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The geometric programming problem

Yancheng Lin

To the Graduate Council:

I am submitting herewith a thesis written by Yancheng Lin entitled "The geometric programming problem." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

Yueh-er Kuo, Major Professor

We have read this thesis and recommend its acceptance:

Ben Fitzpatrick, Tadeusz Janik

Accepted for the Council:
Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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ABSTRACT

The aim of this thesis is to investigate the theory and methods of the geometric programming problem. The stress of the presentation is placed on the methods for solving the problem as well as the proofs of the theorems, which reveal the nature of the geometric programming problem and form the bases of the methods.

The first two chapters discuss the properties of the (posynomial) geometric programming problem and the methods for solving the problem.

Chapter I gives a discussion on the dual method, which solves the geometric programming problem via an indirect approach, and its theoretic basis.

Chapter II presents the primal method, which finds the solutions of the geometric programming problem directly but approximately, and its theoretic basis.

The last two chapters discuss the properties of the generalized geometric programming problem, that is, signomial geometric programming problem, and the methods for solving the problem.

Chapter III presents the general features of the signomial geometric programming problem and the method for solving the problem by way of transforming the problem to complementary geometric programming.

Chapter IV gives a discussion about the properties of reversed geometric programming, which is a special case of signomial geometric programming, and the method for solving the problem. In fact, this method, which can be seen as a supplement for that presented in chapter III, shows another approach to solve the signomial geometric programming problem.

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CHAPTER I

THE DUAL METHOD

In this chapter, we consider the dual method for solving the (posynomial) geometric programming problem. In order to present the method clearly, we will discuss some of the related properties of the problem first. The basic results of this chapter can be found in [6], [7], [9] and [11].

Geometric programming is known as

$$\begin{aligned} &\text{minimize} \quad g_0(\underline{t}), \\ &\text{subject to} \quad \underline{t} \in F_P, \end{aligned} \tag{1.1}$$

where

$$F_P = \{ \underline{t} \mid g_k(\underline{t}) \le 1, \quad \text{for } k = 1, 2, \ldots, p \ \text{(forced constraint)},$$
$$t_j > 0, \quad \text{for } j = 1, 2, \ldots, m \ \text{(natural constraint)} \},$$
$$g_k(\underline{t}) = \sum_{i \in [k]} u_i(\underline{t}) = \sum_{i \in [k]} c_i \prod_{j=1}^{m} t_j^{a_{ij}}, \quad \text{for } c_i > 0,\ a_{ij} \in R.$$

Throughout this paper we use index set $[k] \subset I = \{1, 2, \ldots, n\}$ for $k = 0, 1, \ldots, p$ to stand for numbers of the terms of $g_k(\underline{t})$. It is implied here that $g_k(\underline{t}) > 0$, in which case $g_k(\underline{t})$ is called posynomial (positive polynomial).

The associated dual problem of (1.1) can be reached via appropriate rearrangement of Lagrange multiplier (see Appendix). That is,

$$\begin{aligned} &\text{maximize} \quad V(\underline{d}) = \prod_{i \in I} (c_i / d_i)^{d_i} \prod_{k=1}^{p} (\lambda_k)^{\lambda_k}, \\ &\text{subject to} \quad \underline{d} \in F_D, \end{aligned} \tag{1.2}$$

where

$$F_D = \{ \underline{d} \mid \sum_{i \in I} a_{ij} d_i = 0, \quad \text{for } j = 1, 2, \ldots, m \ \text{(orthogonality condition)},$$

$\lambda_0 = 1$ (normality condition),

$d_i \geq 0$, for $i = 1, 2, ..., n$ (positivity condition) },

$\lambda_k \geq 0$, for $k = 1, 2, ..., p$,

$$\lambda_k = \sum_{i \in [k]} d_i , \quad \text{for } k = 0, 1, ..., p.$$

Note the orthogonality condition here indicates that the exponential matrix A = $(a_{ij})_{nxm}$ is orthogonal to the dual vector $\underline{d} = (d_1, d_2, ..., d_n)^T$ (T denotes the transpose).

In general, $g_k(\underline{t})$ is not necessarily a convex function (e.g., $t_j^{a_{ij}}$ is not convex for $a_{ij} \in (0,1)$), so problem (1.1) is not necessarily a convex program. However, by convexity theory, it is easy to show that the problem that

$$\begin{aligned} &\text{minimize} \quad f_0(\underline{x}) , \\ &\text{subject to} \quad \underline{x} \in F_{P1} , \end{aligned} \tag{1.3}$$

where

$$F_{P1} = \{ \underline{x} \mid f_k(\underline{x}) \leq 1, \quad \text{for } k = 1, 2, ..., p \},$$

$$f_k(\underline{x}) = g_k(\underline{t}) = \sum_{i \in [k]} c_i \exp \{ \sum_{j=1}^{m} a_{ij} x_j \},$$

$$x_j = \log t_j , \quad \text{for } j = 1, 2, ..., m$$

is a convex program. So, any of its local minimum is its global minimum. In view of the fact that (1.3) is obtained through an one-to-one corresponding transformation we have the following important result.

Theorem 1.1. Any of the local minimum of (1.1) is its global minimum.|

Similarly, the fact that

$$\log V(\underline{d}) = \sum_{i=1}^{n} d_i (\log c_i - \log d_i) + \sum_{k=1}^{p} \lambda_k(\underline{d}) \log \lambda_k(\underline{d})$$

is a concave function (see [6]) and F_D is a convex set leads to:

Theorem 1.2. Any of the local maximum of (1.2) is its global maximum.|

Theorem 1.3. For $u_i > 0$, $d_i \in (0,1)$, $i = 1, 2, ..., n$ and $\sum_{i=1}^{n} d_i = 1$, it holds that

$$\prod_{i=1}^{n} (u_i / d_i)^{d_i} \le \sum_{i=1}^{n} u_i , \tag{1.4}$$

where "=" holds if and only if $u_i = u_j$, for $i \ne j$, $i, j = 1, 2, ..., n$.

Proof. Let $y_i = \log x_i$, for $i = 1, 2, ..., n$. Then

$$d^2y_i / dx_i^2 = (-1) / x_i^2 < 0, \quad \text{for } x_i \in S = (0,\infty) \text{ (convex set)}.$$

By convexity theory, it is clear that y_i is a concave function on S and

$$\sum_{i=1}^{n} d_i \log x_i \le \log \left(\sum_{i=1}^{n} d_i x_i \right),$$

that is,

$$\log \left(\prod_{i=1}^{n} x_i^{d_i} \right) \le \log \left(\sum_{i=1}^{n} d_i x_i \right).$$

where "=" holds if and only if $x_i = x_j$, for $i \ne j$, $i, j = 1, 2, ..., n$. Thus,

$$\prod_{i=1}^{n} x_i^{d_i} \le \sum_{i=1}^{n} d_i x_i \quad \text{(monotony of logarithmic function)},$$

where "=" holds if and only if $x_i = x_j$, for $i \ne j$, $i, j = 1, 2, ..., n$.

Let $x_i = u_i / d_i$, for $i = 1, 2, ..., n$. Then

$$\prod_{i=1}^{n} (u_i / d_i)^{d_i} \le \sum_{i=1}^{n} u_i ,$$

where "=" holds if and only if $u_i = u_j$, for $i \ne j$, $i, j = 1, 2, ..., n$. |

The inequality shown in (1.4) is called geometric inequality, which is the foundamental inequality of geometric programming. As we will see later, it plays a root role in the derivation of the basic properties of the geometric programming problem.

Corollary 1.1. For $\lambda = \sum_{i=1}^{n} d_i$,

$$\prod_{i=1}^{n} (u_i / d_i)^{d_i} \lambda^{\lambda} \le \left(\sum_{i=1}^{n} u_i \right)^{\lambda}, \qquad (1.4')$$

and "=" holds if and only if

$$d_j \sum_{i=1}^{n} u_i = u_j \sum_{i=1}^{n} d_i, \quad \text{for } i, j = 1, 2, ..., n.$$

Proof. We know that for $e_i > 0$ and $\sum_{i=1}^{n} e_i = 1$,

$$\prod_{i=1}^{n} (u_i / e_i)^{e_i} \le \sum_{i=1}^{n} u_i .$$

Let $e_i = d_i / \lambda$ where $\lambda = \sum_{i=1}^{n} d_i$ and $\sum_{i=1}^{n} e_i = \sum_{i=1}^{n} d_i / \lambda = 1$. Then

$$\prod_{i=1}^{n} (u_i \lambda / d_i)^{d_i / \lambda} \le \sum_{i=1}^{n} u_i ,$$

that is,

$$\prod_{i=1}^{n} (u_i / d_i)^{d_i} \lambda^{\lambda} \le \left(\sum_{i=1}^{n} u_i \right)^{\lambda}.$$

Let $v_i = u_i / e_i$. Then it is equivalent to (1.4') that

$$\prod_{i=1}^{n} (v_i)^{e_i} \le \sum_{i=1}^{n} e_i v_i ,$$

where "=" holds if and only if $v_i = v_j$ (Theorem 1.3), i.e.,

$$u_i / e_i = u_j / e_j, \quad \text{for all } i \ne j, \ i, j = 1, 2, ..., n.$$

Note that $e_i = d_i / \lambda$, we have

$$(u_i / d_i)\lambda = (u_j / d_j)\lambda, \ u_i = d_i (u_j / d_j), \ \sum_{i=1}^{n} u_i = \sum_{i=1}^{n} d_i (u_j / d_j).$$

Hence, "=" holds in (1.4') if and only if

$$d_j \sum_{i=1}^{n} u_i = u_j \sum_{i=1}^{n} d_i, \quad \text{for all } i \ne j, \ i, j = 1, 2, ..., n.$$

Theorem 1.4. (main lemma) If $\underline{t} \in F_P$, $\underline{d} \in F_D$, then

$$V(\underline{d}) \le g_0(\underline{t}) \prod_{k=1}^{p} [g_k(\underline{t})]^{\lambda_k} \le g_0(\underline{t}),$$

and $V(\underline{d}) = g_0(\underline{t})$ if and only if

$$d_i = u_i(\underline{t}) / g_0(\underline{t}), \quad \text{for } i \in [0],$$

$$d_i = \lambda_k u_i(\underline{t}), \quad \text{for } i \in [k], \ k = 1, 2, ..., p.$$

Then $\underline{t}$ and $\underline{d}$ are the optimum for (1.1) and (1.2), respectively.

Proof. From Corollary 1.1 we know that

$$\prod_{i\in[k]} [\, u_i(\underline{t}) / d_i \,]^{d_i} \lambda_k^{\lambda_k} \leq g_0(\underline{t}) [\, g_k(\underline{t}) \,]^{\lambda_k}, \quad \text{for } k = 0, 1, \ldots, p,$$

where $u_i(\underline{t}) = c_i \prod_{j=1}^{m} t_j^{a_{ij}}$. That is,

$$\prod_{i\in I} [\, u_i(\underline{t}) / d_i \,]^{d_i} \prod_{k=1}^{p} (\lambda_k)^{\lambda_k} = \prod_{i\in I} (c_i \prod_{j=1}^{m} t_j^{a_{ij}} / d_i)^{d_i} \prod_{k=1}^{p} (\lambda_k)^{\lambda_k}$$

$$\leq g_0(\underline{t}) \prod_{k=1}^{p} [\, g_k(\underline{t}) \,]^{\lambda_k}.$$

Note that $\sum_{i\in I} a_{ij} d_i = 0$, $g_k(\underline{t}) \leq 1$ ($\underline{t} \in F_P$, $\underline{d} \in F_D$), and

$$\prod_{i=1}^{n} (\, u_i(\underline{t}) / d_i \,)^{d_i} = \prod_{i=1}^{n} (\, c_i \prod_{j=1}^{m} t_j^{a_{ij}} / d_i \,)^{d_i}$$

$$= \prod_{i=1}^{n} (c_i / d_i)^{d_i} \prod_{j=1}^{m} t_j^{y_j} \quad (\text{where } y_j = \sum_{i=1}^{n} a_{ij} d_i = 0)$$

$$= \prod_{i=1}^{n} (c_i / d_i)^{d_i}.$$

Hence,

$$V(\underline{d}) = \prod_{j=1}^{n} (c_i / d_i)^{d_i} \prod_{k=1}^{p} (\lambda_k)^{\lambda_k} = \prod_{i=1}^{n} [\, u_i(\underline{t}) / d_i \,]^{d_i} \prod_{k=1}^{p} (\lambda_k)^{\lambda_k}$$

$$\leq g_0(\underline{t}) \prod_{k=1}^{p} [\, g_k(\underline{t}) \,]^{\lambda_k} \leq g_0(\underline{t}).$$

(For sufficiency) Note that for $i \in [0]$, $d_i = u_i(\underline{t}) / g_0(\underline{t})$,

$$\prod_{i\in[0]} [\, u_i(\underline{t}) / d_i \,]^{d_i} (\lambda_0)^{\lambda_0} = g_0(\underline{t}) \qquad (\lambda_0 = \sum_{i\in[0]} d_i = 1)$$

For $i \in [k]$, $k = 1, 2, \ldots, p$, $d_i = \lambda_k u_i(\underline{t})$,

$$\prod_{i\in[k]} [\, u_i(\underline{t}) / d_i \,]^{d_i} (\lambda_k)^{\lambda_k} = \prod_{i\in[k]} [\, u_i(\underline{t}) / (u_i(\underline{t})\, \lambda_k) \,]^{d_i} (\lambda_k)^{\lambda_k} = 1. \; (\lambda_k = \sum_{i\in[k]} d_i).$$

Hence,

$$V(\underline{d}) = \prod_{i\in[0]} [\, u_i(\underline{t}) / d_i \,]^{d_i} (\lambda_0)^{\lambda_0} \prod_{k=1}^{p} \prod_{i\in[k]} [\, u_i(\underline{t}) / d_i \,]^{d_i} (\lambda_k)^{\lambda_k}$$
$$= g_0(\underline{t}).$$

(For necessity) From the argument above and Corollary 1.1, we know

$$V(\underline{d}) = \prod_{k=0}^{p} \prod_{i\in[k]} [\, u_i(\underline{t}) / d_i \,]^{d_i} (\lambda_k)^{\lambda_k} \le \prod_{k=0}^{p} [\, g_k(\underline{t}) \,]^{\lambda_k} = g_0(\underline{t}) \prod_{k=1}^{p} [\, g_k(\underline{t}) \,]^{\lambda_k}$$

and

$$\prod_{i\in[k]} [\, u_i(\underline{t}) / d_i \,]^{d_i} (\lambda_k)^{\lambda_k} \le [\, \sum_{i\in[k]} u_i(\underline{t}) \,]^{\lambda_k} \quad (\lambda_0 = 1),$$

where "=" holds if and only if

$$d_j \sum_{i\in[k]} u_i(\underline{t}) = u_j(\underline{t}) \sum_{i\in[k]} d_i \qquad \text{(by Corollary1.1).}$$

So, $V(\underline{d}) = g_0(\underline{t})$ implies $g_k(\underline{t}) = 1$ and "=" holds for the inequality above. Note

$$\sum_{i\in[0]} d_i = 1, \quad \sum_{i\in[k]} d_i = \lambda_k, \quad \sum_{i\in[0]} u_i(\underline{t}) = g_0(\underline{t}), \quad \sum_{i\in[k]} u_i(\underline{t}) = g_k(\underline{t}) = 1.$$

Therefore,

$d_i = u_i(\underline{t}) / g_0(\underline{t})$, for $i \in [0]$,

$d_i = \lambda_k u_i(\underline{t})$, for $i \in [k]$, $k = 1, 2, \ldots, p$. |

The following results can be immediately reached from Theorem 1.4.

Corollary 1.2.

(i) Suppose $\underline{t}^* \in F_P$, $\underline{d}^* \in F_D$, and $V(\underline{d}^*) = g_0(\underline{t}^*)$. Then $\underline{t}^*$, $\underline{d}^*$ must be the optimum for problem (1.1) and (1.2), respectively.

(ii) Suppose that F_P and F_D are nonempty. Let

$M_P = \inf g_0(\underline{t})$, for all $\underline{t} \in F_P$,

$M_D = \sup V(\underline{d})$, for all $\underline{d} \in F_D$.

Then

$0 < M_P \leq M_D \leq \infty$.

(iii) Suppose $V(\underline{d}^*) = \max V(\underline{d})$, for all $\underline{d} \in F_D$. Then $\underline{t} \in F_P$ such that

$$\sum_{j=1}^{m} a_{ij} \log t_j = \log (d_i^* V(\underline{d}^*) / c_i), \quad \text{for } i \in [0],$$

$$\sum_{j=1}^{m} a_{ij} \log t_j = \log (d_i^* / c_i \lambda_k), \quad \text{for } i \in [k],\ k = 1, 2, \ldots, p$$

is the optimum solution for (1.1).

Lemma 1.1. (Farkas lemma) For $\underline{\xi} = (\xi_1, \xi_2, \ldots, \xi_m)^T \geq \underline{0}$ and matrix A,

$$A\underline{\xi} = \underline{b}$$

if and only if it holds that for any vector $\underline{x}$, if

$$\underline{x}^T A \geq \underline{0}$$

then

$$\underline{x}^T \underline{b} \geq 0.$$

Theorem 1.5. (Kuhn-Tucker theorem) Suppose that $f_k(\underline{x})$ for $k = 1, 2, \ldots, p$ are convex functions with continuous partial derivatives of first order, and that for the convex program that

$$\text{minimize} \quad f_0(\underline{x}),$$

$$\text{subject to} \quad \underline{x} \in F, \tag{1.5}$$

where

$$F = \{ \underline{x} \mid f_k(\underline{x}) \leq 0, \quad \text{for } k = 1, 2, \ldots, p \},$$

there exists $\underline{x}$ such that

$$f_k(\underline{x}) < 0, \quad \text{for } k = 1, 2, \ldots, p \quad \text{(Slater condition)}.$$

Then $\underline{x}^* \in F$ is the optimum for the program (1.5) if and only if there exists Lagrange multiplier $\underline{\mu}^* = (\mu_1^*, \mu_2^*, \ldots, \mu_p^*)^T \geq \underline{0}$ such that

$$\nabla L(\underline{x}^*, \underline{\mu}^*) = \underline{0},$$

$$\mu_k^* f_k(\underline{x}^*) = 0, \quad \text{for } k = 1, 2, \ldots, p,$$

where

$$L(\underline{x}, \underline{\mu}) = f_0(\underline{x}) + \sum_{k=1}^{p} \mu_k f_k(\underline{x})$$

is Lagrangian function.

Proof. (For necessity) Suppose that

$$f_0(\underline{x}^*) = \min f_0(\underline{x}), \quad \text{for all } \underline{x} \in F.$$

For k such that $f_k(\underline{x}^*) < 0$, let $\mu_k^* = 0$. So, we can assume for $k = 1, 2, ..., p$,

$$f_k(\underline{x}^*) = 0.$$

Let $\underline{x}$ be any point such that

$$(\underline{x} - \underline{x}^*)^T \nabla_{\underline{x}} f_k(\underline{x}^*) \leq 0, \quad \text{for } k = 1, 2, ..., p.$$

Since (1.5) satisfies Slater condition, that is, $\exists\, \underline{x}^1$ such that for $k = 1, 2, ..., p$,

$$f_k(\underline{x}^1) < 0.$$

Note that $f_k(\underline{x})$ are convex functions with continuous partial derivatives of first order, then

$$(\nabla_{\underline{x}} f_k(\underline{x}^*))^T (\underline{x}^1 - \underline{x}^*) = (\nabla_{\underline{x}} f_k(\underline{x}^*))^T (\underline{x}^1 - \underline{x}^*) + f_k(\underline{x}^*) \quad (f_k(\underline{x}^*) = 0)$$

$$\leq f_k(\underline{x}^1) < 0. \quad (\text{convexity of } f_k(\underline{x})).$$

Let

$$\underline{y} = (1-\theta)\, \underline{x} + \theta\, \underline{x}^1, \quad \text{for } \theta \in (0, 1).$$

Then for $k = 1, 2, ..., p$,

$$(\nabla_{\underline{x}} f_k(\underline{x}^*))^T (\underline{y} - \underline{x}^*) < 0.$$

Let

$$\underline{z} = \underline{x}^* + \tau (\underline{y} - \underline{x}^*), \quad \text{for } \tau > 0.$$

Then for sufficiently small τ,

$$(\nabla_{\underline{x}} f_k(\underline{z}))^T (\underline{y} - \underline{x}^*) < 0. \quad (\text{continuity of } \nabla_{\underline{x}} f_k(\underline{x})).$$

$$f_k(\underline{z}) = f_k(\underline{z}) - f_k(\underline{x}^*) \quad (f_k(\underline{x}^*) = 0)$$

$$\leq (\nabla_{\underline{x}} f_k(\underline{z}))^T (\underline{y} - \underline{x}^*) < 0.$$

That is, $\underline{z} \in F$.

For the objective function, we have

$$(\nabla_{\underline{x}} f_0(\underline{z}))^T (\underline{y} - \underline{x}^*) \geq f_0(\underline{z}) - f_0(\underline{x}^*) \quad \text{(continuity of } f_0(\underline{x})\text{)}$$

$$\geq 0. \quad (f_0(\underline{x}^*) = \min f_0(\underline{x}), \quad \text{for all } \underline{x} \in F).$$

That is, as $\tau \rightarrow 0$,

$$(\nabla_{\underline{x}} f_0(\underline{x}^*))^T ((1-\theta)\, \underline{x} + \theta \underline{x}^1 - \underline{x}^*) \geq 0,$$

and as $\theta \rightarrow 0$,

$$(\nabla_{\underline{x}} f_0(\underline{x}^*))^T (\underline{x} - \underline{x}^*) \geq 0,$$

that is,

$$(\underline{x} - \underline{x}^*)^T [- \nabla_{\underline{x}} f_0(\underline{x}^*)] \leq 0.$$

That means for any $\underline{x}$ such that

$$(\underline{x} - \underline{x}^*)^T [\nabla_{\underline{x}} f_k(\underline{x}^*)] \leq 0, \quad \text{for } k = 1, 2, ..., p,$$

it holds that

$$(\underline{x} - \underline{x}^*)^T [- \nabla_{\underline{x}} f_0(\underline{x}^*)] \leq 0.$$

So, by Lemma 1.1, for $\mu_k^* \geq 0$,

$$\sum_{k=1}^{p} \mu_k^* \; \nabla_{\underline{x}} f_k(\underline{x}^*) = - \nabla_{\underline{x}} f_0(\underline{x}^*).$$

That is,

$$\nabla_{\underline{x}} f_0(\underline{x}^*) + \sum_{k=1}^{p} \mu_k^* \; \nabla_{\underline{x}} f_k(\underline{x}^*) = \underline{0},$$

$$\nabla L (\underline{x}^*, \mu_k^*) = \underline{0}.$$

(For sufficiency) For all $\underline{x} \in F$, we have

$$\mu_k^* f_k(\underline{x}) \leq 0, \quad \text{for } k = 1, 2, ..., p. \quad (\mu_k^* \geq 0, \; f_k(\underline{x}) \leq 0).$$

Note that in view of the convexity of $f_k(\underline{x})$,

$$f_k(\underline{x}) \geq f_k(\underline{x}^*) + (\underline{x} - \underline{x}^*)^T [\nabla_{\underline{x}} f_k(\underline{x}^*)], \quad \text{for } k = 0, 1, ..., p.$$

Therefore,

$$f_0(\underline{x}) \geq f_0(\underline{x}) + \sum_{k=1}^{p} \mu_k^* f_k(\underline{x}) \quad (\mu_k^* f_k(\underline{x}) \leq 0, \text{ for } k = 1, 2, ..., p).$$

$$\geq [f_0(\underline{x}^*) + (\underline{x} - \underline{x}^*)^T (\nabla_{\underline{x}} f_0(\underline{x}^*))] + \sum_{k=1}^{p} \mu_k^* f_k(\underline{x})$$

$$= f_0(\underline{x}^*) + [(\underline{x} - \underline{x}^*)^T (\nabla_{\underline{x}} f_0(\underline{x}^*)) + \sum_{k=1}^{p} \mu_k^* (\underline{x} - \underline{x}^*)^T \nabla_{\underline{x}} f_k(\underline{x}^*)]$$

$$- \sum_{k=1}^{p} \mu_k^* (\underline{x} - \underline{x}^*)^T \nabla_{\underline{x}} f_k(\underline{x}^*) + \sum_{k=1}^{p} \mu_k^* f_k(\underline{x})$$

$$= f_0(\underline{x}^*) + (\underline{x} - \underline{x}^*)^T \nabla L(\underline{x}^*, \mu_k^*) + \sum_{k=1}^{p} \mu_k^* [f_k(\underline{x}) - (\underline{x} - \underline{x}^*)^T (\nabla_{\underline{x}} f_k(\underline{x}^*)]$$

$$\geq f_0(\underline{x}^*) + (\underline{x} - \underline{x}^*)^T \nabla L(\underline{x}^*, \mu_k^*) + \sum_{k=1}^{p} \mu_k^* f_k(\underline{x}^*).$$

Note that it is given that

$$\nabla L(\underline{x}^*, \mu_k^*) = \underline{0},$$

$$\sum_{k=1}^{p} \mu_k^* f_k(\underline{x}^*) = 0.$$

Therefore, $f_0(\underline{x}) \geq f_0(\underline{x}^*)$, for all $\underline{x} \in F$.

Definition 1.1.

(i) A program (primal or dual) is said to be *consistent* if there is at least one vector satisfying its constraints.

(ii) Primal program is said to be *superconsistent* if there is at least one vector $\underline{t}$ such that

$$g_k(\underline{t}) < 1, \quad \text{for } t_j > 0, \; j = 1, 2, ..., m, \; k = 1, 2, ..., p.$$

|

Theorem 1.6. (duality theorem) Suppose that (1.1) is superconsistent and has an optimum solution $\underline{t}^*$. Then there exists $\underline{\mu}^* = (\mu_1^*, \mu_2^*, ..., \mu_p^*)^T \geq 0$ such that

$$\nabla g_0(\underline{t}^*) + \sum_{k=1}^{p} (\mu_k^*) \nabla g_k(\underline{t}^*) = \underline{0},$$

$$\mu_k^* [g_k(\underline{t}^*) - 1] = 0, \quad \text{for } k = 1, 2, ..., p.$$

And $\underline{d}^* = (d_1^*, d_2^*, ..., d_n^*)^T$ such that

$$d_i^* = u_i(\underline{t}^*) / g_0(\underline{t}^*), \quad \text{for } i \in [0],$$

$$d_i^* = (\mu_k^*)\, u_i(\underline{t}^*) / g_0(\underline{t}^*), \quad \text{for } i \in [k],\ k = 1, 2, \dots, p$$

is the optimum solution for (1.2) and satisfies $V(\underline{d}^*) = g_0(\underline{t}^*)$.

Proof. Since (1.1) is superconsistent and $\underline{t}^*$ is the optimum for (1.1), (1.3) is convex program satisfying Slater condition and $\underline{x}^*$ is the optimum solution for (1.3). Then by Theorem 1.5, we have

$$\nabla f_0(\underline{x}^*) + \sum_{k=1}^{p} \mu_k^* \ \nabla f_k(\underline{x}^*) = \underline{0}, \tag{1,6}$$

$$\mu_k^* \,[\, f_k(\underline{x}^*) - 1 \,] = 0, \quad \text{for } k = 1, 2, \dots, p. \tag{1.7}$$

Note that $g_k(\underline{t}^*) = f_k(\underline{x}^*)$, so (1.7) implies that

$$\mu_k^* \,[\, g_k(\underline{t}^*) - 1 \,] = 0, \quad \text{for } k = 1, 2, \dots, p.$$

Assume that $\mu_0^* = 1$, then (1.6) becomes

$$\sum_{k=0}^{p} (\mu_k^*)\ \nabla f_k(\underline{x}^*) = \underline{0}.$$

Since $x_j^* = \log t_j^*$ and

$$\frac{\partial}{\partial x_j} f_k(\underline{x}) = \sum_{i \in [k]} c_i\, a_{ij} \exp\{ \sum_{j=1}^{m} a_{ij} x_j \}, \quad \text{for } j = 1, 2, \dots, m,$$

we have

$$\sum_{k=0}^{p} \mu_k^* \sum_{i \in [k]} c_i\, a_{ij} \prod_{j=1}^{m} (t_j^*)^{a_{ij}} = 0, \quad \text{for } j = 1, 2, \dots, m.$$

Then for $\varphi = 1, 2, \dots, m$, $t_\varphi^* > 0$,

$$\frac{\partial}{\partial t_\varphi} [\, g_0(\underline{t}^*) + \sum_{k=1}^{p} (\mu_k^*)\, g_k(\underline{t}^*) \,] = (1 / t_\varphi^*) \,[\, \sum_{k=0}^{p} \mu_k^* \sum_{i \in [k]} c_i\, a_{ij} \prod_{j=1}^{m} (t_j^*)^{a_{ij}} \,] = 0.$$

That is,

$$\nabla g_0(\underline{t}^*) + \sum_{k=1}^{p} \mu_k^* \ \nabla g_k(\underline{t}^*) = \underline{0}.$$

For $\underline{d}^*$ such that

$$d_i^* = u_i(\underline{t}^*) / g_0(\underline{t}^*), \quad \text{for } i \in [0],$$

$$d_i^* = (\mu_k^*)\, u_i(\underline{t}^*) / g_0(\underline{t}^*), \quad \text{for } i \in [k],\ k = 1, 2, \dots, p,$$

it is clear to see that

$$\underline{d}^* \geq 0 \quad \text{(positivity)} \quad (\text{since } \mu_k^*,\ u_i(\underline{t}^*) \geq 0),$$

$$\lambda_0 = \sum_{i \in [0]} d_i^* = 1 \quad \text{(normality)},$$

$$\sum_{i \in I} a_{ij}\, d_i^* = [\, 1 / g_0(\underline{t}^*)\,]\, [\, \sum_{k=0}^{p} \mu_k^* \sum_{i \in [k]} c_i\, a_{ij} \prod_{j=1}^{m} (t_j^*)^{a_{ij}}\,] = 0 \quad \text{(orthogonality)}.$$

That means $\underline{d}^* \in F_D$.

Note that

$$\lambda_k^* = (\mu_k^*)\, [\, \sum_{i \in [k]} u_i(\underline{t}^*)\,] / g_0(\underline{t}^*) = \mu_k^* / g_0(\underline{t}^*) \qquad (\, \mu_k^*\, [\, g_k(\underline{t}^*) - 1\,] = 0\,).$$

Note $\mu_k^* = \lambda_k^*\, g_0(\underline{t}^*)$, so,

$$d_i^* = [\, \lambda_k^*\, g_0(\underline{t}^*)\,]\, u_i(\underline{t}^*) / g_0(\underline{t}^*) \;=\; \lambda_k^*\, u_i(\underline{t}^*).$$

Therefore, by Theorem 1.4, $V(\underline{d}^*) = g_0(\underline{t}^*)$, and $\underline{d}^*$ is the optimum for (1.2). |

From Theorem 1.6 we know that for all $i \in [0]$, $u_i(\underline{t}^*) > 0$ implies $d_i^* > 0$. For $i \in [k]$, $k = 1, 2, ..., p$, $d_i^* = 0$ if and only if $\mu_k^* = 0$. That $g_k(\underline{t}^*) < 1$ implies $\mu_k^* = 0$ for $1 \leq k \leq p$ since $(\mu_k^*)[\, g_k(\underline{t}^*) - 1\,] = 0$, so it holds that $d_i^* = 0$ for all $i \in [k]$, $k = 1, 2, ..., p$. That $d_q^* = 0$ for some $q \in [k]$ also implies $\mu_k^* = 0$, so $d_i^* = 0$ for all $i \in [k]$, $k = 1, 2, ..., p$. As a result, we reach the following important fact.

Corollary 1.3. Suppose that

$$g_0(\underline{t}^*) = \min g_0(\underline{t}), \quad \text{for all } \underline{t} \in F_P,$$

$$V(\underline{d}^*) = \max V(\underline{d}), \quad \text{for all } \underline{d} \in F_D.$$

Then for all $i \in [k]$, $k = 1, 2, ..., p$,

$$d_i^* > 0, \quad \text{if } g_k(\underline{t}^*) = 1,$$

$$d_i^* = 0, \quad \text{if } g_k(\underline{t}^*) < 1.$$

|

From Corollary 1.3, we know that for $k = 1, 2, ..., p$, in the case that $d_i^* = 0$ for some $i \in [k]$, $k = 1, 2, ..., p$, $g_k(\underline{t})$ must be inactive constraint in (1.1) ($g_k(\underline{t}^*) < 1$) and can be revoked, so can d_i for $i \in [k]$, $k = 1, 2, ..., p$. Clearly, this simplifies solving (1.2).

Note that the existence of an optimum solution for (1.1) is a necessary condition for Theorem 1.6. The following theorem gives a sufficient condition on which (1.1) has an optimum solution.

Theorem 1.7. If (1.1) is consistent and there exists $d_i > 0$ for some i, then (1.1) has an optimum solution.

Proof. Recall that

$$f_k(\underline{x}) = g_k(\underline{t}) = \sum_{i\in[k]} c_i \exp\{\sum_{j=1}^{m} a_{ij} x_j\}, \quad \text{for } k = 0, 1, \ldots, p.$$

Let

$$y_i^n = \sum_{j=1}^{m} a_{ij} x_j^n, \quad \text{for } i = 1, 2, \ldots, n.$$

Suppose rank A = r, where $A = (a_{ij})_{n\times m}$, and the first r columns of A are linearly independent. Then $\exists\ \underline{\alpha}_t = (\alpha_{1t}, \alpha_{2t}, \ldots, \alpha_{rt})^T$, for t = r+1, r+2, ..., m, such that

$$a_{it} = \sum_{s=1}^{r} \alpha_{st} a_{is}.$$

So,

$$y_i^n = \sum_{j=1}^{m} a_{ij} x_j^n = \sum_{s=1}^{r} a_{is} x_s^n + \sum_{t=r+1}^{m} x_t^n \left(\sum_{s=1}^{r} \alpha_{st} a_{is} \right)$$

$$= \sum_{s=1}^{r} a_{is} x_s^n + \sum_{s=1}^{r} a_{is} \left(\sum_{t=r+1}^{m} \alpha_{st} x_t^n \right)$$

$$= \sum_{s=1}^{r} a_{is} \left(x_s^n + \sum_{t=r+1}^{m} \alpha_{st} x_t^n \right).$$

Let

$$z_s^n = x_s^n + \sum_{t=r+1}^{m} \alpha_{st} x_t^n.$$

Then

$$y_i^n = \sum_{s=1}^{r} a_{is} z_s^n, \quad \text{for } i \in [k],\ k = 0, 1, \ldots, p.$$

That (1.1) is consistent implies (1.3) is also consistent. So, $\exists\ \{\underline{x}^n\}$ where $\underline{x}^n = (x_1^{\,n},\ x_2^{\,n},\ \ldots,\ x_m^{\,n})^T$, n = 1, 2, ..., such that

$$f_k(\underline{x}^n) \le 1, \quad \text{for } k = 1, 2, \ldots, p,$$

and

$$f_0(\underline{x}^n) \to M_P \quad \text{as } n \to \infty.$$

In view of the fact that

$$f_k(\underline{x}^n) \ = \sum_{i\in[k]} c_i \exp\{\, y_i^{\,n} \,\} \ \le 1,$$

$$\sum_{i=1}^{n} d_i\, y_i^{\,n} = \sum_{i=1}^{n} d_i \left(\sum_{j=1}^{m} a_{ij}\, x_j^{\,n} \right)$$

$$= \sum_{j=1}^{m} x_j^{\,n} \left(\sum_{i=1}^{n} a_{ij}\, d_i \right) = 0 \quad \left(\sum_{i=1}^{n} a_{ij}\, d_i = 0,\ \exists\ d_i > 0 \text{ for some } i \right),$$

we know that $y_i^{\,n}$ must have upper and lower bounds. Note that $A_{.s}$ (the s-th column of matrix A), s = 1, 2, ..., r is independent of $A_{.t}$, s, t = 1, 2, ..., r, $s \ne t$, so $z_s^{\,n}$ must also be bounded. Hence, $\exists\ z_s^{\,*}$, for s = 1, 2, ..., r, such that $z_s^{\,n} \to z_s^{\,*}$ as $n \to \infty$.

Let

$$x_j^{\,*} = z_j^{\,*}, \quad \text{for } j = 1, 2, \ldots, r,$$

$$x_j^{\,*} = 0, \quad \text{for } j = r+1,\ r+2, \ldots, m.$$

Then $\exists\ y_i^{\,*}$ corresponding to $z_i^{\,*}$ such that

$$y_i^{\,*} = \sum_{s=1}^{r} a_{is}\, z_s^{\,*} = \sum_{j=1}^{m} a_{ij}\, x_j^{\,*}.$$

Now that

$$f_0(\underline{x}^*) \ = \sum_{i\in[0]} c_i \exp\{\, y_i^{\,*} \,\} = M_P\,,$$

$$f_k(\underline{x}^*) \ = \sum_{i\in[k]} c_i \exp\{\, y_i^{\,*} \,\} \le 1, \quad \text{for } k = 1, 2, \ldots, p,$$

thus, $t_j^{\,*} = \exp\{\, x_j^{\,*} \,\}$, for j = 1, 2, ..., m are the optimum for the problem (1.1). |

As a matter of fact, Corollary 1.2 (iii) claims if (1.2) has the optimum solution $\underline{d}^*$, then $\underline{t}$ that appears in Corollary 1.2 (iii) must be the the optimum solution for (1.1). These provide an indirect way for solving (1.1), that is, the dual method, in which the solution of (1.1) is found via solving (1.2). That is nice, because in general (1.2) is of only linear constraints and the solution for (1.2) is far easier to find than that of (1.1), which is highly nonlinear.

Definition 1.2. $\delta = n - (m + 1)$ is called degree of difficulty for (1.1) and (1.2). |

Normality and orthogonality conditions of (1.2) form a system of linear equations including n variables and (m+1) equations. So, if $\delta = 0$ and rank $A = m + 1$, where A is the coefficient matrix of the system, the system has unique solution $\underline{d}^*$. If $d_i^* < 0$ for some i ($1 \leq i \leq n$), which means d_i^* does not satisfy positivity condition, then $F_D = \phi$, and (1.1) has no constrained minimum solution. If $d_i^* \geq 0$ for $i = 1, 2, ..., n$, then $\underline{d}^* \in F_D \neq \phi$ is also the optimum solution of (1.2), and so the constrained minimum solution of (1.1), $\underline{t}^*$, can be found by Corollary 1.2 (iii). Conversely, what can be found by Corollary 1.2 (iii) is $\underline{t}^*$. Thus, for $\delta = 0$, there is no difficulty in solving (1.2).

However, for $\delta > 0$, the solution is not so easy to find. The system and (1.2) may have infinite many groups of feasible solutions, from which it is difficult to find the optimum. And generally, the larger δ is, the more difficult it is. That is why δ is called degree of difficulty.

For the case that $\delta > 0$, the general solution to the dual constraints can be written as

$$\underline{d}(\underline{y}) = \underline{\beta}^{(0)} + \sum_{r=1}^{\delta} \underline{\beta}^{(r)} y_r \geq \underline{0} ,$$

or in component form,

$$d_i(\underline{y}) = \beta_i^{(0)} + \sum_{r=1}^{\delta} \beta_i^{(r)} y_r \geq 0, \quad \text{for } i = 1, 2, ..., n,$$

where $\underline{y} = (y_1, y_2, ..., y_\delta)^T$, and y_r for $r = 1, 2, ..., \delta$ are basic variables. Vectors $\underline{\beta}^{(r)}$ for $r = 0, 1, ..., \delta$ are basic vectors, of which $\underline{\beta}^{(0)}$ is called normality vector, and $\underline{\beta}^{(r)}$ for $r = 1, 2, ..., \delta$ are called nullity vectors. So, in tems of basic variables and vectors the dual function has the form that

$$V(\underline{d}) = V(\underline{y}) = \prod_{i \in I} (c_i / d_i)^{d_i} \prod_{k=1}^{p} (\lambda_k)^{\lambda_k}$$

$$= \prod_{i=1}^{n} c_i^{\beta_i^{(0)}} \prod_{r=1}^{\delta} (\prod_{i=1}^{n} c_i^{\beta_i^{(r)}})^{y_r} \prod_{i=1}^{n} (d_i)^{-d_i} \prod_{k=1}^{p} [\lambda_k(\underline{d})]^{\lambda_k(\underline{d})} .$$

Let

$$C_0 = \prod_{i=1}^{n} c_i^{\beta_i^{(0)}},$$

$$C_r = \prod_{i=1}^{n} c_i^{\beta_i^{(r)}},$$

where C_r is called basic constant. Then

$$V(\underline{y}) = C_0 \prod_{r=1}^{\delta} C_r^{y_r} \prod_{i=1}^{n} [d_i(\underline{y})]^{-d_i(\underline{y})} \prod_{k=1}^{p} [\lambda_k(\underline{y})]^{\lambda_k(\underline{y})},$$

or in logarithmic form,

$$\log V(\underline{y}) = \log C_0 + \sum_{r=1}^{\delta} y_r \log C_r - \sum_{i=1}^{n} (d_i(\underline{y}) \log d_i(\underline{y})) + \sum_{k=1}^{p} \lambda_k(\underline{y}) (\log \lambda_k(\underline{y})).$$

Note that $\max V(\underline{y}) = V(\underline{y}^*)$ if and only if $\max \log V(\underline{y}) = \log V(\underline{y}^*)$, so the dual program becomes a concave program:

$$\text{maximize} \quad \log V(\underline{y}) = \log V (\underline{\beta}^{(0)} + \sum_{r=1}^{\delta} \underline{\beta}^{(r)} y_r),$$

$$\text{subject to} \quad \underline{y} \in F_{D1}, \qquad (1.8)$$

where

$$F_{D1} = \{ \underline{y} \mid \underline{d} = \underline{\beta}^{(0)} + \sum_{r=1}^{\delta} \underline{\beta}^{(r)} y_r \geq \underline{0} \}.$$

Theorem 1.8. (maximizing equations) Suppose $\underline{d} \in F_D$, $d_i \geq 0$, for i = 1, 2, ..., n. Then $\underline{d}(\underline{y}^*)$ is a maximizing point for (1.2) if and only if

$$C_r = \prod_{i=1}^{n} d_i(\underline{y}^*)^{\beta_i^{(r)}} \prod_{k=1}^{p} [\lambda_k(\underline{y}^*)]^{-\lambda_k^{(r)}}$$

where

$$C_r = \prod_{i=1}^{n} c_i^{\beta_i^{(r)}}, \quad r = 1, 2, ..., \delta.$$

Moreover,

$$V(\underline{y}^*) = C_0 \prod_{i=1}^{n} d_i(\underline{y}^*)\, d_i^{-\beta_i^{(0)}} \prod_{k=1}^{p} [\lambda_k(\underline{y}^*)]^{\lambda_k^{(0)}}.$$

Proof. Note the facts that

$$d_i(\underline{y}) = \beta_i^{(0)} + \sum_{r=1}^{\delta} \beta_i^{(r)} y_r,$$

$$\frac{\partial d_i(\underline{y})}{\partial y_r} = \beta_i^{(r)},$$

$$\frac{\partial}{\partial y_r}[\log d_i(\underline{y})] = \beta_i^{(r)} / d_i(\underline{y});$$

$$\lambda_k = \sum_{i\in[k]} d_i(\underline{y}) = \lambda_k^{(0)} + \sum_{r=1}^{\delta} \lambda_k^{(r)} y_r, \quad \text{where } \lambda_k^{(r)} = \sum_{i\in[k]} \beta_i^{(r)},$$

$$\frac{\partial \lambda_k(\underline{y})}{\partial y_r} = \sum_{i\in[k]} \beta_i^{(r)},$$

$$\frac{\partial}{\partial y_r}[\log \lambda_k(\underline{y})] = [\sum_{i\in[k]} \beta_i^{(r)}] / \lambda_k(\underline{y}).$$

So,

$$\frac{\partial}{\partial y_r}[\log V(\underline{y})] = \log C_r - \frac{\partial}{\partial y_r}[\sum_{i=1}^{n} d_i(\underline{y}) \log d_i(\underline{y})] + \frac{\partial}{\partial y_r}[\sum_{k=1}^{p} \lambda_k(\underline{y}) \log \lambda_k(\underline{y})],$$

where

$$\frac{\partial}{\partial y_r}\{\sum_{i=1}^{n}(d_i(\underline{y}) \log [d_i(\underline{y})]\} = \sum_{i=1}^{n}\{\frac{\partial}{\partial y_r}[d_i(\underline{y})] \log d_i(\underline{y}) + d_i(\underline{y})\frac{\partial}{\partial y_r}[\log d_i(\underline{y})]\}$$

$$= \sum_{i=1}^{n}\beta_i^{(r)} + \log [\prod_{i=1}^{n} d_i(\underline{y})^{\beta_i^{(r)}}],$$

$$\frac{\partial}{\partial y_r}[\sum_{k=1}^{p}\lambda_k(\underline{y}) \log \lambda_k(\underline{y})] = \sum_{k=1}^{p}\{\frac{\partial}{\partial y_r}[\lambda_k(\underline{y})] \log \lambda_k(\underline{y}) + \lambda_k(\underline{y})\frac{\partial}{\partial y_r}[\log \lambda_k(\underline{y})]\}$$

$$= \sum_{i=1}^{n}\beta_i^{(r)} + \log [\prod_{k=1}^{p} d_i(\underline{y})^{\lambda_k^{(r)}}].$$

That is,

$$\frac{\partial}{\partial y_r}[\log V(\underline{y})] = \log C_r - \log [\prod_{i=1}^{n} d_i(\underline{y})^{\beta_i^{(r)}} \prod_{i=1}^{n} \lambda_k(\underline{y})^{-\lambda_k^{(r)}}].$$

Suppose that $d_i(\underline{y}) > 0$ and $\frac{\partial}{\partial y_r}[\log V(\underline{y}^*)] = 0$. Then $\underline{y}^*$ is the stationary point for $\log [V(\underline{y})]$, and the maximum solution because of the concavity of (1.7). That is,

$$C_r = \prod_{i=1}^{n} d_i(\underline{y}^*)^{\beta_i^{(r)}} \prod_{k=1}^{p}[\lambda_k(\underline{y}^*)]^{-\lambda_k^{(r)}}$$

where

$$C_r = \prod_{i=1}^{n} c_i^{\beta_i^{(r)}}, \text{ for } r = 1, 2, ..., \delta.$$

Furthermore,

$$V(\underline{y}^*) = C_0 \prod_{r=1}^{\delta}[\prod_{i=1}^{n} d_i(\underline{y}^*)^{\beta_i^{(r)}} \prod_{k=1}^{p}[\lambda_k(\underline{y}^*)]^{-\lambda_k^{(r)}}]^{y_r^*}$$

$$\prod_{i=1}^{n}[d_i(\underline{y}^*)]^{-d_i(y^*)} \prod_{k=1}^{p}[\lambda_k(\underline{y}^*)]^{\lambda_k(\underline{y}^*)}$$

$$= C_0 \prod_{i=1}^{n} [\, d_i(\underline{y}^*) \,]^{-d_i(\underline{y}^*) + S_i} \prod_{k=1}^{p} [\, \lambda_k(\underline{y}^*) \,]^{\lambda_k(\underline{y}^*) - T_k}.$$

where $S_i = \sum_{r=1}^{\delta} \beta_i^{(r)} y_r$, $T_k = \sum_{r=1}^{\delta} \lambda_k^{(r)} y_r$.

Note that

$$d_i(\underline{y}^*) = \beta_i^{(0)} + \sum_{r=1}^{\delta} \beta_i^{(r)} y_r = \beta_i^{(0)} + S_i,$$

$$\lambda_k(\underline{y}^*) = \lambda_k^{(0)} + \sum_{r=1}^{\delta} \lambda_k^{(r)} y_r = \lambda_k^{(0)} - T_k.$$

So

$$- d_i(\underline{y}^*) + S_i = -\beta_i^{(0)},$$

$$\lambda_k(\underline{y}^*) - T_k = \lambda_k^{(0)}.$$

Therefore,

$$V(\underline{y}^*) = C_0 \prod_{i=1}^{n} [\, d_i(\underline{y}^*) \,]^{-\beta_i^{(0)}} \prod_{k=1}^{p} [\, \lambda_k(\underline{y}^*) \,]^{\lambda_k^{(0)}}.$$

|

In order to find basic vector $\underline{\beta}^{(r)}$, for r = 0, 1, ..., δ, consider the dual constraints that

$$\lambda_0 = \sum_{i \in [0]} d_i = 1$$

and

$$\sum_{i \in I} a_{ij} d_i = 0,$$

where

$$d_i(\underline{y}) = \beta_i^{(0)} + \sum_{r=1}^{\delta} \beta_i^{(r)} y_r.$$

That is,

$$\lambda_0 = \sum_{i\in[0]} \beta_i^{(0)} + \sum_{r=1}^{\delta} [\sum_{i\in[0]} \beta_i^{(r)}] \, y_r$$

$$= \lambda_0^{(0)} + \sum_{r=1}^{\delta} \lambda_0^{(r)} y_r = 1$$

and

$$\sum_{i\in I} a_{ij} [\beta_i^{(0)} + \sum_{r=1}^{\delta} \beta_i^{(r)} y_r] = 0.$$

So, we need

$$\sum_{i\in[0]} \beta_i^{(0)} = 1,$$

$$\sum_{i\in[0]} \beta_i^{(r)} = 0, \quad \text{for } r = 1, 2, \ldots, \delta,$$

and

$$\sum_{i\in I} a_{ij} \beta_i^{(r)} = 0, \quad \text{for } r = 1, 2, \ldots, \delta,\ j = 1, 2, \ldots, m.$$

Let P_{kl} denote the elementary matrix got by exchanging row k and row l with $I = (e_{ij}) = 1$ if $i = j$, or $= 0$ if $i \neq j$, and P_e denote elementary matrix with proper row transformation with I such that

$$P_e \, A^T P_{kl} = A_1^T = (I \quad A_2^T).$$

Let

$$B_1 = \begin{pmatrix} -A_2^T \\ I \end{pmatrix}.$$

Then

$$A_1^T B_1 = (I \quad A_2^T) \begin{pmatrix} -A_2^T \\ I \end{pmatrix} = 0.$$

Let

$$B = P_{kl} B_1.$$

Then

$$(P_e \, A^T P_{kl}) (P_{kl} B) = 0 \quad (P_{kl}^{\,2} = I),$$

$$A^T B = 0.$$

So, if let

$$B = (b_{ij}) = (\underline{B}_{.1}, \underline{B}_{.2}, ..., \underline{B}_{.\delta}),$$

then

$$A^T \underline{B}_{.r} = \underline{0}, \quad \text{for } r = 1, 2, ..., \delta,$$

i.e., that $\sum_{i=1}^{n} a_{ij} b_{ir} = 0$ is satisfied. If let

$$k_r = \sum_{i\in[0]} b_{ir}, \quad k_1 = \sum_{i\in[0]} b_{i1} \quad (\text{for } r = 1),$$

$$\underline{\beta}^{(0)} = \underline{B}_{.1} / k_1, \quad \text{or } \beta_i^{(0)} = b_{i1} / k_1,$$

$$\underline{\beta}^{(r)} = \underline{B}_{.(r+1)} - k_{r+1} \underline{\beta}^{[0]}, \quad \text{or } \beta_i^{(r)} = b_{i,(r+1)} - k_{r+1} \beta_i^{(0)},$$

then $\sum_{i\in[0]} \beta_i^{(0)} = 1$, $\sum_{i\in[0]} \beta_i^{(r)} = 0$, and $\sum_{i=1}^{n} a_{ij} \beta_i^{(r)} = 0$, for $r = 1, 2, ..., \delta, j = 1, 2, ...,$ m are satisfied.

The following three illustrative examples are given to show how the dual method is used to solve the geometric programming problem.

Example 1.1. Find the solution of the problem that

minimize $g_0(\underline{t}) = (1/2)\, t_1 t_2^2 t_3 + 3\, (t_1)^{-1} t_2 t_3^2,$

subject to $g_1(\underline{t}) = 3\, t_1 (t_3)^{-2} + 2\, (t_2)^{-2} (t_3)^{-1} \le 1,$

$t_j > 0, \quad \text{for } j = 1, 2, 3.$

Solution. The degree of difficulty of the problem is $\delta = 4 - (3 + 1) = 0$.

The dual function is:

$$V(\underline{d}) = \left(\frac{1/2}{d_1}\right)^{d_1} \left(\frac{3}{d_2}\right)^{d_2} \left(\frac{3}{d_3}\right)^{d_3} \left(\frac{2}{d_4}\right)^{d_4} (d_3 + d_4)^{d_3 + d_4}$$

and the dual constraints are:

$$d_1 + d_2 = 1,$$
$$d_1 - d_2 + d_3 = 0,$$
$$2d_1 + d_2 - 2d_4 = 0,$$
$$d_1 + 2d_2 - 2d_3 - d_4 = 0.$$
$$d_3 + d_4 = \lambda_1 .$$

So, $d_1^* = 1/5$, $d_2^* = 4/5$, $d_3^* = 3/5$, $d_4^* = 3/5$, $\lambda_1 = 6/5$, and

$$g_0(\underline{t}^*) = V(\underline{d}^*)$$

$$= \left(\frac{1/2}{1/5}\right)^{1/5} \left(\frac{3}{4/5}\right)^{4/5} \left(\frac{3}{3/5}\right)^{3/5} \left(\frac{2}{3/5}\right)^{3/5} (6/5)^{6/5} = 5\,(3^{7/5}).$$

By Corollary 1.3 (iii), we know that

$$u_i(\underline{t}) = d_i^* \; V(\underline{d}^*), \quad \text{for } i \in [0],$$
$$u_i(\underline{t}) = d_i^* / \lambda_k , \quad \text{for } i \in [k].$$

That is,

$$u_i(\underline{t}) = (1/2)\, t_1\, t_2^2\, t_3 = (1/5)\, 5\, (3^{7/5}) = 3^{7/5} ,$$

$$u_i(\underline{t}) = 3\, (t_1)^{-1}\, t_2\, t_3^2 = (4/5)\, (3^{7/5}) = 4\, (3^{7/5}) ,$$

$$u_i(\underline{t}) = 3\, t_1\, (t_3)^{-2} = \frac{3/5}{6/5} = 1/2,$$

$$u_i(\underline{t}) = 2\, (t_2)^{-2}\, (t_3)^{-1} = \frac{3/5}{6/5} = 1/2.$$

Thus, the solution for the problem is $t_1^* = (1/2)(3^{7/5})$, $t_2^* = 2\,(3^{-3/5})$, $t_3^* = 3^{6/5}$.

Example 1.2. Solve the problem that

minimize $g_0(\underline{t}) = 6\,(t_1)^{-2}\,(t_2)^3\,(t_3)^{-1} + (1/2)\,t_1\,(t_3)^{1/2} + 4\,(t_1)^{-1/2}\,(t_2)^{-1}(t_3)^{-1/4}$,

subject to $g_1(\underline{t}) = 3\,(t_1)^{-1}\,(t_2)^2 + 2\,(t_2)^{1/2}\,(t_3)^{-1} \le 1$,

$t_j > 0$, for $j = 1, 2, 3$.

Solution. $\delta = 5 - (3 + 1) = 1$.

$$V(\underline{d}) = \left(\frac{6}{d_1}\right)^{d_1}\left(\frac{1/2}{d_2}\right)^{d_2}\left(\frac{4}{d_3}\right)^{d_3}\left(\frac{3}{d_4}\right)^{d_4}\left(\frac{2}{d_5}\right)^{d_5}(d_4+d_5)^{d_4+d_5}.$$

$$A^T = \begin{pmatrix} -2 & 1 & -1/2 & -1 & 0 \\ 3 & 0 & -1 & 2 & 1/2 \\ -1 & 1/2 & -1/4 & 0 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & -1/3 & 3/4 & 0 \\ 0 & 1 & -7/6 & 1/2 & 0 \\ 0 & 0 & 0 & -1/2 & 1 \end{pmatrix} = (I \quad A_2^T) = A \cdot$$

$$B_1 = \begin{pmatrix} -A_2^T \\ I \end{pmatrix} = \begin{pmatrix} -3/4 & 1/3 \\ -1/2 & 7/6 \\ 1/2 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} -3/4 & 1/3 \\ -1/2 & 7/6 \\ 0 & 1 \\ 1 & 0 \\ 1/2 & 0 \end{pmatrix},$$

$k_1 = -5/4, \quad k_2 = 5/2.$

$$\underline{\beta}^{(0)} = B_{\cdot 1}/k_1 = (-4/5)\begin{pmatrix} -3/4 \\ -1/2 \\ 0 \\ 1 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 2/5 \\ 0 \\ -4/5 \\ -2/3 \end{pmatrix},$$

$$\underline{\beta}^{(1)} = B_{\cdot 2} - k_2\underline{\beta}^{(0)} = \begin{pmatrix} 1/3 \\ 7/6 \\ 1 \\ 0 \\ 0 \end{pmatrix} - (5/2)\begin{pmatrix} 3/5 \\ 2/5 \\ 0 \\ -4/5 \\ -2/3 \end{pmatrix} = \begin{pmatrix} -7/6 \\ 1/6 \\ 1 \\ 2 \\ 1 \end{pmatrix},$$

$$\underline{d}(y) = \underline{\beta}^{(0)} + \underline{\beta}^{(1)} y = \begin{pmatrix} 3/5-(7/6)y \\ 2/5-(1/6)y \\ y \\ -4/5+2y \\ -2/6+y \end{pmatrix}.$$

For $\underline{d} \geq \underline{0}$, we have $y \in [2/5, 18/35]$.

The dual problem becomes:

$$\text{maximize } V(y) = \left(\frac{6}{3/5-7y/6}\right)^{3/5-7y/6}\left(\frac{1/2}{2/5+y/6}\right)^{2/5+y/6}\left(\frac{4}{y}\right)^{y}$$

$$\left(\frac{3}{-4/5+2y}\right)^{-4/5+2y}\left(\frac{2}{-2/5+y}\right)^{-2/5+y}(-6/5+3y)^{-6/5+3y},$$

subject to $y \in [\,2/5, 18/35\,]$.

Trial. Let

$y_1 = 2/5 = 0.4$, $\underline{d}(y_1) = (2/15, 7/15, 2/5, 0, 0)^T$,

$y_2 = 17/35 = 0.486$, $\underline{d}(y_2) = (1/30, 101/210, 17/35, 6/35, 3/35)^T$,

$y_3 = 1/2 = 0.5$, $\underline{d}(y_3) = (1/60, 29/60, 1/2, 1/5, 1/10)^T$,

$y_4 = 18/35 = 0.514$, $\underline{d}(y_4) = (0, 17/35, 18/35, 8/35, 4/35)^T$.

Then

$V(y_1) = V(\underline{d}(y_1)) = 4.308$,

$V(y_2) = V(\underline{d}(y_2)) = 5.091$,

$V(y_3) = V(\underline{d}(y_3)) = 5.123$ (max),

$V(y_3) = V(\underline{d}(y_3)) = 5.041$.

Let $t_1 = t_3 = 4$. Then

$g_1(t_2) = (3/4)\, t_2^{\,2} + (1/2)\, t_2^{\,1/2} < 1$ if $t_3 \in (0, 0.845]$,

$g_0(t_2) = (0.094)\, t_2^{\,3} + 4 + (1.4142)\, t_2^{\,-1}$.

Take $t_2 = 0.5, 0.6, 0.825, 0.83, 0.84, 0.845$. Then $g_0(t_2) = 6.84, 6.377, 5.767, 5.758, 5.739$ (min), 5.829, respectively. Note that by Theorem 1.4 we know for all the feasible solution $\underline{t}$ and $\underline{d}$, it holds that

$V(\underline{d}) \leq g_0(\underline{t}^*) \leq g_0(\underline{t})$.

Therefore, the trial above gives the upper and lower bounds for $g_0(\underline{t}^*)$, that is,

$5.123 \leq g_0(\underline{t}^*) \leq 5.739$,

or,

$g_0(\underline{t}^*) \in [\, 5.431-0.308,\ 5.431+0.308\,]$.

Example 1.3. Consider the solution of the problem that

minimize $g_0(\underline{t}) = 12\, t_1 t_2 + 5\, t_1 (t_2)^{2/3} (t_3)^3 + 8\, t_1 (t_2)^{2/3} (t_3)^3 (t_4)$,

subject to $g_1(\underline{t}) = 75\, (t_1)^{-1} (t_2)^{-1} (t_3)^{-1} (t_4)^{-1} + 2\, (t_2)^{-1/3} (t_3)^2 \leq 1$,

$g_2(\underline{t}) = 67\, t_2 t_3 t_4 + 3\, t_4 \leq 1$,

$t_j > 0$, for $j = 1, 2, 3, 4$.

Solution. $\delta = 7 - (4 + 1) = 2$.

$$A^T = \begin{pmatrix} 1 & 1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 2/3 & 2/3 & -1 & -1/3 & 1 & 0 \\ 0 & 3 & 3 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -7 & 0 \\ 0 & 1 & 0 & 0 & 0 & 6 & -1 \\ 0 & 0 & 1 & 0 & 1 & -9 & 1 \\ 0 & 0 & 0 & 1 & 1 & -10 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 7 & 0 \\ 0 & -6 & 1 \\ -1 & 9 & -1 \\ -1 & 10 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$k_1 = -1, \quad k_2 = 10, \quad k_3 = 0.$

$\underline{\beta}^{(0)} = (0, 0, 1, 1, -1, 0, 0))^T,$
$\underline{\beta}^{(1)} = (7, -6, -1, 0, 10, 1, 0)^T,$
$\underline{\beta}^{(2)} = (0, 1, -1, 0, 0, 0, 1)^T.$

$$\underline{d}(\underline{y}) = \begin{pmatrix} 7y_1 \\ -6y_1+y_2 \\ 1-y_1-y_2 \\ 1 \\ -1+10y_1 \\ y_1 \\ y_2 \end{pmatrix} \geq \underline{0}.$$

The dual program becomes:

$$\text{maximize} \quad V(\underline{y}) = \left(\frac{12}{7y_1}\right)^{7y_1} \left(\frac{5}{-6y_1+y_2}\right)^{-6y_1+y_2} \left(\frac{8}{1-y_1-y_2}\right)^{-y_1-y_2} \quad (75)$$

$$\left(\frac{2}{-1+10y_1}\right)^{-1+10y_1} \left(\frac{67}{y_1}\right)^{y_1} (3/y_2)^{y_2} (10y_1)^{10y_1} (y_1+y_2)^{y_1+y_2},$$

subject to $y_1 \in [\, 1/10, (1/6)\, y_2 \,]$,

$(y_1 + y_2) \leq 1.$

CHAPTER II

THE PRIMAL METHOD

This chapter is devoted to the primal method for solving the posynomial geometric programming problem. The presentation concentrates on condensation (an effective way of transforming problem (1.1) to a monomial geometric program) and "cutting-plane" algorithm. The main content can be found in [1] and [3].

Consider the inequality that for $e_i(t) \geq 0$, $\sum_{i \in [k]} e_i(t) = 1$, $[k] \in I = \{1, 2, \ldots, n\}$,

$$\prod_{i \in [k]} (u_i(t) / e_i(t))^{e_i(t)} \leq \sum_{i \in [k]} u_i(t),$$

where

$$u_i(t) = c_i \prod_{j=1}^{m} t_j^{a_{ij}}, \quad \text{for } t_j > 0,\ j = 1, 2, \ldots, m.$$

Note that the left side of the inequality is monomial. Furthermore, we have

$$\begin{aligned}
\prod_{i \in [k]} [\, u_i(t) / e_i(t) \,]^{e_i(t)} &= \prod_{i \in [k]} [\, (c_i / e_i(t)) \prod_{j=1}^{m} t_j^{a_{ij}} \,]^{e_i(t)} \\
&= \prod_{i \in [k]} [\, c_i / e_i(t) \,]^{e_i(t)} \prod_{j=1}^{m} \prod_{i \in [k]} t_j^{a_{ij} e_i(t)} \\
&= c_k(t) \prod_{j=1}^{m} t_j^{a_{kj}(t)},
\end{aligned}$$

where

$$c_k(t) = \prod_{i \in [k]} [\, c_i / e_i(t) \,]^{e_i(t)},$$

$$a_{kj}(t) = \sum_{i \in [k]} a_{ij}\, e_i(t).$$

Let

$$g_k(t) = \sum_{i \in [k]} u_i(t),$$

$$g_k(t, e(t)) = \prod_{i \in [k]} [\, u_i(t) / e_i(t) \,]^{e_i(t)} = c_k(t) \prod_{j=1}^{m} t_j^{a_{kj}(t)}, \quad \text{for } k = 0, 1, \ldots, p.$$

Then we get a monomial geometric program related to (1.1). That is,

$$\text{minimize} \quad g_0(t, e(t)),$$

$$\text{subject to} \quad t \in F_{Pe}, \tag{2.1}$$

where

$$F_{Pe} = \{\, t \mid g_k(t, e(t)) \le 1, \quad \text{for } k = 1, 2, \ldots, p,$$
$$t_j > 0, \quad \text{for } j = 1, 2, \ldots, m \,\},$$

which is called the condensed program of (1.1).

Let

$$x_j = \log t_j, \quad \text{for } j = 1, 2, \ldots, m,$$

$$C_k(t) = \log c_k(t) = \sum_{i \in [k]} e_i(t) [\, \log c_i - \log e_i(t) \,], \quad \text{for } k = 0, 1, \ldots, p,$$

$$G_k(x, e(t)) = \log g_k(t, e(t)) = C_k(t) + \sum_{j=1}^{m} a_{kj}(t)\, x_j, \quad \text{for } k = 0, 1, \ldots, p.$$

Then the transformed linear program of (2.1) can be written as:

$$\text{minimize} \quad G_0(x, e(t)),$$

$$\text{subject to} \quad x \in F_{PL}, \tag{2.2}$$

where

$$F_{PL} = \{\, x \mid G_k(x, e(t)) \le 0, \quad \text{for } k = 1, 2, \ldots, p \,\}.$$

The associated dual program of (2.1) is:

$$\text{maximize} \quad V(\lambda) = \prod_{k=0}^{n} (c_k(t))^{\lambda_k},$$

$$\text{subject to} \quad \lambda \in F_{De}, \tag{2.3}$$

where

$$F_{De} = \{\, \lambda \mid \sum_{k=1}^{p} a_{kj}(t)\, \lambda_k = 0, \ \lambda_0 = 1, \ \lambda_k \ge 0, \quad \text{for } k = 1, 2, \ldots, p \,\},$$

$$c_k(\underline{t}) = \prod_{i\in[k]} [\, c_i / e_i(\underline{t}) \,]^{\, e_i(\underline{t})},$$

$$a_{kj}(\underline{t}) = \sum_{i\in[k]} a_{ij}\, e_i(\underline{t}).$$

The transformed linear program of (2.3) can be obtained by taking logarithmic transformation of the objective function. That is,

$$\text{maximize} \quad \log V(\underline{\lambda}) = \sum_{k=0}^{n} \lambda_k \log c_k(\underline{t}),$$

$$\text{subject to} \quad \lambda_k \in F_{DL}, \qquad (2.4)$$

where

$$F_{DL} = \{\, \underline{\lambda} \mid \sum_{k=1}^{n} a_{kj}(\underline{t})\, \lambda_k = 0,\ \lambda_0 = 1,\ \lambda_k \geq 0,\ \text{for } k = 1, 2, \ldots, p \,\},$$

$$c_k(\underline{t}) = \prod_{i\in[k]} [\, c_i / e_i(\underline{t}) \,]^{\, e_i(\underline{t})},$$

$$a_{kj}(\underline{t}) = \sum_{i\in[k]} a_{ij} e_i(\underline{t}).$$

Let

$$F_P = \{\, \underline{t} \mid g_k(\underline{t}) \leq 1, \ \text{for } t_j > 0,\ j = 1, 2, \ldots, m \,\},$$

$$F_{Pe} = \{\, \underline{t} \mid g_k(\underline{t}, \underline{e}(\underline{t})) \leq 1, \ \text{for } t_j > 0,\ j = 1, 2, \ldots, m \,\},$$

$$M_P = \inf g_0(\underline{t}), \ \text{for all } \underline{t} \in F_P,$$

$$M_{Pe} = \inf g_0(\underline{t}, \underline{e}(\underline{t})), \ \text{for all } \underline{t} \in F_{Pe}.$$

Then we have the following theorems that reveal some useful properties for using (2.1) to find the solution of (1.1).

Theorem 2.1. $F_P \subset F_{Pe}$.

Proof. By geometric inequality, for all $\underline{t} \geq \underline{0}$,

$$g_k(\underline{t}, \underline{e}(\underline{t})) \leq g_k(\underline{t}), \ \text{for } k = 1, 2, \ldots, p,$$

that is,

$$\prod_{i\in[k]} [\, u_i(\underline{t}) / e_i(\underline{t})\,]^{\,e_i(\underline{t})} \le \sum_{i\in[k]} u_i(\underline{t}).$$

Clearly,

$$g_k(\underline{t}, \underline{e}(\underline{t})) \le 1, \quad \text{if } g_k(\underline{t}) \le 1,$$

and the converse is not necessarily true. That is, $\underline{t} \in F_{P_e}$ for all $\underline{t} \in F_P$ and it is possible that $\exists\ \underline{t}^1 \in F_{P_e}$ such that $\underline{t}^1 \notin F_P$. Hence, $F_P \subset F_{P_e}$. |

Theorem 2.2. $M_{P_e} \le M_P$.

proof. Suppose that $M_p < M_{pe}$, and there exists $\underline{t}^* \in F_P$ such that $g_0(\underline{t}^*) = \inf g_0(\underline{t}) = M_P$ for all $\underline{t} \in F_P$. Then $g_0(\underline{t}^*) < \inf g_0(\underline{t}, \underline{e}(\underline{t})) = M_{P_e}$ for all $\underline{t} \in F_{P_e}$. Note that $\underline{t}^* \in F_{P_e}$. Since $F_P \subset F_{P_e}$, we get

$$g_0(\underline{t}^*) < g_0(\underline{t}, \underline{e}(\underline{t})), \quad \text{for } \forall\ \underline{t} \in F_{P_e}.$$

That is,

$$\sum_{i\in[0]} u_i(\underline{t}^*) < \prod_{i\in[0]} [\, u_i(\underline{t}) / e_i(\underline{t})\,]^{\,e_i(t)}, \quad \text{for } \forall\ \underline{t} \in F_{P_e}.$$

That contradicts Theorem 1.3. |

Theorem 2.3. For $e_i(\underline{t}') = u_i(\underline{t}') / g_k(\underline{t}')$, $i \in k$, $k = 1, 2, \ldots, p$, it holds that

$$g_k(\underline{t}, \underline{e}(\underline{t}')) = g_k(\underline{t}'), \quad \text{for } \underline{t} = \underline{t}'.$$

Proof. Now that

$$e_i(\underline{t}') = u_i(\underline{t}') / g_k(\underline{t}'),$$

$$\sum_{i\in[k]} e_i(\underline{t}') = \sum_{i\in[k]} u_i(\underline{t}') / g_k(\underline{t}') = 1,$$

so,

$$\begin{aligned} g_k(\underline{t}', \underline{e}(\underline{t}')) &= g_k(\underline{t}, \underline{e}(\underline{t}'))\,|_{\underline{t}=\underline{t}'} \\ &= \prod_{i\in[k]} [\, u_i(\underline{t}) / e_i(\underline{t}')\,]^{\,e_i(\underline{t}')}\,|_{\underline{t}=\underline{t}'} \\ &= \prod_{i\in[k]} [\, u_i(\underline{t})\, g_k(\underline{t}') / u_i(\underline{t}')\,]^{\,e(\underline{t}')}\,|_{\underline{t}=\underline{t}'} \qquad (e_i(\underline{t}') = u_i(\underline{t}') / g_k(\underline{t}')) \\ &= \prod_{i\in[k]} [\, u_i(\underline{t}')\, g_k(\underline{t}') / u_i(\underline{t}')\,]^{\,e(\underline{t}')} \end{aligned}$$

$$= [\, g_k(\underline{t}') \,]^{E} \quad (\text{where } E = \sum_{i \in [k]} e_i(\underline{t}) = 1)$$

$$= g_k(\underline{t}')$$

Lemma 2.1. Suppose that

$$e_i(\underline{t}') = u_i(\underline{t}') \,/\, g_k(\underline{t}') = u_i(\underline{t}') \,/ \sum_{i \in [k]} u_i(\underline{t}'), \quad \text{for } i \in [k],$$

$$g_k(\underline{t}, \underline{e}(\underline{t}')) = \prod_{i \in [k]} [\, u_i(\underline{t}) \,/\, e_i(\underline{t}') \,]^{e_i(\underline{t}')},$$

then for $j = 1, 2, \ldots, m$, $\underline{t} = (t_1, t_2, \ldots, t_m)^T$,

$$\frac{\partial}{\partial t_j} [g_k(\underline{t}, \underline{e}(\underline{t}'))] = \sum_{i \in [k]} \{\, [\, e_i(\underline{t}') \,/\, u_i(\underline{t}) \,]\, [\, g_k(\underline{t}, \underline{e}(\underline{t}')) \,]\, \frac{\partial u_i(\underline{t})}{\partial t_j} \}.$$

Proof. For $[k] = \{1\}$,

$$\frac{\partial}{\partial t_j} [g_k(\underline{t}, \underline{e}(\underline{t}'))] = \frac{\partial}{\partial t_j} [\, u_i(\underline{t}) \,/\, e_i(\underline{t}') \,]^{e_i(\underline{t}')}$$

$$= [\, e_i(\underline{t}') \,]^{-e_i(\underline{t}')} \frac{\partial}{\partial t_j} [\, u_i(\underline{t}) \,]^{e_i(\underline{t}')}$$

$$= [\, e_i(\underline{t}') \,]^{1 - e_i(\underline{t}')} [\, u_i(\underline{t}) \,]^{e_i(\underline{t}') - 1} \frac{\partial u_i(\underline{t})}{\partial t_j}$$

$$= [\, e_i(\underline{t}') \,/\, u_i(\underline{t}) \,]\, [\, u_i(\underline{t}) \,/\, e_i(\underline{t}') \,]^{e_i(\underline{t}')} \frac{\partial u_i(\underline{t})}{\partial t_j}.$$

For $[k] = \{1, 2\}$,

$$\frac{\partial}{\partial t_j} [g_k(\underline{t}, \underline{e}(\underline{t}'))] = \frac{\partial}{\partial t_j} [\prod_{i=1}^{2} (\, u_i(\underline{t}) \,/\, e_i(\underline{t}') \,)^{e_i(\underline{t}')} \,]$$

$$= [\, u_2(\underline{t}) \,/\, e_2(\underline{t}') \,]^{e_2(\underline{t}')} \frac{\partial}{\partial t_j} [\, (u_1(\underline{t}) \,/\, e_1(\underline{t}'))^{e_1(\underline{t}')} \,]$$

$$+ \; [\, u_1(\underline{t}) \,/\, e_1(\underline{t}') \,]^{e_1(\underline{t}')} \frac{\partial}{\partial t_j} [\, (u_2(\underline{t}) \,/\, e_2(\underline{t}'))^{e_2(\underline{t}')} \,]$$

$$= \sum_{i=1}^{2} \{ [e_i(\underline{t}') / u_i(\underline{t})] [u_i(\underline{t}) / e_i(\underline{t}')]^{e_i(\underline{t}')} \frac{\partial u_i(\underline{t})}{\partial t_j} \}$$

$$[u_\xi(\underline{t}) / e_\xi(\underline{t}')]^{e_\xi(\underline{t}')} \qquad (\text{for } \xi \in [k],\ \xi \neq i)$$

$$= \sum_{i=1}^{2} \{ [e_i(\underline{t}') / u_i(\underline{t})] [\prod_{i=1}^{2} (u_i(\underline{t}) / e_i(\underline{t}'))^{e_i(\underline{t}')}] \frac{\partial u_i(\underline{t})}{\partial t_j} \}.$$

Assume that for $[k] = \{1, 2, \ldots, n-1\}$, it is true that

$$\frac{\partial}{\partial t_j} [g_k(\underline{t}, \underline{e}(\underline{t}'))] = \sum_{i=1}^{n-1} \{ [e_i(\underline{t}') / u_i(\underline{t})] [\prod_{i=1}^{n-1} (u_i(\underline{t}) / e_i(\underline{t}'))^{e_i(\underline{t}')}] \frac{\partial u_i(\underline{t})}{\partial t_j} \}.$$

Then for $[k] = \{1, 2, \ldots, n-1, n\}$,

$$\frac{\partial}{\partial t_j} [g_k(\underline{t}, \underline{e}(\underline{t}'))] = \frac{\partial}{\partial t_j} [\prod_{i=1}^{n} (u_i(\underline{t}) / e_i(\underline{t}'))^{e_i(\underline{t}')}]$$

$$= (u_n(\underline{t}) / e_n(\underline{t}'))^{e_n(\underline{t}')} \frac{\partial}{\partial t_j} [\prod_{i=1}^{n-1} (u_i(\underline{t}) / e_i(\underline{t}'))^{e_i(\underline{t}')}]$$

$$+ \prod_{i=1}^{n-1} (u_i(\underline{t}) / e_i(\underline{t}'))^{e_i(\underline{t}')} \frac{\partial}{\partial t_j} [(u_n(\underline{t}) / e_n(\underline{t}')]^{e_n(\underline{t}')}]$$

$$= \sum_{i=1}^{n-1} \{ [e_i(\underline{t}') / u_i(\underline{t})] [\prod_{i=1}^{n} (u_i(\underline{t}) / e_i(\underline{t}'))^{e_i(\underline{t}')}] \frac{\partial u_i(\underline{t})}{\partial t_j} \}$$

$$+ \{ [e_n(\underline{t}') / u_n(\underline{t})] [\prod_{i=1}^{n} (u_i(\underline{t}) / e_i(\underline{t}'))^{e_i(\underline{t}')}] \frac{\partial u_i(\underline{t})}{\partial t_j} \}$$

$$= \sum_{i=1}^{n} \{ [e_i(\underline{t}') / u_i(\underline{t})] [\prod_{i=1}^{n} (u_i(\underline{t}) / e_i(\underline{t}'))^{e_i(\underline{t}')}] \frac{\partial u_i(\underline{t})}{\partial t_j} \}.$$

Therefore, we obtain that for $j = 1, 2, \ldots, m$,

$$\frac{\partial}{\partial t_j} [g_k(\underline{t}, \underline{e}(\underline{t}'))] = \sum_{i \in [k]} \{ [e_i(\underline{t}') / u_i(\underline{t})] g_k(\underline{t}, \underline{e}(\underline{t}')) \frac{\partial u_i(\underline{t})}{\partial t_j} \}.$$ |

Theorem 2.4. $\nabla_{\underline{t}} [g_k(\underline{t}', \underline{e}(\underline{t}'))] = \nabla_{\underline{t}}\, g_k(\underline{t}')$.

Proof. Note that $e_i(\underline{t}') = u_i(\underline{t}') / g_k(\underline{t}')$, for $i \in [k]$,

$$[\, e_i(\underline{t}') / u_i(\underline{t}) \,] \, [\, \prod_{i \in [k]} (u_i(\underline{t}) / e_i(\underline{t}'))^{\,e_i(\underline{t}')} \,] \, \frac{\partial u_i(\underline{t})}{\partial t_j} \Big|_{\underline{t} = \underline{t}'} =$$

$$= [\, e_i(\underline{t}') / u_i(\underline{t}') \,] \, [\, g_k(\underline{t}') \,]^{E'} \frac{\partial u_i(\underline{t}')}{\partial t_j} \qquad (\text{where } E' = \sum_{i \in [k]} e_i(\underline{t}') = 1)$$

$$= \frac{\partial u_i(\underline{t}')}{\partial t_j} \qquad (e_i(\underline{t}') / u_i(\underline{t}') = 1 / g_k(\underline{t}')).$$

That is, by Lemma 2.1, for all $j = 1, 2, \ldots, m$,

$$\frac{\partial}{\partial t_j} [\, g_k(\underline{t}', \underline{e}(\underline{t}')) \,] = \sum_{i \in [k]} [\, \frac{\partial}{\partial t_j} u_i(\underline{t}') \,] = \frac{\partial}{\partial t_j} [\, \sum_{i \in [k]} u_i(\underline{t}') \,]$$

$$= \frac{\partial}{\partial t_j} [\, g_k(\underline{t}') \,].$$

Therefore, it holds that

$$\nabla_{\underline{t}} [\, g_k(\underline{t}', \underline{e}(\underline{t}')) \,] = \nabla_{\underline{t}} \, g_k(\underline{t}'). \qquad \blacksquare$$

The next theorem gives the relation between the optimum solution of (2.1) and that of (1.2).

Theorem 2.5. Suppose F_{Pe} is nonempty, then $M_{Pe} > 0$ if and only if there exists $\underline{d}^* \in F_D$ such that for $e_i \in (0,1)$,

$$e_i = d_i^* / \sum_{i \in [k]} d_i^*,$$

$$M_{Pe} = V(\underline{d}^*).$$

Proof. (For necessity) Suppose that $F_{Pe} \neq \phi$, and $M_{Pe} > 0$, then

$$M_{PL} = \inf G_0(\underline{x}, \underline{e}(\underline{t})) \qquad (\text{for all } \underline{x} \in F_{PL})$$

$$= \inf \log g_0(\underline{t}) \qquad (\text{for all } \underline{t} \in F_{Pe})$$

$$= \log \inf g_0(\underline{t}) \qquad (\text{for all } \underline{t} \in F_{Pe})$$

$= \log M_{Pe}.$

$M_{DL} = \sup \log V(\lambda) \quad (\text{for all } \lambda \in F_{DL})$

$= \log M_{De}.$

By the duality theorem for linear programming, we know $M_{DL} = M_{PL}$, So,

$$M_{De} = \sup V(\lambda) = M_{Pe} \quad (\text{for all } \lambda \in F_{De})$$

Since

$$V(\lambda^*) = \sup V(\lambda) \quad (\text{for all } \lambda \in F_{De})$$
$$= \prod_{k=0}^{p} (c_k(t))^{\lambda_k^*},$$

we have

$$M_{Pe} = \prod_{k=0}^{p} (c_k(t))^{\lambda_k^*}$$
$$= \prod_{k=0}^{p} \prod_{i \in [k]} (c_i / e_i)^{e_i \lambda_k^*},$$

where λ_k^* is implied to be consistent optimum solution for (2.3) and (2.4).

Define $\underline{d}^*$ as

$$d_i^* = e_i \lambda_k^*, \quad \text{for } i \in [k],\ k = 0, 1, \ldots, p,$$

$$\sum_{i \in [k]} d_i^* = \sum_{i \in [k]} e_i \lambda_k^* = \lambda_k^* \qquad (\sum_{i \in [k]} e_i = 1 \text{ for } k = 0, 1, \ldots, p).$$

That is,

$$d_i^* = e_i \sum_{i \in [k]} d_i^*, \text{ i.e., } e_i = d_i^* \Big/ \sum_{i \in [k]} d_i^*.$$

Hence,

$$M_{Pe} = \prod_{k=0}^{p} \prod_{i \in [k]} (c_i / e_i)^{d_i^*}$$

$$= \prod_{k=0}^{p} [\prod_{i \in [k]} (c_i \sum_{i \in [k]} d_i^* / d_i^*)^{d_i^*}$$

$$= \prod_{k=0}^{p} \prod_{i \in [k]} (c_i / d_i^*)^{d_i^*} \prod_{k=0}^{p} [\prod_{i \in [k]} (\sum_{i \in [k]} d_i^*)^{d_i^*}]$$

$$= \prod_{k=0}^{p} \prod_{i\in[k]} (c_i/d_i^*)^{d_i^*} \prod_{k=0}^{p} (\lambda_k^*)^{\lambda_k^*}$$

$$= V(\underline{d}^*) .$$

Since $e_i \geq 0,\ \lambda_k^* \geq 0,\ \lambda_0^* = 1$, we have

$$d_i^* \geq 0, \quad \sum_{i\in[0]} d_i^* = 1,$$

$$\sum_{i=1}^{n} a_{ij} d_i^* = \sum_{k=1}^{p} \Big(\sum_{i\in[k]} a_{ij} e_i \Big) \lambda_k^* \quad (d_i^* = e_i \lambda_k^*)$$

$$= \sum_{k=1}^{p} a_{kj}(\underline{t})\, \lambda_k^* = 0, \ \text{for } j = 1, 2, \ldots, m.$$

Hence, $\underline{d}^* \in F_D$.

(For sufficiency) Suppose that $\underline{d}^* \in F_D$, $d_i^* = e_i \sum_{i\in[k]} d_i^*$, for $i \in [k]$, $k = 0, 1, \ldots, p$, and $M_{Pe} = V(\underline{d}^*)$. Let

$$\lambda_k^* = \sum_{i\in[k]} d_i^*, \quad \text{for } k = 0, 1, \ldots, p.$$

Then $\lambda_0^* = 1,\ \lambda_k^* \geq 0,\ \sum_{i=1}^{n} a_{ij}(\underline{t})\, d_i^* = 0$. So,

$$\sum_{k=0}^{p} a_{kj}(\underline{t})\, \lambda_k^* = \sum_{k=0}^{p} \Big(\sum_{i\in[k]} a_{ij} e_i \Big) \lambda_k^*$$

$$= \sum_{k=0}^{p} \Big(\sum_{i\in[k]} a_{ij} d_i^* \Big) \quad (d_i^* = e_i \lambda_k^* \ \text{ for } i \in [k])$$

$$= 0.$$

Thus, $\lambda_k^* \in F_{De}$. Since F_{Pe} is nonempty, (2.1) and (2.3) are consistent. Therefore, by Corollary 1.2 (ii), $M_{Pe} > 0$. |

The theorem below provides the relation between (1.1) and (2.1), which forms the main basis for the primal algorithm that will be presented later in this chapter.

Theorem 2.6. Suppose that (1.1) is superconsistent and that

$$g_0(\underline{t}^*) = \min g_0(\underline{t}), \quad \text{for all } \underline{t} \in F_P,$$

$$e_i(\underline{t}^*) = u_i(\underline{t}^*) / g_k(\underline{t}^*), \quad \text{for } i \in [k], k = 0, 1, ..., p.$$

Then it holds that for all $\underline{t} \in F_{Pe}$,

$$g_0(\underline{t}^*, \underline{e}(\underline{t}^*)) = \min g_0(\underline{t}, \underline{e}(\underline{t}^*)),$$

$$M_{Pe} = M_P.$$

Proof. By theorem 2.2, we know that $M_{Pe} \leq M_P$. Now suppose $M_{Pe} < M_P$, then $\exists\ \underline{t}^1 \in F_{Pe}$ such that

$$g_k(\underline{t}^1, \underline{e}(\underline{t})) \leq 1, \quad \text{for } k = 1, 2, ..., p,$$

$$g_0(\underline{t}^1, \underline{e}(\underline{t})) < M_P.$$

Assume that $\exists\ \underline{t}^2 \in F_P \subset F_{Pe}$, such that

$$g_k(\underline{t}^2) < 1, \quad \text{for } k = 1, 2, ..., p.$$

Then

$$g_k(\underline{t}^2, \underline{e}(\underline{t})) \leq g_k(\underline{t}^2) < 1. \quad (\text{"=" holds } \Leftrightarrow \underline{t} = \underline{t}^2)$$

Let

$$\underline{\hat{t}} = (\underline{t}^1)^{1-\theta} (\underline{t}^2)^{\theta}, \quad \text{for } \theta \in (0,1),$$

i.e.

$$\hat{t}_j = (t_j^1)^{1-\theta} (t_j^2)^{\theta}, \quad \text{for } j = 1, 2, ..., m, \ \theta \in (0,1).$$

Then for k = 1, 2, ..., p,

$$g_k(\underline{\hat{t}}, \underline{e}(\underline{t})) = c_k(\underline{t}) \prod_{j=1}^{m} \hat{t}_j^{\,a_{kj}(\underline{t})}$$

$$= c_k(\underline{t}) \prod_{j=1}^{m} [\, (t_j^1)^{1-\theta} (t_j^2)^{\theta} \,]^{a_{kj}(\underline{t})}$$

$$= [\, (c_k(\underline{t}))^{1-\theta} \prod_{j=1}^{m} (t_j^1)^{(1-\theta)\, a_{kj}(\underline{t})} \,] [\, (c_k(\underline{t}))^{\theta} \prod_{j=1}^{m} (t_j^2)^{\theta} \,]^{a_{kj}(\underline{t})}$$

$$= [\, g_k(\underline{t}^1, \underline{e}(\underline{t})) \,]^{1-\theta} [\, g_k(\underline{t}^2, \underline{e}(\underline{t})) \,]^{\theta}.$$

Note that $g_k(\underline{t}, \underline{e}(\underline{t}))$ are monomials, and $g_k(\underline{t}^1, \underline{e}(\underline{t})) \le 1$, $g_k(\underline{t}^2, \underline{e}(\underline{t})) < 1$, so,

$$g_k(\underline{\hat{t}}, \underline{e}(\underline{t})) < 1, \quad \text{for } k = 1, 2, \ldots, p,$$

and note that $g_0(\underline{t}^1, \underline{e}(\underline{t})) < M_P$, so,

$$g_0(\underline{\hat{t}}, \underline{e}(\underline{t})) < M_P, \quad \text{as } \theta \to 0 \quad (\text{i.e. as } \underline{\hat{t}} \in N(\underline{t}^1, \theta)).$$

For $\xi \in (0,1)$, let

$$\underline{t} = (\underline{\hat{t}})^{\xi} (\underline{t}^*)^{1-\xi}.$$

Then as $\xi \to 0$ (i.e., $\underline{t} \to \underline{t}^*$, $\underline{t} \in N(\underline{t}^*, \xi)$, ξ is sufficiently small))

$$g_k(\underline{t}) = \sum_{i\in[k]} u_i(\underline{t}) = \sum_{i\in[k]} [\, c_i \prod_{j=1}^{m} t_j^{a_{ij}} \,]$$

$$= \sum_{i\in[k]} \{\, c_i \prod_{j=1}^{m} [\, (\hat{t}_j)^{\xi} (t_j^*)^{1-\xi} \,]^{a_{ij}} \}$$

$$= \sum_{i\in[k]} \{\, [\, c_i \prod_{j=1}^{m} (t_j^*)^{a_{ij}} \,]\, [\, \prod_{j=1}^{m} (\hat{t}_j / t_j^*)^{a_{ij}} \,]^{\xi} \}$$

$$= \sum_{i\in[k]} \{\, u_i(\underline{t}^*)\ [\, \prod_{j=1}^{m} (\hat{t}_j / t_j^*)^{a_{ij}} \,]^{\xi} \}.$$

So, as $\xi \to 0$ (i.e., $\underline{t} \to \underline{t}^*$, $\underline{t} \in N(\underline{t}^*, \xi)$, ξ is sufficiently small))

$$\frac{d}{d\xi}[\, g_k(\underline{t}) \,]\,|_{\xi=0} = \frac{d}{d\xi}[\, \sum_{i\in[k]} u_i(\underline{t}^*) \prod_{j=1}^{m} (\hat{t}_j / t_j^*)^{a_{ij}\xi} \,]\,|_{\xi=0}$$

$$= \sum_{i\in[k]} \{\, u_i(\underline{t}^*)\ \frac{d}{d\xi}[\, \prod_{j=1}^{m} (\hat{t}_j / t_j^*)^{a_{ij}} \,]^{\xi} \}\,|_{\xi=0}$$

$$= \sum_{i\in[k]} \{\, u_i(\underline{t}^*) \prod_{j=1}^{m} (\hat{t}_j / t_j^*)^{a_{ij}\xi}\,|_{\xi=0} \log [\, \prod_{j=1}^{m} (\hat{t}_j / t_j^*)^{a_{ij}} \,] \}$$

$$= \sum_{i\in[k]} \{\, e_i(\underline{t}^*)\, [\, \sum_{i\in[k]} u_i(\underline{t}^*) \,] \sum_{j=1}^{m} a_{ij} \log (\hat{t}_j / t_j^*) \}$$

$$(u_i(t^*) = e_i(t^*) \sum_{i \in [k]} u_i(t^*) = e_i(t^*)\, g_k(t^*))$$

$$= g_k(t^*) \sum_{i \in [k]} [\, e_i(t^*) \sum_{j=1}^{m} a_{ij} \log (\hat{t}_j / t_j^*) \,]$$

$$= g_k(t^*) \sum_{j=1}^{m} \{ \sum_{i \in [k]} [\, e_i(t^*)\, a_{ij} \,] \log (\hat{t}_j / t_j^*) \}$$

$$= g_k(t^*) \sum_{j=1}^{m} a_{kj}(t^*) \log (\hat{t}_j / t_j^*) \quad (a_{kj}(t^*) = e_i(t^*)\, a_{ij}).$$

Note that

$$g_k (t, \underline{e}(t^*)) = c_k(t^*) \sum_{j=1}^{m} t_j^{\,a_{kj}(t^*)},$$

$$\log g_k(t, \underline{e}(t^*)) = \log c_k(t^*) + \log [\, \sum_{j=1}^{m} t_j^{\,a_{kj}(t^*)} \,].$$

Subsequently,

$$\frac{d}{d\xi} [\, g_k(t) \,] \,|_{\xi=0} = g_k(t^*) [\, \sum_{j=1}^{m} \log (\hat{t}_j)^{a_{kj}(t^*)} - \sum_{j=1}^{m} \log (t_j^*)^{a_{kj}(t^*)} \,]$$

$$= g_k(t^*) [\, \log (\prod_{j=1}^{m} (\hat{t}_j)^{a_{kj}(t^*)}) - \log (\prod_{j=1}^{m} (t_j^*)^{a_{kj}(t^*)}) \,]$$

$$= g_k(t^*) \{ [\, \log c_k(t^*) + \log (\prod_{j=1}^{m} (\hat{t}_j)^{a_{kj}(t^*)}) \,]$$

$$- [\, \log c_k(t^*) + \log (\prod_{j=1}^{m} (t_j^*)^{a_{kj}(t^*)}) \,] \}$$

$$= g_k(t^*) [\, \log g_k(\hat{\underline{t}}, \underline{e}(t^*)) - \log g_k(t^*, \underline{e}(t^*)) \,]$$

$$= g_k(t^*) \log [\, g_k(\hat{\underline{t}}, \underline{e}(t^*)) / g_k(t^*, \underline{e}(t^*)) \,].$$

By Theorem 2.3, we know that

$$g_k(\underline{\hat{t}}, \underline{e}(\underline{t}^*)) < g_k(\underline{t}^*) = g_k(\underline{t}^*, \underline{e}(\underline{t}^*)), \quad \text{for } \underline{\hat{t}} \neq \underline{t}^*,$$

then we obtain

$$\frac{d}{d\xi}[\, g_k(\underline{t}) \,]\, |_{\xi=0} < 0, \quad \text{for } k = 0, 1, \ldots, p.$$

That implies that for k = 1, 2, ..., p,

$$g_k(\underline{t}) < g_k(\underline{t}^*) \leq 1, \quad \text{for } \underline{t} \in F_P \text{ and } \underline{t} \in N(\underline{t}^*, \xi),$$

and for k = 0,

$$g_0(\underline{t}) < g_0(\underline{t}^*) = M_P .$$

This contradicts the assumption that

$$g_0(\underline{t}^*) = \min g_0(\underline{t}), \quad \text{for all } \underline{t} \in F_P .$$

Hence, $M_{P_e} = M_P$. |

Based on the properties stated above we know that the optimum for (1.1) must be the optimum for (2.1) and $F_P \subset F_{P_e}$. Therefore, we can find such an optimum point sequence $\{\underline{t}^{(n)}\}$ that $\{\underline{t}^{(n)}\} \in F_{P_e}$ for (2.1), and $\{\underline{t}^{(n)}\} \in F_P$ as $n \to \infty$, that is, $\{\underline{t}^{(n)}\}$ remains within a region condensed progressively from F_{P_e} to F_P. Thus, this primal method can be seen as a sort of exterior point method. Before stating the algorithm, let us consider transforming problem (1.1) to a program that is of the form facilitating operation.

Theorem 2.7. The problem that

$$\text{minimize} \quad t_0,$$

$$\text{subject to} \quad \underline{t} \in F_{P1}, \qquad (2.5)$$

where

$$F_{P1} = \{\underline{t} \mid t_0^{-1} g_0(\underline{t}) \leq 1,$$

$$g_k(\underline{t}) \leq 1, \quad \text{for } k = 1, 2, \ldots, p,$$

$$t_0, \; t_j > 0, \quad \text{for } j = 1, 2, \ldots, m \}$$

is equivalent to (1.1).

Proof. Let $(t_0^*, \underline{t}^*)$ be the minimum of (2.5). Then for $\forall\, \underline{t} \in F_{P1}$,

$$(t_0^*)^{-1}\, g_0(\underline{t}^*) \le 1,$$

where

$$t_0^* = \min\,(t_0) \le t_0.$$

Now suppose $\underline{t}^*$ is not the minimum of (1.1), then we can find a point $\underline{\hat{t}} \in F_P$ such that $g_0(\underline{\hat{t}}) < g_0(\underline{t}^*) \le t_0^*$. Let

$$\hat{t}_0 = g_0(\underline{\hat{t}}) \quad \text{(the upper bound of } g_0(\underline{t})\text{)}.$$

Then

$$t_0^* > \hat{t}_0.$$

That contradicts the assumption that $t_0^* \le t_0$. Thus, $\underline{t}^*$ must be the minimum of (1.1).

Similarly, let $\underline{t}^*$ be the minimum of (1.1), that is, $g_0(\underline{t}^*) \le g_0(\underline{t})$ for $\forall\, \underline{t} \in F_P$.

And let

$$t_0^* = g_0(\underline{t}^*) \le g_0(\underline{t}),$$

$$t_0 \ge g_0(\underline{t}) \ge t_0^*. \quad \text{(upper bound of } g_0(\underline{t})\text{)}.$$

Suppose that $(t_0, \underline{t}^*)$ is not the minimum of (2.5). Then $\exists\, (\hat{t}_0, \underline{\hat{t}})$, $\hat{t}_0 \ge g_0(\underline{\hat{t}})$ such that

$$\hat{t}_0 < t_0^* = g_0(\underline{t}^*).$$

That is,

$$g_0(\underline{\hat{t}}) \le \hat{t}_0 < t_0^* = g_0(\underline{t}^*)$$

This contradicts the assumption that $g_0(\underline{t}^*) \le g_0(\underline{t})$. ∎

Since the range of the optimum solution can usually be estimated in practice, it is reasonable to set lower and upper bounds for $\underline{t}$ in advance. So, we have the program that

$$\text{minimize} \quad t_0 ,$$
$$\text{subject to} \quad \underline{t} \in F_b , \tag{2.6}$$

where

$$F_b = \{ \underline{t} \mid g_k(\underline{t}) \leq 1, \quad \text{for } k = 0, 1, ..., p,$$
$$0 < t_j^l \leq t_j \leq t_j^u, \quad \text{for } j = 1, 2, ..., m \}.$$

And (2.1) can be rewritten as the program that for the operation vector $\underline{t}^1$,

$$\text{minimize} \quad t_0 ,$$
$$\text{subject to} \quad \underline{t} \in F_{b1} , \tag{2.7}$$

where

$$F_{b1} = \{ \underline{t} \mid g_k(\underline{t}, \underline{e}(\underline{t}^1)) \leq 1, \quad \text{for } k = 0, 1, ..., p,$$
$$0 < t_j^l \leq t_j \leq t_j^u, \quad \text{for } j = 0, 1, ..., m \},$$

$$g_k(\underline{t}, \underline{e}(\underline{t}^1)) = c_k(\underline{t}^1) \prod_{j=0}^{m} t_j^{a_{kj}(\underline{t}^1)}.$$

The logarithmic transformation of (2.7) leads to the problem that

$$\text{minimize} \quad \log t_0 ,$$
$$\text{subject to} \quad \underline{t} \in F_{bl} , \tag{2.8}$$

where

$$F_{bl} = \{ \underline{t} \mid \log g_k(\underline{t}, \underline{e}(\underline{t}^1)) \leq 0, \quad \text{for } k = 0, 1, ..., p,$$
$$\log t_j^l \leq \log t_j \leq \log t_j^u, \quad \text{for } j = 0, 1, ..., m \},$$

$$\log g_k(\underline{t}, \underline{e}(\underline{t}^1)) = \log c_k(\underline{t}^1) + \sum_{j=0}^{m} a_{kj}(\underline{t}^1) \log t_j .$$

In order to use simplex method for which positive variables are needed, define

$$x_j = \log t_j - \log t_j^l \geq 0,$$

$$x_j^u = \log t_j^u - \log t_j^l$$

$$= \log (t_j^u / t_j^l), \quad \text{for } j = 0, 1, \ldots, m.$$

Then we obtain such an approximating linear program that

$$\text{minimize} \quad x_0 + \log t_0^l,$$

$$\text{subject to} \quad \underline{x} \in F_{LP}, \tag{2.9'}$$

where

$$F_{LP} = \{ \underline{x} \mid \log g_k(\underline{x}, \underline{e}(\underline{t}^1)) \leq 0, \ 0 \leq x_j \leq x_j^u, \ \text{for } j = 0, 1, \ldots, m \},$$

$$\log g_k(\underline{x}, \underline{e}(\underline{t}^1)) = \log c_k(\underline{t}^1) + \sum_{j=0}^{m} a_{kj}(\underline{t}^1) \log t_j^l + \sum_{j=0}^{m} a_{kj}(\underline{t}^1)\, x_j$$

$$= \log g_k(\underline{t}^l, \underline{e}(\underline{t}^1)) + \sum_{j=0}^{m} a_{kj}(\underline{t}^1)\, x_j.$$

That is,

$$\text{minimize} \quad x_0,$$

$$\text{subject to} \quad \underline{x} \in F_{LP}, \tag{2.9}$$

where

$$F_{LP} = \{ \underline{x} \mid \sum_{j=0}^{m} a_{kj}(\underline{t}^1)\, x_j \leq -\log g_k(\underline{t}^l, \underline{e}(\underline{t}^1)), \ \text{for } k = 0, 1, \ldots, p,$$

$$0 \leq x_j \leq x_j^u, \ \text{for } j = 0, 1, \ldots, m \}.$$

Now let us consider the following "cutting-plane" algorithm.

Step 1. Arbitrarily take a starting operation point $\underline{t}^0 > \underline{0}$, and let

$$F^0 = \{ \underline{t} \mid g_k(\underline{t}, \underline{e}(\underline{t}^0)) \leq 1, \ \text{for } k = 1, 2, \ldots, p, \ \underline{t}^l \leq \underline{t} \leq \underline{t}^u \}.$$

Then $F_P \subset F^0$. Form program (2.9) for $\underline{t}^1 = \underline{t}^0$. Set $n = 1$.

Step 2. Solve (2.9) for $\underline{t}^1 = \underline{t}^{n-1}$. Find the solution $\underline{x}^n$ and compute $\underline{t}^n$. Note that $\underline{t}^n$ is the optimum for (2.9) and not necessarily feasible for (2.6). If (2.9) has no solution, then there is no solution for (2.6).

Step 3. For program (2.6), let

$$g_{k(a)}(\underline{t}^n) = \max g_k(\underline{t}^n) \leq 1, \quad \text{for all } 1 \leq k \leq p. \tag{2.10}$$

Then $\underline{t}^n$ is the optimum for (2.6), and stop. Otherwise, define

$$g_{k(n)}(\underline{t}^n) = \max \{ g_k(\underline{t}^n) \mid g_k(\underline{t}^n) > 1 \}, \quad \text{for all } 1 \leq k \leq p,$$

and turn to step 4.

Step 4. Condense $g_{k(n)}(\underline{t})$ at $\underline{t}^n$ (new operation point) to obtain a linear constratint for (2.9) (new one added to the current (2.9)) that

$$\log g_k(\underline{x}, \underline{e}(\underline{t}^n)) \leq 0,$$

and define

$$C^n = \{ \underline{t} \mid \log g_k(\underline{x}, \underline{e}(\underline{t}^n)) \leq 0 \ \text{(known as cuts)} \},$$

$$F^n = F^{n-1} \cap C^n.$$

Let n increase by 1. Turn to step 2. |

At each iteration Step 4 produces a new constraint adding to (2.9) and the feasible region of (2.9) reduces, in other words, part of which is cut off. It is worth noting that since $g_k(\underline{t}, \underline{e}(\underline{t}^n)) \leq g_k(\underline{t})$, $g_k(\underline{t}) \leq 1$ implies that $g_k(\underline{t}, \underline{e}(\underline{t}^n)) \leq 1$ for $\forall\, \underline{t} \in F_b$, that is, $\underline{t} \in C^n$, and such $\underline{t} \in F_b$ must also satisfy the constraint added. Consequently, any part of the feasible region of (2.6) can be reserved (not cut off).

(2.10) gives a criterion for judging the optimum solution for (2.6). The following theorem provides the convergence property of the algorithm.

Theorem 2.8. If the algorithm stop within finite steps, that is, (2.10) holds, then t^n must be the optimum solution for (2.6). Otherwise, any limit point of sequence $\{ t^n \}$ is the optimum solution for (2.6).

Proof. From the argument above we know that for $\forall$ n, $F_b \subset C^n$. It is also true that $F_b \subset F^0$. So, $F_b \subset F^n$, where $F^n = F^{n-1} \cap C^n$, n = 1, 2, ...

If (2.10) holds, then $t^n \in F_b$ is the optimum for (2.6) in F^{n-1} ($\supset F^n$) (by Theorem 2.6) and is the optimum for (2.6) in F_b.

Suppose $\{ t^n \}$ for n = 1, 2, ... is an infinite point sequence, and note that $t \in [t^l, t^u]$, $t^l > 0$ means t^n is bounded. So, for a subsequence of $\{ t^n \}$, $\{ t^{n_k} \}$, there exists a limit point t^* such that $t^{n_k} \to t^*$ as $k \to \infty$. So, $t_0^{n_k} \le \min t_0$ for all $t \in F_b$, and $t_0^* \le \min t_0$ for all $t \in F_b$.

In fact, we have

$$g_{k(n_k)}(t^{n_k}) = g_{k(n_k)}[t^{n_k}, e(t^{n_k})] > 1,$$

so,

$$t^{n_k} \notin C^{n_k}.$$

For $n_{n+1} > n_k$,

$$t^{n_{k+1}} \in F^{n_{k+1}-1} \subset F^{n_{k+1}-2} \subset \ldots \subset F^{n_k} \subset C^{n_k}.$$

Let $d (t^{n_k}, C^{n_k})$ denote the distance from t^{n_k} to C^{n_k}. Then by the definition,

$$d (t^{n_k}, C^{n_k}) = \min d (t^{n_k}, t), \quad \text{for all } t \in C^{n_k}.$$

Note that $t^{n_{k+1}} \in C^{n_k}$, so, $d (t^{n_k}, C^{n_k}) \le d (t^{n_k}, t^{n_{k+1}})$, and

$$d (t^{n_k}, t^{n_{k+1}}) \to 0 \quad \text{as } k \to \infty \quad (\text{since } t^{n_k} \to t^* \text{ as } k \to \infty).$$

Thus,

$$d (t^{n_k}, C^{n_k}) \to 0 \quad \text{as } k \to \infty.$$

Now suppose that $t^* \notin F_b$. Then $\exists$ k $(1 \le k \le p)$ such that $g_k(t^*) > 1$. Let

$$\epsilon = (1/2) [g_k(t^*) - 1] > 0.$$

Then $\exists$ N > 0 such that for $n_k > N$,

$$g_{k(n_k)}(\underline{t}^{n_k}, \underline{e}(\underline{t}^{n_k})) = g_{k(n_k)}(\underline{t}^{n_k}) > 1 + \varepsilon \quad (\varepsilon > 0, \underline{t}^{n_k} \in N(\underline{t}^*, \varepsilon)) \tag{2.11}$$

On the other hand, $g_k(\underline{t}, \underline{e}(\underline{t}))$ is continuous uniformly (continuous on $[\underline{t}^l, \underline{t}^u]$).

Hence, for $\underline{t}, \underline{\tau} \in [\underline{t}^l, \underline{t}^u]$, $\exists\, \delta > 0$ such that for $\varepsilon > 0$,

$$| g_k(\underline{t}', \underline{e}(\underline{\tau}') - g_k(\underline{t}'', \underline{e}(\underline{\tau}'') | < \varepsilon$$

if $|| \underline{t}' - \underline{t}''|| + || \underline{\tau}' - \underline{\tau}''|| < \delta$. But for $\forall\, \underline{t} \in C^{n_k}$, we have

$$g_{k(n_k)}(\underline{t}, \underline{e}(\underline{t}^{n_k})) \le 1, \tag{2.12}$$

so, for $\forall\, \underline{t} \in C^{n_k}$,

$$| g_{k(n_k)}(\underline{t}^{n_k}, \underline{e}(\underline{t}^{n_k})) - g_{k(n_k)}(\underline{t}, \underline{e}(\underline{t}^{n_k})) | \ge |1 + \varepsilon - 1| = \varepsilon.$$

Thus, it must be true that

$$|| \underline{t} - \underline{t}^{n_k} || > \delta, \quad \text{for } \forall\, \underline{t} \in C^{n_k},$$

that is,

$$\lim_{k \to \infty} d(\underline{t}^{n_k}, C^{n_k}) \ne 0.$$

That contradicts the consequence that

$$d(\underline{t}^{n_k}, C^{n_k}) \to 0 \text{ as } k \to \infty.$$

Therefore, $\underline{t}^* = \lim_{k \to \infty} \underline{t}^n$, $\underline{t}^* \in F_b$ is the optimum for program (2.6).

The following illustrative example shows the application of the cutting plane algorithm.

Example 2.1. Find the solution of the problem that

minimize t_1,

subject to $g_1(\underline{t}) = 0.574\, t_1^{0.388}\, t_2^{-0.755} + 1.148\, t_1^{-0.612}\, t_2^{0.225} \le 1$,

$g_2(\underline{t}) = 0.083\, t_1^{1.529}\, t_2^{-0.529} + 0.083\, t_1^{-0,471}\, t_2^{1.471}$

$+ 1.169\, t_1^{-0.471}\, t_2^{-0.529} \le 1$,

$1.0 \le t_1 \le 5.5$, $1.0 \le t_2 \le 5.5$.

Solution. Set $\underline{t}^{(0)} = (t_1^{(0)}, t_2^{(0)})^T = (4, 4.5)^T$.

Suppose the problem is linearized at the initial operation point $\underline{t}^0 = (4, 4.5)^T$. The condensed constraints are

$$g_1(\underline{t}, \underline{e}(\underline{t}^{(0)})) = 1.720\, t_1^{-0.305}\, t_2^{-0.083} \le 1,$$

$$g_2(\underline{t}, \underline{e}(\underline{t}^{(0)})) = 0.516\, t_1^{0.166}\, t_2^{0.277} \le 1.$$

$$x_1 = \log t_1 - \log t_1^{\,l} = \log t_1, \quad (\log t_1^{\,l} = \log 1 = 0),$$

$$x_2 = \log t_2 - \log t_2^{\,l} = \log t_2.$$

Consider solving the problem that

$$\begin{aligned} \text{minimize} \quad & x_1, \\ \text{subject to} \quad & -0.305\, x_1 - 0.082\, x_2 \le -0.524, \\ & 0.166\, x_1 + 0.277\, x_2 \le 0.661, \\ & 0 \le x_1 \le \log 5.5, \\ & 0 \le x_2 \le \log 5.5. \end{aligned} \tag{2.13}$$

The solution of problem (2.13) is:

$$x_1^{(1)} = 1.350, \quad x_2^{(1)} = 1.579,$$

$$t_1^{(1)} = 3.858, \quad t_2^{(1)} = 4.852,$$

$$\begin{aligned} g_1(\underline{t}^{(1)}) &= (0.574)\,(3.858)^{0.388}\,(4.852)^{-0.755} \\ &\quad + (1.148)\,(3.858)^{-0.612}\,(4.852)^{0.225} \\ &= 0.717, \end{aligned}$$

$$\begin{aligned} g_2(\underline{t}^{(1)}) &= (0.083)(3.858)^{1.529}(4.852)^{-0.529} \\ &\quad + (0.083)(3.858)^{-0.471}(4.852)^{1.472} \\ &\quad + (1.169)(3.858)^{-0.471}(4.852)^{-0.529} \\ &= 1.001. \end{aligned}$$

The second constraint is violated, so linearizing $g_2(\underline{t})$ at

$\underline{t}^{(1)} = (3.858, 4.852)^T$ (new operation point)

leads to

$$g_2\,(\underset{\sim}{t},\ \underset{\sim}{e}(\underset{\sim}{t}^{(1)})) = (0.498)\, t_1^{0.097}\ t_2^{0.369} \le 1.$$

Hence,

$$g_2\,(\underset{\sim}{x},\ \underset{\sim}{e}(\underset{\sim}{t}^{(1)})) = 0.097\, x_1 + 0.369\, x_2 \le 0.697. \qquad (2.14)$$

Adding (2.14) to the problem (2.13) leads to the program that

$$\begin{aligned} &\text{minimize} \quad x_1, \\ &\text{subject to} \quad -0.305\, x_1 - 0.082\, x_2 \le -0.524, \\ &\qquad 0.166\, x_1 + 0.277\, x_2 \le 0.661, \\ &\qquad 0.097\, x_1 + 0.369\, x_2 \le 0.697, \\ &\qquad 0 \le x_1 \le \log 5.5, \\ &\qquad 0 \le x_2 \le \log 5.5. \end{aligned} \qquad (2.15)$$

The solution of (2.15) is that

$$x_1^{(2)} = 1.354, \quad x_1^{(2)} = 1.566.$$

Hence,

$$t_1^{(2)} = 3.872, \quad t_1^{(2)} = 4.786.$$

$$g_1(\underset{\sim}{t}^{(2)}) < 1, \quad g_2(\underset{\sim}{t}^{(2)}) < 1.$$

However, conducting more iterations can produce a better computation accuracy. That is,

$$\underset{\sim}{t}^{(0)} = (4.0,\ 4.5), \quad \underset{\sim}{t}^{(1)} = (3.872,\ 4.786),$$

$$\underset{\sim}{t}^{(2)} = (3.824,\ 4.824), \quad \underset{\sim}{t}^{(3)} = (3.823,\ 4.823),$$

where $\underset{\sim}{t}^{(3)}$ is sufficiently close to $\underset{\sim}{t}^*$ in the iteration. |

CHAPTER III

THE GENERALIZED GEOMETRIC PROGRAMMING PROBLEM

In this chapter, we discuss the general features of the signomial geometric programming problem (SGP) and the method for solving the problem via transforming the problem to the complementary geometric programming problem (CGP). The basic results can be found in [2], [4], [8] and [12].

The problem that

$$\text{minimize } g_0(\underline{t}),$$
$$\text{subject to } \underline{t} \in F_{PS}, \tag{3.1}$$

where

$$F_{PS} = \{ \underline{t} \mid g_k(\underline{t}) \leq \sigma_k, \ \text{ for } \sigma_k = 1 \text{ or } -1, k = 1, 2, \ldots, p \},$$

$$g_k(\underline{t}) = \sum_{i\in[k]} s_i c_i \prod_{j=1}^{m} t_j^{a_{ij}}, \ \text{for } s_i = 1 \text{ or } -1, c_i, t_j > 0, k = 0, 1, \ldots, p \ \ \text{(signomial)}$$

is known as signomial geometric programming.

The associated dual program of (3.1) can seemingly be written as

$$\text{maximize } V(\underline{d}),$$
$$\text{subject to } \underline{d} \in F_{DS}, \tag{3.2}$$

where

$$F_{DS} = \{ \underline{d} \mid \sum_{i\in[0]} s_i d_i = \sigma_0, \ \sigma_0 = 1 \text{ or } -1,$$

$$\lambda_k = \sigma_k \sum_{i\in[k]} s_i d_i \geq 0, \ \text{ for } k = 1, 2, \ldots, p, \ \lambda_0 = 1,$$

$$\sum_{i=1}^{n} s_i a_{ij} d_i = 0, \ \text{ for } j = 1, 2, \ldots, m,$$

$$d_i \geq 0, \ \text{ for } i = 1, 2, \ldots, n \},$$

$$V(\underline{d}) = \sigma_0 [\prod_{k=0}^{p} \prod_{i \in [k]} (c_i \lambda_k / d_i)^{s_i d_i}]^{\sigma_0}, \quad \text{for } \sigma_0 = 1 \text{ or } -1, \lambda_0 = 1.$$

For (3.1) and (3.2), we can set up the duality theory similar to that of posynomial geometric programming, and the conclusion is weaker and the proof (omitted) is more complicated. Before presenting the consequence of the theory we define the signum function such that

$$\sigma_0 = 1, \quad \text{if } g_0(\underline{t}^*) \geq 0,$$

$$\sigma_0 = -1, \quad \text{if } g_0(\underline{t}^*) < 0,$$

where $\underline{t}^*$ is the minimum for (3.1).

Theorem 3.1. For a local minimum point $\underline{t}^* \in F_{PS}$, there exists $\underline{d}^* \in F_{DS}$ such that $V(\underline{d}^*) = g_0(\underline{t}^*)$. Furtherfore, there is the relation between $\underline{d}^*$ and $\underline{t}^*$ that

$$u_i(\underline{t}^*) = d_i^* \sigma_0 g_0(\underline{t}^*), \quad \text{for } i \in [0],$$

$$u_i(\underline{t}^*) = d_i^* / \lambda_k, \quad \text{for } i \in [k], \ k = 1, 2, ..., p, \quad (u_i(\underline{t}) = c_i \prod_{j=1}^{m} t_j^{a_{ij}}) \tag{3.3}$$

and

$$d_i^* [g_k(\underline{t}^*) - 1] = 0, \quad \text{for } k = 1, 2, ..., p, \tag{3.4}$$

or by taking the logarithmic form, (3.3) becomes

$$\sum_{j=1}^{m} a_{ij} \log t_j = \log (d_i^* \sigma_0 g_0 (\underline{t}^*) / c_i), \quad \text{for } i \in [0],$$

$$\sum_{j=1}^{m} a_{ij} \log t_j = \log (d_i^* / c_i \lambda_k), \quad \text{for } i \in [k], k = 1, 2, ..., p.$$

|

It is important to note that the fact revealed by Theorem 3.1 is the relation between an arbitrary minimum point $\underline{t}^*$ of (3.1) and only a stationary point $\underline{d}^*$ of (3.2). In general, (3.1) is not a convex program, so, its minimum point is not necessarily unique and $\underline{t}^*$ is not necessarily a global minimum point. Similarly, (3.2) is generally not a concave program, therefore, the dual vector $\underline{d}^*$ that satisfies $V(\underline{d}^*) = g_0(\underline{t}^*)$ is not necessarily the maximum of (3.2) and is only a stationary point. In other words, since (3.1) does not have convexity property, it lacks the duality property that posynomial geometric programming possesses. In order to facilitate the discussion for (3.1), the concepts of quasiextreme point and quasiduality relation are introduced.

Definition 3.1. For (3.1), $\underline{t}^* \in F_{PS}$ is called *quasiminimum point* if $\underline{t}^*$ is the Kuhn-Tucker point that satisfies the necessary condition for the local minimum, and the value of $g_0(\underline{t}^*)$ is called *quasiminimum value*, written as $g_0(\underline{t}^*)$ = quasimin $g_0(\underline{t})$ for all $\underline{t} \in F_{PS}$. Accordingly, $\underline{d}^* \in F_{DS}$ is called *quasimaximum point*, and $V(\underline{d}^*)$ is called *quasimaximum value* for (3.2), written as $V(\underline{d}^*)$ = quasimax $V(\underline{d})$ for all $\underline{d} \in F_{DS}$. |

Since Kuhn-Tucker point is not necessarily the optimum, by Definition 3.1, quasiminimum / quasimaximum point is not necessarily the local minimum / maximum point (only the candidate).

Theorem 3.2. Suppose that $\underline{t}^*$ is the quasiminimum point of (3.1). Then there exists a quasimaximum point $\underline{d}^*$ such that for all $\underline{t} \in F_{PS}$ and $\underline{d} \in F_{DS}$,

$$\text{quasimin } g_0(\underline{t}) = \text{quasimax } V(\underline{d}),$$

that is,

$$g_0(\underline{t}^*) = V(\underline{d}^*).$$ |

The duality relation concerning quasiminimum / quasimaximum is called quasiduality relation. Based on the relations shown by Theorem 3.1 and 3.2, we can use the duality method in solving (3.1), that is, solve (3.1) via finding the stationary points of (3.2).

Assume that F_{PS} is nonempty. Then for $\delta = 0$, the unique solution exists. It is a stationary point and needed to check whether it is an extremum point. For $\delta > 0$, the optimum solution is harder to find because δ independent variables are involved. The following example shows how the quasiduality relation can be used to solve the signomial geometric programming problem.

Example 3.1. Solve the problem that

minimize $g_0(\underline{t}) = -3\, t_1^{-1} t_2^{-3} t_3 + (1/2)\, t_1^{-2} t_2^{-2} t_3$,

subject to $g_1(\underline{t}) = (1/4)\, t_1 t_2^2 t_3^{-1} - (1/3)\, t_1^{3/4} t_2^{5/2} t_3^{-1} \leq -1$,

$t_j > 0$, for $j = 1, 2, 3$.

Solution. The degree of difficulty of the problem is: $\delta = 4 - (3+1) = 0$. Note that $s_1 = -1$, $s_2 = 1$, $s_3 = 1$, $s_4 = -1$, $\sigma_1 = -1$, $\lambda_0 = 1$. Let $\sigma_0 = -1$, so,

$$
\begin{aligned}
-d_1 + d_2 \qquad\qquad\qquad &= -1,\\
d_1 - 2d_2 + d_3 - (3/4)\, d_4 &= 0,\\
3d_1 - 2d_2 + 2d_3 - (5/2)\, d_4 &= 0,\\
-d_1 + d_2 - d_3 + d_4 &= 0.
\end{aligned}
$$

That is,

$$
\begin{aligned}
d_1 - (1/2)\, d_4 &= 0,\\
d_3 - (3/4)\, d_4 &= 0,\\
d_2 - (1/4)\, d_4 &= 0,\\
-d_1 + d_2 \qquad &= -1.
\end{aligned}
$$

So, we have $\underline{d}^* = (d_1^*,\ d_2^*,\ d_3^*,\ d_4^*)^T = (2, 1, 3, 4)^T$, and

$$\lambda_1 = s_1 \sum_{i=3}^{4} s_i d_i = (-1)\,[\,3 + (-1)(4)\,] = 1 > 0.$$

Thus, for stationary point $\underline{t}^*$ and $\underline{d}^*$, the objective function value is:

$$
\begin{aligned}
g_0(\underline{t}^*) = V(\underline{d}^*) &= (-1)\,[\,(3/2)^{-2}\,(\frac{1/2}{1})\,(\frac{1/4}{3})^{3}\,(\frac{1/3}{4})^{-4}\,]^{-1}\\
&= (-1)\,(3^{\,2+3-4}\,2^{\,-2+1+6-8}) = -\,3/8.
\end{aligned}
$$

From (3.3) we know that

$3\, t_1^{-1} t_2^{-3} t_3 = (2)\,(-1)\,(-3/8) = 3/4$,

$(1/2)\, t_1^{-2}\, t_2^{-2} t_3 = (1)\,(-1)\,(-3/8) = 3/8,$

$(1/4)\, t_1\, t_2^2\, t_3^{-1} = 3,$

$(1/3)\, t_1^{3/4}\, t_2^{5/2}\, t_3^{-1} = 4,$

That is,

$t_1^{-1}\, t_2^{-3}\, t_3 = 1/4,$

$t_1^{-2}\, t_2^{-2} t_3 = 3/4,$

$t_1\, t_2^2\, t_3^{-1} = 12,$

$t_1^{3/4}\, t_2^{5/2}\, t_3^{-1} = 12.$

The solution for this equation system is: $\underline{t}^* = (t_1^*, t_2^*, t_3^*)^T = (3^{-1}, 3^{-2}, 3^{-5}/4)^T$.

That $\delta = 0$ implies $\underline{t}^*$ is the only stationary point. The Hessian matrix of $g_0(\underline{t})$ is:

$$H(\underline{t}^*) = H(\underline{t})\,|_{\underline{t}=\underline{t}^*} = (h_{ij}(\underline{t}))\,|_{\underline{t}=\underline{t}^*} = \left(\frac{\partial^2 g_0(\underline{t})}{\partial t_i \partial t_j}\right)|_{\underline{t}=\underline{t}^*},$$

where

$h_{11}(\underline{t}) = (-6)\, t_1^{-3} t_2^{-3} t_3 + 3\, t_1^{-4} t_2^{-2} t_3,$

$h_{12}(\underline{t}) = h_{21}(\underline{t}) = (-9)\, t_1^{-2} t_2^{-4} t_3 + 2\, t_1^{-3} t_2^{-3} t_3,$

$h_{13}(\underline{t}) = h_{31}(\underline{t}) = 3\, t_1^{-2} t_2^{-3} - t_1^{-3} t_2^{-2},$

$h_{22}(\underline{t}) = (-36)\, t_1^{-1} t_2^{-5} t_3 + 3\, t_1^{-2} t_2^{-4} t_3,$

$h_{23}(\underline{t}) = h_{32}(\underline{t}) = 9\, t_1^{-1} t_2^{-4} - t_1^{-2} t_2^{-3},$

$h_{33}(\underline{t}) = 0.$

Thus,

$$H(\underline{t}^*) = H(\underline{t})|_{\underline{t}=\underline{t}^*} = \begin{pmatrix} -3^{-5}/4 & -3^{-4}/4 & 0 \\ -3^{-4}/4 & -3^{-5}/4 & 2(3^7) \\ 0 & 2(3^7) & 0 \end{pmatrix}.$$

It is clear that $H(\underline{t}^*)$ is not positive definite. Therefore, $\underline{t}^*$ is not a minimum point and is only a saddle point. |

Definition 3.2. The problem

$$\text{minimize } t_0,$$
$$\text{subject to } \underset{\sim}{t} \in F_{PC}, \tag{3.3}$$

where

$$F_{PC} = \{\underset{\sim}{t} \mid Q_k(\underset{\sim}{t}) / P_k(\underset{\sim}{t}) \le 1, \quad \text{for } k = 1, 2, \ldots, p\},$$

$$Q_k(\underset{\sim}{t}) = \sum_{i\in[k]} c_i \prod_{j=0}^{m} t_j^{a_{ij}} \text{ and } P_k(\underset{\sim}{t}) = \sum_{i\in[k]} d_i \prod_{j=0}^{m} t_j^{b_{ij}} \text{ are posynomials,}$$

is called complementary geometric programming (CGP).

Theorem 3.3. (3.1) can be transformed to (3.3) equivalently.

Proof. By theorem 2.6, (3.1) is equivalent to the following problem

$$\text{minimize } t_0,$$
$$\text{subject to } \underset{\sim}{t} \in F_{PS2}, \tag{3.4}$$

where

$$F_{PS2} = \{\underset{\sim}{t} \mid t_0^{-1} g_0(\underset{\sim}{t}) \le 1,$$
$$g_k(\underset{\sim}{t}) \le \sigma_k, \quad \text{for } \sigma_k = 1 \text{ or } -1, k = 1, 2, \ldots, p,$$
$$t_j > 0, \quad \text{for } j = 0, 1, \ldots, m\},$$

$$g_k(\underset{\sim}{t}) = \sum_{i\in[k]} s_i c_i \prod_{j=1}^{m} t_j^{a_{ij}}, \quad \text{for } s_i = 1 \text{ or } -1, c_i > 0, \ k = 0, 1, \ldots, p,$$

In standard form, (3.4) can be written as:

$$\text{minimize } t_0,$$
$$\text{subject to } \underset{\sim}{t} \in F_{PS1}, \tag{3.5}$$

where

$$F_{PS1} = \{\underset{\sim}{t} \mid h_k(\underset{\sim}{t}) \le \sigma_k, \quad \text{for } \sigma_k = 1 \text{ or } -1, \ \sigma_0 = 1, \ k = 0, 1, \ldots, p,$$
$$t_j > 0, \quad \text{for } j = 0, 1, \ldots, m\},$$

$$h_k(\underset{\sim}{t}) = \sum_{i\in[k]} s_i c_i \prod_{j=0}^{m} t_j^{a_{ij}}, \quad \text{for } s_i = 1 \text{ or } -1, c_i > 0, \ k = 0, 1, \ldots, p \text{ (signomial).}$$

Note that $h_k(\underline{t})$ can be expressed as:

$$h_k(\underline{t}) = h_k^+(\underline{t}) - h_k^-(\underline{t}) \le \sigma_k,$$

where $h_k^+(\underline{t})$, $h_k^-(\underline{t})$ are posynomials. So, for $\sigma_k = 1$, we have

$$h_k^+(\underline{t}) - h_k^-(\underline{t}) \le 1,$$

that is,

$$h_k^+(\underline{t}) / [\, 1 + h_k^-(\underline{t})\,] \le 1.$$

Let $Q_k(\underline{t}) = h_k^+(\underline{t})$ and $P_k(\underline{t}) = 1 + h_k^-(\underline{t})$. Then we reach

$$Q_k(\underline{t}) / P_k(\underline{t}) \le 1, \quad \text{for } k = 0, 1, \ldots, p.$$

For $\sigma_k = -1$, we have

$$h_k^+(\underline{t}) - h_k^-(\underline{t}) \le -1,$$

that is,

$$(\, 1 + h_k^+(\underline{t})\,) / h_k^-(\underline{t}) \le 1.$$

Let $Q_k(\underline{t}) = 1 + h_k^+(\underline{t})$ and $P_k(\underline{t}) = h_k^+(\underline{t})$. Then we also reach

$$Q_k(\underline{t}) / P_k(\underline{t}) \le 1, \quad \text{for } k = 0, 1, \ldots, p.$$

So, via condensing $P_k(\underline{t})$, we obtain the approximating problem of (3.5) that

$$\text{minimize } \; t_0,$$
$$\text{subject to } \; \underline{t} \in F_{PA}, \qquad (3.6)$$

where

$$F_{PA} = \{\, \underline{t} \mid Q_k(\underline{t}) / P_k(\underline{t}, \underline{e}(\underline{t})) \le 1, \quad \text{for } k = 0, 1, \ldots, p \,\},$$

$$Q_k(\underline{t}) = \sum_{i\in[k]} c_i \prod_{j=1}^{m} t_j^{a_{ij}},$$

$$P_k(\underline{t}, \underline{e}(\underline{t})) = \prod_{i\in[k]} \{\, [\, d_i / e_i(\underline{t})\,] \prod_{j=0}^{m} t_j^{b_{ij}} \}^{e_i(\underline{t})},$$

$$e_i(\underline{t}) \in (0,1), \quad \text{for } i = 1, 2, \ldots, n, \quad \sum_{i\in[k]} e_i(\underline{t}) = 1,$$

which is a posynomial geometric program with smaller degree of difficulty than that of (3.1) since the number of terms of $Q_k(\underline{t})$ equals that of $(1+h_k^+(\underline{t}))$. |

Theorem 3.4. If $\underline{t}^*$ is the optimum solution of (3.6), then $\underline{t}^*$ must be a feasible solution of (3.3).

Proof. By Theorem 2.1 we know that

$$\mathcal{P}_k(\underline{t}, \underline{e}(\underline{t})) \le P_k(\underline{t}).$$

So,

$$Q_k(\underline{t}) / P_k(\underline{t}) \le Q_k(\underline{t}) / \mathcal{P}_k(\underline{t}, \underline{e}(\underline{t})).$$

For all $\underline{t} > 0$ such that

$$Q_k(\underline{t}) / \mathcal{P}_k(\underline{t}, \underline{e}(\underline{t})) \le 1,$$

it must hold that

$$Q_k(\underline{t}) / P_k(\underline{t}) \le 1,$$

and the converse is not necessarily true. Hence, if $\underline{t} \in F_{PA}$, then $\underline{t} \in F_{PC}$, that is, $F_{PA} \subset F_{PC}$. Therefore, $\underline{t}^* \in F_{PA}$ implies $\underline{t}^* \in F_{PC}$. |

Theorem 3.5. Suppose that $\underline{t}^* \in F_{PC}$ is the optimum solution of (3.6) with $\mathcal{P}_k(\underline{t}, \underline{e}(\underline{t}^*))$ ($\underline{t}^*$ acts as the operation point here). Then $\underline{t}^*$ is the optimum solution of (3.3).

Proof. Suppose that $\underline{t}^*$ is not the optimum solution of (3.3) and that $\underline{t}^1 \ne \underline{t}^*$ is the optimum solution of (3.3). Then $\underline{t}^1 \in F_{PC}$ and $\underline{t}^1 \notin F_{PA}$. By Theorem 2.3 we have

$$\mathcal{P}_k(\underline{t}^1, \underline{e}(\underline{t}^1)) = P_k(\underline{t}^1).$$

Note that

$$F_{PA} \subset F_{PC}.$$

So, $\underline{t}^1 \in F_{PA}$ and $\underline{t}^1$ is the optimum solution of (3.6). This contracts $\underline{t}^1 \notin F_{PA}$. Thus, $\underline{t}^1 = \underline{t}^*$. |

Based on Theorem 3.4 and 3.5, we can constructe an algorithm for solving (3.1) as follows.

Step 1. Transform (3.1) to (3.5). Fix an initial feasible solution $\underline{t}^0 \in F_{PC}$, and give $\varepsilon > 0$ as tolerant error for an acceptable computation accuracy. Set $n = 1$.

Step 2. Condense $P_k(\underline{t})$ at $\underline{t}^{n-1}$ to form (3.6) with $P_k(\underline{t}, \underline{e}(\underline{t}^{n-1}))$.

Step 3. Solve (3.6) that is a posynomial geometric program and find its optimum solution $\underline{t}^*$.

Step 4. Check whether $\| \underline{t}^{n-1} - \underline{t}^* \| < \varepsilon$. If so, $\underline{t}^*$ is the optimum for (3.5) and for (3.1). Otherwise, increase n by 1 and let $\underline{t}^{n-1} = \underline{t}^*$. Turn to Step 2. |

The following example shows the application of the algorithm to solving SGP problem.

Example 3.2. Solve the problem that

minimize $g_0(\underline{t}) = 2\, t_1 t_2^{1/2} + t_2 t_3^{-1} t_4^{2} + t_1^{-2} t_2^{-1} t_3^{2}$,

subject to $g_1(\underline{t}) = t_1 t_2^{1/2} t_3 - t_2^{-1} t_4^{2} \le 1$,

$g_2(\underline{t}) = t_1 t_2 t_3 \ge 5$,

$t_j > 0$, for $j = 1, 2, 3, 4$.

Solution. The problem can be transformed to:

minimize t_0,

subject to $2\, t_0^{-1} t_1 t_2^{1/2} + t_0^{-1} t_2 t_3^{-1} t_4^{2} + t_0^{-1} t_1^{-2} t_2^{-1} t_3^{2} \le 1$,

$(t_1 t_2^{1/2} t_3) / (1 + t_2^{-1} t_4^{2}) \le 1$,

$5\, t_1^{-1} t_2^{-1} t_3^{-1} \le 1$,

$t_j > 0$, for $j = 0, 1, 2, 3, 4$.

$$P_2(\underline{t}, \underline{e}(\underline{t}^0)) = [\, 1 + (t_2^0)^{-1} (t_4^0)^2 \,] \left(\frac{t_2}{t_2^0} \right)^{(-1)(t_2^0)^{-1}(t_4^0)^2 / [1 + (t_2^0)^{-1}(t_4^0)^2}$$

$$\left(\frac{t_4}{t_4^0}\right)^{2\,(t_2^0)^{-1}\,(t_4^0)^2 / [\,1+(t_2^0)^{-1}\,(t_4^0)^2\,]},$$

Let $\underline{t}^0 = (10,\ 0.1,\ 5,\ 2)^T$, and $t_0 = 8.905$.

Then,

$$\underline{t}^0 \in F = \{\underline{t} \downarrow t_1 t_2^{1/2} t_3 - t_2^{-1} t_4^2 \le 1,$$
$$t_1 t_2 t_3 \ge 5,$$
$$t_j > 0, \quad \text{for } j = 1, 2, 3, 4\}.$$

$$P_2(\underline{t}, \underline{e}(\underline{t}^0)) = (1.12)\, t_2^{-0.9756}\, t_4^{1.95}.$$

The corresponding problem of (3.6) is that

minimize t_0,

subject to $2 t_0^{-1} t_1 t_2^{1/2} + t_0^{-1} t_2 t_3^{-1} t_4^2 + t_0^{-1} t_1^{-2} t_2^{-1} t_3^2 \le 1$,

$(0.89)\, t_1 t_2^{1.4756}\, t_3 t_4^{-1.95} \le 1$,

$5 t_1^{-1} t_2^{-1} t_3^{-1} \le 1$,

$t_j > 0, \quad \text{for } j = 0, 1, 2, 3, 4.$

The degree of difficulty of the problem is $\delta = 6 - (5 + 1) = 0$. We can find the only solution for the problem and the value of the objective function, that is,

$t_1^* = 3.27,\ t_2^* = 0.54,\ t_3^* = 2.82,\ t_4^* = 1.9,$

$t_0^* = g_0(\underline{t}^*) = 6.8852.$

Since $\| \underline{t}^* - \underline{t}^2 \| > \varepsilon$, more iterations are needed. Let

$\underline{t}^1 = \underline{t}^* = (3.27, 0.54, 2.82, 1.9, 6.8852)^T.$

Then computation shows that

$\underline{t}^2 = (3.066, 0.593, 2.75, 1.81, 6.7836)^T.$

$\underline{t}^3 = (3.047, 0.598, 2.744, 1.805, 6.7786)^T.$

$\underline{t}^4 = (3.051, 0.597, 2.745, 1.808, 6.7815)^T.$

And $\| \underline{t}^* - \underline{t}^4 \| < \varepsilon$. Thus, $\underline{t}^4$ is the optimum solution for the problem.

CHAPTER IV

THE REVERSED GEOMETRIC PROGRAMMING PROBLEM

In this chapter, we consider the reversed geometric programming problem (RGP), which is a special case of the signomial geometric programming problem (SGP). Since the SGP problem can be transformed to the RGP problem, this chapter can be considered as an extension of the preceding one in providing an alternative approach to solve the SGP problem other than that presented in the previous chapter. The main content can be found in [5] and [11].

The reversed geometric programming problem is of the form that

$$\begin{aligned} &\text{minimize} \quad g_0(\underline{t}), \\ &\text{subject to} \quad \underline{t} \in F_{PR}, \end{aligned} \tag{4.1}$$

where

$$F_{PR} = \{ \underline{t} \mid g_k(\underline{t}) \le 1, \ \text{for } k = 1, 2, \ldots, l, \quad g_k(\underline{t}) \ge 1, \ \text{for } k = l+1, l+2, \ldots, p, \quad t_j > 0, \ \text{for } j = 1, 2, \ldots, m \},$$

$$g_k(\underline{t}) = \sum_{i \in [k]} u_i(\underline{t}), \quad \text{for } k = 0, 1, \ldots, p,$$

$$u_i(\underline{t}) = c_i \prod_{j=1}^{m} t_j^{a_{ij}}, \quad \text{for } i \in [k], \ k = 0, 1, \ldots, l, \ c_i \ge 0, \ a_{ij} \in R,$$

$$u_i(\underline{t}) = c_i \prod_{j=1}^{m} t_j^{-a_{ij}}, \quad \text{for } i \in [k], \ k = l+1, l+2, \ldots, p, \ c_i \ge 0, \ a_{ij} \in R.$$

Note the exponents in $u_i(\underline{t})$ for $i \in [k]$, $k = l+1, l+2, \ldots, p$, are written as $(-a_{ij})$ here to facilitate the discussion later.

The associated dual problem of (4.1) can be expressed as:

$$\begin{aligned} &\text{maximize} \quad V(\underline{d}), \\ &\text{subject to} \quad \underline{d} \in F_{DR}, \end{aligned} \tag{4.2}$$

where

$$F_{DR} = \{ \underline{d} \mid \lambda_0 = 1,$$

$$\lambda_k = \sum_{i \in [k]} d_i \,, \quad \text{for } k = 0, 1, \ldots, p,$$

$$\sum_{i=1}^{n} a_{ij} d_i = 0 \,, \quad \text{for } j = 1, 2, \ldots, m, \quad (\text{i.e., } A^T \underline{d} = \underline{0}),$$

$$d_i \geq 0 \,, \quad \text{for } i = 1, 2, \ldots, n,$$

$$V(\underline{d}) = \prod_{k=0}^{l} \prod_{i \in [k]} (c_i / d_i)^{d_i} \prod_{k=l+1}^{p} \prod_{i \in [k]} (c_i / d_i)^{-d_i} \prod_{k=1}^{l} (\lambda_k)^{\lambda_k} \prod_{k=l+1}^{p} (\lambda_k)^{-\lambda_k} .$$

As shown in the following way, problem (3.1) can be transformed to problem (4.1).

Let $M_{PS} = \inf g_0(\underline{t})$ for all $\underline{t} \in F_{PS}$. Then for $M_{PS} \geq 0$, (3.1) is equivalent to the problem that

$$\begin{aligned} &\text{minimize} \quad t_0, \\ &\text{subject to} \quad \underline{t} \in F_{PR1}, \end{aligned} \tag{4.3}$$

where

$$F_{PR1} = \{ \underline{t} \mid (t_0)^{-1} g_0(\underline{t}) \leq 1,$$

$$g_k(\underline{t}) \leq \sigma_k \,, \quad \text{for } \sigma_k = 1 \text{ or } (-1), \; k = 1, 2, \ldots, p,$$

$$t_j > 0 \,, \quad \text{for } j = 1, 2, \ldots, m \}.$$

Otherwise, for $M_{PS} < 0$, equivalent to:

$$\begin{aligned} &\text{minimize} \quad (t_0)^{-1}, \\ &\text{subject to} \quad \underline{t} \in F_{PR2}, \end{aligned} \tag{4.4}$$

where

$F_{PR2} = \{ \underline{t} \mid (t_0)^{-1} g_0(\underline{t}) \leq -1,$

$g_k(\underline{t}) \leq \sigma_k$, for $\sigma_k = 1$ or (-1), $k = 1, 2, ..., p$,

$t_j > 0$, for $j = 1, 2, ..., m \}$.

If it is hard to judge the sign of M_{PS} in advance, we need to solve (4.3) and (4.4).

In general, $g_k(\underline{t})$ (the constraint of (3.1)) can be expressed as:

$$g_k(\underline{t}) = h_1(\underline{t}) - h_2(\underline{t}),$$

where $h_1(\underline{t})$, $h_2(\underline{t})$ are posynomials. So, for $g_k(\underline{t}) \leq \sigma_k$, if $\sigma_k = -1$, then

$$h_1(\underline{t}) - h_2(\underline{t}) \leq -1,$$

$$h_1(\underline{t}) + 1 \leq h_2(\underline{t}).$$

By introducing $t_{m+1} > 0$ such that

$$h_1(\underline{t}) + 1 \leq t_{m+1} \leq h_2(\underline{t}),$$

we get

$$(t_{m+1})^{-1} \left(h_1(\underline{t}) + 1 \right) \leq 1,$$

$$(t_{m+1})^{-1} h_2(\underline{t}) \geq 1.$$

If $\sigma_k = 1$, then

$$h_1(\underline{t}) \leq t_{m+1} \leq h_2(\underline{t}) + 1,$$

we get

$$(t_{m+1})^{-1} h_1(\underline{t}) \leq 1,$$

$$(t_{m+1})^{-1} \left(h_2(\underline{t}) + 1 \right) \geq 1.$$

As an illustrative example, consider the transformation of the problem that

$$\text{minimize } \left[\frac{(f_1(\underline{t}))^{1/2} + f_2(\underline{t})}{(f_3(\underline{t}))^{1/2} + f_4(\underline{t})} \right]^{1/2},$$

$$\text{subject to } t_j > 0, \text{ for } j = 1, 2, ..., m \tag{4.5}$$

where $f_i(\underline{t}) > 0$ for $i = 1, 3$, $(f_1(\underline{t}))^{1/2} + f_2(\underline{t}) > 0$, $(f_3(\underline{t}))^{1/2} + f_4(\underline{t}) > 0$.

Clearly, (4.5) is equivalent to the problem that

$$\begin{aligned}
&\text{minimize} \quad (t_0)^{1/2}, \\
&\text{subject to} \quad f_1(\underline{t}) > 0, \ \ f_3(\underline{t}) > 0, \\
&\qquad (f_1(\underline{t}))^{1/2} + f_2(\underline{t}) > 0, \\
&\qquad (f_3(\underline{t}))^{1/2} + f_4(\underline{t}) > 0, \\
&\qquad (f_1(\underline{t}))^{1/2} + f_2(\underline{t}) \le t_0 (f_3(\underline{t}))^{1/2} + t_0 f_4(\underline{t}), \\
&\qquad t_j > 0, \quad \text{for } j = 0, 1, \ldots, m.
\end{aligned} \tag{4.6}$$

That is a SGP problem.

By introducing t_{m+1}, t_{m+2}, $t_{m+3} > 0$ as bound variables, (4.6) can be transformed to the problem that

$$\begin{aligned}
&\text{minimize} \quad (t_0)^{1/2}, \\
&\text{subject to} \quad t_{m+1} \le f_1(\underline{t}) \le t_{m+2}, \\
&\qquad t_{m+3} \le f_3(\underline{t}), \\
&\qquad (t_{m+1})^{1/2} + f_2(\underline{t}) \ge 0, \\
&\qquad (t_{m+3})^{1/2} + f_4(\underline{t}) \ge 0, \\
&\qquad -(t_{m+2})^{1/2} - f_2(\underline{t}) + t_0(t_{m+3})^{1/2} + t_0 f_4(\underline{t}) \ge 0, \\
&\qquad t_j \ge 0, \quad \text{for } j = 0, 1, \ldots, m, m+1, m+2, m+3.
\end{aligned} \tag{4.7}$$

Suppose that

$$f_2(\underline{t}) = h_2'(\underline{t}) - h_2''(\underline{t}),$$

$$f_4(\underline{t}) = h_4'(\underline{t}) - h_4''(\underline{t}),$$

where $h_i'(\underline{t})$, $h_i''(\underline{t})$ for $i = 2, 4$ are posynomials. So,

$$(t_{m+1})^{1/2} + h_2'(\underline{t}) \ge h_2''(\underline{t}),$$

$$(t_{m+1})^{1/2} + h_2'(\underline{t}) \ge t_{m+4} \ge h_2''(\underline{t}),$$

then we have

$$(t_{m+1})^{1/2} (t_{m+4})^{-1} + (t_{m+4})^{-1} h_2'(\underline{t}) \ge 1,$$

$$(t_{m+4})^{-1} h_2''(\underline{t}) \le 1.$$

For the same reason, we have

$$(t_{m+3})^{1/2} (t_{m+5})^{-1} + (t_{m+5})^{-1} h_4'(\underline{t}) \ge 1,$$

$$(t_{m+5})^{-1} h_4''(\underline{t}) \le 1.$$

Therefore, for $t_{m+6} > 0$, we have

$$t_0 (t_{m+3})^{1/2} (t_{m+6})^{-1} + t_0 (t_{m+6})^{-1} h_4'(\underline{t}) + (t_{m+6})^{-1} h_2''(\underline{t}) \ge 1,$$

$$(t_{m+2})^{1/2} (t_{m+6})^{-1} + (t_{m+6})^{-1} h_2'(\underline{t}) + t_0 (t_{m+6})^{-1} h_4''(\underline{t}) \le 1.$$

Thus, we reach the following reversed geometric programming problem, which is equivalent to (4.5), (4.6) and (4.7). That is,

$$\begin{aligned}
\text{minimize} \quad & (t_0)^{1/2}, \\
\text{subject to} \quad & (t_{m+1})^{-1} f_1(\underline{t}) \ge 1, \\
& (t_{m+2})^{-1} f_1(\underline{t}) \le 1, \\
& (t_{m+3})^{-1} f_3(\underline{t}) \ge 1, \\
& (t_{m+1})^{1/2} (t_{m+4})^{-1} + (t_{m+4})^{-1} h_2'(\underline{t}) \ge 1, \\
& (t_{m+4})^{-1} h_2''(\underline{t}) \le 1, \\
& (t_{m+3})^{1/2} (t_{m+5})^{-1} + (t_{m+5})^{-1} h_4'(\underline{t}) \ge 1, \\
& (t_{m+5})^{-1} h_4''(\underline{t}) \le 1, \\
& t_0 (t_{m+3})^{1/2} (t_{m+6})^{-1} + t_0 (t_{m+6})^{-1} h_4'(\underline{t}) + (t_{m+6})^{-1} h_2''(\underline{t}) \ge 1, \\
& (t_{m+2})^{1/2} (t_{m+6})^{-1} + (t_{m+6})^{-1} h_2'(\underline{t}) + t_0 (t_{m+6})^{-1} h_4''(\underline{t}) \le 1, \\
& t_j > 0, \quad \text{for } j = 1, 2, \ldots, m, m+1, \ldots, m+6.
\end{aligned}$$

Consider the Lagrange problem of (4.1) that find $\underline{t}^* \in F_{PR}$ and Lagrange multiplier $\underline{\mu}^* = (\mu_1^*, \mu_2^*, \ldots, \mu_p^*)^T$ such that

$$\mu_k^* (g_k(\underline{t}^*) - 1) = 0,$$

$$\nabla_{\underline{t}} L(\underline{t}^*, \underline{\mu}^*) = 0,$$

where

$$L(\underline{t}, \underline{\mu}) = g_0(\underline{t}) + \sum_{k=1}^{l} \mu_k (g_k(\underline{t}) - 1) + \sum_{k=l+1}^{p} \mu_k (1 - g_k(\underline{t})).$$

is Lagrange function of (4.1),

For (4.1), let

$$S = \{ k \mid g_k(\underline{t}^*) = 1, \quad \text{for } 1 \le k \le p \},$$

$$A_{kj} = \sum_{i\in[k]} a_{ij} u_i(\underline{t}^*), \quad \text{for } k \in S, \ j = 1, 2, \ldots, m, \tag{4.8}$$

$$F_t = \{ \underline{t} \mid \sum_{j=1}^{m} A_{kj} (\log t_j - \log t_j^*) \le 0, \quad \text{for } k \in S \}.$$

Theorem 4.1. Suppose $\underline{t}^* \in F_{PR}$. Then there exists $\underline{\mu}^*$ such that $(\underline{t}^*, \underline{\mu}^*)$ is the solution of the Lagrange problem of (4.1) if and only if for all $\underline{t} \in F_t$,

$$g_0(\underline{t}^*) \le g_0(\underline{t}).$$

Proof. (For necessity) Note that for $k = 0, 1, \ldots, p$,

$$\frac{\partial g_k(\underline{t})}{\partial t_j} = (1/t_j) \frac{\partial}{\partial t_j} (\sum_{i\in[k]} c_i \prod_{l=1}^{m} t_l^{a_{il}}) \, a_{ij} = (1/t_j) \sum_{i\in[k]} u_i(\underline{t}) \, a_{ij}.$$

So, for $k = 0$,

$$\sum_{i\in[0]} u_i(\underline{t}) \, a_{ij} = t_j \frac{\partial g_0(\underline{t})}{\partial t_j}.$$

For $k = 1, 2, \ldots, p$,

$$A_{kj} = \sum_{i\in[k]} u_i(\underline{t}^*) \, a_{ij} = t_j \frac{\partial g_k(\underline{t}^*)}{\partial t_j}.$$

So

$$\frac{\partial}{\partial t_j} [\mu_k (g_k(\underline{t}) - 1)] = (\mu_k / t_j) A_{kj}, \quad \text{for } k = 1, 2, \ldots, l,$$

$$\frac{\partial}{\partial t_j} [\mu_k (1 - g_k(\underline{t}))] = (\mu_k / t_j) A_{kj}, \quad \text{for } k = l+1, l+2, \ldots, p,$$

Thus, for $k \in S$,

$$\frac{\partial}{\partial t_j}[L(\underline{t}^*, \underline{\mu}^*)] = \frac{\partial g_0(\underline{t})}{\partial t_j} + (1/t_j^*) \sum_{k\in S} \mu_k^* A_{kj}.$$

Let $x_j = \log t_j$. Then for $k = 0, 1, ..., p$, we know

$$g_k(\underline{t}) = \sum_{i\in[k]} u_i(\underline{t}) = \sum_{i\in[k]} c_i \prod_{j=1}^{m} t_j^{a_{ij}},$$

so,

$$f_k(\underline{x}) = \sum_{i\in[k]} V_i(\underline{x}) = \sum_{i\in[k]} c_i \exp\{\sum_{j=1}^{m} a_{ij} x_j\},$$

and for all $i \in [k]$, $k = 0, 1, ..., p$,

$$u_i(\underline{t}) = V_i(\underline{x}), \quad g_k(\underline{t}) = f_k(\underline{x}).$$

Hence, for $j = 1, 2, ..., m$,

$$\frac{\partial f_k(\underline{x})}{\partial x_j} = \sum_{i\in[k]} c_i a_{ij} \exp\{\sum_{j=1}^{m} a_{ij} x_j\} = \sum_{i\in[k]} V_i(\underline{x})\, a_{ij} = \sum_{i\in[k]} u_i(\underline{t})\, a_{ij}.$$

That means that

$$\nabla_{\underline{x}} f_0(\underline{x}) = \left(t_1 \frac{\partial g_0(\underline{t})}{\partial t_1}, t_2 \frac{\partial g_0(\underline{t})}{\partial t_2}, ..., t_m \frac{\partial g_0(\underline{t})}{\partial t_m}\right)^T, \quad \text{for } k = 0,$$

$$\nabla_{\underline{x}} f_k(\underline{x}^*) = (A_{k1}, A_{k2}, ..., A_{km})^T, \quad \text{for } k = 1, 2, ..., l,$$

and

$$\nabla_{\underline{x}} f_k(\underline{x}^*) = (-A_{k1}, -A_{k2}, ..., -A_{km})^T, \quad \text{for } k = l+1, l+2, ..., p.$$

By the definition we know that $\exists\, \underline{\mu}^*$ such that $(\underline{t}^*, \underline{\mu}^*)$ forms a solution of the Lagrange problem if and only if for $k \in S$,

$$\nabla_{\underline{t}} L(\underline{t}^*, \underline{\mu}^*) = 0.$$

That is,

$$\frac{\partial g_0(\underline{t}^*)}{\partial t_j} + (1/t_j^*) \sum_{k\in S} (\mu_k^*) A_{kj} = 0, \quad \text{for } t_j^* > 0,$$

i.e.,

$$t_j^* \frac{\partial g_0(\underline{t}^*)}{\partial t_j} = - \sum_{k \in S} (\mu_k^*) A_{kj}, \quad \text{for } \mu_k^* \geq 0, \; j = 1, 2, ..., m.$$

By Lemma 1.1, for $\underline{t} \in F_t$, that is, for all $\underline{t} > 0$, such that for j =1, 2, ..., m,

$$-\sum_{j=1}^{m} A_{kj} (\log t_j - \log t_j^*) \geq 0, \quad \text{(the tangentially optimal property)},$$

i.e.,

$$[\nabla_{\underline{x}} f_k(\underline{x}^*)]^T (\underline{x} - \underline{x}^*) \leq 0, \quad \text{for } k = 1, 2, ..., l,$$

$$[\nabla_{\underline{x}} f_k(\underline{x}^*)]^T (\underline{x} - \underline{x}^*) \geq 0, \quad \text{for } k = l+1, l+2, ..., p.$$

It holds that

$$\sum_{j=1}^{m} [t_j^* \frac{\partial g_0(\underline{t}^*)}{\partial t_j}] (\log t_j - \log t_j^*) \geq 0 \quad \text{(by Farkas lemma)},$$

that is,

$$[\nabla_{\underline{x}} f_0(\underline{x}^*)]^T (\underline{x} - \underline{x}^*) \geq 0.$$

Note that $g_0(\underline{t})$ is posynomial and so $f_0(\underline{x})$ is convex, i.e.,

$$g_0(\underline{t}) - g_0(\underline{t}^*) = f_0(\underline{x}) - f_0(\underline{x}^*) \geq [\nabla_{\underline{x}} f_0(\underline{x}^*)]^T (\underline{x} - \underline{x}^*).$$

Thus, if

$$\nabla_{\underline{t}} L (\underline{t}^*, \underline{\mu}^*) = 0, \quad \text{for } k \in S,$$

($\Leftrightarrow \exists \, \underline{\mu}^*$ such that $(\underline{t}^*, \underline{\mu}^*)$ forms Lagrange problem solution), then

$$g_0(\underline{t}) \geq g_0(\underline{t}^*), \quad \text{for all } \underline{t} \in F_t.$$

(For sufficiency) Conversely, suppose for all $\underline{t} \in F_t$, it holds that

$$g_0(\underline{t}^*) \leq g_0(\underline{t}),$$

i.e.,

$$f_0(\underline{x}^*) \leq f_0(\underline{x}),$$

for $\underline{x}$ such that

$$[\nabla_{\underline{x}} f_k(\underline{x}^*)]^T (\underline{x} - \underline{x}^*) \leq 0, \quad \text{for } k = 1, 2, ..., l,$$

$$[\nabla_{\underline{x}} f_k(\underline{x}^*)]^T (\underline{x} - \underline{x}^*) \geq 0, \quad \text{for } k = l+1, l+2, ..., p.$$

Then, let $\underline{y} = \underline{x}^* + \xi(\underline{x} - \underline{x}^*)$, $\xi > 0$. Then since

$$[\nabla_{\underline{x}} f_k(\underline{x}^*)]^T (\underline{y} - \underline{x}^*) = [\nabla_{\underline{x}} f_k(\underline{x}^*)]^T [\underline{x}^* + \xi(\underline{x} - \underline{x}^*) - \underline{x}^*]$$
$$= \xi [\nabla_{\underline{x}} f_k(\underline{x}^*)]^T (\underline{x} - \underline{x}^*),$$

we have

$$[\nabla_{\underline{x}} f_k(\underline{x}^*)]^T (\underline{y} - \underline{x}^*) \le 0, \quad \text{for } k = 1, 2, \ldots, l,$$
$$[\nabla_{\underline{x}} f_k(\underline{x}^*)]^T (\underline{y} - \underline{x}^*) \ge 0, \quad \text{for } k = l+1, l+2, \ldots, p.$$

Subsequently, $f_0(\underline{x}^*) \le f_0(\underline{y})$ (for such a $\underline{y}$), and take $\xi \to 0$ ($\underline{y} \to \underline{x}^*$), then

$$\lim_{\xi \to 0} \frac{f_0(\underline{y}) - f_0(\underline{x}^*)}{\underline{y} - \underline{x}^*} = \nabla_{\underline{x}} f_0(\underline{x}^*),$$

$$[\nabla_{\underline{x}} f_0(\underline{x}^*)]^T (\underline{x} - \underline{x}^*) = (1/\xi)[\nabla_{\underline{x}} f_0(\underline{x}^*)]^T (\underline{y} - \underline{x}^*) \ge 0,$$

i.e.,

$$\sum_{j=1}^{m} [\,(t_j^*) \frac{\partial g_0(\underline{t}^*)}{\partial t_j}\,] (\log t_j - \log t_j^*) \ge 0, \quad \text{for all } \underline{t} \in F_t.$$

That is equivalent to that $\nabla_{\underline{t}} L(\underline{t}^*, \underline{\mu}^*) = 0$ ($k \in S$), and ($\underline{t}^*, \underline{\mu}^*$) forms a Lagrange solution. |

Since (4.1) lacks convexity, it can not maintain the properties that (1.1) possesses, for instance, any local optimum solution is its global optimum solution, $M_D \le M_P$, etc. For (4.1), like the solution(s) for other nonlinear programming problems, its local optimum solution(s) must be a part of its Lagrange problem solution(s), so we can find its local optimum solution(s) from within the solution(s) of its Lagrange problem. Conversely, in general (for $p > l$), the solution of Lagrange problem $\underline{t}^*$ does not necesarily mean the global or even local optimum solution. However, Theorem 4.1 shows that although $\underline{t}^*$ is not the local optimum solution for the problem, yet within a certain range, that is, for all $\underline{t} \in F_t$, $\underline{t}^*$ remains the optimum. That property is called the tangential optimality.

Definition 4.1. $\underline{t}^* \in F_{PR}$, $\underline{d}^* \in F_{DR}$ are called *primal equilibrium solution* and *dual equilibrium solution*, respectively, if it holds that

$$d_i^* \, g_0(\underline{t}^*) = u_i(\underline{t}^*), \quad \text{for } i \in [0],$$
$$d_i^* = \lambda_k^* \, u_i(\underline{t}^*), \quad \text{for } i \in [k],\ k = 1, 2, \dots, p, \tag{4.9}$$

where

$$\lambda_k^* = \sum_{i \in [k]} d_i^* .$$

Here, $g_0(\underline{t}^*)$ and $V(\underline{d}^*)$ are called *primal equilibrium value* and *dual equilibrium value,* respectively. |

There exists the following relation between the equilibrium solution and the Lagrange problem solution.

Theorem 4.2. Suppose that $\underline{t}^*$ and $\underline{\mu}^*$ are a pair of solutions of Lagrange problem. Then $\underline{t}^*$ and $\underline{d}^*$ defined by

$$d_i^* = u_i(\underline{t}^*) / g_0(\underline{t}^*), \quad \text{for } i \in [0],$$
$$d_i^* = (\mu_k^*) \, u_i(\underline{t}^*) / g_0(\underline{t}^*), \quad \text{for } i \in [k],\ k = 1, 2, \dots, p, \tag{4.10}$$

form a pair of primal and dual equilibrium solutions. Conversely, if $\underline{t}^*$ and $\underline{d}^*$ are a pair of primal and dual equilibrium solutions, then $\underline{t}^*$ and $\underline{\mu}^*$ defined by

$$\mu_k^* = \lambda_k^* \, g_0(\underline{t}^*), \quad \text{for } k = 1, 2, \dots, p, \tag{4.11}$$

where

$$\lambda_k^* = \sum_{i \in [k]} d_i^*,$$

is a pair of solutions of Lagrange problem.

Proof. Suppose $\underline{t}^*$ and $\underline{\mu}^*$ are a pair of solutions of Lagrange problem, then $\underline{d}^*$ defined by (4.10) satisfies that $\underline{d}^* \geq \underline{0}$ (positivity). By Definition 4.1, $\lambda_0^* = 1$ (normality). Let $\mu_0^* = 1$. Then

$$\sum_{k=0}^{p} \sum_{i\in[k]} a_{ij}\, d_i^* = \sum_{k=0}^{p} \mu_k^* \sum_{i\in[k]} a_{ij}\, u_i(\underline{t}^*) \,/\, g_0(\underline{t}^*)$$

$$= [\, t_j^* / g_0(\underline{t}^*) \,]\, [\, (1/t_j^*)\, (\mu_k^*) \sum_{i\in[k]} a_{ij}\, u_i(\underline{t}^*) \,]$$

$$= [\, t_j^* / g_0(\underline{t}^*) \,]\, [\, \frac{\partial}{\partial t_j} L\, (\underline{t}^*, \underline{\mu}^*) \,]$$

$$= 0 \quad \text{(orthogonality)}.$$

So, $\underline{d}^* \in F_{DR}$.

Note that $d_i^* = (\mu_k^*)\, u_i(\underline{t}^*) / g_0(\underline{t}^*)$ and $(\mu_k^*)\, [\, g_k(\underline{t}^*) - 1 \,] = 0$, so,

$$\lambda_k^* = \sum_{i\in[k]} d_i^* = \mu_k^* / g_0(\underline{t}^*),$$

i.e.,

$$\mu_k^* = \lambda_k^*\, g_0(\underline{t}^*).$$

So,

$$d_i^* = \lambda_k^*\, u_i(\underline{t}^*), \quad \text{for } i \in [k],\ k = 1, 2, \dots, p.$$

Note that $d_i^*\, g_0(\underline{t}^*) = u_i(\underline{t}^*)$ for $i \in [0]$ and by Definition 4.1, $\underline{t}^*$ and $\underline{d}^*$ are a pair of primal and dual equilibrium solutions.

Suppose that $\underline{t}^*$ and $\underline{d}^*$ are a pair of primal and dual equilibrium solutions. Since $\underline{d}^* \geq 0$, $d_i^* = \lambda_k^*\, u_i(\underline{t}^*)$ and $\lambda_k^* = \sum_{i\in[k]} d_i^* = \lambda_k^*\, g_k(\underline{t}^*)$, so, $g_k(\underline{t}^*) = 1$. Note that

$$t_j^* \frac{\partial}{\partial t_j} L\, (\underline{t}^*, \underline{\mu}^*) = t_j^*\, (1/t_j^*) \sum_{k=0}^{p} \mu_k^* \sum_{i\in[k]} a_{ij}\, u_i(\underline{t}^*)$$

$$= g_0(\underline{t}^*) \sum_{k=0}^{p} \sum_{i\in[k]} a_{ij} (\lambda_k^*) u_i(\underline{t}^*)$$

$$= g_0(\underline{t}^*) \sum_{k=0}^{p} \sum_{i\in[k]} a_{ij} d_i^* = 0.$$

This leads to $\frac{\partial}{\partial t_j} L(\underline{t}^*, \underline{\mu}^*) = 0$ since $t_j^* > 0$. Hence, $\underline{t}^*$ and $\underline{\mu}^*$ are a pair of solutions of Lagrange problem. |

From Theorem 4.1 and 4.2, we reach the following theorem.

Theorem 4.3. Suppose $\underline{t}^* \in F_{PR}$. Then $\underline{t}^*$ is the primal equilibrium solution if and only if for all $\underline{t} > 0$, $\underline{t} \in F_t$, $g_0(\underline{t}^*) \le g_0(\underline{t})$. |

Theorem 4.4. Suppose $\underline{t}^*$ and $\underline{d}^*$ are a pair of primal and dual equilibrium solutions. Then

(i) For all $k = 0, 1, ..., p$, either $d_i^* = 0$ for all $i \in [k]$, $\lambda_k^* = 0$, or $d_i^* > 0$ for all $i \in [k]$, $\lambda_k^* > 0$, especially $\lambda_0^* > 0$.

(ii) For all $\underline{\tau}$ such that $A^T\underline{\tau} = \underline{0}$, i.e., $\sum_{k=0}^{p} \sum_{i\in[k]} a_{ij}\tau_i = 0$, and $\tau_i = 0$ if $d_i = 0$ (for $(d_i^*)^{\tau_i}$ definable),

$$[g_0(\underline{t}^*)]^{\omega_0} = \prod_{k=0}^{l} \prod_{i\in[k]} (c_i/d_i)^{\tau_i} \prod_{k=l+1}^{p} \prod_{i\in[k]} (c_i/d_i)^{-\tau_i} \prod_{k=1}^{l} (\lambda_k^*)^{\omega_k} \prod_{k=l+1}^{p} (\lambda_k^*)^{-\omega_k}$$

where $\omega_0 = \sum_{i\in[k]} \tau_i$, $\omega_k = \sum_{i\in[k]} \tau_i$, for $k = 1, 2, ..., p$. (4.12)

(iii) $g_0(\underline{t}^*) = V(\underline{d}^*)$.

Proof. (i) Note $d_i^* = \lambda_k^* u_i(\underline{t}^*)$, $u_i(\underline{t}^*) > 0$, for $i \in [k]$, $k = 1, 2, ..., p$. If $\exists\ i \in [k]$ such that $d_i^* = 0$, then it must be true that $\lambda_k^* = 0$, so for all $i \in [k]$, $d_i^* = 0$.

(ii) We know that $d_i^* g_0(\underline{t}^*) = u_i(\underline{t}^*)$ for $i \in [0]$, $d_i^* = \lambda_k^* u_i(\underline{t}^*)$ for $i \in [k]$, $k = 1$, 2, ..., p. So,

$$(d_i^* / c_i)^{\tau_i} [g_0(\underline{t}^*)]^{\tau_i} = [u_i(\underline{t}^*) / c_i]^{\tau_i}, \quad \text{for } i \in [0],$$

$(d_i^* / c_i)^{\tau_i} = [\lambda_k^*]^{\tau_i} [u_i(t^*) / c_i]^{\tau_i}$, for $\in [k]$, $k = 1, 2, ..., l$,

$(d_i^* / c_i)^{-\tau_i} = [\lambda_k^*]^{-\tau_i} [u_i(t^*) / c_i]^{-\tau_i}$, for $i \in [k]$, $k = l + 1, l + 2, ..., p$.

Since $A^T \underline{\tau} = \underline{0}$, i.e., $\sum_{k=0}^{p} \sum_{i\in[k]} a_{ij} \tau_i = \eta_j = 0$, for $j = 1, 2, ..., m$,

$$[g_0(t^*)]^{\omega_0} \prod_{k=0}^{l} \prod_{i\in[k]} (d_i/c_i)^{\tau_i} \prod_{k=l+1}^{p} \prod_{i\in[k]} (d_i/c_i)^{-\tau_i} = \prod_{k=1}^{l} (\lambda_k^*)^{\omega_k} \prod_{k=l+1}^{p} (\lambda_k^*)^{-\omega_k} [\prod_{j=1}^{m} (t_j^*)^{\eta_j}]$$

where

$$\omega_0 = \sum_{i\in[0]} \tau_i ,$$

$$\omega_k = \sum_{i\in[k]} \tau_i , \quad \text{for } k = 1, ..., p,$$

Thus,

$$[g_0(t^*)]^{\omega_0} = \prod_{k=0}^{l} \prod_{i\in[k]} (c_i/d_i)^{\tau_i} \prod_{k=l+1}^{p} \prod_{i\in[k]} (c_i/d_i)^{-\tau_i} \prod_{k=1}^{l} (\lambda_k^*)^{\omega_k} \prod_{k=l+1}^{p} (\lambda_k^*)^{-\omega_k},$$

where

$$\omega_0 = \sum_{i\in[0]} \tau_i ,$$

$$\omega_k = \sum_{i\in[k]} \tau_i , \quad \text{for } k = 1, 2, ..., p.$$

(iii) Let $\underline{\tau} = \underline{d}^*$, $\lambda_0^* = 1$. Then we get $g_0(t^*) = V(\underline{d}^*)$. |

Definition 4.2. (nullity vector) $\underline{v} = (v_1, v_2, ..., v_n)^T$ is a *nullity vector* if

$$\sum_{i\in[0]} v_i = 0,$$

$$A^T \underline{v} = \underline{0}.$$ |

It is clear to see that if take $\underline{\tau}$ of (4.12) as $\underline{v}$ ($\underline{\tau} = \underline{v}$), then we obtain the following theorem that gives a necessary condition on which $\underline{d}^*$ may be a dual equilibrium solution, especially for $l = p$, $\underline{d}^*$ is a dual optimum solution.

Theorem 4.5. Suppose that $\underline{d}^*$ is a dual equilibrium solution. Then for all $\underline{v}$ such that $\underline{v} = \underline{0}$ if $\underline{d}^* = \underline{0}$,

$F(\underline{d}^*, \underline{v}) = G(\underline{c}, \underline{v})$ (equilibrium identity),

where

$$F(\underline{d}^*, \underline{v}) = \prod_{k=0}^{l} \prod_{i\in[k]} (d_i^*)^{v_i} \prod_{k=l+1}^{p} \prod_{i\in[k]} (d_i^*)^{-v_i} \prod_{k=1}^{l} [\sum_{i\in[k]} d_i^*]^{-w_k} \prod_{k=l+1}^{p} [\sum_{i\in[k]} d_i^*]^{w_k}$$

$(w_0 = \sum_{i\in[0]} v_i, \; w_k = \sum_{i\in[k]} v_i, \;$ for $k = 1, 2, ..., p)$,

$$G(\underline{c}, \underline{v}) = \prod_{k=0}^{l} \prod_{i\in[k]} (c_i)^{v_i} \prod_{k=l+1}^{p} \prod_{i\in[k]} (c_i)^{-v_i}.$$ |

Theorem 4.6. Suppose that $\underline{d}$ is a dual consistent solution, i.e., $\underline{d} \in F_{DR}$, $\underline{v}$ is a nullity vector such that $v_i = 0$ if $d_i = 0$, for $i = 1, 2, ..., n$. Then $\underline{d} + \varepsilon\underline{v}$ is a dual consistent solution as $|\varepsilon|$ is sufficiently small ($\varepsilon \to 0$), and

$$D_v V(\underline{d}) = [\log G(\underline{c}, \underline{v}) - \log F(\underline{d}, \underline{v})] V(\underline{d}),$$

where $D_v V(\underline{d})$ is the directional derivative of $V(\underline{d})$ at $\underline{d}$ along the direction $\underline{v}$.

Especially, as $\underline{d} = \underline{d}^*$ (the dual equilibrium solution),

$$D_v [V(\underline{d}^*)] = 0.$$

Proof. Note that $\underline{v} = \underline{0}$ if $\underline{d} = \underline{0}$ and by the definition we know

$$\underline{d} + \varepsilon\underline{v} \in F_{DR}, \text{ as } \varepsilon \to 0.$$

Since

$$\log V(\underline{d}) = \sum_{k=0}^{l} \sum_{i\in[k]} d_i (\log c_i - \log d_i) + \sum_{k=1}^{l} \lambda_k \log \lambda_k$$

$$- \sum_{k=l+1}^{p} \sum_{i\in[k]} d_i (\log c_i - \log d_i) - \sum_{k=l+1}^{p} \lambda_k \log \lambda_k,$$

$$D_v [\log V(\underline{d})] = D_v [V(\underline{d})] / V(\underline{d})$$

$$= \frac{d}{d\varepsilon} \{\log [V(\underline{d} + \varepsilon\underline{v})]\} |_{\varepsilon=0}$$

$$= \sum_{k=0}^{l} \sum_{i\in[k]} (\log c_i - \log d_i - 1) v_i + \sum_{k=1}^{l} \lambda_k (\log \lambda_k + 1)$$

$$- \sum_{k=l+1}^{p} \sum_{i \in [k]} d_i (\log c_i - \log d_i - 1) v_i - \sum_{k=l+1}^{p} \lambda_k (\log \lambda_k + 1)$$

$$= \log G (\underline{c}, \underline{v}) - \log F (\underline{d}, \underline{v}) - \sum_{i \in [0]} v_i .$$

Note that $\sum_{i \in [0]} v_i = 0$, so we have

$$D_v [V(\underline{d})] = [\log G (\underline{c}, \underline{v}) - \log F (\underline{d}, \underline{v})] V(\underline{d}) .$$

By Theorem 4.5, as $\underline{d} = \underline{d}^*$ (the dual equilibrium solution), we have

$$D_v [V(\underline{d}^*)] = 0.$$

Lemma 4.1. Suppose $u_i > 0$, $d_i > 0$, for $i = 1, 2, ..., n$, and $\sum_{i=1}^{n} d_i = 1$. Then

$$(\sum_{i=1}^{n} u_i)^{-1} \le \prod_{i=1}^{n} (d_i / u_i)^{d_i} \le \sum_{i=1}^{n} d_i^2 / u_i$$

and "=" holds if and only if $u_i = d_i \sum_{i=1}^{n} u_i$.

Proof. From Corollary 1.1 we know that for $u_i > 0$, $d_i > 0$, $i = 1, 2, ..., n$,

$$\prod_{i=1}^{n} (u_i / d_i)^{d_i} \lambda^{\lambda} \le (\sum_{i=1}^{n} u_i)^{\lambda},$$

where $\lambda = \sum_{i=1}^{n} d_i$ and "=" holds if and only if $u_i \sum_{i=1}^{n} d_i = d_i \sum_{i=1}^{n} u_i$. That is,

$$(\sum_{i=1}^{n} u_i)^{-1} \le \prod_{i=1}^{n} (d_i / u_i)^{d_i} \quad (\lambda = 1).$$

Now let $u_i = d_i^2 / u_i$. Then

$$[\sum_{i=1}^{n} (d_i^2 / u_i)]^{-1} \le \prod_{i=1}^{n} (\frac{d_i}{d_i^2/u_i})^{d_i},$$

i.e.,

$$\prod_{i=1}^{n} (d_i / u_i)^{d_i} \le \sum_{i=1}^{n} d_i^2 / u_i .$$

Therefore,

$$(\sum_{i=1}^{n} u_i)^{-1} \le \prod_{i=1}^{n} (d_i / u_i)^{d_i} \le \sum_{i=1}^{n} (d_i^2 / u_i) ,$$

so, "=" holds if and only if $u_i = d_i \sum_{i=1}^{n} u_i \quad (\sum_{i=1}^{n} d_i = 1)$.

For $\underline{\omega} > \underline{0}$ (weight vector) such that $\sum_{i=1}^{n} \omega_i = 1$ and posynomial

$$g_k(\underline{t}) = \sum_{i \in [k]} u_i(\underline{t}) = \sum_{i \in [k]} c_i \prod_{j=1}^{m} t_j^{a_{ij}}, \quad \text{for } c_i \geq 0,\ a_{ij} \in R,$$

define

$$g_k'(\underline{t}, \underline{\omega}) = \prod_{i \in [k]} (\, \omega_i / u_i(\underline{t})\,)^{\omega_i},$$

$$g_k''(\underline{t}, \underline{\omega}) = \sum_{i \in [k]} \omega_i^2 / u_i(\underline{t}) \quad \text{(harmonized mean).}$$

Then by Lemma 4.1, for all $\underline{t} > \underline{0}$,

$$1 / g_k(t) \leq g_k'(\underline{t}, \underline{\omega}) \leq g_k''(\underline{t}, \underline{\omega}).$$

For $k = l+1, l+2, \ldots, p$ in (4.1) and $\underline{\omega} > \underline{0}$ such that $\sum_{i=1}^{n} \omega_i = 1$, we can form the harmonized program (4.13) corresponding to (4.1), which is a posynomial geometric programming program. That is,

$$\text{minimize} \quad g_0(\underline{t}) ,$$

$$\text{subject to} \quad \underline{t} \in F_{PW}, \tag{4.13}$$

where

$$F_{PW} = \{\, \underline{t} \mid g_k(\underline{t}) \leq 1, \ \text{for } k = 1, 2, \ldots, l,$$
$$g_k''(\underline{t}, \underline{\omega}) \leq 1, \ \text{for } k = l+1, l+2, \ldots, p,$$
$$t_j > 0, \ \text{for } j = 1, 2, \ldots, m \,\},$$

$$g_k(\underline{t}) = \sum_{i \in [k]} u_i(\underline{t}) = \sum_{i \in [k]} c_i \prod_{j=1}^{m} t_j^{a_{ij}}, \quad \text{for } i \in k],\ k = 0, 1, \ldots, l,\ c_i \geq 0,\ a_{ij} \in R,$$

$$g_k''(\underline{t}, \underline{\omega}) = \sum_{i \in [k]} \omega_i^2 / u_i(\underline{t}) = \sum_{i \in [k]} (\, \omega_i^2 / c_i\,) \prod_{j=1}^{m} t_j^{a_{ij}} \quad \text{(harmonized mean).}$$

The associated dual program of (4.13) is:

$$\text{maximize} \quad V'(\underline{d}) ,$$

subject to $\underline{d} \in F_{DW}$, (4.14)

where

$$F_{DR} = \{ \underline{d} \mid \lambda_0 = \sum_{i \in [0]} d_i = 1,$$

$$\sum_{k=1}^{p} \sum_{i \in [k]} a_{ij} d_i = 0, \quad \text{for } j = 1, 2, \ldots, m,$$

$$d_i \geq 0, \quad \lambda_k = \sum_{i \in [k]} d_i \geq 0, \quad \text{for } i = 1, 2, \ldots, n, \; k = 0, 1, \ldots, p \},$$

$$V'(\underline{d}) = \prod_{k=0}^{l} \prod_{i \in [k]} (c_i / d_i)^{d_i} \prod_{k=l+1}^{p} \prod_{i \in [k]} [\omega_i^2 / (c_i d_i)]^{d_i} \prod_{k=1}^{p} (\lambda_k)^{\lambda_k}.$$

Theorem 4.8. Suppose $\underline{t} \in F_{PW}$ for $\underline{\omega}(\underline{t}) > \underline{0}$ such that $\sum_{i \in [k]} \omega_i(\underline{t}) = 1$, $k = l+1, l+2, \ldots, p$, then $\underline{t} \in F_{PR}$. Conversely, if $\underline{t} \in F_{PR}$, then $\underline{t} \in F_{PW}$ for $\underline{\omega}(\underline{t}) > \underline{0}$ such that $\omega_i(\underline{t}) = u_i(\underline{t}) / g_k(\underline{t})$, $i \in [k]$, $k = l+1, l+2, \ldots, p$.

Proof. Note that

$$1/g_k(t) \leq g_k''(\underline{t}, \underline{\omega}(\underline{t})) \leq 1, \quad \text{for } k = l+1, l+2, \ldots, p,$$

i.e., for all $\underline{t} \in F_{PW}$, it must be true that $\underline{t} \in F_{PR}$, and the converse is not necessarily true. So,

$$F_{PW} \subset F_{PR}.$$

But for $\underline{\omega}(\underline{t}) > \underline{0}$ such that

$$\omega_i(\underline{t}) = u_i(\underline{t}) / g_k(\underline{t}), \quad \text{for } i \in [k], \; k = l+1, l+2, \ldots, p,$$

$$\sum_{i \in [k]} \omega_i(\underline{t}) = 1,$$

by Lemma 4.1, we know that

$$1/g_k(\underline{t}) = g_k''(\underline{t}, \underline{\omega}(\underline{t}))$$

if and only if

$$u_i(\underline{t}) = \omega_i(\underline{t}) \sum_{i \in [k]} u_i(\underline{t}) = \omega_i \, g_k(t).$$

That is, $F_{PW} = F_{PR}$, and $\underline{t} \in F_{PR}$ implies $\underline{t} \in F_{PW}$. |

Note that $F_{DW} = F_{DR}$. So, we reach the following two theorems.

Theorem 4.9. $\underline{d} \in F_{DR}$ if and only if $\underline{d} \in F_{DW}$. And if $\underline{d} \in F_{DR}$ for $\underline{\omega} > \underline{0}$ such that $\sum_{i \in [k]} \omega_i = 1$, then $\underline{d} \in F_{DW}$. |

Lemma 4.2. If (1.1) is consistent and $M_P > 0$, then (1.2) is consistent and $M_D = M_P \in (0, \infty)$. |

Theorem 4.10. If (4.1) is consistent, i.e., $F_{PR} \neq \phi$ and $M_{PR} = \inf g_0(\underline{t}) > 0$ for $\underline{t} \in F_{PR}$, then (4.2) must be consistent, i.e., $F_{DR} \neq \phi$.

Proof. Suppose that $\underline{t} \in F_{PR}$. Then by Theorem 4.8, for $\underline{\omega}(\underline{t}) > \underline{0}$ such that $\omega_i(\underline{t}) = u_i(\underline{t}) / g_k(\underline{t})$, for $i \in [k]$, $k = l+1, l+2, \ldots, p$, $\underline{t} \in F_{PW}$, i.e., $F_{PW} \neq \phi$. Since $F_{PW} \subset F_{PR}$, so $0 < M_{PR} \leq M_{PW}$. Note that (4.13) is a posynomial program, so by Lemma 4.2 it must be consistent. Thus, by Theorem 4.9, (4.2) is consistent. |

Theorem 4.11. Suppose $\underline{t}^* \in F_{PR}$, $\underline{d}^* \in F_{DR}$ and define $\underline{\omega}^*$ as

$$\omega_i(\underline{t}^*) = u_i(\underline{t}^*) / g_k(\underline{t}^*), \text{ for } i \in [k],\ k = l+1, l+2, \ldots, p.$$

Then $\underline{t}^*$, $\underline{d}^*$ are equilibrium solutions of (4.1) and (4.2), respectively, and $g_0(\underline{t}^*) = V(\underline{d}^*, \underline{\omega}^*)$ if and only if $\underline{t}^*$ and $\underline{d}^*$ are a pair of optimum solutions of (4.13) and (4.14).

Proof. By Theorem 4.8 and 4.9, that $\underline{t}^* \in F_{PR}$, $\underline{d}^* \in F_{DR}$ and $\omega_i(\underline{t}^*) = u_i(\underline{t}^*) / g_k(\underline{t}^*)$ implies $\underline{t}^* \in F_{PW}$, $\underline{d}^* \in F_{DW}$. Then by Theorem 1.4, $\underline{t}^*$, $\underline{d}^*$ are the optimum solutions of (4.13) and (4.14), respectively, if and only if

$$d_i^* = u_i(\underline{t}^*) / g_0(\underline{t}^*), \quad \text{for } i \in [0],$$

$$d_i^* = (\lambda_k^*)\, u_i(\underline{t}^*), \quad \text{for } i \in [k],\ k = 1, 2, \ldots, l,$$

$$d_i^* = (\lambda_k^*)\, (\omega_i^*)^2 / u_i(\underline{t}^*), \quad \text{for } i \in [k],\ k = l+1, +2, \ldots, p.$$

Since $\omega_i^* = u_i(\underline{t}^*) / g_k(\underline{t}^*)$, so, for $i \in [k]$, $k = l+1, l+2, \ldots, p$,

$$d_i^* = (\lambda_k^*)\,(\omega_i^*)^2 / u_i(\underline{t}^*) = (\lambda_k^*)\,[\,u_i(\underline{t}^*)\,]^2 / [\,g_k(\underline{t}^*)\,]^2\, u_i(\underline{t}^*)\,,$$

$$= (\lambda_k^*)\, u_i(\underline{t}^*) / [\,g_k(\underline{t}^*)\,]^2. \tag{4.15}$$

For $\lambda_k^* = 0$, $d_i^* = (\lambda_k^*)\, u_i(\underline{t}^*) / [\,g_k(\underline{t}^*)\,]^2 = 0$, and also $d_i^* = (\lambda_k^*)\, u_i(\underline{t}^*) = 0$, for $i \in [k]$, $k = l+1, l+2, \ldots, p$. For $\lambda_k^* > 0$,

$$\sum_{i\in[k]} d_i^* = (\,\lambda_k^*)\, \sum_{i\in[k]} u_i(\underline{t}^*) / [\,g_k(\underline{t}^*)\,]^2 \Rightarrow g_k(\underline{t}^*) = 1,$$

and also,

$$\sum_{i\in[k]} d_i^* = (\lambda_k^*)\, \sum_{i\in[k]} u_i(\underline{t}^*) \;\Rightarrow g_k(\underline{t}^*) = 1,$$

so (4.15) is equivalent to $d_i^* = (\lambda_k^*)\, u_i(\underline{t}^*)$, for $i \in [k]$, $k = l+1, l+2, \ldots, p$. By the definition of equilibrium solution, $\underline{t}^*$, $\underline{d}^*$ are equilibrium solutions of (4.1) and (4.2), respectively. That means $\underline{t}^*$, $\underline{d}^*$ are the optimum solutions of (4.13) and (4.14), respectively, if and only if they are a pair of primal and dual equilibrium solutions for (4.1) and (4.2). So, by Theorem 4.4 (iii), we get $g_0(\underline{t}^*) = V(\underline{d}^*)$. Thus, $g_0(\underline{t}^*) = V(\underline{d}^*, \underline{\omega}^*)$. |

Suppose F_{DR} is nonempty, i.e. (4.1) and (4.2) are canonical (i.e., (4.2) has $\underline{d} > \underline{0}$, $\underline{d} \in F_{DR}$; otherwise, degenerate). Then for all $\underline{\omega}(\underline{t})$ such that $\sum_{i\in[k]} \omega_i(\underline{t}) = 1$ for $k = l+1, l+2, \ldots, p$, $F_{DR} = F_{DW}$, F_{DW} is nonempty, i.e. (4.13) and (4.14) are canonical. Then by Theorem 1.7, (4.13) must have optimum solution.

Based on Theorem 4.11 and 4.8, we have the following algorithm.

Suppose $\underline{t}^0 \in F_{PR}$. Let $n = 1$.

Step 1. Let $\omega_i^n = u_i(\underline{t}^{n-1}) / g_k(\underline{t}^{n-1})$, for $k = l+1, l+2, \ldots, p$, and form (4.13^n) (i.e., (4.13) for $\underline{\omega} = \underline{\omega}^n$).

Step 2. Suppose $\underline{t}^n$ is the optimum solution for (4.13^n). Then

$g_0(\underline{t}^n) \le g_0(\underline{t}^{n-1})$.

(by Theorem 4.8, $t^n, t^{n-1} \in F$, where F is the feasible region for (4.13^n)).

Step 3. If $g_0(t^n) = g_0(t^{n-1})$, then stop. Otherwise, go to Step 4.

Step 4. Increase n by 1 and turn to Step 1. ∎

By Theorem 4.8, we know $t^{n-1} \in F_{PR}$. Conversely,

$$F_{PWn} \subset F_{PR} \Rightarrow t^n \in F_{PR}, \quad n = 0, 1, \ldots \quad (F_{PWn} \text{ denotes } F_{PW} \text{ for } \omega = \omega^n).$$

So, $\{t^n\}$ must be a consistent point sequence for (4.1).

Theorem 4.12.

(i) Suppose (4.13^n) is superconsistent. If there exists n such that

$$g_0(t^n) = g_0(t^{n-1}),$$

then t^{n-1} is the primal equilibrium solution of (4.1).

(ii) Suppose (4.13) is superconsistent. If for n = 1, 2, ...,

$$g_0(t^n) < g_0(t^{n-1}),$$

$$t^n \to t^* \text{ as } n \to \infty.$$

and define that

$$\omega_i^* = u_i(t^*) / g_k(t^*), \quad \text{for } i \in [k], k = l+1, l+2, \ldots, p,$$

then t^* is the primal equilibrium solution of (4.1).

Proof. (i) Since t^n is the optimum solution of (4.13^n) and $g_0(t^n) = g_0(t^{n-1})$, t^{n-1} is the optimum solution of (4.13^n). Note (4.13^n) is superconsistent, so, by Theorem 1.6, (4.14^n) has an optimum solution. By Theorem 4.11, we know that t^{n-1} and any of the optimum solution of (4.14^n) forms a pair of equilibrium solutions of (4.1) and (4.2).

(ii) Suppose $\{t^{n'}\}$ is a subsequence of $\{t^n\}$ and $t^{n'} \to t^*$ as $n' \to \infty$.

Use t' to express a superconsistent solution of (4.13^*) ((4.13) for $\omega = \omega^*$), i.e.,

$$g_k(t') < 1, \quad \text{for } k = 1, 2, \ldots, l,$$

$g_k''(\underline{t}', \underline{\omega}^*) < 1$,   for $k = l+1, l+2, ..., p$,

where

$$g_k''(\underline{t}', \underline{\omega}^*) = \sum_{i\in[k]} [\, (\omega_i^*)^2 / c_i \,] \prod_{j=1}^{m} (t_j')^{a_{ij}}.$$

Since $g_k(\underline{t})$, $u_i(\underline{t})$ are continuous and $\underline{t}^{n'} \rightarrow \underline{t}^*$ as $n' \rightarrow \infty$, so, $\underline{\omega}^{n'} \rightarrow \underline{\omega}^*$ as $n' \rightarrow \infty$, $\exists$ N such that for $n' \geq N$, $g_k''(\underline{t}', \underline{\omega}^{n'}) < 1$, for $k = l+1, l+2, ..., p$. That means $\underline{t}'$ is also a superconsistent solution for ($4.13^{n'}$) (for $n' > N$). And by Theorem 1.6, ($4.14^{n'}$) ($n' > N$) also has an optimum solution $\underline{d}^{n'}$.

For $\varepsilon > 0$ (sufficient small),

$(1 + \varepsilon)\, g_k(\underline{t}') < 1$,   for $k = 1, 2, ..., l$,

$(1 + \varepsilon)\, g_k'(\underline{t}', \underline{\omega}^*) < 1$,   for $k = l+1, l+2, ..., p$

That is, for the problem that

$$\begin{aligned} &\text{minimize} \quad (1+\varepsilon)\, g_0(\underline{t}), \\ &\text{subject to} \quad \underline{t} \in F_{P\varepsilon Wn'}, \end{aligned} \tag{4.16}$$

where

$$F_{P\varepsilon Wn'} = \{\, \underline{t} \mid (1+\varepsilon)\, g_k(\underline{t}) < 1, \quad \text{for } k = 1, 2, ..., l, \quad (1+\varepsilon)\, g_k''(\underline{t}, \underline{\omega}^*) < 1, \quad \text{for } k = l+1, l+2, ..., p \,\}.$$

$\exists\, \underline{t} \in F_{P\varepsilon Wn'}$, that is, (4.16) has a superconsistent solution $\underline{t}'$,

$$M_{P\varepsilon Wn'} \leq (1+\varepsilon)\, g_0(\underline{t}').$$

where

$$M_{P\varepsilon Wn'} = \inf\, (1+\varepsilon)\, g_0(\underline{t}), \quad \text{for all } \underline{t} \in F_{P\varepsilon Wn'}.$$

Note the dual problem of (4.16) has the same constraint set as that which ($4.14^{n'}$) has, so it has consistent solution $\underline{d}^{n'}$, and its corresponding objective function value is

$$(1+\varepsilon)^{D}\, V'(\underline{d}^{n'}, \underline{\omega}^{n'}), \quad \text{where } D = \sum_{k=0}^{p} \sum_{i\in[k]} d_i^{n'}.$$

By Theorem 1.4, $V'(\underline{d}^{n'}, \underline{\omega}^{n'}) = g_0(\underline{t}^{n'})$ and we know that

$$(1+\varepsilon)^{D}\, g_0(\underline{t}^{n'}) \leq (1+\varepsilon)\, g_0(\underline{t}').$$

Since $g_0(\underline{t}^{n'}) \rightarrow g_0(\underline{t}^*) > 0$ (bounded), so $\{\underline{d}^{n'}\}$ must be bounded. Let $n' \rightarrow \infty$, we obtain

$$V'(\underline{d}^*, \underline{\omega}^*) = g_0(\underline{t}^*).$$

Note that $\underline{t}^{n'} \in F_{PWn'}$, $\underline{d}^{n'} \in F_{DWn'}$. By taking limit we have

$$\underline{t}^* \in F_{PW^*},\ \underline{d}^* \in F_{DW^*}.$$

(where F_{PW^*}, F_{DW^*} denote F_{PW}, F_{DW} for $\underline{\omega} = \underline{\omega}^*$, respectively). Therefore, $\underline{t}^*$ and $\underline{d}^*$ are the optimum solutions for (4.13*) and (4.14*), respectively. By Theorem 4.11, $\underline{t}^*$ and any limit point of $\{\underline{d}^{n'}\}$ are the equilibrium solutions for (4.1) and (4.2). ∎

The following example is an illustration for the properties presented above.

Consider the reversed geometric programming problem that

$$\begin{aligned} &\text{minimize} \quad g_0(\underline{x}) = 3x_1 + 2x_2, \\ &\text{subject to} \quad \underline{x} \in F_{PR1}, \end{aligned} \tag{4.17}$$

where

$$F_{PR1} = \{\underline{x} \mid g_1(\underline{x}) = x_1^2 + x_2^2 \geq 1,\ x_1, x_2 > 0\}.$$

Let $x_1 = e^{u_1}$, $x_2 = e^{u_2}$. Then (4.17) can be transformed to

$$\begin{aligned} &\text{minimize} \quad f_0(\underline{u}) = 3e^{u_1} + 2e^{u_2}, \\ &\text{subject to} \quad \underline{u} \in F_{PR2}, \end{aligned} \tag{4.18}$$

where

$$F_{PR2} = \{\underline{u} \mid f_1(\underline{u}) = e^{2u_1} + e^{2u_2} \geq 1\}.$$

Note that

$$\underline{u}^1 = (0, -1)^T, \ \underline{u}^2 = (-1, 0)^T \in F_{PR2}.$$

But

$$\underline{u}^3 = (1/2)\,\underline{u}^1 + (1/2)\,\underline{u}^2 = (-1/2, -1/2)^T \notin F_{PR2}.$$

So, (4.18) is not a convex program.

It is clear that

$$\inf g_0(\underline{x}) = 2, \quad \text{for } \underline{x} \in F_{PR1}.$$

The associated dual program of (4.17) is that

$$\text{maximize} \quad V(\underline{d}) = (3/d_1)^{d_1} (2/d_2)^{d_2} (1/d_3)^{-d_3} (1/d_4)^{-d_4} (d_3+d_4)^{-(d_3+d_4)},$$

$$\text{subject to} \quad \underline{d} \in F_{DR1}, \qquad (4.19)$$

where

$$F_{DR1} = \{ \underline{d} \mid d_1 + d_2 = 1,$$
$$d_1 - 2d_3 = 0,$$
$$d_2 - 2d_4 = 0,$$
$$d_i \geq 0, \quad \text{for } i = 1, 2, 3, 4 \}.$$

Clearly, $\underline{d}^1 = (1, 0, 1/2, 0)^T \in F_{DR1}$ and $V(\underline{d}^1) = 3$, that is,

$$\inf g_0(\underline{x}) = 2 \ \text{ for } \underline{x} \in F_{PR1} < V(\underline{d}^1) = 3 \leq \sup V(\underline{d}) \quad \text{for } \underline{d} \in F_{DR1}.$$

Consider that

$$\underline{x}^* = (3/\sqrt{13}, 2/\sqrt{13})^T,$$

$$\underline{\mu}^* = \sqrt{13}/2.$$

Since

$$g_1(\underline{x}^*) - 1 = (9/13 + 4/13) - 1 = 0,$$

$$\nabla_{\underline{x}} L(\underline{x}^*, \underline{\mu}^*) = (3, 2)^T - (\sqrt{13}/2)\,(2(3)/\sqrt{13}, 2(2)/\sqrt{13})^T = \underline{0},$$

$(\underline{x}^*, \underline{\mu}^*)$ forms the solution for Lagrange problem of (4.17). And $\underline{x}^*$ is not the local optimum of (4.17) since $g_0(\underline{x}^*) = \sqrt{13} > 2$.

$$A_{11} = \sum_{i=3}^{4} a_{i1}\, u_i(\underline{x}^*) = (-2)\,(3/\sqrt{13})^2 = -18/13,$$

$$A_{12} = \sum_{i=3}^{4} a_{i2}\, u_i(\underline{x}^*) = (-2)\,(2/\sqrt{13})^2 = -8/13.$$

Then by Theorem 4.1, for

$$\underline{x} \in F_t = \{\underline{x} \mid \sum_{j=1}^{2} A_{1j}\,(\log x_j - \log x_j^*) \le 0\}$$

$$= \{\underline{x} \mid (-18/13)\,(\log x_1 - \log (3/\sqrt{13})$$

$$+ (-8/13)\,(\log x_2 - \log (2/\sqrt{13}))\,) \le 0\}$$

$$= \{\underline{x} \mid x_1^{\,9}\, x_2^{\,4} \ \ge\ 2^4\, 3^9 / 13^{13/2}\},$$

we have

$$g_0(\underline{x}^*) = \sqrt{13} \le g_0(\underline{x}) = 3x_1 + 2x_2 .$$

That is, $\underline{x}^*$ is the solution for the Lagrange problem and is the tangentially optimal solution.

By Theorem 4.2, we know

$$d_1 = u_1(\underline{x}^*) / g_0(\underline{x}^*) = \frac{9/\sqrt{13}}{\sqrt{13}} = 9/13,$$

$$d_2 = u_2(\underline{x}^*) / g_0(\underline{x}^*) = \frac{4/\sqrt{13}}{\sqrt{13}} = 4/13,$$

$$d_3 = \mu^*\, u_3(\underline{x}^*) / g_0(\underline{x}^*) = (\sqrt{13}/2)\,(9/13)/\sqrt{13} = 9/26,$$

$$d_4 = \mu^*\, u_4(\underline{x}^*) / g_0(\underline{x}^*) = (\sqrt{13}/2)\,(4/13)/\sqrt{13} = 2/13,$$

That is, $\underline{d}^* = (9/13, 4/13, 9/26, 2/13)^T$ is the dual equilibrium solution.

By Theorem 4.5, for $\underline{v} = (v_1, v_2, \ldots, v_n)^T$ such that $\sum_{i \in [0]} v_i = 0$ and $A^T \underline{v} = \underline{0}$, i.e.

$$\begin{aligned} v_1 + v_2 \qquad\qquad\qquad &= 0, \\ v_1 \qquad - 2v_3 \qquad\quad &= 0, \\ v_2 \qquad\qquad - 2v_4 &= 0. \end{aligned}$$

$\underline{y} = (2, -2, 1, -1)^T\ t,\quad t \in R.$

For equilibrium identity, we have

$$F(\underline{d}^*, \underline{y}) = (d_1^*)^{2t} (d_2^*)^{-2t} (d_3^*)^{-t} (d_4^*)^{t}$$

$$= (9/13)^{2t} (4/13)^{-2t} (9/26)^{-t} (2/13)^{t} = (3/2)^{2t}, \quad t \in R,$$

$$G(\underline{c}, \underline{y}) = (c_1)^{2t} (c_2)^{-2t} (c_3)^{-t} (c_4)^{t}$$

$$= (3)^{2t} (2)^{-2t}\ (1)^{-t} (1)^{t} = (3/2)^{2t}, \quad t \in R.$$

Suppose that $\underline{\omega}^* = u_i(\underline{x}^*) / g_k(\underline{x}^*)$, so,

$\omega_1^* = 9/13,$

$\omega_2^* = 4/13.$

Then problem (4.13) corresponding to (4.1) is that

$$\begin{aligned} &\text{minimize} \quad g_0(\underline{x}) = 3x_1 + 2x_2, \\ &\text{subject to} \quad g_1(\underline{x}) = (81/169)\, x_1^{-2} + (16/169)\, x_2^{-2} \le 1, \\ &\qquad\qquad x_1,\ x_2 > 0, \end{aligned} \tag{4.13'}$$

problem (4.14) corresponding to (4.2) is that

$$\begin{aligned} &\text{maximize} \quad V'(\underline{d}, \underline{\omega}^*) = (3/d_1)^{d_1} (2/d_2)^{d_2} (81/169d_3)^{-d_3} \\ &\qquad\qquad (16/169d_4)^{-d_4} (\lambda_1)^{\lambda_1}, \\ &\text{subject to} \quad \lambda_0 = 1, \\ &\qquad\qquad d_1 - 2d_3 = 0, \\ &\qquad\qquad d_2 - 2d_4 = 0, \\ &\qquad\qquad d_i \ge 0, \quad \text{for } i = 1, 2, 3, 4. \end{aligned} \tag{4.14'}$$

By Theorem 4.1, we know that $\underline{x}^*$, $\underline{d}^*$ are the optimum solutions for (4.13) and (4.14), respectively.

Let

$\underline{x}^0 = (3, 2)^T \in F_{PR}.$

Then

$$\underline{\omega}^1 = (9/13,\ 4/13)^T.$$

The optimum solution of (4.13') for $\underline{\omega} = \underline{\omega}^1$ is that

$$\underline{x}^* = (3/\sqrt{13},\ 2/\sqrt{13})^T,$$

and we have

$$\underline{\omega}^2 = \underline{\omega}^1 = (9/13,\ 4/13)^T.$$

So, (3.13') for $\underline{\omega} = \underline{\omega}^2$ is the same as (3.13') for $\underline{\omega} = \underline{\omega}^1$. Therefore, by Theorem 4.12, $\underline{x}^*$ is the equilibrium solution for the problem.

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APPENDIX

DERIVATION OF DUAL PROBLEM

The associated dual problem can be derived from the primal problem by way of proper rearrangement of Lagrange multiplier [10].

Consider the problem that

$$\text{minimize} \quad g_0(\underset{\sim}{t}),$$

$$\text{subject to} \quad \underset{\sim}{t} \in F_P, \qquad \text{(A 1.1)}$$

where

$$F_P = \{ \underset{\sim}{t} \mid s_k [\, 1 - g_k(\underset{\sim}{t}) \,] \geq 0, \;\; s_k = 1 \text{ or } -1, \;\; \text{for } k = 1, 2, \ldots, p,$$

$$t_j > 0, j = 1, 2, \ldots, m \},$$

$$g_k(\underset{\sim}{t}) = \sum_{i \in [k]} u_{ki}(\underset{\sim}{t}) = \sum_{i \in [k]} c_{ki} \prod_{j=1}^{m} t_j^{a_{ij}}, \quad \text{for } k = 0, 1, \ldots, p.$$

Suppose that

$$\omega_{0i} = u_{0i}(\underset{\sim}{t}) \,/\, g_0(\underset{\sim}{t}), \quad \text{for } i \in [0],$$

$$\omega_{ki} = u_{ki}(\underset{\sim}{t}), \quad \text{for } i \in [k], k = 1, 2, \ldots, p.$$

Then it is clear that

$$\sum_{i \in [0]} \omega_{0i}(\underset{\sim}{t}) = 1,$$

$$\sum_{i \in [k]} \omega_{ki}(\underset{\sim}{t}) = g_k(\underset{\sim}{t}), \quad \text{for } k = 1, 2, \ldots, p.$$

Let

$$x_j = \log t_j, \quad \text{for } j = 1, 2, \ldots, m,$$

$$x_0 = \log g_0(\underset{\sim}{t}).$$

Then for $k = 0$, note that $u_{0i}(\underset{\sim}{t}) = \omega_{0i}\, g_0(\underset{\sim}{t})$ and $u_{0i}(\underset{\sim}{t}) = c_{0i} \prod_{j=1}^{m} t_j^{a_{ij}}$, we have

$$\log u_{0i}(\underset{\sim}{t}) = \log \omega_{0i} + \log g_0(\underset{\sim}{t})$$

$$= \log c_{0i} + \sum_{j=1}^{m} a_{ij} x_j .$$

That is,

$$x_0 + \log (\omega_{0i} / c_{0i}) - \sum_{j=1}^{m} a_{ij} x_j = 0. \qquad (A\ 1.2)$$

And for k = 1, 2, ..., p, note that $u_{ki}(\underline{t}) = \omega_{ki}$ and $u_{ki}(\underline{t}) = c_{ki} \prod_{j=1}^{m} t_j^{a_{ij}}$, we have

$$\log \omega_{ki} = \log c_{ki} + \sum_{j=1}^{m} a_{ij} x_j .$$

That is,

$$s_k [\log (\omega_{ki} / c_{ki}) - \sum_{j=1}^{m} a_{ij} x_j] = 0. \qquad (A\ 1.3)$$

Let

$$h_k = s_k (1 - \sum_{i \in [k]} \omega_{ki}(\underline{t})) , \quad \text{for } k = 1, 2, ..., p.$$

Then (A 1.1) is equivalent to the following problem

$$\text{minimize} \quad x_0 ,$$

$$\text{subject to} \quad \underline{\omega}, \underline{x} \in F_{P1} , \qquad (A\ 1.4)$$

where

$$F_{P1} = \{ \underline{\omega}, \underline{x} \mid 1 - \sum_{i \in [0]} \omega_{0i}(\underline{t}) = 0,$$

$$h_k \geq 0 , \quad \text{for } k = 1, 2, ..., p,$$

$$x_0 + \log (\omega_{0i} / c_{0i}) - \sum_{j=1}^{m} a_{ij} x_j = 0 , \quad \text{for } i \in [0],$$

$$s_k [\log (\omega_{ki} / c_{ki}) - \sum_{j=1}^{m} a_{ij} x_j] = 0 , \quad \text{for } i \in [k], k = 1, 2, ..., p \}.$$

By defining Lagrange multiplier λ_k', for k = 1, 2, ..., p, such that

$$\lambda_k' h_k = 0 ,$$

that is called Kuhn-Tucker complementary slackness condition, we can construct a proper Lagrangian function for (A 1.4) as follows.

$$L (\underline{\omega}, \underline{x} ; \underline{\lambda}, \underline{\lambda}') = x_0 - \lambda_0' (1 - \sum_{i \in [0]} \omega_{0i})$$

$$- \sum_{k=1}^{p} \lambda_k' \, h_k$$

$$- \sum_{i\in[0]} \lambda_{0i} \, [\, x_0 + \log (\omega_{0i} / c_{0i}) - \sum_{j=1}^{m} a_{ij} \, x_j \,]$$

$$- \sum_{k=1}^{p} s_k \, \lambda_{ki} \, [\, \log (\omega_{ki} / c_{ki}) - \sum_{j=1}^{m} a_{ij} \, x_j \,]. \qquad \text{(A 1.5)}$$

Let

$$\frac{\partial L}{\partial x_0} = 1 - \sum_{i\in[0]} \lambda_{0i} = 0,$$

$$\frac{\partial L}{\partial x_j} = \sum_{k=0}^{p} \; \sum_{i\in[0]} s_k \, \lambda_{ki} \, a_{ij} = 0, \quad \text{for } j = 1, 2, \ldots, m,$$

$$\frac{\partial L}{\partial \omega_{0i}} = \lambda_0' - \lambda_{0i} / \omega_{0i} = 0, \quad \text{for } i \in [0],$$

$$\frac{\partial L}{\partial \omega_{ki}} = s_k \, [\, \lambda_0' - \lambda_{0i} / \omega_{0i} \,] = 0, \quad \text{for } i \in [k], \; k = 1, 2, \ldots, p.$$

Then for the optimal Lagrange multiplier $\underline{\lambda}$,

$$\sum_{i\in[0]} \lambda_{0i} = 1,$$

$$\lambda_{0i} = \lambda_0' \, \omega_{0i},$$

$$\lambda_{ki} = \lambda_k' \, \omega_{ki}.$$

Note that

$$\sum_{i\in[0]} \lambda_{0i} = \lambda_0' \, (\sum_{i\in[0]} \omega_{0i}) = \lambda_0' = 1 \quad (\sum_{i\in[0]} \omega_{0i} = 1),$$

$$g_k(\underline{t}) = \sum_{i\in[k]} \omega_{ki} = 1, \quad \text{for } \lambda_k' > 0 \quad (\lambda_k' > 0 \Rightarrow h_k = 0 \text{ since } \lambda_k' \, h_k = 0),$$

$$\sum_{i\in[k]} \lambda_{ki} = \lambda_k' \, (\sum_{i\in[k]} \omega_{ki}) = \lambda_k' = 1, \quad \text{for } \lambda_k' > 0,$$

we have

$$\omega_{0i} = \lambda_{0i},$$

$$\omega_{ki} = \lambda_{ki} / \lambda_k' = \lambda_{ki} / \sum_{i \in [k]} \lambda_{ki},$$

$$\sum_{k=0}^{p} \sum_{i \in [0]} s_k \lambda_{ki} a_{ij} = 0, \quad \text{for } j = 1, 2, \ldots, m.$$

Let $s_0 = 1$ and note that

$$\sum_{i \in [0]} \omega_{0i} = 1, \quad \omega_{0i} = \lambda_{0i},$$

$$\begin{aligned} x_0 - \sum_{i \in [0]} \lambda_{0i} x_0 &= x_0 \left(1 - \sum_{i \in [0]} \lambda_{0i} \right) - 1 + 1 \\ &= x_0 \left(1 - \sum_{i \in [0]} \lambda_{0i} \right) - \left(1 - \sum_{i \in [0]} \lambda_{0i} \right) \\ &= (x_0 - 1) \left(1 - \sum_{i \in [0]} \lambda_{0i} \right). \end{aligned}$$

So,

$$\begin{aligned} L (\underline{\lambda}, \underline{\lambda}'; \underline{x}, \underline{h}) &= L (\underline{\omega}, \underline{x} ; \underline{\lambda}, \underline{\lambda}') \\ &= \sum_{k=0}^{p} \sum_{i \in [0]} s_k \lambda_{ki} \log \left[c_{ki} \left(\sum_{i \in [k]} \lambda_{ki} / \lambda_{ki} \right) \right] \\ &\quad + \prod_{j=1}^{m} x_j \left(\sum_{k=0}^{p} \sum_{i \in [0]} s_k \lambda_{ki} a_{ij} \right) \\ &\quad - \sum_{k=1}^{p} h_k \lambda_k' + (x_0 - 1) \left(1 - \sum_{i \in [0]} \lambda_{0i} \right). \end{aligned}$$

This Lagrangian function can be taken as another constrained maximization problem. That is,

$$\text{maximize} \quad f_0(\underline{\lambda}) = \sum_{k=0}^{p} \sum_{i \in [k]} s_k \lambda_{ki} \log \left[c_{ki} \left(\sum_{i \in [k]} \lambda_{ki} / \lambda_{ki} \right) \right],$$

$$\text{subject to} \quad \underline{\lambda} \in F_D, \qquad \text{(A 1.6)}$$

where

$$\begin{aligned} F_D = \{ \underline{\lambda} \mid & \sum_{i \in [0]} \lambda_{0i} = 1, \\ & \sum_{i \in [k]} \lambda_{ki} \geq 0, \quad \text{for } k = 1, 2, \ldots, p, \\ & \sum_{k=0}^{p} \sum_{i \in [0]} s_k \lambda_{ki} a_{ij} = 0, \quad \text{for } j = 1, 2, \ldots, m \}. \end{aligned}$$

Let

$$d_i = \lambda_{ki}, \quad \text{for } i \in [k],\ k = 0, 1, ..., p, \quad \sum_{i \in [k]} d_i = \lambda_k = \sum_{i \in [k]} \lambda_{ki},$$

and

$$V(\underline{d}) = \exp\{ f_0(\underline{\lambda}) \}$$

$$= \prod_{k=0}^{p} \prod_{i \in [k]} [(c_{ki} / \lambda_{ki})^{s_k \lambda_{ki}}] \prod_{k=1}^{p} [(\lambda_k)^{s_k \lambda_k}] \quad (\lambda_0 = 1)$$

$$= \prod_{k=0}^{p} \prod_{i \in [k]} [(c_{ki} / d_i)^{s_k d_i}] \prod_{k=1}^{p} [(\lambda_k)^{s_k \lambda_k}] .$$

Then for $s_k = 1$, (A 1.6) becomes

$$\text{maximize} \quad V(\underline{d}) = \prod_{i=1}^{n} (c_i / d_i)^{d_i} \prod_{k=1}^{p} [(\lambda_k)^{\lambda_k}],$$

$$\text{subject to} \quad \underline{d} \in F_D, \tag{A 1.7}$$

where

$$F_D = \{ \underline{d} \mid \lambda_0 = 1,$$

$$\prod_{i=1}^{n} a_{ij} d_i = 0, \quad \text{for } j = 1, 2, ..., m,$$

$$d_i,\ \lambda_k \geq 0, \quad \text{for } i = 1, 2, ..., n,\ k = 1, 2, ..., p \} .$$

That is, the associated dual problem of (A 1.1) for $s_k = 1$, $k = 1, 2, ..., p$.

VITA

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