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Smoothing in Inventory Processes - Notes

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Notes

SMOOTHING IN INVENTORY PROCESSES

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Discrete Servo-Stochastic Inventory Processes

We define a discrete servo-stochastic inventory process to be a sequence of random variables,

$$P = (i_1, r_1, s_1, i_2, r_2, s_2, \ldots,) \quad ,$$

where we interpret

i_t as <u>inventory</u> in an operation at the beginning of a time period t,

r_t as inventory added to the operation, or <u>requisitions</u>, during period t,

s_t as inventory deleted from the operation, or <u>sales</u>, during period t,

such that P, in conjunction with a history of numbers,

$$I = (\ldots, i_{-1}, r_{-1}, s_{-1}, i_0, r_0, s_0)$$

is characterized by the following properties, for $t = 0, 1, 2, \ldots$,

(1) $$i_{t+1} = i_t + r_t - s_t$$

(2) $$r_{t+1} = R(\ldots, s_{t-1}, i_t, r_t, s_t, i_{t+1})$$

(3) $$\text{Prob } \{ s_{t+1} \leq x \} = S(x)$$

where R is an arbitrary function, and S is an arbitrary distribution.

We also use the following notation,

$$\bar{i}_t = E(i_t), \quad \bar{r}_t = E(r_t), \quad \bar{s}_t = E(s_t) \;,$$

$$\sigma_i^2(t) = E[(i_t - \bar{i}_t)^2], \quad \text{etc.} \;,$$

$$\bar{i} = \lim_{t \to \infty} \bar{i}_t \;, \quad \sigma_i^2 = \lim_{t \to \infty} \sigma_i^2(t), \text{ etc.} \;,$$

(4) $$k_i = \sigma_i / \sigma_s \;, \quad k_r = \sigma_r / \sigma_s \;.$$

Illustrations

"k Month's of Supply" Requisition Policy

Consider

(5) $$r_t = k(\frac{1}{n}\sum_{j=1}^{n} s_{t-j}) - i_t, \quad k, n \text{ positive integers;}$$

(1) and (5) have the solution

$$i_t = -s_{t-1} + \frac{k}{n}\sum_{j=1}^{n} s_{t-1-j} ,$$

$$r_t = (1 + \frac{k}{n})\, s_{t-1} - \frac{k}{n}\, s_{t-n-1} ,$$

with the resulting statistical characterization

$$k_i^2 = 1 + \frac{k^2}{n} , \quad k_r^2 = (1 + \frac{k}{n})^2 + (\frac{k}{n})^2 .$$

A "Linear-Servo" Requisition Policy

Consider

(6) $$r_t - r^* = -\alpha(i_t - i^*) , \quad 0 < \alpha \leq 1, \quad r^*, i^* \text{ arbitrary;}$$

(1) and (6) have the solution (for large t)

$$i_t = i^* + \frac{r^*}{\alpha} - \sum_{j=0}^{\infty} (1-\alpha)^j s_{t-1-j} ,$$

$$r_t = \alpha \sum_{j=0}^{\infty} (1-\alpha)^j s_{t-1-j} ,$$

with the resulting statistical characterization

$$k_i^2 = \frac{1}{1-(1-\alpha)^2} , \quad k_r^2 = \frac{\alpha^2}{1-(1-\alpha)^2} .$$

Numerical Comparisons

	k_i	k_r
k month's of supply:		
a. $k = 5, \; n = 10$	1.8	1.6
b. $k = 10, \; n = 5$	4.6	3.6
linear-servo:		
c. $\alpha = .2$	1.7	0.3
d. $\alpha = .5$	1.2	0.6
e. $\alpha = .8$	1.02	0.8

Basic Theorems

Theorem 1. (Smoothing Capacity) In a discrete servo-stochastic inventory process for any requisition policy R, with bounded inventory deviation,

$$k_i \geq \frac{1}{2}(k_r + \frac{1}{k_r}) \quad .$$

Theorem 2. (Optimal Policy Class) Given α such that $0 < \alpha \leq 1$, the requisition policy

$$R(\alpha): \quad r_t - r^* = -\alpha(i_t - i^*)$$

r^*, i^* being arbitrary constants, determines a discrete servo-stochastic inventory process such that

$$k_i = \frac{1}{2}(k_r + \frac{1}{k_r}) \text{ (and } k_r = \alpha k_i) \text{ .}$$

Theorem 3. (Information Delay) Let R^T be identical with policy R except that information is delayed T periods, i.e.,

$$R^T(\ldots, r_t, s_t, i_{t+1}) = R(\ldots, r_{t-T}, s_{t-T}, i_{t-T+1}) ,$$

then if R determines a discrete servo-stochastic inventory process, R^T also determines a discrete servo-stochastic inventory process with

$$k_i(R^T) = \sqrt{k_i^2(R) + T} \quad , \quad k_r(R^T) = k_r(R) \quad .$$

NOTES

SMOOTHING IN INVENTORY PROCESSES*

Summary

At a typical decentralized inventory point in a multi-echelon supply system, a requisition policy, viewed as a servo, or statistical decision function, transforms statistical time series of sales into derived statistical time series of inventory levels and requisitions on the next echelon. It is often desired to formulate requisition policies which maintain high degrees of stability in, or smooth, these derived time series.

An "uncertainty principle" is deduced in discrete servo-stochastic inventory processes satisfying a material balance relation

$$i_{t+1} = i_t + r_{t-T} - s_t$$

for inventory levels i_t with independent identically distributed sales s_t and requisitions r_{t-T} lagged T periods. Ratios of "inventory deviation to sales deviation," $k_i = \sigma_i/\sigma_s$, and "requisition deviation to sales deviation", $k_r = \sigma_r/\sigma_s$, are shown to satisfy a smoothing capacity relation,

$$(k_i)^2 \geq \frac{1}{4}(k_r + \frac{1}{k_r})^2 + T ,$$

for any requisition policy with bounded inventory deviations (in particular $k_i k_r \geq \frac{1}{2}$). Thus, not both k_i and k_r can be decreased indefinitely by improving requisition policy design.

A family of requisition policies

$$r_t = - \alpha i_t + \beta \qquad 0 < \alpha \leq 1, \quad \beta \text{ arbitrary}$$

is shown to be a complete optimal class, providing maximal smoothing (achieving the equality above).

That so simple a class of requisition policies is optimal seems rather fortuitous in view of its competition with all possible requisition policies with bounded inventory deviations.

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