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The proofs in a quantum mechanical d'Alembert Principle

Jordan E. Cahn

To the Graduate Council:

I am submitting herewith a thesis written by Jordan E. Cahn entitled "The proofs in a quantum mechanical d'Alembert Principle." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

Boris A. Kupershmidt, Major Professor

We have read this thesis and recommend its acceptance:

K. C. Reddy, Horace Crater

Accepted for the Council:
Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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THE PROOFS IN A QUANTUM MECHANICAL D'ALEMBERT PRINCIPLE

A Thesis
Presented for the
Master of Science
Degree
The University of Tennessee, Knoxville

JORDAN E. CAHN

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ABSTRACT

The d' Alembert principle states that if a particle is constrained to a manifold, no work can be done normal to the manifold, however, quantum mechanics forbids the restraint of a particle. The constraint is replaced by an infinite potential and the Schödinger equation can be seperated to produce a potential field on the manifold which is a function of the manifold's curvature. This is done for a one-dimensional curve and then for a general manifold. New work is presented as the case of a particle on a circle and the case of a product manifold are investigated.

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SECTION I

INTRODUCTION

The d'Alembert Principle states that the total work done by effective forces in a reversible infinitesimal virtual displacement within the constraints of a dynamical system, is zero [1]. This principle for the total work is valid for holonomic, time independent constraints. Such a dynamical system can be described by generalized coordinates allowable by the constraints. The geometrical interpretation of this is the embedding of a manifold, on which the particle is constrained, into the original configuration space, i.e., the embedding $i : M \rightarrow \mathbf{R}^n$ where M is defined as the locus of constraints [2]. This affords two points of view on the system: intrinsic (working on M) and extrinsic (working on $i(M)$ in $\mathbf{R}^n$).

Unlike in classical mechanics, the Heisenberg uncertainty relations of quantum mechanics make it impossible to entirely eliminate the constraints in this way. However, it is expected that Schrödinger's equation can be separated, one part depending only on the generalized Lagrangian coordinates q^b, $b = 1, \ldots, D$ of the manifold M and a second part on the redundant coordinates $q^y, y = D+1, \cdots, n$ [3] which would take a particle outside its constraints. This second part would describe a wave function with an arbitrarily rapid oscillation whose amplitude is small in comparison with the amplitude of the wave function on the manifold of constraints M. While quantum mechanics forbids freezing the oscillation, it is possible to investigate the Schrödinger equation in the limit as the amplitude of the oscillation approaches the limit of zero.

In the note by J. Tolar [4], "On a Quantum Mechanical d′ Alembert Principle," this situation was modeled as a quantum mechanical system in $\mathbf{R}^n$ confined within a neighborhood of given radius of the constraint submanifold M by a strong potential force. Since the system cannot be localized on M in quantum mechanics, he investigated whether the presence of the restoring forces, which replace the classical restraints, in the neighborhood of M would affect the motion of a particle on M. The quantum mechanical coupling of the motion on the constrained manifold with the motion off it produces " a peculiar dependence of the constrained Schrödinger equation on the internal as well as external curvature of the submanifold M in $\mathbf{R}^n$." What Tolar found was an additional "quantum potential" in the constrained Schrödinger equation. This potential vanishes in the classical limit and so represents a purely quantal effect.

Similar quantum potentials have been derived from path integrals in quantum mechanics [5], and with Dirac brackets from the Hamiltonian [6]. J. Tolar derived a potential for general manifolds by differential-geometric methods; however, he did not include the majority of the proofs. The following are proofs of all the equations found in the paper by J. Tolar, presented at the 16th International Colloquium on Group Theoretical Methods at Varna, Bulgaria in 1987 and printed in the Lecture Notes in Physics #313, along with some elucidation of the text. All the numbered equations appeared originally in Tolar's paper.

The quantum potential is found first for the simple case of a particle on a planar curve, i.e., one dimensional constraint manifold, the curve, embedded in a two dimensional space, the plane. Once the method is familiar, the general case of a D–dimensional manifold in an n–dimensional space is developed. New work is presented at the end of section I where the particle is constrained to a circle and in section IV where the quantum potential for a product manifold is derived.

SECTION II

Quantum Potential for a Particle Bound to a Planar Curve

We first consider a particle moving along a planar (one dimensional) curve. Let the curve C be (infinitely) differentiable. The curve C in $\mathbf{R}^2$ can be given in a parametized form with

$$\boldsymbol{r} = (x_1, x_2) = \boldsymbol{a}(q^1). \tag{1}$$

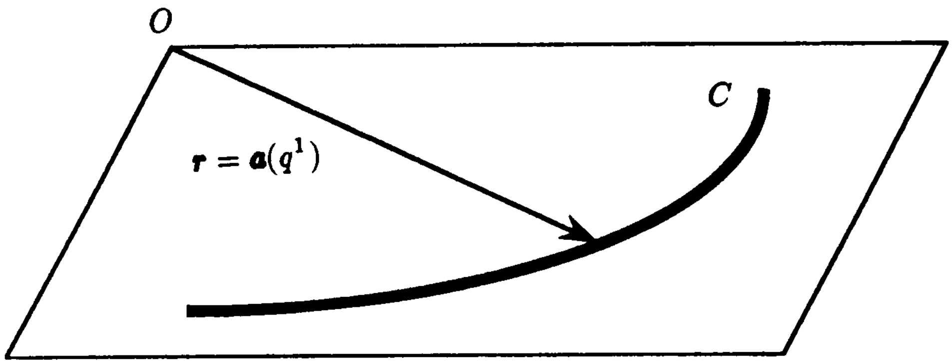

Figure 1. Parametrized curve C in $\mathbf{R}^2$

where q^1 is the Euclidian arc length (i.e., $(d\boldsymbol{r})^2 = (dq^1)^2$). Let us also assume that C admits a "tubular" neighborhood, called W_ε, of constant radius ε. The point of the neighborhood W_ε can be parametrized by (q^1, q^2) with $|q^2| < \varepsilon$. If we call the unit normal to the curve at $q^1, \boldsymbol{n}(q^1)$ then the points in the neighborhood W_ε can be given as

$$\boldsymbol{r} = \boldsymbol{a}(q^1) + q^2\boldsymbol{n}(q^1) \qquad |q^2| < \varepsilon. \tag{2}$$

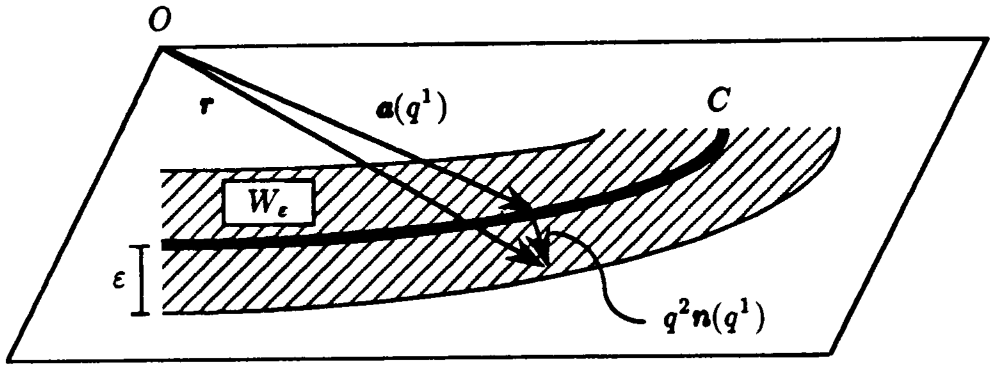

Figure 2. Curve with neighborhood W_ε

Recalling that the particle is to be bound to the curve C, let us set up a constraining potential with infinitely high walls at a distance $d < \varepsilon$ from the curve

$$V(q^1, q^2) = \begin{cases} 0 & |q^2| < d < \varepsilon \\ \infty & |q^2| > d \end{cases} \tag{3}$$

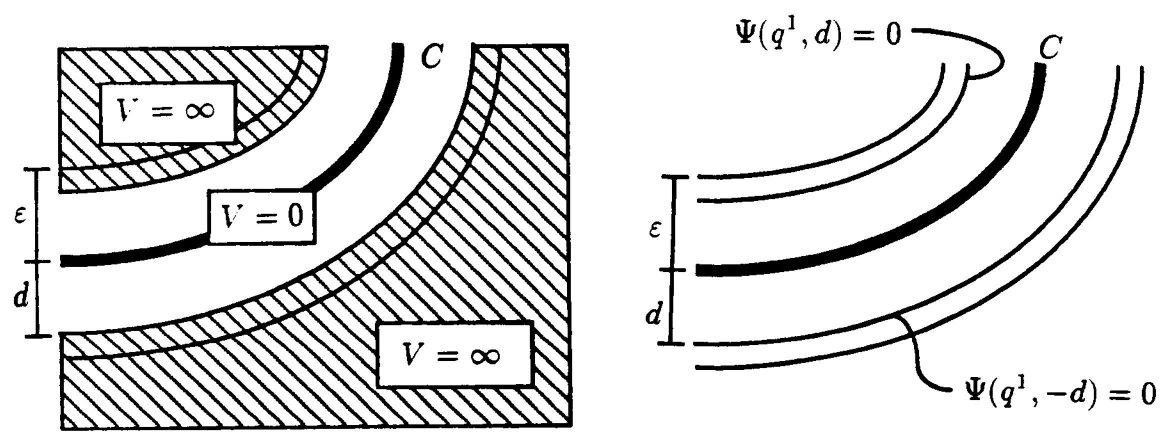

Figure 3. (a) Constraining potential around curve C (b) Boundary conditions on Ψ

This potential is equivelant to the boundary conditions on the wave function of the particle

$$\Psi(q^1, d) = 0 = \Psi(q^1, -d) \tag{4}$$

where Ψ is a solution to the Schrödinger equation

$$-\frac{h^2}{2m}\Delta\Psi = E\Psi \qquad (5)$$

Ψ inside W_ε must be square integrable in the Lebesgue measure ($\Psi \in L^2(W_\varepsilon, dx_1 dx_2)$) since it is in Euclidian space [7]. The Lebesgue measure $dx_1 dx_2$ and the Laplacian $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial}{\partial x_2^2}$ must be transformed from the Cartesian x_1, x_2 coordinates to the new generalized cordinates q^1, q^2 in W_ε.

To do this we must calculate the metric tensor g_{ij} in the generalized coordinates.

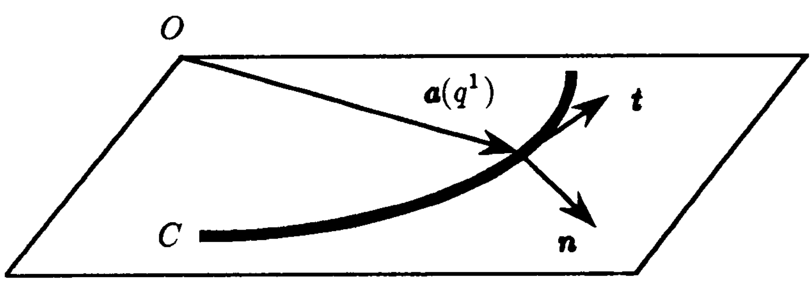

Figure 4. Vectors $\boldsymbol{a}(q_1)$, $\boldsymbol{n}$, $\boldsymbol{t}$

Using the differentials of the tangent and normal unit vectors where the tangent vector $\boldsymbol{t}$ is given by $\boldsymbol{t} = \frac{d\boldsymbol{a}(q^1)}{dq^1}$, we get

$$\frac{dt}{dq^1} = -\eta\boldsymbol{n}, \qquad \frac{d\boldsymbol{n}}{dq^1} = \eta\boldsymbol{t}, \qquad \boldsymbol{t}\cdot\boldsymbol{n} = 0, \qquad \boldsymbol{t}\cdot\boldsymbol{t} = \boldsymbol{n}\cdot\boldsymbol{n} = 1, \qquad (6)$$

where $\eta = \frac{1}{R}$, with R being the radius of curvature for the curve C at point q^1.

Recalling that $\boldsymbol{r} = \boldsymbol{a}(q^1) + q^2\boldsymbol{n}(q^1)$ we solve for the components of the metric tensor

$$g_{ij} = \left(\frac{\partial \boldsymbol{r}}{\partial q^i}\right) \cdot \left(\frac{\partial \boldsymbol{r}}{\partial q^j}\right)$$

$$g_{11} = \left(\frac{\partial \boldsymbol{r}}{\partial q^1}\right)^2 = \left(\frac{\partial \boldsymbol{a}}{\partial q^1} + q^2 \frac{\partial \boldsymbol{n}}{\partial q^1}\right)^2 = (\boldsymbol{t} + q^2 \eta \boldsymbol{t})^2 = (1 + q^2 \eta)^2 \boldsymbol{t} \cdot \boldsymbol{t}.$$

Since $\boldsymbol{t}$ is a unit vector $\boldsymbol{t} \cdot \boldsymbol{t} = 1$, we find

$$g_{11} = (1 + q^2 \eta)^2 \tag{7a}$$

$$g_{22} = \left(\frac{\partial \boldsymbol{r}}{\partial q^2}\right)^2 = (\boldsymbol{n})^2 = 1 \tag{7b}$$

$$g_{12} = \left(\frac{\partial \boldsymbol{r}}{\partial q^1}\right) \left(\frac{\partial \boldsymbol{r}}{\partial q^2}\right) = g_{21} = (1 + q^2 \eta) \boldsymbol{t} \cdot \boldsymbol{n},$$

so that

$$g_{12} = g_{21} = 0. \tag{7c}$$

Then defining $g = \det(g_{ij})$ with

$$g_{ij} = \begin{bmatrix} (1 + q^2 \eta)^2 & 0 \\ 0 & 1 \end{bmatrix}$$

we find

$$g = (1 + q^2 \eta)^2 \text{ and } g^{\frac{1}{2}} = 1 + q^2 \eta.$$

Now,

$$dxdy = g^{\frac{1}{2}} dq^1 dq^2 = (1 + q^2 \eta) dq^1 dq^2 \tag{8a}$$

and

$$\Delta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^1} \frac{\sqrt{g}}{g_{11}} \frac{\partial}{\partial q^1} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^2} \frac{\sqrt{g}}{g_{22}} \frac{\partial}{\partial q^2}.$$

But

$$\frac{\sqrt{g}}{g_{11}} = \frac{1 + q^2 \eta}{(1 + q^2 \eta)^2} = \frac{1}{1 + q^2 \eta} = \frac{1}{\sqrt{g}} \; ; \qquad \frac{\sqrt{g}}{g_{22}} = \frac{\sqrt{g}}{1}$$

so

$$\Delta = g^{-\frac{1}{2}}\frac{\partial}{\partial q^1}\left(g^{-\frac{1}{2}}\frac{\partial}{\partial q^1}\right) + g^{-\frac{1}{2}}\frac{\partial}{\partial q^2}\left(g^{\frac{1}{2}}\frac{\partial}{\partial q^2}\right). \tag{8b}$$

Performing the transformation

$$\Phi = g^{\frac{1}{4}}\Psi = \sqrt{1+q^2\eta}\,\Psi \qquad\qquad \Psi = g^{-\frac{1}{4}}\Phi = \frac{1}{\sqrt{1+q^2\eta}}\Phi \tag{9}$$

we find that

$$\Delta\Psi = E\Psi \text{ becomes } \Delta g^{-\frac{1}{4}}\Phi = Eg^{-\frac{1}{4}}\Phi.$$

Hence

$$g^{\frac{1}{4}}\Delta g^{-\frac{1}{4}}\Phi = E\Phi.$$

Evaluating the left-hand side, we get

$$g^{\frac{1}{4}}\Delta g^{-\frac{1}{4}}\Phi = g^{\frac{1}{4}}\left[g^{-\frac{1}{2}}\frac{\partial}{\partial q^1}\left(g^{-\frac{1}{2}}\frac{\partial(g^{-\frac{1}{4}}\Phi)}{\partial q^1}\right) + g^{-\frac{1}{2}}\frac{\partial}{\partial q^2}\left(g^{\frac{1}{2}}\frac{\partial(g^{-\frac{1}{4}}\Phi)}{\partial q^2}\right)\right]$$

$$= g^{-\frac{1}{4}}\left[\frac{\partial}{\partial q^1}\left(g^{-\frac{1}{2}}\left(\frac{-g^{-\frac{5}{4}}}{4}\Phi\frac{\partial g}{\partial q^1} + g^{-\frac{1}{4}}\frac{\partial\Phi}{\partial q^1}\right)\right) + \frac{\partial}{\partial q^2}\left(g^{\frac{1}{2}}\left(\frac{-g^{-\frac{5}{4}}}{4}\Phi\frac{\partial g}{\partial q^2} + g^{-\frac{1}{4}}\frac{\partial\Phi}{\partial q^2}\right)\right)\right].$$

Now

$$g = (1+q^2\eta)^2$$

with

$$\eta = \frac{1}{\text{radius of curvature}} = \eta(q^1)$$

$$g = 1 + 2q^2\eta(q^1) + \left(q^2\eta(q^1)\right)^2$$

so

$$\frac{\partial g}{\partial q^1} = 2q^2\frac{\partial\eta}{\partial q^1} + 2(q^2)^2\eta\frac{\partial\eta}{\partial q^1} = 2q^2(1+q^2\eta)\frac{\partial\eta}{\partial q^1}$$

and

$$\frac{\partial g}{\partial q^2} = 2\eta + 2\eta^2q^2 = 2\eta(1+\eta q^2).$$

Since η is a function of q^1 only,

$$\frac{\partial \eta}{\partial q^1} = \eta'$$

so

$$\frac{\partial g}{\partial q^1} = 2q^2 g^{\frac{1}{2}} \eta'$$

and

$$\frac{\partial g}{\partial q^2} = 2\eta g^{\frac{1}{2}}.$$

Continuing

$$g^{\frac{1}{4}} \Delta g^{-\frac{1}{4}} \Phi = g^{-\frac{1}{4}} \left\{ \frac{\partial}{\partial q^1} \left(-\frac{g^{-\frac{7}{4}}}{4} 2q^2 g^{\frac{1}{2}} \eta' \Phi + g^{-\frac{3}{4}} \frac{\partial \Phi}{\partial q^1} \right) + \frac{\partial}{\partial q^2} \left(-\frac{g^{-\frac{3}{4}}}{4} 2\eta g^{\frac{1}{2}} \Phi + g^{\frac{1}{4}} \frac{\partial \Phi}{\partial q^2} \right) \right\}$$

$$= g^{-\frac{1}{4}} \left\{ \frac{5}{8} g^{-\frac{9}{4}} \frac{\partial g}{\partial q^1} q^2 \eta' \Phi - \frac{g^{-\frac{5}{4}}}{2} q^2 \eta'' \Phi - \frac{g^{-\frac{5}{4}}}{2} q^2 \eta' \frac{\partial \Phi}{\partial q^1} - \frac{3}{4} g^{-\frac{7}{4}} \frac{\partial g}{\partial q^1} \frac{\partial \Phi}{\partial q^1} \right.$$

$$\left. + g^{-\frac{3}{4}} \frac{\partial^2 \Phi}{(\partial q^1)^2} + \frac{g^{-\frac{5}{4}}}{8} \frac{\partial g}{\partial q^2} \eta \Phi - \frac{g^{-\frac{1}{4}}}{2} \eta \frac{\partial \Phi}{\partial q^2} + \frac{g^{-\frac{3}{4}}}{4} \frac{\partial g}{\partial q^2} \frac{\partial \Phi}{\partial q^2} + g^{\frac{1}{4}} \frac{\partial^2 \Phi}{(\partial q^2)^2} \right\}$$

$$= \frac{5}{8} g^{-\frac{5}{2}} 2q^2 g^{\frac{1}{2}} \eta' q^2 \eta' \Phi - \frac{g^{-\frac{3}{2}}}{2} q^2 \eta'' \Phi - \frac{g^{-\frac{3}{2}}}{2} q^2 \eta' \frac{\partial \Phi}{\partial q^1} - \frac{3}{4} g^{-2} 2q^2 g^{\frac{1}{2}} \eta' \frac{\partial \Phi}{\partial q^1}$$

$$+ g^{-1} \frac{\partial^2 \Phi}{(\partial q^1)^2} + \frac{g^{-\frac{3}{2}}}{8} 2\eta g^{\frac{1}{2}} \eta \Phi - \frac{g^{-\frac{1}{2}}}{2} \eta \frac{\partial \Phi}{\partial q^2} + \frac{g^{-1}}{4} 2\eta g^{\frac{1}{2}} \frac{\partial \Phi}{\partial q^2} + \frac{\partial^2 \Phi}{(\partial q^2)^2}$$

$$= \frac{5}{4} g^{-2} (q^2)^2 \eta'^2 \Phi - \frac{g^{-\frac{3}{2}}}{2} q^2 \eta'' \Phi - \frac{g^{-\frac{3}{2}}}{2} q^2 \eta' \frac{\partial \Phi}{\partial q^1} - \frac{3}{2} g^{-\frac{3}{2}} q^2 \eta' \frac{\partial \Phi}{\partial q^1}$$

$$+ g^{-1} \frac{\partial^2 \Phi}{(\partial q^1)^2} + \frac{g^{-1}}{4} \eta^2 \Phi - \frac{g^{-\frac{1}{2}}}{2} \eta \frac{\partial \Phi}{\partial q^2} + \frac{g^{-\frac{1}{2}}}{2} \eta \frac{\partial \Phi}{\partial q^2} + \frac{\partial^2 \Phi}{(\partial q^2)^2}.$$

Thus

$$g^{\frac{1}{4}} \Delta g^{-\frac{1}{4}} \Phi = \frac{5}{4} g^{-2} (q^2)^2 (\eta')^2 \Phi - \frac{g^{-\frac{3}{2}}}{2} q^2 \eta'' \Phi - 2g^{-\frac{3}{2}} q^2 \eta' \frac{\partial \Phi}{\partial q^1}$$

$$+ g^{-1} \frac{\partial^2 \Phi}{(\partial q^1)^2} + \frac{g^{-1}}{4} \eta^2 \Phi + \frac{\partial^2 \Phi}{(\partial q^2)^2}. \tag{10}$$

Now, if we take the limit when the particle is constrained to the curve C we would take the limit $d \to 0$. Since $|q^2| < d$, $q^2 \to 0$ and the metric $g = (1 + q^2\eta)^2$ becomes $g = 1$. Before this limit is taken our Schrödinger equation read

$$-\frac{\hbar^2}{2m}\Delta\Psi = E\Psi \quad \text{or} \quad -\frac{\hbar^2}{2m}g^{\frac{1}{4}}\Delta g^{-\frac{1}{4}}\Phi = E\Phi.$$

After the limit we have an approximation of the Schrödinger equation

$$\lim_{\substack{q^2 \to 0 \\ g \to 1}} -\frac{\hbar^2}{2m}\left\{\frac{5}{4}g^{-2}(q^2)^2(\eta')^2\Phi - \frac{g^{-\frac{3}{2}}}{2}q^2\eta''\Phi - 2g^{-\frac{3}{2}}q^2\eta'\frac{\partial\Phi}{\partial q^1}\right.$$
$$\left. + g^{-1}\frac{\partial^2\Phi}{(\partial q^1)^2} + \frac{g^{-1}}{4}\eta^2\Phi + \frac{\partial^2\Phi}{(\partial q^2)^2}\right\} = E\Phi$$

or

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\Phi}{(\partial q^1)^2} + \frac{1}{4}\left(\eta(q^1)\right)^2\Phi + \frac{\partial^2\Phi}{(\partial q^2)^2}\right) = E\Phi \qquad (11)$$

Solving this by a separation of variables $\Phi(q^1, q^2) = \chi(q^1)\varphi(q^2)$ with the boundary condition $\varphi(\pm d) = 0$ we obtain

$$-\frac{\hbar^2}{2m}\left[\frac{\partial^2\left(\chi(q^1)\varphi(q^2)\right)}{(\partial q^1)^2} + \frac{1}{4}\left(\eta(q^1)\right)^2\chi(q^1)\varphi(q^2) + \frac{\partial^2\left(\chi(q^1)\varphi(q^2)\right)}{(\partial q^2)^2}\right] = E\chi(q^1)\varphi(q^2)$$

which becomes

$$-\frac{\hbar^2}{2m}\left[\chi''(q^1)\varphi(q^2) + \frac{1}{4}\left(\eta(q^1)\right)^2\chi(q^1)\varphi(q^2) + \chi(q^1)\varphi''(q^2)\right] = E\chi(q^1)\varphi(q^2).$$

Dividing both sides by $\chi\varphi$, we get

$$-\frac{\hbar^2}{2m}\left(\frac{\chi''(q^1)}{\chi(q^1)} + \frac{1}{4}\eta^2(q^1) + \frac{\varphi''(q^2)}{\varphi(q^2)}\right) = E,$$

or

$$-\frac{\hbar^2}{2m}\frac{\varphi''(q^2)}{\varphi(q^2)} = E + \frac{\hbar^2}{2m}\frac{\chi''(q^1)}{\chi(q^1)} + \frac{\hbar^2}{8m}\eta^2(q^1).$$

Since the left-hand side is entirely a function of q^2 and the right-hand side is a function of q^1, they must be equal to a constant, call it E_2

$$-\frac{\hbar^2}{2m}\frac{\varphi''(q^2)}{\varphi(q^2)} = E_2$$

so

$$-\frac{\hbar^2}{2m}\varphi''(q^2) = E_2\varphi(q^2). \tag{12a}$$

Also

$$E_2 = E + \frac{\hbar^2}{2m}\frac{\chi''(q^1)}{\chi(q^1)} + \frac{\hbar^2}{8m}\eta^2(q^1)$$

or rearranging

$$-\frac{\hbar^2}{2m}\frac{\chi''(q^1)}{\chi(q^1)} - \frac{\hbar^2}{8m}\eta^2(q^1) = E - E_2.$$

Let $E = E_1 + E_2$ so $E - E_2 = E_1$ and multiplying through by $\chi(q^1)$

$$-\frac{\hbar^2}{2m}\chi''(q^1) - \frac{\hbar^2}{8m}\eta^2(q^1)\chi(q^1) = E_1\chi(q^1). \tag{12b}$$

Notice that this equation is in the form of the Schrödinger equation with a potential, the elusive quantum potential

$$-\frac{\hbar^2}{2m}(\Delta + U)\chi(q^1) = E_1\chi(q^1)$$

where

$$-\frac{\hbar^2 U}{2m} = -\frac{\hbar^2\eta^2(q^1)}{8m}. \tag{13}$$

So the quantum treatment of the system has uncovered a "quantum" potential which is dependent on the curvature of C. Going a bit further and solving the equation

$$-\frac{\hbar^2}{2m}\varphi''(q^2) = E_2\varphi(q^2)$$

using the ansatz

$$\varphi(q^2) = ae^{i\alpha q^2} + be^{-i\beta q^2} \qquad a, b, \alpha, \beta \in \mathbf{R}$$

we find

$$\varphi'' = -a\alpha^2 e^{i\alpha q^2} - b\beta^2 e^{-i\beta q^2}$$

and

$$\frac{\hbar^2}{2m}\left(\alpha^2 a e^{i\alpha q^2} + \beta^2 b e^{-i\beta q^2}\right) = E_2\left(a e^{i\alpha q^2} + b e^{-i\beta q^2}\right).$$

Imposing the boundary conditions

$$\varphi(\pm d) = 0 = a e^{i\alpha(\pm d)} + b e^{-i\beta(\pm d)}$$

$$a e^{i\alpha(\pm d)} = -b e^{-i\beta(\pm d)}$$

we see that either $a = -b$ and $\alpha = -\beta$ or $a = b$, $\alpha = \beta$ and $\alpha = \beta = \frac{n\pi}{2d}$ with n an odd integer. If

$$a = -b, \quad \alpha = -\beta$$

then

$$\varphi = a e^{i\alpha q^2} - a e^{i\alpha q^2} = 0$$

so this solution is trivial. In the second case,

$$\varphi(q^2) = a\left(e^{i\alpha q^2} + e^{-i\alpha q^2}\right) = 2a\cos\alpha q^2 = 2a\cos\left(\frac{n\pi}{2d}q^2\right).$$

Normalizing φ, we must have

$$\int_{-d}^{d} \varphi\varphi dq^2 = 1 = \int_{-d}^{d} 4a^2\cos^2\left(\frac{n\pi}{2d}q^2\right)dq^2 = 4a^2\left(\frac{1}{2}a^2 + \frac{2d}{4n\pi}\sin\frac{n\pi}{d}q^2\right)\Big|_{-d}^{d}$$

or

$$1 = 4a^2\left(\frac{1}{2}d + \frac{2d}{4n\pi}\sin(n\pi) - \left(-\frac{1}{2}d\right) - \frac{2d}{4n\pi}\sin(-n\pi)\right)$$

i.e.

$$1 = 4a^2\left(\frac{1}{2}d + \frac{1}{2}d\right) = 4a^2 d$$

so that

$$a^2 = \frac{1}{4d}$$

and thus

$$a = \frac{1}{2}d^{-\frac{1}{2}}.$$

Hence

$$\varphi(q^2) = d^{-\frac{1}{2}} \cos\left(\frac{n\pi}{2d}q^2\right). \tag{14}$$

Differentiating φ twice we get

$$\varphi'' = -d^{-\frac{1}{2}}\left(\frac{n\pi}{2d}\right)^2 e^{i\frac{n\pi}{2d}q^2} - d^{-\frac{1}{2}}\left(\frac{n\pi}{2d}\right)^2 e^{-i\frac{n\pi}{2d}q^2} = -d^{-\frac{1}{2}}\left(\frac{n\pi}{2d}\right)^2 \cos\left(\frac{n\pi}{2d}q^2\right).$$

Thus

$$\varphi'' = \left(\frac{n\pi}{2d}\right)^2 \varphi.$$

And the Schrödinger equation

$$-\frac{\hbar^2}{2m}\varphi'' = E_2\varphi$$

gives

$$E_2 = -\frac{\hbar^2}{2m}\left(\frac{n\pi}{2d}\right)^2. \tag{15}$$

Now suppose we wanted to solve the Schrödinger equation for the wave function of a particle on a circle. The position vector in Euclidean coordinates is

$$\boldsymbol{a} = r\cos\theta\boldsymbol{i} + r\sin\theta\boldsymbol{j}.$$

Then

$$d\boldsymbol{a} = -r\sin\theta d\theta\boldsymbol{i} + r\cos\theta d\theta\boldsymbol{j}$$

and

$$(d\boldsymbol{a})^2 = (dq^1)^2 = r^2(d\theta)^2$$

so

$$dq^1 = r d\boldsymbol{\theta}.$$

By integrating both sides and solving for $\boldsymbol{\theta}$ it is found

$$\theta = \frac{q^1}{r}$$

thus

$$\boldsymbol{a} = r\cos\left(\frac{q^1}{r}\right)\boldsymbol{i} + r\cos\left(\frac{q^1}{r}\right)\boldsymbol{j}.$$

The tangent vector is found as

$$\boldsymbol{t} = \frac{\partial \boldsymbol{a}}{\partial q^1} = -\sin\left(\frac{q^1}{r}\right)\boldsymbol{i} + \cos\left(\frac{q^1}{r}\right)\boldsymbol{j}.$$

Then

$$\frac{\partial \boldsymbol{t}}{\partial q^1} = -\eta\boldsymbol{n} = -\frac{1}{r}\cos\left(\frac{q^1}{r}\right)\boldsymbol{i} - \frac{1}{r}\sin\left(\frac{q^1}{r}\right)\boldsymbol{j}$$

hence

$$\boldsymbol{n} = \cos\left(\frac{q^1}{r}\right)\boldsymbol{i} + \sin\left(\frac{q^1}{r}\right)\boldsymbol{j}.$$

and

$$\eta = \frac{1}{r}$$

as we expected. Substituting this into equation (12b)

$$-\frac{\hbar^2}{2m}\chi''(q^1) - \frac{\hbar^2}{8mr^2}\chi(q^1) = E_1\chi(q^1)$$

it is found that

$$-\chi'' = \left(\frac{1}{4r^2} + E_1\frac{2m}{\hbar^2}\right)\chi.$$

The solution to this, subject to normalization is

$$\chi(q^1) = a\sin\left(\left(\frac{1}{4r^2} + E_1\frac{2m}{\hbar^2}\right)^{\frac{1}{2}} q^1\right) + b\cos\left(\left(\frac{1}{4r^2} + E_1\frac{2m}{\hbar^2}\right)^{\frac{1}{2}} q^1\right).$$

By insisting that $\chi(0) = \chi(2\pi r)$ it is necessary that

$$\left(\frac{1}{4r^2} + E_1\frac{2m}{\hbar^2}\right)^{\frac{1}{2}} = \frac{n}{r}$$

where n is an integer. Solving this for the values of E_1 it is found that

$$E_1 = \frac{\hbar^2}{2m}\left(\frac{n^2}{r^2} + \frac{1}{4r^2}\right).$$

Normalizing χ to solve for the factors a and b in the wave function

$$1 = \int_0^{2\pi r} \chi^* \chi dq^1$$

or

$$1 = \int_0^{2\pi r} \left\{ a^2 \sin^2\left(\left(\frac{1}{4r^2} + E_1\frac{2m}{\hbar^2}\right)^{\frac{1}{2}} q^1\right) \right.$$
$$+2ab\sin\left(\left(\frac{1}{4r^2} + E_1\frac{2m}{\hbar^2}\right)^{\frac{1}{2}} q^1\right)\cos\left(\left(\frac{1}{4r^2} + E_1\frac{2m}{\hbar^2}\right)^{\frac{1}{2}} q^1\right)$$
$$\left. +b^2\cos^2\left(\left(\frac{1}{4r^2} + E_1\frac{2m}{\hbar^2}\right)^{\frac{1}{2}} q^1\right)\right\} dq^1$$

so

$$1 = 2\pi r(a^2 + b^2) + \frac{a^2 + 2ab + b^2}{4\left(\frac{1}{4r^2} + E_1\frac{2m}{\hbar^2}\right)^{\frac{1}{2}}}\sin^2\left(\left(\frac{1}{4r^2} + E_1\frac{2m}{\hbar^2}\right)^{\frac{1}{2}} q^1\right)\Big|_0^{2\pi r}$$

hence

$$1 = \pi r(a^2 + b^2)$$

thus

$$a^2 + b^2 = \frac{1}{\pi r}.$$

SECTION III

Quantum Potential for a General Submanifold

For the general treatment of this problem, the curve C will be replaced by a manifold M. Once again it will be infinitely differentiable. Let the dimension of M be D. Now M can be embedded in a bigger configuration space $\mathbf{R}^n$ with $n > D$. Also let our larger configuration space $\mathbf{R}^n$ be equipped with a Euclidian metric. Also assume that M admits a tubular neighborhood W_ε of constant radius ε. This assumption is true for ε, for arbitrary M.

Let x_μ, $\mu = 1, \ldots, n$ be the Cartesian coordinates in $\mathbf{R}^n$. Then if M is covered by coordinate neighborhoods, the points of M can be represented locally by

$$x_\mu = a_\mu(q^b) \quad \text{or} \quad \boldsymbol{r} = \boldsymbol{a}(q^b) \quad \text{with} \quad b = 1, \ldots, D. \tag{16}$$

Vector fields in M will be denoted X and their images under the embedding $i : M \to \mathbf{R}^n$ will be $i_* X$. Let $X \in \chi(M)$, where $\chi(M)$ is the set of all vector fields on the manifold M. If X is given locally as $X = X^a \dfrac{\partial}{\partial q^a}$, then $i_*(X)$ can also be expressed in $\chi(\mathbf{R}^n)$ as

$$i_*(X) = \frac{\partial a_\mu}{\partial q^b} \, X^b \frac{\partial}{\partial x_\mu}$$

or, defining

$$B_b^\mu = \frac{\partial a_\mu}{\partial q^b},$$

as

$$i_*(X) = B_b^\mu \; X^b \frac{\partial}{\partial x_\mu}. \tag{17}$$

Note that the indices a, b, c run from 1 to D, that μ, ν, λ, i, j run from 1 to n. From this point onwards x without a subscript will be an indice and the indices x, y will run from $D+1$ to n while x_μ with a subscript will remain a cartesian coordinate. Now, for the Euclidian metric $\delta_{\mu\nu}$, the induced metric $\dot{g}$ on M is

$$\dot{g}_{ba} = \delta_{\mu\lambda}\frac{\partial a_\mu}{\partial q^b}\frac{\partial a_\lambda}{\partial q^a} = \delta_{\mu\lambda}B^\mu_b B^\lambda_a \tag{18a}$$

or in vector form

$$\dot{g}_{ba} = \frac{\partial \boldsymbol{a}}{\partial q^b}\cdot\frac{\partial \boldsymbol{a}}{\partial q^a} = \boldsymbol{B}_b\cdot\boldsymbol{B}_a. \tag{18b}$$

Returning to the tubular neighborhood W_ϵ around M, the local coordinates are

$$x_\mu = a_\mu(q^a) + q^y \underset{y}{n}_\mu \qquad (a=1,\ldots,D;\;\; y=D+1,\ldots,n) \tag{19}$$

where $\underset{y}{\boldsymbol{n}}$ is a moving orthonormal frame in the normal bundle over M.

Since $\boldsymbol{a}(q^a)$ is the position vector for points on M, $\dfrac{\partial \boldsymbol{a}}{\partial q^a}$ will be a tangent vector to M and since $\underset{y}{\boldsymbol{n}}$ is normal to M

$$\underset{y}{\boldsymbol{n}}\cdot\frac{\partial \boldsymbol{a}}{\partial q^a} = 0. \tag{20}$$

There may be more than one normal vector at any point but these normal vectors are chosen to be mutually orthogonal

$$\underset{x}{\boldsymbol{n}}\cdot\underset{y}{\boldsymbol{n}} = 0 \qquad x \neq y.$$

The elements of the metric tensor are given as

$$g_{ji} = \frac{\partial x_\mu}{\partial q^i}\frac{\partial x_\nu}{\partial q^j}\delta_{\mu\nu}$$

with x_μ, x_ν given in equation (19). For i,j from $D+1$ to n

$$g_{xy} = \left(\frac{\partial a_\mu(q^a)}{\partial q^x} + \frac{\partial(q^x \underset{x}{n}_\mu)}{\partial q^x}\right)\left(\frac{\partial a_\mu(q^b)}{\partial q^y} + \frac{\partial(q^y \underset{y}{n}_\mu)}{\partial q^y}\right)\delta_{\mu\nu}.$$

Since $\boldsymbol{a}(q^a)$ is not a function of q^x (recall a goes from 1 to D and x goes form $D+1$ to n), $\dfrac{\partial a_\mu(q^a)}{\partial q^x} = 0$. Also the components of the normal vectors on M are functions the coordinates on M, i.e., q^a, so that

$$\frac{\partial \underset{x}{n}_\mu}{\partial q^x} = 0$$

Therefore,

$$\frac{\partial a_\mu(q^a)}{\partial q^x} + \frac{\partial (q^x \underset{x}{n}_\mu)}{\partial q^x} = \underset{x}{n}_\mu, \qquad \frac{\partial a_\nu(q^b)}{\partial q^y} + \frac{\partial (q^y \underset{x}{n}_\nu)}{\partial q^y} = \underset{y}{n}_\nu,$$

and

$$g_{xy} = \underset{x}{n}_\mu \, \underset{y}{n}_\nu \, \delta_{\mu\nu} = \underset{x}{\boldsymbol{n}} \cdot \underset{y}{\boldsymbol{n}} = \delta_{xy}$$

as $\boldsymbol{n}$ vectors are orthonormal.

For i, j from 1 to D

$$g_{ba} = \frac{\partial x_\mu}{\partial q^b}\frac{\partial x_\nu}{\partial q^a}\delta_{\mu\nu} = \frac{\partial}{\partial q^b}\left(a_\mu(q^b) - q^y \underset{y}{n}_\mu \right) \frac{\partial}{\partial q^a}\left(a_\nu(q^a) - q^x \underset{x}{n}_\nu \right)\delta_{\mu\nu}$$

$$= \left(\frac{\partial a_\mu(q^b)}{\partial q^b} + q^y \frac{\partial \underset{y}{n}_\mu}{\partial q^b} \right) \left(\frac{\partial a_\nu(q^a)}{\partial q^a} + q^x \frac{\partial \underset{x}{n}_\nu}{\partial q^a} \right)\delta_{\mu\nu}$$

$$g_{ba} = \left(\frac{\partial \boldsymbol{a}}{\partial q^b} + q^y \frac{\partial \underset{y}{\boldsymbol{n}}}{\partial q^b} \right) \cdot \left(\frac{\partial \boldsymbol{a}}{\partial q^a} + q^x \frac{\partial \underset{x}{\boldsymbol{n}}}{\partial q^a} \right). \tag{21}$$

For i from 1 to D and j from $D+1$ to n

$$g_{ya} = \frac{\partial x_\mu}{\partial q^x}\frac{\partial x_\nu}{\partial q^a}\delta_{\mu\nu} = \left(\underset{y}{n}_\mu \right) \left(\frac{\partial a_\mu}{\partial q^a} + q^x \frac{\partial \underset{x}{n}_\nu}{\partial q^a} \right)\delta_{\mu\nu}$$

$$= \underset{y}{\boldsymbol{n}} \cdot \frac{\partial \boldsymbol{a}}{\partial q^a} + \underset{y}{\boldsymbol{n}} \cdot q^x \frac{\partial \underset{x}{\boldsymbol{n}}}{\partial q^a}.$$

So $g_{ya} = 0$. The same is true for j from 1 to D and i from $D+1$ to n.

Therefore the metric tensor takes the form

$$(g_{ji}) = \begin{pmatrix} g_{ba} & 0 \\ 0 & \delta_{xy} \end{pmatrix} \tag{22}$$

with g_{ba} given in equation (21).

If the function $\Psi(q^i)$ is the solution to the Schrödinger equation

$$-\frac{\hbar}{2m}\Delta\Psi = E\Psi \tag{23}$$

then a constraining potential with infinitely high equidistant walls can be replaced with the following boundary condition on Ψ:

$$\Psi(q^i) = 0 \quad \text{whenever} \quad \|q^y\| = d \quad \text{for} \quad d < \varepsilon \tag{24}$$

The Laplacian is given by

$$\Delta\Psi = \sum_{i,j} g^{-\frac{1}{2}} \frac{\partial}{\partial q^j} \left(\frac{g^{\frac{1}{2}}}{g_{ji}} \frac{\partial \Psi}{\partial q^i} \right)$$

where once again $g = \det(g_{ji})$.

Dropping the summation, since repeated indices are summed over, and separating the terms into $i, j \leq D$ and $i, j > D$, we get

$$\Delta\Psi = g^{-\frac{1}{2}} \frac{\partial}{\partial q^b}\, g^{\frac{1}{2}} g^{ba}\, \frac{\partial\Psi}{\partial q^a} + g^{-\frac{1}{2}} \frac{\partial}{\partial q^x}\, g^{\frac{1}{2}} g^{xy}\, \frac{\partial\Psi}{\partial q^y}$$

with

$$g^{ba} = (g_{ba})^{-1} \quad \text{and} \quad g^{xy} = (g_{xy})^{-1}.$$

For the second summand on the right-hand side, we have

$$g^{\frac{1}{2}} g^{xy} = g^{\frac{1}{2}} \delta_{xy}$$

so that

$$\Delta\Psi = g^{-\frac{1}{2}} \frac{\partial}{\partial q^b} \left(g^{\frac{1}{2}} g^{ba}\, \frac{\partial\Psi}{\partial q^a} \right) + g^{-\frac{1}{2}} \frac{\partial}{\partial q^y} \left(g^{\frac{1}{2}} \frac{\partial\Psi}{\partial q^y} \right). \tag{25a}$$

The Lebesgue measure in $\mathbf{R}^n$ is $d^n x$. Hence,

$$d^n x = dx_1 dx_2 \cdots dx_n = d\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial q^1} dq^1 \frac{\partial \boldsymbol{r}}{dq^2} dq^2 \cdots \frac{\partial \boldsymbol{r}}{\partial q^n} dq^n = \sqrt{g} d^n q. \qquad (25b)$$

Factorizing $g = \dot{g}\gamma$, we set

$$\Phi = \gamma^{\frac{1}{4}} \Psi \qquad (26)$$

where $\dot{g}$ comes from the metric induced on M and is once again $\dot{g}(q_1, \ldots, q_D)$. Checking for consistent probabilistic interpretation, we have

$$\int |\Psi|^2 d^n x = \int |\Phi \gamma^{-\frac{1}{4}}|^2 g^{\frac{1}{2}} d^n q = \int |\Phi|^2 \gamma^{-\frac{1}{2}} \dot{g}^{\frac{1}{2}} \gamma^{\frac{1}{2}} d^n q$$

$$= \int |\Phi|^2 \dot{g}^{\frac{1}{2}} d^n q = \int \left(\int_{g^x q_x \leq d^2} |\Phi|^2 dq_{D+1} \cdots dq_n \right) \dot{g} dq_1 \cdots dq_D. \qquad (27)$$

Using equation (25a) in the Schrödinger equation

$$-\frac{\hbar^2}{2m} \Delta \Psi = E \Psi$$

we get

$$-\frac{\hbar^2}{2m} \left[g^{-\frac{1}{2}} \frac{\partial}{\partial q^b} \left(g^{ba} g^{\frac{1}{2}} \frac{\partial \Psi}{\partial q^a} \right) + g^{-\frac{1}{2}} \frac{\partial}{\partial q^y} \left(g^{\frac{1}{2}} \frac{\partial \Psi}{\partial q^y} \right) \right] = E\Psi.$$

Substituting in $g = \dot{g}\gamma$ and $\Psi = \gamma^{-\frac{1}{4}} \Phi$, we obtain

$$-\frac{\hbar^2}{2m} \left[\dot{g}^{-\frac{1}{2}} \gamma^{-\frac{1}{2}} \frac{\partial}{\partial q^b} \left(g^{ba} \dot{g}^{\frac{1}{2}} \gamma^{\frac{1}{2}} \frac{\partial}{\partial q^a} (\gamma^{-\frac{1}{4}} \Phi) \right) \right.$$

$$\left. + \dot{g}^{-\frac{1}{2}} \gamma^{-\frac{1}{2}} \frac{\partial}{\partial q^y} \left(\dot{g}^{\frac{1}{2}} \gamma^{\frac{1}{2}} \frac{\partial}{\partial q^y} (\gamma^{-\frac{1}{4}} \Phi) \right) \right] = E \gamma^{-\frac{1}{4}} \Phi.$$

Eliminating a factor of $\gamma^{-\frac{1}{4}}$ from both sides and expanding the partial differentials of products in the second term, we get

$$E\Phi = -\frac{\hbar^2}{2m} \left\{ \dot{g}^{-\frac{1}{2}} \gamma^{-\frac{1}{4}} \frac{\partial}{\partial q^b} \left[g^{ba} \dot{g}^{\frac{1}{2}} \gamma^{\frac{1}{2}} \frac{\partial}{\partial q^a} (\gamma^{-\frac{1}{4}} \Phi) \right] \right.$$

$$\left. + \dot{g}^{-\frac{1}{2}} \gamma^{-\frac{1}{4}} \frac{\partial}{\partial q^y} \left[\dot{g}^{\frac{1}{2}} \gamma^{\frac{1}{2}} \left(\gamma^{-\frac{1}{4}} \frac{\partial \Phi}{\partial q^y} + \Phi \frac{\partial \gamma^{-\frac{1}{4}}}{\partial q^y} \right) \right] \right\}.$$

Working only with the second term, we have

$$\dot{g}^{-\frac{1}{2}}\gamma^{-\frac{1}{4}}\frac{\partial}{\partial q^y}\left[\dot{g}^{\frac{1}{2}}\gamma^{\frac{1}{2}}\left(\gamma^{-\frac{1}{4}}\frac{\partial\Phi}{\partial q^y}+\Phi\frac{\partial\gamma^{-\frac{1}{4}}}{\partial q^y}\right)\right] = \dot{g}^{-\frac{1}{2}}\gamma^{-\frac{1}{4}}\left\{\dot{g}^{\frac{1}{2}}\left[\frac{\partial\gamma^{\frac{1}{4}}}{\partial q^y}\frac{\partial\Phi}{\partial q^y}+\gamma^{\frac{1}{4}}\frac{\partial^2\Phi}{(\partial q^y)^2}\right.\right.$$

$$\left.\left.+\frac{\partial\gamma^{\frac{1}{2}}}{\partial q^y}\Phi\frac{\partial\gamma^{-\frac{1}{4}}}{\partial q^y}+\gamma^{\frac{1}{2}}\frac{\partial\Phi}{\partial q^y}\frac{\partial\gamma^{-\frac{1}{4}}}{\partial q^y}+\gamma^{\frac{1}{2}}\Phi\frac{\partial^2\gamma^{-\frac{1}{4}}}{(\partial q^y)^2}\right]\right\}$$

$$=\gamma^{-\frac{1}{4}}\left\{\frac{1}{4}\gamma^{-\frac{3}{4}}\frac{\partial\gamma}{\partial q^y}\frac{\partial\Phi}{\partial q^y}+\gamma^{\frac{1}{4}}\frac{\partial^2\Phi}{(\partial q^y)^2}+\frac{1}{2}\gamma^{-\frac{1}{2}}\frac{\partial\gamma}{\partial q^y}\Phi\left(-\frac{1}{4}\right)\gamma^{-\frac{5}{4}}\frac{\partial\gamma}{\partial q^y}\right.$$

$$\left.+\gamma^{\frac{1}{2}}\frac{\partial\Phi}{\partial q^y}\left(-\frac{1}{4}\right)\gamma^{-\frac{5}{4}}\frac{\partial\gamma}{\partial q^y}+\gamma^{\frac{1}{2}}\Phi\left(-\frac{1}{4}\right)\frac{\partial}{\partial q^y}\left(\gamma^{-\frac{5}{4}}\frac{\partial\gamma}{\partial q^y}\right)\right\}.$$

The last term expands to

$$-\frac{1}{4}\gamma^{\frac{1}{2}}\Phi\frac{\partial}{\partial q^y}\left(\gamma^{-\frac{5}{4}}\frac{\partial\gamma}{\partial q^y}\right)=-\frac{1}{4}\gamma^{\frac{1}{2}}\Phi(-\frac{5}{4})\gamma^{-\frac{9}{4}}\frac{\partial\gamma}{\partial q^y}\frac{\partial\gamma}{\partial q^y}-\frac{1}{4}\gamma^{\frac{1}{2}}\Phi\gamma^{-\frac{5}{4}}\frac{\partial^2\gamma}{(\partial q^y)^2}.$$

So the second term is equal to, with minor reductions and rearrangements

$$\frac{1}{4}\gamma^{-1}\frac{\partial\gamma}{\partial q^y}\frac{\partial\Phi}{\partial q^y}+\frac{\partial^2\Phi}{\partial {q^y}^2}-\frac{1}{8}\gamma^{-2}\Phi\left(\frac{\partial\gamma}{\partial q^y}\right)^2$$

$$-\frac{1}{4}\gamma^{-1}\frac{\partial\gamma}{\partial q^y}\frac{\partial\Phi}{\partial q^y}+\frac{5}{16}\gamma^{-2}\Phi\left(\frac{\partial\gamma}{\partial q^y}\right)^2-\frac{1}{4}\gamma^{-1}\Phi\frac{\partial^2\gamma}{(\partial q^y)^2}.$$

Grouping like terms, this equals to

$$\frac{\partial^2\Phi}{(\partial q^y)^2}+\left(\frac{1}{4}\gamma^{-1}-\frac{1}{4}\gamma^{-1}\right)\frac{\partial\gamma}{\partial q^y}\frac{\partial\Phi}{\partial q^y}+$$

$$\left(-\frac{1}{8}\gamma^{-2}\Phi+\frac{5}{16}\gamma^{-2}\Phi\right)\left(\frac{\partial\gamma}{\partial q^y}\right)^2-\frac{1}{4}\gamma^{-1}\Phi\frac{\partial^2\gamma}{(\partial q^y)^2}$$

Performing the substitutions

$$\frac{\partial ln\gamma}{\partial q^y}=\gamma^{-1}\frac{\partial\gamma}{\partial q^y}, \qquad \left(\frac{\partial ln\gamma}{\partial q^y}\right)^2=\gamma^{-2}\left(\frac{\partial\gamma}{\partial q^y}\right)^2,$$

we have

$$\frac{\partial^2 ln\gamma}{(\partial q^y)^2}=\frac{\partial}{\partial q^y}\left(\gamma^{-1}\frac{\partial\gamma}{\partial q^y}\right)=-\gamma^{-2}\frac{\partial\gamma}{\partial q^y}\frac{\partial\gamma}{\partial q^y}+\gamma^{-1}\frac{\partial^2\gamma}{(\partial q^y)^2},$$

So that

$$\gamma^{-1}\frac{\partial^2\gamma}{(\partial q^y)^2} = \frac{\partial^2 ln\gamma}{(\partial q^y)^2} + \left(\frac{\partial ln\gamma}{\partial q^y}\right)^2 .$$

Thus, the second term becomes

$$\frac{\partial^2\Phi}{(\partial q^y)^2} + \frac{3\Phi}{16}\left(\frac{\partial ln\gamma}{\partial q^y}\right)^2 - \frac{1}{4}\Phi\left[\frac{\partial^2 ln\gamma}{(\partial q^y)^2} + \left(\frac{\partial ln\gamma}{\partial q^y}\right)^2\right] =$$

$$\frac{\partial^2\Phi}{(\partial q^y)^2} - \frac{\Phi}{4}\frac{\partial^2 ln\gamma}{(\partial q^y)^2} - \frac{\Phi}{16}\left(\frac{\partial ln\gamma}{\partial q^y}\right)^2 .$$

Hence the Schrödinger equation is

$$-\frac{\hbar^2}{2m}\dot{g}^{-\frac{1}{2}}\gamma^{-\frac{1}{4}}\frac{\partial}{\partial q^b}\left[g^{ba}\dot{g}^{\frac{1}{2}}\gamma^{\frac{1}{2}}\frac{\partial}{\partial q^a}(\gamma^{-\frac{1}{4}}\Phi)\right]$$

$$-\frac{\hbar^2}{2m}\left\{\frac{\partial^2\Phi}{(\partial q^y)^2} + \left[-\frac{1}{4}\frac{\partial^2 ln\gamma}{(\partial q^y)^2} - \frac{1}{16}\left(\frac{\partial ln\gamma}{\partial q^y}\right)^2\right]\Phi\right\} = E\Phi \qquad (28)$$

with the first term summed over a, b from 1 to D and the second term summed over y from $D+1$ to n.

Now

$$\gamma = \dot{g}^{-1}g = \dot{g}^{-1}\det\left[\left(\frac{\partial \boldsymbol{a}}{\partial q^b} + q^y\frac{\partial \underset{y}{\boldsymbol{n}}}{\partial q^b}\right)\cdot\left(\frac{\partial \boldsymbol{a}}{\partial q^a} + q^x\frac{\partial \underset{x}{\boldsymbol{n}}}{\partial q^b}\right)\right] . \qquad (29)$$

In the limit $q_x \to 0$

$$\gamma = \dot{g}^{-1}\det\left[\frac{\partial \boldsymbol{a}}{\partial q^b}\cdot\frac{\partial \boldsymbol{a}}{\partial q^a}\right] .$$

Recalling equation ($18b$)

$$\dot{g}_{ba} = \frac{\partial \boldsymbol{a}}{\partial q^b}\cdot\frac{\partial \boldsymbol{a}}{\partial q^a}$$

we get

$$\lim_{q^x\to 0}\gamma = \dot{g}^{-1}\det\dot{g}_{ba} = \dot{g}^{-1}\dot{g} = 1.$$

Thus in the limit $q^x \to 0$ equation (28) becomes

$$-\frac{\hbar}{2m}\dot{\Delta}\Phi - \frac{\hbar^2}{2m}U\Phi - \frac{\hbar^2}{2m}\sum_y\frac{\partial^2\Phi}{(\partial q^y)^2} = E\Phi \qquad (30)$$

where

$$\dot{\Delta}\Phi = \dot{g}^{-\frac{1}{2}} \frac{\partial}{\partial q^b} \left(g^{ba} \dot{g}^{\frac{1}{2}} \frac{\partial \Phi}{\partial q^a} \right)$$

and the quantum potential is

$$-\frac{\hbar^2}{2m} U = -\frac{\hbar^2}{8m} \lim_{q^y \to 0} \sum \left[\frac{\partial^2 ln\gamma}{(\partial q^y)^2} + \frac{1}{4} \left(\frac{\partial ln\gamma}{\partial q^y} \right)^2 \right]. \quad (31)$$

For further calculation of the quantum potential the Weingarten formula will be used [8]. The vector field X, which was introduced at the beginning of this section, is tangent to the manifold M. Let ξ be a vector field perpendicular to the manifold M. Decomposing $\nabla_x \xi$ uniquely into its tangent and normal parts

$$\nabla_x \xi = -A_\xi(X) + D_x \xi \quad (32)$$

with

$$\dot{g}(A_x i(X), Y) = g(H(X,Y), \xi) \quad (33)$$

and, for $Y \in \chi(M)$, $H(X,Y) = (\nabla_x Y)_n$, being the second fundamental form of M in $\mathbf{R}^n$ such that

$$H : TM \times TM \to TM^\perp$$

Thus the vector $\boldsymbol{H}(= H(X,Y))$ is normal to the manifold and can be written as

$$\boldsymbol{H}_{ba} = \underset{y}{H_{ba}} \underset{y}{\boldsymbol{n}} \quad (34)$$

Now in equation (32) let $\xi = \underset{y}{\boldsymbol{n}}$. In our coordinate system

$$\nabla_x \xi = \nabla_x \underset{y}{\boldsymbol{n}} .$$

Since all normal fields can be chosen parallel to the normal bundle we must have $\nabla_x \underset{y}{\boldsymbol{n}}$ purely tangent, or $D_x \xi = D_x \underset{y}{\boldsymbol{n}} = 0$. Returning to equation (32)

$$\nabla_x \xi = -A_\xi(X) = \frac{\partial \underset{y}{\boldsymbol{n}}}{\partial q^c}.$$

Now using Weingartens formula [9]

$$\frac{\partial \underset{y}{\boldsymbol{n}}}{\partial q^a} = -g^{bc} b_{ba} \frac{\partial \boldsymbol{a}}{\partial q^c}$$

and

$$b_{ba} = \frac{\partial}{\partial q^b} \left(\frac{\partial \boldsymbol{a}}{\partial q^a} \right) \cdot \underset{y}{\boldsymbol{n}} .$$

So

$$\frac{\partial \underset{y}{\boldsymbol{n}}}{\partial q^a} = - \left(\frac{\partial}{\partial q^b} \left(\frac{\partial \boldsymbol{a}}{\partial q^a} \right) \cdot \underset{y}{\boldsymbol{n}} \right) g^{bc} \frac{\partial \boldsymbol{a}}{\partial q^c}$$

and

$$\frac{\partial \underset{y}{\boldsymbol{n}}}{\partial q^a} = - \left(\frac{\partial}{\partial q^b} \left(\frac{\partial \boldsymbol{a}}{\partial q^a} \right) \cdot \underset{y}{\boldsymbol{n}} \right) g^{bc} \boldsymbol{B}_c .$$

Hence

$$\frac{\partial \underset{y}{\boldsymbol{n}}}{\partial q^a} = - \left(\frac{\partial}{\partial q^b} \left(\frac{\partial \boldsymbol{a}}{\partial q^a} \right) \cdot \underset{y}{\boldsymbol{n}} \right) \boldsymbol{B}^b$$

or

$$\frac{\partial \underset{y}{\boldsymbol{n}}}{\partial q^a} = - \underset{y}{H}_{ba} \boldsymbol{B}^b \qquad (35)$$

where the coefficients of the second fundamental form are given by

$$\underset{y}{H}_{ba} = \frac{\partial}{\partial q^b} \left(\frac{\partial \boldsymbol{a}}{\partial q^a} \right) \cdot \underset{y}{\boldsymbol{n}} . \qquad (36)$$

Recalling that

$$\frac{\partial ln\gamma}{\partial q^y} = \frac{1}{\gamma} \frac{\partial \gamma}{\partial q^y} \quad \text{and} \quad \gamma = \dot{g}^{-1} \det(g_{ba}),$$

we find

$$\frac{\partial ln\gamma}{\partial q^y} = \frac{\dot{g}}{\det(g_{ba})} \frac{\partial(\dot{g}^{-1} \det(g_{ba}))}{\partial q^y} = \frac{\dot{g}}{g} \left[\left(\frac{\partial \dot{g}^{-1}}{\partial q^y} \right) g + \dot{g}^{-1} \frac{\partial g}{\partial q^y} \right]$$

Now $\dot{g} = \det(\dot{g}_{ba})$ where $\dot{g}_{ba} = \frac{\partial \boldsymbol{a}}{\partial q^b} \cdot \frac{\partial \boldsymbol{a}}{\partial q^a}$. Since $\boldsymbol{a}$ is a function of $\dot{g}$ for a from 1 to D. $\dot{g}_{ba}$ is a function of q^a also and

$$\dot{g} = \dot{g}(q^a)$$

So

$$\frac{\partial \dot{g}}{\partial q^y} = 0$$

Thus

$$\frac{\partial ln\gamma}{\partial q^y} = \frac{\dot{g}}{g}\dot{g}^{-1}\frac{\partial g}{\partial q^y} = g^{-1}\frac{\partial g}{\partial q^y} = \frac{\partial ln(\det(g_{ba}))}{\partial q^y},$$

$$\frac{\partial lng}{\partial q^y} = \frac{1}{g}\frac{\partial g}{\partial q^y} = \frac{1}{g}\frac{\partial g}{\partial g_{ba}}\frac{\partial g_{ba}}{\partial q^y}.$$

Since $\frac{\partial g}{\partial g_{ba}} = \frac{\partial \det g_{ba}}{\partial g_{ba}}$ = cofactor of g_{ba} in g, we get

$$\frac{\partial lng}{\partial q^y} = \frac{\text{cofactor of } g_{ba} \text{ in } g}{g}\ \frac{\partial g_{ba}}{\partial q^y}.$$

Now $\frac{\text{cofactor of } g_{ba} \text{ in } g}{g}$ is the definition of g^{ba}, so

$$\frac{\partial lng}{\partial q^y} = g^{ba}\frac{\partial g_{ba}}{\partial q^y}.$$

Then

$$\frac{\partial^2 ln\gamma}{(\partial q^y)^2} = \frac{\partial^2 lng}{(\partial q^y)^2} = \frac{\partial g^{ba}}{\partial q^y}\ \frac{\partial g_{ba}}{\partial q^y} + g^{ba}\frac{\partial^2 g_{ba}}{(\partial q^y)^2}.$$

But

$$g_{ba}\ g^{ba} = 1,$$

so

$$\frac{\partial(g_{ba}g^{ba})}{\partial q^y} = 0$$

or

$$\frac{\partial g^{ba}}{\partial q^y}g^{ba} + g_{ba}\frac{\partial g_{ba}}{\partial q^y} = 0,$$

or

$$\frac{\partial g^{ba}}{\partial q^y} = -g^{ba}\frac{\partial g_{ba}}{\partial q^y}g^{ba}.$$

Hence

$$\frac{\partial^2 ln\gamma}{(\partial q^y)^2} = -g^{ba}\frac{\partial g_{ba}}{\partial q^y}g^{ba}\frac{\partial g_{ba}}{\partial q^y} + g^{ba}\frac{\partial^2 g_{ba}}{(\partial q^y)^2}$$

Evaluating

$$\frac{\partial g_{ba}}{\partial q^z} = \sum_{xy}\frac{\partial}{\partial q^z}\left[\left(\frac{\partial \boldsymbol{a}}{\partial q^b} + q^y\frac{\partial \underset{y}{\boldsymbol{n}}}{\partial q^b}\right)\cdot\left(\frac{\partial \boldsymbol{a}}{\partial q^a} + q^x\frac{\partial \underset{x}{\boldsymbol{n}}}{\partial q^a}\right)\right]$$

$$= \sum_{xy}\left\{\left[\frac{\partial}{\partial q^z}\left(\frac{\partial \boldsymbol{a}}{\partial q^b} + q^y\frac{\partial \underset{y}{\boldsymbol{n}}}{\partial q^b}\right)\right]\cdot\left(\frac{\partial \boldsymbol{a}}{\partial q^a} + q^x\frac{\partial \underset{x}{\boldsymbol{n}}}{\partial q^a}\right) + \left(\frac{\partial \boldsymbol{a}}{\partial q^b} + q^y\frac{\partial \underset{y}{\boldsymbol{n}}}{\partial q^b}\right)\right.$$

$$\left.\left[\frac{\partial}{\partial q^z}\left(\frac{\partial \boldsymbol{a}}{\partial q^a} + q^x\frac{\partial \underset{x}{\boldsymbol{n}}}{\partial q^a}\right)\right]\right\}$$

Recall that $\underset{y}{\boldsymbol{n}}$ is a function of q^a's (a from 1 to D) only, hence $\underset{y}{\boldsymbol{n}}=\underset{y}{\boldsymbol{n}}\,(q^a)$, so $\dfrac{\partial \underset{y}{\boldsymbol{n}}(q^a)}{\partial q^b}$ is also a function of q^a's only. Thus $\dfrac{\partial}{\partial q^z}\left(\dfrac{\partial \underset{y}{\boldsymbol{n}}}{\partial q^b}\right) = 0$. The same is true for $\boldsymbol{a}$ as for $\underset{y}{\boldsymbol{n}}$ so $\dfrac{\partial}{\partial q^z}\left(\dfrac{\partial \boldsymbol{a}}{\partial q^a}\right) = 0$.

So we have

$$\frac{\partial g_{ba}}{\partial q^z} = \sum_{xy}\left\{\left(\frac{\partial q^y}{\partial q^z}\frac{\partial \underset{y}{\boldsymbol{n}}}{\partial q^b}\right)\cdot\left(\frac{\partial \boldsymbol{a}}{\partial q^a} + q^x\frac{\partial \underset{x}{\boldsymbol{n}}}{\partial q^a}\right) + \left(\frac{\partial \boldsymbol{a}}{\partial q^a} + q^x\frac{\partial \underset{x}{\boldsymbol{n}}}{\partial q^a}\right)\cdot\left(\frac{\partial q^x}{\partial q^z}\frac{\partial \underset{x}{\boldsymbol{n}}}{\partial q^a}\right)\right\}$$

But

$$\frac{\partial q^x}{\partial q^z} = \delta_{xz}$$

so that

$$\frac{\partial g_{ba}}{\partial q^z} = \sum_{xy}\left\{\frac{\partial \underset{z}{\boldsymbol{n}}}{\partial q^b}\cdot\left(\frac{\partial \boldsymbol{a}}{\partial q^a} + q^x\frac{\partial \underset{x}{\boldsymbol{n}}}{\partial q^a}\right) + \left(\frac{\partial \boldsymbol{a}}{\partial q^b} + q^y\frac{\partial \underset{y}{\boldsymbol{n}}}{\partial q^b}\right)\cdot\frac{\partial \underset{z}{\boldsymbol{n}}}{\partial q^a}\right\}$$

Since no terms remain that are dependent on both x and y, and they vary over the same range, one can be replace by the other.

$$\frac{\partial g_{ba}}{\partial q^z} = \frac{\partial \underset{z}{\boldsymbol{n}}}{\partial q^b}\cdot\frac{\partial \boldsymbol{a}}{\partial q^a} + \frac{\partial \boldsymbol{a}}{\partial q^b}\cdot\frac{\partial \underset{z}{\boldsymbol{n}}}{\partial q^a} + \sum_x q^x\left(\frac{\partial \underset{z}{\boldsymbol{n}}}{\partial q^b}\cdot\frac{\partial \underset{x}{\boldsymbol{n}}}{\partial q^a} + \frac{\partial \underset{x}{\boldsymbol{n}}}{\partial q^b}\cdot\frac{\partial \underset{x}{\boldsymbol{n}}}{\partial q^a}\right)$$

Recalling equation (20):

$$\underset{z}{\boldsymbol{n}} \cdot \frac{\partial \boldsymbol{a}}{\partial q^a} = 0,$$

we have

$$\frac{\partial}{\partial q^b}\left(\underset{z}{\boldsymbol{n}} \cdot \frac{\partial \boldsymbol{a}}{\partial q^a}\right) = 0 = \frac{\partial \underset{z}{\boldsymbol{n}}}{\partial q^b} \cdot \frac{\partial \boldsymbol{a}}{\partial q^a} + \underset{z}{\boldsymbol{n}} \cdot \frac{\partial}{\partial q^b}\left(\frac{\partial \boldsymbol{a}}{\partial q^a}\right),$$

so that

$$\frac{\partial \underset{z}{\boldsymbol{n}}}{\partial q^b} \cdot \frac{\partial \boldsymbol{a}}{\partial q^a} = -\left[\frac{\partial}{\partial q^b}\left(\frac{\partial \boldsymbol{a}}{\partial q^a}\right)\right] \cdot \underset{z}{\boldsymbol{n}} = -\underset{z}{H}_{ba}$$

Likewise

$$\frac{\partial \boldsymbol{a}}{\partial q^b} \cdot \frac{\partial \underset{x}{\boldsymbol{n}}}{\partial q^a} = -\frac{\partial}{\partial q^a}\left(\frac{\partial \boldsymbol{a}}{\partial q^b}\right) \cdot \underset{z}{\boldsymbol{n}} = -\underset{z}{H}_{ab}$$

but since M is infinitely differentiable

$$\frac{\partial}{\partial q^a}\left(\frac{\partial \boldsymbol{a}}{\partial q^b}\right) = \frac{\partial}{\partial q^b}\left(\frac{\partial \boldsymbol{a}}{\partial q^a}\right)$$

so that

$$\underset{z}{H}_{ab} = \left[\frac{\partial}{\partial q^a}\left(\frac{\partial \boldsymbol{a}}{\partial q^b}\right)\right] \cdot \underset{z}{\boldsymbol{n}} = \left[\frac{\partial}{\partial q^b}\left(\frac{\partial \boldsymbol{a}}{\partial q^a}\right)\right] \cdot \underset{z}{\boldsymbol{n}} = \underset{z}{H}_{ba}$$

Therefore

$$\frac{\partial g_{ba}}{\partial q^z} = -2\underset{z}{H}_{ba} + \sum_x q^x \left(\frac{\partial \underset{z}{\boldsymbol{n}}}{\partial q^b} \cdot \frac{\partial \underset{x}{\boldsymbol{n}}}{\partial q^a} + \frac{\partial \underset{x}{\boldsymbol{n}}}{\partial q^b} \cdot \frac{\partial \underset{z}{\boldsymbol{n}}}{\partial q^a}\right)$$

From equation (35),

$$\frac{\partial \underset{x}{\boldsymbol{n}}}{\partial q^a} = \underset{x}{H}_{ab}\boldsymbol{B}^b$$

$$\frac{\partial g_{ba}}{\partial q^z} = -2\underset{z}{H}_{ba} + \sum_x q^x \left(\underset{z}{H}_{bc}\boldsymbol{B}^c \cdot \underset{x}{H}_{ad}\boldsymbol{B}^d + \underset{x}{H}_{be}\boldsymbol{B}^e \cdot \underset{z}{H}_{af}\boldsymbol{B}^f\right)$$

$$= -2\underset{z}{H}_{ba} + \sum_x q^x \left(\underset{z}{H}_{bc}\dot{g}^{cc}\underset{x}{H}_{ac} + \underset{x}{H}_{be}\dot{g}^{ee}\underset{z}{H}_{ae}\right).$$

Since c and e enter separately and run over the same range, we get

$$\frac{\partial g_{ba}}{\partial q^z} = -2\underset{z}{H}_{ba} + \sum_x q^x \left(\underset{z}{H}_b{}^c\underset{x}{H}_{ac} + \underset{x}{H}_b{}^c\underset{z}{H}_{ac}\right). \qquad (37a)$$

Differentiating again, we find

$$\frac{\partial^2 g_{ba}}{(\partial q^z)^2} = \sum_x \frac{\partial q^x}{\partial q^z}\left(\underset{z}{H}{}_b{}^c \underset{x}{H}{}_{ac} + \underset{x}{H}{}_b{}^c \underset{z}{H}{}_{ac}\right)$$

$$= \delta_{xz}\left(\underset{z}{H}{}_b{}^c \underset{x}{H}{}_{ac} + \underset{x}{H}{}_b{}^c \underset{z}{H}{}_{ac}\right).$$

Thus,

$$\frac{\partial^2 g_{ba}}{(\partial q^z)^2} = 2\underset{z}{H}{}_b{}^c \underset{z}{H}{}_{ac}. \tag{37b}$$

Hence,

$$\lim_{q^y \to 0} \frac{\partial ln\gamma}{\partial q^y} = \lim_{q^y \to 0} \frac{\partial lng}{\partial q^y} = \lim_{q^y \to 0} g^{ba}\frac{\partial g_{ba}}{\partial q^y} = -2g^{ba}\underset{y}{H}{}_{ba}$$

$$= -2\underset{y}{H}{}_b{}^b = -2Tr\underset{y}{H}. \tag{38a}$$

Then

$$\lim_{q^y \to 0}\sum_y\left(\frac{\partial ln\gamma}{\partial q^y}\right)^2 = 4\sum_y\left(Tr\underset{y}{H}\right)^2 \tag{38b}$$

where Tr is Trace and $\underset{y}{H} = \underset{y}{H}{}_{ba}$. Also

$$\frac{\partial^2 ln\gamma}{(\partial q^y)^2} = -g^{ba}\frac{\partial g_{ba}}{\partial q^y}g^{ba}\frac{\partial g_{ba}}{\partial q^y} + g^{ba}\frac{\partial^2 g_{ba}}{(\partial q^y)^2}$$

or

$$\lim_{q^y \to 0}\frac{\partial^2 ln\gamma}{(\partial q^y)^2} = -g^{ba}\left(-2\underset{y}{H}{}_{ba}\right)g^{ba}\left(-2\underset{y}{H}{}_{ba}\right) + g^{ba}2\underset{y}{H}{}_b{}^c\underset{y}{H}{}_{ac}$$

so

$$\sum_y \lim_{q^y \to 0}\frac{\partial^2 ln\gamma}{(\partial q^y)^2} = \sum_y\left(-4\underset{y}{H}{}^{ba}\underset{y}{H}{}_{ba} + 2\underset{y}{H}{}^{ac}\underset{y}{H}{}_{ac}\right) = -2\sum_y \underset{y}{H}{}^{ba}\underset{y}{H}{}_{ba}$$

$$= \sum_y -2Tr\left(\underset{y}{H}{}^2\right) \tag{38c}$$

Inserting equation $(38b, c)$ into equation (31) the quantum potential is finally found to be

$$-\frac{\hbar^2}{2m}U = -\frac{\hbar^2}{2m}\sum_y\left[\frac{1}{2}Tr\left(\underset{y}{H}{}^2\right) - \frac{1}{4}\left(Tr\underset{y}{H}\right)^2\right] \tag{39}$$

Backtracking a step, the quantum potential can also be written as

$$-\frac{\hbar^2}{2m}U = \frac{\hbar^2}{8m}\sum_y \left(-2\underset{y}{H}^{ba}\underset{y}{H}_{ba} + \underset{y}{H}_b{}^b\underset{y}{H}_a{}^a\right)$$

The potential can be expressed in terms of the intrinsic scalar curvature of M which is defined as

$$R = \underset{y}{H}_b{}^b\underset{y}{H}_a{}^a - \underset{y}{H}^{ba}\underset{y}{H}_{ba} \tag{40}$$

and a mean curvature vector which produces an extrinsic mean curvature of M in $\mathbf{R}^n$

$$\boldsymbol{\eta} = \frac{1}{D}\underset{y}{H}_a{}^a\underset{y}{\boldsymbol{n}}$$

$$\eta = (\boldsymbol{\eta}\cdot\boldsymbol{\eta})^{\frac{1}{2}} = \left(\frac{1}{D}\underset{y}{H}_a{}^a\underset{y}{\boldsymbol{n}}\cdot\frac{1}{D}\underset{x}{H}_b{}^b\underset{x}{\boldsymbol{n}}\right)^{\frac{1}{2}} = \left(\frac{1}{D^2}\underset{y}{H}_a{}^a\underset{y}{H}_b{}^b\right)^{\frac{1}{2}}$$

So equation (30) is now

$$\begin{aligned} -\frac{\hbar^2}{2m}U &= \frac{\hbar^2}{8m}\left(2R - 2\underset{y}{H}_b{}^b\underset{y}{H}_a{}^a + \underset{y}{H}_b{}^b\underset{y}{H}_a{}^a\right) \\ &= \frac{\hbar^2}{8m}(2R - D^2\eta^2) \\ -\frac{\hbar^2}{2m}U &= \frac{\hbar^2}{2m}\left(\frac{1}{2}R - \frac{1}{4}D^2\eta^2\right) = \frac{\hbar^2}{4m}R - \frac{\hbar^2D^2\eta^2}{8m} \end{aligned} \tag{41}$$

If the manifold M in question is of D dimension embedded in $\mathbf{R}^{D+1}$ then $n = D + 1$ and there is only one normal coordinate, thus the summation over y can be dropped in equation (39). It should be noted that the trace of a matrix is equal to the sum of its eigenvalues. Likewise the trace of a matrix squared is equal to the sum of the square of its eigenvalues.

$$TrA = \sum \lambda_A$$

$$Tr(A^2) = \sum(\lambda_A^2)$$

Since the matrix H is made of the coefficients of the second fundamental form for manifold M, its eigenvalues are the principle curvatures for each dimension [10]. Denoting the principle curvatures as R_b then

$$\lambda_H = \frac{1}{R_b}$$

and

$$TrH = \sum \frac{1}{R_b}$$

and also

$$Tr(H^2) = \sum \left(\frac{1}{R_b}\right)^2$$

with the summation over the dimensions of the manifold. Thus equation (39) becomes

$$-\frac{\hbar^2}{2m}U = -\frac{\hbar^2}{2m}\left[\frac{1}{2}\sum_{b=1}^{D}\left(\frac{1}{R_b}\right)^2 - \frac{1}{4}\left(\sum_{b=1}^{D}\frac{1}{R_b}\right)^2\right]$$

or

$$-\frac{\hbar^2}{2m}U = -\frac{\hbar^2}{4m}\sum_{b=1}^{D}\frac{1}{(R_b)^2} + \frac{\hbar^2}{8m}\left(\sum_{b=1}^{D}\frac{1}{R_b}\right)^2 . \tag{42a}$$

Using this equation with $n = 3$, $D = 2$

$$\begin{aligned}-\frac{\hbar^2}{2m}U &= -\frac{\hbar}{4m}\left(\frac{1}{R_1{}^2} + \frac{1}{R_2{}^2}\right) + \frac{\hbar}{8m}\left(\frac{1}{R_1{}^2} + \frac{2}{R_1R_2} + \frac{1}{R_2{}^2}\right), \\ &= -\frac{\hbar^2}{8m}\left(\frac{1}{R_1{}^2} - \frac{2}{R_1R_2} + \frac{1}{R_2{}^2}\right)\end{aligned}$$

hence,

$$-\frac{\hbar^2}{2m}U = -\frac{\hbar^2}{8m}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)^2 . \tag{42b}$$

If the manifold M is a sphere then all of the principle curvatures are equal to

$$\frac{1}{R_b} = \frac{1}{R}.$$

So for a sphere of dimension $D = n - 1$ equation (42a) is

$$\begin{aligned}
-\frac{\hbar^2}{2m}U &= -\frac{\hbar^2}{4m}\sum_{b=1}^{D}\frac{1}{R^2} + \frac{\hbar}{8m}\left(\sum_{b=1}^{D}\frac{1}{R}\right)^2 \\
&= -\frac{\hbar}{4m}\frac{D}{R^2} + \frac{\hbar^2}{8m}\left(\frac{D}{R}\right)^2 \\
&= \frac{\hbar^2}{8m}\frac{1}{R^2}(D^2 - 2D) \\
&= \frac{\hbar^2}{8mR^2}(D)(D-2),
\end{aligned}$$

or

$$-\frac{\hbar^2}{2m}U = -\frac{\hbar^2}{8mR^2}(n-1)(n-3). \tag{42c}$$

For $n = 3$, $M = S^2$ the potential is

$$-\frac{\hbar^2}{2m}U = 0. \tag{42d}$$

Table 1. Quantum Potential in Special Cases

(R_b = principal radii of curvature)

D	n	$-\frac{\hbar^2}{2m}U$
1	n	$-\frac{\hbar^2}{8m}\eta^2$
2	3	$-\frac{\hbar^2}{8m}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)^2$
$n-1$	n	$-\frac{\hbar^2}{4m}\sum_{b=1}^{D}\frac{1}{R_b^2}+\frac{\hbar^2}{8m}\left(\sum_{b=1}^{D}\frac{1}{R_b}\right)^2$
S^{n-1}	R^n	$(n-1)(n-3)\frac{\hbar^2}{8mR^2}$
S^2	R^3	0

As in the previous section the Schrödinger equation is separable. Using the ansatz $\Phi = \chi(q^a)\varphi(q^y)$ in the Schrödinger equation (30)

$$-\frac{\hbar^2}{2m}\dot{\Delta}\Phi - \frac{\hbar^2}{2m}U\Phi - \frac{\hbar^2}{2m}\sum_y \frac{\partial^2}{(\partial y)^2}\Phi = E\Phi$$

then

$$-\frac{\hbar^2}{2m}\dot{\Delta}(\chi\varphi) - \frac{\hbar^2}{2m}U\chi\varphi - \frac{\hbar^2}{2m}\sum_y \frac{\partial^2}{(\partial q^y)^2}\chi\varphi = E\chi\varphi$$

and

$$-\frac{\hbar^2}{2m}\left[\dot{g}^{\frac{1}{2}}\frac{\partial}{\partial q^a}\left[g^{ba}\dot{g}^{\frac{1}{2}}\frac{\partial}{\partial q^a}(\chi(q^a)\varphi(q^y))\right] + U\chi\varphi + \sum_y \frac{\partial^2}{(\partial q^y)^2}(\chi(q^a)\varphi(q^y))\right] = E\chi\varphi$$

hence

$$-\frac{\hbar^2}{2m}\left[(\dot{\Delta}\chi)\varphi + U\chi\varphi + \chi\sum_y \frac{\partial^2\varphi}{(\partial q^y)^2}\right] = E\chi\varphi.$$

Dividing this equation by $\chi\varphi$,

$$-\frac{\hbar^2}{2m}\left[\frac{\dot{\Delta}\chi}{\chi} + U + \sum_y \frac{\partial^2\varphi}{(\partial q^y)^2}\frac{1}{\varphi}\right] = E.$$

An examination of the quantum potential U derived in the previous pages shows it to be a function of $q^a; a = 1, \cdots, D$ thus

$$-\frac{\hbar^2}{2m}\left(\sum_y \frac{\partial^2\varphi(q^y)}{(\partial q^y)^2}\right)\frac{1}{\varphi(q^y)} = \frac{\hbar^2}{2m}\frac{\dot{\Delta}\chi(q^a)}{\chi(q^a)} + \frac{\hbar^2}{2m}U(q^a) + E.$$

Since the left-hand side is strictly a function of q^y and the right a function of q^a, which are independent, both sides must be equal to a constant, call it E_2,

$$-\frac{\hbar^2}{2m}\left(\sum_y \frac{\partial^2\varphi}{(\partial q^y)^2}\right)\frac{1}{\varphi} = E_2$$

then

$$-\frac{\hbar}{2m}\sum_y \frac{\partial^2\varphi}{(\partial q^y)^2} = E_2\varphi \qquad (43a)$$

and

$$\frac{\hbar^2}{2m}\frac{\dot{\Delta}\chi}{\chi} + \frac{\hbar^2}{2m}U + E = E_2.$$

Thus

$$-\frac{\hbar^2}{2m}\dot{\Delta}\chi - \frac{\hbar^2}{2m}U\chi = E_1\chi \qquad (43b)$$

for

$$E = E_1 + E_2$$

where U is given in equation (39) or (41) or from Table 1.

SECTION IV

Quantum Potential for a Product Manifold

Inspecting equation (42a) it is found that the quantum potential for a cylinder is equal to the quantum potential for a line added to the quantum potential for a circle. This is an almost trivial case as the potential for a line is zero; however, it may lead one to believe that the potential for a product manifold is simply the sum of the potentials for the separate manifolds. To investigate we start with two manifolds. M and N which have the dimensions of μ and ν, respectively. Let M be imbedded in $\mathbf{R}^m$ and N be embedded in $\mathbf{R}^n$. Let us describe the parameters of the product manifold with the notation q^a with $a = 1$ to μ and Q^A with $A = \mu + 1$ to $\mu + \nu$ and q^x with $x = \mu + \nu + 1$ to $M + \nu$ and Q^X with $X = M + \nu + 1$ to $M + N$. Then the position vector can be written as

$$\boldsymbol{r} = \begin{pmatrix} f(q^a) \\ F(Q^A) \\ q^x \\ Q^X \end{pmatrix}.$$

This immediately leads to tangent vectors of the form

$$\frac{\partial \boldsymbol{a}}{\partial q^a} = \begin{pmatrix} \frac{\partial f}{\partial q^a} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\frac{\partial \boldsymbol{a}}{\partial Q^A} = \begin{pmatrix} 0 \\ \frac{\partial F}{\partial Q^A} \\ 0 \\ 0 \end{pmatrix}$$

and also

$$\frac{\partial^2 \boldsymbol{a}}{\partial q^a \partial q^b} = \begin{pmatrix} \frac{\partial^2 f}{\partial q^a \partial q^b} \\ 0 \\ 0 \\ 0 \end{pmatrix};$$

$$\frac{\partial^2 \boldsymbol{a}}{\partial Q^A \partial Q^B} = \begin{pmatrix} 0 \\ \frac{\partial^2 F}{\partial Q^A \partial Q^B} \\ 0 \\ 0 \end{pmatrix}$$

and

$$\frac{\partial^2 \boldsymbol{a}}{\partial q^a \partial Q^B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

As we define the normal vectors, all but $\nu+\mu$ of them can be in the dimensions indiced by $i, j > \nu+\mu$, and thus they will not contribute to the quantum potential. The remainimg $\nu+\mu$ normal vectors can be defined such that

$$\underset{a}{\boldsymbol{n}} = \begin{pmatrix} \alpha_a \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad a = 1, ..., \mu$$

where $\alpha_a = \alpha_a(q^b)$ with $b = 1, ..., \mu$ and $\underset{a}{\boldsymbol{n}}$ are the normal vectors to the M manifold, and

$$\underset{C}{\boldsymbol{n}} = \begin{pmatrix} 0 \\ \beta_C \\ 0 \\ 0 \end{pmatrix} \qquad C = \mu+1, ..., \mu+\nu$$

where $\beta_C = \beta_D(Q^D)$ with $D = \mu+1, ..., \mu+\nu$ and $\underset{C}{\boldsymbol{n}}$ are the normal vectors to the N manifold. Hence we will have

$$\underset{i}{\boldsymbol{n}} \cdot \underset{j}{\boldsymbol{n}} = 0 \qquad \underset{i}{\boldsymbol{n}} \cdot \underset{i}{\boldsymbol{n}} = 1.$$

Thus our second fundamental form is given as

$$\underset{x}{H}= \begin{pmatrix} \frac{\partial^2 \boldsymbol{a}}{\partial q^a \partial q^b} \cdot \underset{y}{\boldsymbol{n}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad x = 1, ..., \mu;$$

$$\underset{Y}{H}= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\partial^2 \boldsymbol{a}}{\partial Q^A \partial Q^B} \cdot \underset{Y}{\boldsymbol{n}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad Y = \mu + 1, ..., \mu + \nu$$

and

$$\underset{z}{H}= [0] \qquad z = \mu + \nu + 1, ..., M + N.$$

It is clear now that $Tr(\underset{x}{H}) = Tr(\underset{x\ \mu}{H})$ and $Tr(\underset{x}{H}^2) = Tr(\underset{x\ \mu}{H}^2)$ where $\underset{x\ \mu}{H}$ is the second fundamental form for the manifold M . Likewise $Tr(\underset{Y}{H}) = Tr(\underset{Y\ \nu}{H})$ and $Tr(\underset{Y}{H}^2) = Tr(\underset{Y\ \nu}{H}^2)$ where $\underset{Y\ n}{H}$ is the second fundamental form for the manifold N. Now using equation (39) as our description of the quantum potential

$$-\frac{\hbar^2}{2m}U = -\frac{\hbar^2}{2m} \sum_y \left[\frac{1}{2} Tr \left(\underset{y}{H}^2 \right) - \frac{1}{4} \left(Tr \underset{y}{H} \right)^2 \right]$$

it is clear that

$$U_{M \times N} = U_M + U_N.$$

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VITA

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