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The fuzzy linear programming problem

Kan Araki

To the Graduate Council:

I am submitting herewith a thesis written by Kan Araki entitled "The fuzzy linear programming problem." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

Yueh-er Kuo, Major Professor

We have read this thesis and recommend its acceptance:

Steve Serbin, Tadeusz Janik

Accepted for the Council:
Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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Date December, 1991

THE FUZZY LINEAR PROGRAMMING PROBLEM

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ABSTRACT

The purpose of this paper is to investigate the theory and methods of Fuzzy Linear Programming problems. The stress of the presentation is placed on the transformation methods from fuzzy models to crisp models, and on the sensitivity analysis on fuzzy parameters. Moreover, we extend sensitivity analysis in general cases and we show a practical method to solve the problems with nonlinear membership functions. In addition to these researches, we propose the new concept in order to relate the method of Fuzzy Linear Programming with conventional methods.

The introduction discusses originality of this thesis and proposes several items for future study.

Chapter 1 discusses the basic theory of Fuzzy Sets and Fuzzy Linear Programming problems and their transformation methods to crisp models.

Chapter 2 presents sensitivity analysis on the two types of Fuzzy Linear Programming problems as well as stability study on fuzzy parameters.

Chapter 3 presents solving the problems with nonlinear membership functions by using piecewise linear functions.

Chapter 4 discusses relationship between a fuzzy programming method and a goal programming method.

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INTRODUCTION

Many systems in the real world have some complicated requirements to solve the problems. Goals and constraints might be competitive with each other, therefore, we have to deal with multiple objective linear programming problems. We have already known some methods such as weighted-sum approach methods which can be found in [12] and goal programming methods in [13]. However, it is hard for us to determine reasonable weighting coefficients or target values prior to executing computation. Moreover, we can't necessarily obtain all the exact data to make mathematical models. Therefore, in order to solve these ambiguous models, we have to consider the problems with flexible goals and constraints which are called Fuzzy Linear Programming problems.

The theory on decision-making in a fuzzy environment was established by R.E.Bellman and L.A.Zadeh in 1970, which is written in [1]. On the basis of their theory, some linear programming methods were introduced by C.V.Negoita and H.J.Zimmerman successively in 1975 and in 1978, which are written in [10] and [18]. Moreover, the method of the sensitivity analysis of Fuzzy Linear Programming problems was found by H.Hamacher, H.Leberling and H.J.Zimmermann in 1978, which is written in [5]. Some current research on Fuzzy Linear Programming problems are on sensitivity analysis in the case of the problems whose system coefficients are including some fuzzy numbers, which can be found in [15]. Other research is on the problems with nonlinear membership functions.

In this thesis, we investigate the important theorems and elaborate these proofs and show the new interpretation on the difference between Type 1 problems and Type 2 problems. In addition to these studies, we execute some examples to clarify the procedure of methods on different types of Fuzzy Linear Programming problems in Chapter 1 and Chapter 2. We also extend the sensitivity analysis in the case that we have more than two perturbed parameters at the same time and introduce the new concept of the standardized sensitivity analysis with the Information Value in Chapter 2. In Chapter 3, we establish a practical method to solve the problem with a nonlinear membership function under some conditions. This method is available to computation, since the problem can be expressed as a simple problem by using the vector of choice without any concave-convex transformation even if it is not a convex problem. Moreover, we investigate the relationship between Fuzzy Linear Programming and Goal Programming and we propose the new problem in order to relate both methods in Chapter 4.

Through this original research, we can propose some items for future study as follows:

(i) We have to consider how to extend the problem in the case that we need the solution which is expressed by fuzzy numbers, which we call Type 3 problem.

(ii) We have to investigate whether we can guarantee the stability on parameter's deviations or not, if a membership function is nonlinear.

(iii) We would like to establish the simple method to compute standardized sensitivity analysis and prove its convergence, if we make the number of grid points on the Information Value infinite.

(iv) We have to extend the approximation method for the case of more than two fuzzy goals and constraints.

(v) We have to study Mixed Programming problem in order to investigate the relationship between Fuzzy Linear Programming and Goal Programming .

CHAPTER 1

FUZZY METHOD OF MULTIPLE OBJECTIVE LINEAR PROGRAMMING

1.1 INTRODUCTION TO FUZZY SETS

In Boolean Algebra, we can express union or intersection of sets as the maximum or the minimum of characteristic functions which takes only the values 1 or 0. In Fuzzy Set Theory, a fuzzy set is a class of objects (points) in which there is no sharp boundary between those objects that belong to the class and those that do not. Characteristic functions of fuzzy sets are called membership functions and can take any values on [0,1]. A more precise definition is as follows. The basic definitions of this section can be found in [1].

Definition 1.1 Let $X=\{x\}$ denote a collection of objects (points) on Euclidian space $\mathbf{R}^n$. A *fuzzy set* A in X is a set of ordered pairs.

$$A = \{(x, \mu_A(x))\} \quad , \; x \in X ,$$

where $\mu_A(x)$ is termed the grade of the membership of x in A , and $\mu_A:X \rightarrow M$ is a function from X to a space $M=[0,1]$ called *membership space.*

We turn next to the definitions of some basic concepts which we shall need latter.

Definition 1.2(a) A fuzzy set A is *normal* if $\sup_x \mu_A(x) = 1$, $x \in X$.

Definition 1.2(b) Two fuzzy sets are *equal*, written as A = B, if and only if $\mu_A(x) = \mu_B(x)$, $x \in X$.

Definition 1.2(c) A' is said to be the *complement* of A if and only if $\mu_{A'}(x) = 1 - \mu_A(x)$, $x \in X$.

Definition 1.2(d) The *intersection* $A \cap B$ of A and B is defined as the fuzzy set whose membership function satisfies $\mu_{A \cap B}(x) = \text{Min}\,(\mu_A(x), \mu_B(x))$, $x \in X$.

Definition 1.2(e) The *union* $A \cup B$ of A and B is defined as the fuzzy set whose membership function satisfies $\mu_A(x) = \text{Max}\,(\mu_A(x), \mu_B(x))$, $x \in X$.

From the above definitions, the union of fuzzy sets is not necessarily complemented in the sense that the maximum on membership functions must be 1 in Boolean Algebra. This means that we can not determine whether an element belongs to the set or not.

We sometimes use the notation of $\wedge$ and $\vee$ instead of Min and Max . For any real numbers a and b, $a \vee b \equiv \text{Max}\,\{a , b\}$ and $a \wedge b \equiv \text{Min}\,\{a , b\}$.

1.2 FUZZY GOALS, CONSTRAINTS AND A DECISION

A Decision-making in a fuzzy environment means a decision process in which the goals and/or constraints are fuzzy in nature, namely, these constitute some classes of alternatives whose boundaries are not sharply defined. The following definition suggests that a fuzzy decision is defined as the fuzzy set of alternatives resulting from the intersection of all the goals and constraints. In this section, we prove several theorems in order to obtain the most important theorem which guarantees existence of the optimal solutions of Fuzzy Linear Programming problems. The main content can be found in [1] and [10].

Definition 1.3(a) Assume that we are given a fuzzy goal Z and a fuzzy constraint R, both of which have membership functions $\mu_Z(x)$ and $\mu_R(x)$ in a space of X. Then a *decision* D is defined as a fuzzy set resulting from intersection of Z and R:

$$D = Z \cap R,$$
$$\mu_D(x) = \mu_{Z \cap R}(x).$$

The following definition can be extended to a decision-making which has many goals and constraints.

Definition 1.3(b) A *fuzzy decision* D is defined as the fuzzy set of all those elements which belong to fuzzy sets Z_l , $l=1,\cdots,k$ describing goals and to fuzzy sets R_i , $i=1,\cdots,m$ describing constraints:

$$D=\bigcap_{l=1}^{k} Z_l \cap \bigcap_{i=1}^{m} R_i .$$

Choosing the optimal element means electing the element x_0 which has the highest degree of the membership to the fuzzy set decision as follows:

$$\begin{aligned}\alpha_0 &\equiv \mu_D(x_0)\\ &= \operatorname*{Max}_{x\in X} \mu_D(x)\\ &= \operatorname*{Max}_{x\in X} \operatorname{Min}\left\{ \operatorname*{Min}_{l=1,\cdots,k} \mu_{Z_l}(x) , \operatorname*{Min}_{i=1,\cdots,m} \mu_{R_i}(x) \right\} .\end{aligned}$$

If each $\mu(x)$ has a lower or a upper bound,we can also denote this as follows:

$$\begin{aligned}\alpha_0 &\equiv \mu_D(x_0)\\ &= \sup_{x\in X} \mu_D(x)\\ &= \sup_{x\in X} \inf\left\{ \inf_{l=1,\cdots,k} \mu_{Z_l}(x) , \inf_{i=1,\cdots,m} \mu_{R_i}(x) \right\} .\end{aligned}$$

Next, we will show the existence of x_0 and α_0. Before we prove this existence, we have to introduce some definitions and to prove some propositions and their corresponding results.

Definition 1.4(a) Let μ_R be a membership function of a fuzzy constraint. $\mathbf{C}_\alpha \equiv \{ x\in X : \mu_R(x) \geq \alpha \}$ is called the α - *cut* of μ_R, where $0 \leq \alpha \leq 1$.

Definition 1.4(b) χ_α is called the *characteristic function* of $\mathbf{C}_\alpha$:

$$\chi_\alpha = \begin{cases} 1, & \text{if } x \in C_\alpha \\ 0, & \text{if } x \notin C_\alpha \end{cases}.$$

Theorem 1.1 The following relation is valid:

$$\sup_{x \in X} \mu_D(x) = \sup_{\alpha \in [0,1]} \left[\alpha \wedge \sup_{x \in C_\alpha} \mu_Z(x) \right].$$

where C_α is the α - cut of μ_R.

Proof: We shall use the following method of decomposition of a fuzzy subset. (See [10]) In this method, we can express every convex fuzzy set A as follows:

$$A = \bigcup_{\alpha \in [0,1]} \alpha \cdot C_\alpha,$$

where $\alpha \cdot C_\alpha$ means that this set has the membership function of $\alpha \cdot \chi_\alpha$.

The membership function can be written by the following method:

$$\mu_R = \bigvee_{\alpha \in [0,1]} (\alpha \cdot \chi_\alpha)$$

$$= \bigvee_{\alpha \in [0,1]} (\alpha \wedge \chi_\alpha),$$

where χ_α is the characteristic function of C_α.

We have

$$\mu_D(x) = \mu_Z(x) \wedge \mu_R(x)$$

$$= \left[\bigvee_\alpha (\alpha \wedge \chi_\alpha(x)) \right] \wedge \mu_Z(x)$$

$$= \bigvee_\alpha \left[\alpha \wedge \chi_\alpha(x) \wedge \mu_Z(x) \right],$$

but

$$\bigvee_{x\in X}\left[\chi_\alpha(x)\wedge\mu_Z(x)\right] = \bigvee_{x\in C_\alpha}\left[\chi_\alpha(x)\wedge\mu_Z(x)\right]\vee\bigvee_{x\notin C_\alpha}\left[\chi_\alpha(x)\wedge\mu_Z(x)\right]$$

$$= \bigvee_{x\in C_\alpha}\mu_Z(x)$$

$$= \sup_{x\in C_\alpha}\mu_Z(x)\,.$$

We may write

$$\sup_{x\in X}\mu_D(x) = \sup_{\alpha\in[0,1]}\left[\alpha\wedge\sup_{x\in C_\alpha}\mu_Z(x)\right]. \qquad \text{Q.E.D.}$$

Theorem 1.2 The following relation is valid:

$$\sup_{x\in X}\mu_D(x) = \sup_{\alpha_1,\cdots\alpha_p}\left[\alpha_1\wedge\cdots\wedge\alpha_p\wedge\sup_{x\in C^1_{\alpha_1}\cap\ldots\cap C^p_{\alpha_p}}\mu_Z(x)\right],$$

where $C^j_{\alpha_j}$ is the α - cut of μ_{R_j}.

Proof: We have

$$\mu_{C^j}(x) = \bigvee_{\alpha\in[0,1]}\left[\alpha\wedge\chi_{C^j_\alpha}(x)\right]\ ,\ 1\le j\le p\ ,$$

and thus

$$\mu_D(x) = \bigvee_{\alpha}\left[\alpha\wedge\chi_{C^1_\alpha}(x)\right]\wedge\cdots\wedge\left[\alpha\wedge\chi_{C^p_\alpha}(x)\right]\wedge\mu_Z(x)$$

$$= \bigvee_{\alpha_1,\cdots,\alpha_p}\left[\alpha_1\wedge\cdots\wedge\alpha_p\wedge\chi_{C^1_{\alpha_1}}(x)\wedge\cdots\wedge\chi_{C^p_{\alpha_p}}(x)\wedge\mu_Z(x)\right].$$

Then, we take the supremum of $\mu_D(x)$ as follows:

$$\sup_{x\in X}\mu_D(x) = \bigvee_{x\in X}\ \bigvee_{\alpha_1,\cdots,\alpha_p}\left[\alpha_1\wedge\cdots\wedge\alpha_p\wedge\chi_{C^1_{\alpha_1}}(x)\wedge\cdots\wedge\chi_{C^p_{\alpha_p}}(x)\wedge\mu_Z(x)\right].$$

Since

$$\bigvee_{x\in X}\left\{\left(\chi_{C^1_{\alpha_1}}(x)\wedge\cdots\wedge\chi_{C^p_{\alpha_p}}(x)\right)\wedge\mu_Z(x)\right\}$$

$$=\left[\bigvee_{x\in C^1_\alpha\cap\cdots\cap C^p_\alpha}\left\{\chi_{C^1_{\alpha_1}}(x)\wedge\cdots\wedge\chi_{C^p_{\alpha_p}}(x)\wedge\mu_Z(x)\right\}\right]\vee\left[\bigvee_{x\notin C^1_\alpha\cap\cdots\cap C^p_\alpha}\left\{\chi_{C^1_{\alpha_1}}(x)\wedge\cdots\wedge\chi_{C^p_{\alpha_p}}(x)\wedge\mu_Z(x)\right\}\right]$$

$$=\bigvee_{x\in C^1_\alpha\cap\cdots\cap C^p_\alpha}\mu_Z(x)$$

$$=\sup_{x\in C^1_\alpha\cap\cdots\cap C^p_\alpha}\mu_Z(x)\ .$$

Hence,

$$\sup_{x\in X}\mu_D(x) = \sup_{\alpha_1,\cdots,\alpha_p}\left[\alpha_1\wedge\cdots\wedge\alpha_p\wedge\sup_{x\in C^1_{\alpha_1}\cap\cdots\cap C^p_{\alpha_p}}\mu_Z(x)\right]. \qquad \text{Q.E.D.}$$

Next, we show that we can replace the problem of maximizing a decision by an extremum problem of a scalar function by introducing the following functions:

$$\varphi(\alpha)\equiv\sup_{x\in C_\alpha}\mu_Z(x)\,,$$

$$\psi(\alpha)\equiv\alpha\wedge\varphi(\alpha)\,,$$

where $\varphi:[0,1]\to[0,1]$ and $\psi:[0,1]\to[0,1]$.

Theorem 1.3 If $\alpha\le\beta$, then $\varphi(\alpha)\ge\varphi(\beta)$.

proof: From the definition of φ, $\varphi(\alpha)=\sup_{x\in C_\alpha}\mu_Z(x)$ and $\varphi(\beta)=\sup_{x\in C_\beta}\mu_Z(x)$.

Since

$$C_\beta=\{\ x\in X:\mu_R(x)\ge\beta\ \}$$

$$= \{ x \in X : \mu_R(x) \geq \beta \geq \alpha \}$$
$$\subseteq \{ x \in X : \mu_R(x) \geq \alpha \}$$
$$= \mathbf{C}_\alpha .$$

Therefore,

$$\varphi(\beta) = \sup_{x \in C_\beta} \mu_Z(x) \leq \sup_{x \in C_\alpha} \mu_Z(x) = \varphi(\alpha).$$

Q.E.D

Theorem 1.4 - If φ is continuous on [0,1], then φ has the fixed point which satisfies $\varphi(\overline{\alpha}) = \overline{\alpha}$.

Proof: Since φ is continuous and [0,1] is a compact space, there exists a sequence $\{\alpha_n\}$ which converges to $\overline{\alpha}$. If $\alpha_n \to \overline{\alpha}$, then $\varphi(\alpha_n) \to \varphi(\overline{\alpha})$ for $\forall \overline{\alpha} \in [0,1]$. Moreover, from Theorem 1.3., $\varphi(\alpha)$ is a monotonically decreasing function. Hence the solution $\overline{\alpha}$ is unique.

Q.E.D.

Theorem 1.5 If there exists $\overline{\alpha}$ which satisfies $\sup_{x \in c_{\overline{\alpha}}} \mu_Z(x) = \overline{\alpha}$, then $\sup_{x \in X} \mu_D(x) = \overline{\alpha}$.

Proof: Since $\sup_{x \in X} \chi_D(x) = \sup_{\alpha \in [0,1]} \psi(\alpha)$ by Theorem 1.1., we only have to show $\sup_{\alpha \in [0,1]} \psi(\alpha) = \overline{\alpha}$. Since $\psi(\overline{\alpha}) = \overline{\alpha} \wedge \varphi(\overline{\alpha}) = \overline{\alpha}$, we show that $\psi(\alpha) \leq \psi(\overline{\alpha})$ for $\forall \alpha \in [0,1]$.

If $\alpha > \overline{\alpha}$, then $\varphi(\alpha) \leq \varphi(\overline{\alpha}) = \overline{\alpha} < \alpha$ by Theorem 1.3. If $\alpha < \overline{\alpha}$, then $\varphi(\alpha) \geq \varphi(\overline{\alpha}) = \overline{\alpha} > \alpha$.

Therefore,

$$\psi(\alpha) = \alpha < \overline{\alpha} = \psi(\overline{\alpha})\,.$$

Hence $\psi(\overline{\alpha})$ is the supremum of $\psi(\alpha)$. Q.E.D.

Theorem 1.6. If φ is a continuous function on $[0,1]$, then Fuzzy Linear Programming has a unique solution.

Proof: From Theorem 1.5., there exists a unique solution which satisfies

$$\sup_{x \in X} \mu(x) = \overline{\alpha}\,.$$

Q.E.D.

1.3 FUZZY LINEAR PROGRAMMING (TYPE 1)

In this type of Fuzzy Linear Programming, we introduce some assumptions as follows. The basic results can be found in [5] and [17].

(A) There exist fuzzy constraints and crisp (non fuzzy) constraints and a fuzzy objective function.

(B) The membership functions of the fuzzy sets representing a fuzzy objective function and fuzzy constraints are linear.

(C) All fuzzy sets are normalized conveniently.

We consider the partially Fuzzy Linear Programming problem (We call this as F.L.P. problem).

F.L.P. problem : (Type 1)

$$\widetilde{\text{Max}}\ \mathbf{C}^t\mathbf{X}$$
$$\text{subject to } \mathbf{AX} \precsim \mathbf{b} \text{ or } \mathbf{AX} \succsim \mathbf{b}, \qquad (1.1)$$
$$\mathbf{DX} \le \mathbf{e}\,,$$
$$\mathbf{X} \ge \mathbf{0},$$

where the notation, $\widetilde{\text{Max}}\ \mathbf{C}^t\mathbf{X}$, is an objective function with upper bound and lower bound of fuzzy goals, $\mathbf{X} \in \mathbf{R}^n$, $\mathbf{C}$ is a 1 by n matrix, $\mathbf{A}$ is a m_1 by n matrix, $\mathbf{D}$ is a m_2 by n matrix, $\mathbf{b}$ is a m_1 by 1 vector, $\mathbf{e}$ is a m_2 by 1 vector and $\mathbf{AX} \precsim \mathbf{b}$ means ambiguous or flexible goals and constraints such that $\mathbf{AX}$ is nearly or roughly less than $\mathbf{b}$.

In order to solve this problem, we rewrite the above problem as follows:

$$\text{F.L.P. problem :} \quad \widetilde{\text{Max}}\ c_i^t\mathbf{X}, \quad i = 1, \cdots, l,$$
$$\text{s.t. } a_j^t\mathbf{X} \tilde{<} b_j, \quad j = 1, \cdots, m_1, \tag{1.2}$$
$$d_j^t\mathbf{X} \le e_j, \quad j = m_1+1, \cdots, m_1+m_2,$$
$$\mathbf{X} \ge \mathbf{0}.$$

We make a decision in such a way:

(1) The value of each objective function $c_i^t X$ has predetermined target of c_l^u and c_l^l.

(2) Each restriction $a_j^t\mathbf{X} \tilde{<} b_j$ is satisfied as well as possible.

(3) Each restriction $d_j^t\mathbf{X} \le e_j$ is strictly satisfied.

(4) For the first m_1 restrictions, the decision maker is prepared to tolerate violation, $t_j > 0$, up to $p_j > 0$ such that $a_j^t\mathbf{X} \le b_j + t_j$, $t_j \le p_j$, $j = 1, \cdots, m_1$.

First of all, we make the decision for the goal of $\mathbf{C}$; namely,

$$\alpha' = \text{Min}\{\mu_{z_1}(x), \cdots, \mu_{z_l}(x)\}$$

where,

$$\mu_{z_i}(x) = \begin{cases} 0 & , \text{if } \sum_j c_{i_j}x_j < c_i^l \\ \dfrac{\sum_j c_{i_j}x_j - c_i^l}{c_i^u - c_i^l} & , \text{if } c_i^l < \sum_j c_{i_j}x_j < c_i^u \\ 1 & , \text{if } c_i^u \le \sum_j c_{i_j}x_j \end{cases},$$

and where, c_i^u is the i-th individual objective function value which satisfies the goal with 100%. c_i^l is the i-th lowest value which is accepted for this goal. Hence,

$$\alpha' \le \mu_{z_i}(x) = \frac{\sum_j c_{ij}x_j - c_i^l}{c_i^u - c_i^l},$$

$$\alpha'(c_i^u - c_i^l) - \sum_j c_{ij}x_j \le -c_i^l, \quad i = 1, \cdots, l.$$

Second of all, we make the decision for constraints of $a_j^t\mathbf{x} \lesssim b_j$ as follows:

$$\alpha'' = \mathrm{Min}\{\mu_{R_1}(x), \cdots, \mu_{R_l}(x)\},$$

where

$$\mu_{R_i}(x) = \begin{cases} 0 & , \text{ if } \sum_j a_{ij}x_j > b_i^u \\ \dfrac{b_i^u - \sum_j a_{ij}x_j}{b_i^u - b_i^l} & , \text{ if } b_i^u > \sum_j a_{ij}x_j \ge b_i^l \\ 1 & , \text{ if } b_i^l \ge \sum_j a_{ij}x_j \end{cases}.$$

Hence,

$$\alpha'' \le \mu_{R_i}(x) = \frac{b_i^u - \sum_j a_{ij}x_j}{b_i^u - b_i^l}$$

$$\alpha''(b_i^u - b_i^l) + \sum_j a_{ij}x_j \le b_i^u, \quad i = 1, \cdots, m_1,$$

where b_i^l satisfies the i-th constraint with 100 %, b_i^u satisfies it with 0 %. All we have to do is to make the final decision by using Definition 1.3.

$$\alpha = \mu_d(x_0)$$

$$= \underset{x \in X}{\text{Max}}\ \mu_d(x)$$

$$= \underset{x \in X}{\text{Max}} \left\{ \text{Min} \left\{ \alpha' , \alpha'' \right\} \right\}$$

The equivalent crisp model for Fuzzy Linear Programming (1.2) can be expressed as follows. We call the Equivalent Linear Programming problem as E.L.P. problem.

E.L.P.problem : Max α

$$\text{s.t.} \ \left(c_i^u - c_i^l\right) \alpha \ - \ \sum_j c_{ij}^t x_j \ \leq \ - c_i^l \ , \ i = 1, \cdots ,l \ , \ j = 1, \cdots , n \ ,$$

$$\left(b_i^u - b_i^l\right) \alpha + \sum_j a_{ij} x_j \ \leq \ b_i^u \ , \ i = 1, \cdots ,m_1, \ j = 1, \cdots , n \ , \qquad (1.3)$$

$$d_j^t \leq e_j \ , \ j = 1, \cdots , m_2,$$

$$\mathbf{X} \geq \mathbf{0} .$$

This model is linear; therefore, we can solve this by The Simplex Method.(See [4]) We show it by the following example.

Example 1.1 Consider the following problem.

$$\widetilde{\text{Max}} \ \mathbf{C}^t\mathbf{X}$$

$$\text{s.t.} \ \mathbf{AX} \lessapprox \mathbf{b},$$

$$\mathbf{DX} \leq \mathbf{e},$$

$$\mathbf{X} \geq \mathbf{0},$$

where

$$\mathbf{C} = [\,1 \ \ 2 \ \ 0\,] \ , \ \mathbf{X} = [x_1 \ x_2 \ x_3\,]^t \ , \ \mathbf{c}^u = [16.5] \ , \ \mathbf{c}^l = [11.5]$$

$$\mathbf{A} = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & 2 \end{bmatrix}$$

$$\mathbf{b}^u = [\ 23 \quad 17 \quad 21\ ]^t \ , \ \mathbf{b}^l = [\ 17 \quad 9 \quad 16\ ]^t \ , \ \mathbf{D} = [\,2 \ \ 7 \ \ 1\,]$$

$\mathbf{e} = [35]$.

An equivalent model to this problem is written as follows:

E.L.P. problem :

$$
\begin{aligned}
\text{Max } & \alpha \\
\text{s.t. } & 5\alpha - (x_1 + 2x_2) \leq -11.5 , \\
& 6\alpha + (-x_1 + 3x_2 + 2x_3) \leq 23 , \\
& 8\alpha + (x_1 + 2x_2 - 3x_3) \leq 17 , \\
& 5\alpha + (2x_1 + x_2 + 2x_3) \leq 21 , \\
& 2x_1 + 7x_2 + x_3 \leq 35 .
\end{aligned}
$$

We can solve the above linear programming directly by LINDO (Linear, Interactive, Discrete, Optimizer). (See [8]) The Optimal solution is that

$$
\begin{aligned}
\alpha^o &= 0.600 , \\
\mathbf{X}^o &= [x_1^o \, x_2^o \, x_3^o]^t = [\, 10.5 \quad 2 \quad 0 \,]^t .
\end{aligned}
$$

From the result, we can satisfy the goal with 60.0%.

1.4 FUZZY LINEAR PROGRAMMING (TYPE 2)

The concept of the goal which has upper bound and lower bound of target values may be more useful than that of maximizing an objective function, because many constraints and goals lead to a criterion. First of all, we consider, in this section, constraints and goals which are given by sets. Constraints and goals are just identical concepts in the sense that a decision must be given so as to satisfy both sets. Second of all, we use Zadeh's principle which is equivalent to Theorem 1.7 in order to deal with fuzzy linear functions. Lastly we have to define the inequality on fuzzy numbers to lead the formula by using (1.3a). Without any distinction of constraints and goals, we can write a linear programming problem in the following form. The basic concept can be found in [14] and [15].

$$\mathbf{AX} \lesssim \mathbf{b} \text{ or } \mathbf{AX} \gtrsim \mathbf{b} \quad , \text{ for goals and constraints.}$$
$$(\mathbf{a}_i\mathbf{X} \lesssim b_i \text{ or } \mathbf{a}_i\mathbf{X} \gtrsim b_i , \quad i = 1 , \cdots , m).$$

We consider a more general case of the Fuzzy Linear Programming problem with the goal concept. We can describe a problem by expressing as follows:

$$\mathbf{AX} \gtrsim \mathbf{b}$$
$$(\mathbf{a}_i\mathbf{X} \gtrsim b_i \quad , \; i = 1 , \cdots , m) ,$$

which means each restriction $\mathbf{a}_i\mathbf{X} \gtrsim b_i$ is satisfied as well as possible. We suppose a decision maker is prepared to tolerate violation, s_i , up to $d_i > 0$. We can rewrite the above problem as "classical" form which doesn't mean the goal concept but an objective function.

Theorem 1.6. The Fuzzy Linear Programming problem with the goal concept

F.L.P. problem : $\mathbf{AX} \gtrsim \mathbf{b}$,
$$(\mathbf{a}_i\mathbf{X} \gtrsim b_i \quad , \; i = 1, \cdots ,m),$$

is equivalent to the following Equivalent Linear Programming problem.

E.L.P. problem : Max λ

$$\text{s.t. } \mathbf{a}_i\mathbf{X} + s_i \geq b_i , \quad i = 1, \cdots ,m , \tag{1.3b}$$
$$d_i\lambda + s_i \leq d_i .$$

Proof: Let λ be the value of the membership function of min{ $\mu_1(x)$,$\cdots$, $\mu_m(x)$ } where $\mu_i(x)$ is a membership function of a constraint or a goal. Assume that a decision maker is prepared to tolerate violation to restriction for both constraints and goals, s_i , up to $d_i = b_i^u - b_i^l$, which is shown in Figure 1.1.

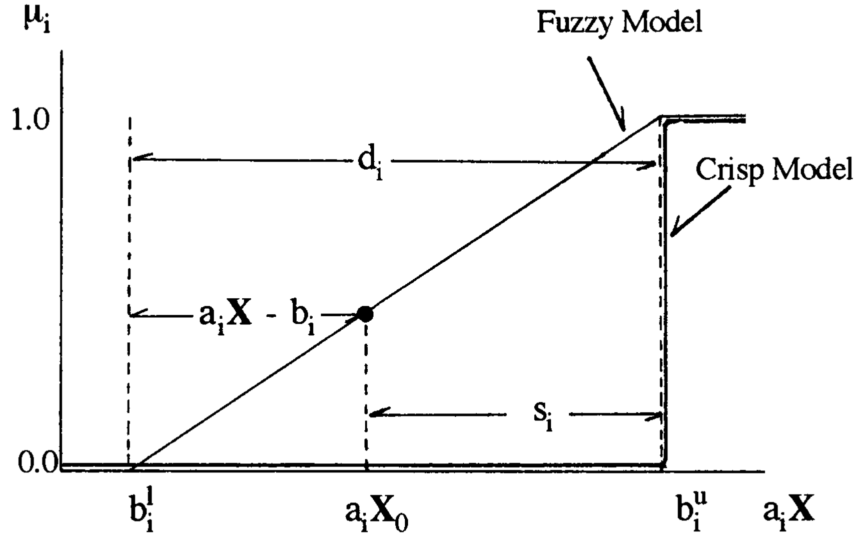

Figure 1.1 A Membership Function of a Fuzzy Goal or a Constraint

We are given a linear function as a membership function.

$$\mu_i(x) = \begin{Bmatrix} 0 & , \text{ if } \sum_j b_i^l > a_{i_j} x_j \\ \dfrac{\sum_j a_{i_j} x_j - b_i^l}{b_i^u - b_i^l} & , \text{ if } b_i^u > \sum_j a_{i_j} x_j \geq b_i^l \\ 1 & , \text{ if } \sum_j a_{i_j} x_j \geq b_i^u \end{Bmatrix} .$$

From Definition 1.3 (b),

$$\lambda \leq \mu_i(x) = \frac{\sum_j a_{i_j} x_j - b_i^u}{b_i^u - b_i^l} ,$$

$$\lambda\left(b_i^u - b_i^l\right) \leq \sum_j a_{ij}x_j - b_i^l ,$$

$$d_i\lambda \leq \sum_j a_{ij}x_j - b_i^l = d_i - s_i .$$

Hence,

$$d_i\lambda + s_i \leq d_i , \quad i = 1,\cdots,m .$$

Since s_i is the gap toward 100% satisfaction, the first formula of $\mathbf{a}_i\mathbf{X} + s_i \geq b_i$ is obviously understood. Q.E.D.

The formula of (1.3a) and the formula of (1.3b) are equivalent to each other, if we take the same direction of inequalities on constraints or goals. (1.3b) is more available than (1.3a), because we have to develop the forms of all the membership functions according to Definition 1.3(b), namely, we have to deal with membership functions directly in (1.3a). On the other hand, we may only deal with the parameters of upper and lower target levels in (1.3b). We can solve the problem of Example 1.1 by (1.3b). Since d_i is the gap between upper and lower bound of a goal or a constraint, we can rewrite the form of Example 1.1 simply as follows:

E.L.P. problem : Max λ

$$\begin{aligned}
\text{s.t. } & 5\lambda + s_0 \leq 5, \\
& 6\lambda + s_1 \leq 6, \\
& 8\lambda + s_2 \leq 8, \\
& 5\lambda + s_3 \leq 5, \\
& -x_1 - 2x_2 - s_0 \leq -16.5, \\
& -x_1 + 3x_2 + 2x_3 - s_1 \leq 23, \\
& x_1 + 2x_2 - 3x_3 - s_2 \leq 17, \\
& 2x_1 + x_2 + 2x_3 - s_3 \leq 21, \\
& 2x_1 + 7x_2 + x_3 \leq 35.
\end{aligned}$$

We can solve the above linear programming problem by LINDO. The optimal solution is

$$\alpha^o = 0.600,$$
$$\mathbf{X}^o = [x_1^o\ x_2^o\ x_3^o]^t = [\,10.5 \quad 2 \quad 0\,]^t,$$
$$\mathbf{S}^o = [s_0\ s_1\ s_2\ s_3]^t = [\,2\ 0\ 0\ 2\,]^t.$$

The most interesting discussion in this section is generalizing the above problem; namely, we consider the following Fuzzy Linear Programming which is called Type 2.

F.L.P. problem : $\tilde{\mathbf{A}}\mathbf{X} \tilde{\leq} \mathbf{b}$

$$(\tilde{\mathbf{a}}_i\mathbf{X} \tilde{\leq} b_i \ , \quad i = 1, \cdots, m)$$

where each coefficient $\widetilde{a_{ij}}$ is the *fuzzy number* which is defined as follows.

Definition 1.6 (a) A fuzzy set $\tilde{A}$ which satisfies the following conditions is called a *fuzzy number.*

(1) $\max_{x \in X} \mu_{\tilde{A}}(x) = 1$, where $\mu_{\tilde{A}}$ is the membership function of $\tilde{A}$.

(2) $\mu_{\tilde{A}}(x) \geq \mu_{\tilde{A}}(a) \wedge \mu_{\tilde{A}}(b)$, $\forall x \in [a\ b]$

(3) $\mu_{\tilde{A}}$ is a piecewise continuous function on X.

The membership function of a fuzzy number $\widetilde{a_{ij}}$ can be regarded as a probability distribution of a_{ij}. We assume the membership function of a fuzzy number $\widetilde{a_{ij}}$ is a triangular membership function for simplicity which is shown in Figure 1.2. On this Figure, α_{ij} is the center value of the possible distribution of a_{ij} and the width w_{ij} is regarded as fuzziness of $\widetilde{a_{ij}}$.

Before we apply the above problem to fuzzy coefficients, we have to introduce some definitions concerning elementary operations called *extension principles.*

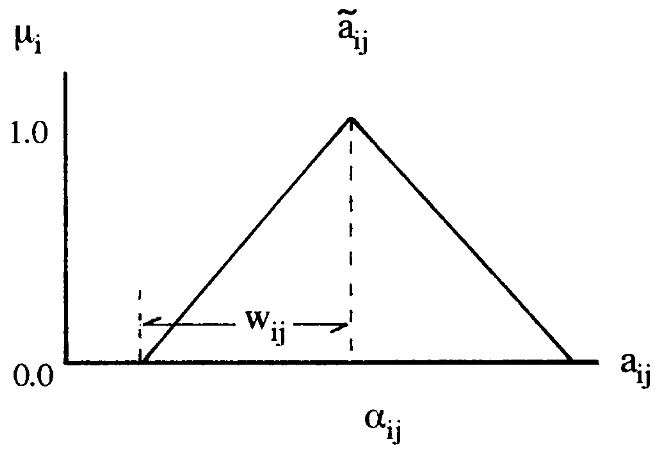

Figure 1.2 A Membership Function of a Fuzzy Number $\tilde{a}_{ij}$ with a Triangular Function

Definition 1.6 (b) We define the *integral form* for a fuzzy set to express the relationship between x and $\mu_{\tilde{A}}(x)$.

$$\Lambda = \int_X \mu_{\tilde{A}}(x) / x .$$

For example, if we consider the following relationship,

x	1	2	3	4	5
$\mu_{\tilde{A}}(x)$	0.8	0.8	0.9	1.0	0.9 ,

then Λ can be written as follows:

$$\Lambda = 0.8 / 1 + 0.8 / 2 + 0.9 / 3 + 1.0 / 4 + 0.9 / 5.$$

If x is in a continuous case, Λ can be expressed as the above integral form.

Definition 1.7 (a) If f is a function $f : X \rightarrow Y$, then $f(A)$ can be expressed as an integral form.

$$f(A) = \int_Y \mu_{\tilde{A}}(x) / f(x).$$

Moreover, the membership function of $f(A)$ is defined by

$$\mu_{f(A_A)} = \underset{y=f(x)}{\text{Max}}\ \mu_{\tilde{A}}(x),$$

and this definition is called the *extension principle* .

In general, f is not always a one to one function, therefore, there may exist several solutions x for fixed y. We take the maximum value of $\mu_{f(A_A)}(x)$ corresponding to the solutions x for y as $\mu_{f(A_A)}(y)$.

Definition 1.7(b) Let us extend the above case for two variable functions. $g : X_1, X_2 \rightarrow Z$. $\mu_{g(A_C)}$ can be expressed as

$$g(A_C) = \int_Z \left[\mu_{\tilde{A}_1}(x_1) \wedge \mu_{\tilde{A}_2}(x_2)\right] / g(x_1, x_2).$$

The membership function of $\mu_{g(A_C)}$ is defined by

$$\mu_{g(A_C)} = \underset{z=g(x_1,x_2)}{\text{Max}} \left[\mu_{\tilde{A}_1}(x_1) \wedge \mu_{\tilde{A}_2}(x_2)\right],$$

where the right hand side means that we take the maximum among some x_1, x_2 which result in the same fixed z.

Theorem 1.7 We suppose the membership functions, both $\mu_{\tilde{A}_1}$ and $\mu_{\tilde{A}_2}$ are triangular functions which are shown in Figure 1.3, then

$$(1)\ \tilde{A}_1 + \tilde{A}_2 = (a_1 + a_2\ ,\ w_1 + w_2)$$
$$(2)\ \tilde{A}_1 - \tilde{A}_2 = (a_1 - a_2\ ,\ w_1 + w_2)$$
$$(3)\ \lambda \tilde{A}_i = (\lambda a_i\ ,|\lambda| w_i)\ ,\quad i = 1, 2$$

where $\tilde{A}_i = (a_i\ ,\ w_i)$, a_i is the center value and w_i is the width (fuzziness) of the membership function $\mu_{\tilde{A}_i}$.

Proof (i): From assumption of a triangular function and Definition 1.6(b),

$$\tilde{A}_i = \int \frac{x_i - (a_i - w_i)}{w_i} \Big/ x_i\ ,\ \text{if}\ 0 \le x_i - (a_i - w_i) \le w_i\ ,\ i = 1, 2,$$

$$\tilde{A}_i = \int \frac{(a_i - w_i) - x_i}{w_i} \Big/ x_i\ ,\ \text{if}\ 0 \le (a_i - w_i) - x_i \le w_i$$

From Definition 1.7(b),

$$g(\Lambda_c) = \int_Z \left[\mu_{\tilde{A}_1}(x_1) \wedge \mu_{\tilde{A}_2}(x_2)\right] / g(x_1\ ,\ x_2).$$

Hence, we can obtain the membership function of $g(\Lambda_c)$ as follows:

$$\mu_{g(\Lambda_c)} = \underset{z=g(x_1,x_2)}{\text{Max}} \left[\mu_{A_1}(x_1) \wedge \mu_{A_2}(x_2)\right]$$
$$= \underset{z=g(x_1,x_2)}{\text{Max}} \left[\frac{x_1 - (a_1 - w_1)}{w_1} \wedge \frac{x_2 - (a_2 - w_2)}{w_2}\right]$$
$$= \underset{z=x_1+x_2}{\text{Max}} \left[\frac{x_1 - (a_1 - w_1)}{w_1} \wedge \frac{z - x_1 - (a_2 - w_2)}{w_2}\right],$$

if $I_1(x_1\ ,\ x_2) = \{0 \le x_1 - (a_1 - w_1) \le w_1\ \text{and}\ 0 \le x_2 - (a_2 - w_{21}) \le w_2\}$.

Since $\mu_{\tilde{A}_1}$ and $\mu_{\tilde{A}_2}$ are monotonically increasing linear functions on I_1, $\mu_{g(\Lambda_c)}$ takes the maximum value when $\frac{x_1 - (a_1 - w_1)}{w_1} = \frac{z - x_1 - (a_2 - w_2)}{w_2}$. We substitute x_1 into $\mu_{g(\Lambda_c)}$, then,

$$\mu_{g(\Lambda_c)} = \underset{z=x_1+x_2}{\mathrm{Max}} \left[\frac{x_1 - (a_1 - w_1)}{w_1} \wedge \frac{z - x_1 - (a_2 - w_2)}{w_2} \right]$$

$$= \frac{x_1 - (a_1 - w_1)}{w_1} \Big|_{x_1}$$

$$= 1 - \frac{(a_1 + a_2) - z}{w_1 + w_2} .$$

On $I_2(x_1, x_2) = \{ 0 \le (a_1 - w_1) - x_1 \le w_1$ and $0 \le (a_2 - w_{21}) - x_2 \le w_2 \}$, we can obtain the following result like the case of I_1 .

$$\mu_{g(\Lambda_c)} = 1 - \frac{z - (a_1 + a_2)}{w_1 + w_2} .$$

If z is fixed, then the maximum value of $\mu_{A_1}(x_1) \wedge \mu_{A_2}(x_2)$ exists in I_1 and I_2. Hence, $\mu_{g(\Lambda_c)}$ has the membership function, where the center value is $a_1 = a_2$ and the width of fuzziness is $w_1 = w_2$. $\mu_{g(\Lambda_c)}$ is as follows:

$$\mu_g(\Lambda_C) = \begin{cases} 1 - \frac{(a_1 + a_2) - z}{w_1 + w_2} , & \text{if } 0 \le z - \{(a_1 + a_2) - (w_1 + w_2)\} \le w_1 + w_2 \\ 1 - \frac{z - (a_1 + a_2)}{w_1 + w_2} , & \text{if } 0 \le \{(a_1 + a_2) - (w_1 + w_2)\} - z \le w_1 + w_2 \\ 0 , & \text{otherwise} \end{cases} .$$

proof (ii): Since $\mu_{\tilde{-A_2}}(x_2) \equiv \mu_{\tilde{A_2}}(-x_2)$, the center value of $\mu_{\tilde{A_2}}(-x_2)$ comes to $-a_2$ and the width of fuzziness doesn't change. Nonzero area of this membership function changes from $[a_2 - w_2 , a_2 + w_2]$ to $[-(a_2 + w_2), -(a_2 - w_2)]$. We write the membership function of μ_{A_1} and μ_{A_2} in terms of some characteristic parameters a and w as follows:

$$\mu_{A_1} = (a_1 , w_1) ,$$

$$\mu_{-A_2} = (-a_2\ ,\ w_2)\ ,$$

$$\widetilde{A_1} + \widetilde{-A_2} = (a_1\ ,\ w_1) + (-a_2\ ,\ w_2) = (a_1 - a_2\ ,\ w_1 + w_2).$$

Proof: (iii) From definition 1.7(a), we can express $f(\Lambda_A)$ $\left(f(x) = \lambda x\ ,\ \lambda \in \mathbf{R} \right)$ as follows:

$$f(\Lambda_A) = \int_Y \mu_{\widetilde{A}}(x)\ /\ f(x)$$

$$= \int_Y \mu_{\widetilde{A}}(x)\ /\ \lambda x\ ,$$

$$\mu_{f(\Lambda_A)} = \underset{y=\lambda x}{\mathrm{Max}}\ \mu_{\Lambda_A}(x)$$

$$= \begin{cases} \underset{y=\lambda x}{\mathrm{Max}} \dfrac{x - (a - w)}{w}\ , \text{ if } 0 \le x - (a - w) \le w \\ \underset{y=\lambda x}{\mathrm{Max}} \dfrac{(a - w) - x}{w}\ , \text{ if } 0 \le (a - w) - x \le w \end{cases}.$$

If $\lambda \ge 0$, then $y = |\lambda| x$.

$$\mu_{f(\Lambda_A)}(y) = \underset{y=|\lambda| x}{\mathrm{Max}} \frac{\frac{y}{\lambda} - (a - w)}{w}$$

$$= \underset{y}{\mathrm{Max}} \frac{y - (\lambda a - \lambda w)}{|\lambda| w}\ .$$

If $\lambda < 0$, then $y = -|\lambda| x$.

$$\mu_{-f(\Lambda_A)}(y) = \underset{y}{\mathrm{Max}} \frac{y - \{-(\lambda a - \lambda w)\}}{|\lambda| w}\ .$$

Q.E.D.

By Theorem 1.7, we can deal with a fuzzy linear function, namely, a linear combination which is written with coefficients given as fuzzy numbers.

Theorem 1.8 We define a *fuzzy linear function* as follows:

$$\tilde{y}_i = \tilde{\mathbf{A}}_i \mathbf{X}$$
$$= \tilde{a}_{i_1} x_1 + \cdots + \tilde{a}_{i_n} x_n,$$

where

$$\tilde{\mathbf{A}}_i = \left(\tilde{a}_{i_1}, \cdots, \tilde{a}_{i_n} \right).$$

The membership function of a fuzzy linear function is represented as

$$\mu(y_i) = \begin{cases} 1 - \dfrac{|\boldsymbol{\alpha}_i \mathbf{X} - y_i|}{\mathbf{w}_i |\mathbf{X}|_1}, & \text{if } \mathbf{X} \neq \mathbf{0},\ |\boldsymbol{\alpha}_i \mathbf{X} - y_i| \leq \mathbf{w}_i |\mathbf{X}|_1 \\ 1, & \text{if } \mathbf{X} \neq \mathbf{0},\ y_i = 0 \\ 0, & \text{otherwise} \end{cases},$$

where $\boldsymbol{\alpha}_i = \left(\alpha_{i_1}, \cdots, \alpha_{i_n} \right)$ is a vector of each fuzzy number's center value and $|\mathbf{X}|_1 = \left(|x_1|, \cdots, |x_n| \right)^t$.

Proof: By Theorem 1.7., we can obtain the membership function as follows:

$$\tilde{y}_i = \sum_{j=1}^{n} \tilde{a}_{ij} x_j$$
$$= \sum_{j=1}^{n} (\alpha_{ij}, w_{ij}) x_j$$
$$= \sum_{j=1}^{n} (x_j \alpha_{ij}, |x_j| w_{ij})$$
$$= (\boldsymbol{\alpha}_i \mathbf{X}, \mathbf{w}_i |\mathbf{X}|_1).$$

Q.E.D.

Next, we introduce another important definition to deal with the inequality including both a fuzzy number and a nonfuzzy number.

Definition 1.8 For $h \in [0,1]$, $\tilde{y}_i \geq^h b_i$ is defined as $\tilde{y}_i = \tilde{A}_i X \geq^h b_i$, if and only if $\mu_{\tilde{y}_i}(b_i) \leq h$ and $\alpha_i X \geq b_i$, which means that all the elements of the h-level set of the fuzzy number $\tilde{y}_i$ are greater than b_i. Here h is given before we solve a problem.

Figure 1.3. explains that all the elements of the h-level set of the fuzzy number $\tilde{y}_i = \tilde{A}_i X$ are greater than $b_i - s_i$. We may choose b_i^u as b_i, because a crisp constraint of b_i is the most strict condition among the fuzziness interval on $[b_i^l, b_i^h]$. From the crisp model of Theorem 1.6, we can obtain $\lambda = 1 - \frac{s_i}{d_i}$. We suppose we are given h level so that it satisfies $\tilde{y}_i \geq^h b_i - s_i$. In this inequality, the left hand side is a fuzzy number and the right hand side is a nonfuzzy number; therefore, we can compute this directly. Because a fuzzy number is something like a spread value and can not be represented by one specified value like a nonfuzzy number in the case of Type 1, we have to select which part of this spread domain satisfies inequality in the sense of a nonfuzzy inequality.

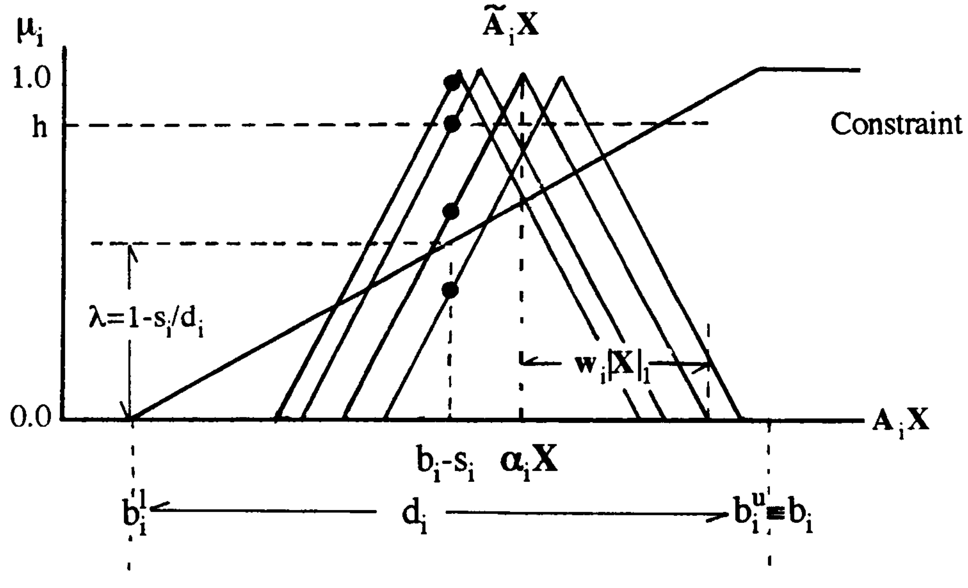

Figure 1.3 Geometrical Interpretation of a Membership Function and a Constraint with H-level.

In Figure 1.3, the fuzzy numbers which are placed in the right hand side of the center value of $\boldsymbol{\alpha}_i\mathbf{X}$ satisfy two requirements of Definition 1.8; however, the numbers which exist in the left side do not satisfy them. By using this definition, we can transform a Fuzzy Linear Programming problem with fuzzy numbers into a crisp Linear Programming problem as follows.

We suppose the following Fuzzy Linear Programming problem doesn't distinguish goals from constraints.

F.L.P. Problem : $\tilde{\mathbf{A}}\mathbf{X} \tilde{\geq} \mathbf{b}$,

(Type 2) $\left(\tilde{A}_i\mathbf{X} \tilde{\geq} b_i \quad , \quad i = 1, \cdots, m\right)$

$\mathbf{X} \geq \mathbf{0}$.

By Definition 1.8,

$$\mu_{\tilde{y}_i}(b_i - s_i) \leq h.$$

Since $\mu_{\tilde{y}_i}$ is given by Theorem 1.8,

$$\mu_{\tilde{y}_i}(b_i - s_i) = 1 - \frac{|\boldsymbol{\alpha}_i \mathbf{X} - (b_i - s_i)|}{\mathbf{w}_i |\mathbf{X}|_1} .$$

Since $\boldsymbol{\alpha}_i \mathbf{X} \geq b_i - s_i$,

$$1 - h \leq \frac{\boldsymbol{\alpha}_i \mathbf{X} - (b_i - s_i)}{\mathbf{w}_i |\mathbf{X}|_1} .$$

We develop the above formula subsequently,

$$b_i \leq \{\boldsymbol{\alpha}_i - \mathbf{w}_i(1 - h)\}\mathbf{X} + s_i .$$

We can obtain λ for s_i . When we have a constraint gap s_i toward b_i, we can satisfy the i-th constraint with ratio λ which is equal to the value of the membership function corresponding to the i-th constraint. Hence, the relationship between s_i and λ is given by

$$d_i\lambda + s_i \leq d_i \quad , \quad i = 1, \cdots, m .$$

Hence, we can obtain a crisp model of a Fuzzy Linear Programming problem (Type 2).

F.L.P. problem : Max λ

[For F.L.P. problem (Type 2)]

$$\begin{aligned} \text{s.t. } & \{\boldsymbol{\alpha}_i - (1-h)\mathbf{w}_i\}\mathbf{X} + s_i \geq b_i, \quad i = 1, \cdots, m, \\ & d_i\lambda + s_i \leq d_i, \\ & \mathbf{X} \geq \mathbf{0}. \end{aligned} \tag{1.4a}$$

If h=0, then (1.4a) is reduced as follows:

F.L.P. problem : Max λ

$$\begin{aligned} \text{s.t. } & \{\boldsymbol{\alpha}_i - \mathbf{w}_i\}\mathbf{X} + s_i \geq b_i, \qquad i = 1, \cdots, m, \\ & d_i\lambda + s_i \leq d_i, \\ & \mathbf{X} \geq \mathbf{0} \end{aligned} \tag{1.4b}$$

This formula of (1.4b) is the same form as that of (1.3b) except $\boldsymbol{\alpha}_i - \mathbf{w}_i$ appears instead of $\mathbf{a}_i$. The difference between $\boldsymbol{\alpha}_i - \mathbf{w}_i$ and $\mathbf{a}_i$ is that the former is the center value of fuzzy number and the latter is the specified number in a nonfuzzy number. Therefore, we can consider that the other constraints are concealed in (1.4b) as follows:

$$\begin{aligned} & \{\boldsymbol{\alpha}_i - (1-h)\mathbf{w}_i\}\mathbf{X} + s_i \geq b_i, \\ & \{\boldsymbol{\alpha}_i - (1-h+\Delta h)\mathbf{w}_i\}\mathbf{X} + s_i \geq b_i, \\ & \{\boldsymbol{\alpha}_i - (1-h+2\Delta h)\mathbf{w}_i\}\mathbf{X} + s_i \geq b_i, \\ & \qquad \vdots \\ & \{\boldsymbol{\alpha}_i - \mathbf{w}_i\}\mathbf{X} + s_i \geq b_i, \end{aligned}$$

where each constraint is corresponding to each pulse, namely, the specified number in the sense of a nonfuzzy number like the case of Type 1 problems. If the last constraint satisfies its inequality, all the concealed constraints satisfy inequalities, because $\mathbf{X}$, h and $\mathbf{w}_i$ are nonnegative. Hence, we can regard Type 1 problems as the particular cases of Type 2 problems.

Example 1.2. Consider the problem.

$$\mathbf{AX} \gtrsim \mathbf{b},$$

$$\mathbf{DX} \le \mathbf{e},$$

where

$$\tilde{\mathbf{A}} = [\ \tilde{A}_1\ \ \tilde{A}_2\ \ \tilde{A}_3\ \ \tilde{A}_4\ ]^t,$$

$$\mathbf{b} = [\ 15\ \ 12\ \ 7\ \ 11\ ]^t,\ \ \mathbf{D} = [\ 2\ 3\ 1\ ],$$

$$\mathbf{X} = [x_1\ x_2\ x_3]^t \ge \mathbf{0}\ ,\ \tilde{A}_1 = \{\ \boldsymbol{\alpha}_1 = (6.3\ ,\ 10.1\ ,\ -3)\ ,\ \mathbf{w}_1 = (5\ ,3\ ,4)\ \},$$

$$\tilde{A}_2 = \{\ \boldsymbol{\alpha}_2 = (-1.8\ ,\ 7.9\ ,\ 2.5)\ ,\ \mathbf{w}_2 = (2\ ,5\ ,3)\ \},$$

$$\tilde{A}_3 = \{\ \boldsymbol{\alpha}_3 = (\ 1\ ,\ -1.5\ ,\ 8)\ ,\ \mathbf{w}_3 = (4\ ,3\ ,4)\ \},$$

$$\tilde{A}_4 = \{\ \boldsymbol{\alpha}_4 = (\ 5\ ,\ 3\ ,\ 2)\ ,\ \mathbf{w}_4 = (6\ ,4\ ,2)\ \}.$$

Before we solve this problem, we are given h-level as follows.

$$h = 0.5$$

By Theorem 1.8 and Definition 1.8, we can make an equivalent linear programming problem. Since constraints for fuzzy numbers can be expressed as

$$\{\boldsymbol{\alpha}_i - (1 - h)\mathbf{w}_i\}\mathbf{X} + s_i \ge b_i,\quad i = 1,\cdots,4\ ,$$

A crisp model is as follows:

E.L.P. Problem :

$$\begin{aligned} &\text{Max } \lambda \\ &\text{s.t. } 3.8x_1 + 8.6x_2 - 5x_3 + s_1 \ge 15, \\ &\quad -2.8x_1 + 5.4x_2 + x_3 + s_2 \ge 12, \\ &\quad -x_1 - 3x_2 + 6x_3 + s_3 \ge 7, \\ &\quad 2x_1 + x_2 + x_3 + s_4 \ge 11, \\ &\quad 6\lambda + s_1 \le 6, \\ &\quad 4\lambda + s_2 \le 4, \\ &\quad 5\lambda + s_3 \le 5, \\ &\quad 4\lambda + s_4 \le 4, \\ &\quad 2x_1 + 3x_2 + x_3 \le 11. \end{aligned}$$

CHAPTER 2

SENSITIVITY ANALYSIS OF FUZZY LINEAR PROGRAMMING

2.1 SENSITIVITY ANALYSIS (TYPE 1)

In chapter 1 we assumed that all the parameters of fuzziness are given. However, for many problems, the width of fuzziness is estimated or the parameters vary in each case because of their ill-defined form. In our problems a solution depends on the width of the fuzziness, since the smaller the fuzziness, the more satisfactory an exact solution is expected to be obtained. We consider the sensitivity analysis for Fuzzy Linear Programming (Type 1) in this section. The definition of the sensitivity analysis can be expressed as the variation of the value of the optimal λ to the deviation of the value of the fuzzy parameter such as the difference between upper or lower aspiration level, namely, to the tolerance value. We want to seek this value of the sensitivity by using matrices which are used in order to obtain the original optimal solution λ. The basic results content can be found in [2] and [5]. In addition to these studies, we extend the sensitivity analysis, if we have more than two deviations at the same time.

Fuzzy Linear Programming (Type 1) can be written as the following form of a crisp model by the method of (1.4). We do not distinguish goals and constraints from each other.

$$\begin{aligned} &\text{F.L.P. problem :} \quad \widetilde{\text{Max}}\ \mathbf{C}^t\mathbf{X} \\ &\text{(Type 1)} \qquad\quad \text{s.t. } \mathbf{A}'\mathbf{X} \precsim \mathbf{b}', \\ &\qquad\qquad\qquad\quad \mathbf{G}\mathbf{X} \le \mathbf{e}, \\ &\qquad\qquad\qquad\quad \mathbf{X} \ge \mathbf{0}, \end{aligned} \tag{2.1}$$

where $\mathbf{C} = (\mathbf{c}_1, \cdots, \mathbf{c}_l)$, $\mathbf{A}' = (\mathbf{a}_1, \cdots, \mathbf{a}_{l'})^t$, $\mathbf{G} = (\mathbf{g}_1, \cdots, \mathbf{g}_{m_2})^t$, $l+l' = m_1+l$, $m_1+l+m_2 = m$.

This problem is equivalent to

E.L.P. problem : Max λ

$$\begin{aligned} \text{s.t. } \mathbf{a}_i \mathbf{X} + s_i &\geq b_i, & i = 1, \cdots, m_1+1, \\ d_i\lambda + s_i &\leq d_i, & i = 1, \cdots, m_1+1, \\ \mathbf{g}_i\mathbf{X} &\leq e_i, & i = 1, \cdots, m_2, \\ \mathbf{X} &\geq \mathbf{0}, \end{aligned} \tag{2.2}$$

where $\mathbf{a}_1 \equiv \mathbf{c}_1, \cdots, \mathbf{a}_l \equiv \mathbf{c}_l, \mathbf{a}_{l+1} \equiv A_1', \cdots, \mathbf{a}_{m_1} \equiv A_{l'}'$ and $b_1 \equiv c_1^u, \cdots, b_l \equiv c_l^u$, $b_{l+1} \equiv b_1', \cdots, b_{m_1} \equiv b_{l'}'$ and $d_1 \equiv c_1^u - c_1^l, \cdots, d_l \equiv c_l^u - c_l^l$, $d_{l+1} \equiv b_1'^u - b_1'^l, \cdots, d_{m_1} \equiv b_{l'}'$. We denote the matrix of $(\mathbf{a}_1, \cdots, \mathbf{a}_{m_1})$ as $\mathbf{A}''$ and also do the vector of $(b_1, \cdots, b_{m_1})^t$ as $\mathbf{b}''$.

We reformulate the above formula by using an augmented matrix as follows:

E.L.P. problem : Max λ

$$\text{s.t. } \mathbf{A}^{\#}(\lambda\ \mathbf{X}\ \mathbf{s})^t \leq \mathbf{b}^{\#}, \tag{2.3 a}$$

where let $\lambda \in \mathbf{R}$, $\mathbf{X} \in \mathbf{R}^n$, $\mathbf{s} \in \mathbf{R}^{n+1}$, $\mathbf{b}^{\#} \in \mathbf{R}^{3(m_1+1)+m_2}$.

$$\mathbf{b}^{\#} \equiv [\mathbf{d}\ \mathbf{b}''\ \mathbf{d}\ \mathbf{e}]^t,$$

$$\mathbf{A}^{\#} \equiv \begin{bmatrix} \mathbf{d}_{m_1+1,1} & \mathbf{0}_{m_1+1,n} & \mathbf{E}_{m_1+1,m_1+1} \\ \mathbf{0}_{m_1+1,1} & \mathbf{A}_{m_1+1,n} & -\mathbf{E}_{m_1+1,m_1+1} \\ \mathbf{0}_{m_1+1,1} & \mathbf{0}_{m_1+1,n} & \mathbf{E}_{m_1+1,m_1+1} \\ \mathbf{0}_{m_2,1} & \mathbf{D}_{m_2,n} & \mathbf{0}_{m_2,m_1+1} \end{bmatrix},$$

where $\mathbf{0}$ is a zero-matrix and $\mathbf{E}$ is an identity-matrix.

We define

$$\mathbf{e}_j^* \equiv (0_{1\text{-st}}, \cdots, 1_{j\text{-th}}, \cdots, 0_{(m_1+1)\text{-th}} \vdots \mathbf{0}_{(m_1+1)} \vdots 0_{1\text{-st}}, \cdots 0, 1_{j\text{-th}}, 0, \cdots, 0_{(m_1+1)\text{-th}} \vdots \mathbf{0}_{(m_2+1)})^t$$

and let $\overline{\mathbf{b}} \equiv \mathbf{b}^{\#} + \Delta d \cdot \mathbf{e}_j^*$, where Δd is a deviation of fuzziness and $\mathbf{e}^*$ designates the perturbed element.

$$\overline{\mathbf{A}} \equiv \mathbf{A}^{\#} + \begin{bmatrix} \Delta d \cdot \mathbf{e}_j & \\ & \mathbf{0}_{3(m_1+1)+m_2, n+m+1} \\ \mathbf{0}_{2(m_1+1)+m_2} & \end{bmatrix},$$

where $\mathbf{e}_j \equiv \left(0_{1\text{-st}}, \cdots, 1_{j\text{-th}}, \cdots, 0_{(m_1+1)\text{-th}}\right)^t$.

We can rewrite (2.3) by the augmented matrix and vectors as follows:

E.L.P. problem : Max λ

$$\text{s.t. } \overline{\mathbf{A}} \left(\lambda \, \mathbf{X} \, \mathbf{s} \right)^t \le \overline{\mathbf{b}}, \qquad (2.3\text{ b})$$

$$\mathbf{X}, \mathbf{s} \ge \mathbf{0},$$

We want to obtain $\lambda_{max}(\Delta d)$ which is the optimal solution for the above problem when a deviation occurs from d to d + Δd . We suppose there is no exchange of basis columns on the matrix **B** which is the basis matrix corresponding to independent column vectors on matrix $\mathbf{A}^{\#}$. We solve this problem by the Revised Simplex Method.

Since λ is a basic variable and then the first column of $\overline{\mathbf{A}}$ is a column of the basis matrix **B**, **B** changes as d changes d + Δd.
In the above

$$\overline{\mathbf{B}} \equiv \mathbf{B} + \Delta d \cdot \overline{\mathbf{E}}_j, \qquad (2.4)$$

and

$$\overline{\mathbf{E}}_j = \begin{bmatrix} \mathbf{0} & \\ 1_{j\text{-th}} & \mathbf{0}_{3(m_1+1)+m_2, 3(m_1+1)+m_2-1} \\ \mathbf{0} & \end{bmatrix}.$$

The λ_{max} is obtained by the Revised Simplex Method.

$$\lambda_{max} = \mathbf{C}_B^t \mathbf{B}^{-1} \mathbf{b}^{\#}, \quad \mathbf{B}^{-1} = \left(\beta_{ij}\right), \quad i,j = 1, \ldots, 3(m_1+1)+m_2,$$

where $\mathbf{C}_B$ is a unit vector in this case.

Similarly,

$$\lambda_{max}(\Delta d) = \mathbf{C}^{t}_{\overline{B}} \overline{\mathbf{B}}^{-1} \overline{\mathbf{b}} \tag{2.5}$$

and $\mathbf{C_B} = \mathbf{C}_{\overline{\mathbf{B}}}$, because of no exchange on basis columns.

$$\begin{aligned}\overline{\mathbf{B}}^{-1} &= \left(\mathbf{B} + \Delta d \cdot \overline{\mathbf{E}}_j\right)^{-1} \\ &= \left(\mathbf{B}\left(\mathbf{E} + \mathbf{B}^{-1} \cdot \Delta d \cdot \overline{\mathbf{E}}_j\right)\right)^{-1} \\ &= \left(\mathbf{E} + \mathbf{B}^{-1} \cdot \Delta d \cdot \overline{\mathbf{E}}_j\right)^{-1} \mathbf{B}^{-1}.\end{aligned}$$

We have to obtain $\mathbf{E} + \mathbf{B}^{-1} \cdot \Delta d \cdot \overline{\mathbf{E}}_j$.

$$\mathbf{B}^{-1} \cdot \Delta d \cdot \overline{\mathbf{E}}_j = \mathbf{B}^{-1} \begin{bmatrix} \mathbf{0} & \\ \Delta d_{j\text{-th}} & \mathbf{0}_{3(m_1+1)+m_2,\,3(m_1+1)+m_2-1} \\ \mathbf{0} & \end{bmatrix}$$

$$= \begin{bmatrix} \beta_{1j}\Delta d & \\ \beta_{2j}\Delta d & \\ \vdots & \mathbf{0}_{3(m_1+1)+m_2,\,3(m_1+1)+m_2-1} \\ \beta_{3(m_1+1)+m_2,\,j}\Delta d & \end{bmatrix},$$

$$\mathbf{E} + \mathbf{B}^{-1} \cdot \Delta d \cdot \overline{\mathbf{E}}_j = \begin{bmatrix} \beta_{1j}\Delta d+1 & 0 & \cdots & 0 \\ \beta_{2j}\Delta d & 1 & & \mathbf{0} \\ \vdots & & \ddots & \\ \beta_{3(m_1+1)+m_2,j}\Delta d & \mathbf{0} & & 1 \end{bmatrix},$$

$$\left(\mathbf{E} + \mathbf{B}^{-1} \cdot \Delta d \cdot \overline{\mathbf{E}}_j\right)^{-1}$$

$$= \begin{bmatrix} \frac{1}{\beta_{1j}\Delta d+1} & 0 & \cdots & 0 \\ \frac{-\beta_{2j}\Delta d}{\beta_{1j}\Delta d+1} & 1 & & \mathbf{0} \\ \vdots & & \ddots & \\ \frac{-\beta_{3(m_1+1)+m_2,j}\Delta d}{\beta_{1j}\Delta d+1} & \mathbf{0} & & 1 \end{bmatrix}$$

$$\equiv \mathbf{M} + \mathbf{E}.$$

$$\mathbf{M} \equiv \begin{bmatrix} \frac{1}{\beta_{1j}\Delta d+1} - 1 & \\ \frac{-\beta_{2j}\Delta d}{\beta_{1j}\Delta d+1} & \mathbf{0}_{3(m_1+1)+m_2,3(m_1+1)+m_2-1} \\ \vdots & \\ \frac{-\beta_{3(m_1+1)+m_2,j}\Delta d}{\beta_{1j}\Delta d+1} & \end{bmatrix}.$$

From (2.5),

$$\begin{aligned} \lambda_{max}(\Delta d) &= \mathbf{C}_{\overline{B}}^{t}\,\overline{\mathbf{B}}^{-1}\,\overline{\mathbf{b}} \\ &= \mathbf{C}_{B}^{t}\,\overline{\mathbf{B}}^{-1}\,\overline{\mathbf{b}} \\ &= \mathbf{C}_{B}^{t}\,(\mathbf{M} + \mathbf{E})\,\mathbf{B}^{-1}\left(\mathbf{b}^{\#} + \Delta d \cdot \mathbf{e}_j^{*}\right) \\ &= \mathbf{C}_{B}^{t}\,\mathbf{B}^{-1}\,\mathbf{b}^{\#} + \mathbf{C}_{B}^{t}\left\{\mathbf{M}\,\mathbf{B}^{-1}\mathbf{b}^{\#} + (\mathbf{M} + \mathbf{E})\,\mathbf{B}^{-1}\Delta d \cdot \mathbf{e}_j^{*}\right\} \end{aligned}$$

$$= \lambda_{max} + C_B^t \left\{ \mathbf{M}\ \mathbf{B}^{-1}\mathbf{b}^{\#} + (\mathbf{M} + \mathbf{E})\ \mathbf{B}^{-1}\Delta d \cdot \mathbf{e}_j^{\ *} \right\}.$$

Theorem 2.1 If we suppose there is no exchange of the basis column of **B** with the optimal solution, then the sensitivity to the width of fuzziness can be obtained as follows:

$$\Delta\lambda(\Delta d) \equiv \lambda_{max}(\Delta d) - \lambda_{max} = C_B^t \left\{ \mathbf{M}\ \mathbf{B}^{-1}\mathbf{b}^{\#} + (\mathbf{M} + \mathbf{E})\ \mathbf{B}^{-1}\Delta d \cdot \mathbf{e}_j^{\ *} \right\}. \quad (2.6)$$

Example 2.1 We reconsider Example 1 as a problem of a sensitivity analysis. We have already got a crisp model as follows:

$$\begin{aligned}
&\text{Max } \lambda \\
&\text{s.t. } 5\lambda + s_0 \le 5, \\
&\quad 6\lambda + s_1 \le 6, \\
&\quad 8\lambda + s_2 \le 8, \\
&\quad 5\lambda + s_3 \le 5, \\
&\quad -x_1 - 2x_2 - s_0 \le -16.5, \\
&\quad -x_1 + 3x_2 + 2x_3 - s_1 \le 23, \\
&\quad x_1 + 2x_2 - 3x_3 - s_2 \le 17, \\
&\quad 2x_1 + x_2 + 2x_3 - s_3 \le 21, \\
&\quad 2x_1 + 7x_2 + x_3 \le 35,
\end{aligned}$$

where $\mathbf{s} = [s_0\ s_1\ s_2\ s_3]^t \le [5\ \ 6\ \ 8\ \ 5]^t$.

Let's examine the sensitivity analysis about fuzziness of the first constraint. We have to satisfy at least 11.5 for the goal, since the fuzziness of this goal is 5 and the complete aspiration value is 16.5. We suppose this fuzziness increases from 5 to 6. We use the following notation to designate an element . Let

$$\Delta d_1 = +1,$$

which means that fuzziness of the first constraint increases by 1.

First, we solve this problem directly by LINDO. Second, we solve this by using Theorem 2.1. Last, we examine the feasibility and the optimality of the new solution after we change fuzziness of d_1.

In order to make sure that s_i takes a reasonable value, we add some restrictions as follows:

$$\mathbf{s} = [s_0\ s_1\ s_2\ s_3]^t \leq [5\ \ 6\ \ 8\ \ 5]^t.$$

By the formula of (2.3), the system matrix of the original problem can be expressed as the following:

$$\mathbf{b}^{\#} \equiv \left[\mathbf{d}\ \mathbf{b}''\ \mathbf{d}\ \mathbf{e}\right]^t$$

$$= [\ 5 \quad 6 \quad 8 \quad 5 \quad \vdots \quad -16.5 \quad 23 \quad 17 \quad 21 \quad \vdots \quad 5 \quad 6 \quad 8 \quad 5 \quad \vdots \quad 35\ ]^t$$

$$\mathbf{A}^{\#} = \begin{bmatrix} 5 & 6 & 8 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 2 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -2 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 2 & -3 & 2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^t.$$

Hence, the equivalent form is given as

E.L.P. Problem : Max λ

$$\text{s.t. } \overline{\mathbf{A}}\ [\lambda\ \mathbf{X}\ \mathbf{s}]^t \leq \overline{\mathbf{b}},$$

where $\mathbf{X} = [x_1\ x_2\ x_3] \geq \mathbf{0}$ and $\mathbf{s} = [s_0\ s_1\ s_2\ s_3] \geq \mathbf{0}$.

In the process for solving this equation by LINDO, $\mathbf{A}^{\#}$ is extended with slack variables and rewritten as

$$\left[\begin{array}{ccc} \mathbf{A}^{\#} & \vdots & \mathbf{I}_{13,13} \end{array}\right],$$

where **I** is an identity matrix.

LINDO can pick up the basis columns for the optimal solution as follows:

$$\mathbf{B} = \begin{bmatrix} 5&0&0&0&1&0&0&0&0&0&0&0&0 \\ 6&1&0&0&0&0&0&0&0&0&0&0&0 \\ 8&0&1&0&0&0&0&0&0&0&0&0&0 \\ 5&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&-1&0&0&-1&0&0&0&0&-2 \\ 0&0&0&0&0&1&0&-1&0&0&0&0&3 \\ 0&0&0&0&0&0&1&1&0&0&0&0&2 \\ 0&0&0&-1&0&0&0&2&0&0&0&0&1 \\ 0&0&0&0&1&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&2&0&0&0&0&7 \end{bmatrix}.$$

We have already obtained the objective function value of the original problem; therefore, we have to seek only the solution after changing the fuzziness of the first row of $\mathbf{A}^{\#}$ and the first element of $\mathbf{b}^{\#}$ increasing by 1. By LINDO, we can obtain the optimal solution as $\lambda = 0.6552$. Since the basis columns for this optimal solution are the same as those of the original problem, we can keep the assumption that no basis column changes, as in Theorem 2.1. Hence, by the definition of the sensitivity analysis, we can obtain the result as follows:

$$\Delta\lambda(\Delta d_1) = \lambda^{o}(\Delta d_1) - \lambda^{o} = 0.6552 - 0.600 = 0.0552.$$

Second of all, we seek the same results directly by using the matrices on Theorem 2.1.

$$\Delta\lambda(\Delta d_1) = \mathbf{C}_B^t\{\mathbf{M}\ \mathbf{B}^{-1}\mathbf{b}^{\#} + (\mathbf{M} + \mathbf{E})\ \mathbf{B}^{-1}\Delta d_1 \cdot \mathbf{e}_1^{*}\}. \qquad (2.7)$$

Since $\mathbf{C}_B^t$ is the cost coefficient vector corresponding to basis columns,

$$\mathbf{C}_B^t = [1\ 0\ 0\ 0 \ \vdots\ 0\ 0\ 0\ 0 \ \vdots\ 0\ 0\ 0\ 0 \ \vdots\ 0]^t,$$

and $\mathbf{B}^{-1}$ can be obtained from the tableau of LINDO.

$$\mathbf{B}^{-1} = \begin{bmatrix}
0.16 & 0 & 0 & 0.04 & 0.16 & 0 & 0 & 0.04 & 0 & 0 & 0 & 0 & 0.04 \\
-0.96 & 1 & 0 & -0.24 & -0.96 & 0 & 0 & -0.24 & 0 & 0 & 0 & 0 & -0.24 \\
-1.28 & 0 & 1 & -0.32 & -1.28 & 0 & 0 & -0.32 & 0 & 0 & 0 & 0 & -0.32 \\
-0.8 & 0 & 0 & 0.8 & -0.8 & 0 & 0 & -0.2 & 0 & 0 & 0 & 0 & -0.2 \\
0.2 & 0 & 0 & -0.2 & -0.8 & 0 & 0 & -0.2 & 0 & 0 & 0 & 0 & -0.2 \\
-0.867 & 0 & 0 & 0.867 & -0.867 & 1 & 0 & 0.867 & 0 & 0 & 0 & 0 & -0.8 \\
0.2 & 0 & 0 & -0.2 & 0.2 & 0 & 1 & -0.2 & 0 & 0 & 0 & 0 & -0.2 \\
-0.467 & 0 & 0 & 0.467 & -0.467 & 0 & 0 & 0.467 & 0 & 0 & 0 & 0 & -0.2 \\
-0.2 & 0 & 0 & 0.2 & 0.8 & 0 & 0 & 0.2 & 1 & 0 & 0 & 0 & 0.2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0.8 & 0 & 0 & -0.8 & 0.8 & 0 & 0 & 0.2 & 0 & 0 & 0 & 1 & 0.2 \\
0.133 & 0 & 0 & -0.133 & 0.133 & 0 & 0 & -0.133 & 0 & 0 & 0 & 0 & 0.2
\end{bmatrix},$$

$$\mathbf{M} = \begin{bmatrix}
-0.138 & \\
0.828 & \\
1.103 & \\
0.69 & \\
-0.172 & \\
0.747 & \\
-0.172 & \mathbf{0}_{13,12} \\
0.403 & \\
0.172 & \\
0 & \\
0 & \\
-0.69 & \\
-0.115 &
\end{bmatrix},$$

$$\mathbf{e}_1^* = [1\ 0\ 0\ 0 \ \vdots\ 0\ 0\ 0\ 0 \ \vdots\ 1\ 0\ 0\ 0 \ \vdots\ 0]^t.$$

We substitute these matrices into (2.7). Then we obtain the first term and the second term.

$$\mathbf{C}_B^t \mathbf{M}\ \mathbf{B}^{-1}\mathbf{b}^{\#} = -\ 0.0825,$$

$$\mathbf{C}_B^t (\mathbf{M} + \mathbf{E})\ \mathbf{B}^{-1} \Delta d_1 \cdot \mathbf{e}_1^* = 0.138.$$

Hence,

$$\Delta\lambda(\Delta d_1) = -\ 0.0825 + 0.138 = 0.0555.$$

The last thing which we have left is checking this result on optimality and feasibility conditions. To preserve the feasibility condition (see [4]), we must have

$$\overline{\mathbf{X}}^o = [\ \overline{\lambda}\quad \overline{x}_1^o\quad \overline{x}_2^o\quad \overline{x}_3^o\quad \overline{s}_0\quad \overline{s}_1\quad \overline{s}_2\quad \overline{s}_3\ ]^t$$
$$= \overline{\mathbf{B}}^{-1}\,\overline{\mathbf{b}} \geq \mathbf{0},$$

where $\overline{\mathbf{X}^o}$ is the optimal solution after changing fuzziness.

In our case, $\overline{\mathbf{B}}^{-1}$ is obtained by LINDO and $\overline{\mathbf{b}}$ is written as follows:

$$\overline{\mathbf{b}} = [\ 5{+}1\quad 6\quad 8\quad 5\quad \vdots\quad -16.5\quad 23\quad 17\quad 21\quad \vdots\quad 5{+}1\quad 6\quad 8\quad 5\quad \vdots\quad 35\ ]^t.$$

Hence,

$$\overline{\mathbf{X}}^o = \overline{\mathbf{B}}^{-1}\,\overline{\mathbf{b}}$$
$$= [\ 0.655\quad 10.33\quad 2.04\quad 0\quad 2.069\quad 0\quad 0\quad 1.724\ ]^t \geq \mathbf{0}$$

To preserve the optimality condition (see [4]), the reduced cost for the maximum problem must be satisfied.

$$\overline{z}_j - \overline{c}_j = \mathbf{C}_{\overline{B}}^t\,\overline{\mathbf{B}}^{-1}\,\overline{\mathbf{P}}_j - \overline{c}_j \geq 0 \qquad \text{for all j element,}$$

where $\overline{\mathbf{P}_j}$ is a vector on the optimal basis and $\overline{c_j}$ is the j-th element of $\mathbf{C}^T$. In this case

$$\mathbf{C}^t = [\,1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\,]^t, \text{and } \mathbf{C}_{\overline{B}}^t = [\,1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\,]^t,$$

therefore,

$$\bar{z}_1 - \bar{c}_1 = C_B^t \bar{B}^{-1} \bar{P}_1 - \bar{c}_1 = 0,$$

where

$$\bar{P}_1 = [\ 5+1 \quad 6 \quad 8 \quad 5 \quad \vdots \quad 0 \quad 0 \quad 0 \quad 0 \quad \vdots \quad 0 \quad 0 \quad 0 \quad 0 \quad \vdots \quad 0\ ]^t$$

and $\bar{c}_1 = 1$.

Results for other elements can be obtained similarly as follows:

$$\bar{z}_j - \bar{c}_j = 0, \quad j = 2, \cdots, 12,$$
$$\bar{z}_{13} - \bar{c}_{13} = 0.1034\ .$$

Hence,

$$\bar{z}_j - \bar{c}_j \geq 0, \text{ for all j element.}$$

We show results with changing every element of fuzziness in Figure 2.1.(See Page 40) From this figure , we can find that the relationships between $\Delta\lambda$ and Δd_i is not necessarily linear as we know from Theorem 2.1. An infeasible point appears on this figure, since intersection area disappears as fussiness is lost on all the goals or constraints.

We have discussed the case in which we have just one deviation of fuzziness so far. We can extend the above procedure for more than two deviations of fuzziness at the same time. We show this as follows:

$$\bar{\mathbf{b}} \equiv \mathbf{b}^{\#} + \sum_{p_i \in N_0} \Delta d_{p_i} \cdot \mathbf{e}_{p_i}^{*}, \tag{2.8}$$

where $N_0 = \{p_i \mid p_i \text{ is the index number with respect to the deviation of fuzziness}\}$.

$$\bar{\mathbf{A}} \equiv \mathbf{A}^{\#} + \begin{bmatrix} \sum_{p_i \in N_0} \Delta d_{p_i} \cdot \mathbf{e}_{p_i}^{*} & \mathbf{0}_{3(m_1+1)+m_2, n+m+1} \\ \mathbf{0}_{2(m_1+1)+m_2} & \end{bmatrix}.$$

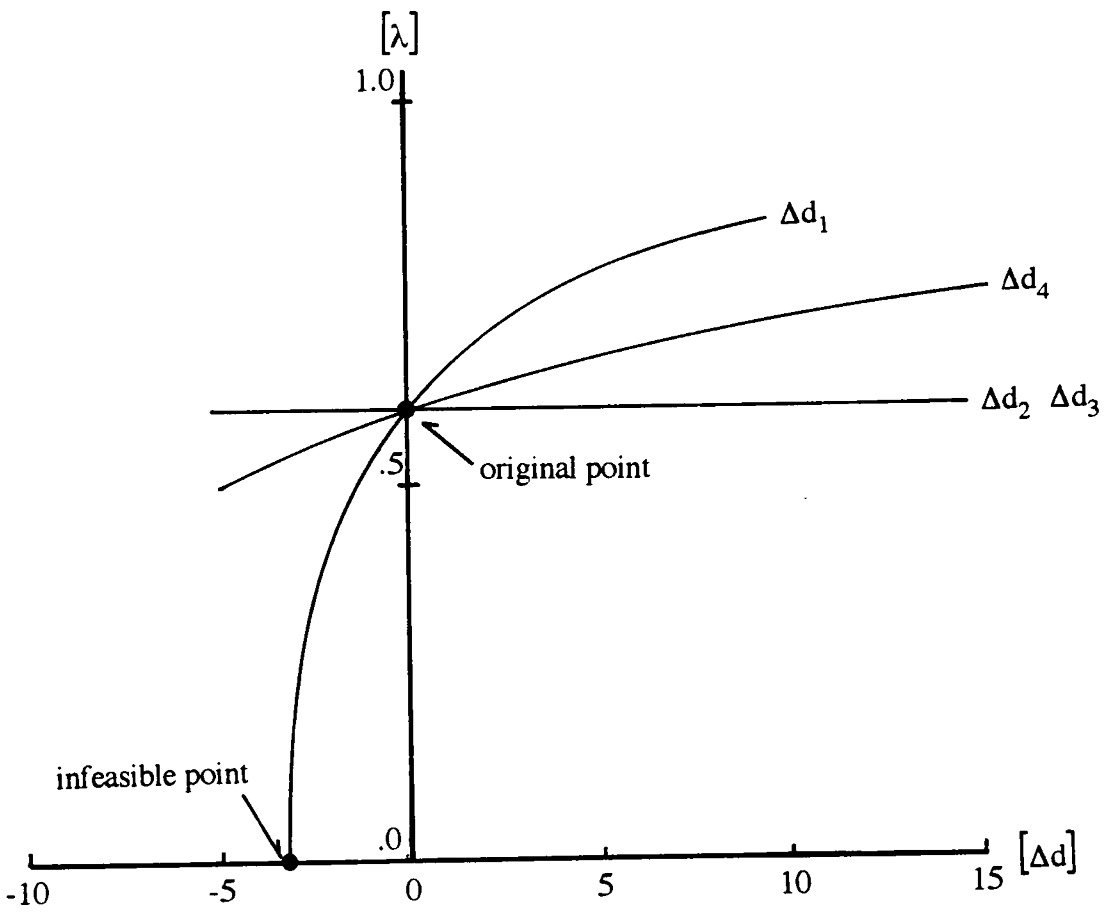

Figure 2.1 The Result of the Sensitivity Analysis of Example 2.1

As the previous case of (2.4), we assume there exist no exchange in the basis matrix **B**. Let

$$\overline{\mathbf{B}} \equiv \mathbf{B} + \sum_{p_i \in N_0} \Delta d_{p_i} \cdot \mathbf{e}_{p_i}^{*}. \tag{2.9}$$

We apply the Revised Simplex Method to solve this parametric problem. On the other hand, we can obtain the following relation like the case of just one deviation.

$$\lambda_{max} = \mathbf{C}_B^t \mathbf{B}^{-1} \mathbf{b}^{\#},$$

$$\lambda_{max}\left(\Delta d_{p_1}, \cdots, \Delta d_{p_{n_0}}\right) = \mathbf{C}_{\overline{B}}^t \overline{\mathbf{B}}^{-1} \overline{\mathbf{b}},$$

where

$$\overline{\mathbf{B}}^{-1} \equiv \left(\mathbf{B} + \sum_{p_i \in N_0} \Delta d_{p_i} \cdot \overline{\mathbf{E}}_{p_i}\right)^{-1}$$

$$= \left(\mathbf{E} + \mathbf{B}^{-1} \cdot \sum_{p_i \in N_0} \Delta d_{p_i} \cdot \overline{\mathbf{E}}_{p_i}\right)^{-1} \mathbf{B}^{-1},$$

$$\overline{\mathbf{E}}_{p_i} \equiv \begin{bmatrix} 0 & \\ \vdots & \\ 1_{p\text{-th}} & \mathbf{0}_{3(m_1+1)+m_2,\, 3(m_1+1)+m_2-1} \\ \vdots & \\ 0 & \end{bmatrix},$$

$$\mathbf{B}^{-1}\sum_{p_i} \Delta d_{p_i} \cdot \overline{\mathbf{E}}_{p_i} = \mathbf{B}^{-1} \begin{bmatrix} 0 & \\ \vdots & \\ \Delta d_{p_1} & \\ 0 & \\ \vdots & \\ \Delta d_{p_2} & \mathbf{0}_{3(m_1+1)+m_2,\, 3(m_1+1)+m_2-1} \\ 0 & \\ \vdots & \\ \Delta d_{p_{n_0}} & \\ \vdots & \\ 0 & \end{bmatrix}$$

$$
= \begin{bmatrix}
0 & \\
\vdots & \\
\sum_{p_i \in N_0} \beta_{1p_i} \Delta d_{p_i} & \\
0 & \\
\vdots & \\
\sum_{p_i \in N_0} \beta_{2p_i} \Delta d_{p_i} & \mathbf{0}_{3(m_1+1)+m_2,\, 3(m_1+1)+m_2-1} \\
0 & \\
\vdots & \\
\sum_{p_i \in N_0} \beta_{3(m_1+1)+m_2, p_i} \Delta d_{p_i} & \\
\vdots & \\
0 &
\end{bmatrix},
$$

$$
\left(\mathbf{E} + \mathbf{B}^{-1} \cdot \sum_{p_i \in N_0} \Delta d_{p_i} \cdot \overline{\mathbf{E}}_{p_i}\right)^{-1} = \mathbf{M}_p + \mathbf{E} ,
$$

where

$$\mathbf{M}_p \equiv \begin{bmatrix} \dfrac{1}{\sum_{p_i \in N_0} \beta_{1p_i} \Delta d_{p_i} + 1} - 1 & \\ \dfrac{-\sum_{p_i \in N_0} \beta_{2p_i} \Delta d_{p_i}}{\sum_{p_i \in N_0} \beta_{1p_i} \Delta d_{p_i} + 1} & \mathbf{0}_{3(m_1+1)+m_2,\,3(m_1+1)+m_2-1} \\ \vdots & \\ \dfrac{-\sum_{p_i \in N_0} \beta_{3(m_1+1)+m_2,p_i} \Delta d_{p_i}}{\sum_{p_i \in N_0} \beta_{1p_i} \Delta d_{p_i} + 1} & \end{bmatrix}.$$

Since $\overline{\mathbf{B}}^{-1} = \left(\mathbf{M}_p + \mathbf{E}\right) \mathbf{B}^{-1}$,

$$\begin{aligned} \lambda_{max}(\Delta d) &= \mathbf{C}_{\overline{B}}^{t} \overline{\mathbf{B}}^{-1} \overline{\mathbf{b}} \\ &= \mathbf{C}_{B}^{t} \overline{\mathbf{B}}^{-1} \overline{\mathbf{b}} \\ &= \mathbf{C}_{B}^{t} \left(\mathbf{M}_p + \mathbf{E}\right) \mathbf{B}^{-1} \left(\mathbf{b}^{\#} + \sum_{p_i \in N_0} \Delta d_{p_i} \cdot \mathbf{e}_{p_i}^{*}\right) \\ &= \mathbf{C}_{B}^{t} \mathbf{B}^{-1} \mathbf{b}^{\#} + \mathbf{C}_{B}^{t} \left\{\mathbf{M}_p \mathbf{B}^{-1} \mathbf{b}^{\#} + \left(\mathbf{M}_p + \mathbf{E}\right) \mathbf{B}^{-1} \sum_{p_i \in N_0} \Delta d_{p_i} \cdot \mathbf{e}_{p_i}^{*}\right\} \\ &= \lambda_{max} + \mathbf{C}_{B}^{t} \left\{\mathbf{M}_p \mathbf{B}^{-1} \mathbf{b}^{\#} + \left(\mathbf{M}_p + \mathbf{E}\right) \mathbf{B}^{-1} \sum_{p_i \in N_0} \Delta d_{p_i} \cdot \mathbf{e}_{p_i}^{*}\right\}. \end{aligned}$$

Theorem 2.2 If we suppose there is no exchange of the basis column of **B** and we have more than two deviations of the width of fuzziness, then we can obtain the following relation:

$$\Delta\lambda\left(\Delta d_{p_1}, \Delta d_{p_2}, \cdots, \Delta d_{p_{n_0}}\right) = \lambda_{max} + \mathbf{C}_B^t\left\{\mathbf{M}_p\ \mathbf{B}^{-1}\mathbf{b}^{\#} + (\mathbf{M}_p + \mathbf{E})\ \mathbf{B}^{-1}\sum_{p_i \in N_0} \Delta d_{p_i}\cdot \mathbf{e}_{p_i}^*\right\}. \tag{2.10}$$

Example 2.2. We will show the case of the sensitivity analysis concerning more than two deviations of fuzziness on the problem of Example 2.1. We consider the case that

$$\Delta d_1 = +2 \ , \ \Delta d_2 = -1 \ , \ \Delta d_4 = +3 .$$

We can solve this problem by Theorem 2.2.

$$\Delta\lambda(\Delta d_1, \Delta d_2, \Delta d_4) \equiv \lambda^{o}(\Delta d_1, \Delta d_2, \Delta d_4) - \lambda^{o}$$

$$= \mathbf{C}_B^t\left\{\mathbf{M}_p\ \mathbf{B}^{-1}\mathbf{b}^{\#} + (\mathbf{M}_p + \mathbf{E})\ \mathbf{B}^{-1}\left(\Delta d_1\cdot \mathbf{e}_1^* + \Delta d_2\cdot \mathbf{e}_2^* + \Delta d_4\cdot \mathbf{e}_4^*\right)\right\}.$$

The only difference from the case of Example 2.1 is on the matrix $\mathbf{M}_p$, where

$$\mathbf{M}_p \equiv \begin{bmatrix} \dfrac{1}{\beta_{11}\Delta d_1 + \beta_{12}\Delta d_2 + \beta_{14}\Delta d_4 + 1} - 1 & \\ \dfrac{-(\beta_{21}\Delta d_1 + \beta_{22}\Delta d_2 + \beta_{24}\Delta d_4)}{\beta_{11}\Delta d_1 + \beta_{12}\Delta d_2 + \beta_{14}\Delta d_4 + 1} & \mathbf{0}_{3(m_1+1)+m_2,\,3(m_1+1)+m_2-1} \\ \vdots & \\ \dfrac{-(\beta_{13,1}\Delta d_1 + \beta_{13,2}\Delta d_2 + \beta_{13,4}\Delta d_4)}{\beta_{11}\Delta d_1 + \beta_{12}\Delta d_2 + \beta_{14}\Delta d_4 + 1} & \end{bmatrix},$$

$$\mathbf{M}_p = \begin{bmatrix} -0.305 & \\ 2.582 & \\ -0.555 & \\ 0.138 & \\ -0.602 & \\ 0.138 & \mathbf{0}_{13,12} \\ -0.324 & \\ -0.138 & \\ 0 & \\ 0 & \\ 0.555 & \\ 0.092 & \end{bmatrix}.$$

Since we may use the previous result of $\mathbf{B}^{-1}$ and $\mathbf{b}^{\#}$, the first term of $\Delta\lambda$ is as follows:

$$\mathbf{C}_B^t \mathbf{M}_p \mathbf{B}^{-1}\mathbf{b}^{\#} = -0.1833 .$$

We seek the second term of $\Delta\lambda$ as follows:

$$\Delta d_1 \cdot \mathbf{e}_1^* + \Delta d_2 \cdot \mathbf{e}_2^* + \Delta d_4 \cdot \mathbf{e}_4^*$$

$$= +2[1\,0\,0\,0 \;\vdots\; 0\,0\,0\,0 \;\vdots\; 1\,0\,0\,0 \;\vdots\; 0]^t - 1[0\,1\,0\,0 \;\vdots\; 0\,0\,0\,0 \;\vdots\; 0\,1\,0\,0 \;\vdots\; 0]^t$$

$$+3[0\,0\,0\,1 \;\vdots\; 0\,0\,0\,0 \;\vdots\; 0\,0\,0\,1 \;\vdots\; 0]^t$$

$$= [2\,{-1}\,0\,3 \;\vdots\; 0\,0\,0\,0 \;\vdots\; 2\,{-1}\,0\,3 \;\vdots\; 0]^t,$$

and then

$$\mathbf{C}_B^t (\mathbf{M}_p + \mathbf{E})\, \mathbf{B}^{-1}(\Delta d_1 \cdot \mathbf{e}_1^* + \Delta d_2 \cdot \mathbf{e}_2^* + \Delta d_4 \cdot \mathbf{e}_4^*) = 0.3055.$$

Hence,

$$\Delta\lambda = -0.1833 + 0.3055 = 0.1222 .$$

On the other hand, we can obtain the same result by LINDO as follows:

$$\Delta\lambda = \overline{\lambda^o} - \lambda^o = 0.7222 - 0.6000 = 0.1222 .$$

2.2 SENSITIVITY ANALYSIS (TYPE 2)

We consider the sensitivity analysis of Fuzzy Linear Programming (Type 2) in this section. The definition of the sensitivity analysis is the same as that of Type 2; however, in this case, the deviation is concerning parameters of fuzzy numbers such as the width of fuzziness. We have already obtained an Equivalent Linear Programming problem on Type 2 as (1.4). The main content can be found in [8] and [15].

Problem: Max λ

$$\text{s.t } \{\alpha_i - (1-h)\mathbf{w}_i\}\mathbf{X} + s_i \geq b_i , \qquad i = 1, \cdots, m,$$
$$d_i\lambda + s_i \leq d_i ,$$
$$\mathbf{X} \geq \mathbf{0}.$$

We introduce some nonnegative slack variables u_i and t_i , then we can obtain the following L.P. problem in a standard form.

Problem: Max λ

$$\text{s.t. } \{\alpha_i - (1-h)\mathbf{w}_i\}\mathbf{X} + s_i - u_i = b_i \quad , \quad i = 1, \cdots, m$$
$$d_i\lambda + s_i + t_i = d_i , \tag{2.11}$$
$$\lambda , s_i , u_i , d_i \geq 0, \qquad \mathbf{X} \geq \mathbf{0} , \qquad i = 1, \cdots, m.$$

The extended coefficient matrix **L** of the above problem is represented as follows:

L =

$$\begin{bmatrix} 1 & -1 & \mathbf{0}_{1,n} & \mathbf{0}_{1,m} & \mathbf{0}_{1,m} & \mathbf{0}_{1,m} \\ \mathbf{0}_{m,1} & \mathbf{0}_{m,1} & \{\alpha - (1-h)\mathbf{w}\}_{m,n} & \mathbf{I}_{m,m} & -\mathbf{I}_{m,m} & \mathbf{0}_{m,m} \\ \mathbf{0}_{m,1} & \mathbf{d}_{m,1} & \mathbf{0}_{m,n} & \mathbf{I}_{m,m} & \mathbf{0}_{m,m} & \mathbf{I}_{m,m} \end{bmatrix} , \tag{2.12}$$

where **0** is a zero matrix and **I** is an identity matrix with some dimension.

We can write the system equation as

$$\mathbf{L}\,\mathbf{X} = \mathbf{g},$$

where

$$\mathbf{X} = \left[\lambda_{1,1} \;\; \lambda_{1,1} \;\; \mathbf{X}_{n,1} \;\; \mathbf{s}_{m,1} \;\; \mathbf{u}_{m,1} \;\; \mathbf{t}_{m,1} \right]^t \text{ and } \mathbf{g} = \left[0_{1,1} \;\; \mathbf{b}_{m,1} \;\; \mathbf{d}_{m,1} \right]^t.$$

We suppose there exist 2m +1 independent columns among $\mathbf{L}$. We can pick up 2m+1 columns from the extended matrix of $\mathbf{L}$ and make the basis matrix $\mathbf{B}$ in order to solve the system equation of $\mathbf{L}\,\mathbf{X} = \mathbf{g}$ with the method of the revised simplex computational procedure. Let the width of fuzziness w_{kl} be reduced by Δw_{kl} and the corresponding column vector varies as follows:

$$\bar{\mathbf{e}}^{l+1} = \mathbf{e}^{l+1} + (1\text{-}h)\,\Delta w_{kl}\,\mathbf{I}_k,$$

where $\mathbf{e}^{l+1}$ is the l+1-th column vector of $\mathbf{L}$ and $\mathbf{I}_k \equiv [\,0,\cdots,0,\;\; 1_{k\text{-th}}\;\; ,0,\cdots,0\,]^t$. We number columns and rows from zero. Let

$$\gamma^j \equiv \mathbf{B}^{-1}\,\mathbf{e}^j,$$

then,

$$\bar{\gamma}^{l+1} = \mathbf{B}^{-1}\,\bar{\mathbf{e}}^{l+1}$$
$$= \mathbf{B}^{-1}\,\mathbf{e}^{l+1} + (1\text{-}h)\,\Delta w_{kl}\,\mathbf{B}^{-1}\,\mathbf{I}_k\,.$$

Let $\mathbf{B}^{-1}\,\mathbf{e}^{l+1} = \mathbf{I}_p$, namely, $\mathbf{e}^{l+1}$ is the p-th independent vector of $\mathbf{B}$. This procedure is necessarily to find out which columns of matrix $\mathbf{L}$ can become basis columns. $\mathbf{e}^{l+1}$ is not necessarily to be the l+1-th basis column. We can express the above formula by using a unit vector as follows:

$$\bar{\gamma}^{l+1} = \mathbf{I}_p + (1\text{-}h)\,\Delta w_{kl}\,\mathbf{B}^{-1}\,\mathbf{I}_k, \quad \mathbf{B}^{-1}\,\mathbf{I}_k \equiv \Pi^k = [\Pi_{0,k}, \cdots, \Pi_{2m,k}]^t.$$

We can choose $\mathbf{e}^j$ so that $\mathbf{B}^{-1}\,\mathbf{e}^j = \mathbf{I}_{v\neq p}$ and we can make the rest of $\mathbf{I}_v$. However, we have to select as many basis columns from $\mathbf{L}$ not corresponding to slack variables as we can, since we would like to avoid losing information contained in a system. We make the inverse matrix from all the basis columns which are transformed the original columns into unit vectors in $\mathbf{L}$ including the perturbed column as follows:

$$\overline{\mathbf{B}} \equiv \left[\mathbf{I} + (1-h)\,\Delta w_{kl}\,\mathbf{B}^{-1}\,\mathbf{I}_{kp}\right]^{-1}.$$

Perturbation term is expressed as follows:

$$\Delta w_{kl}\,\mathbf{B}^{-1}\,\mathbf{I}_{kp} = \begin{bmatrix} 0\cdots 0 & \Pi_{0k}\Delta w_{kl} & 0\cdots 0 \\ & \Pi_{1k}\Delta w_{kl} & \\ & \vdots & \\ & \vdots & \\ 0\cdots 0 & \Pi_{2m,k}\Delta w_{kl} & 0\cdots 0 \end{bmatrix},$$

where the matrix $\mathbf{I}_{kp}$ is a null matrix except for the element (k, l) which equals a unity and only the p-th column is nonzero in the above matrix.

All the basis columns gathered after transformation into unit vectors can be written as follows:

$$\mathbf{I} + (1-h)\,\Delta w_{kl}\,\mathbf{B}^{-1}\,\mathbf{I}_{kp} = \begin{bmatrix} 1\,0\cdots & (1-h)\Pi_{0k}\Delta w_{kl} & \cdots 0 \\ & (1-h)\Pi_{1k}\Delta w_{kl} & \\ & \vdots & \\ & 1+(1-h)\Pi_{p,k}\Delta w_{kl} & \\ & \vdots & \\ & (1-h)\Pi_{p+1,k}\Delta w_{kl} & \\ 0\cdots & (1-h)\Pi_{2m,k}\Delta w_{kl} & \cdots 0\,1 \end{bmatrix}.$$

The inverse of this matrix is given by

$$\left(\mathbf{I} + (1\text{-h})\,\Delta w_{kl}\,\mathbf{B}^{-1}\,\mathbf{I}_{kp}\right)^{-1} = \begin{bmatrix} 1\;0\cdots & \dfrac{-(1\text{-h})\Pi_{0k}\Delta w_{kl}}{1+(1\text{-h})\Pi_{p,k}\Delta w_{kl}} & \cdots 0 \\ & \dfrac{-(1\text{-h})\Pi_{1k}\Delta w_{kl}}{1+(1\text{-h})\Pi_{p,k}\Delta w_{kl}} & \\ & \vdots & \\ & \dfrac{1}{1+(1\text{-h})\Pi_{p,k}\Delta w_{kl}} & \\ & \vdots & \\ & \dfrac{-(1\text{-h})\Pi_{p+1,k}\Delta w_{kl}}{1+(1\text{-h})\Pi_{p,k}\Delta w_{kl}} & \\ 0\cdots & \dfrac{-(1\text{-h})\Pi_{2m,k}\Delta w_{kl}}{1+(1\text{-h})\Pi_{p,k}\Delta w_{kl}} & \cdots 0\;1 \end{bmatrix},$$

where Π_{ij} is the (i,j) element of the basis inverse matrix $\mathbf{B}^{-1}$. We assume that the deviation of w_{kl} does not change the basic variable as the case of Type 1. We want to seek the optimal solution after the deviation of w_{kl} . First of all, we can obtain the optimal solution before the deviation Δw_{kl} as follows:

$$\mathbf{X}^{o} = \mathbf{B}^{-1}\,\mathbf{g}. \tag{2.13}$$

The new basis matrix is given by

$$\hat{\mathbf{B}} = \mathbf{B} + (1\text{-h})\,\Delta w_{kl}\,\mathbf{I}_{kp}, \tag{2.14}$$

where we have to keep the following condition, to preserve the feasibility

$$\widehat{\mathbf{X}^{o}} = \mathbf{B}^{-1}\,\mathbf{g} \geq \mathbf{0},$$

and to preserve the optimality

$$z_j - c_j = \mathbf{c}_0 \widehat{\mathbf{B}}^{-1} \mathbf{P}_j - c_j \leq 0,$$

where $\mathbf{c}_0 = [\,1 \;\; 1 \;\; 0 \cdots 0\,]$ and $\mathbf{P}_j$ is the vector of the optimal basis. On the other hand, from (2.14),

$$\begin{aligned}\widehat{\mathbf{B}} &= \mathbf{B} + (1\text{-}h)\,\Delta w_{kl}\,\mathbf{I}_{kp} \\ &= \mathbf{B}\left(\mathbf{I} + (1\text{-}h)\,\Delta w_{kl}\,B^{-1}\,\mathbf{I}_{kp}\right),\end{aligned}$$

and then,

$$\widehat{\mathbf{B}}^{-1} = \left(\mathbf{I} + (1\text{-}h)\,\Delta w_{kl}\,\mathbf{B}^{-1}\,\mathbf{I}_{kp}\right)^{-1}\mathbf{B}^{-1}.$$

Therefore,

$$\begin{aligned}\widehat{\mathbf{B}}^{-1}\mathbf{g} &= \left(\mathbf{I} + (1\text{-}h)\,\Delta w_{kl}\,\mathbf{B}^{-1}\,\mathbf{I}_{kp}\right)^{-1}\mathbf{B}^{-1}\mathbf{g} \\ &= \left(\mathbf{I} + (1\text{-}h)\,\Delta w_{kl}\,\mathbf{B}^{-1}\,\mathbf{I}_{kp}\right)^{-1}\mathbf{X}^{o} \\ &= \left(\mathbf{I} + (1\text{-}h)\,\Delta w_{kl}\,\Pi^{kp}\right)^{-1}\mathbf{X}^{o},\end{aligned}$$

where $\Pi^{kp} = \left[\mathbf{0}_{2m+1,1}, \cdots, \Pi^{k}{}_{p\text{-th}}, \cdots, \mathbf{0}_{2m+1,1}\right].$

Hence, we can obtain the new optimal solution as follows:

$$\widehat{\mathbf{X}^{o}} = \left(\mathbf{I} + (1\text{-}h)\,\Delta w_{kl}\,\Pi^{kp}\right)^{-1}\mathbf{X}^{o}$$

$$
= \begin{bmatrix}
1\,0\cdots & \frac{-(1-h)\Pi_{0k}\Delta w_{kl}}{1+(1-h)\Pi_{p,k}\Delta w_{kl}} & \cdots 0 \\
& \frac{-(1-h)\Pi_{1k}\Delta w_{kl}}{1+(1-h)\Pi_{p,k}\Delta w_{kl}} & \\
& \vdots & \\
& \frac{1}{1+(1-h)\Pi_{p,k}\Delta w_{kl}} & \\
& \vdots & \\
& \frac{-(1-h)\Pi_{p+1,k}\Delta w_{kl}}{1+(1-h)\Pi_{p,k}\Delta w_{kl}} & \\
0\cdots & \frac{-(1-h)\Pi_{2m,k}\Delta w_{kl}}{1+(1-h)\Pi_{p,k}\Delta w_{kl}} & \cdots 0\,1
\end{bmatrix}
\begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \vdots \\ \beta_{2m} \end{bmatrix} .
$$

We rewrite the basic solution of (2.13) as $\boldsymbol{\beta} = \left[\beta_0 \cdots \beta_{2m}\right]^t$.

$$
\hat{\boldsymbol{\beta}} = \overline{\mathbf{B}}\,\boldsymbol{\beta} ,
$$

$$
\beta_i' = \beta_i - \frac{(1-h)\Pi_{ik}\Delta w_{kl}}{1+(1-h)\Pi_{p,k}\Delta w_{kl}}\beta_p ,
$$

where β_i' is the i-th element of $\hat{\boldsymbol{\beta}}$.

From these expressions,

$$
\Delta\lambda = \beta_0' - \beta_0 = \frac{-(1-h)\Pi_{0k}\Delta w_{kl}}{1+(1-h)\Pi_{p,k}\Delta w_{kl}}\beta_p .
$$

Theorem 2.3 We suppose there is no exchange of the basis column of **B** with the optimal solution, then the sensitivity to the width of fuzziness can be obtained as follows:

$$\Delta\lambda(\Delta w_{kl}) = \beta_0' - \beta_0 = \frac{-(1-h)\Pi_{0k}\Delta w_{kl}}{1+(1-h)\Pi_{p,k}\Delta w_{kl}}\beta_p. \qquad (2.15)$$

Example 2.3. We consider the sensitivity analysis on the following problem.

F.L.P. problem : $\tilde{5}\,x_1 + \tilde{3}\,x_2 \tilde{\lesssim} 13,$

(Type 2) $\tilde{2}\,x_1 + 7\,x_2 \tilde{\lesssim} 17,$

$-x_1 - x_2 \tilde{\lesssim} -3\ ,$

which is expressed by the following vector forms of fuzzy numbers.

$$\widetilde{\mathbf{A}_1}\mathbf{X} \tilde{\lesssim} b_1 \ ,\quad \widetilde{\mathbf{A}_1} = \{\boldsymbol{\alpha}_1 = (5,3)\ ,\ \mathbf{w}_1 = (2,1)\}\ ,$$
$$\widetilde{\mathbf{A}_2}\mathbf{X} \tilde{\lesssim} b_2 \ ,\quad \widetilde{\mathbf{A}_2} = \{\boldsymbol{\alpha}_2 = (2,7)\ ,\ \mathbf{w}_2 = (1,0)\}\ ,$$
$$\widetilde{\mathbf{A}_3}\mathbf{X} \tilde{\lesssim} b_3 \ ,\quad \widetilde{\mathbf{A}_3} = \{\boldsymbol{\alpha}_3 = (-1,-1)\ ,\ \mathbf{w}_3 = (0,0)\}\ .$$

This problem can be rewritten in a standard form by (2.11).

$$\begin{aligned}
&\text{Max } \lambda \\
&\text{s.t. } 4.2\,x_1 + 2.6\,x_2 + s_1 - u_1 = 13, \\
&\quad 1.6\,x_1 + 7\,x_2 + s_2 - u_2 = 17, \\
&\quad -x_1 - x_2 + s_3 - u_3 = -3, \\
&\quad 3\,\lambda + s_1 + t_1 = 3, \\
&\quad \lambda + s_2 + t_2 = 1, \\
&\quad \lambda + s_3 + t_3 = 1,
\end{aligned}$$

where h-level is given 0.6 as the prior information.

Let's examine the sensitivity analysis, if we change the width of fuzziness of $\mathbf{w}_1$'s first element which is reduced by $\Delta w_{11} = -1.5$, namely, decreasing fuzziness on a fuzzy number $\tilde{5}$. First, we can directly solve this problem to compare the result before changing fuzziness with that after decreasing fuzziness by LINDO. The result of the sensitivity analysis is as follows:

$$\Delta\lambda = \lambda' - \lambda^o = 0.597 - 0.496 = 0.101 .$$

Next, we solve this problem by Theorem 2.3. We need some information in order to use this method, such as the extended matrix **L** and the basis matrix **B** and the basic solution of the original problem. The extended coefficient matrix **L** is obtained from (2.12).

$$\mathbf{L} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4.2 & 2.6 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.6 & 7 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The basis matrix **B** can be obtained from the results by LINDO.

$$\mathbf{B} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4.2 & 2.6 & 1 & 0 & 0 \\ 0 & 0 & 1.6 & 7 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The inverse matrix of **B** is

$$B^{-1} = \begin{bmatrix} 1 & -0.125 & -0.037 & -0.586 & 0.125 & 0.037 & 0.586 \\ 0 & -0.125 & -0.037 & -0.586 & 0.125 & 0.037 & 0.586 \\ 0 & 0.185 & -0.130 & -0.427 & -0.185 & 0.130 & 0.427 \\ 0 & -0.060 & 0.167 & 0.013 & 0.060 & -0.167 & -0.013 \\ 0 & 0.376 & 0.111 & 1.759 & 0.623 & -0.111 & -1.759 \\ 0 & 0.125 & 0.037 & 0.586 & -0.125 & 0.962 & -0.586 \\ 0 & 0.125 & 0.037 & 0.586 & -0.125 & -0.037 & 0.423 \end{bmatrix}.$$

The basic solution can also be obtained by LINDO.

$$\begin{aligned} \boldsymbol{\beta}_0 &= [\beta_0\ \beta_1 \cdots \beta_6]^t \\ &= [\lambda^o\ \lambda^o\ x_1\ x_2\ s_1\ s_2\ s_3]^t \\ &= [0.496\ \ 0.496\ \ 1.486\ \ 2.016\ \ 1.511\ \ 0.503\ \ 0.503]^t. \end{aligned}$$

Since $\Delta w_{kl} = \Delta w_{11}$, this case is given as $k = 1$, $l = 1$. We want to seek the number of subscript p to obtain Π_{pk}.

$$\begin{aligned} \mathbf{I}_p &\equiv \mathbf{B}^{-1}\mathbf{e}^{l+1} \\ &= \mathbf{B}^{-1}\mathbf{e}^{2} \\ &= [0\ 0\ 1\ 0\ 0\ 0\ 0]^t. \end{aligned}$$

Therefore, $p = 2$. We obtained all the data to execute the sensitivity analysis as follows:

$$h = 0.6\ ,\ \Delta w_{11} = -1.5\ ,\ \Pi_{01} = -0.125\ ,\ \Pi_{21} = 0.185\ ,\ \beta_2 = 1.486\ .$$

Hence, by (2.15),

$$\Delta\lambda(\Delta w_{11} = 1.5) = \frac{-(1-h)\Pi_{01}\Delta w_{11}}{1 + (1-h)\Pi_{21}\Delta w_{11}}\beta_2 = 0.101 .$$

In addition to this result, and by LINDO, we can easily check that the basis vectors do not change by decreasing fuzziness on this case. Finally, we will show the results with some h-level cases in Figure 2.2. (See Page 56)

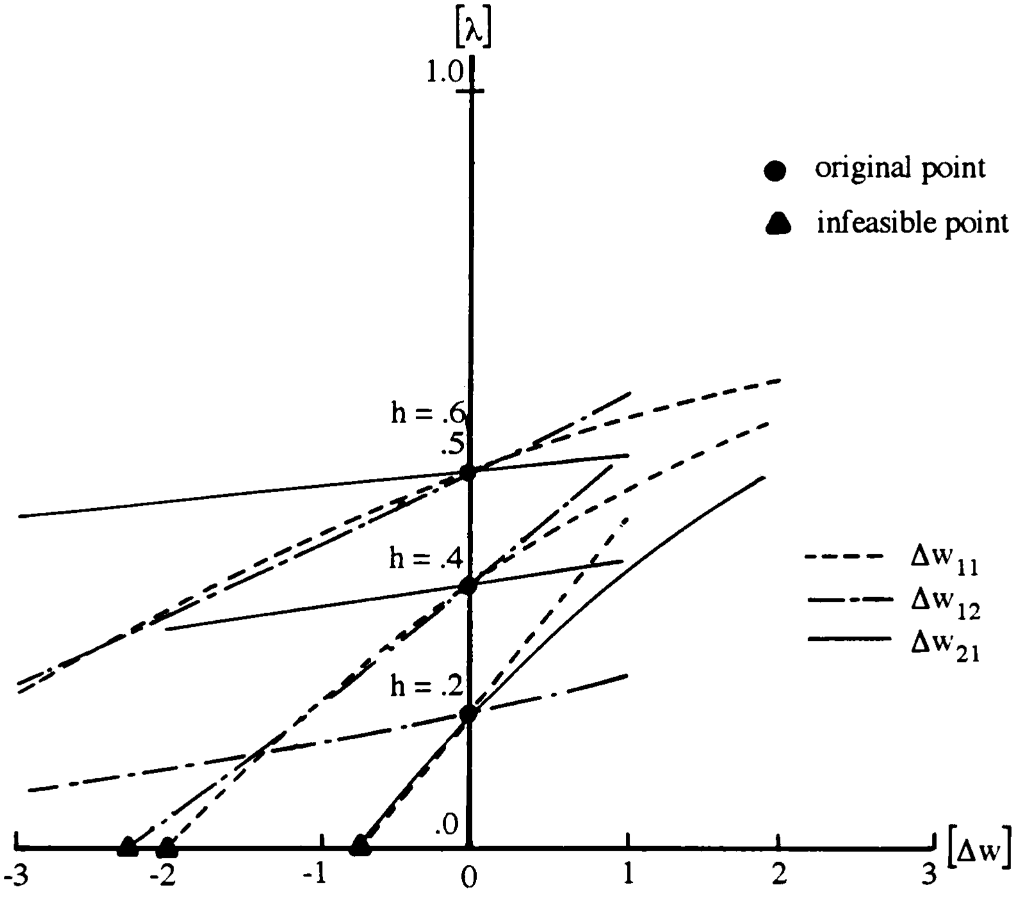

Figure 2.2 The Result of the Sensitivity Analysis of Example 2.3

2.3 STABILITY IN FUZZY LINEAR PROGRAMMING

In this section, we investigate the stability of the solution in Fuzzy Linear Programming with respect to the variation of the fuzzy parameter, α , the center value of a fuzzy number. Stability means that the amount of the deviation of the optimal solution can be bounded within the sufficiently small amount of variation corresponding to the deviation of the perturbed parameter.The basic concept can be found in [9] and [10]. First we set Fuzzy Linear Programming as follows:

$$\text{Problem: } \quad \tilde{\mathbf{a}}_i \mathbf{x} \mathrel{\tilde{<}} b_i \quad , \quad i = 0, 1, \cdots, m, \tag{2.16}$$

where $\tilde{\mathbf{a}}_i = (\tilde{a}_{i_1}, \cdots, \tilde{a}_{i_n})$ and $\mathbf{x} = (x_1, \cdots, x_n)$.

Let us apply Theorem 1.7 for this problem. From the proof of Theorem 1.8, we can rewrite the left hand term of this problem as

$$\tilde{y}_i = \tilde{\mathbf{a}}_i \mathbf{x} = (\boldsymbol{\alpha}_i \mathbf{x}, \mathbf{w}_i |\mathbf{x}|_1), \tag{2.17}$$

where $|\mathbf{x}|_1 = (|x_1|, \cdots, |x_n|)$, and then $\tilde{b}_i = (b_i, d_i)$. By Zadeh's Principle (see [14]),

$$\tilde{y}_i - \tilde{b}_i = (\boldsymbol{\alpha}_i \mathbf{x} - b_i, \mathbf{w}_i |\mathbf{x}|_1 + d_i).$$

Hence, the membership function of this problem is represented as

$$\mu_i(\mathbf{x}) = \begin{cases} 1 & , \text{ if } \boldsymbol{\alpha}_i \mathbf{x} \le b_i \\ 1 - \dfrac{\boldsymbol{\alpha}_i \mathbf{x} - b_i}{\mathbf{w}_i |\mathbf{x}|_1 + d_i} & , \text{ if otherwise} \\ 0 & , \text{ if } \boldsymbol{\alpha}_i \mathbf{x} > b_i + \mathbf{w}_i |\mathbf{x}|_1 + d_i \end{cases}.$$

We assume that there exists the deviation from the exact center value of $\boldsymbol{\alpha}_i$ and b_i, and this deviation is small so that it could satisfy the following conditions.

$$\max_{i,j} \left| \alpha_{ij} - \alpha_{ij}(\delta) \right| \le \delta, \qquad \delta > 0,$$

$$\underset{i}{\text{Max}} \left| b_i - b_i(\delta) \right| \le \delta .$$

We consider the optimal solution of the following problem in which we change the center value of a fuzzy number.

$$\text{Problem : } \tilde{\mathbf{a}}_i^\delta \mathbf{x} \precsim \tilde{b}_i^\delta \quad , \quad i = 0, 1, \cdots, m , \tag{2.18}$$

where $\tilde{\mathbf{a}}_i^\delta = \left(\widetilde{\alpha_i(\delta)} , \mathbf{w}_i \right)$, $\tilde{b}_i^\delta = \left(\widetilde{b_i(\delta)} , d_i \right)$.

From Definition 1.3(b), we can obtain the optimal solution as follows:

$$\mu^\delta(\mathbf{x}) = \underset{i=0,\cdots,m}{\text{Min}} \mu_i^\delta(\mathbf{x}) ,$$

$$\mu^\delta(\mathbf{x}^*) = \underset{X \in R^n}{\text{Max}} \mu^\delta(\mathbf{X}) .$$

Theorem 2.4 Let $\mu(\mathbf{X})$ be the solution of (2.17) and $\mu^\delta(\mathbf{X})$ be the solution of (2.18), and $0 < \mu_i(\mathbf{X}) < 1$, $0 < \mu_i^\delta(\mathbf{X}) < 1$.
Then,

$$\| \mu_i(\mathbf{X}) - \mu_i^\delta(\mathbf{X}) \|_1 = \sup_{X \in R^n} \left| \mu(\mathbf{X}) - \mu^\delta(\mathbf{X}) \right| < \delta \left(\frac{1}{w} + \frac{1}{d} \right),$$

where $d = \underset{i}{\text{Min}}\, d_i$ and $w = \underset{ij}{\text{Min}}\, w_{ij}$, $d_i , w_{ij} > 0$.

Proof:

$$\| \mu_i(\mathbf{X}) - \mu_i^\delta(\mathbf{X}) \|_1 = \left| \mu(\mathbf{X}) - \mu^\delta(\mathbf{X}) \right|$$

$$= \left| 1 - \frac{\alpha_i \mathbf{x} - b_i}{\mathbf{w}_i |\mathbf{x}|_1 + d_i} - \left[1 - \frac{\alpha_i(\delta) \mathbf{x} - b_i(\delta)}{\mathbf{w}_i |\mathbf{x}|_1 + d_i} \right] \right|$$

$$= \frac{\left| b_i - b_i(\delta) + \alpha_i(\delta) \mathbf{X} - \alpha_i \mathbf{X} \right|}{\mathbf{w}_i |\mathbf{x}|_1 + d_i}$$

$$\le \frac{\left| b_i - b_i(\delta) \right| + \left| \alpha_i(\delta) \mathbf{X} - \alpha_i \mathbf{X} \right|}{\mathbf{w}_i |\mathbf{x}|_1 + d_i}$$

$$\leq \frac{\delta + \| \boldsymbol{\alpha}_i(\delta) - \boldsymbol{\alpha}_i \|_\infty |\mathbf{x}|_1}{\mathbf{w}_i |\mathbf{x}|_1 + d_i}$$

$$\leq \delta \left(\frac{1 + |\mathbf{x}|_1}{\mathbf{w}_i |\mathbf{x}|_1 + d_i} \right)$$

$$\leq \delta \left(\frac{1}{w} + \frac{1}{d} \right).$$

Since we can guarantee the stability on the deviation of the center value, we introduce the sensitivity analysis on the center value of a fuzzy number, α. We consider Problem (1.4). The extended coefficient matrix $\mathbf{L}$ of that problem is given in (2.12). We take a small change on $\boldsymbol{\alpha}$ instead on $\mathbf{w}$ in the previous section. We can develop this problem like in the previous Section 2.2. Let the center value α_{kl} deviate by $\Delta\alpha_{kl.}$ An element of the matrix $\mathbf{L}$ changes as follows:

$$\bar{\mathbf{e}}^{l+1} = \mathbf{e}^{l+1} + \Delta\alpha_{kl}\, \mathbf{I}_k,$$

where $\Delta\alpha_{kl}$ is the deviation of the k-th row's element of $\boldsymbol{\alpha}$ in $\mathbf{L}$.

As we have already known the similar procedure in the case of the width of fuzziness, we can find out the following real basis column which is affected by the deviation of $\boldsymbol{\alpha}$.

$$\bar{\boldsymbol{\gamma}}^{l+1} = \mathbf{B}^{-1}\, \bar{\mathbf{e}}^{l+1} + \Delta\alpha_{kl}\, \mathbf{B}^{-1}\, \mathbf{I}_k$$

$$= \mathbf{I}_p + \Delta\alpha_{kl}\, \mathbf{B}^{-1}\, \mathbf{I}_k.$$

Let

$$\overline{\mathbf{B}} \equiv \left[\mathbf{I} + \Delta\alpha_{kl}\, \mathbf{B}^{-1}\, \mathbf{I}_{kp} \right]^{-1}.$$

Since

$$\mathbf{I} + \Delta\alpha_{kl}\, \mathbf{B}^{-1}\, \mathbf{I}_{kp} = \begin{bmatrix} 1 \cdots 0 & \Pi_{0k}\Delta\alpha_{kl} & 0 \cdots 0 \\ & \Pi_{1k}\Delta\alpha_{kl} & \\ & \vdots & \\ & \vdots & \\ 0 \cdots 0 & \Pi_{2m,k}\Delta\alpha_{kl} & 0 \cdots 1 \end{bmatrix},$$

then

$$\left(\mathbf{I} + \Delta\alpha_{kl}\,\mathbf{B}^{-1}\,\mathbf{I}_{kp}\right)^{-1} = \begin{bmatrix} 1\,0\cdots & \dfrac{-\Pi_{0k}\Delta\alpha_{kl}}{1+\Pi_{p,k}\Delta\alpha_{kl}} & \cdots 0 \\ & \dfrac{-\Pi_{1k}\Delta\alpha_{kl}}{1+\Pi_{p,k}\Delta\alpha_{kl}} & \\ & \vdots & \\ & \dfrac{1}{1+\Pi_{p,k}\Delta\alpha_{kl}} & \\ & \vdots & \\ & \dfrac{-\Pi_{p+1,k}\Delta\alpha_{kl}}{1+\Pi_{p,k}\Delta\alpha_{kl}} & \\ 0\cdots & \dfrac{-\Pi_{2m,k}\Delta\alpha_{kl}}{1+\Pi_{p,k}\Delta\alpha_{kl}} & \cdots 0\,1 \end{bmatrix}.$$

Hence, we can get the optimal solution $\widehat{\mathbf{X}^{o}}$ for the deviation of $\boldsymbol{\alpha}$.

$$\widehat{\mathbf{X}^{o}} = \left(\mathbf{I} + \Delta\alpha_{kl}\,\mathbf{B}^{-1}\,\mathbf{I}_{kp}\right)^{-1}\mathbf{X}^{o},$$

$$\beta_i' = \beta_i - \frac{\Pi_{ik}\Delta\alpha_{kl}}{1+\Pi_{p,k}\Delta\alpha_{kl}}\beta_p.$$

Theorem 2.5 We suppose there is no exchange of the basis column of **B** with the optimal solution, then the sensitivity to the center value of a fuzzy number can be obtained as follows:

$$\Delta\lambda(\Delta\alpha_{kl}) = \beta_0' - \beta_0 = \frac{-\Pi_{0k}\Delta\alpha_{kl}}{1+\Pi_{p,k}\Delta\alpha_{kl}}\beta_p.$$

2.4 SENSITIVITY ANALYSIS AND INFORMATION VALUE

If we are given all the coefficients of a system exactly, we are likely thinking that we don't need to use fuzzy numbers, fuzzy constraints and fuzzy goals; however, it is impossible to gather all the information exactly. Moreover, even if a system can be expressed exactly, it is not necessary that a feasible solution will exist. In this section, we consider the relationship between fuzziness and existence of the optimal solution as the original study. Theorem 2.7 gives us the most important reason why we should use the fuzzy programming method. Moreover, we introduce the new method on the sensitivity analysis which we call the *standardized sensitivity analysis*. In this method, we can extend the sensitivity analysis in general cases so that we do not need to specify any perturbed elements and also we can grasp the total sensitivity against the generalized amount of deviations without any consideration on the numbers of perturbed elements or any designation on specified elements in a system by using the *Information Value*.

Definition 2.1 The *Information Value* is defined as

$$I \equiv \frac{\Delta w_{ij}}{w_{ij}}, \qquad 0 \leq \Delta w_{ij} \leq w_{ij},$$

where w_{ij} is the width of fuzziness of each coefficient and Δw_{ij} is the decreasing amount of its fuzziness, namely, the deviation.

The information which gives us the exact data on $\widetilde{A}_{ij}$, namely, $w_{ij} = 0$; $(I = 1)$, is called *perfect information* on $\widetilde{A}_{ij}$, and the information which gives $\Delta w_{ij} = 0$; $(I = 0)$ is called *zero information*. The intermediate information is called *imperfect information.*

Next, we introduce the concept of the generalized sensitivity analysis by the Information Value. A sensitivity analysis, so far, has been performed in the sense that we select one or more deviations of fuzziness of coefficients arbitrarily. Meanwhile, we would like to evaluate the sensitivity analysis for more than two deviations as the standardized sensitivity analysis corresponding to the Information Value.

Definition 2.2. The Information Value for more than two deviations is defined as

$$I \equiv \frac{1}{n}\sum_{i=1}^{n} I_i\,, \text{ for some } n\,,\ 2 \le n \le N\,,$$

where n is the number of the elements of which fuzziness is reduced and N is the total number of coefficients which have fuzziness.

Definition 2.3 The *standardized sensitivity analysis* in which we have more than two deviations is defined as

$$P_n(I\,;\Delta I) = E\begin{Bmatrix} N \\ n \end{Bmatrix} \cdot E_{I=\frac{1}{n'}\sum_{i=1'}^{n'} \frac{\Delta w_{i'}}{w_{i'}}} \Delta\lambda\left(\Delta w_{i'}, \cdots, \Delta w_{n'}\right), \tag{2.20}$$

where we suppose to have N coefficients with fuzziness in a system.

We show the procedure to calculate (2.20). First of all, we pick up n coefficients among N's arbitrarily and change their n's original subscript numbers into new subscript numbers from $1'$ to n'. The special notation, $E_{I=\frac{1}{n'}\sum_{i=1}^{n'} \frac{\Delta w_{i'}}{w_{i'}}}$ means the average operator in the sense that we consider all the cases in order to take a mean value for $\Delta\lambda$ where we can change any Δw_i under fixed n elements subjected to $I = \frac{1}{n'}\sum_{i=1'}^{n'} \frac{\Delta w_{i'}}{w_{i'}}$. We suppose ΔI is the unit of disctized I; therefore, we may only consider the finite number cases to take an average. The notation, $E\begin{Bmatrix} N \\ n \end{Bmatrix}$ is also the average operator corresponding to all the cases which can be selected n coefficients among N.

We can regard the standardized sensitivity analysis as the sensitivity analysis on the total system. We do not need any consideration about the number of perturbed elements and about which elements deviate from their initial data, since these factors are generalized as the amount of the Information Value. This method is more available to investigate the sensitivity in the case that we can not know exactly which element is perturbed.

Theorem 2.7 We can obtain the standardized sensitivity analysis on every element corresponding to the fuzzy parameters such as the center value or the width of fuzziness on a system matrix with perfect or imperfect information as follows:

(i) $P_N(1, \Delta I) = \begin{Bmatrix} 0 & , \text{if } \lambda^o = 1.0 \\ \text{infeasible} & , \text{if } \lambda^o \neq 1.0 \end{Bmatrix}$

(ii) $P_N(0, \Delta I) = 0$

where λ^o is the optimal solution for the original Fuzzy Linear Programming problem in which we are given no information.

Proof (i) : From Definition 2.3,

$$P_N(1, \Delta I) = E\begin{Bmatrix} N \\ N \end{Bmatrix} \cdot E_{I=\frac{1}{N}\sum_{i=1'}^{N'} \frac{\Delta w_{i'}}{w_{i'}}=1\,;\,\Delta I} \Delta\lambda\left(\Delta w_{i'}, \cdots, \Delta w_{N'}\right).$$

Since $0 \leq \Delta w_{i'} \leq w_{i'}$ and $1.0 = I = \frac{1}{N}\sum_{i=1'}^{N} \frac{\Delta w_{i'}}{w_{i'}}$, then $\frac{\Delta w_{i'}}{w_{i'}} = 1.$ Hence, we may consider just one case to take an average value concerning $\Delta w_{i'}$.

$$P_N(1, \Delta I) = E\begin{Bmatrix} N \\ N \end{Bmatrix} \Delta\lambda\left(\Delta w_{i'} = w_{i'}, \cdots, \Delta w_{N'} = w_{N'}\right).$$

Since $\begin{Bmatrix} N \\ N \end{Bmatrix} = 1$, we just consider one case, namely, all the fuzzy numbers are picked up. Hence $P_N(1, \Delta I) = \Delta\lambda(w_{i'}, \cdots, w_{N'})$. On the other hand, $\Delta w_{i'} = w_{i'}$ means $\widetilde{A_{ij}} = \alpha_{ij}$, where α_{ij} is the center value of a fuzzy number, $\widetilde{A_{ij}}$; therefore, Fuzzy Linear Programming becomes just a crisp problem which has no fuzziness. Since $\lambda^o \neq 1.0$, $P_N(1, \Delta I)$ is infeasible.

Proof (ii): Since we have N deviations,

$$P_N(0, \Delta I) = E\begin{Bmatrix} N \\ N \end{Bmatrix} \cdot E_{I=\frac{1}{N}\sum_{i=1'}^{N'} \frac{\Delta w_{i'}}{w_{i'}}=0\,;\,\Delta I} \Delta\lambda\left(\Delta w_{i'}, \cdots, \Delta w_{N'}\right).$$

Since $I = 0$, then every $\Delta w_{i'} = 0$, which means no deviation corresponding to every fuzzy number. Therefore, we can say

$$\begin{aligned}\Delta\lambda &= \lambda\left(\Delta w_{i'},\cdots,\Delta w_{N'}\right)-\lambda^{o}\\ &= \lambda(0,\cdots,0)-\lambda^{o}\\ &= \lambda^{o}-\lambda^{o}\\ &= 0.\end{aligned}$$

Hence,

$$P_N\left(0\,,\Delta I\right) = 0. \qquad \text{Q.E.D.}$$

Theorem 2.7 is important in a sense that it means Fuzzy Linear Programming gives us a compromised solution even if constraints and goals can not be satisfied at the same time. Fuzziness of each coefficient acts an important role to avoid a infeasible solution, because losing the fuzziness causes to vanish the intersection area in which all the conditions are permitted. If there exists this area before losing fuzziness, we do not need to use a fuzzy method, because $\lambda^{o} = 1.0$ means that we do not need to seek the compromised solution. On the other hand, increasing fuzziness decreases exactness of a solution; therefore, how we chose fuzziness for each coefficient is a problem which is left to be solved in the future for us. Moreover, this method is complicated for computation; therefore, we have to study on how to calculate this value as the number of grid points which is corresponding to the difference of the Information Value goes to infinite in the future.

Example 2.4. We consider the same case as Example 2.3. By Definition 2.3, we seek $P_2\left(I\,;\Delta I = 0.1\right)$. Since coefficients with fuzziness are $\widetilde{A_{11}}$, $\widetilde{A_{12}}$ and $\widetilde{A_{21}}$. We can pick up all the cases for $\left\{ {3 \atop 2} \right\}$ as follows:

$$\left\{ {3 \atop 2} \right\} = \{\, (w_{11}\,,w_{12}),(w_{12}\,,w_{21}),(w_{21}\,,w_{11})\,\}.$$

Since the difference is given as $\Delta I = 0.1$, we can determine $\Delta w_{i'}$ for each case. If I is given as I_0, then $\Delta w_{i'}$ can take the following values. We rename $(\,w_{11}\cdot w_{12}\,)$ as $\left(\Delta w_{1'},\Delta w_{2'}\right)$. $\Delta w_{1'}$ can take some values corresponding to $I\,,I-\Delta I\,,I-2\Delta I\,,\cdots,I-\left(\frac{I}{\Delta I}-1\right)\Delta I\,,0$ as $\Delta w_{2'}$ takes some values corresponding to $0\,,\Delta I\,,2\Delta I\,,\cdots,\left(\frac{I}{\Delta I}-1\right)\Delta I\,,I$. We take an average of all the combinations under the fixed Information Value. We can obtain $\lambda(\Delta w_1', \Delta w_2')$ on each fixed value of I by LINDO or vector calculation in Theorem 2.3. The last thing is taking an average on all the cases which have the same value of I. The result of the relationship between $\Delta\lambda$ and I is expressed as follows.

$$\frac{\Delta\lambda}{I} \cong 1.64,$$

$$\lambda \cong 0.5 + 1.64\,I, \quad 0 < I < 1.0.$$

CHAPTER 3

THE APPROXIMATION OF FUZZY NONLINEAR PROGRAMMING

3.1 FUZZY LINEAR PROGRAMMING WITH NONLINEAR MEMBERSHIP FUNCTION

In this chapter, we deal with Fuzzy Linear Programming (Type 1) with nonlinear membership functions. Fuzzy Linear Programming was introduced by Zimmerman in 1978.(See [3]) This programming can be transformed into a conventional linear programming mentioned in Chapter 1; however, Fuzzy Linear Programming has always had a linear membership function. We extend this function in a more generalized form. In our method, even if a problem include a concave part, we can write this problem as a compact form without any transformation from a convex to a concave problem and can compute a concave-convex problem easily by using the vector of choice. Although this method deals with only the case with a simple fuzzy goal or constraint, we will be able to extend this method to general cases by selecting the vector of choice. The other methods can be found in [11] and [18]. Characteristic of membership functions may be nonlinear smoothing curves in the real world. From Definition 1.3 (b), a decision making can be done as follows:

$$\alpha_0 \equiv \mu_D(\mathbf{x}_0) = \max_{x \in X} \left\{ \min_j \mu_{\tilde{A}_j}(\mathbf{A}_j \mathbf{X}) \right\}. \tag{3.1}$$

Let $\tilde{A}_j$ be a fuzzy constraint or a fuzzy goal set in $Y = \mathbf{R}^1$ whose membership function $\mu_{\tilde{A}_j}(y = \mathbf{A}_j \mathbf{X})$ is expressed in a piecewise linear form, and let $\{ \sigma_{j1}(y), \sigma_{j2}(y), \cdots, \sigma_{jn}(y) \}$ be a set of linear functions on a constraint $\tilde{A}_j$. We consider the three types of functions, a convex function, a concave function, and a concave-convex mixed function. A membership function $\mu_{\hat{\tilde{A}}_i}(y)$ denotes a convex part and $\mu_{\tilde{A}_i}^{\vee}(y)$ denotes a concave part. We suppose there exists only one inflection point on a nonlinear interval.

$$\mu(y) = \left[\left\{ \bigvee_{j=1}^{m'} \mu_{\tilde{A}_i}^{\vee}(y) \right\} \wedge \left\{ \bigwedge_{j=1}^{m''} \mu_{\tilde{A}_i}^{\wedge}(y) \right\} \wedge 1 \right] \vee 0 \ , \ y \in Y \ ,$$

where operator, $\bigvee_{j=1,m'}$ means the maximum value among all the concave functions, $\mu_{\tilde{A}_i}^{\vee}(y)$, and an operation, $\bigwedge_{j=1,m''}$ means the minimum value among all the convex functions, $\mu_{\tilde{A}_i}^{\wedge}(y)$. Hence, we can express all the membership functions as piecewise linear functions. We will show some examples for piecewise linear membership functions in Figure 3.1.(See Page 68)

We deal with the following three types of functions: (a) type is a convex, (b) type is a concave and (c) type is a concave-convex function which has only one inflection point on the interval. We show some examples.

(a) type function:

$$\mu^{\wedge}(y) = a\left[1 - \exp\{-b\,(y - y^0)/(y^1 - y^0)\}\right] \ , \ a > 0 \ , \ b > 0 \ ,$$

and $\mu^{\wedge}(y) = 1$, if $y > y^1$, $\mu^{\wedge}(y) = 0$, if $y \leq y^0$, where y^1 or y^0 is a point which satisfies $\mu^{\wedge}(y) = 1$ or $\mu^{\wedge}(y) = 0$.

(b) type function:

$$\mu^{\vee}(y) = a\left[1 - \exp\{-b\,(y - y^0)/(y^1 - y^0)\}\right] \ , \ a > 0 \ , \ b < 0 \ ,$$

where y^1 or y^0 is a point which satisfies $\mu^{\vee}(y) = 1$ or $\mu^{\vee}(y) = 0$.

(c) type function:

$$\tilde{\mu}(y) = 1/\left[1 + (y - c)^p\right] \ , \ p = 2, 4, 6, \cdots, 2m \ .$$

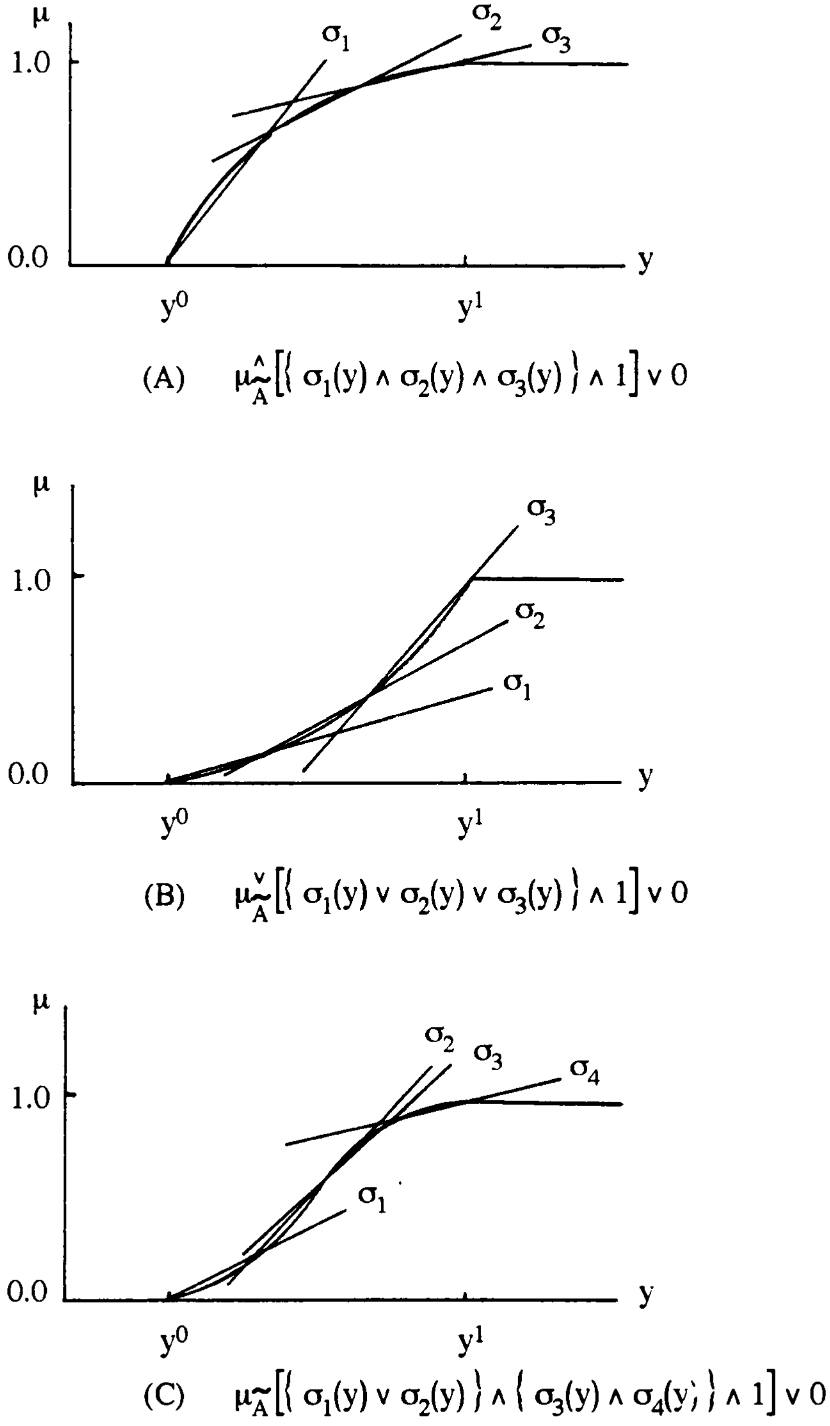

Figure 3.1 Fuzzy Logical Representations of Membership Functions in Piecewise Linear Functions.

(a) type function satisfies

$$\mu^{\wedge}(qy_1 + (1-q)\,y_2) \geq q\mu(y_1) + (1-q)\,\mu(y_2) \quad , \quad 0 \leq q \leq 1 \quad , \quad y_1 < y_2 \ ,$$

(b) type function satisfies

$$\mu^{\vee}(qy_1 + (1-q)\,y_2) \leq q\mu(y_1) + (1-q)\,\mu(y_2) \quad .$$

Since (c) type can be divided by two parts, the one side on the inflection point can be expressed by (b) type, another side can be done by (a) type, all these functions are supposed to be monotonically increasing or decreasing .

In this thesis, to make it easier, we restrict to the case ourselves when there is only one fuzzy goal or constraint and the others are strictly crisp inequalities. The following problem is one of them.

Problem : $\widetilde{\text{Max}}\ \mathbf{CX}$

s.t. $\mathbf{DX} \leq \mathbf{e}$,

$\mathbf{X} \geq \mathbf{0}$.

3.2 APPROXIMATION METHOD WITH PIECEWISE LINEAR FUNCTION

First, we consider how to solve the problem whose goal, $\mathbf{CX}$, is expressed by the membership function of type (a). Before we solve this problem, we have to determine how to select the difference on the interval of the nonlinear part. To make it easier, we divide equally n grid points as follows:

$$\mu^{\wedge}(y) = \left[\left\{\bigwedge_{i=1}^{n} \sigma_i(y)\right\} \wedge 1\right] \vee 0 \ ,$$

$$\sigma_i(y) = \frac{f\left(y^0 + \Delta\, i\right) - f\left(y^0 + \Delta\,(i-1)\right)}{\Delta}\, y + b_{i0} \ , \qquad (3.2)$$

where $\quad y = \mathbf{CX}\ , \quad \Delta \equiv \left(y^1 - y^0\right)/\,n \quad$ and ,

$$b_{i0} \equiv \frac{f\left(y^0 + \Delta(i-1)\right)\cdot\left(y^0 + \Delta i\right) - f\left(y^0 + \Delta i\right)\cdot\left(y^0 + \Delta(i-1)\right)}{\Delta}.$$

We can express the membership function, $\mu^{\wedge}$, by using n distinct linear functions. First, we make a piecewise linear function for each $\mu^{\wedge}$; then we can make a crisp model for each $\mu^{\wedge}$ by (1.3).

Piecewise Linear Programming : Max λ

(We call this as P.L.P.)

$$\begin{aligned} \text{s.t. } & \lambda \leq \sigma_i(y) \quad , \quad i = 1, \cdots, n, \\ & \mathbf{DX} \leq \mathbf{e}, \\ & y = \mathbf{CX}, \end{aligned} \tag{3.3}$$

Since $\sigma_i(y)$ is linear, the equivalent model is given like that on (1.3).

Linear Programming : Max λ

$$\begin{aligned} \text{s.t. } & \mathbf{CX} + s_i \geq b_i, \quad i = 1, \cdots, n, \\ & d_i\lambda + s_i \geq d_i, \\ & \mathbf{DX} \leq \mathbf{e}, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} d_i &= y_i^1 - y_i^0, \\ y_i^1 &= (1 - b_{i0})\Delta / \left\{f\left(y^0 + \Delta i\right) - f\left(y^0 + \Delta(i-1)\right)\right\}, \\ y_i^0 &= -b_{i0}\Delta / \left\{f\left(y^0 + \Delta i\right) - f\left(y^0 + \Delta(i-1)\right)\right\}, \\ b_i &= y_i^1. \end{aligned}$$

Next, we consider the case of (b) type, whose membership function can be expressed as

$$\mu^{\vee}(y) = \left[\left\{\bigvee_{j=1}^{n} \sigma_i(y)\right\} \wedge 1\right] \vee 0.$$

In this case, we can't make the same type of the problem of (3.3) like the case of a convex membership function, because a concave function cannot be expressed as the minimum problem on all σ_i. Even if we can get the optimal solution, λ°, it is not necessarily so that all the constraints on σ_i satisfy $\lambda \leq \sigma_i(y)$. Hence, we have to solve this problem on each interval of i. In order to solve these difficult conditions, we introduce the new expression for this problem.

$$\text{P.L.P.}: \quad \text{Max } \lambda^*$$

$$\begin{aligned} \text{s.t. } & \lambda^* = \boldsymbol{\lambda}\cdot\boldsymbol{\delta}, \\ & \lambda_i \le \sigma_i(y) + L(1-\delta_i), \qquad i = 1, \cdots, n, \\ & y \le (\Delta i + b_0)\delta_i + y(1-\delta_i), \\ & y \ge (\Delta(i-1) + b_0)\delta_i + y(1-\delta_i), \\ & \mathbf{DX} \le \mathbf{e}, \\ & y = \mathbf{CX}, \end{aligned} \tag{3.5}$$

where $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_n)$, $\boldsymbol{\delta} = (\delta_1, \cdots, \delta_n)$ and $\boldsymbol{\delta}$ is a unit vector, i.e., $\boldsymbol{\delta} = (0,\cdots,0,1,0,\cdots,0)$. We call this vector as *the vector of choice*. We select a big positive number L so that constraints which include L satisfy their inequalities completely if $\delta_i = 0$.

In this piecewise linear programming, if we take $\boldsymbol{\delta} = (0,\cdots,0,1,0,\cdots,0)$, where the i-th element is one, this problem can be reduced as follows:

$$\begin{aligned} &\text{Linear Programming}: \quad \text{Max } \lambda \\ &\text{(subproblem)} \qquad \text{s.t. } \lambda_i \le \sigma_i(y), \\ &\qquad\qquad y \le \Delta i + b_{i0}, \\ &\qquad\qquad y \ge \Delta(i-1) + b_{i0}, \\ &\qquad\qquad \mathbf{DX} \le \mathbf{e}, \\ &\qquad\qquad y = \mathbf{CX}. \end{aligned} \tag{3.6}$$

This is equivalent to the following model:

$$\begin{aligned} &\text{Equivalent Model}: \quad \text{Max } \lambda \\ &\text{(subproblem)} \qquad \text{s.t. } \mathbf{CX} + s_i \ge b_i, \qquad i = 1, \cdots, n, \\ &\qquad\qquad d_i\lambda + s_i \ge d_i, \\ &\qquad\qquad \mathbf{CX} \le \Delta i + b_{i0}, \end{aligned} \tag{3.7}$$

$$\mathbf{CX} \geq \Delta(i-1) + b_{i0},$$
$$\mathbf{DX} \leq \mathbf{e} \, .$$

In order to get the optimal solution for the original problem, we apply this process for each i, namely, we consider all the unit vectors as $\boldsymbol{\delta}$. The last thing we have to do is to solve the following problem:

$$\text{Main Problem : } \lambda^{*} = \max_{i} \{\lambda_i^{o}\}, \tag{3.8}$$

where λ_i^{o} is solution for each equivalent model.

For example, if $\max_i \{\lambda_i^{o}\} = \lambda_{i0}^{o}$, the optimal solution is as follows:

$$\lambda^{*} = \lambda_{i0}^{o} \;, \quad \mathbf{x}^{*} = x_{i0}^{o} \;, \quad \delta^{*} = \delta_{i0} \;,$$

$$\mu(y) = \left[\left\{ \bigvee_{j=1}^{n_1} \sigma_i(y) \right\} \wedge \left\{ \bigwedge_{j=n_1+1}^{n} \sigma_i(y) \right\} \wedge 1 \right] \vee 0 \,,$$

where $\widetilde{\mu_{\tilde{A}}}(y)$ denotes the concave-convex membership function and only one inflection point lies on the point which joints $\sigma_{j\, n_1}$ and $\sigma_{j\, n_1+1}$. First, we consider only one σ_j, which is divided into two parts as follows:

$$\sigma^{\vee} \equiv \bigvee_{j=1}^{n_1} \sigma_i(y) \,,$$

$$\sigma^{\wedge} \equiv \bigwedge_{j=n_1+1}^{n} \sigma_i(y) \,.$$

By selecting the vector of choice, $\boldsymbol{\gamma}$, each problem can be divided into a convex part and a concave part and can be solved by the method mentioned in a convex case or a concave case. Moreover, we can designate subproblems by the vector of choice $\boldsymbol{\delta}$. We can express this problem as a maximum problem as follows:

$$\text{P.L.P. : Max } \lambda^{*\wedge}(1-\gamma) + \lambda^{*\vee}\gamma$$

$$
\begin{aligned}
\text{s.t. } \lambda^{*\vee} &= \boldsymbol{\lambda}\boldsymbol{\delta}, \\
\lambda_i &\le \sigma_i(y) + L\left(1 - \delta_i\right) + M(1 - \gamma), \quad i = 1, \cdots, n_1, \\
y &\le \left(\Delta i + b_0\right)\delta_i + y\left(1 - \delta_i\right), \\
y &\ge \left(\Delta(i-1) + b_0\right)\delta_i + y\left(1 - \delta_i\right), \qquad (3.9) \\
\lambda^{*\wedge} &\le \sigma_i(y) + M\gamma, \quad i = n_1 + 1, \cdots, n, \\
y &\ge y_{inflect} + M(1 - \gamma), \\
\mathbf{DX} &\le \mathbf{e}, \\
y &= \mathbf{CX},
\end{aligned}
$$

where $\boldsymbol{\lambda} = \left(\lambda_1, \cdots, \lambda_n\right)$, $\boldsymbol{\delta} = \left(\delta_1, \cdots, \delta_n\right)$ and $\boldsymbol{\delta}$ is a unit vector, i.e., $\boldsymbol{\delta} = (0,\cdots,0,1,0,\cdots,0)$, L and M are very large numbers. $\boldsymbol{\gamma}$ must be 0 or 1 and and $\mathbf{x} \ge \mathbf{0}$.

It is clear that if $\gamma = 0$, then this problem can be reduced to the problem of the convex case and if $\gamma = 1$, then this problem becomes the problem of the concave case. We also call this $\boldsymbol{\gamma}$ as the vector of choice. These expressions of this problem with the vectors of choice, $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$, may be considered as a Mixed Integer Programming problem.(See [4]) In this thesis, we consider all the cases of combinations of δ_i and γ and compare all the solutions of subproblems for all the cases and pick up the maximum value as the optimal solution; however, if n is larger, it is advantageous to solve this as a Mixed Integer Programming problem .

Example 3.1. We consider the following problem.

$$
\begin{aligned}
\text{Problem: } \widetilde{\text{Max}}\ & \mathbf{CX} \\
\text{s.t. } & \mathbf{DX} \le \mathbf{e}, \\
& \mathbf{X} \ge \mathbf{0},
\end{aligned}
$$

where

$$\mathbf{C} = [\ 1.1 \quad 1.4\ ], \ \mathbf{X} = [x_1\ x_2]^t, \ C^u = [15.4], \ C^l = [4.62],$$

$$\mathbf{D} = \begin{bmatrix} 0.65 & 2.3 \\ 1.0 & 2.0 \\ 0.9 & 0.8 \\ 1.3 & 0.7 \end{bmatrix},$$

$$\mathbf{e} = [\ 14.95 \quad 14 \quad 7.2 \quad 9.1\ ]^t.$$

Furthermore, the membership function of the goal is expressed as

$$\mu^{\wedge}(y) = \begin{cases} 1 & , \text{ if } y > 15.4 \\ 1.431\,[1 - \exp\{(-1.2)(y - 4.62)\}] & , \text{ if } 15.4 \geq y > 4.62 \\ 0 & , \text{ if } 4.62 > y \end{cases},$$

We can transform this problem into the following Piecewise Linear Programming problem.

P1: Max λ

$$\begin{aligned} \text{s.t. } & \lambda \leq \sigma_i(y), \quad i = 1, \cdots, n, \\ & \mathbf{DX} \leq \mathbf{e}, \\ & \mathbf{X} \geq \mathbf{0}, \\ & y = \mathbf{CX}. \end{aligned}$$

First of all, we make a piecewise linear function $\{\sigma_i\}$. To make it easier, we set n = 3 , 3 is the number of grid points. Therefore, $\Delta = (C^u - C^l) / n = 3.593$. By (3.2), we obtain each σ_i.

$$\sigma_1(y) = 0.1312\, y - 0.6065,$$

$$\sigma_2(y) = 0.08\, y - 0.251,$$

$$\sigma_3(y) = 0.0569\, y - 0.0915.$$

Then we can express the problem of P1 as the equivalent linear programming problem.

P2: Max λ

$$\begin{aligned} \text{s.t. } & \mathbf{CX} + s_i \geq b_i, \qquad i = 1, 2, 3, \\ & d_i\lambda + s_i \geq d_i, \\ & \mathbf{DX} \leq \mathbf{e}, \end{aligned}$$

$$\mathbf{X} \geq \mathbf{0},$$

where one can determine b_i and d_i from (3.4). Therefore, we may just solve the following linear programming problem.

P3: Max λ

$$\begin{aligned}
\text{s.t.}\quad & 1.1x_1 + 1.4x_2 + s_1 \geq 12.263, \\
& 1.1x_1 + 1.4x_2 + s_2 \geq 14.215, \\
& 1.1x_1 + 1.4x_2 + s_3 \geq 15.4, \\
& 7.616\lambda + s_1 \leq 7.616, \\
& 11.362\lambda + s_1 \leq 11.362, \\
& 16.951\lambda + s_2 \leq 16.951, \\
& 0.65x_1 + 2.3x_2 \leq 14.95, \\
& 1.0x_1 + 2.0x_2 \leq 14.0, \\
& 0.9x_1 + 0.8x_2 \leq 7.2, \\
& 1.3x_1 + 0.7x_2 \leq 9.1, \\
& x_1, x_2 \geq 0.
\end{aligned}$$

We can obtain the optimal solution as follows:

$$\lambda^o = 0.724, \quad \mathbf{X} = [\ 3.2 \quad 5.4\ ].$$

Example 3.2. We consider the case when a membership function can be expressed by a concave function.

$$\mu^v(y) = \begin{cases} 1 & , \text{ if } y > 15.4 \\ -0.431\,[1 - \exp\{(+1.2)(y - 4.62)\}] & , \text{ if } 15.4 \geq y > 4.62 \\ 0 & , \text{ if } 4.62 > y \end{cases},$$

According to the formula of (3.5), this problem is transformed into a Piecewise Linear Programming problem. In this case, since n=3, $\lambda^* = \lambda_1\delta_1 + \lambda_2\delta_2 + \lambda_3\delta_3$. One of the elements of $\boldsymbol{\delta}$ must be 1 and the others are 0. We substitute for each i-th element into (3.5) so that we can obtain subproblems like (3.6). Furthermore, we make piecewise linear function $\{\sigma_i\}$:

$$\sigma_1(y) = 0.0589\, y - 0.2725,$$
$$\sigma_2(y) = 0.088\, y - 0.5108,$$
$$\sigma_3(y) = 0.1312\, y - 1.0219.$$

Since each subproblem for i can be reduced to an equivalent linear programming by the formula of (3.7), we may solve only the following problem, so that we can get suboptimal λ_i^o on each subinterval of $[\Delta(i-1)+b\,, \Delta i+b]$. For $\boldsymbol{\delta} = [\,1\ \ 0\ \ 0\,]$, the reduced subproblem is expressed as

$$\begin{aligned} \text{P4}:\ \text{Max}\ & \lambda_1 \\ \text{s.t.}\ & 1.1x_1 + 1.4x_2 + s_1 \geq 21.571, \\ & 16.95\lambda_1 + s_1 \leq 16.95, \\ & 1.1x_1 + 1.4x_2 \geq 4.62, \\ & 1.1x_1 + 1.4x_2 \leq 8.213, \\ & \mathbf{DX} \leq \mathbf{e}, \\ & \mathbf{X} \geq \mathbf{0}. \end{aligned}$$

The following subproblem is,for $\boldsymbol{\delta} = [\,0\ \ 1\ \ 0\,]$,

$$\begin{aligned} \text{P5}:\ \text{Max}\ & \lambda_2 \\ \text{s.t.}\ & 1.1x_1 + 1.4x_2 + s_2 \geq 17.167, \\ & 11.362\lambda_2 + s_2 \leq 11.362, \\ & 1.1x_1 + 1.4x_2 \geq 8.213, \\ & 1.1x_1 + 1.4x_2 \leq 11.806, \\ & \mathbf{DX} \leq \mathbf{e}, \\ & \mathbf{X} \geq \mathbf{0}. \end{aligned}$$

The following one is, for $\boldsymbol{\delta} = [\,0\ \ 0\ \ 1\,]$, P6: Max λ_3

$$\text{s.t. } 1.1x_1 + 1.4x_2 + s_3 \geq 15.4,$$
$$7.616\lambda_3 + s_3 \leq 7.616,$$
$$1.1x_1 + 1.4x_2 \geq 11.806,$$
$$1.1x_1 + 1.4x_2 + s_3 \leq 15.4,$$
$$\mathbf{DX} \leq \mathbf{e},$$
$$\mathbf{X} \geq \mathbf{0}.$$

We solve the main problem in order to get the optimal solution for the original problem.

P7: $\lambda^* = \text{Max}\{\lambda_1^o, \lambda_2^o, \lambda_3^o\}$.

From the result of P4, P5, P6, $\lambda_1^o = 0.2119$, $\lambda_2^o = 0.4642$ and λ_3^o is an infeasible solution. Hence, $\lambda^* = 0.4642$ and $\mathbf{X} = [\ 3.2 \quad 5.4\ ]^t$.

Example 3.3 We consider the case of a concave-convex membership function.

$$\tilde{\mu}(y) = \begin{cases} 1 & , \text{ if } y > 15.4 \\ \exp\{(-0.06)(y - 15.4)^2\} & , \text{ if } 15.4 \geq y > 4.62 \\ 0 & , \text{ if } 4.62 > y \end{cases}.$$

This membership function can be divided into two parts, one of which is a concave and the other is a convex one. Therefore, we separate and solve the problem by the same method of Example 3.1 or Example 3.2 corresponding to a convex part or a concave part, which means we take $\gamma = 1$ or $\gamma = 0$ in (3.9). First, we have to make a piecewise linear function for $\tilde{\mu}$. Second of all, we find out an inflection point where a coefficient changes from increasing to decreasing; then we separate the problem. Third of all, we solve each part. The last thing we have to do is to pick out the maximum λ which is Max $\lambda^{*\wedge}(1-\gamma) + \lambda^{*\vee}\gamma$. We show the result of this problem in the case of n = 12 in Table.3.1.

Table 3.1 The Solution of Example 3.1

y_{i-1}	y_i	gradient of σ_i	y_0^1	d_i	λ_i or λ	δ	γ	$\lambda^{\wedge *}$	$\lambda^{\vee *}$	λ^*
4.62	5.51	.002	472	468	.003	$\delta_1=1$	1			
5.51	6.41	.005	183	178	.007	$\delta_2=1$	1			
6.41	7.31	.013	81.2	75.4	.019	$\delta_3=1$	1			
7.31	8.21	.028	42.1	35.5	.045	$\delta_4=1$	1			
8.21	9.11	.053	26.0	18.6	.093	$\delta_5=1$	1			
9.11	10.0	.090	19.0	10.9	.175	$\delta_6=1$	1			
10.0	10.9	.137	16.0	7.29	.297	$\delta_7=1$	1			
10.9	11.8	.181	14.7	5.51	.329	$\delta_8=1$	1			
11.8	12.7	**.206**	14.4	4.83	.336	$\delta_9=1$	1	.336		**.336**
12.7	13.6	.197	14.4	5.07	.310		0		.310	
13.6	14.5	.143	14.8	6.97			0			
14.5	15.4	.052	15.4	19.0			0			

It is important to decide how many grid points we should take for an exact solution.

Theorem 3.1 If the maximum value of the approximation error of a nonlinear membership function which is approximated by a piecewise linear function is

$$K = \max_i \max_{\Delta i < y < \Delta(i+1)} |\sigma_i(y) - f(y)|,$$

and $f(y)$ is a convex or a concave monotone function, then

$$\lim_{n \to \infty} K = 0$$

Proof: We show the only convex case here. Since f is convex,

$$f(qy_1 + (1-q)y_2) \geq qf(y_1) + (1-q)f(y_2) \quad , \quad 0 \leq q \leq 1 \quad ,$$

and $y_1 \equiv y^0 + \Delta i$, $y_2 \equiv y^0 + \Delta(i+1)$. Since $y_2 - y_1 = \Delta = \frac{y^1 - y^0}{n}$, $y_2 \rightarrow y_1$ $(n \rightarrow \infty)$, where $f(y^0) = 0.0$ and $f(y^1) = 1.0$. On the other hand, $\sigma_i(y) = q f(y_1) + (1-q) f(y_2)$ and $qy_1 + (1-q)y_2 = y$. Therefore,

$$\begin{aligned}
\lim_{n \to \infty} K &= \lim_{n \to \infty} |\sigma_i(y) - f(y)| \\
&= \lim_{y_1 \to y_2} |q f(y_1) + (1-q) f(y_2) - f(qy_1 + (1-q)y_2)| \\
&= \lim_{y_1 \to y_2} |q\{ f(y_1) - f(y_2) \} + f(y_2) - f\{ q(y_1 - y_2) + y_2 \}| \\
&= 0.
\end{aligned}$$

We can prove a concave case similarly. Q.E.D.

Theorem 3.2 When we take some finite number, n, the optimal solution for a piecewise linear problem is bounded as follows:

$$\lambda_n^u \geq \lambda_\infty \geq \lambda_n^l \ ,$$

where λ_∞ denotes the optimal solution of the membership function value if $n \rightarrow \infty$.

Proof: We assume f is convex. We can solve a problem by (3.3).

L.B.P. : Max λ

(lower bounded problem) s.t. $\lambda \leq \sigma_i(y)$, $i = 1, \cdots, n$.

We regard this solution, λ_n^l ,as the lower bounded solution for the original problem. Now we define the upper bounded solution, λ_n^u.

U.B.P. : Max λ

(upper bounded Problem) s.t. $\lambda \leq \sigma_i(y) + K$, $i = 1, \cdots, n$.

Since $K > 0$ in a convex case and $\sigma_i(y) + K \geq \sigma_i(y)$, we obtain $\sigma_i(y) \leq f(y) \leq \sigma_i(y) + K$ and $f(y) = \lim_{n \to \infty} \sigma_i(y)$. Hence, it is obvious that $\lambda_n^u \geq \lambda_\infty \geq \lambda_n^l$.

Q.E.D.

Although we can obtain an approximation value, the solution is not the solution of the original problem, but of its piecewise linear programming problem. It is difficult to find K and to obtain an exact solution if f has a complicated form; however, Theorem 3.2 guarantees that there exists an approximated solution. We show an example for convergence corresponding to the number of the grid points n.

Example 3.4. We consider the case of Example 3.1. We change the number of the grid points n. In the case that $b = 1.2$, we obtain the following optimal solutions:

$\lambda = 0.7240$, $n = 3$, $\lambda = 0.7304$, $n = 6$, $\lambda = 0.7333$, $n = 12$,
$\lambda = 0.7337$, $n = 24$, $\lambda = 0.7338$, $n = 48$.

In this chapter, we dealt with just one fuzzy goal; however, there exist more difficult problems which include more than two fuzzy goals or constraints in which cases we can't solve problems by simple min-max problems if the membership functions include concave parts. We have to continue researching this problem in the future.

CHAPTER 4

FUZZY LINEAR PROGRAMMING AND GOAL PROGRAMMING METHOD

4.1 GOAL PROGRAMMING METHOD WITH PENALTY WEIGHTS

In this chapter, we consider Fuzzy Linear Programming as Goal Programming under some assumptions. In Fuzzy Linear Programming, it is permitted to compensate the violation for goals or constraints with their fuzziness, which an idea is similar to that of Goal Programming. Both problems are related in the sense that the penalty weights of a Goal Programming problem is corresponding to the membership functions of a Fuzzy Linear Programming problem. The methods of Goal Programming can be found in [7],[9] and [13].

A usual Goal Programming problem can be formulated as

$$\text{Problem : } \quad \text{Min} \sum_{i=1}^{m} w_i^+ d_i^+ + w_i^- d_i^-$$

$$\begin{aligned} \text{s.t.} \quad & \mathbf{g}_i\mathbf{X} - d_i^+ + d_i^- = t_i \quad , \quad i = 1, \dots, m \quad , \\ & \mathbf{DX} \le \mathbf{e}, \\ & \mathbf{X} \ge \mathbf{0}, \\ & d_i^+ , d_i^- \ge 0 , \end{aligned} \tag{4.1}$$

where each $\mathbf{g}_i$ is the row vector of coefficients of a goal function and t_i is a target value. d_i^+ and d_i^- are the differences from a target value. w_i^+ and w_i^- are penalty weights corresponding to d_i^+ and d_i^-.

In this chapter, we consider the problem of Type 1 and define the penalty function corresponding to the membership function of each goal.

Definition 4.1 We define the relationship between Fuzzy Linear Programming and Goal Programming as follows. It is said that if the penalty function can be expressed as follows, then there exists the relationship between Fuzzy Linear Programming and Goal Programming.

$$w_i^+ = 0 \ , \ \text{for any } d_i^+ ,$$

$$w_i^- = \begin{cases} \dfrac{w_0 \, d_i^-}{c^u - c^l} & , \text{ if } c^u - c^l \geq d_i^- \\ L & , \text{ if } c^u - c^l < d_i^- \end{cases} ,$$

where the value of the membership function becomes 1.0 at c^u and 0 at c^l in Fuzzy Linear Programming. L is a very large number and w_0 is constant.

In Fuzzy Linear Programming, each value of goals must satisfy at least the lowest target, which means $\lambda = 0$ at c^l. On the other hand, in Goal Programming, the goal value which is less than c^l is given the big amount of penalty; therefore, we must avoid such a big penalty value in order to obtain a possible solution.

Example 4.1. We compare a Fuzzy Linear Programming problem with a Goal Programming problem.

P1: Fuzzy Linear Programming : $\widetilde{\text{Max}} \ \mathbf{CX}$

$$\text{s.t. } \mathbf{AX} \leq \mathbf{b},$$
$$\mathbf{X} \geq \mathbf{0},$$

where

$$\mathbf{X} = [x_1 \, x_2]^t , \ C^u = [15.4 \ \ 12.0]^t , \ C^l = [4.62 \ \ 7.2]^t ,$$

$$\mathbf{C} = \begin{bmatrix} 1.1 & 1.4 \\ 1.5 & 0.8 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0.65 & 2.3 \\ 1.0 & 2.0 \\ 0.9 & 0.8 \\ 1.3 & 0.7 \end{bmatrix}.$$

The optimal solution for this problem is $\lambda^0 = 0.5473$, $\mathbf{X}_F = [\ 4.378 \quad 4.074\ ]^t$.

$$\text{P2: Goal Programming: } \quad \text{Min } \frac{w_0\, d_1^-}{c_1^u - c_1^l} + \frac{w_0\, d_2^-}{c_2^u - c_2^l} \equiv P$$

$$\text{s.t. } \mathbf{CX} + \mathbf{d} = \mathbf{C}^u,$$

$$\mathbf{AX} \leq \mathbf{b},$$

$$\mathbf{X} \geq \mathbf{0},$$

where $\mathbf{d} = [d_1^- \; d_2^-]$ and $\mathbf{C}^u = [c_1^u \; c_2^u]^t$, $\mathbf{C}^l = [c_1^l \; c_2^l]^t$.

The optimal solution for this problem is $\mathbf{X}_G = [\ 5.463 \quad 2.853\ ]^t$, $P_G = 81.71$. We substitute this optimal solution of $\mathbf{X}_F$ or $\mathbf{X}_G$ into another problem each other in order to obtain λ or P. We substitute $\mathbf{X}_F$ into Problem P2, then we solve the following problem.

$$\text{Problem: Min } \quad P = \frac{w_0\, d_1^-}{c_1^u - c_1^l} + \frac{w_0\, d_2^-}{c_2^u - c_2^l}$$

$$\text{s.t. } \mathbf{CX}_F + \mathbf{d} = \mathbf{C}^u.$$

The solution is $\mathbf{X}_F = [\ 4.378 \quad 4.074\ ]^t$, $P_F = 90.53$. We substitute $\mathbf{X}_G$ into the problem of P1, then we solve the following problem:

$$\text{Problem: } \lambda = \text{Min}\,\{\mu_1(\mathbf{X}_G),\ \mu_2(\mathbf{X}_G)\}..$$

The solution is $\mathbf{X}_G = [\ 5.463 \quad 2.853\ ]^t$, $\lambda_G = 0.4995$.

We used the following penalty function:

$$W_1 = \begin{cases} 0 & ,\ \text{if } d_1^- = 0 \\ \dfrac{w_0\, d_1^-}{10.78} & ,\ \text{if } 0 < d_1^- \leq 10.78 \\ L & ,\ \text{if } 10.78 < d_1^- \end{cases},$$

$$W_2 = \begin{cases} 0 & , \text{ if } d_2^- = 0 \\ \dfrac{w_0 d_2^-}{4.8} & , \text{ if } 0 < d_2^- \le 4.8 \\ L & , \text{ if } 4.8 < d_2^- \end{cases},$$

where $w_0 = 100$, $L = 10^8$.

The membership function is as follows:

$$\mu_1(y_1) = \begin{cases} 1 & , \text{ if } y_1 > c_1^u \\ \dfrac{y_1}{10.78} & , \text{ if } c_1^u \ge y_1 > c_1^l \\ 0 & , \text{ if } c_1^l > y_1 \end{cases},$$

$$\mu_2(y_2) = \begin{cases} 1 & , \text{ if } y_2 > c_2^u \\ \dfrac{y_2}{4.8} & , \text{ if } c_2^u \ge y_2 > c_2^l \\ 0 & , \text{ if } c_2^l > y_2 \end{cases},$$

where $[y_1\ y_2]^t \equiv \mathbf{CX}$.

From the result, if we want to decrease P, then λ is decreased. If we want to increase λ, then P is increased. The best result, if it is possible, is to satisfy decreasing P and increasing λ; however, we can't obtain such a fortunate result as we showed in Example 4.1. We show that Fuzzy Linear Programming is different from Goal Programming entirely even if the form of the penalty function exactly matches the form of the membership function as mentioned above.

Goal Programming can be considered if we have m goals.

$$\text{Min } P = \underset{\mathbf{x}}{\text{Min}} \left\{ \sum_{i=1}^{m} W_i \right\}$$

$$= \underset{\mathbf{x}}{\text{Min}} \left\{ w_0 \sum_{i=1}^{m} (1 - \mu_i(y_i)) \right\},$$

where $\mathbf{y}_i = [y_1, \cdots, y_m]^t = \mathbf{CX}$.

This problem is equivalent to the following problem:

Problem : $\underset{\mathbf{x}}{\text{Max}} \left\{ k_1 + k_2 \sum_{i=1}^{m} \mu_i(y_i) \right\}$,

where k_1 , k_2 are constant and $k_2 > 0$.

On the other hand, a Fuzzy Linear Programming problem is a problem to seek

$$\underset{\mathbf{x}}{\text{Max}} \left[\underset{i}{\text{Min}} \{ \mu_1(y_1), \cdots, \mu_m(y_m) \} \right].$$

Hence, if both problems are equivalent, there must exist two constant values, k_1 and k_2, which can satisfy the following condition:

$$\underset{\mathbf{x}}{\text{Max}} \left\{ k_1 + k_2 \sum_{i=1}^{m} \mu_i(y_i) \right\} = \underset{\mathbf{x}}{\text{Max}} \left[\underset{i}{\text{Min}} \{ \mu_i(y_i) \} \right]. \tag{4.2}$$

However, there does not exist those constant values to satisfy the condition of (4.2) for every membership function μ_i.

Let's consider the following problem:

$$\begin{aligned} \text{Mixed Program : } & \widetilde{\text{Max}} \ \mathbf{C}^t\mathbf{X} \\ & \text{Min} \ \ P \\ & \text{s.t. } \mathbf{AX} \le \mathbf{b}, \\ & \qquad \mathbf{X} \ge \mathbf{0}. \end{aligned} \tag{4.3}$$

This problem doesn't make sense until both objective functions are added to each other with some weighting. Therefore, we introduce some definitions to make this problem possible to be solved.

Mixed Program : Max $\alpha\lambda - \beta q$

$$\text{s.t. } \mathbf{Cx} + \mathbf{s} \geq \mathbf{C}^{u},$$
$$(\mathbf{C}^{u} - \mathbf{C}^{l})\lambda + \mathbf{s} \leq \mathbf{C}^{u} - \mathbf{C}^{l}, \quad (4.4)$$
$$\mathbf{Cx} + \mathbf{d} = \mathbf{C}^{u},$$
$$\mathbf{AX} \leq \mathbf{b},$$
$$\mathbf{X} \geq \mathbf{0},$$

where $q = \frac{1}{w_0}P$ and s is a compensation vector which is the same as that in Chapter I . α , $\beta > 0$ are some weighting coefficients which are given before we solve a problem. This objective function means that satisfaction on λ is reduced by the amount of the penalty. We have left some ambiguity on this definition in the sense that λ and q can be compared by the same dimension or same meaning. We show some examples corresponding to some weighting coefficients in Table 4.1.

Table 4.1 Mixing Programming Problem Solutions

Weighting							
α	0.0	0.5	1.0	1.0	2.0	2.0	3.0
β	1.0	1.5	0.0	1.0	1.0	3.0	2.0
Objective Function							
F = Max $\alpha\lambda$-βq	-0.8175	-0.3633	0.9053	0.0946	0.6420	-0.2272	0.7366
Variables							
λ	0.4995	0.4995	0.5473	0.5473	0.5473	0.4995	0.5473
x_1	5.4634	5.4634	4.3786	4.3786	4.3786	5.4634	4.3786
x_2	2.8536	2.8536	4.0740	4.0740	4.0740	2.8536	4.0740
d_1	5.3951	5.3951	4.8798	4.8798	4.8798	5.3951	4.8798
d_2	1.5219	1.5219	2.1728	2.1728	2.1728	1.5219	2.1728

In this Table, the case that $\alpha = 0$, $\beta = 1$ is the same as that of a Goal Programming problem and the case that $\alpha = 1$, $\beta = 0$ is reduced to a Fuzzy Linear Programming problem. From the result, the optimal solution is always that of Fuzzy Linear Programming or that of Goal Programming corresponding to sign of $F\left(= \text{Max } \alpha\lambda - \beta q\right)$. This result gives us very important items for future study. We will clarify whether there exist some relationships between Goal and Fuzzy Linear Programming or not by progressing the research on this Mixed Programming problem.

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