A comparison of the analytical, numerical, and experimental solutions to the classic dangling chain problem

Shane G. Gahagan
To the Graduate Council:

I am submitting herewith a thesis written by Shane G. Gahagan entitled "A comparison of the analytical, numerical, and experimental solutions to the classic dangling chain problem." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Aviation Systems.

Uwe Peter Solies, Major Professor

We have read this thesis and recommend its acceptance:

Charles Paludon,

Accepted for the Council:

Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)
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A COMPARISON OF THE
ANALYTICAL, NUMERICAL, AND EXPERIMENTAL SOLUTIONS TO THE
CLASSIC DANGLING CHAIN PROBLEM

A Thesis
Presented for the
Master of Science
Degree
The University of Tennessee, Knoxville

Shane G. Gahagan
May 1994
ABSTRACT

The incorporation of trailing wire antennas in military aircraft has been the result of unique mission requirements of HF (High Frequency), LF (Low Frequency), and VLF (Very Low Frequency) strategic communications. The complex physical dynamics of these trailing wires necessitate accurate modeling in order to determine the unique characteristics associated with trailing wire antennas. Accurate modeling of trailing wires stems from the need to determine the time dependent cable wire shape with respect to aircraft maneuvers and/or environmental characteristics that produce direct forces to the trailing wire. The knowledge of wire shape with respect to various input forces will therefore result in practical engineering solutions to limit the adverse effects caused by such forces. As an invaluable instructional tool for the development of dynamic models of trailing wires, the fidelity of lab experiments that correlate with both the analytical and numerical modeling of complex physical systems are essential for complete understanding. This thesis demonstrates the close correlation possible for the classic dangling chain problem. A full appreciation model of the governing equation for the classic dangling chain problem with a dead weight attached to the end was developed and solved analytically. The analytical solutions were then used to validate the solutions found in the numerical simulation and actual experiment. The solutions were then compared to determine the
utility of using applied numerical and experimental techniques for solving the dangling chain governing equation. Comparison of the solutions demonstrated the close correlation of simple experiments with both the analytical and numerical models of physical dynamic systems that can provide a robust instructional tool in the development of trailing wires in military aircraft.
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GLOSSARY, LIST OF SYMBOLS, ABBREVIATIONS

GLOSSARY

Catwalk  
A narrow walkway.

Dead weight  
An unrelieved weight of a mass.

SYMBOLS

\( A \)  
Constant

\( A_0 \)  
Amplitude of incident and reflected waves

\( A_m \)  
Wave amplitude

\( B \)  
Constant

\( B.C. \)  
Boundary Conditions

\( c \)  
Change in variable

\( d\alpha, d\sigma \)  
Derivative of \( \alpha, \sigma \)

\( e \)  
Exponential

\( F(x,t) \)  
Forcing function

\( F_y \)  
Forces in the \( y \) direction

\( g \)  
Gravitational constant

\( h \)  
Distance between distinct points

\( i \)  
Imaginary number \(-1\)

\( I.C. \)  
Initial Conditions

\( J_0 \)  
Bessel's function of zero order

\( k \)  
Wave number

\( L \)  
Total length of cable wire

\( L_{eq} \)  
Equivalent length of weight with respect

\( q \)  
Change in variable

\( t \)  
Time

\( T \)  
Period of wave motion

\( T(x) \)  
Tension of cable wire to cable wire

\( v \)  
Magnitude of propagation velocity

\( W \)  
Weight attached to end of cable wire

\( x \)  
Coordinate along cable wire

\( y \)  
Lateral displacement of cable wire

\( Y(x) \)  
Solution form of \( y(x,t) \)

\( Y_0 \)  
Bessel's function of the second kind

\( \alpha \)  
Change in variable

\( \partial \)  
Partial derivative

\( \alpha \)  
Definition of Bessel's function values

\( \beta \)  
Definition of Bessel's function values

\( \Delta \)  
Elemental change

\( \delta \)  
Angle between \( T(0) \) and \( T'(0) \)

\( \lambda \)  
Wavelength

\( \mu \)  
Mass per unit length

\( \pi \)  
Pi

\( \Sigma \)  
Summation

\( \tau \)  
Phase constant
\[ \phi \quad \text{Angle cable wire makes with vertical axis at point } x_0 \]
\[ \gamma \quad \text{Angle cable wire makes with vertical axis at point } x_2 \]
\[ \omega \quad \text{Angular frequency} \]

**SUBSCRIPTS**

\( i \) Element of \( x \)

\( m \) Time step index

\( n,N \) Numerical gridpoint along the wire

**ABBREVIATIONS**

- **HF** High Frequency
- **LF** Low Frequency
- **TACAMO** Take Charge And Move Out
- **VLF** Very Low Frequency
CHAPTER 1
INTRODUCTION

The use of long cable or wires, such as trailing wire antennas, in military aircraft has expanded in recent years with the expansion of mission requirements. Initial requirements for the direct communication of military aircraft with strategic missile submarines has expanded to theater and tactical level battlegroup communications. Although the length of trailing wire antennas is determined by mission requirements (namely communication frequency), the approach to modeling the time dependent response of wires under the influence outside forces is unique. For example, VLF communication antennas require extremely long trailing wire antennas (the Navy's E-6A TACAMO incorporates 30,000 feet of trailing wire antenna) which respect to certain HF communication trailing wire antennas (Navy's E-2C Hawkeye has a 30 foot antenna for HF communication) but the dynamic modeling approach of both antennas is similar. This thesis concentrates on the modeling of dynamic systems such as long cables and wires in which a complete understanding of the effects of various force inputs can have on the wire shape. To develop a full appreciation of both analytic and numerical modeling of physical dynamic systems, such as trailing wires, requires hands-on experiments which relate the empirical measurements with both the analytical and
Numerical development and solutions of the dynamic system. Experiments are needed which demonstrate the close correlation possible between the developed analytical and numerical wire model. The analytical, numerical, and experimental solutions can be compared to demonstrate the ease, flexibility, and accuracy of these simple techniques. The experiment developed here fulfills this need. The intent of this thesis is to develop and conduct an experiment that demonstrates the utility of analytical models of complex physical systems including their closed form solutions and then to emphasize the utility of numerical techniques for solving the same governing equations. Engineers that develop the dynamic system modeling of trailing wire antennas can use this approach to effectively develop engineering solutions to limit the adverse effects of certain forces (inputs) upon long cable antennas.

The candidate experiment that was chosen was the classic dangling chain problem. The experiment consisted of a long wire hung from the catwalk of a hangar with a dead weight attached to the end. The development of the analytical model for the dangling chain problem with boundary conditions was provided by Clifton [1] and Volterra and Zachmanoglou [2]. The analytical model was adapted to the dead weight and wire combination used in this thesis and then solved in closed form for the homogeneous solution
including the eigenvalue and eigenvector structure. A computer program based upon Clifton’s work was developed to obtain solutions to the governing equations for representative forcing functions. Lastly, an experiment with a hanging wire with a fixed dead weight at one end was designed and performed. The wire was excited with numerous forcing functions at the free end and recorded on film to complete the thesis. The solutions from the analytical, numerical analysis (computer algorithm), and experiment were compared to determine their correlation. The comparison was based upon the fidelity of the model solutions when compared to the experimental results and the overall feasibility of the exercise.

Chapter 2 of this thesis contains a discussion of the mechanics and forms of waves in suspended wires. Chapter 3 is a development of the governing equations for the dangling chain as well as the analytical solution of the homogeneous form of these governing equations. It will be seen that the particular solution is not necessary to validate the model results. Chapter 4 is a development of the time dependent numerical solution to the dangling chain model and Chapter 5 is a discussion of the suspended dangling chain experiment. Finally, Chapter 6 is a discussion of the results from the models and experiments.
CHAPTER 2
PROPAGATION FORCES IN SUSPENDED WIRES

This Chapter discusses the physical properties of wave motion; in particular, the wave motion in deformable elastic media. The properties and equations that describe the properties of wave motion in elastic media was performed by Halliday and Resnick [3] and are repeated and elaborated upon (with respect to the dangling chain problem) in this text. Wave motion in deformable elastic media, among which propagating waves in a vertical wire under tension are an example, are called mechanical waves. Mechanical waves originate due to the displacement or disturbance of an element of an elastic medium (such as a string or wire) from its equilibrium position, causing it to vibrate or oscillate about its equilibrium point. Likewise, because of the elastic nature of the medium, the disturbance is transmitted from one element to the next through the medium. Physically, the medium does not move as a whole, rather, the various elements oscillate in some coordinated fashion. Thus, mechanical waves are characterized by the transport of disturbances (energy) through a medium without the corresponding bulk motion of the medium itself. This thesis concentrates upon the motion of mechanical waves through a vertically dangling cable wire with a concentrated mass
attached to one end.

Types of Waves

Types of mechanical waves in cable are distinguished by how the variation of the exciting force (disturbance) is related to the direction of wave propagation in the wire. If the direction of the disturbance is perpendicular to the direction of the propagation of the wave in the wire, then it is defined as a transverse wave. A classical example of transverse wave propagation is the dangling string in which the elements of the string vibrate at right angles to the direction in which the wave itself is propagated (Figure 2.1).

Figure (2.1)

Transverse Wave Propagation in Dangling String

Another type of mechanical wave is distinguished by the motion of a disturbance in which direction of the wave through the medium itself is along the same axis. These waves are called longitudinal waves due to the fact that both the disturbance and travelling wave travel longitudinally down the medium. An example is shown in Figure 2.2 in which a vertical spring is disturbed along the spring axis and the propagating wave travels up and down the same longitudinal axis.

![Figure 2.2](image-url)

**Figure (2.2)**

*Longitudinal Wave Propagation in a Spring*


Additionally, mechanical waves can also be classified according to the number of dimensions in which they
propagate energy. Waves moving along a vertical string (Figure 2.1) or spring (Figure 2.2) can be assumed to be one-dimensional for analysis purposes. Surface waves on water, because of their elliptical path are characterized as two-dimensional, and sound waves or waves that emanate radially from a small source are three-dimensional.

Mechanical waves can be additionally classified according to the type of disturbance applied to the medium. For example, a pulse wave can be produced by a single step input to the medium. Each element is at equilibrium until the waves reaches it and returns to an equilibrium state after the pulse passes. However, if the disturbance is periodic, like a sinusoidal continuous input (simple harmonic wave), then a periodic train of waves is produced and each element in the medium has a characteristic periodic motion. Surfaces where all points are in the same phase of motion are called wavefronts. For homogenous media, the direction of propagation is at right angles to the wavefront. Wavefronts can be characterized by the direction of the disturbance; some of which are plane waves for single direction propagation and spherical waves for propagation from a point source.

**Traveling Waves**

Limiting the discussion to one-dimensional transverse mechanical waves and wavefronts, consider a vertically
dangling chain at some instant in time, say, \( t=0 \). The shape of the wire can be represented by \( y=f(x) \) at \( t=0 \), where \( y \) is the lateral displacement of the wire and \( x \) is the coordinate along the wire, as shown in Figure 2.3.

Assuming frictional losses are small, the transverse wave will travel along the chain without significantly changing its waveform. The wave will travel a distance down the wire, \( vt \), where \( v \) is the magnitude of the propagation velocity of the wave along the string, assumed constant. Therefore, the equation for the curved chain at some time \( t \) is \( y=f(x - vt) \) for all times \( t \). For a particular harmonic waveform, the equation of the wave at time \( t \) is expressed in equation (2.1), where \( A_m \) is the
amplitude, and \( \lambda \) is the wavelength [3:p. 295].

\[
y = A_m \sin \frac{2\pi}{\lambda} (x - vt)
\]  

(2.1)

Multiplying the \( 2\pi/\lambda \) term through and substituting the definition \( v=\lambda/T \), where \( T \) is the period of the wave motion, into equation (2.1) results in equation (2.2).

\[
y = A_m \sin \left( \frac{2\pi x}{\lambda} - \frac{2\pi vt}{T} \right)
\]  

(2.2)

Defining two quantities, the wave number, \( k \), and the angular frequency, \( \omega \), yields equation (2.3) [3:p. 296].

\[
y = A_m \sin (kx - \omega t)
\]  

(2.3)

where \( k = \frac{2\pi}{\lambda}, \quad \omega = \frac{2\pi}{T} \)

The travelling wave expressions of equations (2.1) and (2.3) assumed that the displacement \( y \) was zero at the position \( x=0 \) at time \( t=0 \). However, this need not be the case. The general expression for a sinusoidal wave travelling in the \( +x \) direction is shown in equation (2.4) [3:p. 296].
where the quantity in parentheses is called the phase of the wave and \( \tau \) is the phase constant. Moreover, the general expression for the lateral displacement of a given element of the wire, say \( x = \pi / k \), about its equilibrium position as a result of some harmonic input disturbance can be expressed in equation (2.5) [3:p 297].

\[
y = A_m \sin(\omega t + \tau) \tag{2.5}
\]

Standing Waves

Standing waves are the result of two wavetrains of the same frequency, speed, and amplitude which are traveling in opposite directions along a wire. For example, in a one-dimensional wire held relatively taut at both ends a distance \( L \) apart, traveling waves in the wire are reflected from the boundaries of the wire. Consequently, the reflected waves add to the incident waves according to the principle of superposition. The equation for a standing wave is expressed in equation (2.6) [3:p. 305].

\[
y = 2A_0 \sin(kx) \cos(\omega t) \tag{2.6}
\]

where \( A_0 \) is the amplitude of the incident and reflected
waves. Notice that the motion at any point $x$ along the wire can be described as simple harmonic motion and that all points oscillate at the same frequency. The amplitude varies with each point $x$ along the wire. Positions along the wire where the amplitude, $2A_0\sin(kx)$, has the maximum value of $2A_0$ are called antinodes, while, positions where the amplitude is minimized (zero) are called nodes. Since energy cannot flow past the nodal points, which are permanently at rest (equilibrium position), it is clear that energy is not transported along the wire, hence the energy remains "standing" in the wire and alternates between vibrational kinetic energy and elastic potential energy. Like for all elastic media, there exist forcing functions with certain special frequencies, called natural frequencies, that generate standing waves. The natural frequencies are functions of wire elasticity and length. This thesis deals primarily with the standing wave structure and the natural frequencies that produce the effects.

**Mechanical Wave Speed Characteristics**

Mechanical wave propagation speeds are based primarily upon the properties of the medium through which the wave is travelling. The medium's inertia and elasticity characteristics are prominent in the propagation wave speeds. The elasticity of a medium determines the restoring forces on any element of a vertically hanging wire displaced
from its equilibrium position; inertia of a medium indicates how the displaced portion of the elements will respond to restoring forces. Taken together, these two forces determine the wave propagation speed through the cable wire.

Elasticity is measured by the tension in the medium; the greater the tension, the greater the elastic restoring force on any one element. Inertia is a function of the mass per unit length, \( \mu \). To determine the restoring force of a dangling wire, one must consider the lateral (one-dimensional) forces acting upon the wire at some fixed time, \( t \). Consider a homogenous, flexible wire of length \( L \) and of density \( \mu \), as shown in Figure (2.4).

![Homogeneous Cable Wire](image)

Figure 2.4
Homogeneous Cable Wire

To determine the equation of motion for the wire with
respect to transverse vibrations (waves) the following assumptions are made:

1. The equilibrium position for the dangling wire is along the x-axis from 0 to L, which is completely vertical.
2. The motion of every element x along the wire oscillates in a direction perpendicular to the x-axis in the xy plane, and its position is specified by the coordinates x and y.
3. The maximum value of y is very small compared to the overall length L of the wire, the slope of the wire is small, and the vertical displacement of the wire can be neglected.
4. The tension of the wire T(x) is sufficiently great so that the tangent to the curve at element Δx makes a very small angle, φ with the x-axis.
5. Damping and frictional forces affecting the travelling transverse wave are negligible. The cable wire is perfectly flexible; it offers no resistance to bending and the wire is considered non extensible.

Under the above assumptions, the force exerted by the rest of the wire on the end points of any element of the wire is in the direction tangent to the wire, and the tension distribution T(x) of the wire is assumed to be constant and equal to its value at equilibrium, see Figure (2.3). The displacement from equilibrium, y, of each element of the wire is a function of wire length x and time t as shown in equation (2.7).
\( y = y(x,t) \) \hspace{1cm} (2.7)

As shown in Figure (2.4), consider the motion of an element \( x_i, x_i + \Delta x \) of wire length \( \Delta x \) with its midpoint at a height \( x_i + \Delta x/2 \). Assuming small displacements, where \( L \), the overall length of the wire is much greater than \( y(x,t) \) and, then \( \partial y(x,t)/\partial x \) is approximately equal to the tangent of the angle made by the direction of the tension force \( T(x) \) with respect to the vertical at time \( t \).

Applying Newton's second law to the wire element \( x_i, x_i + \Delta x \) shown in figure (2.4), the sum of the lateral force components acting upon an incremental element \( \Delta x \) of the wire is equal to the lateral inertial force component as shown in equation (2.8), where \( \mu \) is defined as the mass per unit length of wire.

\[ \Sigma F_y = (\mu \Delta x) \frac{\partial y^2(x,t)}{\partial t^2} \] \hspace{1cm} (2.8)

The following discussion will detail the forces acting upon a dangling cable wire. First, consider the wire segment \( x_0, x_2 \) as one element of the displaced cable wire shown in Figure (2.5). The two tensions \( T(x_0) \) and \( T(x_2) \) acting at ends \( x_0 \) and \( x_2 \) of the wire element \( \Delta x \) make angles \( \phi \) and \( \gamma \) with the vertical axis.
From the assumptions defined in Chapter 2, the element $\Delta x$ equals the length of the wire from $x_0$ to $x_1$ and using the small angle approximation, $\sin \gamma = \tan \gamma$ and $\sin \phi = \tan \phi$. The lateral displacements of the wire along the $y$-axis of the wire are defined by points $y_0$, $y_1$, and $y_2$. Defining $T(x_i)$ as the tension at the center point $x_i$ of the element and summing the lateral tension forces acting upon the element results in equation (2.9). Note from the assumptions in Chapter 2, the wire tension force $T(x)$ was not a function of the lateral displacement along the $y$-axis. These forces
represent the lateral component of the tensile forces acting upon the wire length \( \Delta x \). Note the tension distribution along the wire length increases linearly from \( x=0 \) to \( x=L \) and the distance from points \( x_i \) to both \( x_0 \) and \( x_2 \) is equal to \( \Delta x/2 \).

\[
\Sigma F_y = T(x_i + \frac{\Delta x}{2}) \left( \frac{y_1-y_2}{\Delta x/2} \right) - T(x_i - \frac{\Delta x}{2}) \left( \frac{y_0-y_1}{\Delta x/2} \right)
\]

Therefore, combining equations (2.8) and (2.9) and dividing through by \( \Delta x \) yields equation (2.10).

\[
\mu \frac{\partial y^2(x,t)}{\partial t^2} = \frac{T(x_i + \frac{\Delta x}{2}) \left( \frac{y_1-y_2}{\Delta x/2} \right) - T(x_i - \frac{\Delta x}{2}) \left( \frac{y_0-y_1}{\Delta x/2} \right)}{\Delta x}
\]

As \( \Delta x \) is reduced in size, the tension values \( T(x_0) \) and \( T(x_2) \) become more similar, as do the angles \( \phi \) and \( \gamma \). The latter means that \( y_1-y_2 \) becomes more similar to \( y_0-y_1 \), or \( \Delta y/2 \). Taking the limit as \( \Delta x \) goes to zero produces equation (2.11) at \( x=x_1 \).
Equation (2.11) defines the generic form of the net lateral forces acting upon a cable wire. The following Chapter will use the development of the generic form of the net lateral forces acting upon a dangling cable wire shown in equation (2.11) to define the analytical model with initial and boundary conditions.

\[ \mu \frac{\partial^2 y(x, t)}{\partial t^2} = \frac{\partial}{\partial x} \left( T(x) \frac{\partial y(x, t)}{\partial x} \right) \]
CHAPTER 3
DANGLING CHAIN MODEL DEVELOPMENT

The derivation of the classical dangling chain problem in the case of the vertically hanging chain to include a mass at the end of the chain wire was performed by Clifton [1]. An understanding of the derivation and development of the dangling chain problem was critical to the process of analytically solving the resulting governing equation, so much of Clifton’s development of the equations was repeated and elaborated upon in the following text. As stated in Chapter 2, the vibrations inherent in suspended wires are dominated by the lateral standing wave structure.

This dangling chain model is developed by first considering the chain without a fixed mass at the end and then incorporating the attachment of a dead weight. The final characteristic governing equation is then solved analytically to determine the appropriate eigenvalues and eigenvectors.

As indicated in figure (3.1), the origin of the coordinate system is established at the free end or dead weight at x=0 with the fixed upper end at x=L. The lateral displacement from equilibrium is defined as y(x,t). The assumption that the displacement from equilibrium, y(x,t) is small in comparison to overall length of the wire is valid and will be seen in both the analytical model output and
empirical data. The tension distribution for any point along the wire is defined by $T(x)$. Clifton discussed the validity of using the time independent tension distribution $T(x)$ for the dangling chain problem vice the time dependent $T(x,t)$. [1, p. 74].

![Figure (3.1)](image)

**Dangling Chain with Dead Weight**

The Greek letter, $\mu$, defines the mass per unit length of the wire. The net lateral force acting upon any portion of the chain was defined in Chapter 2 and is reproduced in equation (3.1) for convenience.

The initial form of the hanging chain equation of motion developed in Chapter 2, is repeated in equation (3.1). It was derived by equating the net lateral forces acting upon the wire to the inertial reaction.
The forcing function $F(x,t)$ is defined as the perpendicular input to the wire and has units of force per unit length at time $t$. This force is added to the right hand side of equation (3.1) to complete the balance of forces. Equation (3.2) defines the expression for the final form of the hanging chain equation of motion with forcing function $F(x,t)$ \[1:p. 173\].

\[\mu \frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial}{\partial x} (T(x) \frac{\partial y(x,t)}{\partial x}) + F(x,t) \] (3.2)

Volterra and Zachmanoglou detailed the boundary conditions for the classical dangling chain or vibrating string problem with one fixed and one free boundary condition \[2:pp. 418-420\]. The initial conditions assumes the dangling chain begins at rest and at the equilibrium state and shape (vertically hanging). Combining the initial conditions, the boundary conditions provided by Volterra and Zachmanoglou and equation (3.2), results in equation (3.3), the expression for the classical dangling chain problem with boundary and initial conditions \[1:p. 174\]. The boundedness condition imposed upon $y(L,t)$ precludes the physically impossible case of infinite displacements. Additionally,
the lateral displacement at the top of the wire element \( y(0,t) \) was assumed to be zero. Although in the experimental solution the upper gridpoint of the wire was used to input a sinusoidal displacement, the displacement was 1.1% of the overall length of the cable wire. Therefore, the assumption that the upper gridpoint was zero was completely valid as indicated by the results (see Chapter 6).

\[
\mu \frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial}{\partial x} \left( T(x) \frac{\partial y(x,t)}{\partial x} \right) + F(x,t)
\]

\[\text{B.C. } y(L,t) = 0 \quad y(0,t) \rightarrow \text{Bounded} \quad (3.3)\]

\[\text{I.C. } y(x,0) = 0 \quad \frac{\partial y}{\partial t}(x,0) = 0\]

Next, the boundary conditions of the model for the classical dangling chain is modified to incorporate the presence of the dead weight at the bottom of the wire. The addition of a dead weight, \( W \), at the end of the wire requires the addition of a third boundary condition. Applying Newton’s Second Law, the mass times acceleration of the weight has to balance the lateral force components on it. As depicted in Figure (3.2), the lateral force components acting upon the weight is equal to the lateral component of the tension, \( T(0) \), plus the forcing function \( F(0,t) \), at the dead weight attachment point, \( x=0 \). The addition of the third boundary conditions is shown in

21
equation (3.4). Equation (3.4) represents the modified governing equation with the initial and boundary conditions [1:p. 175].

\[ \mu \frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial}{\partial x} \left( T(x) \frac{\partial y(x,t)}{\partial x} \right) + F(x,t) \]

**B.C.** \( y(L,t) = 0 \quad y(0,t) \rightarrow \text{Bounded} \)

\[ F(0,t) + T(0) \frac{\partial y(0,t)}{\partial x} = \frac{W}{g} \frac{\partial^2 y(0,t)}{\partial t^2} \]

**I.C.** \( y(x,0) = 0 \quad \frac{\partial y}{\partial t}(x,0) = 0 \)

Figure (3.2)
Free-Body Force Diagram for the Dead Weight
The rest of this Chapter will discuss the analytical solutions for the dangling chain problem with and without the dead weight at the end of the wire, $x=0$. The case without the dead weight at the end of the wire was solved by applying the model for the dangling chain governing equation (3.3). The solution assumes a uniform tension distribution, $T(x)=\mu gx$, where $g$ is defined as the gravitational acceleration. Setting $F(x,t)$ to zero provides the homogenous solution. Substituting for $T(x)=\mu gx$ and $F(x,t)=0$ in equation (3.3) results in equation (3.5) [1:p. 176]. Note that $\mu$ was cancelled from both sides of the equation after the substitution for $T(x)$.

\[
\frac{\partial^2 y(x,t)}{\partial t^2} = g \frac{\partial}{\partial x} (x \frac{\partial y(x,t)}{\partial x})
\]  

(3.5)

By limiting the analysis of equation (3.5) to the homogenous solution, and applying the concept of linearity [4: pp. 29-30], further inspection of equation (3.3) reveals that the expression is a linear partial differential equation. Additionally, inspection of the boundary conditions of equation (3.3) reveals that they are linear and homogenous expressions as well. As a result, equation (3.5) is solved by the method of separation of variables. Assuming a solution in the form of equation (3.6) and substituting this expression into equation (3.5) results in equation (3.7)
\[ y(x, t) = Y(x) e^{i\omega t} \] \hspace{1cm} (3.6)

\[-Y(x) \omega^2 e^{i\omega t} = g \frac{\partial}{\partial x} \left( x \frac{\partial (Y(x) e^{i\omega t})}{\partial x} \right) \] \hspace{1cm} (3.7)

Applying the product rule to the right hand side of equation (3.7) and simplifying, results in equation (3.8), which has the initial form of Bessel's equation [1:p. 176].

\[ x \frac{d^2 Y(x)}{dx^2} + \frac{dY(x)}{dx} + \frac{\omega^2}{g} Y(x) = 0 \] \hspace{1cm} (3.8)

In order to change equation (3.8) into the exact form of Bessel's equation, a change of variable is performed in terms of \( z = 2q x^{1/2} \), where \( q^2 = \omega^2 / g \). Additionally, equation (3.8) is divided through by \( x \) to yield (3.8a).

\[ \frac{d^2 Y(x)}{dx^2} + \frac{1}{x} \frac{dY(x)}{dx} + \frac{\omega^2}{xg} Y(x) = 0 \] \hspace{1cm} (3.8a)

The change of variable is accomplished first by expanding each derivative in equation (3.8) in terms of \( z \). For example, the term \( dY(x)/dx \) was expanded as shown in equations (3.9) through (3.9b).
\[ \frac{dY(x)}{dx} = \frac{dY(x)}{dz} \frac{dz}{dx} \] (3.9)

where \[ \frac{dz}{dx} = \frac{q}{x^{1/2}} \] (3.9a)

yields \[ \frac{dY(x)}{dx} = \frac{dY(x)}{dz} \frac{q}{x^{1/2}} \] (3.9b)

The expansion then was completed for the second derivative term in (3.8) using the results from equation (3.9b) as shown in equations (3.9c) and (3.9d).

\[ \frac{d^2Y(x)}{dx^2} = -\frac{1}{2} qx^{-3/2} \frac{dY(x)}{dz} + qx^{-1/2} \frac{d}{dx} \frac{dY(x)}{dz} \] (3.9c)

where \[ dx = dz qx^{-1/2} \] (see equation 3.9a) (3.9d)

substituting equation (3.9d) for \( dx \) in equation (3.9c) and rearranging terms yields equation (3.9e).

\[ \frac{d^2Y(x)}{dx^2} = -\frac{1}{2} qx^{-3/2} \frac{dY(x)}{dz} + \frac{q^2}{x} \frac{d^2Y(x)}{dz^2} \] (3.9e)

Therefore, substituting equations (3.9b) and (3.9e) into equation (3.8a) yields equation (3.9f).

\[ -\frac{1}{2} \frac{q}{x^{3/2}} \frac{dY(x)}{dz} + \frac{q^2}{x} \frac{d^2Y(x)}{dz^2} + \frac{q}{x^{3/2}} \frac{dY(x)}{dz} + \frac{q^2}{x} Y(x) = 0 \] (3.9f)
Combining and rearranging terms produces equation (3.9g).

\[ \frac{q^2}{x} \frac{d^2 Y(x)}{dz^2} + \frac{q}{2x^{3/2}} \frac{dY(x)}{dz} + \frac{q^2}{x} Y(x) = 0 \]  
(3.9g)

Equation (3.9g) is then multiplied through by \(x/q^2\) to produce equation (3.9h).

\[ \frac{d^2 Y(x)}{dz^2} + \frac{1}{2q x^{1/2}} \frac{dY(x)}{dz} + Y(x) = 0 \]  
(3.9h)

Substituting the definition for \(z\) into equation (3.9h) transforms it into the form of Bessel’s equation (3.10) for which the tabulated solutions are well known [1:p. 177].

\[ \frac{d^2 Y(x)}{dz^2} + \frac{1}{z} \frac{dY(x)}{dz} + Y(x) = 0 \]  
(3.10)

The solution to equation (3.10) is in terms of Bessel functions of the second kind of zero order and therefore the general solution is written in the form of equation (3.11) [5:pp 6-30].

\[ Y(x) = AJ_0(z) + BY_0(z) \]  
(3.11)
where A and B are arbitrary constants and \( J_0(x) \) and \( Y_0(x) \) are Bessel's functions of zero order of the first and second kind, respectively. Therefore, equation (3.10) solutions for \( Y(x) \) are expressed by equation (3.12), where \( z \) is replaced by \( 2qx^{1/2} \), and \( q^2 = \omega^2/g \) [1:p. 177].

\[
Y(x) = AJ_0(2\omega \sqrt{\frac{x}{g}}) + BY_0(2\omega \sqrt{\frac{x}{g}}) \tag{3.12}
\]

The boundedness condition of equation (3.3) limited \( y(0,t) \) to physically realizable values while the Bessel function \( Y_0(z) \) becomes infinite at \( z=0 \). Consequently, the constant \( B \) has to equal zero to satisfy the boundary conditions. The simplified expression is included in equation (3.13). For the boundary condition \( y(L,t)=0 \) to be satisfied, \( \omega \) had to take on only those values where the Bessel function has a zero crossing. The number of possible frequencies are infinite, but countable, and thus \( \omega \) is annotated with the subscript \( n \).

\[
Y(x) = AJ_0(2\omega_n \sqrt{\frac{x}{g}}) \tag{3.13}
\]

The solutions of \( \omega_n \) for zero crossings can be obtained through numerous sources. Extensive tables of
values of $J_n(x)$, especially $J_0(x)$ and $J_1(x)$, have been calculated due to their applications to many physical processes. Additionally, various software packages - this author used MathCAD version 2.0 - are available for providing Bessel function values.

As mentioned above, there are an infinite set of zero crossing points corresponding to the infinite set of $\omega_n$ values. During the work of this thesis, these characteristic values of $\omega_n$, called eigenvalues, were substituted into equation (3.13) to obtain solutions for $Y(x)$. Thus each eigenvalue returned an eigenfunction. The homogeneous solution of equation (3.13) consisted of the infinite set of eigenvalues and associated eigenfunctions. Appendix A shows the calculations for the three lowest eigenvalues for the corresponding cable length and the eigenfunctions. Although there are an infinite number of eigenvalues and eigenfunctions, it is typical for the bulk of the information concerning the dynamics of a physical system to be contained in the first few modes.

Next, the homogeneous solution for the case of the dangling chain where a dead weight is attached at the end of the cable wire is developed. The dead weight, $W$, attached at the end of the cable wire is of mass $W/g$. Clifton and Volterra and Zachmanoglou compensate for the dead weight by reexpressing the weight as an equivalent length of cable wire with mass per unit length, $\mu$, as presented in equation
Although the dead weight can be modeled differently mathematically, expressing it in terms of cable wire length is conducive to previous solution techniques.

\[ L_{eq} = \frac{W}{\mu g} \quad (3.14) \]

Likewise, the tension distribution, \( T(x) \), is adjusted for the influence of the dead weight at the end of the cable wire as described in equation (3.15). The equation applies for \( 0 < x < L \).

\[ T(x) = (\mu x + \frac{W}{g}) g = \mu g (x + L_{eq}) \quad (3.15) \]

As was done for the dangling chain problem without a dead weight attached to the end, the solutions for the influence of the dead weight can be determined through the use of separation of variables. Beginning with equation (3.3) and limiting the analysis to the homogenous solution implies the forcing function, \( F(x) \), to equal zero. Substituting for \( T(x) \), and eliminating \( \mu \) from the equation, results in equation (3.16).

\[ \frac{\partial^2 y(x, t)}{\partial t^2} = g \frac{\partial}{\partial x} \left( (x + L_{eq}) \frac{\partial y(x, t)}{\partial x} \right) \quad (3.16) \]
Assuming a solution of the form of equation (3.6), substituting into equation (3.16) and differentiating the left hand side results in the modified governing differential equation (3.17) [1:p. 179]. Note the cancellation of the exponential terms from both sides of the equation.

\[
\frac{d}{dx} \left( (x+L_{eq}) \frac{dY(x)}{dx} \right) + \frac{\omega^2}{g} Y(x) = 0 \tag{3.17}
\]

Equation (3.17) does not apply to the portion of the cable wire beyond the original length, \(L\). A change of variable can be performed by defining \(c=x+L_{eq}\) and \(dx=dc\) to produce equation (3.18). The boundary conditions are stated in terms of the variable \(c\), incorporating the requirement that the chain is fixed at the upper end [1:p. 179].

\[
\frac{d}{dc} \left( c \frac{dY(c)}{dc} \right) + \frac{\omega^2}{g} Y(c) = 0 \tag{3.18}
\]

for \(L_{eq} < c < L + L_{eq}\)

**B.C.** \(Y(c) = 0\) at \(c = L + L_{eq}\)

\[\frac{dY(c)}{dc} + \frac{\omega^2}{g} Y(c) = 0 \quad \text{at} \quad c = L_{eq}\]
The right hand side of equation (3.18) can be simplified (product rule) to produce equation (3.19).

\[ c \frac{d^2 Y(c)}{dc^2} + \frac{dY(c)}{dc} + \frac{\omega^2}{g} Y(c) = 0 \]  

(3.19)

A change of variable is performed in terms of \( z_i = 2q^2 c^2 \) where \( q^2 = \omega^2 / g \) (see change of variable derivation equations (3.9) through (3.10) to yield the form of Bessel's equation (3.20).

\[ \frac{d^2 Y(c)}{dz_i^2} + \frac{1}{z_i} \frac{dY(c)}{dz_i} + Y(c) = 0 \]  

(3.20)

As with the case of the dangling chain without the dead weight attached to one end, the solution is in the form of Bessel functions of the second kind of zero order in which the solution is expressed in equation (3.21). Note \( z_i \) has been replaced by its definition.

\[ Y(c) = A J_0 \left( 2 \omega \sqrt{\frac{c}{g}} \right) + B Y_0 \left( 2 \omega \sqrt{\frac{c}{g}} \right) \]  

(3.21)

Unlike the previous dangling chain problem, the boundary conditions for equation (3.18) do not allow that the constant B equals zero for the elimination of the \( Y_0 \).
Bessel function. The requirement for the satisfaction of the boundary conditions in equation (3.18) and the general solution expressed in equation (3.21) allows the problem to be formulated as the coupled equations shown in (3.22) and (3.23) [1:p. 180].

\[ AJ_0(\beta) + BY_0(\beta) = 0 \]  
\[ A[J_1(\alpha) - \frac{1}{2} \alpha J_0(\alpha)] + B[Y_1(\alpha) - \frac{1}{2} \alpha Y_0(\alpha)] = 0 \]

where \( \beta = 2\omega \sqrt{\frac{L + L_{eq}}{g}} \)

where \( \alpha = 2\omega \sqrt{\frac{L_{eq}}{g}} \)

Equation (3.22) can be solved for the constant \( B \), which is substituted into equation (3.23) which is divided through by constant \( A \) resulting in expression (3.24).

\[ [J_1(\alpha) - \frac{1}{2} \alpha J_0(\alpha)] - \frac{J_0(\beta)}{Y_0(\beta)} [Y_1(\alpha) - \frac{1}{2} \alpha Y_0(\alpha)] = 0 \]

Equation (3.24) defines the characteristic equation for the solution to the dangling chain with dead weight attached to one end. The software package MathCAD version 2.0 was used to solve equation (3.24) for the zero crossing points (eigenvalues). Appendix A details the calculations.
The first three eigenvalues are shown in equation (3.25).

\[
\begin{align*}
\omega_1 &= 0.85 \text{ (rad/sec)} \\
\omega_2 &= 3.22 \text{ (rad/sec)} \\
\omega_3 &= 6.04 \text{ (rad/sec)}
\end{align*}
\]
CHAPTER 4
NUMERICAL SOLUTIONS FOR THE DANGLING CHAIN PROBLEM

This Chapter calculates the solutions to the classical dangling chain problem by applying numerical methods. One of the major engineering applications of numerical methods is the solution of differential equations, ordinary and partial. Solutions for differential equations involve estimating derivatives. There are three different approaches to finding derivatives from a limited data set. The first method uses Lagrange interpolation techniques to fit smooth curves through a data set and differentiate the function to determine the required results. Another method is to use Taylor series expansions to form linear combinations of terms to solve for the derivative desired. This approach yields the same results of the interpolation techniques, however, an estimate of the error of the result can be made. The third method involves numerical integration techniques where interpolation techniques are used to solve integration problems.

Finite-differencing techniques, derived from the truncated Taylor Series expansion of the governing equation, were used to solve the dangling chain problem. Specifically, the second order accurate, central differencing scheme, was applied to the governing equation.
and boundary conditions, repeated in equation (4.1).

\[
\mu \frac{\partial^2 y(x, t)}{\partial t^2} = \frac{1}{\partial x} \left( T(x) \frac{\partial y(x, t)}{\partial x} \right) + F(x, t)
\]

**B.C.** \[y(L, t) = 0 \quad y(0, t) = \text{Bounded}\]

\[
T(0) \frac{\partial}{\partial x} y(0, t) = \frac{h}{g} \frac{\partial^2 y(0, t)}{\partial t^2}
\]

**I.C.** \[y(x, 0) = 0 \quad \frac{\partial}{\partial t} (x, 0) = 0\]

The derivation of the finite-difference approximation of equation (4.1) involved the approximation of derivatives in terms of values at a discrete element or point. One of the simplest methods for obtaining finite difference approximations to derivatives is by using the Taylor Series expansions presented in equation (4.2).

\[
f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots + \frac{h^n}{n!} f^{(n)}(x) + \ldots = \sum \frac{h^k}{k!} f^{(k)}(x)
\]

\[
f(x - h) = f(x) - hf'(x) + \frac{h^2}{2!} f''(x) - \ldots + \frac{h^n}{n!} f^{(n)}(x) + \ldots = \sum \frac{h^k}{k!} f^{(k)}(x)
\]

where \( h \) is the spacing between points and \( f^{(k)}(x) \) denotes the \( k \)th derivative of \( f(x) \). For the expansion to have any practical benefit the series must be truncated with some
finite number of terms. Taylor's Theorem, presented in equation (4.3) defined the nature of the error involved after truncation.

\[ f(x+h) = \sum_{k=0}^{n} \frac{h^k}{k!} f^{(k)}(x) + E \tag{4.3} \]

where \( E = h^{n+1} f^{(n+1)}(\xi) / (n+1)! \), \( x \leq \xi \leq x+h \).

Solving for \( f'(x) \) for the first Taylor expansion presented in equation (4.2) resulted in equation (4.4), which can be approximated as in equation (4.5).

\[ f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{1}{2} h^2 f''(x) - \ldots - \frac{h^n}{n!} f^{(n)}(x) \tag{4.4} \]

\[ f'(x) \approx \frac{f(x+h) - f(x)}{h} \tag{4.5} \]

Additionally, \( f'(x) \) was solved for with the subtraction of the second Taylor expansion from the first which resulted in equation (4.6).

\[ f'(x) = \frac{f(x+h) - f(x-h)}{2h} \tag{4.6} \]

Similarly, the second derivative of function \( f(x) \) was
approximated using the equation shown in (4.7).

\[ f''(x) = \frac{[f(x+h)-2f(x)+f(x-h)]}{2h^2} \] (4.7)

Several assumptions were made in the numerical solution of the dangling chain problem. First, as in the analytical solution, the movement of the cable in response to harmonic inputs were assumed to be one-dimensional in the lateral direction.

Note that the analysis could have been extended to account for element motion in two dimensions by applying the exact same technique to the orthogonal coordinate direction within the plane of motion. The two dimensional analysis was not needed here and so the single dimensional analysis was continued.

The wire was divided into 200 grid segments (199 equal lengths) from the top of the wire to the bottom of the concentrated weight. Using equation (4.7), the central difference approximation of the time derivative component of equation (4.1) was developed with \( n \) defined as the grid segment and \( m \) as the time step index as shown in equation (4.8). Note \( y = y(x,t) \).

\[ \frac{\partial^2 y_{n,m}}{\partial t^2} = \frac{y_{n,m+1} - 2y_{n,m} + y_{n,m-1}}{2\Delta t^2} \] (4.8)
The central difference approximation of $y$ with respect to $x$ was determined by referring to figure (4.1) as shown in equation (4.9). Note the derivative approximation was determined for the point between the grid segments; in other words, taken about the half step.

$$\frac{\partial y_{n+\frac{1}{2},m}}{\partial x} = \frac{y_{n+1,m} - y_{n,m}}{2 \left( \frac{1}{2} \Delta x \right)}$$  \hspace{1cm} (4.9)$$

The technique was then repeated for the approximation with respect to $x$ of the product of $T(x)$ and the derivative of $y$ with respect to $x$ as shown in equation (4.10) and simplified to get equation (4.11). Note the tension at $x$, $T(x)$ was determined between two successive grid elements and the derivative was taken again about the half step.

$$\frac{\partial}{\partial x} (T(x) \frac{\partial y_{n,m}}{\partial x}) = \frac{T_{n+\frac{1}{2}} \left( \frac{y_{n+1,m} - y_{n,m}}{\Delta x} \right) - T_{n-\frac{1}{2}} \left( \frac{y_{n,m} - y_{n-1,m}}{\Delta x} \right)}{2 \left( \frac{1}{2} \Delta x \right)}$$  \hspace{1cm} (4.10)$$

$$\frac{\partial}{\partial x} (T(x) \frac{\partial y_{n,m}}{\partial x}) = \frac{T_{n+\frac{1}{2}} \left( \frac{y_{n+1,m} - y_{n,m}}{\Delta x} \right) - T_{n-\frac{1}{2}} \left( \frac{y_{n,m} - y_{n-1,m}}{\Delta x} \right)}{\Delta x^2}$$  \hspace{1cm} (4.11)$$

Substituting equations (4.8) and (4.11) into equation (4.1) resulted in the finite difference approximation of the
governing PDE for the dangling chain with a dead weight attached to one end as shown in equation (4.12). The forcing function \( F(n,m) \), was assumed zero and was eliminated from the equation.

\[
\mu \left( \frac{y_{n,m+1} - 2y_{n,m} + y_{n,m-1}}{2\Delta t^2} \right) = \frac{T_n \cdot \frac{1}{2} (y_{n+1,m} - y_{n,m}) - T_{n-\frac{1}{2}} (y_{n,m} - y_{n-1,m})}{\Delta x^2} + F_{n,m}
\]

(4.12)

As shown in equation (4.12) the entire central difference setup was taken about the point \( n,m \).
Additionally, since \( T(x) \) was constant over time, the expression was fully explicit in the variable \( y_{n,m+1} \) as shown in equation (4.13). Note that this explicit equation required knowledge of the two previous gridpoints in both space and time. This was only a difficulty at the beginning of the chain and as the model was started. The solution was provided through the use of the boundary and initial conditions.

\[
y_{n,m+1} = \frac{2\Delta t^2}{\mu} \left[ \frac{T_n \cdot \frac{1}{2} (y_{n+1,m} - y_{n,m}) - T_{n-\frac{1}{2}} (y_{n,m} - y_{n-1,m})}{\Delta x^2} \right] + 2y_{n,m} - y_{n,m-1}
\]

(4.13)
Reviewing the boundary conditions of equation (4.1) showed that the first grid segment (top of the wire) was essentially pinned and therefore $y_{1,m}$ equaled zero and as described in Chapter 3, the physical system was bounded. However, the two boundary conditions were not helpful to determine a solution at the bottom of the wire with a concentrated dead weight, grid segment $y_{N,m+1}$. As shown in equation (4.13), in order to calculate $y_{N,m+1}$, $y_{N+1,m}$ was required. The problem was solved with the application of the third boundary condition of equation (4.1). The boundary condition showed that the tension $T(x)$, times the derivative of $y$ with respect to $x$ at the bottom grid segment at any time $t$ was equal to the acceleration at the grid segment times the weight of the concentrated mass. The boundary condition ensured force equilibrium at the end grid segment, $N$. Application of the central difference formula resulted in the expression (4.14).

$$
T_N \left( \frac{y_{N-1,m} - y_{N-1,m} - y_{N+1,m}}{2\Delta x} \right) = \frac{W}{g} \left( \frac{y_{N+1,m} - 2y_{N,m} + y_{N-1,m}}{2\Delta t^2} \right) \quad (4.14)
$$

Equation (4.14) described the balance between the tension forces and the lateral acceleration of the concentrated mass. An assumption was made that the forces acting upon the dead weight were much greater than the forces on the last bit of wire. Based upon the assumption,
the wire slope was nearly constant, since the shape of the wire was dependent upon the dead weight forces and not the distributed wire forces. Therefore, the arbitrarily defined point \( y_{N+1} \), beyond the end grid segment, was used to approximate the wire slope at gridpoint \( N \), using the central difference formula [1:p. 77]. As shown in equation (4.15) this was done by equating the slope at the half step before and after the dead weight.

\[
\frac{y_{N,m} - y_{N-1,m}}{\Delta x} = \frac{y_{N+1,m} - y_{N,m}}{\Delta x}
\]  

(4.15)

Additionally, since the forces upon the dead weight were assumed to be much greater than the wire, the tension forces were assumed to be equal at each half step, \( T_N = T_{N+1/2} = T_{N-1/2} \). Finally, to determine \( y_{N,m+1} \), equation (4.15) was solved for \( y_{N+1,m} \) and substituted into equation (4.14). Next, \( T_N \) in equation (4.14) was replaced by \( T_{N-1/2} \) and the resulting expression was solved for \( y_{N,m+1} \) as shown in equation (4.16). Equation (4.16) was in terms of previously known quantities.

The numerical technique was begun from steady-state, equilibrium condition (no motion, vertically hanging) so that the two previous time step position sets were known.
\[ y_{N,m+1} = \frac{2\Delta t^2 g}{W} \left[ T_{N'-\frac{1}{2}} \left( \frac{y_{N-1,m} - y_{N,m}}{\Delta x} \right) \right] + 2y_{N,m} - y_{N,m-1} \]  

(4.16)

As shown previously, the boundary conditions determine the two end grid segments and equation (4.7) was used to determine the internal gridpoints. Appendix B includes the Fortran program that was used to calculate the solutions for the dangling chain problem.
CHAPTER 5

EXPERIMENTAL APPROACH TO THE DANGLING CHAIN PROBLEM

Chapter 3 discussed the analytical approach toward developing solutions to the classical dangling chain problem. Chapter 4 was the numerical approach. The numerical analysis involved the use of finite differencing computational techniques to approximate derivatives and calculate the motion of the dangling chain. This Chapter will provide a hands-on experimental approach for determining the time-dependent motion of the dangling chain.

The development of an experimental technique necessitated the construction of a long cable and a location from which to hang the wire. Table (5.1) lists the material/resources used for the construction of the cable. The cable length was made long enough to take advantage of the available space and to maximize the validity of assumptions made during the model development (see Chapter 3). The six inch hollow tube was attached to the upper end of the cable using the 1/4 inch clip and thimble. The hollow tube allowed for easier grasping of the cable when inputting forcing functions manually. The barbell weight was attached to the attachment hook with a 1/4 inch nut and stabilized with washers on each side. A 1/4 inch clip and thimble was used to attach the weighted hook to the cable.
which completed the construction process.

Table 5.1
CABLE MATERIALS

<table>
<thead>
<tr>
<th>MATERIAL</th>
<th>SIZE</th>
</tr>
</thead>
<tbody>
<tr>
<td>CABLE (STAINLESS STEEL)</td>
<td>43.7 FT</td>
</tr>
<tr>
<td>CABLE CLIPS &amp; THIMBLE</td>
<td>1/4 IN</td>
</tr>
<tr>
<td>BARBELL WEIGHT (SOFT)</td>
<td>3.0 LB</td>
</tr>
<tr>
<td>ATTACHMENT HOOK</td>
<td>6.0 IN</td>
</tr>
<tr>
<td>HOLLOW TUBING</td>
<td>6.0 IN</td>
</tr>
<tr>
<td>ENAMEL PAINT</td>
<td>ORANGE</td>
</tr>
</tbody>
</table>

Note 1: Includes 1/4" nut, 2 1/4" washers

The cable was hung from a catwalk inside of one of the hangars at Force Warfare Aircraft Test Directorate, NAS Patuxent River, Maryland. The dead weight was attached to the dangling end of the cable. The cable thickness was selected due to its required elasticity and the potential visibility of the cable when filmed. The amount of weight attached to the end of the cable was the minimal necessary to keep the cable taut, without kinks or bending. Visibility was enhanced by spray painting the whole cable with an orange color enamel paint. Inputs were facilitated by
manually grasping the hollow tube at the top of the wire with both hands and moving the cable horizontally. The initial setup consisted of dangling the cable wire with attached weight from the catwalk inside the Force Warfare hangar. The distance from the catwalk to the hangar floor was approximately 50 feet which provided clearance for the cable wire with attached weight which was 43.625 feet in length. The cable wire was completely vertical in the steady-state condition \( \frac{\partial y(x,t)}{\partial t} = 0 \) for each initial setup. From the initial steady-state condition, sinusoidal displacements to the upper gridpoint were initiated. The initial forcing function corresponded with the analytical solutions (eigenvalues). The time required to complete a period for each eigenvalue was used as the basis for standardizing the manual input. Additionally, to standardize the amplitude of the input, the displacement was limited to approximately 1/2 foot either side of the origin. The upper gridpoint sinusoidal displacements were inputted manually. Detailed attention was placed on maintaining the proper input frequency and amplitude; however, the frequency was adjusted when required in order to achieve resonance. The cable wire shape was filmed on VHS format and photographed to present the appropriate eigenvector shape. The experiment was conducted for each eigenvalue calculated through the analytical process (see Chapter 3). The eigenvalue frequency was obtained by visually counting the
number of times the cable wire passed through a designated point on the hangar wall (behind the wire) divided by the elapsed time. The calculations are depicted in Chapter 6, Table 6.1.
CHAPTER 6
RESULTS AND DISCUSSION

In this Chapter, comparison of the results from the analytical, numerical, and experimental solutions will be discussed to include accuracy of the solutions and the utility of the exercise as an instructional tool for the modeling approach of trailing wire antennas. The premise of the exercise was to develop solutions through the analytical and numerical approaches and then to compare the results to the lab experiment.

The accuracy of the modeling was validated by comparison of the eigenvalues and eigenvectors calculated in the analytical solution to those found in the numerical simulation and the empirical experiment. The analytically derived eigenvalues were used to determine the frequency for the forcing functions in the numerical model and cable wire experiment. The inputs consisted of sinusoidal displacements of the upper gridpoint at the selected eigenfrequency. This was done for both the numerical model (see Appendix B) and for the experiment as discussed in Chapter 5. The sinusoidal displacement for the upper gridpoint was at the analytically calculated eigenvalue from the analytical solutions with an amplitude of approximately 0.5 ft. Graphs of specific eigenvectors from the numerical model and cable
wire experiment were produced to show correlation as shown in figures B.1 through B.3 in Appendix B. Plots of the analytical solutions were beyond the scope of this thesis due to the complexity in solving the time dependent constants A and B (see Chapter 3, equation (3.21) for the Bessel function of the second kind of zero order). However, the eigenfrequencies (0.85, 3.22, and 6.04 rad/sec) define the first, second, and third modes that describe the motion of the cable wire. These analytically derived eigenvalues (also known as eigenfrequencies) returned the corresponding eigenvectors. The distinctive shape of eigenvectors, depending on the specific eigenvalues, are characterized by the zeros or crossings (excluding the endpoints). For example, if the nth mode (eigenfrequency) is calculated, the corresponding eigenvectors will have n-1 crossings (zeros). For the analytically calculated second mode (3.22 rad/sec) the corresponding eigenvector should have one zero crossing. Therefore, the solutions from the analytical derivation were used to test the validity of the numerical and experimental models. The numerical and experimental models were validated by plotting the largest lateral displacement eigenvectors. Instantaneous snapshots of the numerical model and experimental wire shape at various times during one period of the eigenfrequency were plotted and the corresponding eigenvectors were seen to exist (see Appendix B). Since the dynamics of the model were completely
described by the eigenvalues and eigenvectors when forced at a single eigenfrequency, the numerical model was validated.

The three largest eigenvalues were tested. The eigenfrequency was validated by plotting the time history of a single gridpoint and examining the period. The experimentally derived period of the eigenfrequency was measured directly by noting the period of the sinusoidal displacement of points on the wire as discussed in Chapter 5. The inputs were provided at frequencies approximating the analytically derived eigenfrequencies. The period was then adjusted to achieve resonance. Some technique was involved but the resonance at the eigenfrequency was quite distinct and returned the eigenvector quite clearly. Table (6.1) shows the solutions from the analytical, numerical, and experimental derivations.

<table>
<thead>
<tr>
<th>Eigenvalues</th>
<th>Analytical (rad/sec)</th>
<th>Numerical (rad/sec)</th>
<th>Experiment (rad/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>First</td>
<td>0.85</td>
<td>0.87</td>
<td>0.8</td>
</tr>
<tr>
<td>Second</td>
<td>3.22</td>
<td>3.19</td>
<td>3.3</td>
</tr>
<tr>
<td>Third</td>
<td>6.04</td>
<td>6.10</td>
<td>6.2</td>
</tr>
</tbody>
</table>

As shown in table (6.1) the first three eigenvalues for the numerical and experimental solutions varied at most 5.8%
(first eigenvalue difference between analytical and experimental solutions) from the analytical solutions.

The development of analytical solutions and the corresponding numerical models coupled with hands-on experiments provide a robust teaching tool. As shown in this thesis, this instruction technique has unique qualities that can provide a well rounded and complete analysis for applied numerical analysis, advanced dynamics or experimental course material.

Based upon the work from this thesis, the following follow-on projects are possible for further research.
1. Fully investigate the stability and accuracy of the numerical approximation.
2. Determine the validity of the models for a range of cable lengths and diameters, weight sizes and test configurations.
3. Study the limits of lateral displacements for which the analytical and numerical models are valid.
4. Investigate the characteristics of the travelling lateral wave, to include the determination of the wave propagation speed.
5. Expand the analysis for the two dimensional case.
6. Investigate the usefulness of other lab experiments that can be used as an instructional tool that do not require extensive requirements for lab space.
LIST OF REFERENCES


APPENDICES
APPENDIX A
ANALYTICAL SOLUTIONS FOR DANGLING CHAIN
WITH ATTACHED WEIGHT
The model development of the dangling chain was discussed in Chapter 3 of the text. This development was further modified to include a concentrated mass attached at the end of the chain. Equation (3.5) represented the governing equation for the classical dangling chain problem. The equation was in the form of a homogeneous linear partial differential equation with linear boundary conditions. As described in Chapter 3, there existed an infinite set of unique nontrivial solutions called eigenvalues that satisfied the boundary value problem. Likewise, a nontrivial function called an eigenfunction existed for each corresponding eigenvalue. Equation (3.24) described the characteristic equation for the dangling chain with an attached dead weight. The specific values of $\omega$ (eigenvalues) which satisfied the equation were determined with the aid of the MathCad 2.0 application program. The solutions of the characteristic equation were determined for omega values ranging from 0.1 - 10 rad/sec. Additionally, ASCII matrix files were created in order to create a plot of $S(\omega)$ vs $\omega$, as shown in figure A.2. Solutions which equaled zero (zero crossings) corresponded to eigenvalues that satisfied the characteristic equation. As shown in figure A.2, the three lowest eigenvalues in rad/sec were 0.85, 3.22, and 6.04. Table (6.1) in Chapter 6 compared the
analytical, numerical, and experimental solutions of the classical dangling chain problem.
The following mathematical equations for the classical dangling chain problem were derived and developed in Chapter 3 of the text. The derivation describes the analytical solutions to the cable wire with a weight attached to the free end. All definitions and measurements from the actual experimental setup described in Chapter 5. The solutions were the three lowest eigenvalues for the cable wire length. \( S(w) \) defines the eigenvector for each individual eigenvalue. A plot of \( S(w) \) versus omega, \( w \), is shown in figure A.2. The zero crossings of the plot \( (S(w) \text{ vs } w) \) determined the eigenvalues.

Length of cable wire (ft) \( L := 43.625 \)
Weight of cable wire (lb) \( W_c := 5.2 \)
Weight of mass at end of wire (lb) \( W := 3.8 \)
Specific gravity \( g := 32.174 \)
Specific density of wire (lb/ft) \( u_g := 0.119 \)
Equivalent length of cable with mass (ft) \( \frac{W}{u_g} \)
Omega (radians/sec) \( w := .1,.102,.10 \)

Definition of Bessel function variables
\[
a(w) := 2 \cdot \frac{\sqrt{\frac{L_{eq}}{g}}}{w} \\
B(w) := 2 \cdot \sqrt{\frac{L + L_{eq}}{g}}
\]

Characteristic equation (see equation (3.23) in Chapter 3.)
\[
S(w) := \left[ \frac{J_1(a(w)) - \frac{1}{2} a(w) \cdot J_0(a(w))}{J_0(B(w)) \cdot Y_1(a(w)) - \frac{1}{2} a(w) \cdot Y_0(a(w))} \right]
\]

Characteristic equation solution matrix file

\( \text{WRITEPRN[SOLEQN MAT]} := S(w) \)

Omega (rad/sec) matrix file

\( \text{WRITEPRN[OMEGA MAT]} := w \)

Figure A.1

Mathematical Equations for Dangling Chain Problem
Figure A.2
Characteristic Equation Zero Crossings
APPENDIX B
NUMERICAL SOLUTION FOR THE DANGLING CHAIN
WITH ATTACHED WEIGHT
NUMERICAL SOLUTION FOR THE DANGLING CHAIN WITH ATTACHED WEIGHT

THIS PROGRAM IS A NUMERICAL COMPUTATION PROGRAM WRITTEN IN FORTRAN TO SOLVE THE CLASSICAL DANGLING CHAIN PROBLEM AS DEVELOPED IN CHAPTER 4 OF THE TEXT. ALL THE SCALARS AND VARIABLES ARE DEFINED AND EXPLAINED IN DETAIL. THE BASIC COMPUTATIONAL EQUATION FOR THE DETERMINATION OF THE INTERNAL GRID POINTS IS IDENTICAL TO EQUATION (4.13); EQUATION (4.16) SOLVED THE BOUNDARY CONDITION FOR THE END GRID POINT (EQUATION (4.1)); AND THE TOP BOUNDARY CONDITION FROM EQUATION (4.1) IS USED TO DETERMINE THE TOP GRID POINT SOLUTION.

THE PROGRAM WAS RUN THREE TIMES FOR EACH OF THE EIGENVALUES THAT WERE CALCULATED FROM THE ANALYTICAL SOLUTION. EACH VALUE WAS ADJUSTED AFTER THE INITIAL RUN TO REFINE THE EIGENVECTOR PLOT AND TO SHOW THE DESIRED EIGENVECTOR STRUCTURE. THE PROGRAM WAS BASED UPON CLIFTON'S WORK [REF 1] WITH RESPECT TO THE PLOT STRUCTURE.

DECLARE AND DIMENSION VARIABLES.

SCALARS

D IS THE DIAMETER OF THE WIRE.

REAL D

DELTAL IS THE INCREMENT OF WIRE LENGTH AT THE N'TH GRIDPOINT

REAL DELTAL

DELTAT IS THE TIME STEP INCREMENT.

REAL DELTAT

G IS THE ACCELERATION DUE TO GRAVITY.

REAL G

MHU IS THE MASS OF WIRE PER UNIT LENGTH.

REAL MHU

EQN1, EQN2, EQN3 ARE DUMMY VARIABLES USE TO BREAK UP EQUATIONS.

REAL EQN1, EQN2, EQN3

THEDOT IS THE STEP INPUT IN RADIANS PER SECOND (HARMONIC INPUT).

THE VALUES OF THEDOT WERE CALCULATED FROM THE ANALYTICAL SOLUTION AND WERE USED AS THE INITIAL VALUES TO BEGIN THE NUMERICAL ANALYSIS. THE VALUES WERE VARIED ACCORDING TO ESTABLISH RESONANCE AND TO OBTAIN THE DESIRED GRAPHS THAT SHOWED CLEARLY THE EIGENVECTOR STRUCTURE.

REAL THEDOT

WD IS THE WEIGHT OF THE MASS ATTACHED TO THE END OF THE WIRE.

REAL WD

INTEGERS.

I IS AN INDEX USED FOR VARIOUS DO LOOP COMPUTATIONS.

INTEGER I

K IS THE NUMBER OF TIME STEPS DESIRED.

INTEGER K

M IS THE TIME STEP INDEX.

INTEGER M

N IS THE GRIDPOINT INDEX FROM THE TOP TO BOTTOM ALONG THE CABLE WIRE.

INTEGER N

ARRAYS.

DEFINES ARRAY FOR EACH GRID POINT WITH THREE TIME STEPS.

REAL X(200,3)

DEFINES THE TENSION DISTRIBUTION FOR THE ENTIRE LENGTH OF THE CABLE WIRE.

REAL T(200)

REAL 2200(40000)

60
C INITIALIZE CONSTANTS.
C
C D IS THE DIAMETER OF THE WIRE (FT).
D=.25/12.0
C DELTAT IS THE TIME STEP INCREMENT (SECONDS).
DELTAT=0.0025
C DELTAL IS THE INCREMENT OF WIRE LENGTH AT THE N'TH GRIDPOINT (FT).
DELTAL=0.21922
C G IS THE ACCELERATION DUE TO GRAVITY (FT/SEC**2).
G=32.174
C MHU IS THE MASS OF WIRE PER UNIT LENGTH (LBF/FT*G).
MHU=0.119/G
C THEDOT IS THE STEP INPUT IN RADIANS PER SECOND (HARMONIC INPUT).
C INPUTTED VALUES WERE 0.85, 3.22, AND 6.04. AS DISCUSSED EARLIER,
C THESE VALUES WERE SOLUTIONS TO THE ANALYTICAL SOLUTION.
THEDOT=0.85
C REAL WD IS THE WEIGHT OF THE MASS ATTACHED TO THE END OF THE WIRE (LBF).
WD=3.8
WRITE(6,'*')'NUMBER OF TIME STEPS DESIRED:'
READ(5,*)K
Z200(1)=0.0
Z200(2)=0.0
C
C INITIALIZE WIRE AT THE STEADY STATE EQUILIBRIUM POSITION
C
DO 100 1=1,200
X(I,1)=0.0
X(I,2)=0.0
100 CONTINUE
C
C DETERMINE THE TENSION DISTRIBUTION OF THE WIRE AT EACH GRID POINT;
C THE TENSION IS EQUAL TO THE WEIGHT OF THE DROGUE PLUS THE AMOUNT
C OF WIRE BENEATH THE NTH GRID POINT.
C
DO 110 1=1,200
T(201-1)=WD+MHU*DELTAL*(I -1.0)*G
110 CONTINUE
C
C THE OUTER LOOP DETERMINES THE NUMBER OF TIME STEPS ENTERED.
C
DO 5000 M=3,K
C
C CALCULATE THE TOP END POSITION.
C
X(1,3)=1*SIN(THEDOT*(M-2)*DELTAT)
DO 1000 N=2,199
C
C THE DANGLING CHAIN CALCULATION DEVELOPED IN CHAPTER 4. EQN1
C DEFINES THE FIRST PORTION OF EQUATION (4.13); EQN2 CORRESPONDS
C TO THE MIDDLE PORTION OF EQUATION (4.13); AND EQN3 DEFINES
C THE LAST TWO MATHEMATICAL COMPUTATIONS OF EQUATION (4.13).
C
EQN1=DELTAT**2/MHU
EQN2=((T(N+1)+T(N))/2)*((X(N+1,2)-X(N,2))-
((T(N)+T(N-1))/2)*((X(N,2)-X(N-1,2))
EQN3=2*X(N,2)-X(N,1)
XN=EQN1*(EQN2/DELTAL**2)+EQN3
X(N,3)=XN
1000 CONTINUE
C
C DETERMINES THE END WEIGHT GRID POINT AS PER EQUATION (4.14)
C
EQN1=DELTAT**2*G/WD
EQN2=1*(X(200,2)-X(200,1))
EQN3=2*X(200,2)-X(200,1)
XN=EQN1*(EQN2/DELTAL)+EQN3
X(200,3)=XN
C
C
COMPUTATIONAL SCHEME THAT SAVES THE LAST THREE TIME STEPS
AT A NTH GRID POINT AND DISCARDS THE M-1 TIME STEP THAT IS NOT
REQUIRED FOR FURTHER CALCULATIONS.

DO 1800 I=1,200
X(I,1)=X(I,2)
X(I,2)=X(I,3)
1800 CONTINUE

SAVES CERTAIN ARRAYS SIGNIFYING THE CABLE WIRE POSITION
AT EACH GRID POINT AT CERTAIN TIME STEP INTERVALS. THE
INPUT WAS 10000 STEP INTERVALS. THE FINAL PLOTS WERE A
COMPILATION OF THE CABLE SHAPE FOR THE LAST PERIOD INPUT WHERE
THE 10000TH TIME STEP WAS PLOTTED AND EXACTLY ONE PERIOD BACK
WAS PLOTTED TO SHOW THE SHAPE OVER A COMPLETE PERIOD. THE LAST
TEN POINTS WERE SAVED BUT ONLY THE FIVE EIGENVECTORS WERE
OVER THE PREVIOUS PERIOD. PLOTS FOR EACH EIGENVALUES (0.85
RAD/SEC, 3.22 RAD/SEC, AND 6.04 RAD/SEC) ARE SHOWN IN FIGURES
B-1 THROUGH B-6.

Z200(M)=X(200,3)
IF (M .EQ. 9622) THEN
OPEN (UNIT=11,FILE='DATA1.MAT')
DO 1850 I=1,200
WRITE(11,*) X(I,3)
1850 CONTINUE
CLOSE(11)
ENDIF

IF (M .EQ. 9664) THEN
OPEN (UNIT=12,FILE='DATA2.MAT')
DO 1860 I=1,200
WRITE(12,*) X(I,3)
1860 CONTINUE
CLOSE(12)
ENDIF

IF (M .EQ. 9706) THEN
OPEN (UNIT=13,FILE='DATA3.MAT')
DO 1870 I=1,200
WRITE(13,*) X(I,3)
1870 CONTINUE
CLOSE(13)
ENDIF

IF (M .EQ. 9748) THEN
OPEN (UNIT=14,FILE='DATA4.MAT')
DO 1880 I=1,200
WRITE(14,*) X(I,3)
1880 CONTINUE
CLOSE(14)
ENDIF

IF (M .EQ. 9790) THEN
OPEN (UNIT=15,FILE='DATA5.MAT')
DO 1890 I=1,200
WRITE(15,*) X(I,3)
1890 CONTINUE
CLOSE(15)
ENDIF

IF (M .EQ. 9832) THEN
OPEN (UNIT=16,FILE='DATA6.MAT')
DO 1900 I=1,200
WRITE(16,*) X(I,3)
1900 CONTINUE
CLOSE(16)
ENDIF

62
IF (M .EQ. 9874) THEN
OPEN (UNIT=17,FILE='DATA7.MAT')
DO 1910 I=1,200
WRITE(17,*) X(I,3)
1910 CONTINUE
CLOSE(17)
ENDIF

C
IF (M .EQ. 9916) THEN
OPEN (UNIT=18,FILE='DATA8.MAT')
DO 1920 I=1,200
WRITE(18,*) X(I,3)
1920 CONTINUE
CLOSE(18)
ENDIF

C
IF (M .EQ. 9958) THEN
OPEN (UNIT=19,FILE='DATA9.MAT')
DO 1930 I=1,200
WRITE(19,*) X(I,3)
1930 CONTINUE
CLOSE(19)
ENDIF

C
IF (M .EQ. 10000) THEN
OPEN (UNIT=20,FILE='DATA10.MAT')
DO 1940 I=1,200
WRITE(20,*) X(I,3)
1940 CONTINUE
CLOSE(20)
ENDIF

C
5000 CONTINUE
C
OPEN(UNIT=21,FILE='Z100.MAT')
DO 2000 I=1,M
WRITE(21,*) Z(I)
2000 CONTINUE
CLOSE(21)
END
Figure B.1

Numerical and Experimental Eigenvectors at Eigenfrequency 0.85 rad/sec
Figure B.2

Numerical and Experimental Eigenvectors at Eigenfrequency 3.22 rad/sec
Figure B.3
Numerical and Experimental Eigenvectors at Eigenfrequency 6.04 rad/sec
VITA

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He is presently a project officer stationed at the Naval Air Warfare Center, Nas Patuxent River, Maryland.