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Application of the Green's Function for Solutions of Third Order Nonlinear Boundary Value Problems

Shannon Mathis Morrison
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To the Graduate Council:

I am submitting herewith a thesis written by Shannon Mathis Morrison entitled "Application of the Green's Function for Solutions of Third Order Nonlinear Boundary Value Problems." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

Don Hinton, Major Professor

We have read this thesis and recommend its acceptance:

Suzanne Lenhart, Jochen Denzler

Accepted for the Council:
Dixie L. Thompson

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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Accepted for the Council:

Carolyn R. Hodges
Vice Chancellor and Dean of
Graduate Studies

APPLICATION OF THE GREEN'S FUNCTION FOR SOLUTIONS OF THIRD ORDER NONLINEAR BOUNDARY VALUE PROBLEMS

A Thesis
Presented for the
Master of Science
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Abstract

Green's functions are used to prove a collection of existence and uniqueness theorems for third order nonlinear boundary value problems. Several examples of Green's functions for both second and third order boundary value problems are given. Various applications of the existence theorems are presented in detail.

Contents

1 Introduction and Preliminaries

1.1 Introduction

The solutions of third order boundary value problems are the subject of this work. In particular, the focus will be on applications of the corresponding Green's functions and the resulting qualitative properties of the solutions. After introducing the basic definitions, the first step is to guarantee that solutions to the equations in question exist and are unique. Thus, a series of uniqueness and existence theorems for third order differential equations with homogeneous boundary conditions are established. Next, an extensive collection of Green's functions is derived, many of which are used later on to illustrate various applications. Finally, several specific existence and uniqueness theorems are proved by applying the previous results.

1.2 Preliminary Definitions and Theorems

This section will introduce the basic definitions, theorems, and constructions that will be used throughout.

Definition 1.1. *A Banach space, $(X, \|\cdot\|)$, is a complete normed linear space.*

Let $B = C[a, b]$ with the supremum norm, denoted by $\|\cdot\|_\infty$. Then B is a Banach space. It will be beneficial, however, to use a variation of this norm on some subspace of B. An example of such a space that will be used frequently is given by the following.

Theorem 1.1. *Let $w \in C[a, b]$ be a fixed function such that $w(a) = w(b) = 0$ and $w(x) > 0$ for $a < x < b$. Let*

$$B^* = \{u \in B : |u(x)| \leq Cw(x) \text{ for some } C > 0\}.$$

For $u \in B^*$, define

$$\|u\|^* = \sup_{a<x<b} \frac{|u(x)|}{w(x)}.$$

Then $\|\cdot\|^*$ is a norm on B^* and $(B^*, \|\cdot\|^*)$ is a Banach space.

Proof. It is easy to see that B^* is a subspace of B. By the definition of B^*, $u(a) = u(b) = 0$ for any $u \in B^*$. Moreover, if $\|u\|^* = 0$, then

$$\frac{|u(x)|}{w(x)} \leq \sup_{a<x<b} \frac{|u(x)|}{w(x)} = 0 \Rightarrow u(x) = 0.$$

The triangle inequality and the fact that scalars can be factored out of $\|\cdot\|^*$ follow easily from the definition. Thus, $\|\cdot\|^*$ is a norm on B^*. Now, it must be shown that B^* is complete. Let $\{u_n\}_{n=1}^\infty$ be a Cauchy sequence in $(B^*, \|\cdot\|^*)$. Let $M =$

$\sup_{a \le x \le b} w(x)$. Let $\epsilon > 0$ be given. There exists an $N \in \mathbb{N}$ such that for any n, $m \ge N$, $\|u_n - u_m\|^* < \frac{\epsilon}{M}$. If n, $m \ge N$, then for $a < x < b$ we have $1 \le \frac{M}{w(x)}$, so that

$$|u_n(x) - u_m(x)| \le \sup_{a<x<b} \frac{M|u_n(x) - u_m(x)|}{w(x)} = M\|u_n - u_m\|^* < \epsilon.$$

Thus, $|u_n(x) - u_m(x)| < \epsilon \ \forall\, x \in [a, b] \Rightarrow \|u_n - u_m\|_\infty < \epsilon \ \forall\, n, m \ge N$. Since $\epsilon > 0$ is arbitrary, $\{u_n\}$ is a Cauchy sequence in B, showing that a Cauchy sequence in B^* is also Cauchy in B. B is complete, implying that there exists a $u \in C[a, b]$ such that $u_n \to u$ in $\| \cdot \|_\infty$. It remains to show that u is in B^* and that $u_n \to u$ in $(B^*, \| \cdot \|^*)$.

Choose an $x \in (a, b)$. Since $u_n \to u$ in $\| \cdot \|_\infty$, there exists an $N \in \mathbb{N}$ such that $n \ge N \Rightarrow \|u_n - u\|_\infty < w(x)$. For any $n \ge N$,

$$|u_n(x) - u(x)| \le \sup_{a \le x \le b} |u_n(x) - u(x)| = \|u_n - u\|_\infty < w(x)$$

which, along with the reverse triangle inequality, implies

$$\frac{|u(x)|}{w(x)} < \frac{|u_n(x)|}{w(x)} + 1 \le \sup_{a<x<b} \frac{|u_n(x)|}{w(x)} + 1 = \|u_n\|^* + 1.$$

So, $\frac{|u(x)|}{w(x)} < \|u_n\|^* + 1$. Since Cauchy sequences are bounded there exists a constant $K > 0$ such that $\frac{|u(x)|}{w(x)} < K + 1$. This holds for all $x \in (a, b)$, which gives

$$\sup_{a<x<b} \frac{|u(x)|}{w(x)} \le K + 1 \Rightarrow \|u\|^* \le K + 1 < \infty \Rightarrow u \in B^*.$$

Finally, for $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for $n, m \ge N$ and for all $x \in (a, b)$,

$$\frac{|u_n(x) - u_m(x)|}{w(x)} \le \|u_n - u_m\|^* < \epsilon.$$

Let $m \to \infty$ in the previous in equality. Then for $n \ge N$ and $\forall\, x$, it follows that

$$\frac{|u_n(x) - u(x)|}{w(x)} \le \epsilon.$$

Thus, $\|u_n - u\|^* \le \epsilon$ for $n \ge N$, showing that $u_n \to u$ in $(B^*, \| \cdot \|^*)$. □

Theorem 1.1 can be generalized for a non-identically zero function $w \in C[a, b]$ such that $w(x) \ge 0$ on $[a, b]$. Let $B_w = B^*$ in Theorem 1.1. A norm on B_w can be defined by $\|u\|^* = \sup_{x \in S_w} \frac{|u(x)|}{w(x)}$ where $S_w = \{x : w(x) \ne 0\}$. The preceding proof applies without change to show that B_w is complete under $\| \cdot \|^*$. Typically in the following applications, however, $w(x) > 0$ on $a < x < b$.

Contraction Mapping Theorem *Let $T : B \to B$ be a continuous map from the*

Banach space, B, into itself such that for all $u, v \in B$,

$$\|T(u) - T(v)\| \leq \theta \|u - v\|$$

for some fixed $\theta \in (0,1)$. Then T has a unique fixed point u_0; i.e. $T(u_0) = u_0$ and $T(u) = u$ if and only if $u = u_0$.

The Contraction Mapping Theorem is an important tool in proving existence and uniqueness of solutions to ordinary differential equations, as will be seen later.

The following is some basic material from the theory of ordinary differential equations and boundary value problems. The definitions and the proofs of Theorems 1.2 and 1.3 are given in Walter ([Wa], Ch. 6). For notational purposes, arbitrary differential operators will be denoted by D.

Definition 1.2. *The linear second order boundary value problem with separated boundary conditions is defined as*

$$(Du)(x) := (p(x)u'(x))' + q(x)u(x) = g(x), \quad x \in [a,b] \tag{1.1}$$

with linearly independent boundary conditions

$$\begin{aligned} R_1 u &:= \alpha_1 u(a) + \alpha_2 p(a) u'(a) = \eta_1 \\ R_2 u &:= \beta_1 u(b) + \beta_2 p(b) u'(b) = \eta_2, \end{aligned} \tag{1.2}$$

assuming that $p \in C^1[a,b]$ and $q, g \in C^0[a,b]$ are real-valued functions, that $p(x) > 0$ in $[a,b]$, and that $\alpha_i, \beta_i, \eta_i$, $i = 1, 2$, are real constants satisfying $\alpha_1^2 + \alpha_2^2 > 0$ and $\beta_1^2 + \beta_2^2 > 0$. The corresponding homogeneous boundary value problem is given by

$$Du = 0 \text{ on } [a,b] \tag{1.3}$$

$$\begin{aligned} \alpha_1 u(a) + \alpha_2 p(a) u'(a) &= 0 \\ \beta_1 u(b) + \beta_2 p(b) u'(b) &= 0. \end{aligned} \tag{1.4}$$

Theorem 1.2. *Let $u_1(x)$, $u_2(x)$ be a fundamental system of solutions to the homogeneous differential equation $Du = 0$. The inhomogeneous boundary value problem,* (1.1)*, with boundary conditions,* (1.2)*, is uniquely solvable if and only if the homogeneous problem,* (1.3), (1.4)*, has only the zero solution $u \equiv 0$. The latter is true if and only if the determinant of $\begin{bmatrix} R_1 u_1 & R_1 u_2 \\ R_2 u_1 & R_2 u_2 \end{bmatrix}$ is nonzero. Moreover, the determinant condition does not depend on the choice of fundamental system.*

Consequently, it is sufficient to solve (1.1) with the homogeneous boundary conditions, (1.4), instead of (1.1), (1.2). To illustrate this, suppose a function, $w(x)$ in

$C^2[a,b]$ can be found that satisfies (1.2). If v satisfies $Dv = g(x) - Dw$ and (1.4), then $u = v + w$ satisfies $Du = Dv + Dw = g(x)$ and (1.2). Finding such a function, w, is typically not difficult. For example, if $\alpha_2 = \beta_2 = 0$, then a linear $w(x)$ which satisfies $\alpha_1 u(a) = eta_1$, $\beta_1 u(b) = eta_2$ will work. Thus, the homogeneous boundary conditions, (1.4), will be used in most discussions.

Definition 1.3. *Let $Du(x) := u''(x) = f(x, u(x), u'(x))$, for $x \in [a,b]$, be a nonhomogeneous second-order ODE. A* **fundamental solution** *to this equation is any function, u, satisfying the corresponding homogeneous problem, $Du = 0$. The* **Green's function** *corresponding to the nonhomogeneous problem is a function $G(x,s)$, for $a \le x, s \le b$, such that $G(x,s)$ is a fundamental solution of $Du = 0$ for $x \ne s$ and $R_1 G = R_2 G = 0$ for each $s \in (a,b)$.*

Green's function is given explicitly by

$$G(x,s) = \frac{1}{c}\begin{cases} u_1(s)u_2(x) & \text{in } a \le s \le x \le b \\ u_1(x)u_2(s) & \text{in } a \le x \le s \le b \end{cases} \tag{1.5}$$

where u_1 satisfies the first boundary condition, u_2 satisfies the second boundary condition, and $c \ne 0$ is determined by the Wronskian

$$c = \begin{bmatrix} u_1 & u_2 \\ pu_1' & pu_2' \end{bmatrix}$$

on the interval (a,b), which is constant.

Theorem 1.3. *Assume $p \in C^1[a,b]$ and $q, g \in C^0[a,b]$ are real valued functions, $p(x) > 0$ in $[a,b]$ and $\alpha_1^2 + \alpha_2^2 > 0$, $\beta_1^2 + \beta_2^2 > 0$. If the homogeneous boundary value problem*

$$Du = 0 \text{ on } [a,b], \quad R_1 u = R_2 u = 0, \tag{1.6}$$

has only the trivial solution (i.e. if the determinant given in Theorem 1.2 is nonzero) then the Green's function for this boundary value problem exists and is unique. It is explicitly given by (1.5) and is symmetric,

$$G(x,s) = G(s,x).$$

The unique solution of the "semihomogeneous" boundary value problem

$$Lu = g(x) \text{ on } [a,b], \quad R_1 u = R_2 u = 0 \tag{1.7}$$

is given by

$$u(x) = \int_a^b G(x,s)g(s)ds. \tag{1.8}$$

The uniqueness follows from Theorem 1.2.

The focus of this work will be on third order ordinary differential equations, $y''' =$

$f(x, y(x), y'(x), y''(x))$, satisfying a Lipschitz condition of the form

$$|f(x, u(x), u'(x), u''(x)) - f(x, v(x), v'(x), v''(x))| \leq \\ L|u(x) - v(x)| + K|u'(x) - v'(x)| + M|u''(x) - v''(x)|, \tag{1.9}$$

where K, L, and M are fixed positive constants. At a later point, these constants will be replaced by functions $p(x), q(x)$, and $r(x)$, giving a more general Lipschitz condition.

Theorems 1.2 and 1.3 have extensions for differential systems of arbitrary order. In particular, if D is a linear differential operator of order n, then the nonhomogeneous problem $Du = g$ with n linearly independent linear boundary conditions has a unique solution if and only if the corresponding homogeneous problem has only the zero solution. In this case, the solution of the nonhomogeneous problem has the representation (1.8), although the Green's function is in general not symmetric.

2 Existence and Uniqueness

The following two theorems will form the foundation for what is to come. The remainder of this work will consist of applications of the following existence and uniqueness theorems. The proofs are generalizations of the corresponding second order theorems, which are proved in Bailey, ([Ba], Ch. 3).

Consider the third order differential equation

$$u'''(x) = f(x, u(x), u'(x), u''(x)) \tag{2.1}$$

with linearly independent boundary conditions

$$\begin{aligned} R_1 u &:= \alpha_1 u(a) + \beta_1 u'(a) + \gamma_1 u''(a) = 0 \\ R_2 u &:= \alpha_2 u(a) + \beta_2 u'(a) + \gamma_2 u''(a) = 0 \\ R_3 u &:= \alpha_3 u(b) + \beta_3 u'(b) + \gamma_3 u''(b) = 0. \end{aligned} \tag{2.2}$$

A test for linear independence is given by Coddington ([Co]) to be

$$\text{rank} \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & 0 & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} = 3.$$

The boundary condition space, denoted by S, consists of all u in $C^3[a, b]$ that satisfy the boundary conditions. Various norms similar to that given in Theorem 1.3 will be assigned throughout to make S a subspace of a Banach space. Note that for $u'''(x) = g(x)$, the Green's function exists for many cases of (2.2) by an analogue of Theorem 1.3 for third order boundary value problems. However, examples of functions satisfying (2.2) can be constructed for which a nontrivial solution of $u'''(x) = 0$ exists. One such example is

$$u(x) = 1 + x + x^2$$

with boundary conditions

$$u(0) - u'(0) = 0, \;\; 2u'(0) - u''(0) = 0, \;\; u(1) - u'(1) = 0.$$

The following theorem is proved for third order equations, though the conclusion holds for second order equations as well.

Theorem 2.1. *Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies*

$$|f(x, u_2(x)) - f(x, u_1(x))| \le p(x)|u_2(x) - u_1(x)|$$

for some nonnegative continuous function p. Suppose the Green's function G for the boundary value problem $u'''(x) = g(x)$ and (2.2) exists. Define the operator, $T : C[a, b] \to S \subset C[a, b]$, by

$$(Tu)(x) = \int_a^b G(x,s)f(s,u(s))ds.$$

Suppose w is a fixed nontrivial element of $C[a,b]$ with $w(x) \geq 0$. Suppose also $T : B_w \to B_w$ where the Banach space B_w is described in the remarks following Theorem 1.1.

a) *If the Green's function, G is of constant sign, and*

$$\sup_{x \in S_w} \left[\frac{z(x)}{w(x)} \right] < 1,$$

where z is defined by $z(x) = \int_a^b |G(x,s)|p(s)w(s)ds$ and $S_w = \{x \in [a,b] : w(x) \neq 0\}$, then (2.1), (2.2) has a unique solution. Further z satisfies $z'''(x) = sign(G)p(x)w(x)$ with boundary conditions (2.2).

b) *If G is possibly not of constant sign and*

$$\sup_{x \in S_w} \left[\frac{1}{w(x)} \int_a^b |G(x,s)|p(s)w(s)ds \right] < 1,$$

then (2.1), (2.2) has a unique solution.

Proof. **(a)** Consider the case where G is negative (the proof for positive G is similar). Let $\|\cdot\|^*$ denote the norm that was defined in Theorem 1.1, but with the maximum taken over S_w. Then

$$\begin{aligned} |(Tu)(x) - (Tv)(x)| &= \left| \int_a^b G(x,s)[f(s,u(s)) - f(s,v(s))]ds \right| \\ &\leq \int_a^b |G(x,s)||u(s) - v(s)|p(s)ds \\ &\leq \int_a^b \|u - v\|^* |G(x,s)|p(s)w(s)ds \\ &= \|u - v\|^* z(x). \end{aligned}$$

From the definition of $z(x)$ and the fact that G is a Green's function, it follows that $z'''(x) = -p(x)w(x)$ with boundary conditions (2.2). Now for $x \in S_w$

$$\frac{|(Tu)(x) - (Tv)(x)|}{w(x)} \leq \frac{z(x)\|u - v\|^*}{w(x)}.$$

This implies

$$\|Tu - Tv\|^* \leq \|u - v\|^* \max_{x \in S_w} \frac{z(x)}{w(x)}$$

where $\max_{x\in S_w} \frac{z(x)}{w(x)} < 1$ by hypothesis, proving that T is a contraction on B_w which yields a unique fixed point that is the solution of (2.1)- (2.2). This proves part a.

(b) If G is possibly not of one sign, then for $x \in S_w$,

$$\frac{|(Tu)(x) - (Tv)(x)|}{w(x)} \leq \|u - v\|^* \frac{1}{w(x)} \int_a^b |G(x,s)|p(s)w(s)ds.$$

Thus,

$$\|Tu - Tv\|^* \leq \|u - v\|^* \max_{x\in S_w} \frac{1}{w(x)} \int_a^b |G(x,s)|p(s)w(s)ds.$$

The maximum is less than 1 by hypothesis, so T is a contraction, which yields a unique fixed point that is a solution of (2.1)-(2.2). □

Two cases of Theorem 2.1 are needed because if G is of constant sign, the function $z(x)$ is much easier to compute by solving the differential equation

$$z'''(x) = sign(G)p(x)w(x)$$

.

One of the obstacles that can arise in applying Theorem (2.1) is confirming the hypothesis that T maps B_w into B_w. In many cases, $w(x) = 1$, in which case $B_w = C[a,b]$ and $T : B_w \to B_w$ clearly holds. For $w(x) > 0$ on $[a,b]$, the resulting norm is equivalent to the norm for $w(x) = 1$, so $T : B_w \to B_w$ holds in that case, as well. In general, though, for w having zeros in $[a,b]$, the properties of the corresponding Green's functions must be used to establish that this hypothesis holds.

It will be beneficial, particularly for the examples and applications that will be presented later, to state the analog of Theorem 2.1 for second order equations and was found in Bailey [(Ba)].

Theorem 2.1* *Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies*

$$|f(x,u_2(x)) - f(x,u_1(x))| \leq p(x)|u_2(x) - u_1(x)|$$

for some nonnegative continuous function p. Suppose the Green's function G for the boundary value problem $u''(x) = g(x)$ and (1.4) *exists. Define the operator, $T : C[a,b] \to C[a,b]$, by*

$$(Tu)(x) = \int_a^b G(x,s)f(s,u(s))ds.$$

Suppose w is a fixed nontrivial element of $C[a,b]$ with $w(x) \geq 0$. Suppose also $T : B_w \to B_w$ where the Banach space B_w is described in the remarks following Theorem 1.1.

a) *If the Green's function, G is of constant sign, and*

$$\sup_{x\in S_w}\left[\frac{z(x)}{w(x)}\right] < 1,$$

where z is defined by $z(x) = \int_a^b |G(x,s)|p(s)w(s)ds$ *and* $S_w = \{x \in [a,b] : w(x) \neq 0\}$, *then* $Du = g(x)$, *with boundary conditions* (1.4), *has a unique solution.*

b) *If G is possibly not of constant sign and*

$$\sup_{x\in S_w}\left[\frac{1}{w(x)}\int_a^b |G(x,s)|p(s)w(s)ds\right] < 1,$$

then $Du = g(x)$ *with* (1.4) *has a unique solution. Further, z satisfies*

$$z''(x) = sign(G)p(x)w(x)$$

with boundary conditions (1.4).

In Chapter 4, examples will be given showing how the function w may be chosen and how the existence of solutions depends on this choice.

Theorem 2.2. *Let* $f : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ *satisfy* (1.9). *Suppose* $G(x,s)$, $a \le x, s \le b$, *the Green's function for the boundary value problem* $u'''(x) = g(x)$ *and* (2.2), *exists. Suppose further that there exist constants* M_1, M_2, M_3, *such that*

$$\int_a^b |G(x,s)|ds \le M_1, \quad \int_a^b |G_x(x,s)|ds \le M_2, \quad \int_a^b |G_{xx}(x,s)|ds \le M_3.$$

Assume also that $LM_1 + KM_2 + MM_3 < 1$. *Then there exists a unique solution to the boundary value problem*

$$y'''(x) = f(x, y(x), y'(x), y''(x)), \qquad x \in [a,b],$$

with boundary conditions (2.2).

Proof. Let $\|u\| = \max_{a\le x\le b}[L|u(x)| + K|u'(x)| + M|u''(x)|]$ be the norm on $C^2[a,b]$ so that $C^2[a,b]$ is a Banach space. Define the operator $T : C^2[a,b] \to C^3[a,b]$ by

$$y = Ty = \int_a^b G(x,s)f(s, y(s), y'(s), y''(s))ds.$$

To see that T does, indeed, map into $C^3[a,b]$, note first that that the differentiability of G allows differentiation under the integral sign. Hence,

$$(Tu)'(x) = \int_a^b G_x(x,s) f(s,u(s),u'(s),u''(s))ds,$$

$$(Tu)''(x) = \int_a^b G_{xx}(x,s) f(s,u(s),u'(s),u''(s))ds$$

and,

$$(Tu)'''(x) = \int_a^b G_{xxx}(x,s) f(s,u(s),u'(s),u''(s))ds.$$

Now it must be shown that T is a contraction map.

$$\begin{aligned} |(Tu)(x)-(Tv)(x)| &\leq \int_a^b |G(x,s)| \big| f(s,u(s),u'(s),u''(s)) - f(s,v(s),v'(s),v''(s)) \big| ds \\ &\leq \int_a^b |G(x,s)| \big(L|u(s)-v(s)| + K|u'(s)-v'(s)| + M|u''(s)-v''(s)| \big) ds \\ &\leq \|u-v\| \int_a^b |G(x,s)| ds \\ &\leq \|u-v\| M_1. \end{aligned}$$

Similarly,

$$|(Tu)'(x)-(Tv)'(x)| \leq \|u-v\| \int_a^b |G_x(x,s)| ds \leq \|u-v\| M_2$$

and

$$|(Tu)''(x)-(Tv)''(x)| \leq \|u-v\| \int_a^b |G_{xx}(x,s)| ds \leq \|u-v\| M_3.$$

Since x is arbitrary in the previous inequalities, it follows that

$$\|Tu - Tv\| \leq \|u-v\|(LM_1 + KM_2 + MM_3).$$

By hypothesis, $LM_1 + KM_2 + MM_3$ is less than 1. Therefore, T is a contraction from the complete space, $C^2[a,b]$, into $C^3[a,b] \in C^2[a,b]$. Consequently, it has unique fixed point, u, which is the desired solution. □

Now that existence and uniqueness of solutions has been established, the next step is to derive the corresponding Green's functions.

3 Green's Functions and First Eigenvalues

This section will begin with a series of examples showing the derivation of Green's functions for second and third order boundary value problems. This will facilitate the transition into the last part of the chapter, where the relationship between these Green's functions and the eigenvalues of the corresponding differential equations is illustrated through more examples. Also in this section, the general interval, $[a, b]$, will be replaced by $[0, a]$ for $a > 0$.

Example 3.1. Beginning with a simple example, the computation of Green's function for $u'' = g(x)$ with antiperiodic boundary conditions $u(0) = -u(a)$ and $u'(0) = -u'(a)$ follows. The general solution of $u''(x) = g(x)$ is

$$u(x) = d_1 + d_2 x + \int_0^x (x - s)g(s)ds.$$

Then $u(0) = -u(a)$ implies

$$d_1 = -\frac{a}{2}d_2 - \frac{1}{2}\int_0^a (a - s)g(s)ds$$

and $u'(0) = -u'(a)$ implies

$$d_2 = -\frac{1}{2}\int_0^a g(s)ds.$$

It follows that

$$u(x) = \int_0^a g(s)\Big[\frac{2s - a - 2x}{4}\Big]ds + \int_0^x (x - s)g(s)ds$$

and

$$u(x) = \int_0^x \Big(\frac{2s - a - 2x}{4}\Big)g(s)ds + \int_x^a \Big(\frac{2s - a - 2x}{4}\Big)g(s)ds + \int_0^x (x - s)g(s)ds$$

Thus, Green's function is

$$G(x, s) = \begin{cases} \frac{2x-a-2s}{4}, & 0 \le s \le x \le a \\ \frac{2s-a-2x}{4}, & 0 \le x \le s \le a. \end{cases} \tag{3.1}$$

The next two examples follow in exactly the same manner as Example 3.1

Example 3.2. Green's function for $Du = u'' = g(x)$ with boundary conditions $u(0) = u(a) = 0$ is given by

$$G(x, s) = \begin{cases} \frac{s}{a}(x - a), & 0 \le s \le x \le a \\ \frac{x}{a}(s - a), & 0 \le x \le s \le a. \end{cases} \tag{3.2}$$

Example 3.3. Green's function for $Du = u'' = g(x)$ with boundary conditions $u(0) = 0$ and $u'(a) = 0$ is

$$G(x,s) = \begin{cases} -s, & 0 \le s \le x \le a \\ -x, & 0 \le x \le s \le a. \end{cases} \tag{3.3}$$

Next is an example arising in steady-state one dimensional heat flow.

Example 3.4. Consider the equation

$$Du = u'' = g(x)$$

with boundary conditions

$$u(0) = u'(0), \;\; u(a) = -u'(a).$$

Following the same computations as before yields

$$u(x) = d_1 + d_2 x + \int_0^x (x-s)g(s)ds$$

$$u(0) = d_1, \; u'(0) = d_2 \Rightarrow d_1 = d_2$$

$$u(a) = d_1 + ad_2 + \int_0^a (a-s)g(s)ds$$

$$u(a) = -u'(a) \Rightarrow d_1 + ad_2 + \int_0^a (a-s)g(s)ds = -d_2 - \int_0^a g(s)ds$$

$$d_1 = d_2 = \frac{-1}{2+a}\int_0^a (a-s+1)g(s)ds$$

$$u(x) = \frac{-(1+x)}{2+a}\int_0^a (a-s+1)g(s)ds + \int_0^x (x-s)g(s)ds.$$

Thus, Green's function is

$$G(x,s) = \begin{cases} -\frac{1+x}{2+a}(a-s+1) + (x-s), & 0 \le s \le x \le a \\ -\frac{1+x}{2+a}(a-s+1), & 0 \le x \le s \le a. \end{cases} \tag{3.4}$$

For $0 < s < x$, some algebra shows

$$-\frac{1+x}{2+a}(a-s+1) + (x-s) = -\frac{1+s}{2+a}(a-x+1)$$

from which we see that G is indeed symmetric.

Now the previous methods will be applied to a third order example.

Example 3.5. Consider the equation

$$Du = u''' = g(x)$$

with boundary conditions

$$u(0) = u'(0) = u(a) = 0.$$

Integrating first to find u gives

$$u'''(x) = g(x)$$
$$u''(x) = d_2 + \int_0^x g(s)ds$$
$$u'(x) = d_1 + d_2 x + \int_0^x (x-s)g(s)ds$$
$$u(x) = d_0 + d_1 x + \frac{1}{2}d_2 x^2 + \int_0^x \frac{(x-s)^2}{2} g(s)ds$$
$$u(0) = u'(0) = 0 \Rightarrow d_0 = d_1 = 0$$
$$u(a) = 0 \Rightarrow d_2 = -\frac{1}{2a^2}\int_0^a (a-s)^2 g(s)ds$$

$$u(x) = \int_0^x \Big[\frac{-x^2(a-s)^2}{2a^2} + \frac{(x-s)^2}{2}\Big] g(s)ds + \int_x^a \Big[\frac{-x^2(a-s)^2}{2a^2}\Big] g(s)ds.$$

The corresponding Green's function is

$$G(x,s) = \begin{cases} \frac{-x^2(a-s)^2}{2a^2} + \frac{(x-s)^2}{2}, & 0 \le s \le x \le a \\ \frac{-x^2(a-s)^2}{2a^2}, & 0 \le x \le s \le a. \end{cases} \tag{3.5}$$

This next example will be used at a later point to illustrate one of the uniqueness theorems.

Example 3.6. Consider the third order ordinary differential equation $Du(x) = u'''(x) = g(x)$ with boundary conditions $u(0) = u'(0) = u''(a) = 0$. Integrate to get

$$u''(x) = d_2 + \int_0^x g(s)ds$$
$$u'(x) = d_1 + d_2 x + \int_0^x (x-s)g(s)ds$$
$$u(x) = d_0 + d_1 x + \frac{1}{2}d_2 x^2 + \frac{1}{2}\int_0^x (x-s)^2 g(s)ds.$$

By the boundary conditions, $u(0) = d_0 = 0$ and $u'(0) = d_1 = 0$. Further calculations give $d_2 = -\int_0^a g(s)ds$. Hence, some algebra shows that

$$u(x) = -\int_0^x \frac{s}{2}(s-x)g(s)ds - \frac{x^2}{2}\int_x^a g(s)ds.$$

Green's function is

$$G(x,s) = \begin{cases} \frac{s}{2}(s-2x), & 0 \le s \le x \le a \\ -\frac{x^2}{2} & 0 \le x \le s \le a. \end{cases} \tag{3.6}$$

Similar calculations yield the following.

Example 3.7. Green's function for $Du = u'''(x) = g(x)$ with boundary conditions $u(0) = u'(0) = u'(a) = 0$ is

$$G(x,s) = \begin{cases} \frac{s}{2}\left(\frac{x^2}{a} - 2x + s\right), & 0 \le s \le x \le a \\ \frac{-x^2}{2a}(a-s), & 0 \le x \le s \le a \end{cases} \tag{3.7}$$

and will also be used at a later point.

The next example is a second order equation defined by an operator that differs slightly from those used in the previous problems.

Example 3.8. As before, the goal is to find Green's function for $Du = (r(x)u'(x))' = g(x)$ with boundary conditions $u(0) = u'(a) = 0$ and where $r(x) > 0$. Integrating the corresponding homogeneous equation generates the following.

$$(r(x)u'(x))' = 0 \Rightarrow r(x)u'(x) = c_0$$

$$u'(x) = \frac{c_0}{r(x)} \Rightarrow u(x) = c_0 \int_0^x \frac{1}{r(s)}ds + c_1.$$

Now define

$$u_1(x) = \int_0^x \frac{1}{r(s)}ds, \;\; u_2(x) = 1.$$

The Wronskian of u_1 and u_2 is $c = -\frac{1}{r(x)}$ ($ru_1'u_2 - ru_2'u_1$ is constant). By the variation of parameters formula, it can be shown that the terms involving $r(x)$ in the expression for u will cancel. Using methods similar to those used previously,

$$u(x) = d_1u_1 + d_2u_2 + \int_0^x \Big[\int_0^x \frac{ds}{r(s)} - \int_0^\xi \frac{ds}{r(s)}\Big] g(\xi)d\xi$$

$$u'(x) = \frac{d_1}{r(x)} + g(x)\Big[\int_0^x \frac{ds}{r(s)} - \int_0^x \frac{ds}{r(s)}\Big] + \int_0^x \frac{g(\xi)}{v(x)}d\xi$$

$$u'(a) = \frac{d_1}{r(a)} + \int_0^a \frac{g(\xi)}{r(a)}d\xi = 0$$

$$d_1 = -\int_0^a g(\xi)d\xi$$

$$u(x) = -\int_0^a g(\xi)d\xi \int_0^x \frac{ds}{r(s)} + \int_0^x \left[\int_0^x \frac{ds}{r(s)} - \int_0^\xi \frac{ds}{r(s)}\right] g(\xi)d\xi.$$

Simplifying this expression yields

$$u(x) = -\int_0^x g(\xi)d\xi \int_0^\xi \frac{ds}{r(s)} - \int_x^a g(\xi)d\xi \int_0^x \frac{ds}{r(s)}.$$

Finally, Green's function is

$$G(x,s) = \begin{cases} -\int_0^\xi \frac{ds}{r(s)}, & 0 \le x \le s \le a \\ -\int_0^x \frac{ds}{r(s)}, & 0 \le s \le x \le a. \end{cases} \tag{3.8}$$

Example 3.9. The final Green's function provided is for $(r(x)u'(x)) = g(x)$ for $r(x) > 0$ and boundary conditions $u(0) = u(a) = 0$. The calculations reveal Green's function to be

$$G(x,s) = \begin{cases} -\int_0^\xi \frac{ds}{r(s)} \int_x^a \frac{ds}{r(s)}, & 0 \le s \le x \le a \\ -\int_0^x \frac{ds}{r(s)} \int_\xi^a \frac{ds}{r(s)}, & 0 \le x \le s \le a. \end{cases} \tag{3.9}$$

The last part of this chapter will focus on the eigenvalues and eigenfunctions associated with the Green's functions that were just given. Recall the problem from Example 3.2, $Du = u'' = f(x, u(x))$ with boundary conditions $u(0) = u(a) = 0$, where f satisfies the conditions of Theorem 2.1*. An eigenvalue of the operator, D, is a constant λ such that $Du = -\lambda p(x)u(x)$ for some nontrivial u satisfying the given boundary conditions. Consider the special case where $f(x, u(x), u'(x)) = Lu(x)$. The corresponding eigenvalue problem is $u''(x) = -\lambda L u(x)$. The general solution for $\lambda > 0$ is

$$u(x) = c_1 \sin(\sqrt{\lambda L}x) + c_2 \cos(\sqrt{\lambda L}x).$$

The boundary conditions reveal that

$$u(0) = 0 \Rightarrow c_2 = 0 \Rightarrow u(a) = c_1 \sin(\sqrt{\lambda L}a) = 0 \Rightarrow \sqrt{\lambda L}a = n\pi.$$

For the smallest positive eigenvalue, let $n = 1$ so that $\lambda_0 = \frac{\pi^2}{La^2}$ with corresponding eigenfunction $\sin\frac{\pi x}{a}$. In Theorem 2.1*, with $f(x, u(x)) = Lu$ and $w(x) = \sin\frac{\pi x}{a}$, the function z is given by $z(x) = \frac{1}{\lambda_0}w(x)$. Note that when $\lambda_0 = 1$, $L = \frac{\pi^2}{a^2}$, leading to the conclusion that there is not a unique solution of $u'' = -Lu$ with boundary conditions $u(0) = u(a) = 0$ since $u(x) = c\sin\frac{\pi x}{a}$ is a solution for all values of c. If $\lambda_0 > 1$, then Theorem 2.1* applies and the resulting bound on L is $L < \frac{\pi^2}{a^2}$. Note that as the size of the interval increases (i.e. as a gets bigger) L must correspondingly decrease. Thus, for larger intervals the class of solutions becomes more restricted.

Example 3.10. A slight variation of Example 3.3 yields the problem $Du = u'' =$

$-\lambda Lu$ with boundary conditions $u(0) = u'(a) = 0$. The first positive eigenvalue is $\lambda_0 = \frac{\pi^2}{4a^2 L}$ with eigenfunction $\sin \frac{\pi x}{2a}$. If $\lambda_0 > 1$, then $L < \frac{\pi^2}{4a^2}$.

The next two examples require some basic numerical computation.

Example 3.11. Consider finding the first positive eigenvalue for $Du = u''$ with boundary conditions $u(0) = u'(0)$ and $u(1) = -u'(1)$. This is the heat flow equation with $a = 1$. Note that the corresponding Green's function, given by (3.4), is constant in sign. Now,

$$u''(x) = -\lambda Ly$$
$$u(x) = c_1 \sin(\sqrt{\lambda L}x) + c_2 \cos(\sqrt{\lambda L}x)$$
$$u(0) = u'(0) \Rightarrow c_2 = \sqrt{\lambda L}c_1.$$

The condition $u(1) = -u'(1)$ along with the previous result implies

$$c_1 \sin(\sqrt{\lambda L}) + 2\sqrt{\lambda L}c_1 \cos(\sqrt{\lambda L}) - \lambda Lc_1 \sin(\sqrt{\lambda L}) = 0$$
$$\Rightarrow \sin(\sqrt{\lambda L}) + 2\sqrt{\lambda L} \cos(\sqrt{\lambda L}) - \lambda L \sin(\sqrt{\lambda L}) = 0$$
$$\Rightarrow \tan \sqrt{\lambda L} = \frac{2\sqrt{\lambda L}}{\lambda L - 1}.$$

Numerical calculations give

$$\lambda_0 \approx \frac{1.307^2}{L} > 1 \Rightarrow L < 1.708.$$

Example 3.12. Finally, consider $Du = u''' = g(x)$ with boundary conditions $u(0) = u'(0)$ and $u(1) = 0$, which yields the eigenvalue problem $u''' = -\lambda Lu$. To find the first positive eigenvalue, λ_0, note that finding solutions of the form $u(x) = e^{rx}$ gives the characteristic equation $r^3 + L\lambda = 0$ and with $\theta = L\lambda$,

$$r = \theta^{\frac{1}{3}} \exp\left(i\pi + 2\pi ki\right), \quad k = 0, 1, 2.$$

The general solution is

$$u(x) = c_1 \exp(-\theta^{\frac{1}{3}}x) + c_2 \exp\Big(\theta^{\frac{1}{3}}(\frac{1}{2} + i\frac{\sqrt{3}}{2})x\Big) + c_3 \exp\Big(\theta^{\frac{1}{3}}(\frac{1}{2} - i\frac{\sqrt{3}}{2})x\Big).$$

Let $\alpha = \frac{\theta^{\frac{1}{3}}}{2}$ to simplify the notation. Applying the boundary conditions to u yields the system of equations

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 e^{-2\alpha} + c_2 e^{\alpha} \cos(\sqrt{3}\alpha) + c_3 e^{\alpha} \sin(\sqrt{3}\alpha) &= 0 \\ -2\alpha c_1 + \alpha c_2 + \sqrt{3}\alpha c_3 &= 0. \end{aligned}$$

Basic linear algebra reduces this system to the single equation

$$e^{\alpha}c_1\big(e^{-3\alpha} - \cos(\sqrt{3}\alpha) + \sqrt{3}\sin(\sqrt{3}\alpha)\big) = 0.$$

Finally, numerical computations show that the first positive root of this equation is

$$\alpha = \frac{\theta^{1/3}}{2} = \frac{L{\lambda_0}^{\frac{1}{3}}}{2} \approx 2.1166 \Longrightarrow \lambda_0 \approx \frac{4.2332^3}{L} \Longrightarrow L < 75.859 \text{ for } \lambda_0 > 1.$$

4 Applications to Boundary Value Problems

In this section, various aspects of the preceding material will be applied to boundary value problems, with the specific goal of illustrating the theorems in Section 2. In particular, with respect to finding the first positive eigenvalue, better bounds on the Lipschitz constant L will be obtained.

4.1 Applications of Theorem 2.1 and Theorem 2.1*

As an application of Theorem 2.1*(a), using the equation $Du = u'' = f(x, u(x))$, where f satisfies $|f(x, u_1(x) - f(x, u_2(x))| \leq L|u_1(x) - u_2(x)|$ on $[0, a]$ with boundary conditions $u(0) = u(a) = 0$, choose a polynomial of degree 2, w, satisfying the boundary conditions. Denote w by

$$w(x) = a_0 + a_1 x + a_2 x^2.$$

Applying the boundary conditions to find a_0, a_1, a_2 yields

$$w(x) = ax - x^2.$$

Solving next for z, where z satisfies $z'' = -Lw$, $z(0) = z(a) = 0$, yields $z''(x) = L(-ax + x^2)$ to get

$$z(x) = c_0 + c_1 x - \frac{1}{6}Lax^3 + \frac{1}{12}Lx^4.$$

Applying the boundary conditions produces,

$$z(x) = \frac{1}{12}La^3 x - \frac{1}{6}Lax^3 + \frac{1}{12}Lx^4.$$

Basic calculus shows that $\max_{0<x<a} \frac{z(x)}{w(x)} = \frac{5La^2}{48}$. Moreover, the Green's function for this problem was computed in Example 3.2, and is easily seen to be of constant sign. Now it must be shown that T maps B_w into B_w. Using equation (3.2) and letting $w(x) = x(a - x)$ it follows that

$$\begin{aligned} Tu &= \int_0^a G(x,s) f(s, u(s)) ds \\ &= \int_0^x \frac{s}{a}(x - a) f(s, u(s)) ds + \int_x^a \frac{x}{a}(s - a) f(s, u(s)) ds. \end{aligned}$$

Then, since G is negative,

$$\begin{aligned}
|Tu(x)| &\leq \frac{a-x}{a}\int_0^x s|f(s,u(s))|ds + \frac{x}{a}\int_x^a (a-s)|f(s,u(s))|ds \\
\frac{|Tu(x)|}{w(x)} &\leq \frac{1}{ax}\int_0^x s|f(s,u(s))|ds + \frac{1}{a-x}\int_x^a |(a-s)f(s,u(s))|ds \\
&\leq \frac{x}{ax}\int_0^x |f(s,u(s))|ds + \frac{a-x}{a(a-x)}\int_x^a |f(s,u(s))|ds \\
&\leq \frac{1}{a}\int_0^a |f(s,u(s)|ds = C(u).
\end{aligned}$$

Thus, the hypotheses of Theorem 2.1$^*(a)$ are satisfied if $L < \frac{48}{5a^2} \approx \frac{9.6}{a^2}$. Comparison of this bound on L to the optimal bound obtained using the first positive eigenvalue, $L < \frac{\pi^2}{a^2} \approx \frac{9.87}{a^2}$ shows that the approximation method of Theorem 2.1(a) gives a bound almost as good as the best possible bound on L .

The previous example showed that choosing w to be a polynomial yields a good but not optimal bound on L. The more w differs from the eigenfunction, the worse the bound on L becomes. Consider the following example on the ODE from the previous problem with the same boundary conditions where w is taken to be the constant 1. Then $z''(x) = -L$. After integrating and applying the boundary conditions,

$$z(x) = \frac{-1}{2}Lx^2 + \frac{Lx}{2}.$$

Computing the $\max_{0<x<a} \frac{z(x)}{w(x)}$ reveals the bound on L to be $L < \frac{8}{a^2}$.

To discuss the optimality of Theorems 2.1 and 2.1*, consider, for ease of computation, the second order case. Suppose λ_0 is the first positive eigenvalue of $w''(x) = -\lambda p(x)w(x)$ with boundary conditions

$$\begin{aligned}
R_1 w := \alpha_1 w(0) + \beta_1 w'(0) &= 0, \\
R_2 w := \alpha_2 w(a) + \beta_2 w'(a) &= 0,
\end{aligned}$$

and suppose also that the Green's function is negative. Let the eigenfunction corresponding to λ_0 be denoted by w_0. By definition of the Green's function,

$$w(x) = \int_a^b -\lambda_0 G(x,s)p(s)w_0(s)ds = \lambda_0 z(x)$$

implying

$$\frac{z(x)}{w_0(x)} = \frac{1}{\lambda_0}.$$

Therefore, if $\lambda_0 > 1$, then T is a contraction. An explicit calculation for $p(x) = L$ with the boundary conditions $w_0(0) = w_0(a) = 0$ yields $w_0(x) = \sin\sqrt{L\lambda}x$. Then

$w_0(a) = 0$ implies that

$$\sqrt{L\lambda}a = \pi$$

$$L\lambda_0 = \frac{\pi^2}{a^2}$$

and

$$\lambda_0 = \frac{\pi^2}{a^2 L}.$$

Now, $\lambda_0 > 1$ if and only if $\frac{\pi^2}{a^2} > L$. This example shows the dependence of the Lipschitz constant, L on the length of the interval.

For the problem $Du = u'' = f(x, u)$ with boundary conditions $u(0) = -u'(0)$ and $u(1) = -u'(1)$, choose, as in the previous case, a polynomial, w, satisfying the boundary conditions. Again, denote w by

$$w(x) = a_0 + a_1 x + a_2 x^2.$$

In order to satisfy the boundary conditions, choose $a_0 = 1$, and use the boundary conditions to compute $a_1 = 1$, and $a_2 = -1$. This gives

$$w(x) = 1 + x - x^2.$$

Note that $w(x) > 0$ on $[0, 1]$, so that $B_w \approx B = C[0, 1]$ and T maps B_w into B_w. Integrating $z''(x) = -Lw(x)$ twice yields

$$z(x) = -\frac{1}{2}Lx^2 - \frac{1}{6}Lx^3 + \frac{1}{12}Lx^4 + c_0 x + c_1.$$

Applying the boundary conditions to $z(x)$ to solve for c_0 and c_1 produces

$$z(x) = L\Big(-\frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{7}{12}x + \frac{7}{12}\Big).$$

Next, computing $\max_{0<x<1} \frac{z(x)}{w(x)}$ numerically shows that it is necessary for L to satisfy $L < 1.7029$ in order to apply Theorem 2.1(a). The Green's function for this problem was also computed previously in Example 3.4, and is constant in sign. Hence, Theorem 2.1(a) does apply as long as the bound on L holds. Again, comparing this bound on L to the bound found in Example 3.11, $L < 1.708$, shows that the approximation method yields a bound very close to the optimal.

Next, a third order example is given. Consider $Lu = u''' = f(x, u)$ and boundary conditions $u(0) = u'(0) = 0$ and $u(1) = 0$. Choose $w(x)$ to be a polynomial that satisfies the boundary conditions,

$$w(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

To satisfy the boundary conditions, let $a_0 = a_1 = 0$, $a_2 = 1$, and $a_3 = -1$. Now,

$w(x) = x^2 - x^3$ and

$$z'''(x) := -Lw(x) \Rightarrow z'''(x) = L(x^3 - x^2).$$

Integrating yields

$$z(x) = -L\frac{x^5}{60} + L\frac{x^6}{120} + c_2\frac{x^2}{2} + c_1 x + c_0.$$

Applying the boundary conditions to $z(x)$ implies

$$c_0 = c_1 = 0, \;\; c_2 = \frac{L}{60}$$

so that

$$z(x) = L\Big(-\frac{x^5}{60} + \frac{x^6}{120} + \frac{x^2}{120}\Big).$$

Finding $\max_{0<x<1} \frac{z(x)}{w(x)}$ by calculus shows that $L < 60$. It is necessary to show that T maps B_w into B_w in order to apply Theorem 2.1(a). The Green's function calculated in (3.5) is negative and simplifies for $a = 1$ to

$$G(x,s) = \begin{cases} \frac{s^2(x^2-1)}{2}, & 0 \le s \le x \le 1 \\ \frac{-x^2(1-s)^2}{2}, & 0 \le x \le s \le 1. \end{cases}$$

Thus,

$$\begin{aligned}
(Tu) &= \int_0^1 G(x,s)f(s,u(s))ds \\
&= \int_0^x \Big[\frac{-s^2(1-x^2)}{2}\Big] f(s,u(s))ds + \int_x^1 \frac{-x^2(1-s)^2}{2} f(s,u(s))ds, \\
|Tu(x)| &\le \int_0^x \frac{s^2(1-x^2)}{2}|f(s,u(s))|ds + \int_x^1 \frac{x^2(1-s)^2}{2}|f(s,u(s))|ds, \\
\frac{|Tu(x)|}{w(x)} &\le \frac{1}{2x^2(1-x)}\int_0^x s^2(1-x^2)|f(s,u(s))|ds \\
&\quad + \frac{x^2}{2x^2(1-x)}\int_x^1 (1-s)^2|f(s,u(s))|ds \\
&\le \frac{x^2(1-x^2)}{2x^2(1-x)}\int_0^x |f(s,u(s))|ds + \frac{(1-x)^2}{2(1-x)}\int_x^1 |f(s,u(s))|ds \\
&= \frac{1+x}{2}\int_0^1 |f(s,u(s))|ds + \frac{1-x}{2}\int_x^1 |f(s,u(s))|ds \\
&\le \int_0^x |f(s,u(s)|ds + \frac{1}{2}\int_x^1 |f(s,u(s))|ds \\
&\le \int_0^1 |f(s,u(s))|ds \le C(u).
\end{aligned}$$

Now the hypotheses of Theorem 2.1 are satisfied. Comparing the bound $L < 60$ to the optimal bound, $L < 75.859$, obtained in Example 3.12 shows that the function w as a polynomial is not a good estimate of the eigenfunction. If w is chosen to be more similar to the eigenfunction, a better bound on L could be achieved.

Coming back to the ODE $Du = u'' = f(x,u)$ with boundary conditions $u(0) = u(a) = 0$, a different approach is given as an application of Theorem 2.1*. Note the fact that T maps B_w into B_w was established in the first example in this section. Since the boundary conditions are the same as in the first example, the proof given that T maps B_w into B_w also applies here. In the previous examples, the function $p(x)$ that appears in the Lipschitz condition in Theorem 2.1*(a) was taken to be a constant, L. Now that restriction is lifted. Solve $z''(x) = -p(x)w(x)$ with the same boundary conditions as above but with $p(x) = Lx$. Using the same function found above for w, let $w(x) = ax - x^2$. Then,

$$\begin{aligned} z''(x) &= Lx(-ax + x^2) \\ z'(x) &= -\frac{1}{3}ax^3L + \frac{1}{4}x^4L + c_1 \\ z(x) &= L\Big(\frac{-1}{12}ax^4 + \frac{1}{20}x^5\Big) + c_1x + c_0. \end{aligned}$$

Applying the boundary conditions produces $c_0 = 0$ and $c_1 = \frac{1}{30}a^4L$. Then

$$z(x) = L\Big(-\frac{1}{12}ax^4 + \frac{1}{20}x^5 + \frac{1}{30}a^4x\Big).$$

To get a bound on L, compute $\max_{0<x<a} \frac{z(x)}{w(x)}$. This is

$$\begin{aligned} \max_{0<x<a} \frac{L\big[-\frac{1}{12}ax^4 + \frac{1}{20}x^5 + +\frac{1}{30}a^4x\big]}{-ax + x^2} &= \max_{0<x<a} \frac{\frac{L}{60}\big(5ax^3 - 3x^4 - 2a^4\big)}{(x-a)} \\ &= \max_{0<x<a} \frac{L}{60}\big(-9x^2 + 4ax + 2a^2\big). \end{aligned}$$

Employing methods from calculus, the maximum is $\frac{3.3La^3}{60}$ which must be less than 1 to apply Theorem 2.1(a). Thus the required bound on L is $L < \frac{18.8}{a^3}$.

A similar application to the same ODE above using $p(x) = \frac{L}{x}$ is given next. Note that $p(x)$ is singular in this example. This time,

$$\begin{aligned} z''(x) &= \frac{L}{x}(-ax + x^2) = (a - x)L \\ z'(x) &= -axL + \frac{1}{2}x^2 + c_1 \\ z(x) &= -\frac{1}{2}ax^2L + \frac{1}{6}x^3L + c_1x + c_0. \end{aligned}$$

Applying the boundary conditions gives $c_0 = 0$ and $c_1 = \frac{1}{3}a^2 L$. So then

$$z(x) = L\Big(\frac{1}{6}x^3 - \frac{1}{2}ax^2 + \frac{1}{3}a^2 x\Big).$$

Finding the bound on L yields

$$\max_{0<x<a} L\Big[\frac{-\frac{1}{6}x^3 + \frac{1}{2}ax^2 - \frac{1}{3}a^2 x}{x(x-a)}\Big] = \frac{La}{3}.$$

Thus $L < \frac{3}{a}$ is the desired bound.

4.2 Applications of Theorem 2.2

The following is an application of Theorem 2.2.

Theorem 4.1. *Let $f : [0, a] \times \mathbb{R}^3 \to \mathbb{R}$ be continuous and satisfy*

$$|f(t, y, y', y'') - f(t, x, x', x'')| \leq K|y - x| + L|y' - x'| + M|y'' - x''|$$

with K, L, and $M > 0$. Assume also that

$$\frac{2a^3}{11}K + \frac{5a^2}{9}L + Ma < 1.$$

Then there exists a unique solution to the ordinary boundary value problem

$$\begin{gathered} y'''(t) = f(t, y(t), y'(t), y''(t)) \\ y(0) = y'(0) = y''(a) = 0. \end{gathered}$$

Proof. Let S be as in Theorem 2.2. Define operator T by

$$y = Ty = \int_a^b G(x, s) f(s, y(s), y'(s), y''(s)) ds.$$

Now Theorem 2.2 will be used to show that T is a contraction map. As in the hypotheses of that theorem, the constants M_1, M_2, and M_3 will be computed as bounds on the integrals of the corresponding Green's function and its partial derivatives. The Green's function for $Du = u'''$ with the given boundary conditions was found in Chapter 3, Example 3.6 and is

$$G(x, s) = \begin{cases} \frac{s}{2}(s - 2x), & 0 \leq s \leq x \leq a \\ -\frac{x^2}{2}, & 0 \leq x \leq s \leq a. \end{cases} \tag{4.1}$$

Notice that the Green's function is negative. To calculate M_1, consider

$$\begin{aligned}
\int_0^a G(x,s)ds &= \int_0^x \frac{s}{2}(x-s)ds - \int_x^a \frac{x^2}{2}ds \\
&= -\frac{1}{12}x^3 - \frac{1}{2}(x^2a - x^3) \\
\int_0^a |G(x,s)|ds &= \int_0^a -G(x,s)ds \\
&= \frac{1}{12}x^3 + \frac{1}{2}(x^2a - x^3).
\end{aligned}$$

The upper bound is calculated via methods from calculus yielding a maximum at $x = \frac{4a}{13}$. Thus $\int_0^a |G(x,s)|ds \leq \frac{2}{11}a^3$ implying $M_1 = \frac{2}{11}a^3$. Next, to calculate M_2,

$$\begin{aligned}
\int_0^a |G_x(x,s)|ds &\leq \int_0^x \frac{s}{2}ds + \int_x^a xds \\
&= ax - \frac{3x^2}{4}
\end{aligned}$$

which yields a maximum at $x = \frac{2a}{3}$. Then

$$\int_0^a |G_x(x,s)|ds \leq \frac{5a^2}{9}$$

so that $M_2 = \frac{5a^2}{9}$. Lastly, to calculate M_3, note that G_{xx} vanishes for $0 \leq s \leq x \leq a$ and is equal to -1 otherwise. Thus,

$$\int_0^a |G_{xx}(x,s)|ds \leq \int_x^a ds = a - x \Rightarrow M_3 = a.$$

Following the proof of Theorem 2.2, the norm can now be calculated.

$$\|Tu - Tv\| = \max_{a \leq x \leq b} \left[K\Big|Tu - Tv\Big| + L\Big|\frac{d}{dx}Tu - \frac{d}{dx}Tv\Big| + M\Big|\frac{d^2}{dx^2}Tu - \frac{d^2}{dx^2}Tv\Big| \right].$$

Applying the Lipschitz condition given in the hypothesis,

$$|Tu - Tv| \leq \|u - v\|\frac{2a^3}{11}.$$

Similarly,

$$\Big|\frac{dTu}{dx} - \frac{dTv}{dx}\Big| \leq \|u - v\|\frac{5a^2}{9},$$

and

$$\Big|\frac{d^2Tu}{dx^2} - \frac{d^2Tv}{dx^2}\Big| \leq \|u - v\|a.$$

Now, $\|Tu - Tv\| \leq \|u - v\|\left[\frac{2a^3}{11}K + \frac{5a^2}{9}L + aM\right]$. By hypothesis, the coefficient of $\|u - v\|$ on the right hand side is less than 1.Therefore, T is a contraction from the complete space, S, into itself. Consequently, it has unique fixed point, u, which is the desired solution.

□

Theorem 4.2. *Let $f : [0, a] \times \mathbb{R}^3 \to \mathbb{R}$ be continuous and satisfy*

$$|f(t, y, y', y'') - f(t, x, x', x'')| \leq K|y - x| + L|y' - x'| + M|y'' - x''|$$

with K, L, and $M > 0$. Assume also that

$$\frac{2Ka^3}{81} + \frac{5La^2}{6} + \frac{4Ma}{3} < 1.$$

Then there exists a unique solution to the ordinary boundary value problem

$$y'''(t) = f(t, y(t), y'(t), y''(t))$$
$$y(0) = y'(0) = y(a) = 0.$$

Proof Let S be as in Theorem 2.2. Define operator T as in the previous theorem. The Green's function for $Lu = u'''$ with the given boundary conditions is calculated in Example 3.5 and is

$$G(x, s) = \begin{cases} -\frac{x^2(a-s)^2}{2a^2} + \frac{(x-s)^2}{2}, & 0 \leq s \leq x \leq a \\ -\frac{x^2(a-s)^2}{2a^2}, & 0 \leq x \leq s \leq a. \end{cases}$$

Integrating G produces

$$\begin{aligned} \int_0^a G(x, s)ds &= \int_0^x \left[\frac{-x^2(a-s)^2}{2a^2} + \frac{(x-s)^2}{2}\right]ds + \int_x^a \frac{-x^2(a-s)^2}{2a^2}ds \\ &= \frac{x^2(x-a)}{6} \end{aligned}$$

so that

$$\int_0^a |G|ds = \int_0^a -Gds = \frac{x^2(x-a)}{6} \leq \frac{2a^3}{81}$$

hence $M_1 = \frac{2a^3}{81}$.

To calculate M_2, differentiating G with respect to x yields

$$G_x(x, s) = \begin{cases} -\frac{x(a-s)^2}{a^2} + (x - s), & 0 \leq s \leq x \leq a \\ -\frac{x(a-s)^2}{a^2}, & 0 \leq x \leq s \leq a. \end{cases}$$

So

$$\begin{aligned}\int_0^a |G_x| ds &\leq \int_0^x \frac{x(a-s)}{a^2} ds + \int_0^x (x-s) ds + \int_x^a \frac{x}{a^2}(a-s)^2 ds \\ &= \frac{xa^3}{3a^3} + \frac{x^2}{2} \\ &= \frac{x}{6}(2a+3x) \\ &\leq \frac{5a^2}{6}\end{aligned}$$

and $M_2 = \frac{5a^2}{6}$. The calculation for G_{xx} leads to M_3. Now,

$$G_{xx}(x,s) = \begin{cases} -\frac{(a-s)^2}{a^2} + 1, & 0 \leq s \leq x \leq a \\ -\frac{(a-s)^2}{a^2}, & 0 \leq x \leq s \leq a. \end{cases}$$

The above calculation gives

$$\begin{aligned}\int_0^a |G_{xx}| ds &\leq \int_0^x -\frac{(a-s)^2}{a^2} ds + \int_0^x ds + \int_x^a \frac{(a-s)^2}{a^2} ds \\ &= \int_0^a \frac{(a-s)^2}{a^2} + \int_0^x ds \\ &= \frac{a}{3} + x \\ &\leq \frac{4a}{3}\end{aligned}$$

so that $M_3 = \frac{4a}{3}$. The rest of the proof follows the proof of Theorem 4.1. Recall the norm

$$\|Tu - Tv\| = \max_{a \leq x \leq b} \left[K\big|Tu - Tv\big| + L\Big|\frac{d}{dx}Tu - \frac{d}{dx}Tv\Big| + M\Big|\frac{d^2}{dx^2}Tu - \frac{d^2}{dx^2}Tv\Big| \right].$$

Applying the Lipschitz condition given in the hypothesis gives the following three inequalities. First,

$$|Tu - Tv| \leq |u - v\|\frac{2a^3}{81},$$

$$\Big|\frac{dTu}{dx} - \frac{dTv}{dx}\Big| \leq \|u - v\|\frac{5a^2}{6},$$

and lastly

$$\Big|\frac{d^2Tu}{dx^2} - \frac{d^2Tv}{dx^2}\Big| \leq \frac{4a}{3}.$$

Thus, these three inequalities imply

$$\|Tu - Tv\| \leq \|u - v\|\Big[\frac{2Ka^3}{3} + \frac{5La^2}{6} + \frac{4Ma}{3}\Big]$$

leading to the desired inequality

$$\frac{2Ka^3}{3} + \frac{5La^2}{6} + \frac{4Ma}{3} < 1.$$

□

Theorem 4.3. *Let $f : [0, a] \times \mathbb{R}^3 \to \mathbb{R}$ be continuous and satisfy*

$$|f(t, y, y', y'') - f(t, x, x', x'')| \leq K|y - x| + L|y' - x'| + M|y'' - x''|$$

with K, L, and $M > 0$. Assume also that

$$\frac{Ka^3}{12} + \frac{La^2}{8} + \frac{Ma}{2} < 1.$$

Then there exists a unique solution to the ordinary boundary value problem

$$y'''(t) = f(t, y(t), y'(t), y''(t))$$
$$y(0) = y'(0) = y'(a) = 0.$$

Proof Let S be as in Theorem 2.2. Define operator T as in the previous theorem. The Green's function for $Lu = u'''$ with the given boundary conditions is shown in Example 3.7 and is

$$G(x, s) = \begin{cases} \frac{s}{2}\Big(\frac{x^2}{a} - 2x + s\Big), & 0 \leq s \leq x \leq a \\ -\frac{x^2}{2a}(a - s), & 0 \leq x \leq s \leq a. \end{cases}$$

Integrating G produces

$$\begin{aligned} \int_0^a G(x, s)ds &= \int_0^x \Big(\frac{sx^2}{2a} - xs + \frac{s^2}{2}\Big)ds - \int_x^a \Big(\frac{x^2}{2a}(a - s)\Big)ds \\ &= \frac{x^3}{6} - \frac{ax^2}{4} \end{aligned}$$

so that

$$\int_0^a |G|ds = \int_0^a -Gds = \frac{-x^4}{6} + \frac{ax^2}{4},$$

hence $M_1 = \frac{a^3}{12}$.

To calculate M_2, differentiating G with respect to x yields

$$G_x(x,s) = \begin{cases} \frac{sx}{a} - s \\ -x + \frac{xs}{a}, & 0 \le x \le s \le a. \end{cases}$$

Since $G_x \le 0$,

$$\begin{aligned} \int_0^a |G_x| ds &= \int_0^x -\Big(\frac{sx}{a} - s\Big) ds + \int_x^a -\big(-x + \frac{xs}{a} ds \\ &= \frac{-x^2}{2} + \frac{xa}{2} \\ &\le \frac{a^2}{8}. \end{aligned}$$

Next,

$$G_{xx}(x,s) = \begin{cases} \frac{s}{a}, & 0 \le s \le x \le a \\ -1 + \frac{s}{a}, & 0 \le x \le s \le a. \end{cases}$$

This gives

$$\begin{aligned} \int_0^a |G_{xx}| ds &= \int_0^x \frac{s}{a} ds + \int_x^a \Big(1 - \frac{s}{a}\Big) ds \\ &= \frac{x^2}{a} + \frac{a}{2} - x \\ &\le \frac{a}{2}. \end{aligned}$$

The rest of the proof follows the proof of Theorem 4.1. Recall the norm

$$\|Tu - Tv\| = \max_{a \le x \le b} \left[K\big|Tu - Tv\big| + L\Big|\frac{d}{dx}Tu - \frac{d}{dx}Tv\Big| + M\Big|\frac{d^2}{dx^2}Tu - \frac{d^2}{dx^2}Tv\Big| \right].$$

Applying the Lipschitz condition given in the hypothesis gives the following three inequalities. First,

$$|Tu - Tv| \le |u - v\|\frac{a^3}{12},$$

$$\Big|\frac{dTu}{dx} - \frac{dTv}{dx}\Big| \le \|u - v\|\frac{a^2}{8},$$

and lastly

$$\Big|\frac{d^2Tu}{dx^2} - \frac{d^2Tv}{dx^2}\Big| \le \frac{a}{2}.$$

Thus,

$$\|Tu - Tv\| \leq \|u - v\| \Big[\frac{Ka^3}{4} + \frac{La^2}{8} + \frac{Ma}{2}\Big]$$

leading to the desired inequality

$$\frac{Ka^3}{4} + \frac{La^2}{8} + \frac{Ma}{2} < 1.$$

□

4.3 A Final Existence Theorem

One last existence/uniqueness theorem is given making use of the Green's function. First, some preliminaries are introduced. Define the norm on S to be

$$\|u\| = \max\left[\max_{a<t<b} \frac{|u(t)|}{w(t)}, \max_{a<t<b} \frac{|u'(t)|}{v(t)}, \max_{a<t<b} \frac{|u''(t)|}{z(t)}\right],$$

where w, v, and z are fixed functions in S, possibly vanishing at the endpoints. A modified Lipschitz condition is given by

$$|f(t, y(t), y'(t), y''(t)) - f(t, x(t), x'(t), x''(t))| \leq p(t)|y-x| + q(t)|y'-x'| + r(t)|y''-x''|,$$

where p, q, and r are nonnegative functions. Using the same operator, T, from the previous theorem with G from Example 3.6, note that

$$\begin{aligned}
\frac{|Ty(t) - Tx(t)|}{w(t)} &\leq \frac{1}{w(t)} \int_a^b \big|G(t,s)\big| |f(s, y(s), y'(s), y''(s)) - f(s, x(s), x'(s), x''(s))| ds \\
&\leq \frac{1}{w(t)} \int_a^b \big|G(t,s)\big| \Big[p(s)w(s)\frac{|y(s) - x(s)|}{w(s)} + q(s)v(s)\frac{|y'(s) - x'(s)|}{v(s)} \\
&\qquad + r(s)z(s)\frac{|y''(s) - x''(s)|}{z(s)}\Big] ds.
\end{aligned}$$

The following three inequalities are needed,

$$\frac{|Ty(t) - Tx(t)|}{w(t)} \leq \frac{\|y-x\|}{w(t)} \int_a^b |G(t,s)|[p(s)w(s) + q(s)v(s) + r(s)z(s)]ds,$$

$$\frac{|\frac{d}{dt}(Ty(t) - Tx(t))|}{v(t)} \leq \frac{\|y-x\|}{v(t)} \int_a^b |G_t(t,s)|[p(s)w(s) + q(s)v(s) + r(s)z(s)]ds, \quad (4.2)$$

and

$$\frac{|\frac{d^2}{dt^2}(Ty(t) - Tx(t))|}{z(t)} \leq \frac{\|y-x\|}{z(t)} \int_a^b |G_{tt}(t,s)|[p(s)w(s) + q(s)v(s) + r(s)z(s)]ds.$$

The goal is to find w, v, and z for which the following hold,

$$\begin{aligned}
\frac{1}{w(t)}\int_a^b |G(t,s)|[p(s)w(s)+q(s)v(s)+r(s)z(s)]ds &\leq \alpha<1 \\
\frac{1}{v(t)}\int_a^b |G_t(t,s)|[p(s)w(s)+q(s)v(s)+r(s)z(s)]ds &\leq \alpha<1 \qquad (4.3)\\
\frac{1}{z(t)}\int_a^b |G_{tt}|(t,s)[p(s)w(s)+q(s)v(s)+r(s)z(s)]ds &\leq \alpha<1.
\end{aligned}$$

Theorem 4.4. *Suppose $f(t,y(t),y'(t),y''(t))$ is continuous on $[0,a]\times\mathbb{R}^3$ and satisfies the above modified Lipschitz condition. If the equation*

$$u'''(t)+r(t)u''(t)+q(t)u'(t)+p(t)u(t)=0$$

has a solution satisfying $u(0)=0$, $u'(0)=0$ on $[0,a]$, and $u''(t)>0$ on $[0,a]$, then

$$y'''(t)=f(t,y(t),y'(t),y''(t))$$

with boundary conditions $y(0)=y'(0)=0$ and $y''(a)=0$ has exactly one solution.

Proof. Consider the comparison problem

$$w'''+\frac{1}{\alpha}[r(t)w''(t)+q(t)w'(t)+p(t)w(t)]=0$$

with

$$w(0)=0,\;\; w'(0)=0,\;\; w''(0)=u''(0).$$

It will be shown that it is possible to find w, $v=w'$, and $z=w''$ so that (4.3) holds for some $\alpha<1$.

Let $m=\min_{0\leq t\leq a}u''(t)$. By continuity of solutions with respect to parameters, given $\epsilon>0$ there exists $\alpha<1$ so that on $[0,a]$,

$$|u(t)-w(t)|<\epsilon,\;\; |u'(t)-w'(t)|<\epsilon,\;\; |u''(t)-w''(t)|<\epsilon.$$

Take $\epsilon<\frac{m}{2}$. Therefore,

$$w''(t)>u''(t)-\epsilon>m-\frac{m}{2}=\frac{m}{2}>0$$

for $t\in[0,a]$. By the Mean Value Theorem, for any $t\in(0,a]$

$$\frac{w'(t)-w'(0)}{t-0}=w''(t^*)>0$$

for some t^* between 0 and t implying that $w'(t) > 0$. A similar argument concludes that $w(t) > 0$ on $(0, a]$. Hence,

$$w'(t),\ w(t)\ > 0 \text{ on } (0, a],\ \ w''(t) > 0 \text{ on } [0, a].$$

The Green's function for $u'''(x) = g(x)$ with boundary conditions $u(0) = u'(0) = u''(a) = 0$ is given by (3.6). Note that G, G_x, and G_{xx} are negative (or zero) and since $w'''(x) = -\frac{1}{\alpha}[rw''' + qw' + pw]$, it follows that h defined by

$$h(x) := \int_0^a |G(x,s)|\big[p(s)w(s) + q(s)w'(s) + r(s)w''(s)\big]ds$$

satisfies the following.

$$\begin{aligned}
h(x) &= \alpha \int_0^a G(x,s)w'''(s)ds \\
&= \alpha\Big(\frac{s}{2}(s-2x)w''(s)\Big|_0^x - \int_0^x (s-x)w''(s)ds - \frac{x^2}{2}w''(s)\Big|_x^a\Big) \\
&= \alpha\Big(\int_0^x w'(s)ds - \frac{x^2}{2}w''(a)\Big) \\
&= \alpha\Big(w(x) - \frac{x^2}{2}w''(a)\Big) \\
&\leq \alpha w(x).
\end{aligned}$$

Thus, $\frac{h(x)}{w(x)} \leq \alpha$. It follows from the expression for h that

$$h'(x) = \alpha[w'(x) - xw"(a)] \leq \alpha w'(x) \text{ and } h''(x) \leq \alpha[w"(x) - w"(a)] \leq \alpha w''(x).$$

Therefore, (4.3) holds implying that T is a contraction mapping. □

4.4 Computations

To illustrate the conclusion of example 3.12 consider the problem $u'''(x) = -62|u| = f(x, u(x))$ with non-homogeneous boundary conditions $u(0) = 2$, $u'(0) = 0$, $u(1) = -1$. Recall from Example 3.12 that the approximation optimal bound on the Lipschitz constant, L, is 75.859. In this case, the Lipschitz constant is chosen to be $L = 62$. Since Theorem 2.1 requires homogeneous boundary conditions, it cannot be applied directly to this example. Define w to be a polynomial satisfying the same boundary conditions, $w(x) = 2-3x^2$. By letting $z(x) = u(x)-w(x)$, z satisfies the homogeneous boundary conditions as stated in 3.12. Moreover,

$$z'''(x) = u'''(x) - w'''(x) = u'''(x) = -62|z(x) + w(x)| := -62f(x, z),$$

so z satisfies the same ODE as u with the same Lipschitz constant. Also note that

$$|f(x,z_2) - f(x,z_1)| = |z_2 + 2 - 3x^2 - z_1 - 2 + 3x^2| = |z_2 - z_1|.$$

Thus, Theorem 2.1 can be applied directly to the ODE $z''' = -62f(x,z)$ with the homogeneous boundary conditions. As long as $L < 75.859$, Theorem 2.1 and Example 3.12 give a unique solution z, satisfying the homogeneous boundary conditions, which in turn, gives a unique solution to the non-homogeneous problem, u. An illustration of this unique solution is given in Figure 1. An iterative process was used to determine the initial value for u'' that corresponds to the boundary conditions $u(0) = 2$, $u'(0) = 0$, $u(1) = -1$. That value was found to be approximately $u''(0) = 113.108$. The graph was then obtained using the standard initial value problem package in MATLAB.

Next, consider the non-homogeneous boundary value problem

$$u''' + \frac{1}{2}\sin u' + \frac{1}{2}|u| = 4$$

with boundary conditions $u(0) = u'(0) = u(1) = 0$. This is an illustration of Theorem 4.2 with $a = 1$ and

$$f(x,u(x),u'(x)) = 4 - \frac{1}{2}\sin u' - \frac{1}{2}|u|.$$

It is necessary to choose K, L, and M so that f satisfies the Lipschitz condition given in the theorem and

$$\frac{2K}{3} + \frac{5L}{6} + \frac{4M}{3} < 1.$$

Since f does not depend on u'', M can be chosen to be 0. Choosing $K = L = \frac{1}{2}$ satisfies the inequality, leaving only the Lipschitz condition to be verified. Note that

$$\begin{aligned}
\big|f(x,u_2(x),u_2'(x)) &- f(x,u_1(x),u_1'(x))\big| \\
&= \big|4 - \frac{1}{2}\sin u_2'(x) - \frac{1}{2}|u_2(x)| - 4 + \frac{1}{2}\sin u_1'(x) + \frac{1}{2}|u_1(x)|\big| \\
&= \big|\frac{1}{2}\sin u_2'(x) - \frac{1}{2}\sin u_1'(x) + \frac{1}{2}|u_2(x)| - \frac{1}{2}|u_1(x)|\big| \\
&\le \frac{1}{2}|\sin u_2'(x) - \sin u_1'(x)| + \frac{1}{2}|u_2(x) - u_1(x)| \\
&\le \frac{1}{2}|u_2'(x) - u_1'(x)| + \frac{1}{2}|u_2 - u_1|.
\end{aligned}$$

Thus, Theorem 4.2 can be applied directly to this problem, yielding a unique solution. A graph of this solution is given in Figure 2. It was obtained using the same methods employed in the previous example using an iterative process to determine the initial value for u'' that corresponds to the boundary conditions $u(0) = u'(0) = u(1) = 0$. That value was found to be approximately $u''(0) = -1.3615$.

Finally, for an illustration of Theorem 4.3 with $a = 1$, consider the problem $u'''(x) + Lu'(x) + K\cos(u(x)) = 0$ with boundary conditions $u(0) = u'(0) = u'(1) = 0$.

The Lipschitz condition holds as follows for any K and L greater than zero.

$$\begin{aligned}|f(x,u_2(x),u_2'(x)) - f(x,u_1(x),u_1'(x))| = \\ &|-Lu_2'(x) - K\cos(u_2(x)) + Lu_1'(x) + K\cos(u_1(x))| \\ &\leq L|u_1'(x) - u_2'(x)| + K|\cos(u_1(x)) - cos(u_2(x))| \\ &\leq L|u_1'(x) - u_2'(x)| + K|u_1(x) - u_2(x)|.\end{aligned}$$

By choosing $K = 2$ and $L = 3$, the inequality in Theorem 4.3 holds. Thus, the hypotheses of Theorem 4.3 hold and a unique solution exists. The graph of this solution is given in Figure 3.

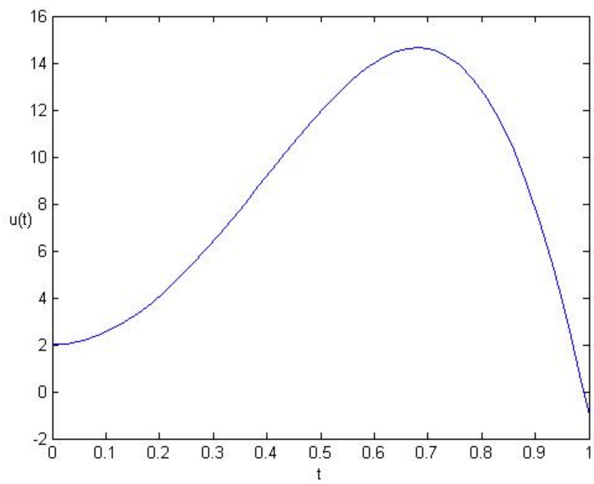

Figure 1: Solution to $u'''(x) = -62|u(x)|$ with $u(0) = 2$, $u'(0) = 0$, $u(1) = -1$

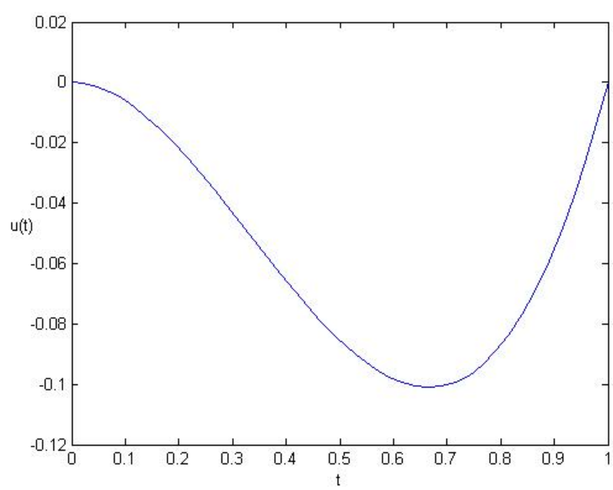

Figure 2: Solution to $u''' + \frac{1}{2}\sin u' + \frac{1}{2}|u| = 4$ with $u(0) = u'(0) = u(1) = 0$

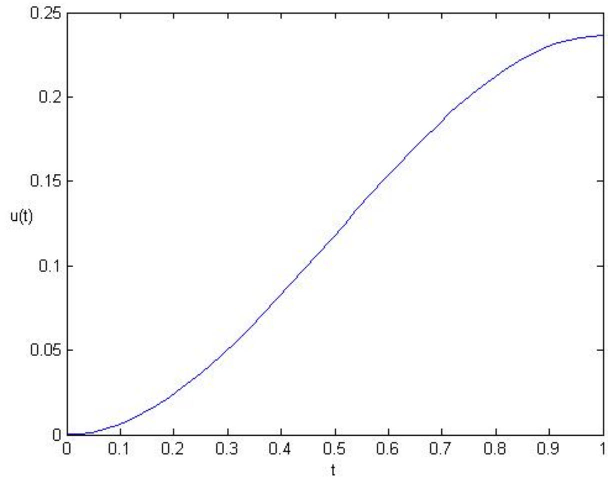

Figure 3: Solution to $u'''(x) + Lu'(x) + K\cos(u(x)) = 0$ with $u(0) = u'(0) = u'(1) = 0$

5 Conclusion

It has been demonstrated that Green's functions have a wide range of applications with regard to boundary value problems. In particular, existence and uniqueness of solutions of a large class of third order boundary value problems has been established. In fact, given any third order ODE with homogeneous boundary conditions, as long as the corresponding Green's function exists and f satisfies an appropriate Lipschitz condition, Theorem 2.1 guarantees the existence of a unique solution under very mild conditions. Similarly, Theorem 2.2 also guarantees such a solution under equally mild conditions. These theorems are contrasted with classical ODE existence theorems in that they circumvent the use of classical convergence analysis by assuming the existence of the Green's function. Banach techniques are still used, but the existence of the Green's function is the primary tool in showing existence and uniqueness. This requires, of course, that the Green's function exists for a particular problem, but the examples in Chapter 3 show that this is usually not a severe restriction.

However, as mild as the restrictions seem to be, one should pay particular detail to the range of values of the Lipschitz constant(s). As Examples 3.10 - 3.12 demonstrate, there are bounds on the Lipschitz constant L that are necessary in order to apply Theorem 2.1. Specifically, as the interval increases in length, the necessary bounds on L become more restrictive, thus lessening the applicability of the existence theorem. Another restriction on the Lipschitz constants arises in Theorem 2.2. The Lipschitz constants corresponding to f must also satisfy an inequality involving bounds on integrals of G and its derivatives, which, if G is badly behaved, may be a severe restriction. The examples of Chapter 4 illustrate these ideas. For example, Theorems 4.1 - 4.3 are specific cases in which Theorem 2.2 is applicable.

As a final application of Green's functions, Theorem 4.4 guarantees a unique solution for a specific third order boundary value problem. This example replaces the standard Lipschitz constants with nonnegative functions, providing a more general Lipschitz condition. However, the existence of a solution under this generality requires more restrictive hypotheses. Thus, the applicability of this theorem is not as broad as that of those in Chapter 2. Nevertheless, this theorem is indicative of how Green's functions can be used in a variety of ways to prove existence and uniqueness of solutions to ODE's. The theorems for third order existence/uniqueness are new and parallel known theorems for second order boundary value problems.

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Vita

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