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American Finance Association

On the Measurement of Fund Performance
Author(s): Harlan D. Mills
Source: *The Journal of Finance*, Vol. 25, No. 5 (Dec., 1970), pp. 1125-1131
Published by: Blackwell Publishing for the American Finance Association
Stable URL: http://www.jstor.org/stable/2325584
Accessed: 09/09/2011 14:03

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ON THE MEASUREMENT OF FUND PERFORMANCE

HARLAN D. MILLS*

I. INTRODUCTION

IN A CELEBRATED PAPER in statistical decision theory, John Milnor[1] laid to rest a variety of subjective arguments at the foundations of economic theory. He formulated a reasonable looking set of criteria for decisions under uncertainty and then proceeded to show that no possible method of choice could satisfy them all. In this way, it becomes apparent that any method of choice among economic alternatives under uncertainty must be based on a pragmatic judgment which omits at least one criterion that seems desirable.

Our purpose here is to show that a similar situation holds in the measurement of fund performance. A reasonable looking set of fund measurement criteria will be shown to be self-contradictory. Thus, any measurement of fund performance must also omit as least one such criterion, and a pragmatic judgment is necessary in evaluating fund performances, as well.

The measurement of fund performance has stimulated a great amount of study and controversy in recent years. There are two major candidates in the financial community—commonly known as "internal rate of return"[2] and "linked rate of return."[3] It is noted that fund evaluation commonly involves risk, as well as return, considerations in looking ahead at a fund's prospects. In the measurement of actual fund performance, however, risk is a meaningless concept, since no uncertainty exists in past events.

Ben-Shahar and Sarnat[4] developed a theoretical distinction between internal rate of return and linked rate of return in terms of reinvestment models. In brief, the internal rate of return model reinvests (or discounts) all cash flows at a single long term rate over a given time period, while the linked rate of return model reinvests such flows at a short term rate subsequent to their creation within the time period. They then argued that linked rate of return has certain theoretical advantages in measuring the performance of common stocks. The internal rate of return, however, is also much used

* Dominick and Dominick, Incorporated, and International Business Machines Corporation. The author is grateful for valuable suggestions from George Wadelton and referees of this Journal.

1. John Milnor, "Games Against Nature," in *Decision Processes*, R. M. Thrall, C. H. Coombs, and R. L. Davis (ed.), Wiley, 1954, pp. 49-59.

2. Also known as "discounted cash flow rate," "bond yield rate," "dollar weighted average rate," etc.

3. Also known as "geometric average rate," "time weighted average rate," etc., discussed in J. H. Lorie *et. al.*, *Measuring the Investment Performance of Pension Funds for the Purpose of Inter-Fund Comparison*, Bank Administration Institute, 1968.

4. Haim Ben-Shahar and Marshall Sarnat, "Reinvestment and the Rate of Return on Common Stocks," *Journal of Finance*, Volume 21, No. 4 (December, 1966), pp. 737-742.

in measuring the performance of funds; in fact, it is the most widely used method today.[5]

Our main result is an "Impossibility Theorem," which shows that not all desirable characteristics in measuring a rate of return can be achieved in a single method. The proof is mathematical. The discussion of the Theorem and its implications is given, initially, in financial terms.

Having posed a necessary dilemma, we suggest a practical escape. The two major candidates are reconsidered in the light of the Impossibility Theorem, and we recognize more clearly what they do—internal rate of return measures *fund performance*, and linked rate of return measures *fund management performance*. In this way, the dilemma at the fund measurement level is transformed into sharpened requirements for choice at the policy level in fund selection and management.

II. The Impossibility Theorem—Financial Formulation

There are two fundamental units in fund operations—value (in dollars, say) and time. Cash flows into (contributions) or out of (withdrawals) a fund take place at definite points in time. Fund valuations are made at definite points in time. These two units are treated differently in the rate of return question. A rate of return is invariant with a change of scale in the value unit, and with a translation in the time unit. That is, two funds which differ only by a constant of proportionality in all their value units (cash flows and valuations) on identical time points will have the same rate of return; two funds which have identical value units occurring (point by point) with a constant displacement in time will also have the same rate of return. And two funds may differ both by a constant of proportionality in value units and by a constant displacement in time, with the same resulting rate of return.

It will be convenient to consider funds existing at identical times, with identical valuations, in certain hypotheses and proofs below. The foregoing observation permits the theoretical results to apply to changes in value scales and translations in time.

We consider the possibility of fund evaluation methods which have certain desirable properties, which we summarize in the following principles.

1. *Principle of Equivalent Cash Flow.* If two funds have identical initial values, identical cash flows (contributions or withdrawals) at identical times, and identical final values over a single evaluation period, then their performance measurements should be identical.
2. *Principle of Equivalent Appreciation.* If the assets of two funds appreciate at identical rates in every subperiod of a single evaluation period, then their performance measurements should be identical.
3. *Principle of Ordinary Return.* If a fund has no cash flows (no contributions or withdrawals), then the fund performance measurement should be that ordinary rate of return which will appreciate the initial value to the final value of the fund over the evaluation period.

5. For example, as used in Edward A. Fox, "Comparing Performance of Equity Pension Trusts," *Financial Analysts Journal*, Vol. 24, No. 5 (1968), pp. 121-129.

Notice, the first two Principles do not require anything of a fund measurement except consistency in dealing with funds which have certain identical characteristics. It is easily verified that internal rate of return satisfies Principle 1, and that linked rate of return satisfies Principle 2. Both these returns satisfy Principle 3 if a fund has no cash flow in a given period. In fact, these Principles appear so simple and natural that it would seem easy enough to satisfy them together.

Yet, as strange as it may be, we find, below, that these three Principles are logically self-contradictory. That is, no matter how ingenious we may be, it is not possible to create a fund evaluation method which does not violate at least one of these three Principles. We restate this as a theorem:

Impossibility Theorem—Financial Formulation: It is not possible to devise a single fund evaluation method which satisfies the three Principles above.

The Impossibility Theorem shows that no "perfect fund measurement" awaits some future insight, and, in fact, rather strengthens the position of the two candidates in measuring aspects of fund performance. Indeed, the Principles they satisfy point up their differences in a precise way. First, we summarize as follows:

Internal rate of return (The Principle of Equivalent Cash Flow) gives the actual rate of return on the dollars which are available to the fund manager. It measures the actual performance of the fund, as it was managed within the constraints of dollar availability. For this reason, we say internal rate of return measures fund performance.

Linked rate of return (The Principle of Equivalent Appreciation) gives the rate of appreciation of the assets of the fund. This rate of appreciation is independent of the cash flows and dollar availabilities of the fund. It depends solely on the asset composition of the fund, which is completely under the discretion of the fund manager. For this reason, we say linked rate of return measures fund management performance.

We restate these observations together:

Rate of Return Characterization: Internal rate of return measures fund performance. Linked rate of return measures fund management performance. Note in the foregoing, we do not imply that internal rate of return is not vitally affected by fund management. It is, indeed. But from the standpoint of measurement, the effect of fund management is confounded with the effect of dollar availabilities of the fund, over which the fund manager, typically, has no control.

Seen in this light, the choice between the two candidates comes down to the reasons for making the measurement in the first place. It cannot be made simply on a methodological basis. Instead, we must distinguish between measuring fund performance and measuring fund management performance in a broader policy context, case by case.

The strategy of proof for the Impossibility Theorem, as developed in the next section, is to assume that a fund evaluation method is possible which satisfies the three Principles, and then to arrive at a contradiction. In carry-

ing out this strategy, a mathematical hypothesis is introduced to correspond to financial common sense—that a fund performance changes smoothly with small changes in the data, and that the changes which do occur themselves change smoothly as well. The two candidates above satisfy the mathematical hypothesis, as would any reasonable scheme.

In more detail, the proof identifies a set of partial differential equations, which is implied by the mathematical formulation of the three Principles—and then shows that no solution can exist for these partial differential equations.

III. The Impossibility Theorem—Mathematical Formulation

For a fund under consideration, consider a sequence of equally spaced instants in time, labeled 0, 1, 2, . . . , $n + 1$, of total duration T between instants 0 and $n + 1$. Suppose the fund has market values $V_0, V_1, \ldots, V_{n+1}$ at each such instant, and that cash flows $F_0, F_1, \ldots, F_n$ occur immediately after each of the first n instants. We do not necessarily know these market values, but assume they exist.

Let us suppose a fund performance measurement is given in the form of a twice differentiable function, X, of the values $V_0, \ldots, V_{n+1}$ and flows $F, \ldots, F_n$.

Principle 1 states that if $V_0, F_0, \ldots, F_n, V_{n+1}$ agree for two funds, then their measurements X must also agree. This implies, therefore, that X must be uniquely defined by just this set of data on which the funds agree; otherwise, if X were to depend, in addition, on some of the intermediate V_i, which need not agree by Principle 1, then their measurements X might not agree as required.

Since, when $V_0, F_0, \ldots, F_n, V_{n+1}$ are fixed, the functional value of X is also fixed, the partial derivatives of X with respect to each of the remaining variables $V_1, \ldots, V_n$ must be zero.

Principle 2 admits a similar development, except that the situation is a little more complex. In this case, X is assumed to be uniquely defined by the appreciation ratios $V_1/(V_0 + F_0), \ldots, V_{n+1}/(V_n + F_n)$. In the proof of the theorem, below, we introduce a new coordinate system in the V_i, F_j space, so that these ratios are coordinates themselves. Then, using the same line of reasoning as above, the remaining coordinates of this new coordinate system cannot effect the value of the measurement X. This means that some new function, the transform of X in this new coordinate system, and denoted as Y in the proof, has the property that its partial derivatives with respect to these remaining coordinates must be zero, also.

Principle 3 gives an explicit evaluation for X when all of the flows $F_0, \ldots, F_n$ are zero, namely, as $(V_{n+1}/V_0)^{1/T}$.

The Impossibility Theorem establishes that no such function X can satisfy the mathematical implications of all three of these Principles. The conditions (1), (2), (3) in the statement of the Theorem correspond exactly to Principles 1, 2, and 3.

Impossibility Theorem—Mathematical Formulation

There exists no function X of variables $V_0, \ldots, V_{n+1}, F_0, \ldots, F_n$, which is twice differentiable, such that its values

$$X(V_0, \ldots, V_{n+1}, F_0, \ldots, F_n),$$

are determined uniquely by

$$V_0, F_0, \ldots, F_n, V_{n+1}, \tag{1}$$

are determined uniquely by

$$\frac{V_1}{V_0 + F_0}, \ldots, \frac{V_{n+1}}{V_n + F_n}, \tag{2}$$

and

$$X(V_0, \ldots, V_{n+1}, 0, \ldots, 0) = (V_{n+1}/V_0)^{1/T}. \tag{3}$$

Proof. By (1), we must have, everywhere,

$$\frac{\partial X}{\partial V_1} = \ldots = \frac{\partial X}{\partial V_n} = 0. \tag{4}$$

Next, we define a new function Y, of variables $W_0, \ldots, W_{n+1}, G_0, \ldots, G_n$ such that whenever

$$W_0 = V_0, W_1 = \frac{V_1}{V_0 + F_0}, \ldots, W_{n+1} = \frac{V_{n+1}}{V_n + F_n}, \tag{5}$$

$$G_0 = F_0, \ldots, G_n = F_n,$$

its values are

$$Y(W_0, \ldots, W_{n+1}, G_0, \ldots, G_n) = X(X_0, \ldots, V_{n+1}, F_0, \ldots, F_n). \tag{6}$$

Then Y is differentiable and, by (2), we must have, everywhere,

$$\frac{\partial Y}{\partial W_0} = \frac{\partial Y}{\partial G_0} = \ldots = \frac{\partial Y}{\partial G_n} = 0. \tag{7}$$

At this point, we have identified two sets of partial differential equations on the same hypersurface (defined by X and Y), referenced to two coordinate systems (V_i, F_j, and W_i, G_j). We will show next that this total system of partial differential equations has only the solution X = constant, which contradicts condition (3). In order to do this, we first reformulate the partial differential equations of (7) in terms of the function X in the coordinate system (V_i, F_j).

We need to solve for variables (V_i, F_j) in terms of variables (W_i, G_j) in what follows. It is easy to see that they are

$$V_0 = W_0, \tag{8}$$

$$V_1 = W_1W_0 + W_1G_0,$$

$$V_2 = W_2W_1W_0 + W_2W_1G_0 + W_2G_1,$$

$$V_3 = W_3W_2W_1W_0 + W_3W_2W_1G_0 + W_3W_2G_1 + W_3G_2,$$

$$\cdots$$

$$V_{n+1} = W_{n+1}\ldots W_0 + W_{n+1}\ldots W_1G_0 + \ldots + W_{n+1}G_n.$$

Using the chain rule to reformulate the first condition of (7), we find

$$\frac{\partial Y}{\partial W_0} = \frac{\partial X}{\partial V_0}\frac{\partial V_0}{\partial W_0} + \ldots + \frac{\partial X}{\partial V_{n+1}}\frac{\partial V_{n+1}}{\partial W_0} + \frac{\partial X}{\partial F_0}\frac{\partial F_0}{\partial W_0} + \ldots + \frac{\partial X}{\partial F_n}\frac{\partial F_n}{\partial W_0} = 0,$$

which, using (4) and the independence of $F_0, \ldots, F_n$ on W_0 in (8) reduces to

$$\frac{\partial Y}{\partial W_0} = \frac{\partial X}{\partial V_0}\frac{\partial V_0}{\partial W_0} + \frac{\partial X}{\partial V_{n+1}}\frac{\partial V_{n+1}}{\partial W_0} = 0,$$

which, using (8), becomes

$$\frac{\partial Y}{\partial W_0} = \frac{\partial X}{\partial V_0} + (W_{n+1}\ldots W_1)\frac{\partial X}{\partial V_{n+1}} = 0,$$

and, finally, using (5)

$$\frac{\partial X}{\partial V_0} + \left(\frac{V_1}{V_0 + F_0}\right)\cdots\left(\frac{V_{n+1}}{V_n + F_n}\right)\frac{\partial X}{\partial V_{n+1}} = 0. \tag{9}$$

In other words, this is a new partial differential equation, in X, with variables V_i, F_j, reformulated from the first partial differential equation of (7).

Since (9) holds everywhere, we can differentiate it with respect to V_1, say, to obtain

$$\frac{\partial^2 X}{\partial V_1 \partial V_0} + \left(\frac{F_1}{V_1(V_1 + F_1)}\right)\left(\frac{V_1}{V_0 + F_0}\right)\cdots\left(\frac{V_{n+1}}{V_n + F_n}\right)\frac{\partial X}{\partial V_{n+1}}$$
$$+ \left(\frac{V_1}{V_0 + F_0}\right)\cdots\left(\frac{V_{n+1}}{V_n + F_n}\right)\frac{\partial^2 X}{\partial V_1 \partial V_{n+1}} = 0.$$

But now, since, from (4),

$$\frac{\partial}{\partial V_0}\left(\frac{\partial X}{\partial V_1}\right) = \frac{\partial^2 X}{\partial V_1 \partial V_0} = 0, \frac{\partial}{\partial V_{n+1}}\left(\frac{\partial X}{\partial V_1}\right) = \frac{\partial^2 X}{\partial V_1 \partial V_{n+1}} = 0,$$

this last equation reduces to

$$\frac{F_1}{V_1(V_1 + F_1)}\left(\frac{V_1}{V_0 + F_0}\right)\cdots\left(\frac{V_{n+1}}{V_n + F_n}\right)\frac{\partial X}{\partial V_{n+1}} = 0. \tag{10}$$

From (10) we can conclude that for any point where all V_i and some F_j are not zero (we have illustrated the case for F_1, but the treatment of any other F_j is similar), then we must have

$$\frac{\partial X}{\partial V_{n+1}} = 0. \tag{11}$$

Next, using (9) and (11), we have

$$\frac{\partial X}{\partial V_0} = 0. \tag{12}$$

Returning to (7), we use the chain rule, again to differentiate Y with respect to G_0, and

$$\frac{\partial Y}{\partial G_0} = \frac{\partial X}{\partial V_0}\frac{\partial V_0}{\partial G_0} + \ldots + \frac{\partial X}{\partial V_{n+1}}\frac{\partial V_{n+1}}{\partial G_0} + \frac{\partial X}{\partial F_0}\frac{\partial F_0}{\partial G_0} + \ldots + \frac{\partial X}{\partial F_n}\frac{\partial F_n}{\partial G_0} = 0,$$

which reduces immediately to

$$\frac{\partial X}{\partial F_0} = 0,$$

and similarly for each other G_j the result is

$$\frac{\partial X}{\partial F_0} = \ldots = \frac{\partial X}{\partial F_n} = 0. \tag{13}$$

Now, assembling the original conditions of (4), with (11), (12), and (13), we find every partial derivative of X is zero, whenever all the V_i and some F_j is not zero; therefore, since X is twice differentiable its partial derivatives must be zero everywhere.

By the foregoing, any function X satisfying conditions (1) and (2) must be of the form X = constant, which contradicts condition (3). This completes the proof of the theorem.