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Harlan D. Mills

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Harlan D. Mills

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Princeton University
Econometric Research Program
92-A Nassau Street
Princeton, N. J.

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Harlan Mills*

INTRODUCTION

Many problems encountered in logistics are those of "organized complexity" [12] — problems of moderately large, but heterogeneous systems. These systems are often too complex for modern analytic techniques. New concepts seem to be in order. Our current shortcomings have more to do with the quality than with the quantity of information produced. As a rule, we develop more detail and less perspective than we would like — the problem is how to trade one for the other in an effective way.

We consider the operation of certain elementary logistic components as stochastic processes. By transforming questions about inventory levels and ordering or production rates into questions about their statistical properties, we seek new sources of macroscopic relationships and perspectives in problems of production and inventory smoothing.

This approach parallels that of Simon [10], Vassian [11], and Pinkham [9] in seeking servo-statistical properties of logistic operations. Another approach of great promise, dynamic programming, has been formulated by Bellman [4] with antecedents in classic papers of Arrow, Harris and Marschak [1] and Dvoretsky, Kiefer and Wolfowitz [5].

ELEMENTARY LOGISTIC COMPONENTS

Consider an operation engaged in storing, shipping (in response to external demand), and requisitioning (on an external source) a single commodity. The operation is described, for our purposes, by a set of

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measurements (nonnegative numbers) at, or between, an ordered set of discrete time points, as indicated:

d_t - demands for the commodity during period t

i_t - inventory level of the commodity at the beginning of period t

r_t - requisition for the commodity made during period t (requisitions are filled c periods later, c being called the "requisition cycle")

s_t - shipments of the commodity made during period t.

Shipments are not allowed to exceed demands in any given period — if the commodity may be backordered, we redefine a "cumulative" demand to include that condition. Abstractly, we characterize such an operation as a logistic component as follows.

A logistic component L is the set of sequences

$$\{(\ldots, d_{t-1}, i_{t-1}, r_{t-1}, s_{t-1}, d_t, i_t, r_t, s_t, \ldots)\}$$

such that for each t,

$$i_{t+1} = i_t + r_{t-c} - s_t,$$
$$s_t \leq \min(d_t, i_t + r_{t-c})$$
$$s_t \geq 0, \; i_t \geq 0, \; r_t \geq 0$$

and dt, $t > 0$, is an outcome of a random experiment.

We assume the random experiments leading to the d_t are independent and identical, and for some number b, and each t, $\text{Prob}\,\{d_t > b\} = 0$ (we say the demand is bounded by b). We also take the requisition cycle $c \geq 1$ — otherwise there is no problem.

At each discrete point in time, given a history $h_t = (\ldots, d_t, i_t, r_t, s_t)$, d_{t+1} and i_{t+1} are determined by the definitions above, while r_{t+1} and s_{t+1} (the requisition and allocation decisions) need to be determined by the agency operating the component. While no natural requirement rules out complete caprice, we shall only consider these decisions as consistently based on past information. This, briefly, is what we mean by a decision policy, defined as follows:

A decision policy P is a function, mapping the set

$$\{h_{t-1}, d_t, i_t\}$$

into the set

$$\{(r, s) | r \geq 0, \; 0 \leq s \leq \min(d_t, i_t + r_{t-c})\} \quad .$$

As a function of past information, P has access to sample statistics associated with the demand, but does not have access to the population statistics of the demand. For example, we do not allow a policy which requisitions "mean demand" etc.

The rules of a logistic component L, a fixed history h_o, and decision policy P select one sided subsequences, $(d_1; i_1, r_1, s_1, d_2, i_2, r_2, s_2, \ldots)$, from L with definite probabilities, i.e., they determine a stochastic process, which we denote by

$$\{(D_1, I_1, R_1, S_1, D_2, I_2, R_2, S_2, \ldots)\}$$

where D_t, I_t, R_t, S_t are random variables. For convenience, we transfer the specifications of L into this process, writing

$$I_{t+1} = I_t + R_{t+c} - S_t$$

$$D_t \geq S_t \geq 0, \; I_t \geq 0; \; R_t \geq 0$$

to mean the relations hold for every possible realization in the process; if D_t is bounded by b, we write $D_t \leq b$. If

$$\bar{D}_t = E(D_t), \; \bar{I}_t, \; \bar{S}_t, \; \bar{R}_t \quad ;$$

$$\bar{D} = \lim_{N \to \infty} \frac{1}{N} \Sigma_{t=1}^{N} \bar{D}_t, \; \bar{I}, \; \bar{S}, \; \bar{R} \quad ;$$

$$\sigma^2_{D_t} = E[(D_t - \bar{D}_t)^2], \; \sigma^2_{I_t}, \; \sigma^2_{S_t}, \; \sigma^2_{R_t}$$

$$\sigma^2_{D} = \lim_{N \to \infty} \frac{1}{N} \Sigma_{t=1}^{N} \sigma^2_{D_t}, \; \sigma^2_{I}, \; \sigma^2_{S}, \; \sigma^2_{R} \quad ,$$

exist we say the process is stable. Other moments may be of interest — these are sufficient for our present development. In a stable process, we have, directly from the material balance

$$\bar{I}_{t+1} = \bar{I}_t + \bar{R}_{t-c} - \bar{S}_t$$
$$\bar{R} = \bar{S} \quad .$$

We use such moments to characterize the performance of a given decision policy. For example,

$\bar{D} - \bar{S}$ indicates unsatisfied demands,
$\bar{I}$ indicates inventory levels,
σ_I indicates inventory variability,
σ_R indicates requisition (production) variability .

These indicators must be used judiciously. Generally they measure about what they seem to. However, the very simple process given by

$$h_0 = (\ldots, 0, 0, 0)$$

$$d_t = 1, \; r_t = d_{t-k}, \; s_t = \min(d_t, s_{t-1}), \; t > 1$$

gives $\bar{D} - \bar{S} = 0$, but k can be chosen to give as large an unsatisfied demand as we please.*

Our general program for studying a logistic component is to search for

1) ultimate boundaries of performance — necessary conditions on measures of performance imposed by the specifications of the logistic component itself,
2) decision policies which approach these ultimate boundaries in performance.

Theorem 1, below, is directed toward task 1), establishing an ultimate boundary in the moment space which characterizes certain contradictory elements in the multifold objectives of minimizing $\bar{D} - \bar{S}$, $\bar{I}$, σ_I, and

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* This example was kindly supplied by a referee.

σ_R. Theorem 2, devoted to task 2), establishes the completeness and optimality of a certain class of decision policies in an asymptotic sense (to be defined) for these objectives. Theorem 3 develops a relationship between requisition cycles and the measures σ_I, σ_R.

THEOREM 1. (Smoothing Capacity). For any decision policy P which determines a stable stochastic process

$$\{(D_t, I_t, R_t, S_t)\}$$

in a logistic component L, it is necessary that

$$\sigma_I \geq \frac{1}{2} (\sigma^R + \frac{a^2}{\sigma_R}), \quad \text{where} \quad a^2 = \sigma_D^2 - (\bar{D} - \bar{S})(2b + 2\bar{I} - \bar{D} + \bar{S}) .$$

Proof. Since

$$I_{t+1} = I_t + R_{t-c} - S_t, \; S_t \leq D_t$$

we have

$$I_{t+1} \geq I_t + R_{t-c} - D_t \quad ,$$

and with probability 1,

$$I_{t+1} + b \geq I_t + R_{t-c} - D_t + b \geq 0 \quad .$$

We square both nontrivial expressions (preserving the inequality) and take expectations, using the hypothesis that the D_t's are uncorrelated, to obtain

$$\sigma_I^2 + (b + \bar{I})^2 \geq \sigma_I^2 + \sigma_R^2 + \sigma_D^2 + 2\nu\sigma_I\sigma_R + (b + \bar{I} + \bar{R} - \bar{D})^2$$

for some ν, $-1 \leq \nu \leq 1$ (the correlation between I_t and R_{t-c}). This can be restated, using $\bar{R} = \bar{S}$ as

$$\sigma_I \geq -\nu\sigma_I \geq \frac{1}{2} (\sigma_R + \frac{a^2}{\sigma_R}), \quad \text{where} \quad a^2 = \sigma_D^2 - (\bar{D} - \bar{S})(2b + 2\bar{I} - \bar{D} + S)$$

as was to be shown. This completes the proof of the theorem.

The boundary

$$U:\ \sigma_I = \frac{1}{2}\left(\sigma_R + \frac{a^2}{\sigma_R}\right), \qquad \text{where} \qquad a^2 = \sigma_D^2 - (\bar{D} - \bar{S})(2b + 2\bar{I} - \bar{D} + \bar{S})$$

of Theorem 1 involves $\bar{D}$, $\bar{S} = \bar{R}$, $\bar{I}$, σ_I, σ_R, σ_D but has been stated in this particular form because we are primarily interested in the relation between σ_I and σ_R, all other moments being fixed. For example, with $\bar{D}$, σ_D given in advance, $\bar{S}$ and σ_R fixed, Theorem 1 relates σ_I and $\bar{I}$ in the form

$$\sigma_I \geq A - B\bar{I} \qquad\qquad B \geq 0 \quad ;$$

that is, at the boundary U, inventory level can be "traded" for inventory stability. We shall be motivated to a large extent by the one displayed initially; when $a^2 > 0$, the region

$$\sigma_I \geq \frac{1}{2}\left(\sigma_R + \frac{a^2}{\sigma_R}\right)$$

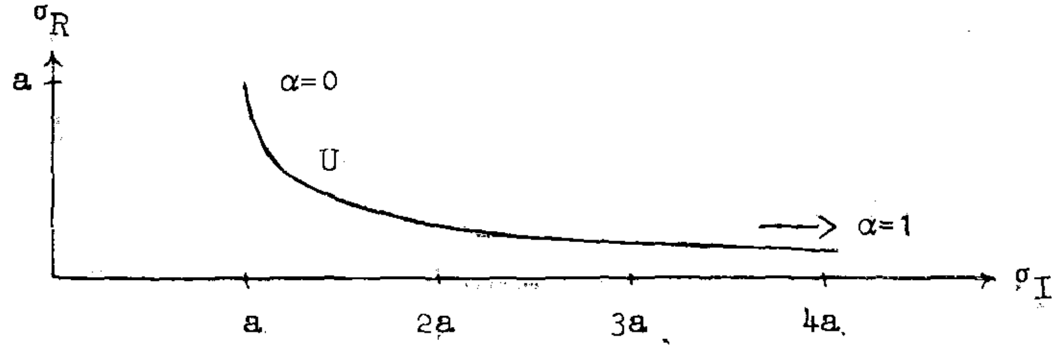
has a boundary U at which inventory stability can be traded for requisition (production) stability (when $a^2 < 0$, no effective boundary exists). This boundary has a convenient parametric form, for stating Theorem 2, using a parameter α, where,

$$\sigma_R = (1 - \alpha)\sigma_I \quad .$$

Then U is given in parametric form by

$$\sigma_I^2 = \frac{1}{1 - \alpha^2}\, a^2, \qquad \sigma_R^2 = \frac{1 - \alpha}{1 + \alpha}\, a^2 \quad .$$

This boundary is indicated in the diagram below, when $a^2 > 0$, $0 \leq \alpha \leq 1$.



Theorem 1 states that any policy P (determining a stationary process) will lead to a point (σ_I, σ_R) on or above the curve. Theorem 2 shows that an "optimal" class of decision policies, in a certain asymptotic sense, sweeps out this curve; given any policy, then, a member of this optimal class can do at least as well in minimizing both σ_I and σ_R.

THEOREM 2. (Optimal Policy Class). Let L be a logistic component with unit requisition cycle (c = 1). Define a decision policy $P(\alpha, B)$ by the relations

$$P(\alpha, B):\ s_t = \min(d_t, i_t + r_t) \qquad t = 1, 2, \ldots$$

$$r_1 = 0$$

$$r_t = \begin{cases} r_{t-1} + (1-\alpha)d_{t-1} & \text{if } i_t \leq B \\ 0 & \text{if } i_t > B \end{cases} \qquad t = 2, 3, \ldots$$

$$B \geq 0, \quad 0 \leq \alpha < 1 \ .$$

Then, if $\sigma_I \leq \bar{I}$ it is necessary that

$$\text{a)}\quad \sigma_R^2 \leq \frac{1-\alpha}{1+\alpha}\sigma_D^2 + (\bar{D} - \bar{S})(\bar{D} + \bar{S})$$

$$\text{b)}\quad \sigma_I^2 \leq \frac{1}{1-\alpha^2}\sigma_D^2 \ .$$

Proof. a) Let T(t) (a random variable) be the least number of periods ago for which either $I_t > B$ or the process began (t = 0). Then, referring to $P(\alpha, B)$, we find

$$R_t = \left(\frac{1-\alpha}{\alpha}\right)\sum_{j=1}^{T(t)} \alpha^j D_{t-j}$$

whence

$$0 \leq R_t \leq \left(\frac{1-\alpha}{\alpha}\right)\sum_{j=1}^{\infty} \alpha^j D_{t-j} \ .$$

where we take D_t, $t \leq 0$ to be replicates of the independent, identical random variable D_t, $t > 0$. We square both nontrivial expressions (preserving the inequality) and take expectations, to obtain

$$\sigma_R^2 + \bar{R}^2 \leq \left(\frac{1 - \alpha}{\alpha}\right)^2 \sum_{j=1}^{\infty} \alpha^{2j} \sigma_D^2 + \left(\frac{1 - \alpha}{\alpha}\right)^2 \left(\sum_{j=1}^{\infty} \alpha^j \bar{D}\right)^2 \quad ;$$

this can be restated in the form (using $\bar{R} = \bar{S}$)

$$\sigma_R^2 \leq \left(\frac{1 - \alpha}{1 + \alpha}\right)\sigma_D^2 + (\bar{D} - \bar{S})(\bar{D} + \bar{S})$$

as was to be shown for case a).

b) Let $S(t)$ (a random variable) be the least number of periods ago for which $D_t \geq I_t + R_t$. Then

$$I_t = \sum_{j=1}^{S(t)} \left(- D_{t-k} + \left(\frac{1 - \alpha}{\alpha}\right) \sum_{j=1}^{T(t-k)} \alpha^j D_{t-k-j}\right) ,$$

and

$$0 \leq I_t \leq \sum_{j=1}^{S(t)} \left(- D_{t-k} + \left(\frac{1 - \alpha}{\alpha}\right) \sum_{j=1}^{\infty} \alpha^j D_{t-k-j}\right) .$$

The right-hand expression can be written, with $S = S(t)$, as

$$0 \leq I_t \leq - \sum_{j=1}^{S-1} \alpha^j D_{t-1-j} + \sum_{j=0}^{\infty} (1 - \alpha^S)\alpha^j D_{t-S-j-1} \quad ;$$

i.e., for each t, this relation holds for some S. But squaring both nontrivial expressions, we find that

$$\begin{aligned} 2\sigma_I^2 &\leq \sigma_I^2 + \bar{I}^2 \\ &\leq \sum_{j=0}^{S-1} \alpha^{2j} \sigma_D^2 + (1 - \alpha^S)^2 \sum_{j=0}^{\infty} \alpha^{2A} \sigma_D^2 \\ &= \left(\frac{1 - \alpha^{2S}}{1 - \alpha^2} + \frac{(1-\alpha^S)^2}{1 - \alpha^2}\right)\sigma_D^2 \\ &= \frac{2(1-\alpha^S)}{1 - \alpha^2}\sigma_D^2 \quad ; \end{aligned}$$

and, hence

$$\sigma_I^2 \leq \frac{1}{1 - \alpha^2}\, \sigma_D^2$$

for all S, as was to be shown for case b). This completes the proof of the theorem.

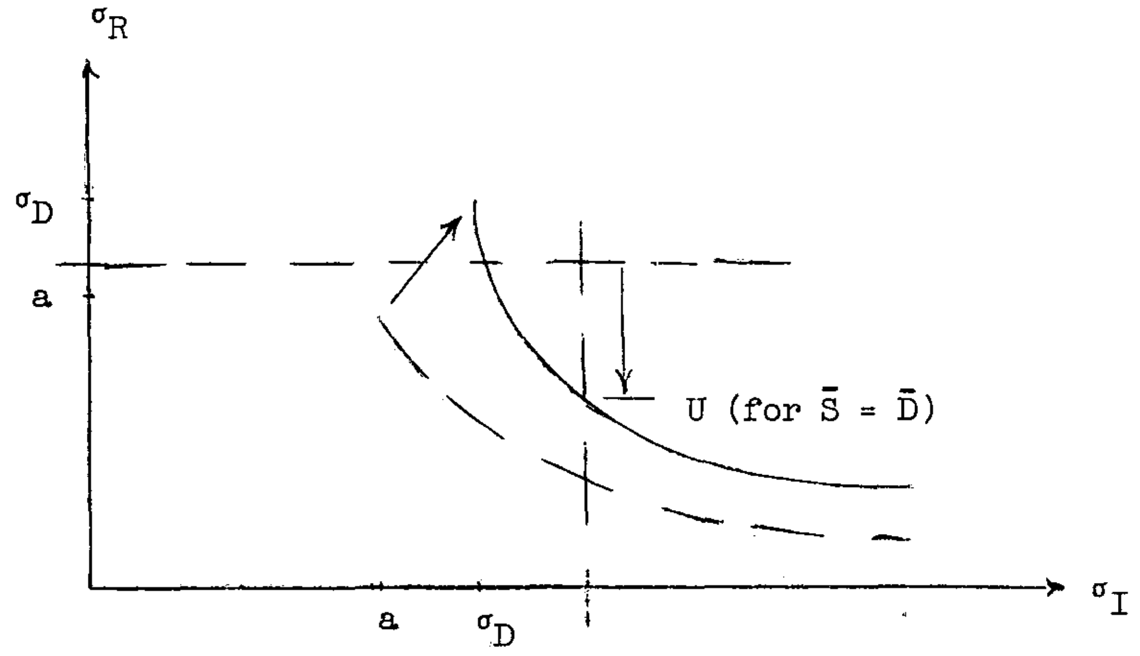
Theorem 1 and Theorem 2 combine to "box in" the point (σ_I, σ_R) induced by a policy $P(\alpha, B)$ to a point on the curve U, as $\bar{S} \longrightarrow \bar{D}$. To see this, notice the three inequalities

$$\sigma_I \geq \frac{1}{2}\left(\sigma_R + \frac{a^2}{\sigma_R}\right), \qquad a^2 = \sigma_D^2 - (\bar{D} - \bar{S})(2b + 2\bar{I} - \bar{D} + \bar{S})$$

$$\sigma_R^2 \leq \frac{1 - \alpha}{1 + \alpha}\, \sigma_D^2 + (\bar{D} - \bar{S})(\bar{D} + \bar{S})$$

$$\sigma_I^2 \leq \frac{1}{1 - \alpha^2}\, \sigma_D^2$$

describe a curvilinear triangle which degenerates to a point on U as $\bar{S} \longrightarrow \bar{D}$, as shown in the diagram.



Whereas we took $c = 1$ (unit requisition cycle) in Theorem 2, we study the very effect of c in Theorem 3. On reflection it is clear that a requisition cycle and an information delay are logically equivalent — a decision maker with a requisition cycle of c and an information delay d (at time t, no data more recent than $t - d$ is known) has the same problem as one with a requisition cycle of c' and an information delay of d' if $c + d = c' + d'$ (they make their decisions at different points in time, but each has the same effective information and prospects). For this reason, we can convert requistion cycles into information delays for convenience.

THEOREM 3. (Information Delay). Let L be a logistic component with unit requisition cycle $(c = 1)$. Suppose P is a decision policy independent of i_t (inventory levels) and determines a stable stochastic process

$$\{(D_t, I_t, R_t, S_t)\}$$

with moments denoted $\bar{D}$, σ_I, etc. Let $L(c)$ be the logistic component with the same demand as L and requisition cycle $c \geq 2$. Then, if

$$I_{t-c+1} - \sum_{k=1}^{c-1} S_{t-k} \geq 0 \quad ,$$

the decision policy P will determine the process

$$\{(D_t, I_{t-c+1} - \sum_{k=1}^{c-1} S_{t-k}, R_t, S_t)\}$$

in $L(c)$. Furthermore, if $S_t = D_t$, and $\sigma_I(c)$, etc. refer to $L(c)$, then,

$$\sigma_I^2(c) = \sigma_I^2 + (c - 1)\sigma_D^2, \quad \sigma_R(c) = \sigma_R \quad .$$

<u>Proof</u>. If P determines $\{(D_t, I_t, R_t, S_t)\}$ with unit requisition

$$I_{t+1} = I_t + R_{t-1} - S_t$$

and this can be rewritten as

$$(I_{t+1} - S_{t+1}) = (I_t - S_t) + R_{t-1} - S_{t+1} \quad ;$$

using the transformation $I_t^2 = I_{t-1} - S_{t-1}$, we write

$$I_{t+1}^2 = I_t^2 + R_{t-1} - S_t \quad .$$

If $I_t^2 \geq 0$, the process

$$\{(D_t,\ I_t^2,\ R_t,\ S_t)\}$$

will satisfy all conditions of P, and hence will be determined by P in $L(2)$. Continuing, we obtain, in $c - 1$ steps, a sequence of transformations,

$$I_t,\ I_{t-1} - S_{t-1},\ \ldots,\ I_{t-c+1} - \sum_{k=1}^{c-1} S_{t-k} \quad .$$

Thus, if

$$I_{t-c+1} - \sum_{k=1}^{c-1} S_{t-k} \geq 0 \quad ;$$

then P will, in fact, determine the process

$$\{(D_t,\ I_{t-c+1} - \sum_{k=1}^{c-1} S_{t-k},\ R_t,\ S_t)\}$$

in $L(c)$ as was to be shown.

If $S_t = D_t$, then $\sigma_I^2(c)$ is the variance of

$$I_{t-c+1} - \sum_{k=1}^{c-1} D_{t-k}$$

(where the D's are independent of I_{t-c}). Then, we have, simply,

$$\sigma_I^2(c) = \sigma_I^2 + (c - 1)\sigma_D^2$$

as was to be shown. This completes the proof of the theorem.

Theorem 3, again, gives asymptotic results — most policies will depend in some way on inventory levels — $P(\alpha, B)$ depends on them, though, it would seem, relatively innocuously. In practical problems of logistic system design the results of all three theorems may be more effectively employed as "rules of thumb" than as exact relationships. Perhaps the most important information contained in them is the general fact that classes of relatively simple "almost linear" policies of the type $P(\alpha, B)$ perform "very well" according to criteria such as σ_I and σ_R, and at any balance between them desired. While the hypotheses of Theorem 3 are restrictive, it would seem that the general relationship between information delay and inventory variance is near what is described in the Theorem. These last remarks are predicted on the fact that in most logistic problems, $\bar{S}$ is to be 90%, 95%, or even 99% of $\bar{D}$.

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