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Quasi-normal Modes of Black Holes in Five-Dimensional Anti-De Spitter Space

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Expansion of quasi-normal modes of black holes in five dimensional anti-de Sitter space

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Abstract

I calculate analytically the asymptotic form of quasi-normal (QN) frequencies for black holes in five-dimensional Anti de-Sitter (AdS) space. I first discuss the three-dimensional case in which exact expressions are obtained. In five dimensions, the wave equation reduces to a Heun equation. I perturbatively solve this equation to obtain the first-order correction to the QN frequencies. The zeroth order frequencies are found by approximating the Heun equation by a hypergeometric equation (valid at large frequencies). My results are in agreement with published numerical results.

1 Review of Normal Modes

Before one can understand or clearly define what a quasi-normal mode is, the definition of normal modes must be made. Simply, a normal mode is an oscillations in which all particles in a system move *in sync* (but not necessarily in the same phase) with each other. For different systems there are different methods by which the normal modes are found.

If a system composed of discrete masses has n degrees of freedom, there will be n normal modes. The number of degrees of freedom depends upon the characteristics of the system; e.g. n-coupled one-dimensional oscillators have n degrees of freedom; a molecule of n atoms has $3n-6$ vibrational degrees of freedom (overall there are $3n$ degrees of freedom, however, 3 are needed for translational motion and 3 for rotational). The general motion of a system is a superposition (often complicated) of the normal modes of the system. One can extend this discussion to include continuous systems (such as waves). By imagining that a string has an infinite number of particles on it, one can see that for a string there exists an infinite number of normal modes.

A basic example of this is two masses connected by a spring to each other and by springs to fixed positions (on a wall, for instance). There are only two normal modes for this system (1-d example with 2 masses). The masses can move in the same direction with the same velocity (the exact same phase), or the two masses can move 180^o out of phase with each other. The general motion of this system can be expressed as a superposition of these 2 normal modes.

If damping is introduced into a system (such as friction), the modes are not exactly normal. The obtained frequencies are no longer purely real but contain a complex part. This tells us that information or energy has been lost. This is basically what defines a quasi-normal mode.

In the case of a black hole, the event horizon causes the damping to occur. This damping is a result of the fact that once information in some form has crossed the event horizon, we can no longer know anything about what happens to it.

2 Introduction

One may at first believe that this project contains no physics, that this project is entirely mathematical without any physical basis. This is not true. This project exemplifies the characteristics of mathematical physics. That is, the solutions of the wave equation I solve are subject to limitations imposed by physical restrictions which are peculiar to my problem (explained later). These limitations take the form of conditions on the solution that must be met at the extremes of the intervals of space and time that are of physical interest [1]. The extremes that are interesting in my problem are at large distances (near infinity) and as the distance approaches the event horizon.

Before Einstein's General Theory of Relativity, all forces of nature were defined *on* spacetime. Einstein proposed that gravity, however, is inherent in spacetime itself, as a manifestation of the curvature of spacetime (due to the presence of matter or energy). This curvature, and the existence of a gravitational field, are impossible to detect locally (in small regions of space), and the laws of physics reduce to those of Einstein's Theory of Special Relativity (flat spacetime) [2].

By studying specific solutions to Einstein's equation, such as in a spherically symmetric gravitational field, we find that due to the coordinate dependent solutions there exist singularities. Singularities in the solution are coefficients that become infinite in certain regimes. By studying singularities (which upon a change of coordinates or other examination may not actually be singularities), such as in the Schwarzschild case, an object referred to as a black hole was hypothesized. A black hole is a region of spacetime that is separated from the rest of the universe by an event horizon, which is a surface beyond which nothing escapes.

By studying the equations that describe the curvature of spacetime assuming maximal symmetry, one finds that there are three different types of spacetime allowed by Einstein's equation (for the three different types of curvature). One notices immediately that the first possibility is one that we all are familiar with, zero curvature. This is the spacetime of special relativity (referred to as Minkowski space, or flat spacetime). The second case, that of positive curvature, is the one many believe our universe is approximately described by now (it is referred to as de Sitter space). Lastly, there is the possibility of negative curvature (anti-de Sitter space or AdS). An interesting feature of anti-de Sitter space is that information from infinity can reach us in a finite amount of time.

The reason for the investigation (and thus the reason for my project) is that in recent years, the study of quasi-normal modes (QNM) has received renewed interest due to the fact that the QN frequencies are in the suggested region of the gravitation wave detectors. The AdS/CFT (Conformal Field Theory) correspondence has also sparked an extensive investigation of the QNM in asymptotically AdS spacetimes. According to the AdS/CFT correspondence, a black hole corresponds to a thermal state in CFT. A perturbation of the black hole corresponds to a perturbation of the thermal state, and the decay of the perturbation corresponds to a return to thermal equilibrium. The QN frequencies, therefore, provide one with the timescale for the return to thermal equilibrium.

Numerical simulations in black hole dynamics, ranging from the collision of two black holes to the formation of black holes in a gravitational collapse, which are just perturbations of the system, show that QNMs dominate the response. QNMs also allow us to test the stability of the event horizon in the face of small perturbations and we are also given information about the global characteristics of the black hole (mass, charge, angular momentum) by analyzing their characteristic waveform.

The QNMs are obtained by solving the wave equation and imposing appropriate boundary conditions. The wave equation is subject to the conditions that the waves are ingoing at the horizon and the wavefunction vanishes at infinity (since the black hole potential diverges at infinity), the boundary of AdS space. Due to these boundary conditions, the QNM are complex.

In this paper, I will first solve for exact solutions of the QNMs. In three dimensions, the wave equation reduces to a hypergeometric equation from which exact expressions can be obtained. In higher dimensions, e.g., five, an analytical solution is not possible and only numerical results can be obtained for the frequencies. The wave equation reduces to a Heun equation in five dimensions, which I solve perturbatively to obtain the corrections, here first order, to the QN frequencies. The zeroth order frequencies are obtained by approximating the wave equation by a hypergeometric equation.

3 Quasi-normal frequencies in 3-dimensions

The metric for a large AdS black hole reads

$$ds^2 = -\frac{1}{L^2}\left(\frac{r^2}{r_h^2} - \frac{r_h^{d-3}}{r^{d-3}}\right) dt^2 + \frac{L^2 dr^2}{\frac{r^2}{r_h^2} - \frac{r_h^{d-3}}{r^{d-3}}} + r^2(dx_1^2 + \ldots + dx_{d-2}^2) \tag{1}$$

where r_h is the radius of the horizon, L is the AdS radius, and d is the number of dimensions in which one is working. Normally, we only experience four dimensions (three spatial and one time). However, due to the nature of the AdS/CFT correspondence (where there is an additional "holographic" dimension to which only gravitons can escape into) one deals with five dimensions. I wish to solve the scalar wave equation for a massless scalar

$$\frac{1}{\sqrt{-g}}\partial_\alpha \sqrt{-g} g^{\alpha\beta}\partial_\beta \Phi = 0 \tag{2}$$

where $g_{\alpha\beta}$ is the metric and $g = \det g_{\alpha\beta}$. Using the metric (2), we obtain

$$\frac{1}{L^2 r^{d-2}}\frac{\partial}{\partial r}\left[r^{d-2}(r^2 - \frac{r_h^{d-1}}{r^{d-3}})\frac{\partial \Psi}{\partial r}\right] + \frac{1}{r^2}\nabla^2\Psi = \frac{L^2}{r^2 - \frac{r_h^{d-1}}{r^{d-3}}}\frac{\partial^2 \Psi}{\partial t^2} \tag{3}$$

This wave equation is exactly the same as the wave equation everyone is familiar with, except in negatively curved spacetime. The wave equation describes an oscillation in a field (electromagnetic, gravitational, etc.).

I start with three dimensions to show the methods by which I will be solving the more difficult five dimensional case. In three dimensions ($d = 3$), the wave equation is given by

$$\frac{1}{L^2 r}\frac{\partial}{\partial r}\left(r(r^2 - r_h^2)\frac{\partial \Psi}{\partial r}\right) - \frac{L^2}{r^2 - r_h^2}\frac{\partial^2 \Psi}{\partial t^2} + \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial x^2} = 0 \tag{4}$$

I will be solving this equation in the interval $r \in [r_h, \infty]$, I solve it inside the horizon ($r \in [0, r_h]$). This is just computationally advantageous. Separating variables, I write the wavefunction as

$$\Psi(t, r, x) = T(t)\psi(r)X(x) \tag{5}$$

I then substitute this into (4) to obtain differential equations in X,T, and R. The resulting separated differential equations read

$$T'' = -\omega^2 T \tag{6}$$

$$X'' = -p^2 X \tag{7}$$

$$r_h^4(\frac{r^2}{r_h^2} - 1)\psi'' + \frac{r_h^4}{r}(3\frac{r^2}{r_h^2} - 1)\psi' - \left[\frac{p^2 r_h^2 L^2}{r^2} - \frac{\omega^2 L^4}{(\frac{r^2}{r_h^2} - 1)}\right]\psi = 0 \tag{8}$$

The solutions in (6) and (7) are easily found. The wavefunction may then be written as

$$\Psi = e^{i(\omega t - px)}\psi(r) \tag{9}$$

The radial equation (8) is a little more difficult to solve. To find its solution, I make the substitution $y = \frac{r^2}{r_h^2}$. My differential equation then becomes

$$y(1-y)\psi''(y) + (1-2y)\psi'(y) + \left(\frac{\hat{\omega}^2}{1-y} + \frac{\hat{p}^2}{y}\right)\psi = 0 \tag{10}$$

where

$$\hat{\omega}^2 = \frac{\omega^2 L^4}{4r_h^2}\ ,\ \hat{p}^2 = \frac{p^2 L^2}{4r_h^2} \tag{11}$$

To find the solution, I must examine the behavior of the differential equation when I impose my boundary conditions. The boundary conditions (that is the extremes of physical interest I mentioned before) are at large distances (as $r \to \infty$, or $y \to \infty$), as one approaches the event horizon ($r \to r_h$, or $y \to 1$), and as $y \to 0$ (the black hole singularity). Mathematically, these singularities are found by examining the coefficient of the ψ'' term. Here, there are three singular points, as I have stated (they are $y = 0, 1, \infty$). By studying the behavior of the wavefunction as $y \to 1$ (near the horizon), I obtain two linearly independent solutions (since it is a second order differential equation).

$$\psi_\pm \sim (1-y)^{\pm i\hat{\omega}} \tag{12}$$

where ψ_+ is outgoing and ψ_- is ingoing at the horizon. By examining the behavior at the black hole singularity ($y \to 0$), I obtain a second set of linearly independent solutions

$$\psi_\pm \sim y^{\pm i\hat{p}} \tag{13}$$

One of my conditions (from the Introduction) my solutions would have to obey to obtain the QNMs is that the wave be purely ingoing at the horizon. This amounts to $\psi \sim \psi_-$ as $y \to 1$. By combining my expressions for the extreme behavior and introducing a function of y to account for other behavior (to obtain a full solution of the radial equation (10)), I write

$$\psi(y) = y^{\pm i\hat{p}}(1-y)^{-i\hat{\omega}}F(y) \tag{14}$$

By substituting this expression into (10), the differential equation reduces to a hypergeometric equation which reads

$$y(1-y)F''(y) + \{1 \pm 2i\hat{p} - (2 - 2i(\hat{\omega} \mp \hat{p}))y\}F'(y) + (\hat{\omega} \mp \hat{p})(\hat{\omega} \mp \hat{p} + i)F(y) = 0 \tag{15}$$

The solution is the hypergeometric function $F(\alpha, \beta; \gamma; y)$ where

$$\alpha = 1 - i(\hat{\omega} \mp \hat{p})$$
$$\beta = -i(\hat{\omega} \mp \hat{p})$$
$$\gamma = 1 \pm 2i\hat{p}$$

We thus have a solution in the form

$$F(y) = F(1 - i(\hat{\omega} \mp \hat{p}), -i(\hat{\omega} \mp \hat{p}); 1 \pm 2i\hat{p}; y) \tag{16}$$

which is a mixture of ingoing and outgoing waves, and it diverges as $y \to \infty$. To study the behavior near the horizon ($y \to 1$), I use the hypergeometric identity

$$F(1-\alpha, \beta; \gamma; y) = \frac{\Gamma(\gamma)\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} F(\alpha, \beta; \alpha+\beta-\gamma+1; 1-y)$$

$$+(1-y)^{\gamma-\alpha-\beta}\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} F(\gamma-\alpha, \gamma-\beta; \gamma-\beta-\alpha+1; 1-y) \tag{17}$$

From this expression, I see that as $y \to 1$, the argument of the hypergeometric functions on the right-hand side becomes zero. When this occurs, the hypergeometric function is equal to one (one can see this by expanding the hypergeometric function as a series in powers of $(1-y)$). Therefore, the first term on the right hand side of (17) approaches a constant as $y \to 1$. The second term is singular as $y \to 1$ and contributes to an incoming wave at the horizon. Therefore, it should not contribute to the quasi-normal modes. This will be the standard way by which I find my quasi normal frequencies. I will observe the behavior of my solutions near the singularities. Any piece that does not remain well behaved (finite) will be removed. This will be accomplished by setting its coefficient equal to zero. From this condition on the coefficient, I will have a way to calculate the quasi-normal frequencies. This implies that the argument of one of the gamma functions in the denominator (chosen here to be $\Gamma(\beta)$) on the second term on the right-hand side of (17) must be equal to a negative integer (since this will cause the gamma function to diverge to infinity), that is, I obtain

$$\hat{\omega} = \pm\hat{p} - in \ , \ n = 1, 2, \ldots \tag{18}$$

This result gives us a discrete set of complex frequencies, as expected [4]. From this, we can rewrite (16)

$$F(y) = F(1-n, -n; 1 \pm 2i\hat{p}; y) \tag{19}$$

which is a polynomial of order $n-1$ that is constant at $y = 1$ and as $y \to \infty$ it behaves as

$$F(y) \sim y^{n-1} \sim y^{i(\hat{\omega}\mp\hat{p})-1}$$

This solution then exhibits the proper behavior as $y \to \infty$ (i.e. $\psi \sim y^{-1}$ and $\Psi \sim 0$).

4 Quasinormal Frequencies in 5-dimensions

In five dimensions ($d = 5$), the wave equation for a massless scalar becomes

$$\frac{1}{L^2 r^3}\frac{\partial}{\partial r}(r^5(1-\frac{r_h^4}{r^4})\frac{\partial \Psi}{\partial r}) - \frac{L^2}{r^2(1-\frac{r_h^4}{r^4})}\frac{\partial^2 \Psi}{\partial t^2} + \frac{1}{r^2}\vec{\nabla}^2\Psi = 0 \tag{20}$$

where $\vec{\nabla}^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$. By assuming the wavefunction is of a separable form (as performed in the three dimensional case)

$$\Psi(t, r, x_1, x_2, x_3) = T(t)R(r)X_1(x)X_2(x)X_3(x) \tag{21}$$

we are led to similar separated differential equations as found in the three-dimensional case. The wavefunction now takes the form

$$\Psi = e^{i(\omega t - \vec{p}\cdot\vec{x})}\psi(r) \tag{22}$$

The radial differential equation, after I make the substitution $y = \frac{r^2}{r_h^2}$, reads

$$y(y^2-1)^2\psi'' + (y^2-1)(3y^2-1)\psi' + \left[\frac{\hat{\omega}^2}{4}y^2 - \frac{\hat{p}^2}{4}(y^2-1)\right]\psi = 0 \tag{23}$$

where

$$\hat{\omega} = \frac{\omega L^2}{r_h} \;,\; \hat{p} = \frac{|\vec{p}|L}{r_h}$$

By studying the behavior of (23) near the singularities (found again by looking at the coefficient of the ψ'' term and noticing where it would become equal to zero or diverge, I isolate the behavior of the wavefunction at the singularities $y = \pm 1$. I then write

$$\psi(r) = (y-1)^{-\frac{i\hat{\omega}}{4}}(y+1)^{\pm\frac{i\hat{\omega}}{4}}F(y)$$

One can also examine the behavior at large y, from which one obtains

$$\psi(y) \sim y^{-h_\pm} \;,\; h_\pm = 1 \pm 1$$

Since I am interested in a solution that vanishes at the boundary of AdS space, the solution must behave as $\psi \sim y^{-h_+}$ as $y \to \infty$. Then equation (23) reduces to

$$y(y^2-1)^2F''(y) + \left[(3 - \frac{(i\mp 1)\hat{\omega}}{2})y^2 - \frac{(i\pm 1)\hat{\omega}}{2}y - 1\right]F'(y) +$$

$$\left[\frac{\hat{\omega}}{2}(\mp\frac{i\hat{\omega}}{4} \pm 1 - i)y - (i-1)\frac{\hat{\omega}}{4} - \frac{\hat{p}^2}{4}\right]F(y) = 0 \tag{24}$$

For large $\hat{\omega}$ (large frequencies), the constant terms in (24) are small when compared to the other terms and can be dropped in the region of physical interest, $y \geq 1$. I then approximate (24) by the hypergeometric equation

$$(y^2-1)F'' + \left[(3 - \frac{(i\mp 1)\hat{\omega}}{2})y - \frac{(i\pm 1)\hat{\omega}}{2}\right]F' + \frac{\hat{\omega}}{2}(\mp\frac{i\hat{\omega}}{4} \pm 1 - i)F = 0 \tag{25}$$

To bring this to the exact form of the Hypergeometric equation, I must then make another substitution

$$z = \frac{1-y}{2} \tag{26}$$

This then reduces (25) to the normal form of the hypergeometric equation which now reads

$$z(1-z)F'' + \left[(\frac{3}{2} - \frac{i\hat{\omega}}{2} - (3 - \frac{(i+1)\hat{\omega}}{2})z\right] F' - \frac{\hat{\omega}}{2}\left(\frac{i\hat{\omega}}{4} - 1 - i\right) F = 0 \tag{27}$$

From which it is easy to determine two linearly independent solutions of (27)

$$u_1(z) = F(a_+, a_-; c; z) \ , \ u_2 = z^{1-c} F(1 + a_+ - c, 1 + a_- - c; 2 - c; z) \tag{28}$$

where a_+, a_-, c are given by

$$a_+ = 2 - \frac{(i+1)\hat{\omega}}{4} \tag{29}$$

$$a_- = -\frac{(i+1)\hat{\omega}}{4} \tag{30}$$

$$c = \frac{3}{2} - \frac{i\hat{\omega}}{2} \tag{31}$$

Since I am interested in solutions which are well behaved at the horizon ($z = 0$), the acceptable solution of (27) is u_1 (Eq (28)). By examining the behavior at the boundary of AdS space ($z \to \infty$), I obtain another solution to (27) (by looking up standard hypergeometric transformations)

$$K = (1-z)^{-a_+} F\left(a_+, c - a_-; a_+ - a_- + 1; \frac{1}{1-z}\right) \tag{32}$$

As $z \to -\infty$, I have $K \sim z^{-a_+}$ which corresponds to $\psi \sim y^{-h_+}$, leading to $\psi \sim 0$. This is the type of behavior that I want. To find the behavior of K near the horizon, I express K as a linear combination of u_1 and u_2. From basic hypergeometric identities, I obtain

$$K = A_0 u_1 + B_0 u_2 \tag{33}$$

where

$$A_0 = \frac{\Gamma(1-c)\Gamma(1 - a_- + a_+)}{\Gamma(1 - a_-)\Gamma(1 - c - a_+)} \ , \ B_0 = \frac{\Gamma(c-1)\Gamma(1 + a_+ - a_-)}{\Gamma(a_+)\Gamma(c - a_-)} \tag{34}$$

For the correct behavior at the horizon (that is, it is well behaved), I demand

$$B_0 = 0 \tag{35}$$

from which I obtain two different conditions,

$$a_+ = -n + 1 \ , \ n = 1, 2, 3, \ldots \tag{36}$$

or

$$c - a_- = -n + 1 \ , \ n = 1, 2, 3, \ldots \tag{37}$$

From (36), I obtain one set of quasi-normal frequencies

$$\hat{\omega}_n = 2(n+1)(1-i) \tag{38}$$

and from (37), I obtain a second set

$$\hat{\omega}_n = -2(1+i)(n+\frac{1}{2}) \tag{39}$$

which was expected. I will be working with the second set (39) of quasi-normal frequencies. Since I am dealing with large frequencies (large n), the two expressions asymptotically (as $n \to \infty$) have the same imaginary part but opposite real parts. This is expected from the wave equation and is similar to the three-dimensional case. It is redundant to study both sets. The above approximation to the Heun equation, K, can be used as the basis for the calculation of corrections to the quasi-normal frequencies. I can thus write (27) as

$$(H_0 + H_1)F = 0 \tag{40}$$

where

$$H_0 = z(1-z)F'' + \left[(\frac{3}{2} - \frac{i\hat{\omega}}{2} - (3 - \frac{(i+1)\hat{\omega}}{2})z\right] F' - \frac{\hat{\omega}}{2}\left(\frac{i\hat{\omega}}{4} - 1 - i\right) F = 0 \tag{41}$$

and

$$H_1 = \frac{1}{2(2z-1)}\left(\frac{d}{dz} + 2(n+\frac{1}{2})\right) \tag{42}$$

The zeroth order equation

$$H_0 F_0 = 0 \tag{43}$$

has already been solved (hypergeometric equation (32)), and the solution was chosen as

$$F_0 = K \tag{44}$$

By treating H_1 as a perturbation, I can expand the wavefunction

$$F = F_0 + F_1 + \ldots \tag{45}$$

and solve (40) perturbatively. That is, by writing out the product between the perturbation (40) and the expansion in the wavefunction (45), I obtain

$$H_0F_0 + H_1F_0 + H_0F_1 + \ldots = 0$$

From (43), I obtain the first-order equation

$$H_0F_1 = -H_1F_0$$

which is an inhomogeneous equation for F_1. Therefore, to find the first order perturbation, I must solve for F_1. The expression I obtain to find F_1 (found using variation of parameters) is

$$F_1(z) = u_2(z)\int_{-\infty}^{z} \frac{u_1H_1F_0}{W} - u_1(z)\int_{-\infty}^{z} \frac{u_2H_1F_0}{W} \tag{46}$$

where W is the Wronskian of u_1, u_2 which is

$$W = (c-1)z^{-\gamma}(1-z)^{-(a_+ + a_- - c+1)} \tag{47}$$

F_1 does not alter the behavior of $F_0 = K$ at infinity. To find its behavior near the horizon, it is convenient to introduce the expression

$$\delta F_1(z) = u_2(z)\int_z^\lambda \frac{u_1 H_1 F_0}{W} - u_1(z)\int_z^\lambda \frac{u_2 H_1 F_0}{W} \tag{48}$$

As $\lambda \to 0$, one sees that δF_1 remains finite near the horizon ($z \sim 0$). By applying the same method as I have done before, by combining all of the terms that remain finite and well behaved at the horizon with each other and then combining all of the terms that do not remain well behaved, I will have an equation I can then solve to find the quasi normal frequencies. Since (48) remains finite, I can add it to (46) without it affecting the pieces that I want to investigate later (the terms that are not well behaved). I thus have (after I allow $\lambda \to 0$),

$$\delta F_1(z) + F_1(z) = u_2(z)\int_{-\infty}^0 \frac{u_1 H_1 F_0}{W} - u_1\int_{-\infty}^0 \frac{u_2 H_1 F_0}{W} \tag{49}$$

I know that u_1 is finite (well behaved) near the horizon and that u_2 is not. By isolating u_2 in my expression (49), I can obtain the corrections to the modes. I thus write

$$F_1(z) \sim A_1 u_1 + B_1 u_2 \tag{50}$$

Where A_1 is the piece that is well behaved and finite near the horizon, and thus the segments that I want (but do not need to calculate). B_1, however, is the segment which is not well behaved at the horizon, and is given by the same expression as in (49). Thus,

$$B_1 = \int_{-\infty}^0 \frac{u_1 H_1 F_0}{W} \tag{51}$$

where

$$u_1 = \frac{1}{A_0}K$$

and A_0 is given by (34). I now have a method by which I can solve for the corrections to the quasi-normal frequencies. To first order, the quasi-normal frequencies are found by solving

$$B_0 + B_1 = 0 \tag{52}$$

where B_0 is given by (34). I set $B_0 = 0$ before, but this zeroth-order equation must be corrected. The correction to B_0 is small. This correction I call ϵ. One can see where the correction is made by correcting (37), that is

$$c - a_- = -n + 1 + \epsilon$$

I first obtain the general expression for epsilon that I will calculate. I can do this by solving for ϵ in B_0. Therefore, by using

$$\Gamma(x+1) = x\Gamma(x) \tag{53}$$

and applying it to the general case, I obtain

$$\Gamma(\epsilon - n + 1) = \frac{(-1)^{n-1}}{\epsilon(n-1)!} \tag{54}$$

From this, B_0 reads

$$B_0 = (-1)^{n-1} \frac{\Gamma(a_- - n)\Gamma(3)(n-1)!}{\Gamma(a_- - 2)} \epsilon + \ldots \tag{55}$$

However, before I calculate B_1, I must make another substitution. It is convenient to reverse the limits of integration in (51). I therefore make the substitution

$$z = -x \tag{56}$$

For any n, the expression for B_1 becomes

$$B_1 = \frac{\Gamma(n - a_-)\Gamma(3)}{\Gamma(n - a_- - 2)\Gamma(1 - a_-)} \int_0^\infty \frac{K(\frac{d}{dx} + 2(n + \frac{1}{2}))K}{2(2x+1)(a_- - (n-1))x^{-a_- + n - 1}(1+x)^{-(a_- + n + 2)}} dx \tag{57}$$

where K is given by (32). Therefore, by substituting (57) and (55) into (52) and solving for ϵ, I obtain

$$\epsilon = -\frac{\Gamma(a_- - 2)}{\Gamma(a_- - n)\Gamma(3)} \frac{\Gamma(1 - a_-)\Gamma(n - 2 - a_-)}{\Gamma(n - a_-)\Gamma(3)}$$
$$\times \int_0^\infty \frac{K(\frac{d}{dx} + 2(n + \frac{1}{2}))K}{2(2x+1)(a_- - (n-1)x^{-a_- + n - 1}(1+x)^{-(a_- + n + 2)}} dx \tag{58}$$

By using these relations, one will be able to determine the numerical value for ϵ for any n. The asymptotic value of n (as $n \to \infty$) is (from numerical analysis [3])

$$\epsilon = 0.002775 - 0.011i \tag{59}$$

I will calculate the first-order correction to $\hat{\omega}_1$. From (37) for $n = 1$ I obtain $c = a_-$, which means

$$K = F_0 = (1 - z)^{-a_+} \tag{60}$$

and

$$u_1 = K \tag{61}$$

Substituting this into (51), I obtain

$$B_1 = \int_0^\infty \frac{(1+x)^{-a_+}(\frac{d}{dx} + 3)(1+x)^{-a_+}}{2(2x+1)(c-1)x^{-c}(1+x)^{-(a_+ + a_- - c + 1)}} dx \tag{62}$$

which I can compute using

$$\int_0^\infty dx \frac{x^\mu (1+x)^{-\nu}}{2x+1} = B(1+\mu, \nu - \mu) F(1, 1+\mu; 1+\nu; -1) \tag{63}$$

where $B(x,y)$ is the Beta function and F is just the hypergeometric function, I obtain

$$B_1 = \frac{\Gamma(a_- - 1)\Gamma(1+a_+ - a_-)}{\Gamma(a_+)}[F(1, 1+a_-; 1+a_+; -1) - F(1, 1+a_-; a_+; -1)] \tag{64}$$

which I can simplify using a Gaussian recursion relation

$$\frac{a_-}{a_+(a_+ + 1)}F(a_+ + 1, a_- + 1; c+2; z) = F(a_+, a_-; c; z) - F(a_+, a_-; c+1; z) \tag{65}$$

Equation (63) then reads

$$B_1 = \frac{a_-^2}{2(c-1)a_+(a_+ + 1)}B(a_- - 1, 1 + a_+ - a_-)F(2, a_- + 1; a_+ + 2; -1) \tag{66}$$

By substituting in my expression for the Beta function I obtain

$$B_1 = \frac{\Gamma(a_- - 1)\Gamma(1+a_+ - a_-)}{\Gamma(a_+)}\frac{a_-^2}{2(c-1)a_+(a_+ + 1)}F(2, a_- + 1; a_+ + 2; -1) \tag{67}$$

By substituting in my expressions for a_+ and a_- into (61), I find that

$$B_1 = -\frac{1}{\frac{3i}{2} - 1}\frac{\Gamma(\frac{3i}{2} - 1)\Gamma(3)}{\Gamma(2 + \frac{3i}{2})}(\frac{-3}{50} + \frac{3i}{25})F(2, 1 + \frac{3i}{2}; 4 + \frac{3i}{2}; -1) \tag{68}$$

To find the correction to the quasi normal frequencies, I also need B_0, which is given by (55) for $n = 1$,

$$B_0 = \frac{\Gamma(c-1)\Gamma(1+a_+ - a_-)}{\Gamma(a_+)\Gamma(\epsilon)} \tag{69}$$

By applying (53) as was done in the general case, I obtain

$$\epsilon\frac{\Gamma(c-1)\Gamma(1+a_+ - a_-)}{\Gamma(a_+)} + B_1 = 0 \tag{70}$$

Now I can solve for the correction, ϵ,

$$\epsilon = -\frac{B_1\Gamma(a_+)}{\Gamma(c-1)\Gamma(1+a_+ - a_-)} \tag{71}$$

By substituting in my expressions for B_1, a_+, and a_-, I obtain

$$\epsilon = \frac{1}{\frac{3i}{2} - 1}(\frac{-3}{50} + \frac{3i}{25})F(2, \frac{3i}{2} + 1; \frac{3i}{2} + 4; -1) \tag{72}$$

which can be calculated numerically by expanding F in a Taylor series. My mentor, Dr. Siopsis, wrote the code that can be used to find the value of ϵ. He obtained

$$\epsilon = 0.038 - 0.021i \tag{73}$$

I can now compare my obtained value of ϵ with that in [3] for $n = 1$. The value obtained through numerical analysis [3] is $\epsilon = 0.0398 - 0.0844i$. My result (73) is close to the number found through numerical analysis. In relation to the asymptotic value of ϵ, one should obtain the correct value of the asymptotic ϵ from my expression (58), by taking the $n \to \infty$ limit.

5 Conclusion

I have found the lowest correction for the first-order quasi-normal modes for a scalar perturbation of an AdS black hole. The numerical value I obtained analytically for the lowest quasi-normal mode in (73) is in agreement with what has already been found through numerical analysis. One should be able to use my expression (58) to find ϵ for all values of n. Future research should involve computing my expression in the asymptotic limit ($n \to \infty$) and checking agreement with the numerical result (59). Though the expressions one would obtain from my expression would be very complex, one should be able to determine the value of ϵ.

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