if at any point a placement subscript exceeds $n - k$. Then, with essentially the same proof, a closed form expression for the C-L numbers is:

Theorem 2.3.4. *For all $n, k \in \mathbb{N}$ and any pair of sequences $\langle a_i \rangle_{i \geq 0}$ and $\langle b_i \rangle_{i \geq 0}$ in $\mathbb{N}$, letting*

$$S_j := \sum_{i=0}^{j} d_i, \tag{2.32}$$

for each $j \in [k]^*$ *with* $S_{-1} := 0$,

$$A(n,k;\langle a_i\rangle,\langle b_i\rangle) = \sum_{\substack{d_0+\cdots+d_k=n-k\\ d_i\in\mathbb{N}}} \prod_{j=0}^{k} \left(\prod_{i=S_{j-1}}^{S_j-1} (a_{i+j}+b_j) \right). \tag{2.33}$$

Proof. Among all sequences in $\mathcal{U}(n,k;\langle a_i\rangle,\langle b_i\rangle)$, the term

$$\prod_{j=0}^{k} \left(\prod_{i=S_{j-1}}^{S_j-1} (a_{i+j}+b_j) \right)$$

counts those in which the urn placement results in the list of labels given in (2.31). $\square$

Notice that the product on the inside of (2.33) is over exactly d_j terms for each $j \in [k]^*$, and when $a_i \equiv 0$ those terms are identical: each is b_j. Hence the expression reduces to (1.31) in Comtet's theorem.

2.4 Bijections and Applications to Structures in the Special Cases

Since two combinatorial interpretations for the C-L numbers are provided above, we connect those two structures before proceeding with the applications special cases. Because of this connection and for brevity, the individual bijections between the structures in the special cases will be given only in terms of the interpretation of "words in special alphabets" with commentary provided concerning the balls-and-urns interpretation when it is apropos.

The general idea behind this connection is that within a word $w \in \mathcal{CL}(n,k;\langle a_i\rangle,\langle b_i\rangle)$, all of the letters from the alphabets labeled B can be reconstrued as letters from extended versions of some of the alphabets labeled A that do not already appear in w. The specific map from $\mathcal{CL}(n,k;\langle a_i\rangle,\langle b_i\rangle)$ to $\mathcal{U}(n,k;\langle a_i\rangle,\langle b_i\rangle)$ can be described by the following process.

First, note that a typical word in $w \in \mathcal{CL}(n,k;\langle a_i\rangle,\langle b_i\rangle)$ has the form, for some $m \in [n-k]^*$,

$$w = a_{\alpha_1,x_1}a_{\alpha_2,x_2}\cdots a_{\alpha_m,x_m}b_{\beta_1,u_1}b_{\beta_2,u_2}\cdots b_{\beta_{n-k-m},u_{n-k-m}}, \tag{2.34}$$

where for any $i \in [m]$, each $x_i \in [a_i]$, and for any $i \in [n-k-m]$, each $u_i \in [b_i]$.

Now, supposing that w is written as in (2.34), replace the letters denoted with b's in w with letters denoted with $\hat{a}$'s, as follows, to get

$$\hat{w} = a_{\alpha_1,x_1}a_{\alpha_2,x_2}\cdots a_{\alpha_m,x_m}\hat{a}_{\alpha_{\beta_1},a_{\alpha_{s_1}}+u_1}\hat{a}_{\alpha_{s_2},a_{\alpha_{s_2}}+u_2}\cdots \hat{a}_{\alpha_{s_{n-k-m}},a_{\alpha_{s_{n-k-m}}}+u_{n-k-m}}, \tag{2.35}$$

where the $\hat{a}$'s are the letters obtained from the b's by letting α_{s_i} be the $(\beta_i+i)^{\text{th}}$ α_j not present among the subscripts of the a's, i.e. the $(\beta_i+i)^{\text{th}}$ element of the set $\{\alpha_1,\ldots,\alpha_m\}^c$, where the complement is taken in $[n-1]^*$ and is written canonically in increasing order. Note $\hat{w}$ is clearly unique to w. Also note that the second subscripts of the letters denoted with $\hat{a}$ exceed those allowed by our convention on the alphabets labeled A, rendering the hats unnecessary for identifying these letters.

Next, rewrite $\hat{w}$ from (2.35) by permuting the letters so that they appear in order of increasing first subscripts, dropping the hats. This provides the word

$$w_\alpha = a_{\alpha_{t_1},x_{t_1}} a_{\alpha_{t_2},x_{t_2}} \cdots a_{\alpha_{t_{n-k}},x_{t_{n-k}}}, \tag{2.36}$$

where $0 \le t_1 < \cdots < t_{n-k} \le n-1$ are the initial subscripts of from the word of form (2.36) written in increasing order. Then w_α is clearly unique to w.

Finally, using w_α from (2.36), identify the sequence of ordered pairs s_w of length $n-k$ obtained by selecting, from among urns labeled 0 through $n-1$, the urns labeled α_{t_j}, $j \in [n-k]$, and then, for each $j \in [n-k]$, choosing the ${x_{t_j}}^{\text{th}}$ ball from the urn labeled α_{t_j}, assuming again that the labels on the red balls begin where the labels on the white balls leave off, i.e. by rewriting the subscripts of the letters in w_α as ordered pairs. Since this sequence is also unique to w, the map from $\mathcal{CL}(n,k;\langle a_i\rangle,\langle b_i\rangle)$ to $\mathcal{U}(n,k;\langle a_i\rangle,\langle b_i\rangle)$ given by $w \mapsto s_w$ is an injection.

Theorem 2.4.1. *The map from $\mathcal{CL}(n,k;\langle a_i\rangle,\langle b_i\rangle)$ to $\mathcal{U}(n,k;\langle a_i\rangle,\langle b_i\rangle)$ described above, taking $w \mapsto s_w$ is a bijection.*

Proof. (Surj.): Let $s \in \mathcal{U}(n,k;\langle a_i\rangle,\langle b_i\rangle)$. Identify in s a word $w_{\alpha,s}$ of the form (2.36) by recording the ordered pairs that provide the label of each urn with the label on the ball chosen from it. Next, mark with a hat each letter in $w_{\alpha,s}$ for which $x_{t_j} > a_{\alpha_{t_j}}$, for $j \in [n-k]$, and move each hat-marked letter to the end of the word, preserving the order in which they appear. This provides a new word $\hat{w}_s$ of the form (2.35). Now compare the leading subscripts of the hatted letters in $\hat{w}_s$ against the complement in $[n-1]^*$ of the set of leading subscripts of the non-hatted letters in $\hat{w}_s$, denoted L. Finally relabel the i^{th} hatted letter $\hat{a}_{i_{r_1},i_{r_2}}$ in $\hat{w}_s$ with $b_{i_{p_1},i_{p_2}}$, where i_{p_1} is i less than the position of i_{r_1} in L and $i_{p_2} = i_{r_2} - a_{\alpha_{i_{r_1}}}$. This provides a word of the form (2.34) since the letters labeled with b's after the final rewrite appear in nondecreasing order and since if there are m letters labeled with an a with no hat in $\hat{w}_s$, the initial subscript of the final b after the final relabeling must be no greater than $n-1-(n-k-m-1)-m = k$. This last point follows since among the $n-1$ possible initial subscripts, m are taken by the a's and $n-k-m-1$ are taken by the previous b's. □

It is useful to see an example of this map, for clarity.

Example 2.4.2. *For this example, suppose that for all $i \in \mathbb{N}$, $a_i = i^2$ and $b_i = 2i+1$ and that $n = 9$ and $k = 3$. Then consider the word:*

$$w = a_{2,1}\, a_{3,7}\, a_{5,13}\, b_{0,1}\, b_{2,3}\, b_{2,3}. \tag{2.37}$$

Here, $\{\alpha_1,\alpha_2,\alpha_3\} = \{2,3,5\}$, and so $\{\alpha_1,\alpha_2,\alpha_3\}^c = \{0,1,4,6,7,8\}$.
Since $\beta_1 + 1 = 0 + 1 = 1$, $\alpha_{s_1} = 0$. Similarly, since $\beta_2 + 2 = 2 + 2 = 4$, $\alpha_{s_2} = 6$, and since $\beta_3 + 3 = 2 + 3 = 5$, $\alpha_{s_3} = 7$.
Also, note that $a_0 + 1 = 0 + 1 = 1$, $a_6 + 3 = 36 + 3 = 39$, and $a_7 + 3 = 49 + 3 = 52$.
Thus, $w \mapsto \hat{w}$, given by

$$\hat{w} = a_{2,1}\, a_{3,7}\, a_{5,13}\, \hat{a}_{0,1}\, \hat{a}_{6,39}\, \hat{a}_{7,52}, \tag{2.38}$$

and $\hat{w} \mapsto w_\alpha$*, given by*

$$w_\alpha = a_{0,1}\, a_{2,1}\, a_{3,7}\, a_{5,13}\, a_{6,39}\, a_{7,52}. \tag{2.39}$$

From w_α *the sequence* s_w *can be extracted, given by:*

$$s_w = ((0,1),(2,1),(3,7),(5,13),(6,39),(7,52)). \tag{2.40}$$

Observe that the ordered pairs $(0,1)$*,* $(6,39)$*, and* $(7,52)$ *can all be identified as being those that arose from letters that were initially from alphabets labeled with* B *because their second term exceeds the square (in this example) of the first term. The reverse map can thus be read by going from the bottom up.*

For convenience in what follows, given a word w_α of the form (2.36) that is uniquely associated with some word $w \in \mathcal{CL}(n,k;\langle a_i\rangle,\langle b_i\rangle)$, let the set $\{\alpha_1+1,\ldots,\alpha_{n-k}+1\}$ be called the *α-signature* of w. Also, in dealing with partitions or permutations in k nonempty blocks or cycles, call the k-element set of smallest elements of the blocks or cycles the *least-element set*, abbreviated *LE-set*, of such a partition or permutation. Finally, since the bijections to familiar structures in this chapter are more difficult than those in the Comtet-numbers cases, the an informal overview of each map will be given before formally defining it.

2.4.1 The Cycle Numbers

When $a_i = i$, for all $i \in \mathbb{N}$, and $b_i \equiv 0$, the recurrence given in (2.15) takes the form, for all $n,k \in \mathbb{P}$,

$$A(n,k;\langle i\rangle,\langle 0\rangle) = A(n-1,k-1;\langle i\rangle,\langle 0\rangle) + (n-1)A(n-1,k;\langle i\rangle,\langle 0\rangle), \tag{2.41}$$

subject to the boundary conditions $A(0,k;\langle i\rangle,\langle 0\rangle) = \delta_{0,k}$, for all $k \in \mathbb{N}$, and $A(n,0;\langle i\rangle,\langle 0\rangle) = 0$, for all $n \in \mathbb{P}$. These are the same recurrence and boundary conditions satisfied by the cycle numbers, given in (2.1), and so $A(n,k;\langle i\rangle,\langle 0\rangle) = c(n,k)$. Thus, the cycle number $c(n,k)$ also enumerates the words in the set $\mathcal{CL}(n,k;\langle i\rangle,\langle 0\rangle)$, i.e. words of length $n-k$ in the strictly ascending alphabets $A_0\cup\cdots\cup A_{n-1}$, with each $A_i = \{a_{i,1},\ldots,a_{i,i}\}$. In addition, $c(n,k)$ enumerates the sequences in the set $\mathcal{U}(n,k;\langle i\rangle,\langle 0\rangle)$, i.e. those sequences obtained from selecting balls from urns, as described in Section 2.3.2, in which the label on each urn describes exactly how many balls it contains, all of which are white.

Recall that for all $n,k \in \mathbb{N}$, the cycle number $c(n,k)$ enumerates $\mathcal{P}(n,k)$, the set of permutations of $[n]$ with exactly k nonempty cycles, which will be taken to be written canonically with the least element of each cycle listed first and the cycles listed in order by increasing least (i.e. initial) elements.

The general idea behind the bijection from $\mathcal{CL}(n,k;\langle i\rangle,\langle 0\rangle)$ to $\mathcal{P}(n,k)$ is to consider a typical word $w \in \mathcal{CL}(n,k;\langle i\rangle,\langle 0\rangle)$ composed of letters from the strictly ascending alphabets $A_{\alpha_1}, A_{\alpha_2},\ldots,A_{\alpha_{n-k}}$, i.e. with α-signature $\{\alpha_1+1,\alpha_2+1,\ldots,\alpha_{n-k}+1\}$, and match it with a certain permutation in $\mathcal{P}(n,k)$ with LE-set $\{\alpha_1+1,\alpha_2+1,\ldots,\alpha_{n-k}+1\}^c$. The particular choice of the letter from each A_{α_j} indicates where within the permutation each particular element α_j+1 of $[n]$ will be placed in the permutation[10].

[10]In terms of balls and urns, the urn selection is the α-signature and the particular balls in each urn correspond directly with the possible letters in the word.

To identify specifically where each element is placed in the permutation, label n positions. Then, starting with the last letter in the word, $a_{\alpha_{n-k},j_{n-k}}$, with $j_{n-k} \in [\alpha_{n-k}]$, place the element $\alpha_{n-k}+1$ of $[n]$ in position $j_{n-k}+1$ among the n spots. Repeat this with the next-to-last letter, placing the element $\alpha_{n-k-1}+1$ in position $j_{n-k-1}+1$ among the remaining $n-1$ spots, and so on, until the element α_1+1 is placed in position j_1+1 among the remaining $n-(n-k-1)=k+1$ spots. Note that the first spot is always left empty by this process. Finally, place the least elements, i.e. those in $\{\alpha_1+1, \alpha_2+1, \ldots, \alpha_{n-k}+1\}^c$, in increasing order in the remaining k spots, closing the previous cycle and opening a new one immediately before each least-element so they are listed first in their respective cycles. It is notationally convenient to refer to the ultimate position of α_i+1, among all of the elements following 1, as y_i.

Formally, the map is given by mapping the word

$$a_{\alpha_1,j_1}a_{\alpha_2,j_2}\cdots a_{\alpha_{n-k},j_{n-k}} \in \mathcal{CL}(n,k;\langle i\rangle,\langle 0\rangle)$$

to the permutation

$$(s_1\ \sigma_1\ \ldots\ \sigma_{j_1})(s_2\ \sigma_{j_1+2}\ \ldots\ \sigma_{j_2})\cdots(s_k\ \sigma_{j_{k-1}+2}\ \ldots\ \sigma_{j_k}) \in \mathcal{P}(n,k), \tag{2.42}$$

where

1. the LE-set $\{s_1,\ldots,s_k\} = \{\alpha_1+1,\ldots,\alpha_{n-k}+1\}^c$, and
2. with y_i as above, $\sigma_{y_i} = \alpha_i+1$, $\forall i \in [n-k]$.

Theorem 2.4.3. *The above map from $\mathcal{CL}(n,k;\langle i\rangle,\langle 0\rangle)$ to $\mathcal{P}(n,k)$ is a bijection.*

Proof. Observe first that each possible α-signature is represented. Then the proof again proceeds in three parts: First, the image of the map will be established; second, the map will be shown to be injective; and third, the map will be shown to be surjective.

To see that the image of the map is $\mathcal{P}(n,k)$ as claimed, it is necessary to show that the "least element" of each cycle is indeed that cycle's smallest element. This follows since there are $\alpha_i-i+1=\alpha_i-(i-1)$ least elements smaller than α_i+1 since there are α_i elements of $[n]$ less than α_i+1, and none of the $i-1$ elements $\alpha_1+1, \ldots, \alpha_{i-1}+1$ is eligible to be a least element, noting that each of these is smaller than α_i+1.. Thus, it is sufficient to show that α_i+1 appears one of the first $\alpha_i-(i-1)$ cycles.

To see this fact, notice that new cycles begin in each spot left unfilled by elements in $\{\alpha_1+1,\ldots,\alpha_{n-k}+1\}$ within the permutation. Thus, consider the situation that creates the largest number of new cycles to begin before the placement of α_i+1, i.e. when $j_i=\alpha_i$, and, $\forall m>i$, $j_m>j_i$. In that case there are α_i-1 spots to the left of the placement of α_i+1, and $i-1$ of those are occupied by the elements $\alpha_1+1, \ldots, \alpha_{i-1}+1$ since $j_i=\alpha_i$ implies that $\forall r<i$, $\alpha_r<\alpha_i$, and hence necessarily $j_r<\alpha_i$. Hence, there are strictly fewer than $\alpha_i-(i-1)$ spots preceding the placement of α_i+1 in which new cycles could begin in the image permutation, and therefore the element α_i+1 appears in one of the first $\alpha_i-(i-1)$ cycles.

(Inj.): Consider two distinct words in $\mathcal{CL}(n,k;\langle i\rangle,\langle 0\rangle)$. There are two cases that describe how the sequences can differ.

First, the α-signatures of the sequences could differ. In this case, the LE-sets of the associated permutations are different, and hence the permutations themselves are different.

Second, when the α-signatures are the same, at least one of the j_i's, $1 \leq i \leq n-k$, must be different from its counterpart in the same position, i, of the other sequence. In this case, while the LE-sets are the same in the associated permutations, at least one of the y_i's is different from its counterpart. Thus, the associated permutations are different.

(Surj): Consider the general permutation given in (2.42) and call it π. Define the set of non-least-elements of π by $\Sigma_\pi := \{\sigma_1, \sigma_2, \ldots, \sigma_{n-k}\}$. Then π is the image of the sequence of the form (2.41) with, $\forall i \in [n-k]$, $\alpha_i = \min(\Sigma_\pi \setminus \{\alpha_1 + 1, \ldots, \alpha_{i-1} + 1\}) - 1$, and j_i is given by the position of the element $\alpha_i + 1$ within the permutation, where the count begins with the element following 1 and ignores the (least) elements $\alpha_{i+1} + 1, \ldots, \alpha_{n-k} + 1$ as well as all of the elements larger than itself. $\square$

An example of this map being applied is extremely useful for clarity.

Example 2.4.4. *Consider the example for $n = 8$ and $k = 4$ with the word*

$$a_{2,2}\, a_{4,4}\, a_{5,2}\, a_{7,5}. \tag{2.43}$$

First notice that $\{\alpha_1+1, \ldots, \alpha_4+1\} = \{3, 5, 6, 8\}$, so the LE-set of the associated permutation is $\{3, 5, 6, 8\}^c = \{1, 2, 4, 7\}$. Now set up the permutation structure by starting it with a 1 and then $n - 1 = 7$ blank spaces:

$$(1\, \underset{1}{\_}\, \underset{2}{\_}\, \underset{3}{\_}\, \underset{4}{\_}\, \underset{5}{\_}\, \underset{6}{\_}\, \underset{7}{\_}).$$

The element 8 is placed first, into position $j_4 = 5$. In the process, the blank spaces after Position 5 change their numbers, since the placement is among those not already occupied:

$$(1\, \underset{1}{\_}\, \underset{2}{\_}\, \underset{3}{\_}\, \underset{4}{\_}\, 8\, \underset{5}{\_}\, \underset{6}{\_}).$$

The element 6 is placed next, into position $j_3 = 2$ among the remaining positions. Again, the numbering changes to only enumerate the remaining empty spots:

$$(1\, \underset{1}{\_}\, 6\, \underset{2}{\_}\, \underset{3}{\_}\, 8\, \underset{4}{\_}\, \underset{5}{\_}).$$

The element 5 is placed next, into position $j_2 = 4$ among the remaining positions. Again the numbering changes:

$$(1\, \underset{1}{\_}\, 6\, \underset{2}{\_}\, \underset{3}{\_}\, 8\, 5\, \underset{4}{\_}).$$

The element 3 is placed next, into position $j_1 = 2$ among the remaining positions. Now the numbering of the positions is unneeded and is dropped:

$$(1\, \_\, 6\, 3\, \_\, 8\, 5\, \_).$$

Finally, the remaining three blank positions are filled with cycle delimiters attached to the least elements 2, 4, and 7, in that order:

$$(1)(2\,6\,3)(4\,8\,5)(7). \tag{2.44}$$

Going in reverse, as described in the surjectivity part of the proof, is also helpful. Starting

with the permutation given in (2.44), first collect the non-least elements from each permutation cycle: 6, 3, 8, and 5 and recopy the list in increasing order. Associate each, written in increasing order, with the corresponding $\alpha_i + 1$ *to yield*
$\alpha_1 = 3 - 1 = 2$, $\alpha_2 = 5 - 1 = 4$, $\alpha_3 = 6 - 1 = 5$, *and* $\alpha_4 = 8 - 1 = 7$.
Hence, examining where within the permutation the non-least elements appear, taking them in increasing order, the j_i*'s can be obtained:*

Since 3 shows up second among those elements bigger than 1 and smaller than itself, being preceded by 2, $j_1 = 2$.
Since 5 appears fourth among those elements bigger than 1 and smaller than itself, being preceded by 2, 3, and 4, $j_2 = 4$.
Since 6 appears second among those elements bigger than 1 and smaller than itself, being preceded by 2, $j_3 = 2$.
And since 8 appears fifth among those elements bigger than 1 and smaller than itself, being preceded by 2, 6, 3, and 4, $j_4 = 5$, *recovering the word (2.43).*

Now notice that if the specific initial subscripts in a word $w \in \mathcal{CL}(n, k; \langle i \rangle, \langle 0 \rangle)$ are $0 < i_1 < i_2 < \cdots < i_{n-k} < n$, then there are $i_1 i_2 \cdots i_{n-k}$ possible manifestations of w, providing the formula, for all $n, k \in \mathbb{N}$,

$$c(n,k) = \sum_{0 < i_1 < i_2 < \cdots < i_{n-k} < n} i_1 i_2 \cdots i_{n-k}. \tag{2.45}$$

Also, recall the two variants on the Hockey Stick Theorem, (2.3) and (2.4): first the diagonal variant

$$c(n,k) = \sum_{j=0}^{k} (n-1-j) c(n-1-j, k-j),$$

and second, respectively, the vertical variant

$$c(n,k) = \sum_{j=k}^{n} c(j-1, k-1)(n-1)^{\underline{n-j}},$$

both subject to the same boundary conditions as (2.1).

In light of the proofs of those formulas, among all elements of $\mathcal{P}(n,k)$,

- the term $(n-1-j)c(n-1-j, k-j)$ in (2.3) counts those in which all of the elements $n-j+1, \ldots, n$ in $[n]$ are fixed points of the permutation, but $n-j$ is not, and
- the term $c(j-1,k-1)(n-1)^{\underline{n-j}}$ in (2.4) counts those in which $j \in [n]$ is the largest fixed point of the permutation.

Observe that these interpretations are also available by direct inspection of (2.3) and (2.4), considering the structure $\mathcal{P}(n,k)$.

Consider now the closed-form expression for the C-L numbers given in Theorem 2.3.4. In the cycle-numbers special case, (2.33) can be expressed as follows:

Theorem 2.4.5. *For all $n, k \in \mathbb{N}$,*

$$c(n,k) = \sum_{\substack{d_1+d_2+\cdots+d_k=n-k \\ d_i\in\mathbb{N}}} \frac{n!}{(d_1+1)(d_1+d_2+2)\cdots(d_1+\cdots+d_k+k)}. \tag{2.46}$$

Proof. This is straightforward by applying Theorem 2.3.4 to the case of $a_i = i$ and $b_i = 0$ and rewriting it in terms of the factors not present.

Alternatively, we offer a proof using $\mathcal{P}(n,k)$, ordered canonically:
In (2.46), each d_i, for $i \in [k]$, indicates the number of elements in cycle $k+1-i$ that are not that cycle's smallest element. This follows from the fact that that there are

$$(n-1)^{\underline{d_k}} = (d_1+\cdots+d_k+k-1)^{\underline{d_k}}$$

ways to select d_k elements of $[n]$ to join 1 in the first cycle. Also, that choice determines the smallest element of the next cycle to be the smallest remaining element of $[n]$. Thus, similarly, there are

$$(d_1+\cdots+d_k+k-1-d_k-1)^{\underline{d_{k-1}}} = (d_1+\cdots+d_{k-1}+k-2)^{\underline{d_{k-1}}}$$

ways to select d_{k-1} elements from those remaining to fill the second cycle, and so on, until there are just

$$(d_1+d_2+1-d_2-1)^{\underline{d_1}} = d_1!$$

ways to place the last d_1 remaining elements of $[n]$ into cycle k. This gives a total of

$$\begin{aligned}(1\cdot 2\cdots d_1)\cdot(d_1+2)(d_1+3)\cdots(d_1+d_2+1)\cdots(d_1+\cdots+d_{k-1}+k)\\ (d_1+\cdots+d_{k-1}+k+1)\cdots(d_1+\cdots+d_k+k-1)\end{aligned}$$

permutations. Factoring out $n!$, noting that $n = d_1+\cdots+d_k+k$, and summing across all possible sets of values for the d_i's yields (2.46). □

Given the interpretation in the above proof, by restricting each d_i in (2.46) to the set $\mathbb{P}$ instead of $\mathbb{N}$, the derangement numbers arise. Specifically, $\forall n, k \in \mathbb{P}$, let

$$d(n,k) := \sum_{\substack{d_1+d_2+\cdots+d_k=n-k \\ d_i\in\mathbb{P}}} \frac{n!}{(d_1+1)(d_1+d_2+2)\cdots(d_1+\cdots+d_k+k)}. \tag{2.47}$$

Within $\mathcal{P}(n,k)$, $d(n,k)$ enumerates those permutations of $[n]$ so that when written in cycle notation, there are exactly k cycles, and no cycle is a singleton, i.e. permutations of $[n]$ into exactly k cycles with no fixed points. Summing across all possible numbers of cycles, then yields the set of permutations of $[n]$ with no fixed points, which are known, see [20], to be counted by the derangement number d_n. Thus,

Theorem 2.4.6. *For all $n \in \mathbb{P}$,*
$\sum_k d(n,k) =$

$$\sum_{k\geq 0} \sum_{\substack{d_1+d_2+\cdots+d_k=n-k \\ d_i\in\mathbb{P}}} \frac{n!}{(d_1+1)(d_1+d_2+2)\cdots(d_1+\cdots+d_k+k)} = d_n. \tag{2.48}$$

Thus, not only are the derangement numbers recovered from this approach, the equation (2.47) provides an expression for the derangements of $[n]$ which, when written in cycle notation, have precisely k cycles. These k-cycle derangement numbers also satisfy a known[11] two-term recurrence relation:

$$d(n,k) = (n-1)(d(n-2,k-1) + d(n-1,k)). \tag{2.49}$$

Proof. First, $(n-1)d(n-2,k-1)$ enumerates, among all derangements of $[n]$ written in exactly k cycles, those in which the element n appears in a cycle of cardinality 2. This is because any of the $n-1$ other elements of $[n]$ can be paired with n, and there are $d(n-2,k-1)$ derangements of $[n-2]$ written in exactly $k-1$ cycles. On the other hand, $(n-1)d(n-1,k)$ enumerates those derangements of $[n]$ written in exactly k cycles in which n appears in a cycle of cardinality at least 3, which can be seen clearly by noting that there are $d(n-1,k)$ permutations of $[n-1]$ written in exactly k cycles, and $n-1$ choices for the element that is chosen to immediately precede n in the relevant derangement of $[n]$. $\square$

Note that summing (2.49) across k recovers the known[12] recurrence relation for d_n, namely $d_n = (n-1)(d_{n-1} + d_{n-2})$.

2.4.2 The Lah Numbers

When $a_i = i$ and $b_i = i$ for all $i \in \mathbb{N}$, the recurrence given in (2.15) takes the form, for all $n, k \in \mathbb{P}$,

$$A(n,k;\langle i\rangle,\langle i\rangle) = A(n-1,k-1;\langle i\rangle,\langle i\rangle) + (n+k-1)A(n-1,k;\langle i\rangle,\langle i\rangle), \tag{2.50}$$

subject to the boundary conditions $A(0,k;\langle i\rangle,\langle i\rangle) = \delta_{0,k}$, for all $k \in \mathbb{N}$, and $A(n,0;\langle i\rangle,\langle i\rangle) = 0$, for all $n \in \mathbb{P}$. These are the same recurrence and boundary conditions satisfied by the Lah numbers, given in (2.6), and so $A(n,k;\langle i\rangle,\langle i\rangle) = L(n,k)$. Thus, the Lah number $L(n,k)$ also enumerates the words in the set $\mathcal{CL}(n,k;\langle i\rangle,\langle i\rangle)$, i.e. words of length $n-k$ in the combined alphabets of the strictly ascending alphabets $A_0 \cup \cdots \cup A_{n-1}$ together with the ascending alphabets $B_0 \cup \cdots \cup B_k$, with each $A_i = \{a_{i,1}, \ldots, a_{i,i}\}$ and with each $B_i = \{b_{i,1}, \ldots, b_{i,i}\}$. In addition, $L(n,k)$ enumerates the sequences in the set $\mathcal{U}(n,k;\langle i\rangle,\langle i\rangle)$, i.e. those sequences obtained from selecting balls from urns, as described in Section 2.3.2, in which the label on each urn describes the number of white balls it contains and the sublabel on each urn describes the number of red balls it contains.

Recall that for all $n, k \in \mathbb{N}$, the Lah number $L(n,k)$ enumerates $\overrightarrow{\Pi}_{n,k}$, the set of partitions of $[n]$ into exactly k nonempty blocks, each equipped with a linear order, which will be taken to be written canonically with the blocks listed in order by increasing least elements.

The general idea used to obtain a bijection from $\mathcal{CL}(n,k;\langle i\rangle,\langle i\rangle)$ to $\overrightarrow{\Pi}_{n,k}$ is to consider a typical word $w \in \mathcal{CL}(n,k;\langle i\rangle,\langle i\rangle)$ in terms of the related word w_α, as in (2.36), here with α-signature specified to be $\{\alpha_{t_1}+1, \alpha_{t_2}+1, \ldots, \alpha_{t_{n-k}}+1\}$. The goal is to match it with a certain partition in $\overrightarrow{\Pi}_{n,k}$ that has LE-set $\{\alpha_{t_1}+1, \alpha_{t_2}+1, \ldots, \alpha_{t_{n-k}}+1\}^c$. For $i \in [n-k]$,

[11] See [25] for instance.
[12] See [20] for instance.

the particular manifestations of each letter $a_{\alpha_{t_i},j_i}$, in w_α, for $j_i \in [2\alpha_{t_i} - t_i + 1]$, will indicate where within the partition each particular element $\alpha_{t_i} + 1$ of $[n]$ will be placed in the partition[13].

To do so, first mark $n+k-1$ spots since the method will consider the partition dividers as objects to be placed along with each of the elements of $[n]$, creating $n+k-1$ objects to be placed. Then, in similar fashion to the analogous bijection in the cycle numbers case, place $\alpha_{t_{n-k}} + 1$ in blank j_{n-k} before placing $\alpha_{t_{n-k-1}} + 1$ in blank j_{n-k-1}, among those remaining, and so on, proceeding from largest initial subscript to smallest. Notice that once all $n-k$ elements of the form $\alpha_{t_i} + 1$ are placed, there are $(n+k-1)-(n-k) = 2k-1$ free positions remaining. Into these, $k-1$ partition dividers will be placed in every second blank that remains[14] to create k blocks, each with one empty space. Finally place the elements of the set $\{\alpha_{t_1} + 1, \alpha_{t_2} + 1, \ldots, \alpha_{t_{n-k}} + 1\}^c$ in the remaining blanks in increasing order. It is notationally convenient to refer to the ultimate position of $\alpha_{t_i} + 1$, among all of the elements and partition dividers, as y_{t_i}.

Formally, consider the map from $\mathcal{CL}(n, k; \langle i\rangle, \langle i\rangle)$ to $\overrightarrow{\Pi}_{n,k}$ given by mapping a word w of the form (2.34) to its variant w_α given in (2.36), specified here with $a_i = i$ and $b_i = i$ for all $i \in \mathbb{N}$, and then mapping the resulting $w_\alpha = a_{\alpha_{t_1},j_{t_1}} a_{\alpha_{t_2},j_{t_2}} \cdots a_{\alpha_{t_{n-k}},j_{t_{n-k}}}$ to the partition

$$\{l_1, \ldots, l_{i_1} | l_{i_1+1}, \ldots, l_{i_2} | \ldots | l_{i_{k-1}+1}, \ldots, l_{i_k}\} = \{L_1 | \ldots | L_k\} \in \overrightarrow{\Pi}_{n,k}, \tag{2.51}$$

where

1. the LE-set $\{\min(L_1), \ldots, \min(L_k)\} = \{\alpha_{t_1} + 1, \ldots, \alpha_{t_{n-k}} + 1\}^c$, and
2. with y_{t_i} as above, $l_{y_{t_i}} = \alpha_{t_i} + 1$, $\forall i \in [n-k]$.

Theorem 2.4.7. *The map described above from $\mathcal{CL}(n, k; \langle i\rangle, \langle i\rangle)$ to $\overrightarrow{\Pi}_{n,k}$ is a bijection.*

Proof. First, the image of the map will be established; second, the map will be shown to be injective; and third, the map will be shown to be surjective.

To see that the image of the map is $\overrightarrow{\Pi}_{n,k}$ as claimed, it is necessary to show that the proposed "least element" in each block is indeed the smallest element in that block. This follows since there are $\alpha_{t_i} - t_i + 1 = \alpha_{t_i} - (t_i - 1)$ least elements smaller than $\alpha_{t_i} + 1$ since there are α_{t_i} elements of $[n]$ less than $\alpha_{t_i} + 1$, and none of the $t_i - 1$ elements $\alpha_{t_1} + 1$, …, $\alpha_{t_{i-1}} + 1$, is eligible to be a least element[15]. Thus, it is sufficient to show that the element $\alpha_{t_i} + 1$ of $[n]$ appears in one of the first $\alpha_{t_i} - (t_i - 1)$ blocks of the partition.

To see this fact, consider the situation which creates the largest number of empty spaces, and hence partition dividers, to appear before the placement of $\alpha_{t_i} + 1$, i.e. when $j_{t_i} = 2\alpha_{t_i} - (t_1 - 1)$, and, $\forall m > i$, $j_{t_m} > j_{t_i}$. In that case, there are $2\alpha_{t_i} - (t_i - 1) - 1$ spots to the left of the placement of $\alpha_{t_i} + 1$, and $t_i - 1$ of those are occupied by the elements $\alpha_{t_1} + 1, \ldots, \alpha_{t_{i-1}} + 1$, since $j_{t_i} = 2\alpha_{t_i} - (t_i - 1)$ implies that for all $r < i$, $\alpha_{t_r} < 2\alpha_{t_i} - (t_i - 1)$ and

[13] Again, via the connection between words and balls and urns, the two cases need not be developed separately

[14] This results in the situation in which partition dividers never appear in the first blank or in two spots consecutive with respect to being empty after the placement of the $n-k$ elements.

[15] though each of these is smaller than $\alpha_{t_i} + 1$.

hence that $j_{t_r} < 2\alpha_{t_i} - (t_i - 1)$ as well. Thus, there are strictly fewer than $2\alpha_{t_i} - 2(t_i - 1) - 1$ empty spots which could precede the placement of $\alpha_{t_i} + 1$ in the image partition. Therefore, since the partition dividers are placed in every second empty spot remaining, at most $\alpha_{t_i} - (t_i - 1) - 1$ partition dividers can fit in the spaces before the placement of $\alpha_{t_i} + 1$. Thus, the element $\alpha_{t_i} + 1$ appears in one of the first $\alpha_{t_i} - (t_i - 1)$ blocks.

It follows that the map is injective by considering two distinct words $u, w \in \mathcal{CL}(n, k; \langle i\rangle, \langle i\rangle)$. The associated words u_α and w_α are different by Theorem 2.4.1. There are two cases that describe how the words can differ.

First, the α-signatures of the words may be different. In this case, the LE-sets of the associated partitions will be different, and hence the partitions themselves will be different.

Second, the α-signatures could be the same and hence at least one of the j_{t_i}'s, $i \in [n-k]$, differs from its counterpart in the same position, call it h_{t_i}, of the other sequence. In this case, while the LE-sets will be the same in the associated partitions, at least one of the $\alpha_{t_i} + 1$'s will appear in a different position in each partition. Thus, the partitions are different.

Finally, to see that the map is surjective, consider the general partition of the form given in (2.51) and call it $\mathcal{L}$. Notice that each $L_j = \{l_{i_{j-1}+1}, \ldots, l_{i_j}\}$, and define the set $M = \{\min(L_1), \ldots, \min(L_k)\}$. Then $\mathcal{L}$ is the image of the word of the form given by the Lah case of (2.36) with, $\forall t_i \in [n-k]$, $\alpha_{t_i} = \min(M \setminus \{\alpha_{t_1} + 1, \ldots, \alpha_{t_{i-1}} + 1\}) - 1$ and j_{t_i} is given by the position of the element $\alpha_{t_i} + 1$ within the partition, where the count ignores the elements $\alpha_{t_{i+1}} + 1, \ldots, \alpha_{t_{n-k}} + 1$ as well as elements larger than itself but includes partition dividers. $\square$

An example of this map being applied is useful for clarity.

Example 2.4.8. *Consider the example where $n = 9$ and $k = 4$ with the word*

$$a_{2,3}\, a_{3,3}\, a_{8,6}\, b_{4,1}\, b_{4,1}. \tag{2.52}$$

This word maps to the "hatted" word, all in a's:

$$a_{2,3}\, a_{3,3}\, a_{8,6}\, \hat{a}_{5,5+1}\, \hat{a}_{6,6+1} = a_{2,3}\, a_{3,3}\, a_{8,6}\, \hat{a}_{5,6}\, \hat{a}_{6,7}, \tag{2.53}$$

since among $\{0, 1, 4, 5, 6, 7\}$, i.e. the subset of $[n-1]^* = [8]^*$ that is the complement of the set of initial subscripts of the a's in the word, 5 occurs fourth and then, once 5 is taken, 6 occurs fourth. Reordering this word in terms of increasing initial subscripts yields a word of the form given by the Lah case of (2.36):

$$a_{2,3}\, a_{3,3}\, a_{5,6}\, a_{6,7}\, a_{8,6}. \tag{2.54}$$

Now the map can be performed to a partition of $[9]$ into 4 nonempty blocks, each equipped with a linear order via the process described above.

First notice that $\{\alpha_{t_1} + 1, \ldots, \alpha_{t_5} + 1\} = \{3, 4, 6, 7, 9\}$, so the LE-set of the associated partition is $\{3, 4, 6, 7, 9\}^c = \{1, 2, 5, 8\}$. Now set up the partition structure by placing $n + k - 1 = 12$ blank spaces inside of set brackets:

$$\{\underline{\ }_1, \underline{\ }_2, \underline{\ }_3, \underline{\ }_4, \underline{\ }_5, \underline{\ }_6, \underline{\ }_7, \underline{\ }_8, \underline{\ }_9, \underline{\ }_{10}, \underline{\ }_{11}, \underline{\ }_{12}\}.$$

The element 9 is placed first, into position $j_{t_5} = 6$*. In the process, the blank spaces after Position 6 change their numbers, since the placement is restricted to those positions not already occupied:*

$$\{\underset{1}{\_},\underset{2}{\_},\underset{3}{\_},\underset{4}{\_},\underset{5}{\_},9,\underset{6}{\_},\underset{7}{\_},\underset{8}{\_},\underset{9}{\_},\underset{10}{\_},\underset{11}{\_}\}.$$

The element 7 is placed next, into position $j_{t_4} = 7$ *among the remaining positions. Again, the numbering changes to only enumerate the remaining empty spots:*

$$\{\underset{1}{\_},\underset{2}{\_},\underset{3}{\_},\underset{4}{\_},\underset{5}{\_},9,\underset{6}{\_},7,\underset{7}{\_},\underset{8}{\_},\underset{9}{\_},\underset{10}{\_}\}.$$

The element 6 is placed next, into position $j_{t_3} = 6$ *among the remaining positions. Again the numbering changes:*

$$\{\underset{1}{\_},\underset{2}{\_},\underset{3}{\_},\underset{4}{\_},\underset{5}{\_},9,6,7,\underset{6}{\_},\underset{7}{\_},\underset{8}{\_},\underset{9}{\_}\}.$$

The element 4 is placed next, into position $j_{t_2} = 3$ *among the remaining positions. Again the numbering changes:*

$$\{\underset{1}{\_},\underset{2}{\_},4,\underset{3}{\_},\underset{4}{\_},9,6,7,\underset{5}{\_},\underset{6}{\_},\underset{7}{\_},\underset{8}{\_}\}.$$

The element 3 is placed next, into position $j_{t_1} = 3$ *among the remaining positions. The numbering of the positions is now unnecessary and is omitted:*

$$\{\_,\_,4,3,\_,9,6,7,\_,\_,\_,\_\}.$$

Now, the partition dividers can be inserted into every second remaining open spot, which implies the first spot is left empty:

$$\{\_\,|\,4,3,\_,9,6,7\,|\,\_\,|\,\_\}.$$

Finally, the least elements can be inserted into the remaining spaces in increasing order, yielding the partition:

$$\{1\,|\,4,3,2,9,6,7\,|\,5\,|\,8\}.$$

A direct bijective map from the words in $\mathcal{CL}(n,k;\langle i\rangle,\langle i\rangle)$ to the appropriate set of Lah restricted growth functions discussed in Section 2.1.2 is also desirable since it would correspond nicely with what was presented in the case of the Stirling numbers above. Of course, composing the map given here with the map Wagner provides in [19] between Laguerre configurations and Lah restricted growth functions, such a bijection can be had indirectly.

Now, recall the two variants on the Hockey Stick Theorem (2.7) and (2.8), first the diagonal variant,

$$L(n,k) = \sum_{j=0}^{k} (n-1+k-2j)L(n-1-j, k-j),$$

and second, respectively, the vertical variant,

$$L(n,k) = \sum_{j=k}^{n} L(j-1, k-1)(n+k-1)^{\underline{n-j}},$$

both subject to the same boundary conditions as (2.6).

In light of the proofs of those formulas, among all elements of $\overrightarrow{\Pi}_{n,k}$,

- the term $(n-1+k-2j)L(n-1-j,k-j)$ in (2.7) counts those in which the largest j elements of $[n]$, $n-j+1$, ..., and n, all appear in their own blocks, necessarily listed last, but $n-j$ does not, and
- the term $L(j-1,k-1)(n+k-1)^{\underline{n-j}}$ in (2.8) counts those in which $j \in [n]$ is the largest least element, i.e. the least element of the k^{th} block of the partition.

These interpretations do not use the bijection from $\mathcal{CL}(n,k;\langle i\rangle,\langle i\rangle)$ to $\overrightarrow{\Pi}_{n,k}$ but rather are had by direct inspection using the interpretation of $\overrightarrow{\Pi}_{n,k}$.

Finally, consider the closed-form expression for the C-L numbers given in Theorem 2.3.4. In the Lah-numbers special case, (2.33) can be expressed as follows:

Theorem 2.4.9. *For all $n,k \in \mathbb{N}$,*

$$L(n,k) = \sum_{\substack{d_1+d_2+\cdots+d_k=n-k \\ d_i\in\mathbb{N}}} \frac{(n+k+1)!}{(d_1+2)^{\overline{2}}(d_1+d_2+4)^{\overline{2}}\cdots(d_1+\cdots+d_k+2k)^{\overline{2}}}. \tag{2.55}$$

Proof. This is straightforward by applying Theorem 2.3.4 to the case of $a_i = i$ and $b_i = i$ and rewriting it in terms of the factors not present.

Alternatively, we offer a proof using $\overrightarrow{\Pi}_{n,k}$:
In (2.55) the term

$$\frac{(n+k+1)!}{(d_1+2)^{\overline{2}}(d_1+d_2+4)^{\overline{2}}\cdots(d_1+\cdots+d_k+2k)^{\overline{2}}}$$

enumerates, among all partitions in $\overrightarrow{\Pi}_{n,k}$, those with LE-set

$$D = \{1, d_1+2, d_1+d_2+3, \ldots, d_1+\cdots+d_{k-1}+k\}.$$

As before, the approach is to place the elements of $[n]$ into $n+k-1$ spots, the additional $k-1$ spots being those what will ultimately hold the partition dividers. Note that there are $(n+k-1)^{\underline{d_k}}$ ways to place the largest d_k elements of $[n]\setminus D$, which are those elements larger than every least-element. Hence each of these is free to be placed anywhere within the

$n+k-1$ spots without being smaller than any block's smallest element. Once those elements are placed, the rightmost remaining free position will receive the element $d_1+\cdots+d_{k-1}+k$, and the subsequently remaining rightmost position can be filled with the partition divider for the last block, leaving $(n+k-1)-d_k-2 = d_1+\cdots+d_{k-1}+2k-3$ positions free. The next d_{k-1} largest elements, those in $\{d_1+\cdots+d_{k-2}+k,\ldots,d_1+\cdots+d_{k-1}+k-1\}$, can then be placed in those spots in $(d_1+\cdots+d_{k-1}+2k-3)^{\underline{d_{k-1}}}$ ways. Again these elements are all sufficiently large not to be smaller than any remaining block's smallest element. Once this is done, the rightmost remaining free position is filled with the element $d_1+\cdots+d_{k-2}+k-1$, and the subsequently remaining rightmost free position is filled with the partition divider for the penultimate block. At this point, the last two blocks are defined entirely and there are $(d_1+\cdots+d_{k-1}+2k-3)-d_{k-1}-2 = d_1+\cdots+d_{k-2}+2k-5$ free positions left in the partition. This process is repeated until the last d_1 remaining elements of $[n]$, those in $\{2,\ldots,d_1+1\}$, are placed into the remaining d_1+1 spots in $(d_1+1)^{\underline{d_1}}$ ways. The final remaining spot is then occupied by 1. This yields a total of

$$\begin{aligned}
&((d_1+\cdots+d_k+2k-1)\cdots(d_1+\cdots+d_{k-1}+2k))\\
&\qquad((d_1+\cdots+d_{k-1}+2k-3)\cdots(d_1+\cdots+d_{k-2}+2k-2))\\
&\qquad\quad\cdots((d_1+1)(d_1)\cdots(2))
\end{aligned}$$

possible arrangements. Factoring out $(n+k+1)!$, noting that

$$(n+k)(n+k+1) = (d_1+\cdots+d_k+2k)(d_1+\cdots+d_k+2k+1),$$

and summing across all possible sets of values for the d_i's yields (2.55). □

Note the rising factorials in the denominator of (2.55), which are easy to overlook. Also, observe that the formulas (2.46) and (2.55) for the cycle and Lah numbers, respectively, underpin a similarity in the two arrays on a structural level, and they offer a potential place to generalize both within a single context. For instance, the arrays created by adding jr to each $S_j = d_0+\cdots+d_j$ in the denominators and looking at the r^{th} rising factorial of each term, adjusting the factorial term in the numerators accordingly, could potentially prove interesting. The Stirling numbers appear not to have a formula of quite the same form, the closest obvious formula being an easy consequence of (1.14):

$$S(n,k) = \sum_{\substack{d_1+\cdots+d_k=n-k\\ d_i\in\mathbb{N}}} \frac{(k!)^{n-k}}{1^{n-k-d_1}2^{n-k-d_2}\cdots k^{n-k-d_k}}, \tag{2.56}$$

for all $n,k\in\mathbb{N}$.

2.4.3 The Binomial Coefficients, Again

When $a_i = 1$, for all $i\in\mathbb{N}$, and $b_i\equiv 0$, the recurrence given in (2.15) takes the form, for all $n,k\in\mathbb{P}$,

$$A(n,k;\langle 1\rangle,\langle 0\rangle) = A(n-1,k-1;\langle 1\rangle,\langle 0\rangle) + A(n-1,k;\langle 1\rangle,\langle 0\rangle), \tag{2.57}$$

subject to the boundary conditions $A(0,k;\langle 1\rangle,\langle 0\rangle) = \delta_{0,k}$, for all $k \in \mathbb{N}$, and $A(n,0;\langle 1\rangle,\langle 0\rangle) = 1$, for all $n \in \mathbb{P}$. These are the same recurrence and boundary conditions satisfied by the binomial coefficients in (1.6), and so

$$A(n,k;\langle 1\rangle,\langle 0\rangle) = \binom{n}{k} = A(n,k;\langle 0\rangle,\langle 1\rangle).$$

Thus, the binomial coefficient $\binom{n}{k}$ also enumerates the words in the set $\mathcal{CL}(n,k;\langle 1\rangle,\langle 0\rangle)$, i.e. words of length $n-k$ in the strictly ascending alphabets $A_0 \cup \cdots \cup A_{n-1}$, with each $A_i = \{a_{i,1}\}$. In addition, $\binom{n}{k}$ enumerates the sequences in the set $\mathcal{U}(n,k;\langle 1\rangle,\langle 0\rangle)$, i.e. those sequences obtained from selecting balls from urns, as described in Section 2.3.2, in which each urn contains exactly one white ball and no red ones, i.e. the possible selections of $n-k$ urns from among n of them.

The bijection to k-element subsets of $[n]$ is obvious by taking the complement, in $[n]$, of the set of initial subscripts of the letters of a word in $\mathcal{CL}(n,k;\langle 1\rangle,\langle 0\rangle)$ after adding one to each. The same is true for the labels on the urns. Notice that as compared with the Comtet case, this construction more naturally provides k-element subsets of $[n]$.

Consider now the closed-form expression for the C-L numbers given in Theorem 2.3.4. In the binomial coefficients special case, nothing new appears:
For all $n, k \in \mathbb{N}$,

$$\sum_{\substack{d_0+d_1+\cdots+d_k=n-k \\ d_i \in \mathbb{N}}} 1.$$

Notice, however, what happens when the sum is restricted to each $d_i \in \mathbb{P}$. When $n \in \mathbb{P}$, by the combinatorial interpretation given in this section, the result counts the k-element subsets of $[n]$ in which

(i) 1 is not an element;

(ii) n is not an element; and

(iii) no two consecutive elements appear,

obtained by using the interpretation that $\forall i \in [k]^*$, d_i is the number of elements of $[n]$ excluded from the subset, when listed in increasing order, between the i^{th} element and the $(i+1)^{\text{th}}$ element, noting that the "0^{th} element" refers to the beginning of the set and the "$(k+1)^{\text{th}}$ element" the end. By mapping from these k-element subsets of $[n]$ onto the k-element subsets of $[n-1]$ in which

(i′) $n-1$ is not an element; and

(ii′) no two consecutive elements appear,

given by subtracting one from each element in the set, a bijection is established. Notice that for all $n \in \mathbb{P}$, the collection of all such subsets of $[n-1]$, regardless of their cardinality, is counted by F_{n-1}, the $(n-1)^{\text{th}}$ Fibonacci number, parameterized by $F_0 = F_1 = 1$. To establish this, the boundary conditions are obvious and the recurrence $F_n = F_{n-1} + F_{n-2}$ is straightforward by considering separately the cases when 1 is not and is, respectively, an element of the subset. Then,

Theorem 2.4.10. *For all $n \in \mathbb{P}$,*

$$\sum_{k\geq 0} \sum_{\substack{d_0+d_1+\cdots+d_k=n-k \\ d_i\in\mathbb{P}}} 1 = F_{n-1}, \tag{2.58}$$

where F_n denotes the n^{th} usual Fibonacci number, taken where $F_0 = F_1 = 1$.

This could be refined by defining for each $n, k \in \mathbb{P}$,

$$F(n,k) \;=\; \sum_{\substack{d_0+d_1+\cdots+d_k=n-k \\ d_i\in\mathbb{P}}} 1, \tag{2.59}$$

which satisfies the two-term recurrence

$$F(n,k) = F(n-2,k-1) + F(n-1,k). \tag{2.60}$$

A quick proof of this recurrence can be given along the same lines as the preceding discussion: $F(n-2,k-1)$ enumerates those structurally relevant k-element subsets of $[n]$ that contain 2, whence they do not contain 1. The map that deletes 2 from the set and subtracts 2 from each remaining element matches it with a structurally relevant $(k-1)$-element subset of $[n-2]$. On the other hand, $F(n-1,k)$ enumerates those k-element subsets which do not contain 2, the matching subsets of $[n-1]$ obtained by subtracting 1 from each element in the set.

A similar treatment on the either of the q-binomial arrays presented in this work yields a q-analogue of the Fibonacci numbers that appears to be of little interest.

2.4.4 The Comtet Case Revisited

When $a_i \equiv 0$, the C-L numbers specialize to the Comtet numbers. That case has already been discussed in terms of words, and here it will get brief treatment in the balls-from-urns interpretation given in Section 2.3.2. Since $a_i \equiv 0$ here, technically the labels α_j on the urns in $\mathcal{U}(n,k;\langle 0\rangle,\langle b_i\rangle)$ play no direct role, serving only as an accessory to obtain the sublabels β_j. The purpose of this section, then, is to describe the set $\mathcal{U}(n,k;\langle 0\rangle,\langle b_i\rangle)$ without the use of the α_j's.

Consider a large number[16] of labeled urns so that amply many bear the label j, for each $j \in [k]^*$, with any two urns with the same label being considered identical. From these, $n-k$ are to be placed into $n-k$ spaces labeled $1,\ldots,n-k$ so that the urns appear in increasing order by their labels, though more than one urn with the same label can be chosen. Refer to the label of the urn in the j^{th} position as β_j. Then, assume b_{β_j} labeled balls are placed inside every urn labeled with β_j for every $j \in [n-k]$, and draw from each urn one ball to create a sequence of ordered pairs of length $n-k$ by recording in order the labels on the selected urns paired with the label on the balls drawn from them. Then the number of such sequences is $C(n,k;\langle b_i\rangle)$, for all $n,k \in \mathbb{N}$. This provides the desired description with proof

[16] Infinitely many, if needed.

of this fact following from the proof of Theorem 2.3.3.

2.5 Comparison to a Similar Structure

John Konvalina presented a unification of the binomial coefficients and the Stirling numbers in [11] and [12], and there is some similarity between his interpretation and those presented here: $\mathcal{CL}(n,k;\langle a_i\rangle,\langle b_i\rangle)$ and, more closely, $\mathcal{U}(n,k;\langle a_i\rangle,\langle b_i\rangle)$. In our notation Konvalina chooses to index the sets B_i on $\mathbb{P}$ rather than on $\mathbb{N}$ and analyzes the numbers

$$\left(\binom{n}{k}\right)_{(b_i)} = \sum_{1\le i_1\le\cdots\le i_k\le n} b_{i_1}b_{i_2}\cdots b_{i_k} = \sum_{\substack{d_1+\cdots+d_n=k\\ d_i\in\mathbb{N}}} b_1^{d_1}\cdots b_n^{d_n}, \tag{2.61}$$

providing the interpretation of these numbers as selections with repetition of k balls from boxes $B_1, \ldots, B_n$, where B_i contains b_i labeled balls with the requirement that if more than one ball is chosen from a box then the order in which the balls are chosen is recorded. Note that this is a variant of the set $\mathcal{U}(n,k;\langle 0\rangle,\langle b_i\rangle)$ presented in Section 2.4.4, indexed differently and ignoring the additional clarity afforded by considering the selections to be sequences of ordered pairs, and it is therefore a related but less general version of $\mathcal{U}(n,k;\langle a_i\rangle,\langle b_i\rangle)$.

Konvalina also makes these observations:

1. $\binom{n}{k}_q = \left(\binom{n-k+1}{k}\right)_{(b_i)}$ when $b_i = q^{i-1}$, $\forall i \ge 1$;
2. $S(n,k) = \left(\binom{k}{n-k}\right)_{(b_i)}$ when $b_i = i$, $\forall i \ge 1$; and
3. $\binom{n}{k} = \left(\binom{n-k+1}{k}\right)_{(b_i)}$ when $b_i \equiv 1$.

Unfortunately, there is dissimilarity in the format of 2 compared with 1 and 3 that Konvalina could have partially remedied by choosing to interchange k and $n-k$ in 1 and 3, justified by symmetry of $\binom{n}{k}_q$ and $\binom{n}{k}$. Had he done so, the result would have written $\binom{n}{k}_q$, $S(n,k)$, and $\binom{n}{k}$ all in the same form:

1′. $\binom{n}{k}_q = \left(\binom{k+1}{n-k}\right)_{(b_i)}$ when $b_i = q^{i-1}$, $\forall i \ge 1$;

2′. $S(n,k) = \left(\binom{k}{n-k}\right)_{(b_i)}$ when $b_i = i$, $\forall i \ge 1$; and

3′. $\binom{n}{k} = \left(\binom{k+1}{n-k}\right)_{(b_i)}$ when $b_i \equiv 1$.

The greater similarity in these would make them more easily identifiable with the k^{th} complete symmetric functions.

Still, even had he done this, there is still an uncomfortable asymmetry between the expressions given in $1'$ and $3'$ as compared with $2'$, specifically presenting $k+1$ in $1'$ and $3'$ versus k in $2'$. This unfortunate situation does not occur, however, when the boxes are indexed instead on $\mathbb{N}$, as is seen in $\mathcal{U}(n, k; \langle 0 \rangle, \langle b_i \rangle)$, with urns playing the role of boxes or in $\mathcal{W}(n, k; \langle b_i \rangle)$, with alphabets fulfilling that role. In addition, indexing on $\mathbb{N}$ allows for extending the results in many cases to the situation in which one or both of $n = 0$ or $k = 0$. Because of his choice and the resulting anomalies presented in 1, 2, and 3 above, Konvalina also fails to point out that the numbers in his investigation act as connection constants, and nothing is mentioned about the algebraic unification provided by Comtet.

Chapter 3

Additional Examples Available Via Statistical Generating Functions

Consider now the method of statistical generating functions as an alternative to direct enumeration for the purposes of deriving q-variants of familiar formulas[1]. Given some combinatorial structure Γ, in which the quantity $g := |\Gamma|$ has some combinatorial meaning[2], a q-generalization of $|\Gamma|$ is accessible by calculating some statistic s on the elements γ of the structure Γ, and then defining $g(q)$ as a polynomial in the indeterminate q by:

$$g(q) = \sum_{\gamma \in \Gamma} q^{s(\gamma)}. \tag{3.1}$$

Many examples of C-L numbers can be defined this way, including examples that do not have an apparent interpretation in terms of q-vector spaces including q-generalizations of the cycle and Lah numbers. Also available, by considering two statistics on a structure at once, are p, q-generalizations of the Stirling numbers as well as the cycle[3] and Lah numbers[4].

3.1 The q-Binomial Coefficients

Using the method of statistical generating functions, the q-binomial coefficients can be defined for q an indeterminate, providing a q-generalization as opposed to a q-analogue. Two approaches are presented here that generate the q-binomial coefficients.

Recall Equation (1.8), valid $\forall n, k \in \mathbb{N}$:

$$\binom{n}{k} = \sum_{\substack{d_0+d_1+\cdots+d_k=n-k \\ d_i \in \mathbb{N}}} 1.$$

[1] Discussed and utilized extensively, for instance, in [17] among many other places. The format presented here is an echo of that in [17].

[2] Typically here, Γ depends on n and k or sometimes just on n, but the situation is not limited to those cases. The quantities g will be indexed in the same way as is Γ.

[3] These two cases being in agreement with those found in [9] and [10].

[4] There is a p, q-generalization of the binomial coefficients as well, see [5], but they are not C-L numbers, rather belonging to an extension of the C-L numbers that can be developed via the balls-and-urns interpretation.

Following [21], observe that each sequence $(d_0, d_1, \ldots, d_k)$ in $\mathbb{N}$, has the property $d_0 + d_1 + \cdots + d_k = n - k$. Thus, each subsequence $(d_1, d_2, \ldots, d_k)$ in $\mathbb{N}$, has the property $d_1 + d_2 + \cdots + d_k \leq n - k$. Each of these can be identified with an integer partition with at most $n - k$ parts where each part is no larger than k by letting each d_i count the number of parts of value i, $1 \leq i \leq k$. Thus, the value $d_1 + 2d_2 + \cdots + kd_k$ is the sum of the parts of that partition. Following Knuth, as presented in [21], letting $p(k, n-k, m)$ denote the partitions of the integer m into $n - k$ parts where each part is no larger than k, it is shown that for all $n, k \in \mathbb{N}$

$$\binom{n}{k}_q = \sum_{m \geq 0} p(k, n-k, m) q^m, \tag{3.2}$$

where q is an indeterminate. Then as in [21], we could instead write (3.2) as

$$\binom{n}{k}_q = \sum_{k \geq m_1 \geq m_2 \geq \cdots \geq m_{n-k} \geq 0} q^{m_1 + m_2 + \cdots + m_{n-k}}. \tag{3.3}$$

Then (3.2) and (3.3) provide two opportunities to describe the q-binomial coefficients from the perspective of statistical generating functions by defining a statistic on the elements σ of the set $P(i, j)$, of integer partitions with at most i parts where each part is no larger than j. The first is given by $s(\sigma)$, which is equal to the sum of the parts of the partition σ. This gives the formula for all $n, k \in \mathbb{N}$,

$$\binom{n}{k}_q = \sum_{m \geq 0} p(k, n-k, m) q^m = \sum_{\sigma \in P(n-k, k)} q^{s(\sigma)}. \tag{3.4}$$

The second can be obtained by noticing that each sequence $k \geq m_1 \geq m_2 \geq \cdots \geq m_{n-k} \geq 0$ corresponds uniquely with a lattice path from $(0, 0)$ to $(n - k, k)$ with area below the path[5] $m_1 + m_2 + \cdots + m_{n-k}$. Thus, denoting the set of lattice paths from $(0, 0)$ to (i, j) by $\mathcal{L}(i, j)$. On the elements $\lambda \in \mathcal{L}(i, j)$, define the statistic $\alpha(\lambda)$ to be the area below λ. Then, for all $n, k \in \mathbb{N}$,

$$\binom{n}{k}_q = \sum_{\lambda \in \mathcal{L}(n-k, k)} q^{\alpha(\lambda)}. \tag{3.5}$$

This fact was first noted by Polya [15], and following his logic, it is particularly easy to verify (3.5) to give a flavor for this style of proof:

Proof. Using (3.5), note that for $n = 0$, the only relevant path is a horizontal line of length k which encloses no area, and for $k = 0$ the lattice paths are only defined when $n = 0$ as well.

For $n, k \in \mathbb{P}$, consider the disjoint and exhaustive cases when the lattice path ends with a vertical step versus when it ends in a horizontal step. When $\lambda \in \mathcal{L}(n - k, k)$ ends in a vertical step, the path up to that step is some path $\lambda' \in \mathcal{L}(n - k, k - 1)$. The final vertical step adds no area. When instead $\lambda \in \mathcal{L}(n - k, k)$ ends in a horizontal step, the path up to that step is some path $\lambda'' \in \mathcal{L}(n - k - 1, k)$. The final horizontal step, occurring at height k creates a $k \times 1$ rectangle of additional area on the path, so the contribution to $\alpha(\lambda)$ is

[5] Meaning also within the box defined by $(0, 0)$ and $(n - k, k)$.

increased by k in all such cases.

Therefore,

$$\sum_{\lambda \in \mathcal{L}(n-k,k)} q^{\alpha(\lambda)}$$

satisfies the same boundary conditions and recurrence as in Theorem 1.1.15. $\square$

From the recurrence, the other results, Theorems 1.1.13, 1.1.16, and 1.1.17 all follow, now for q an indeterminate. Additionally, it is clear from this definition that $\binom{n}{k}_q$ is a polynomial in q with nonnegative integer coefficients. Observe lastly that (1.6) is recovered by choosing $q = 1$.

3.1.1 A Variant on the q-Binomial Coefficients

Consider the following variant of the q-binomial coefficients, for which the notation $\left\langle {n \atop k} \right\rangle_q$ will be used:

Recall that the set of lattice paths from $(0,0)$ to $(n-k,k)$ is denoted $\mathcal{L}(n-k,k)$ and that $|\mathcal{L}(n-k,k)| = \binom{n}{k}$.[6] Now, on the elements $\lambda \in \mathcal{L}(n-k,k)$, mark each step with the numbers in $[n-1]^*$ in increasing order, beginning with the step emanating from $(0,0)$. Then define the statistic $s_h(\lambda)$ to be the sum of the step numbers on the horizontal steps[7], and define for all $n, k \in \mathbb{N}$ and an indeterminate q,

$$\left\langle {n \atop k} \right\rangle_q = \sum_{\lambda \in \mathcal{L}(n-k,k)} q^{s_h(\lambda)}. \tag{3.6}$$

Then,

Theorem 3.1.1. *With boundary conditions $\left\langle {0 \atop k} \right\rangle_q = \delta_{0,k}$ and $\left\langle {n \atop 0} \right\rangle_q = q^{\binom{n}{2}}$ for every $n, k \in \mathbb{N}$ and an indeterminate q,*

$$\left\langle {n \atop k} \right\rangle_q = \left\langle {n-1 \atop k-1} \right\rangle_q + q^{n-1} \left\langle {n-1 \atop k} \right\rangle_q, \quad \forall n, k \in \mathbb{P}. \tag{3.7}$$

Proof. The boundary condition in k is clear. On the other hand, the boundary condition in n follows from the fact that the lattice path when $k = 0$ is from $(0,0)$ to $(n,0)$, so every step is a horizontal step. Therefore, the sum of the horizontal step numbers is $\binom{n}{2}$.

For $n, k \in \mathbb{P}$, observe that those lattice paths in $\mathcal{L}(n-k,k)$ that end in a vertical step are first a lattice path $\lambda' \in \mathcal{L}(n-k,k-1)$, and the final vertical step has no impact on the value of $s_h(\lambda')$. On the other hand, those lattice paths in $\mathcal{L}(n-k,k)$ that end in a horizontal step are first a lattice path $\lambda' \in \mathcal{L}(n-k-1,k)$, and the final horizontal step increases the value of $s_h(\lambda')$ by $n-1$. $\square$

It is clear from (3.7) that $\langle {n \atop k} \rangle_q$ is an array of C-L numbers, in this case having the form $A(n,k;\langle q^i \rangle, \langle 0 \rangle) = \langle {n \atop k} \rangle_q$. This is to be compared with the q-binomial coefficients as they

[6]This can be verified, for instance, by noting that among the n steps, $n-k$ of them are chosen to be horizontal with, necessarily, k vertical.

[7]Alternatively, index the steps on $\mathbb{P}$ and define $s_h(\lambda)$ to be $n-k$ less than the sum of the horizontal step numbers.

are typically given, i.e. $A(n,k;\langle 0\rangle,\langle q^i\rangle) = \binom{n}{k}_q$. While related to the Comtet-numbers case $\binom{n}{k}_q$, it is clear that these q-binomial coefficients are not themselves Comtet numbers. Also unlike their Comtet-case counterparts, these are not usually given an interpretation when q is a power of a prime number in terms of $\mathbb{F}_q$-vector spaces.

Fittingly, there is a relationship between the statistic s_h and the area statistic α used in Section 1.1.3 to obtain the q-binomial coefficients. Since each vertical step augments the step count and height of the lattice path above the x-axis by 1, their contributions to the statistics are equal. On the other hand, each horizontal step augments the step count by 1 and the height of the lattice path by 0. Since there are $n-k$ horizontal steps and the effect is cumulative, though a horizontal step is required to create area, for any $\lambda \in \mathcal{L}(n-k,k)$,

$$s_h(\lambda) = (1+2+\cdots+(n-k-1))\alpha(\lambda) = q^{\binom{n-k}{2}}\alpha(\lambda). \tag{3.8}$$

Thus,

Theorem 3.1.2. *For every $n,k \in \mathbb{N}$ and indeterminate q,*

$$\left\langle {n \atop k} \right\rangle_q = q^{\binom{n-k}{2}} \binom{n}{k}_q. \tag{3.9}$$

Also, since $\langle {n \atop k} \rangle_q$ is an array of C-L numbers, from Lancaster's theorem the following formulas hold:
First,

Theorem 3.1.3. *For every $n \in \mathbb{N}$,*

$$(x+1)(x+q)(x+q^2)\cdots(x+q^{n-1}) = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_q x^k. \tag{3.10}$$

Note that (3.1.3) is a variant on the well-known, see [21], q-binomial theorem.
Second,

Theorem 3.1.4. *For all $n,k \in \mathbb{P}$,*

$$\left\langle {n \atop k} \right\rangle_q = \sum_{j=0}^{k} q^{n-1-j} \left\langle {n-1-j \atop k-j} \right\rangle_q, \tag{3.11}$$

subject to the same boundary conditions given in Theorem 3.1.1.

And third,

Theorem 3.1.5. *For all $n,k \in \mathbb{N}$ and an indeterminate q,*

$$\left\langle {n \atop k} \right\rangle_q = \sum_{j=k}^{n} q^{\binom{n}{2}-\binom{j}{2}} \left\langle {j-1 \atop k-1} \right\rangle_q, \tag{3.12}$$

subject to the same boundary conditions given in Theorem 3.1.1.

In terms of applications of the combinatorial interpretation of the C-L numbers, the alternate q-binomial coefficient $\langle {n \atop k} \rangle_q$ enumerates the words in the set $\mathcal{CL}(n,k;\langle q^i\rangle,\langle 0\rangle)$, i.e. words of length $n-k$ in the strictly ascending alphabets $A_0 \cup \cdots \cup A_{n-1}$, with each $A_i = \{a_{i,1}, \ldots, a_{i,q^i}\}$. Furthermore, $\left\langle {n \atop k} \right\rangle_q$ enumerates the sequences of ordered pairs in the set $\mathcal{U}(n, k; \langle q^i \rangle, \langle 0 \rangle)$, i.e. those sequences obtained from selecting balls from urns, as described in Section 2.3.2, in which for each $j \in [n-k]$, if the urn has label $\alpha_j = i_j$, then it holds q^{i_j} balls, all of which are white.

Notice that if the specific initial subscripts in a word $w \in \mathcal{CL}(n, k; \langle q^i \rangle, \langle 0 \rangle)$ here are $0 \leq i_1 < i_2 < \cdots < i_{n-k} \leq n-1$, then there are $q^{i_1+i_2+\cdots+i_{n-k}}$ possible manifestations of w, providing the formula, for all $n, k \in \mathbb{N}$,

$$\left\langle {n \atop k} \right\rangle_q = \sum_{0 \leq i_1 < i_2 < \cdots < i_{n-k} \leq n-1} q^{i_1+i_2+\cdots+i_{n-k}}. \tag{3.13}$$

Now, recall the two variants on the Hockey Stick Theorem given in Formulas (3.11) and (3.12), first

$$\left\langle {n \atop k} \right\rangle_q = \sum_{j=0}^{k} q^{n-1-j} \left\langle {n-1-j \atop k-j} \right\rangle_q,$$

and second, respectively,

$$\left\langle {n \atop k} \right\rangle_q = \sum_{j=k}^{n} q^{\binom{n}{2}-\binom{j}{2}} \left\langle {j-1 \atop k-1} \right\rangle_q,$$

both subject to the same boundary conditions given in (3.7).

Consider now the closed-form expression for the C-L numbers given in Theorem 2.3.4. Here it can be expressed by:

Theorem 3.1.6. *For all $n, k \in \mathbb{N}$,*

$$\left\langle {n \atop k} \right\rangle_q = \sum_{\substack{d_0+d_1+\cdots+d_k=n-k \\ d_i \in \mathbb{N}}} q^{0d_0+1d_1+\cdots+kd_k} \cdot q^{1+2+\cdots+(d_0+d_1+\cdots+d_k-1)}. \tag{3.14}$$

Proof. Straightforward. $\square$

Note, however, that this formula provides nothing new once it is simplified. In fact, for all $n, k \in \mathbb{N}$,

$$\begin{aligned}
\left\langle {n \atop k} \right\rangle_q &= \sum_{\substack{d_0+d_1+\cdots+d_k=n-k \\ d_i \in \mathbb{N}}} q^{0d_0+1d_1+\cdots+kd_k} \cdot q^{1+2+\cdots+(d_0+d_1+\cdots+d_k-1)} \\
&= \sum_{\substack{d_0+d_1+\cdots+d_k=n-k \\ d_i \in \mathbb{N}}} q^{0d_0+1d_1+\cdots+kd_k} \cdot q^{\binom{d_0+d_1+\cdots+d_k}{2}} \\
&= q^{\binom{n-k}{2}} \sum_{\substack{d_0+d_1+\cdots+d_k=n-k \\ d_i \in \mathbb{N}}} q^{0d_0+1d_1+\cdots+kd_k} \\
&= q^{\binom{n-k}{2}} \binom{n}{k}_q,
\end{aligned}$$

which was shown in Theorem 3.1.2.

Incidentally, the formula for these q-binomial coefficients that generalizes (1.2) is, for all $n, k \in \mathbb{N}$,

$$q^{\binom{k}{2}} \left\langle {n \atop k} \right\rangle_q = q^{\binom{n-k}{2}} \left\langle {n \atop n-k} \right\rangle_q . \tag{3.15}$$

3.2 The Carlitz q and p, q-Stirling Numbers

Now we consider statistics on the canonically ordered sets of partitions of $[n]$ into k nonempty blocks (i.e. $\Pi_{n,k}$). Write the elements π of $\Pi_{n,k}$, as follows: $\pi = \{E_1|\cdots|E_k\}$ put in canonical order so that the elements in each block are listed in increasing order and the blocks are listed in increasing order by their least (here: initial) elements, and note the inversion statistic, $inv(\pi)$, is the number of (block) inversions of π, where a block inversion in π occurs iff, for $1 \leq i < j \leq k$, a pair (x, E_j) appears with $x \in E_i$ and $x > \min E_j$. It is known, see [1], that inv generates the Carlitz q-Stirling numbers.

That is, for all $n, k \in \mathbb{N}$, and an indeterminate q,

$$\tilde{S}_q(n,k) := \sum_{\pi \in \Pi_{n,k}} q^{inv(\pi)}. \tag{3.16}$$

As Carlitz showed, the claim that the definition in (3.16) agrees with the definition for $\tilde{S}_q(n,k)$ given in Section 1.1.4 is justified since it follows from this definition that, for all $n, k \in \mathbb{P}$,

$$\tilde{S}_q(n,k) = \tilde{S}_q(n-1,k-1) + k_q \tilde{S}_q(n-1,k), \tag{3.17}$$

subject to the boundary conditions $\tilde{S}_q(n,0) = \delta_{n,0}$ and $\tilde{S}_q(0,k) = \delta_{0,k}$, for all $n, k \in \mathbb{N}$[8].

Now consider another statistic on the elements of $\Pi_{n,k}$: the number of (block) non-inversions of π, denoted by $ninv(\pi)$, where a block non-inversion in π occurs for each pair (E_h, x) such that $1 \leq h < i \leq k$ and x is a non-initial element of E_i iff $x > \min E_h$[9]. These two statistics together will be described to define a p, q-generalization for the Stirling numbers, and giving the Carlitz q-Stirling numbers in the special case when $p = 1$.

Now define for all $n, k \in \mathbb{N}$ and a pair of indeterminates p and q,

$$\tilde{S}_{p,q}(n,k) := \sum_{\pi \in \Pi_{n,k}} p^{ninv(\pi)} q^{inv(\pi)}. \tag{3.18}$$

Then,

Theorem 3.2.1. *For all $n, k \in \mathbb{P}$ and any pair of indeterminates p and q,*

$$\tilde{S}_{p,q}(n,k) = \tilde{S}_{p,q}(n-1,k-1) + k_{p,q} \tilde{S}_{p,q}(n-1,k), \tag{3.19}$$

[8] This fact is verified below in a more general setting.

[9] By the canonical ordering, the initial elements of the blocks trivially satisfy this condition and are therefore ignored for the count.

subject to the boundary conditions $\tilde{S}_{p,q}(n,0) = \delta_{n,0}$ *for all* $n \in \mathbb{N}$ *and* $\tilde{S}_{p,q}(0,k) = \delta_{0,k}$ *for all* $k \in \mathbb{N}$*, where with* $0_{p,q} = 0$ *and for every* $k \in \mathbb{P}$*,*

$$k_{p,q} := \sum_{j=1}^{k} p^{k-j} q^{j-1}, \tag{3.20}$$

Proof. The boundary conditions are obvious.

Now consider separately the cases within $\Pi_{n,k}$ in which n appears as its own block and when n appears in some otherwise nonempty block. First, choosing some $\pi' \in \Pi_{n-1,k-1}$, observe that adding a k^{th} block that contains only n yields a $\pi \in \Pi_{n,k}$ with n as the sole element of its block, and since n is in the last block, on the one hand, and it is the least element of its block on the other, neither *inv* nor *ninv* is affected. Thus, this case provides $\tilde{S}_{p,q}(n-1,k-1)$.

Now, starting with some $\pi' \in \Pi_{n-1,k}$, note that adding n to the j^{th} block, $j \in [k]$, provides a $\pi \in \Pi_{n,k}$ with n not the sole element in its block, and this increases *inv* by $k-j$ and *ninv* by $j-1$ since n is larger than the least element of every block before and after the block it is inserted into. Summing across all possible $j \in [k]$ in this case provides $k_{p,q}\tilde{S}_{p,q}(n-1,k)$. □

Notice that Theorem 3.2.1 justifies Equations (3.16) and (3.17). Also, observe that due to the p,q-symmetry of $k_{p,q}$, choosing $q = 1$ also provides the same numbers, now in the parameter p. Thus, the Carlitz q-Stirling numbers also arise as a result of the *ninv* statistic on $\Pi_{n,k}$. Furthermore, since choosing $p = 1$ provides the Carlitz q-Stirling numbers, naming these after Carlitz as well is fitting. Since these numbers are Comtet numbers, specifically satisfying the same recurrence and boundary conditions as $\mathcal{C}(n,k;\langle i_{p,q}\rangle)$, other results follow[10]:

First,

Theorem 3.2.2. *For every* $n \in \mathbb{N}$ *and a pair of indeterminates* p *and* q*,*

$$x^n = \sum_{k=0}^{n} \tilde{S}_{p,q}(n,k) x(x-1_{p,q})(x-2_{p,q}) \cdots (x-(k-1)_{p,q}). \tag{3.21}$$

Second,

Theorem 3.2.3. *For every* $k \in \mathbb{N}$ *and a pair of indeterminates* p *and* q*,*

$$\sum_{n \geq 0} \tilde{S}_{p,q}(n,k) x^n = \frac{x^k}{(1-1_{p,q}x)(1-2_{p,q}x)\cdots(1-k_{p,q}x)}. \tag{3.22}$$

And third,

Theorem 3.2.4. *For every* $n, k \in \mathbb{N}$ *and a pair of indeterminates* p *and* q*,*

$$\tilde{S}_{p,q}(n,k) = \sum_{\substack{d_1+\cdots+d_k=n-k \\ d_i \in \mathbb{N}}} (1_{p,q})^{d_1} (2_{p,q})^{d_2} \cdots (k_{p,q})^{d_k}. \tag{3.23}$$

[10]The analogous results for the Carlitz q-Stirling numbers given in Section 1.1.4 could be taken to follow from these as corollaries.

Also, the two Hockey Stick Theorems apply to the case of the Carlitz p,q-Stirling numbers:

$$\tilde{S}_{p,q}(n,k) = \sum_{j=0}^{k}(k-j)_{p,q}\tilde{S}_{p,q}(n-1-j,k-j), \tag{3.24}$$

and

$$\tilde{S}_{p,q}(n,k) = \sum_{j=k}^{n}(k_{p,q})^{n-j}\tilde{S}_{p,q}(j-1,k-1), \tag{3.25}$$

both subject to the same boundary conditions given in (3.19).

Furthermore, by the results of Sections 1.3 and 2.3, $\tilde{S}_{p,q}(n,k)$ has the combinatorial interpretations given by $\mathcal{W}(n,k;\langle i_{p,q}\rangle)$ and $\mathcal{U}(n,k;\langle 0\rangle,\langle i_{p,q}\rangle)$.

In the literature, Pierre Leroux and Anne de Médicis for example (see [9] and [10]) present these p,q-generalizations of the Stirling numbers using statistics on 0-1 tableaux. In those papers all of the above four but the condition concerning connection constants are mentioned.

3.3 The q and p,q-Cycle Numbers

Leroux also presents p,q-cycle numbers, again arising from a pair of statistics on 0-1 tableaux[11]. Above they are defined instead in a manner analogous to the one used here for the p,q-Stirling numbers.

On the elements σ of $\mathcal{P}(n,k)$, written in canonical order so that the cycles are listed in increasing order by their least (here: initial) elements, consider the following two statistics:

1. $inv(\sigma)$, first due to Carlitz in [1], is the number of inversions of σ in a left-to-right scan of the word created by dropping the cycle delimiters from a permutation in cycle notation, where an inversion in $\sigma = (x_1,\ldots,x_n)$ occurs whenever $i<j$ but $x_i > x_j$, and

2. $ninv(\sigma)$ is the number of non-inversions of σ, where a non-inversion in σ for each element other than the element 1 that is smaller than and appearing to the left of some element x that is not the initial element of its cycle[12].

Then define for all $n,k\in\mathbb{N}$ and a pair of indeterminates p and q,

$$c_{p,q}(n,k) := \sum_{\sigma\in\mathcal{P}(n,k)} p^{ninv(\sigma)}q^{inv(\sigma)}. \tag{3.26}$$

Then,

[11] These presentations also appear in [9] and [10].

[12] The non-inversion condition occurs trivially for the initial elements of the cycles by the canonical ordering and is therefore ignored in the count, and since the element 1 always is written first in the canonical ordering, the existence of larger elements to its right never contributes to the non-inversion count.

Theorem 3.3.1. *For all $n, k \in \mathbb{P}$,*

$$c_{p,q}(n,k) = c_{p,q}(n-1,k-1) + (n-1)_{p,q} c_{p,q}(n-1,k), \tag{3.27}$$

subject to the boundary conditions $c_{p,q}(n,0) = \delta_{n,0}$ for all $n \in \mathbb{N}$ and $c_{p,q}(0,k) = \delta_{0,k}$ for all $k \in \mathbb{N}$.

Proof. The boundary conditions are obvious.

Now consider separately the cases within $\mathcal{P}(n,k)$ in which n appears as last as a one-cycle and when n appears in a cycle together with at least one other element. First, choosing some $\sigma' \in \mathcal{P}(n-1,k-1)$, observe that adding a k^{th} cycle that contains only n yields a $\sigma \in \mathcal{P}(n,k)$ with n as the sole element of the last cycle, and since n listed last, on the one hand, and it is the least element of its cycle on the other, neither *inv* nor *ninv* is affected. Thus, this case provides $c_{p,q}(n-1,k-1)$.

Now, starting with some $\sigma' \in \mathcal{P}(n-1,k)$, note that adding n to the position immediately following the j^{th} element in a left-to-right scan and within the same cycle as the element it follows, $j \in [n-1]$, provides a $\sigma \in \mathcal{P}(n,k)$ with n not the sole element in its cycle, and this increases *inv* by $n-1-j$ and *ninv* by $j-1$ since n is larger than every element before and after the place it is inserted into, though the element 1 (before n) does not contribute. Summing across all possible $j \in [n-1]$ in this case provides $(n-1)_{p,q} c_{p,q}(n-1,k)$. □

Definition 3.3.2. *For all $n, k \in \mathbb{P}$, and an indeterminate q,*

$$c_q(n,k) := \sum_{\sigma \in \mathcal{P}(n,k)} q^{inv(\sigma)}. \tag{3.28}$$

Notice that this is obtained by choosing $p = 1$ in (3.26).

It follows that

Corollary 3.3.3. *For all $n, k \in \mathbb{P}$ and an indeterminate q,*

$$c_q(n,k) = c_q(n-1,k-1) + (n-1)_q c_q(n-1,k), \tag{3.29}$$

subject to the boundary conditions $c_q(0,k) = \delta_{0,k}$ for all $k \in \mathbb{N}$ and $c_q(n,0) = 0$ for all $n \in \mathbb{P}$.

Gould, in [7], first proposed this q-generalization of the cycle numbers, where he defined them to be[13] "the sum of the $\binom{n}{k}$ possible products, each with different factors, which may be formed from the first n q-natural numbers 1_q, 2_q, ..., n_q." Although Gould makes no explicit reference to whether q is a prime power, real or complex variable, or indeterminate, since he discusses the theory in terms of q-series, his treatment is likely one of the latter cases. Observe that in the case here, due to the p,q-symmetry of $k_{p,q}$, choosing $q = 1$ also provides the same numbers, now in the parameter p. Thus, the q-cycle numbers also arise as a result of the *ninv* statistic on $\mathcal{P}(n,k)$. Furthermore, since these p,q-cycle numbers satisfy the same recurrence and boundary conditions as $A(n,k;\langle i_{p,q}\rangle,\langle 0\rangle)$ they are C-L numbers and hence satisfy several more identities, which will be presented alongside analogous formulas for the q-cycle numbers.

[13] Using our notation.

First,

Theorem 3.3.4. *For all $n, k \in \mathbb{N}$ and a pair of indeterminates p and q,*

$$x(x+1_{p,q})(x+2_{p,q})\cdots(x+(n-1)_{p,q}) = \sum_{k=0}^{n} c_{p,q}(n,k)x^k. \tag{3.30}$$

Corollary 3.3.5. *For all $n, k \in \mathbb{N}$ and an indeterminate q,*

$$x(x+1_q)(x+2_q)\cdots(x+(n-1)_q) = \sum_{k=0}^{n} c_q(n,k)x^k. \tag{3.31}$$

Second,

Theorem 3.3.6. *For all $n, k \in \mathbb{P}$ and a pair of indeterminates p and q,*

$$c_{p,q}(n,k) = \sum_{j=k}^{n} c_{p,q}(j-1,k-1)(n-1)_{p,q}(n-2)_{p,q}\cdots(n-j)_{p,q}, \tag{3.32}$$

subject to the same boundary conditions as (3.29).

Corollary 3.3.7. *For all $n, k \in \mathbb{P}$ and an indeterminate q,*

$$c_q(n,k) = \sum_{j=k}^{n} c_q(j-1,k-1)(n-1)_q(n-2)_q\cdots(n-j)_q, \tag{3.33}$$

subject to the same boundary conditions as (3.29).

And third,

Theorem 3.3.8. *For all $n, k \in \mathbb{P}$ and a pair of indeterminates p and q,*

$$c_{p,q}(n,k) = \sum_{j=0}^{k} (n-1-j)_{p,q}c_{p,q}(n-1-j,k-j), \tag{3.34}$$

subject to the same boundary conditions as (3.27).

Corollary 3.3.9. *For all $n, k \in \mathbb{P}$ and an indeterminate q,*

$$c_q(n,k) = \sum_{j=0}^{k} (n-1-j)_q c_q(n-1-j,k-j), \tag{3.35}$$

subject to the same boundary conditions as (3.29).

These formulas are similar to those for the Comtet numbers, notably (1.28), (1.29), (1.36), and (1.37), though these are clearly not Comtet numbers. Also it appears that neither $c_{p,q}(n,k)$ nor $c_q(n,k)$ is deeply studied in terms of a structure that is a q- or p,q-analogue of any counted by $c(n,k)$, particularly in the case where $p=1$ and q is a power of a prime number. Hence, such a structure, especially in terms of $\mathbb{F}_q$ vector spaces, is desirable.

In applying the combinatorial interpretation given for the C-L numbers, the p,q-cycle number $c_{p,q}(n,k)$ counts the words in the set $\mathcal{CL}(n,k;\langle i_{p,q}\rangle,\langle 0\rangle)$, i.e. words of length $n-k$ in the strictly ascending alphabets $A_0\cup\cdots\cup A_{n-1}$, with each $A_i=\{a_{i,1},\ldots,a_{i,i_{p,q}}\}$. Furthermore, $c_{p,q}(n,k)$ enumerates the sequences in the set $\mathcal{U}(n,k;\langle i_{p,q}\rangle,\langle 0\rangle)$, i.e. those sequences obtained from selecting balls from urns, as described in Section 2.3.2, in which for $j\in[n-k]$ the urn with label $\alpha_j=i$ holds $i_{p,q}$ balls, all of which are white.

Notice if the specific initial subscripts in a word $w\in\mathcal{CL}(n,k;\langle i_{p,q}\rangle,\langle 0\rangle)$ are $0<i_1<i_2<\cdots<i_{n-k}<n$, then there are $(i_1)_{p,q}(i_2)_{p,q}\cdots(i_{n-k})_{p,q}$ possible manifestations of w, providing the formula, for all $n,k\in\mathbb{N}$,

$$c_{p,q}(n,k)=\sum_{0<i_1<i_2<\cdots<i_{n-k}<n}(i_1)_{p,q}(i_2)_{p,q}\cdots(i_{n-k})_{p,q}. \tag{3.36}$$

This specializes in the case where $p=1$ to the formula, for all $n,k\in\mathbb{N}$,

$$c_q(n,k)=\sum_{0<i_1<i_2<\cdots<i_{n-k}<n}(i_1)_q(i_2)_q\cdots(i_{n-k})_q. \tag{3.37}$$

Remark 3.3.10. *Equation (3.37) is an expression of Gould's definition for $c_q(n,k)$ in [7].*

Now consider the special cases of the C-L closed form given in Theorem 2.3.4. Here they take the forms:

Theorem 3.3.11. *For all $n,k\in\mathbb{N}$ and a pair of indeterminates p and q,*
$c_{p,q}(n,k)=$

$$\sum_{\substack{d_1+d_2+\cdots+d_k=n-k\\ d_i\in\mathbb{N}}}\frac{n\underset{p,q}{!}}{(d_1+1)_{p,q}(d_1+d_2+2)_{p,q}\cdots(d_1+\cdots+d_k+k)_{p,q}}, \tag{3.38}$$

recalling that $0\underset{p,q}{!}:=1$ and for all $n\in\mathbb{P}$, $n\underset{p,q}{!}:=n_{p,q}(n-1)_{p,q}\cdots 1_{p,q}$.

Proof. This is straightforward by applying Theorem 2.3.4 to the case of $a_i=i_{p,q}$, for all $i\in\mathbb{N}$, and $b_i\equiv 0$ and rewriting it in terms of the factors not present. ☐

Corollary 3.3.12. *For all $n,k\in\mathbb{N}$ and $q\in\mathbb{P}$,*

$$c_q(n,k)=\sum_{\substack{d_1+d_2+\cdots+d_k=n-k\\ d_i\in\mathbb{N}}}\frac{n\underset{q}{!}}{(d_1+1)_q(d_1+d_2+2)_q\cdots(d_1+\cdots+d_k+k)_q}. \tag{3.39}$$

Thus, p,q-derangement numbers, and hence q-derangement numbers, can be defined in a straightforward manner analogous to the discussion at the end of Section 2.4.1. By restricting each d_i in (3.38) and (3.39) to the set $\mathbb{P}$ instead of $\mathbb{N}$, q- and p,q-analogues[14] of the derangement numbers arise, valid for all $p,q\in\mathbb{P}$. Specifically, $\forall n,k,p,q\in\mathbb{P}$ let

[14]Due to the way these formulas were derived, p and q will be taken to be in $\mathbb{P}$ here. The formulas do not dictate it, though, and p and q could be indeterminates instead. In that case, the following would be p,q-generalizations of the derangement numbers.

$d_{p,q}(n,k) \ =$

$$\sum_{\substack{d_1+d_2+\cdots+d_k=n-k \\ d_i\in\mathbb{P}}} \frac{n\underset{p,q}{!}}{(d_1+1)_{p,q}(d_1+d_2+2)_{p,q}\cdots(d_1+\cdots+d_k+k)_{p,q}}. \tag{3.40}$$

Summing across all k yields a p,q-analogue of the derangement number d_n, which we denote $d_{p,q}(n)$. Thus,

Theorem 3.3.13. *For all $n,p,q \in \mathbb{P}$, let*
$d_{p,q}(n) := \sum_k d_{p,q}(n,k) =$

$$\sum_{k\geq 0}\sum_{\substack{d_1+d_2+\cdots+d_k=n-k \\ d_i\in\mathbb{P}}} \frac{n\underset{p,q}{!}}{(d_1+1)_{p,q}(d_1+d_2+2)_{p,q}\cdots(d_1+\cdots+d_k+k)_{p,q}}. \tag{3.41}$$

$$d_q(n,k) \ = \sum_{\substack{d_1+d_2+\cdots+d_k=n-k \\ d_i\in\mathbb{P}}} \frac{n\underset{q}{!}}{(d_1+1)_{q}(d_1+d_2+2)_{q}\cdots(d_1+\cdots+d_k+k)_{q}}. \tag{3.42}$$

Summing across all k yields a q-analogue of the derangement number d_n, which we denote $d_q(n)$. Thus,

Theorem 3.3.14. *For all $n,q \in \mathbb{P}$, let*
$d_q(n) := \sum_k d_q(n,k) =$

$$\sum_{k\geq 0}\sum_{\substack{d_1+d_2+\cdots+d_k=n-k \\ d_i\in\mathbb{P}}} \frac{n\underset{q}{!}}{(d_1+1)_{q}(d_1+d_2+2)_{q}\cdots(d_1+\cdots+d_k+k)_{q}}. \tag{3.43}$$

3.4 The Comtet-Lancaster q and p,q-Lah Numbers

A Lah-analogue of the above two cases provides a Comtet-Lancaster variant on p,q-Lah numbers and hence on a q-Lah array as well[15]. These are defined below.

Recall first that the Lah numbers are defined by $L(n,k) = \left|\overrightarrow{\Pi}_{n,k}\right|$. Suppose the elements of $\overrightarrow{\Pi}_{n,k}$ are canonically written. Then consider the linear order on each block to be a permutation of the elements in the block, and write each in cycle notation.

Example 3.4.1. *Suppose $n = 6$ and $k = 2$. Then the partition with ordered blocks $\{6,2,5,1|4,3\}$ would be written $(1\,6)(2)(5)|(3\,4)$. Here, the cycles (2) and (5) are fixed points in their block.*

Call the resulting structure a *partitioned permutation* of $[n]$ into k nonempty blocks. These will be taken to be written canonically as follows:

1. each cycle is written with its least element listed first,

[15]There are several variants on q-Lah numbers appearing in the literature, for example in [17] and [19], but none of those are C-L numbers.

2. within each block the cycles are written in increasing order according to their least (i.e. initial) elements, and

3. the blocks are listed then in increasing order by their least elements, which necessarily appear first.

Denote the set of partitioned permutations of $[n]$ into k nonempty blocks by $\Lambda_{n,k}$, and assume by convention that each element therein is written canonically. Note that the number of cycles is not specified. Also it is clear that $|\Lambda_{n,k}| = L(n,k)$.

On the elements λ of $\Lambda_{n,k}$, written in canonical order so that the cycles are listed in increasing order by their least (here: initial) elements within their blocks and the blocks are listed in increasing order by their least elements, define the following two statistics:

1. $inv^*(\lambda)$ is the number of inversions appearing in λ created by elements that are not initial within their cycles together with the number of "block inversions" created by the initial elements of the cycles with the blocks, where a *block inversion* occurs any time the initial element of some cycle in a block is larger than the smallest element of a block to its right, and

2. $ninv^*(\lambda)$ is the number of non-inversions of λ, where a non-inversion in λ occurs whenever any element other than the smallest element of a cycle is larger than an element to its left, excepting the element 1 which always leads, or whenever the smallest element of some cycle that is not the smallest element of a block is larger than the smallest element of a block to its left.

In Example 3.4.1, i.e. $(1\,6)(2)(5)|(3\,4)$, inv^* is 5. The appearance of 6 in a cycle with another element adds to the inversion count, yielding a contribution of 4 to inv^* since 2, 5, 3 and 4 all appear to its right. The appearance of 5 as a fixed point in the first block adds to the block inversion count, yielding a contribution of 1 since there is one block to the right with a smaller least element. Note that in this example, neither 5 preceding 4 in a left-to-right scan nor the fixed point 2 in the first block have an impact on inv^*. Likewise, $ninv^*$ is 1, the element 4 occurring after 2 being the only contributor.

Then define, for all $n, k \in \mathbb{N}$ and a pair of indeterminates p and q,

$$\bar{L}_{p,q}(n,k) := \sum_{\lambda \in \Lambda(n,k)} p^{ninv^*(\lambda)} q^{inv^*(\lambda)}. \tag{3.44}$$

It follows that

Theorem 3.4.2. *For all $n, k \in \mathbb{P}$ and a pair of indeterminates p and q,*

$$\bar{L}_{p,q}(n,k) = \bar{L}_{p,q}(n-1,k-1) + [(n-1)_{p,q} + k_{p,q}]\bar{L}_{p,q}(n-1,k), \tag{3.45}$$

subject to the boundary conditions $\bar{L}_{p,q}(0,k) = \delta_{0,k}$ for all $k \in \mathbb{N}$ and $\bar{L}_{p,q}(n,0) = \delta_{n,0}$ for all $n \in \mathbb{N}$.

Proof. The boundary conditions are clear. Now let $n, k \in \mathbb{P}$. Consider separately the subsets of $\Lambda_{n,k}$ in which the one-cycle (n) appears as its own block[16] in the partitioned permutation

[16]Necessarily, this block would appear last.

and in which n appears somewhere other than as its own last block, which has two subcases.

In the first case, obtain $\lambda \in \Lambda_{n,k}$ by choosing $\lambda' \in \Lambda_{n-1,k-1}$ and appending a k^{th} block (n). Then there is no change to inv^* or $ninv^*$ since n appears last as a least element of its block. Thus, this case provides $\bar{L}_{p,q}(n-1,k-1)$.

In the second case, consider first the subcase when n appears in a cycle with at least one other element. These λ can be obtained by choosing any $\lambda' \in \Lambda_{n-1,k}$, and inserting n into any of the $n-1$ spots that immediately follow some other element[17]. Doing so increases inv^* by j if n follows the $(n-1-j)^{\text{th}}$ entry in λ', once for each element to the right of that spot, and $ninv^*$ by $j-1$, once for each element to its left other than 1. Summing all possibilities across $j \in [n-1]$ yields $(n-1)_q \bar{L}_{p,q}(n-1,k)$.

Now consider the subcase when λ is obtained by adding the cycle (n) to some $\lambda' \in \Lambda_{n-1,k}$, necessarily listed last in some otherwise nonempty block. There are k choices, and if (n) appears as a fixed point of block j, then inv^* is increased by $k-j$, once for each block to its right since in its cycle, n is listed first. Also, $ninv^*$ is increased by $j-1$, once for each block to its left. Summing all possibilities across $j \in [k]$ yields $k_{p,q}\bar{L}_{p,q}(n-1,k)$. $\square$

From this recurrence, notice that these p,q-Lah numbers are C-L numbers, and so call the array defined by the recurrence (3.45) the *Comtet-Lancaster p,q-Lah numbers.* Specifically, $\bar{L}_{p,q}(n,k) = A(n,k;\langle i_{p,q}\rangle,\langle i_{p,q}\rangle)$ for all $n,k \in \mathbb{N}$, and hence several other results compatible with Lancaster's theorem follow. In the special case where $p=1$, the resulting array will be called *Comtet-Lancaster q-Lah numbers*, which can be defined, for all $n,k \in \mathbb{N}$ and an indeterminate q, by

$$\bar{L}_q(n,k) := \sum_{\lambda \in \Lambda(n,k)} q^{inv^*(\lambda)}. \tag{3.46}$$

It follows that

Corollary 3.4.3. *For all $n,k \in \mathbb{P}$ and an indeterminate q,*

$$\bar{L}_q(n,k) = \bar{L}_q(n-1,k-1) + [(n-1)_q + k_q]\bar{L}_q(n-1,k), \tag{3.47}$$

subject to the boundary conditions $\bar{L}_q(0,k) = \delta_{0,k}$ for all $k \in \mathbb{N}$ and $\bar{L}_q(n,0) = \delta_{n,0}$ for all $n \in \mathbb{N}$.

Observe that due to the p,q-symmetry of $(n-1)_{p,q}$, and $k_{p,q}$, choosing $q=1$ also provides the same numbers, now in the parameter p. Thus, the Comtet-Lancaster q-Lah numbers also arise as a result of the $ninv^*$ statistic on $\Lambda_{n,k}$.

Now consider the formulas for the Comtet-Lancaster p,q-Lah numbers that arise from Lancaster's theorem, along with the analogous formulas for the Comtet-Lancaster q-Lah numbers.

First,

Theorem 3.4.4. *For all $n,k \in \mathbb{N}$ and a pair of indeterminates p and q,*

$$x(x+1_{p,q})(x+2_{p,q})\cdots(x+(n-1)_{p,q}) =$$

[17] Note n cannot be listed first since the canonical representation requires 1 is always listed first

$$\sum_{k=0}^{n} \bar{L}_{p,q}(n,k) x(x-1_{p,q})(x-2_{p,q})\cdots(x-(k-1)_{p,q}). \tag{3.48}$$

Corollary 3.4.5. *For all $n, k \in \mathbb{N}$ and an indeterminate q,*

$$x(x+1_q)(x+2_q)\cdots(x+(n-1)_q) = \sum_{k=0}^{n} \bar{L}_q(n,k) x(x-1_q)(x-2_q)\cdots(x-(k-1)_q). \tag{3.49}$$

Second,

Theorem 3.4.6. *For all $n, k \in \mathbb{P}$ and a pair of indeterminates p and q,*

$$\bar{L}_{p,q}(n,k) = \sum_{j=0}^{k} [(n-1-j)_{p,q} + (k-j)_{p,q}] \bar{L}_{p,q}(n-1-j, k-j), \tag{3.50}$$

subject to the same boundary conditions as (3.45).

Corollary 3.4.7. *For all $n, k \in \mathbb{P}$ and an indeterminate q.*

$$\bar{L}_q(n,k) = \sum_{j=0}^{k} [(n-1-j)_q + (k-j)_q] \bar{L}_q(n-1-j, k-j), \tag{3.51}$$

subject to the same boundary conditions as (3.47).

And third,

Theorem 3.4.8. *For all $n, k \in \mathbb{P}$ and a pair of indeterminates p and q,*

$$\bar{L}_{p,q}(n,k) = \sum_{j=k}^{n} \bar{L}_{p,q}(j-1, k-1) \prod_{i=j}^{n-1} (i_{p,q} + k_{p,q}). \tag{3.52}$$

subject to the same boundary conditions as (3.45).

Corollary 3.4.9. *For all $n, k \in \mathbb{P}$ and an indeterminate q,*

$$\bar{L}_q(n,k) = \sum_{j=k}^{n} \bar{L}_q(j-1, k-1) \prod_{i=j}^{n-1} (i_q + k_q). \tag{3.53}$$

subject to the same boundary conditions as (3.47).

There is similarity in these formulas to those for the Comtet numbers, notably (1.28), (1.29), (1.36), and (1.37), though these too are clearly not Comtet numbers. Also, a structure that is a q-analogue of one counted by $L(n,k)$ in terms of $\mathbb{F}_q$-vector spaces in the case where q is a power of a prime number is desirable.

In terms of the combinatorial interpretation extended to these arrays from the one given in Section 2.3, the C-L p,q-Lah number $\bar{L}_{p,q}(n,k)$ enumerates the words in the set $\mathcal{CL}(n,k;\langle i_q\rangle,\langle i_q\rangle)$, i.e. words of length $n-k$ in both the strictly ascending alphabets $A_0\cup\cdots\cup A_{n-1}$ together with the ascending alphabets $B_0\cup\cdots\cup B_k$, with each $A_i=\{a_{i,1},\ldots,a_{i,i_{p,q}}\}$ and each with each $B_i=\{b_{i,1},\ldots,b_{i,i_{p,q}}\}$. Furthermore, $\bar{L}_{p,q}(n,k)$ enumerates the sequences in the set $\mathcal{U}(n,k;\langle i_{p,q}\rangle,\langle i_{p,q}\rangle)$, i.e. those sequences obtained from selecting balls from urns, as described in Section 2.3.2, in which the number of white balls in any urn is given by the p,q-integer analogue of its label and the number of red balls is given by the p,q-integer analogue of its sublabel.

Consider now the closed-form expression for the C-L p,q-Lah and q-Lah numbers given via Theorem 2.3.4.

Theorem 3.4.10. *For all $n,k\in\mathbb{N}$ and a pair of indeterminates p and q,*

$$\bar{L}_{p,q}(n,k)=\sum\frac{(n+k+1)_{p,q}!}{(d_1+2)_{p,q}(d_1+3)_{p,q}\cdots(d_1+\cdots+d_k+2k)_{p,q}(d_1+\cdots+d_k+2k+1)_{p,q}},\tag{3.54}$$

where the sum runs over all choices of $d_1,\ldots,d_k\in\mathbb{N}$ with $d_1+\cdots+d_k=n-k$.

Proof. This is straightforward by applying Theorem 2.3.4 to the case of $a_i=i_{p,q}$ and $b_i=i_{p,q}$ and rewriting it in terms of the factors not present. □

Corollary 3.4.11. *For all $n,k\in\mathbb{N}$ and an indeterminate q,*

$$\bar{L}_{q}(n,k)=\sum\frac{(n+k+1)_{q}!}{(d_1+2)_{q}(d_1+3)_{q}\cdots(d_1+\cdots+d_k+2k)_{q}(d_1+\cdots+d_k+2k+1)_{q}},\tag{3.55}$$

where the sum runs over all choices of $d_1,\ldots,d_k\in\mathbb{N}$ with $d_1+\cdots+d_k=n-k$.

Summary and Future Directions

The primary goal of the first chapter is to re-introduce and develop in parallel manner and then to unify algebraically and combinatorially several of the examples of the numbers classified as Comtet numbers including the binomial coefficients, Stirling numbers (of the second kind), q-binomial coefficients, and Carlitz q-Stirling numbers (also of the second kind). These are each shown to have similar formulations in terms of their recurrences, closed-form expressions, ordinary (column) generating functions, and roles as connection constants between x^n and various falling-factorial polynomials. Those similarities are unified by Comtet's theorem (Theorem 1.2.1), see [4] and [21], in Formulas (1.28) – two-term recurrence, (1.36) and (1.37) – "hockey stick" recurrences, (1.31) – closed-form expressions, (1.30 – column generating functions, and (1.29) – connection-constant relations. In Section 1.3, these Comtet numbers are given a unifying combinatorial interpretation in terms of words enumerated by a class of complete symmetric polynomials, and then bijections are given between the new interpretations and the known ones given in Section 1.1.

In the second chapter, the cycle and Lah numbers are presented as examples of similar arrays to those unified by Comtet's theorem and yet outside of its umbrella. These have similar formulations in terms of their recurrences and roles as connection constants, this time connecting rising-factorial polynomials with x^n in the case of the cycle numbers and with the (usual) falling-factorial polynomial in the case of the Lah numbers. Those similarities are unified by Lancaster's theorem (Theorem 2.2.2), see [8], in Formulas (2.15) – two term recurrences, (2.17) and (2.16) – "hockey stick" recurrences, and (2.14) – connection-constant relations. In Section 2.3, these Comtet-Lancaster numbers are given a unifying combinatorial interpretation in terms of words enumerated by a class of sums of products of elementary and complete symmetric polynomials as well as a more classical balls-and-urns interpretation, and then bijections are given between the new interpretations and the known ones given in Section 2.1. Furthermore, using the balls-and-urns interpretation, we give the C-L numbers a closed-form expression (2.33) that generalizes (1.31), and using it, new closed-form expressions for the cycle number (2.46) and Lah numbers (2.55) arise that highlight a different direction for potential generalization of these arrays as well as providing a closed-form expression for the derangement numbers (2.48). Finally, the interpretation provided in this chapter is compared with the earlier attempt provided by John Konvalina in [11] and [12] and is seen to be both more salient in composition and more general in construction.

The third chapter endeavors to explore q- and p, q-generalizations of the arrays presented in Chapters 1 and 2 via the method of statistical generating functions, utilizing some new and some known statistics on the various structures presented in Sections 1.1 and 2.1. Using these, p, q- and hence q-generalizations of the Stirling numbers and cycle numbers, in agreement with those of Carlitz [1], Gould [7], Leroux and de Médicis [9] and [10], and Wachs and White [18], are presented using statistics on permutations given in cycle notation. These allow a definition, following the method in Chapter 2 using the closed form expression, of p, q and q-derangement numbers. Also given are p, q- and hence q-generalizations of the Lah numbers that are indeed C-L numbers, a novel pair of arrays that can be added to the list provided for instance by Wagner [19] and Shattuck [17]. In addition, an alternative q-binomial coefficient is defined that is not an array of Comtet numbers but that are C-L numbers using a modification of a well-known statistic, see [15], on lattice paths that generates the q-binomial coefficients. Many of these examples do not have structures, particularly in terms of $\mathbb{F}_q$-vector spaces, that provide q-analogues, and so developing such structures would be of interest. The author has developed in a separate work a structural framework that accomplishes many of these goals, though not in terms of vector spaces.

Another direction for future study includes the development of statistics on the structures given by $\mathcal{W}(n, k; \langle b_i \rangle)$ and $\mathcal{CL}(n, k; \langle a_i \rangle, \langle b_i \rangle)$, particularly in the cases of the binomial coefficients, Stirling numbers, cycle numbers, and Lah numbers so that their q and p, q-generalizations are available. Ideally, those would be analogous to the *inv* and *ninv* statistics given on the structures, though in some cases, that may prove intractable. Furthermore, it would be interesting to derive a more general notion of functions of restricted growth that provide structures for either the Comtet numbers or C-L numbers, in as broad a sense as that is possible, extending the structure presented in Section 1.4.4.

Also, based on the discussion in Section 1.4.4, a development of the idea of restricted growth functions to the generality of the Comtet numbers is already available, and it would be interesting to note whether anything can be said on the matter for the C-L numbers. For instance, the cycle numbers can be described as counting a class of surjective functions that are both forced and restricted in terms of their growth. The balls-and-urns approach to interpreting them seems particularly likely to provide insight in such an endeavor.

Furthermore, direct generalizations of the C-L numbers are also possible, for instance providing a general two-term recurrence that specializes to the C-L numbers in certain cases while also encapsulating the similar Eulerian numbers and p, q-binomial coefficients, see [5], neither of which are C-L numbers. These allow, *inter alia*, Formulas (1.57) and (1.58) to receive similar treatment to the other q-binomial Hockey Stick Theorems as they arise from the p, q-binomial coefficients with $q = 1$, reparameterized in terms of q in place of p. That approach proceeds from generalizing the balls-and-urns interpretation of the C-L numbers and appears to be able to be extended to also capture and generalize multinomial coefficients as well.

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Appendix

Appendix A

Appendix: Partial Tables of Values of Arrays

In this appendix, partial tables and lists of values of the main arrays discussed in this body of work are presented.

The partial table of values for the binomial coefficients, $\binom{n}{k}$ is

$n \backslash k$	0	1	2	3	4	5	6	$\sum$
0	1							1
1	1	1						2
2	1	2	1					4
3	1	3	3	1				8
4	1	4	6	4	1			16
5	1	5	10	10	5	1		32
6	1	6	15	20	15	6	1	64

This is Pascal's triangle and is well-known (see [20]). Observe that the entries in the "sum" column here are given by 2^n. They count, for instance, the subsets of $[n]$.

The partial table of values for the Stirling numbers, $S(n, k)$ is

$n \backslash k$	0	1	2	3	4	5	6	$\sum$
0	1							1
1	0	1						1
2	0	1	1					2
3	0	1	3	1				5
4	0	1	7	6	1			15
5	0	1	15	25	10	1		52
6	0	1	31	90	65	15	1	203

The entries in the "sum" column here, the Bell numbers, are well known (see [20]), those being denoted B_n and named for Eric Temple Bell. They count, for instance, the number of partitions of $[n]$ into any number of nonempty blocks.

The partial table of values for the cycle numbers, $c(n,k)$ is

$n\backslash k$	0	1	2	3	4	5	6	$\sum$
0	1							1
1	0	1						1
2	0	1	1					2
3	0	2	3	1				6
4	0	6	11	6	1			24
5	0	24	50	35	10	1		120
6	0	120	274	225	85	15	1	720

The entries in the "sum" column here are given by $n!$, as is well known (see [20]), counting all of the permutations of $[n]$ (with any number of cycles).

The partial table of values for the Lah numbers, $L(n,k)$ is

$n\backslash k$	0	1	2	3	4	5	6	$\sum$
0	1							1
1	0	1						1
2	0	2	1					3
3	0	6	6	1				13
4	0	24	36	12	1			73
5	0	120	240	120	20	1		501
6	0	720	1800	1200	300	30	1	4051

The entries in the "sum" column here are frequently denoted L_n (see [20]), counting all of the partitions of $[n]$ into any number of nonempty blocks, each equipped with a linear order. Observe that the $k=1$ column here agrees with the "sum" column for the cycle numbers for $n \in \mathbb{P}$, as these obviously count $\overrightarrow{\Pi}_{n,1}$, the set of partitions of $[n]$ in one block with a linear order, i.e. the permutations of the elements of $[n]$.

For the arrays given in terms of indeterminates q, or p and q, instead of presenting a partial table of values, a list is given since the polynomials are notational cumbersome in tabular format. Also, in the cases of the q-arrays, they are presented only to $n=5$ instead of $n=6$. In the cases of the p,q-arrays, they are presented only to $n=4$.

The following list gives some of the values of the q-binomial coefficients:
For $n=0$,

- $k=0:1$;
- $\sum_k \binom{0}{k}_q = 1$.

For $n = 1$,

- $k = 0 : 1$;
- $k = 1 : 1$;
- $\sum_k \binom{1}{k}_q = 2$.

For $n = 2$,

- $k = 0 : 1$;
- $k = 1 : 1 + q$;
- $k = 2 : 1$;
- $\sum_k \binom{2}{k}_q = 3 + q$.

For $n = 3$,

- $k = 0 : 1$;
- $k = 1 : 1 + q + q^2$;
- $k = 2 : 1 + q + q^2$;
- $k = 3 : 1$;
- $\sum_k \binom{3}{k}_q = 4 + 2q + 2q^2$.

For $n = 4$,

- $k = 0 : 1$;
- $k = 1 : 1 + q + q^2 + q^3$;
- $k = 2 : 1 + q + 2q^2 + q^3 + q^4$;
- $k = 3 : 1 + q + q^2 + q^3$;
- $k = 4 : 1$;
- $\sum_k \binom{4}{k}_q = 5 + 3q + 4q^2 + 3q^3 + q^4$.

For $n = 5$,

- $k = 0 : 1$;
- $k = 1 : 1 + q + q^2 + q^3 + q^4$;
- $k = 2 : 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6$;
- $k = 3 : 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6$;
- $k = 4 : 1 + q + q^2 + q^3 + q^4$;
- $k = 5 : 1$;

- $\sum_k \binom{5}{k}_q = 6 + 4q + 6q^2 + 6q^3 + 6q^4 + 2q^5 + 2q^6$.

The sums across k for fixed n here are frequently denoted $G_n(q)$ and are known as the Galois numbers in the parameter q (see [21]), counting all of the vector subspaces of of $\mathbb{F}_q^n$ when q is a power of a prime number.

The following list gives some of the values of the Carlitz q-Stirling numbers:
For $n = 0$,

- $k = 0 : 1$;
- $\sum_k \tilde{S}_q(0, k) = 1$.

For $n = 1$,

- $k = 0 : 0$;
- $k = 1 : 1$;
- $\sum_k \tilde{S}_q(1, k) = 1$.

For $n = 2$,

- $k = 0 : 0$;
- $k = 1 : 1$;
- $k = 2 : 1$;
- $\sum_k \tilde{S}_q(2, k) = 2$.

For $n = 3$,

- $k = 0 : 0$;
- $k = 1 : 1$;
- $k = 2 : 2 + q$;
- $k = 3 : 1$;
- $\sum_k \tilde{S}_q(3, k) = 4 + q$.

For $n = 4$,

- $k = 0 : 0$;
- $k = 1 : 1$;
- $k = 2 : 3 + 3q + q^2$;
- $k = 3 : 3 + 2q + q^2$;
- $k = 4 : 1$;

- $\sum_k \tilde{S}_q(4,k) = 8 + 5q + 2q^2$.

For $n = 5$,

- $k = 0 : 0$;
- $k = 1 : 1$;
- $k = 2 : 4 + 6q + 4q^2 + q^3$;
- $k = 3 : 6 + 8q + 7q^2 + 3q^3 + q^4$;
- $k = 4 : 4 + 3q + 2q^2 + q^3$;
- $k = 5 : 1$;
- $\sum_k \tilde{S}_q(5,k) = 16 + 17q + 13q^2 + 5q^3 + q^4$.

The sums across k for fixed n here are sometimes denoted $\tilde{B}_n(q)$ and are known as the (Carlitz) q-Bell numbers in the parameter q (see [22]).

The following list gives some of the values of the Carlitz p,q-Stirling numbers:
For $n = 0$,

- $k = 0 : 1$;
- $\sum_k \tilde{S}_{p,q}(0,k) = 1$.

For $n = 1$,

- $k = 0 : 0$;
- $k = 1 : 1$;
- $\sum_k \tilde{S}_{p,q}(1,k) = 1$.

For $n = 2$,

- $k = 0 : 0$;
- $k = 1 : 1$;
- $k = 2 : 1$;
- $\sum_k \tilde{S}_{p,q}(2,k) = 2$.

For $n = 3$,

- $k = 0 : 0$;
- $k = 1 : 1$;
- $k = 2 : 1 + p + q$;
- $k = 3 : 1$;

- $\sum_k \tilde{S}_{p,q}(3,k) = 3 + p + q$.

For $n = 4$,

- $k = 0 : 0$;
- $k = 1 : 1$;
- $k = 2 : 1 + p^2 + p + 2pq + q + q^2$;
- $k = 3 : 1 + p^2 + p + pq + q + q^2$;
- $k = 4 : 1$;
- $\sum_k \tilde{S}_{p,q}(4,k) = 4 + 2p^2 + 2p + 3pq + 2q + 2q^2$.

The sums across k for fixed n here could be denoted $\tilde{B}_n(p,q)$ and be called the Carlitz p,q-Bell numbers in the parameters p and q. Notice how the introduction of the parameter p here offers a somewhat more pleasing symmetry to the polynomials than in the Carlitz q-Stirling numbers case.

The following list gives some of the values of the q-cycle numbers:
For $n = 0$,

- $k = 0 : 1$;
- $\sum_k c_q(0,k) = 1$.

For $n = 1$,

- $k = 0 : 0$;
- $k = 1 : 1$;
- $\sum_k c_q(1,k) = 1$.

For $n = 2$,

- $k = 0 : 0$;
- $k = 1 : 1$;
- $k = 2 : 1$;
- $\sum_k c_q(2,k) = 2$.

For $n = 3$,

- $k = 0 : 0$;
- $k = 1 : 1 + q$;
- $k = 2 : 2 + q$;
- $k = 3 : 1$;

- $\sum_k c_q(3,k) = 4+2q$.

For $n=4$,

- $k=0:0$;
- $k=1:1+2q+2q^2+q^3$;
- $k=2:3+4q+3q^2+q^3$;
- $k=3:3+2q+q^2$;
- $k=4:1$;
- $\sum_k c_q(4,k) = 8+8q+6q^2+2q^3$.

For $n=5$,

- $k=0:0$;
- $k=1:1+3q+5q^2+6q^3+5q^4+3q^5+q^6$;
- $k=2:4+9q+12q^2+12q^3+8q^4+4q^5+q^6$;
- $k=3:6+9q+9q^2+7q^3+3q^4+q^5$;
- $k=4:4+3q+2q^2+q^3$;
- $k=5:1$;
- $\sum_k c_q(5,k) = 16+24q+28q^2+26q^3+16q^4+8q^5+2q^6$.

The sums across k for fixed n here exhibit an interesting property. First notice that they are not given by $n!_q$. Instead, if we let $t_i = (1+(i-1)_q)$ for all $i \in \mathbb{P}$, then

$$\sum_k c_q(n,k) = \prod_{i=1}^{n} t_i, \tag{A.1}$$

noting the empty-product convention when $n=0$.

Proof. This is most directly proved by using Theorem 3.31, which states

$$x(x+1_q)(x+2_q)\cdots(x+(n-1)_q) = \sum_{k=0}^{n} c_q(n,k)x^k,$$

and applying $x=1$. □

For an enumerative proof, observe that both sides of (A.1) count the words of any length (naturally limited to being of lengths between 0 and $n-1$ letters) in the strictly ascending alphabets $A_1, \ldots, A_{n-1}$, with each $A_i = \{a_{i,1}, \ldots, a_{i,i_q}\}$. The left-hand side of (A.1) does this by the recognition in Section 3.3 that the q-cycle numbers are C-L numbers.

On the other hand, the right-hand side of (A.1) counts the same words by filling n slots with letters of the same kind as follows: the i^{th} slot, for $i \in [n]$, either takes a letter from A_{i-1} or is left blank, creating $t_i = (1+(i-1)_q)$ possible choices for each block. The word is created by dropping any blank spaces and maintaining the order of the letters otherwise. □

The following list gives some of the values of the p, q-cycle numbers:
For $n = 0$,

- $k = 0 : 1$;
- $\sum_k c_{p,q}(0, k) = 1$.

For $n = 1$,

- $k = 0 : 0$;
- $k = 1 : 1$;
- $\sum_k c_{p,q}(1, k) = 1$.

For $n = 2$,

- $k = 0 : 0$;
- $k = 1 : 1$;
- $k = 2 : 1$;
- $\sum_k c_{p,q}(2, k) = 2$.

For $n = 3$,

- $k = 0 : 0$;
- $k = 1 : p + q$;
- $k = 2 : 1 + p + q$;
- $k = 3 : 1$;
- $\sum_k c_{p,q}(3, k) = 2 + 2p + 2q$.

For $n = 4$,

- $k = 0 : 0$;
- $k = 1 : p^3 + 2p^2q + 2pq^2 + q^3$;
- $k = 2 : p^3 + p^2 + p + 2p^2q + pq + 2pq^2 + q + q^2 + q^3$;
- $k = 3 : 1 + p^2 + p + pq + q + q^2$;
- $k = 4 : 1$;
- $\sum_k c_{p,q}(4, k) = 2 + 2p^3 + 2p^2 + 2p + 4p^2q + 2pq + 4pq^2 + 2q + 2q^2 + 2q^3$.

Notice how again the introduction of the parameter p here offers a somewhat more pleasing symmetry to the polynomials than in the q-cycle numbers case. Also, the sums across k for fixed n exhibit the same property, *mutatis mutandis* as in the case of the q-cycle numbers, i.e. if we let $t_i = (1 + (i - 1)_{p,q})$ for all $i \in \mathbb{P}$, then

$$\sum_k c_{p,q}(n, k) = \prod_{i=1}^{n} t_i, \tag{A.2}$$

noting the empty-product convention when $n = 0$.

Proof. This is most directly proved by using Theorem 3.30, which states

$$x(x+1_{p,q})(x+2_{p,q})\cdots(x+(n-1)_{p,q}) = \sum_{k=0}^{n} c_{p,q}(n,k)x^k,$$

and applying $x = 1$. $\square$

For an enumerative proof, observe that both sides of (A.2) count the words of any length (naturally limited to being of lengths between 0 and $n-1$ letters) in the strictly ascending alphabets A_1, …, A_{n-1}, with each $A_i = \{a_{i,1}, \ldots, a_{i,i_{p,q}}\}$. The left-hand side of (A.2) does this by the recognition in Section 3.3 that the p,q-cycle numbers are C-L numbers.

On the other hand, the right-hand side of (A.2) counts the same words by filling n slots with letters of the same kind as follows: the i^{th} slot, for $i \in [n]$, either takes a letter from A_{i-1} or is left blank, creating $t_i = (1+(i-1)_{p,q})$ possible choices for each block. The word is created by dropping any blank spaces and maintaining the order of the letters otherwise. $\square$

The following list gives some of the values of the Comtet-Lancaster q-Lah numbers:
For $n = 0$,

- $k = 0 : 1;$
- $\sum_k \bar{L}_q(0,k) = 1.$

For $n = 1$,

- $k = 0 : 0;$
- $k = 1 : 1;$
- $\sum_k \bar{L}_q(1,k) = 1.$

For $n = 2$,

- $k = 0 : 0;$
- $k = 1 : 2;$
- $k = 2 : 1;$
- $\sum_k \bar{L}_q(2,k) = 3.$

For $n = 3$,

- $k = 0 : 0;$
- $k = 1 : 4 + 2q;$
- $k = 2 : 4 + 2q;$
- $k = 3 : 1;$
- $\sum_k \bar{L}_q(3,k) = 9 + 4q.$

For $n = 4$,

- $k = 0 : 0;$
- $k = 1 : 8 + 8q + 6q^2 + 2q^3;$
- $k = 2 : 12 + 14q + 8q^2 + 2q^2;$
- $k = 3 : 6 + 4q + 2q^2;$
- $k = 4 : 1;$
- $\sum_k \bar{L}_q(4, k) = 27 + 26q + 16q^2 + 4q^3.$

For $n = 5$,

- $k = 0 : 0;$
- $k = 1 : 16 + 24q + 28q^2 + 26q^3 + 16q^4 + 8q^5 + 2q^6;$
- $k = 2 : 32 + 60q + 62q^2 + 48q^3 + 26q^4 + 10q^5 + 2q^6;$
- $k = 3 : 24 + 34q + 32q^2 + 20q^3 + 8q^4 + 2q^5;$
- $k = 4 : 8 + 6q + 4q^2 + 2q^3;$
- $k = 5 : 1;$
- $\sum_k \bar{L}_q(5, k) = 81 + 124q + 126q^2 + 96q^3 + 50q^4 + 20q^5 + 4q^6.$

The sums across k for fixed n here could be denoted $\bar{L}_n(q)$ and would generalize L_n in a new way. Observe that for $n \in \mathbb{P}$, the values when $k = 1$ match the values of the sums, over k, of the q-cycle numbers, i.e. $\bar{L}_q(n, 1) = \sum_k c_q(n, k)$. This fact underscores a connection between the structures discussed in this work, in particular that there may be a pair of suitable structures, one for the q-cycle numbers and one for the Comtet-Lancaster q-Lah numbers, so that the one analogous to $\overrightarrow{\Pi}_{n,k}$ is composed of "blocks" containing the structures analogous to $\mathcal{P}(n, k)$, as is the case when $q = 1$.

The following list gives some of the values of the Comtet-Lancaster p, q-Lah numbers:
For $n = 0$,

- $k = 0 : 1;$
- $\sum_k \bar{L}_{p,q}(0, k) = 1.$

For $n = 1$,

- $k = 0 : 0;$
- $k = 1 : 1;$
- $\sum_k \bar{L}_{p,q}(1, k) = 1.$

For $n = 2$,

- $k = 0 : 0$;
- $k = 1 : 2$;
- $k = 2 : 1$;
- $\sum_k \bar{L}_{p,q}(2, k) = 3$.

For $n = 3$,

- $k = 0 : 0$;
- $k = 1 : 2 + 2p + 2q$;
- $k = 2 : 2 + 2p + 2q$;
- $k = 3 : 1$;
- $\sum_k \bar{L}_{p,q}(3, k) = 5 + 4p + 4q$.

For $n = 4$,

- $k = 0 : 0$;
- $k = 1 : 2 + 2p^3 + 2p^2 + 2p + 4p^2q + 2pq + 4pq^2 + 2q + 2q^2 + 2q^3$;
- $k = 2 : 2 + 2p^3 + 4p^2 + 4p + 4p^2q + 6pq + 4pq^2 + 4q + 4q^2 + 2q^3$;
- $k = 3 : 2 + 2p^2 + 2p + 2pq + 2q + 2q^2$;
- $k = 4 : 1$;
- $\sum_k \bar{L}_{p,q}(4, k) = 7 + 4p^3 + 8p^2 + 8p + 8p^2q + 10pq + 8pq^2 + 8q + 8q^2 + 4q^3$.

The sums across k for fixed n here could be denoted $\bar{L}_n(p, q)$ and would further generalize L_n. Notice how the introduction of the parameter p here offers a somewhat more pleasing symmetry to the polynomials than in the case of the Comtet-Lancaster q-Lah numbers, and once again, the values when $k = 1$ match the values of the sums, over k, of the p, q-cycle numbers, i.e. $\bar{L}_{p,q}(n, 1) = \sum_k c_{p,q}(n, k)$. This fact underscores a connection between the structures discussed in this work, in particular that there may be a pair of suitable structures, one for the p, q-cycle numbers and one for the Comtet-Lancaster p, q-Lah numbers, so that the one analogous to $\overrightarrow{\Pi}_{n,k}$ is composed of "blocks" containing the structures analogous to $\mathcal{P}(n, k)$, as is the case when $p = q = 1$.

The results (A.1) and (A.2) hold more generally for the C-L numbers cases in which $b_i \equiv 0$. Specifically, if we denote by $A(n; \langle a_i \rangle)$ the row-sum of $A(n, k; \langle a_i \rangle, \langle 0 \rangle)$, then

$$A(n; \langle a_i \rangle) = \prod_{i=0}^{n-1} (1 + a_i), \tag{A.3}$$

noting the empty-product convention when $n = 0$.

Proof. This is most directly proved by using the connection constants variant of Lancaster's Theorem 2.2.2, which states

$$(x+a_0)(x+a_1)(x+a_2)\cdots(x+a_{n-1}) = \sum_{k=0}^{n} A(n,k;\langle a_i\rangle,\langle 0\rangle)x^k,$$

and applying $x = 1$. □

For an enumerative proof, observe that both sides of (A.3) count the words of any length (naturally limited to being of lengths between 0 and $n-1$ letters) in the strictly ascending alphabets A_1, …, A_{n-1}, with each $A_i = \{a_{i,1}, \ldots, a_{i,a_i}\}$. The left-hand side of (A.3) does this by Theorem 2.3.2.

On the other hand, the right-hand side of (A.3) counts the same words by filling n slots with letters of the same kind as follows: the i^{th} slot, for $i \in [n-1]^*$, either takes a letter from A_{i-1} or is left blank, creating $a_i + 1$ possible choices for each block. The word is created by dropping any blank spaces and maintaining the order of the letters otherwise. □

Furthermore, the sums of the rows in many of these arrays exhibit two-term recurrences of their own, notably the cases of 2^n, $n!$, L_n, and $G_n(q)$, while others do not, particularly it is known that B_n does not (see [20] and [21]). It would be interesting if the row-sums of the Comtet-Lancaster q- and p,q-Lah numbers exhibit a two-term recurrence analogous to that of L_n. Observe that the row-sums in the cases of the q- and p,q-cycle numbers each satisfy a two-term recurrence, for all $n \geq 2$,, that can be derived, respectively, from (A.1) and (A.2):

$$\prod_{i=1}^{n} t_i = \prod_{i=1}^{n-1} t_i + t_{n-1}(t_n - 1)\prod_{i=1}^{n-2} t_i,$$

for each case of t_i as given above, subject to the boundary conditions that the row-sums are both 1 when $n = 0$ and $n = 1$.

Thus, it is also of interest to investigate the row-sums of the C-L numbers $A(n,k;\langle a_i\rangle,\langle b_i\rangle)$ to try to determine conditions on the sequences $\langle a_i\rangle_{i\geq 0}$ and $\langle b_i\rangle_{i\geq 0}$ that describe when a such a two-term recurrence is available and when it is not. As with the result in (A.3), a partial result is available: it can be shown that when $b_i \equiv 0$, for all $n \geq 2$,

$$A(n;\langle a_i\rangle) = A(n-1;\langle a_i\rangle) + a_{n-1}(a_{n-2}+1)A(n-2;\langle a_i\rangle), \tag{A.4}$$

subject to the boundary conditions $A(0;\langle a_i\rangle) = 1$ and $A(1;\langle a_i\rangle) = a_0 + 1$.

Proof. Notice first that the right-hand side of (A.4) can be simplified to a one-term recurrence[1]:

$$A(n;\langle a_i\rangle) = (a_{n-1}+1)A(n-1;\langle a_i\rangle), \tag{A.5}$$

valid for all $n \geq 1$ with $A(0;\langle a_i\rangle) = 1$. This is perhaps easiest to see algebraically:

[1]It is given here as a two-term recurrence specifically to have it in that form, not for utmost simplicity of the expression.

Note that[2]

$$A(n) := \sum_{k=0}^{n} A(n,k) = \sum_{k=0}^{n} (A(n-1,k-1) + a_{n-1}A(n-1,k)) = (a_{n-1}+1)A(n-1).$$

Now both sides of (A.5) count the words of length up to n in the strictly ascending alphabets $A_0, \ldots, A_{n-1}$. That the left-hand side does so is clear considering the construction in Section 2.3.1. The right-hand side does so in two disjoint, exhaustive classes: those in $\mathcal{A}_{n-1}$ and those in $\mathcal{A}_{n-1}^c$. □

Notice that this establishes the claim that the row sums of the q-cycle and p,q-cycle numbers satisfy a two-term recurrence. Also, if the "other" q-binomial coefficients, $\langle {n \atop k} \rangle_q$ have row-sums denoted by $\hat{G}_n(q)$, these satisfy the two-term recurrence, for all $n \geq 2$,

$$\hat{G}_n(q) = \hat{G}_{n-1}(q) + q^{n-1}(q^{n-2}+1)\hat{G}_{n-2}(q), \tag{A.6}$$

subject to the boundary conditions $\hat{G}_0(q) = 1$ and $\hat{G}_1(q) = 2$.

[2]Suppressing reference to the sequence $\langle a_i \rangle_{i \geq 0}$ for compactness of notation.

Vita

James Stephen Lindsay was born on, June 8, 1979, in Ogdensburg, New York, and subsequently moved to Maryville, Tennessee, at the age of five. Soon thereafter, he entered the Maryville City Schools system, graduating from Maryville High School in 1997. Four years later, he earned the degree of Bachelor of Science in Physics from Tennessee Technological University in Cookeville, Tennessee, and was recognized with the Key to the City of Cookeville for the organization and execution of a large city-service project. He then entered the Masters of Business Administration program for one semester before deciding that business school was not for him. At that point, he began the Masters of Science program in Mathematics at Tennessee Tech, completing that degree a year and a half later in 2003. That fall, he enrolled as a graduate student at the University of Tennessee at Knoxville, earning a Doctor of Philosophy in Mathematics in 2010. His chief mathematical interests lie in combinatorics and number theory, though he has a particular bent toward teaching elementary reasoning and problem solving via abstract mathematics, particularly to students to whom it is an introduction. His primary non-mathematical interests lie in martial arts, writing, etymology, therapeutic bodywork, culinary arts, and non-formal philosophy and religious studies, particularly stemming from the East.