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Implicit Mathematical Decision Criteria

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REPORT OF THE ANN ARBOR MEETING
AUGUST 29–SEPTEMBER 1, 1955

THE AMERICAN summer meeting of the Econometric Society was held in Ann Arbor, Michigan, August 29–September 1, 1955 in conjunction with related social science organizations. The program committee consisted of Albert W. Tucker, Princeton University (chairman); Merrill M. Flood, Columbia University; Tjalling C. Koopmans, Yale University; John Meyer, Harvard University; Robert M. Thrall, University of Michigan; and Charles C. Holt, Carnegie Institute of Technology. Richard Ruggles, Secretary, the Econometric Society, was an ex-officio member. Robert M. Thrall, University of Michigan, was in charge of the local arrangements.

An index of participants who presented papers precedes the report of the meeting given below. An asterisk denotes papers reported here only by title. Abstracts of the discussions of papers are not published here.

[1] Published as an article in the January, 1956 issue under the title "Dynamic Programming under Uncertainty with a Quadratic Criterion Function."

ECONOMETRIC STUDIES

Monday afternoon, August 29. Richard Ruggles, Yale University, chairman. Papers were discussed by Harold Guthrie, Yale University, and Andrew T. Court, General Motors Corporation.

Determinants of Consumer Demand for Household Furnishings and Equipment, VERNON LIPPITT, Harvard University.

THE PURPOSE OF this investigation was to develop a method of forecasting consumer purchases of house furnishings and equipment. The consumer unit chosen was the family.

Cross-section (budget study) data from the 1935–36 Consumer Purchases Study were analyzed to bring out the influence on percentage of family income spent for furnishings and equipment of: income, family size and composition, occupation of the head, place of residence (city size or region), and home ownership. Multivariate analysis of variance techniques were used, with allowance for unequal numbers in cells. The effects of the variables were assumed independent (or additive).

Next aggregation was carried out to determine the influence on total national expenditures of the distribution of income among families classified by income, family type, occupation, and place of residence. It was assumed that the effect of each family characteristic on a family's expenditure percentage remains constant over time, if income and expenditures are expressed at constant prices. The aggregation was performed for six benchmark years, and a time series was constructed by interpolation.

Finally, the above time series was used along with series for national disposable income, relative price of furnishings and equipment, income of recently married couples, and change in per capita income from previous year to derive an explanatory equation for aggregate expenditures for furnishings and equipment. Residuals were given a reasonable interpretation in terms of influences not included in the explanatory equation.

This method of combining cross-section and time series data in analyzing consumer behavior seems a powerful one and capable of extension to other categories of consumer spending, to saving, and to specific items of equipment.

Automobile Consumption, ROBERT A. BANDEEN, Duke University.

THIS PAPER described the research results of part of the Study of Differences in State Per Capita Incomes, jointly financed by Duke University and the Rockefeller Foundation. The consumption of automobiles is defined as the depreciation of the stock of automobiles for a year. State observations for the two years 1940 and 1950 of the depreciation of both new and used cars, income, and population density were used in a form of cross-sectional analysis. To allow for state differences, relative changes in depreciation, income, and population density between the two years, rather than the data for one year, were related by means of linear regression to arrive at estimates of the parameters. The results indicated that with a 10 per cent increase in the income of a state there would be associated a 9 per cent increase in automobile consumption, while a 10 per cent increase in population density would be associated with a 3 per cent decline in automobile consumption.

The model hypothesized a fixed relation between automobile consumption and income with the amount of automobile consumption arising from new car ownership being the residual between total consumption as determined by the income level and the depreciation of the stock of used cars. The projected 1950 data gave estimates of new car purchases in 1953 within 1 per cent of the actual number purchased.

Firm and Industry Progress Functions, WERNER Z. HIRSCH, Washington University.

THE (UNIT) PROGRESS function of a firm or industry is specified as $U = aX^b$ (1.0 + b), where U is the amount of direct labor input congealed in a specific product unit, X cumulative output, a the intercept of the progress function on the U-axis, and b the slope of the progress function (between 0.0 and -1.0). With the help of this function, estimates of the labor requirement of a given product unit can be made.

Empirical work carried out so far indicates that the progress functions of the textile machine, machine tool, and multiple construction machine industries have a mean slope of about -0.32 which corresponds to a progress ratio of about -20. While it may be useful to know that doubling cumulative output in these industries is on the average associated with about a 20 per cent decline in direct labor requirement, there still remains the question of what is the progress function of individual firms and the products they produce.

Of the firm progress functions estimated by us so far the progress ratios varied from 16.5 to 24.8 per cent. In order to learn whether the sources of differences in the characteristics of progress functions can be identified and measured, a number of hypotheses were established and tested. It was found that progress functions differ depending on (1) whether production is on a piece, lot, or assembly line basis; (2) differential amounts of compounded experience congealed in the model; and (3) the machining-assembling ratio of the model.

The theory underlying time and motion studies, particularly in industries producing in lots of long production periods, must be reconsidered in the light of the learning process. In companies with relatively steep progress functions there is no unequivocal standard of performance which could constitute a norm, particularly not in the first 20–30 lots. Furthermore, it is questionable whether workers' remuneration should be tied to such a norm as long as management's decision as to whether output increases result from larger or more frequent lots appears to affect the figures on workers' productivity.

INPUT-OUTPUT MODELS

Monday evening, August 29. Cuthbert C. Hurd, International Business Machines Corporation, chairman. Papers were discussed by T. M. Whitin, Massachusetts Institute of Technology, and John Fei, Harvard Economic Research Project and Antioch College.

On the Equilibrium of a Linear Economic System with Non-Dominant Outputs,[2] OSKAR MORGENSTERN AND Y. K. WONG, Princeton University.

WE PRESENT some new results for a linear economic system together with their economic meaning. The operations are rational. A linear economic system is said to be in equilibrium when it has a nonnegative solution. The coefficients a_{ij} $(i, j = 1, 2, \cdots, n)$ are nonnegative. *First*, we consider the system of equations (1) $x_i - \sum_{j=1}^{n} a_{ij}x_j = h_i$, where (2) $\sum_{i=1}^{n} a_{ij} \leqslant 1$, (3) h_i is nonnegative. Then (1) has a unique nonnegative solution if and only if (4) $\sum_{i=1}^{j} a_{ij} < 1$ for $j = 1, 2, \cdots, n$, subject to the permutations of rows and the same columns. Instead of summing the elements on and above the diagonal (as in (4)), we may sum the elements on and below the diagonal. (4) yields many interesting properties. A classical result states that the system of equations (1) with properties (2) and (3) has a unique nonnegative solution if the matrix (a_{ij}) is indecomposable and one column sum is less than 1, but not conversely. Condition (4) is stronger than this

[2] Prepared at Princeton University under Office of Naval Research Contract.

classical result. *Second,* we consider the system of equations (1) with condition (3), omitting, however, condition (2), thereby making it a system with non-dominant outputs. Without assuming (2), the system of linear equations (1) has a unique nonnegative solution *only if* (4) holds. The system of equations (1) with condition (3) has a unique nonnegative solution if and only if for $k = 1, 2, \cdots, n$, (5) $a_{kk} + c_k < 1$, where (6) $c_1 = 0$, $c_k = \sum_{i,j=1}^{k-1} a_{ki}(I - A_{k-1})^{-1}(i, j)a_{jk}$, $(k = 2, \cdots, n)$, and A_k is the principal submatrix consisting of the elements in the first k rows and columns of A. (In place of the sequence $\{A_k\}$, the statement remains valid if there exists a sequence of principal submatrices $\{A_{(k)}\}$ of order k such that $A_{(k+1)}$ contains $A_{(k)}$ as its principal submatrix with properties analogous to (5) and (6) above.) In order to make (6) meaningful, it is essential to know the existence of the inverses of $I - A_k$. Property (5) actually gives much more. By induction from (5), we can prove not only the regularity of the submatrices $I - A_k$ with k ranging from 2 to n, but also the nonnegativeness of their inverses. Thus, $I - A_1$ has a nonnegative (actually, positive) inverse if $a_{11} < 1$; if $I - A_{k-1}$ has a nonnegative inverse and if (5) holds, then $I - A_k$ has a nonnegative inverse. In our system, *where* (2) *is not assumed,* the total amount that "sector j" purchases from various sectors may be greater than 1. The quantity a_{kk} is the direct intra-sector transaction whereas c_k represents the indirect intra-sector transaction relative to the other $k - 1$ sectors within some subsystem of k sectors. A few interesting equivalent conditions are stated in the following: (7) there exists a positive vector u such that $u'A < u'$ where u' is the row vector of u. (8) There is a principal submatrix $A_{(n-1)}$ and a positive vector w both of order $n - 1$ such that $w'A_{(n-1)} < w'$ and $\det(I - A) > 0$. The following sufficient condition is very practical in concrete examples: (9) There is a principal submatrix $A_{(n-1)}$ whose column sums are individually at most 1 while the corresponding inequalities (4) hold, and $\det(I - A) > 0$.

Experimental Measurement of Utility by Use of a Linear Programming Model, Donald Davidson and Patrick Suppes, Stanford University.

The purpose of this paper is to report an experiment which was designed to measure the cardinal utility of non-monetary outcomes and to use the computed utilities to predict further choices. The basic experimental design followed that of previous experimental work by Davidson, Siegel, and Suppes on the measurement of the utility of money. A simple one-person 2×2 game is used. In this game subjects are asked to choose between two options, each of which is a two-element probability combination; a chance event with subjective probability one-half is used throughout the game. Let (x, y) be Option 1 and (u, v) be Option 2. Then if Φ is a utility function, we may represent the choice of Option 1 by the inequality:

$$\Phi(x) + \Phi(y) \geqslant \Phi(u) + \Phi(v). \tag{1}$$

(This inequality is obtained from the obvious expected value formulation by multiplying both sides by 2 to eliminate the constant probability of $\frac{1}{2}$.) Once six outcomes are fixed on, subjects are presented with a sequence of games having these outcomes in various combinations. A choice for each game is made, which may be represented by an inequality of the form of (1). In general the set of inequalities in six variables thus obtained will not have a solution; in other words, people are not that consistent in their choices. Therefore, each inequality of the form of (1) is replaced by:

$$\Phi(x) - \Phi(v) + \Theta \geqslant \Phi(u) - \Phi(y). \tag{2}$$

Linear programming methods are now applied to obtain a solution of the set of inequalities of the form of (2) for each subject: the *minimum* Θ is found for which the set of inequalities has a solution, with the normalizing restriction that the smallest interval be of length one. (For computational convenience the dual problem is solved.) Intuitively Θ may be thought of as the *threshold* of preference. If two intervals differ by more than Θ the larger must be chosen. If the difference is within Θ the choice is stochastic.

Seven music students were used as subjects, with long-playing records as the outcomes. Each subject came to three sessions; all testing was done individually. In the first session a utility curve for six records was determined by the method described. In the second session a utility curve was found for another set of six records, two of which were drawn from the set used in the first session (to permit the construction of a joint curve). The joint curve was used to predict choices between new combinations, and these predictions were tested in a third session using three records from each of the previous sessions.

It was found that the predictions made by the linear programming model were somewhat more numerous and accurate than the predictions made by the obvious ordinal model.

PROGRAMMING PROBLEMS

Tuesday morning, August 30. Harold W. Kuhn, Bryn Mawr College, chairman. Papers were discussed by David Gale, Brown University, and A. J. Hoffman, National Bureau of Standards and Office of Naval Research.

Generalizations of the Warehousing Problem, A. Charnes, Purdue University, and W. W. Cooper, Carnegie Institute of Technology.

"The Warehouse Problem" first stated by A. S. Cahn (*Bulletin*, American Mathematical Society, Vol. 54, Oct., 1948) may be summarized as follows:

> "Given a warehouse with fixed capacity and an initial stock of a certain product, which is subject to known seasonal price and cost variations, what is the optimal pattern of purchasing (or production), storage and sales."

A general model is constructed in the present paper and a convenient method of solution devised for handling multiple products and warehouses and (discontinuous) variations in prices. The method involves use of the "dual theorem" of Gale, Kuhn, and Tucker ("Linear Programming and the Theory of Games," *Activity Analysis of Production and Allocation*) and the "regrouping principle" developed by the present authors in collaboration with B. Mellon ("A Model for Programming and Sensitivity Analysis in an Integrated Oil Company," *Econometrica*, 22, 2). A simple and convenient algorithm is obtained which is far more efficient to apply than the simplex method for such problems.

Remarks on extensions of these methods to inclusion of financial constraints are ventured along with observations on the model as a general type of transportation model and its relation to certain dynamic scheduling problems.

Quadratic Programming, Marguerite Frank and Philip Wolfe, Princeton University.

No abstract available.

INVESTMENT STUDIES

Tuesday afternoon, August 30. Tjalling C. Koopmans, Yale University, chairman. Papers were discussed by Michel Verhulst, Princeton University.

Use of Surrogates in Measuring Expectational Variables: An Example of the Prediction of Investment, Paul G. Darling, Carnegie Institute of Technology.

In econometric work based on time series data, expectations held by firms and households have commonly been measured from the "causal" side, i.e., the factors hypothesized to "trigger" the expectations have been included as independent variables in behavioral equations. Klein, for example, has used current and lagged values to measure an anticipated future value of certain variables; Tinbergen has employed rate-of-change of recent price to measure speculative anticipations in the stock market. An alternative approach to the measurement of expectations is, however, possible in many cases. This method makes use of the "tracks" left by such expectations in the sands of statistical data, with these "effects" used as surrogates of the anticipations. The balance of the paper deals with an example of this methodology. It is first demonstrated that business liquidity is largely an expectational variable, i.e., that changes in liquidity take the principal form of changes in a set of expectations concerning future cash *flows* (e.g., future inflows from operations), rather than changes in the *stock* of currently held assets. The hypothesis is then presented that changes in business liquidity will be associated with changes in a key flow-variable, corporate dividend disbursements, after abstracting from other, non-liquidity, determinants of dividends.

An empirical test of this hypothesis is offered. The investment function used is similar to Klein's (*Economic Fluctuations in the United States*, pp. 115–117), incorporating a liquidity variable, L. To provide a surrogative measurement of L, a time series is derived of the differences between actual dividends and those computed as "normal" from a linear function which incorporates the non-liquidity determinants of dividends (mainly, the level of profits). Using time series data for three separate samples of manufacturing corporations, support for the surrogative methodology is secured from the finding that the inclusion of L in the investment function significantly improves estimates of plant and equipment expenditures.

A Capacity Concept and Investment Models,[3] Edward J. Zabel, Princeton University.

The problem of investment in durable production equipment for a profit maximizing firm is explored, and a simple model is presented in which the firm with a given output projection for an indefinite planning horizon has to determine an investment policy for the present period and for each subsequent period. Restrictions on the nature of the production function permit the problem to be transformed into one in which the firm wishes to minimize the total costs of obtaining required services from the durable goods over the planning horizon.

The following parameters are given for the firm in each period: the output projection, the price of equipment, the scrap value of equipment, a production function, an output standard for machinery which permits comparisons to be made as to the efficiency of different machines, maintenance costs for machinery which are the sum of a fixed cost dependent on the period in which durable equipment is installed and a variable cost dependent on the age and the rate of use of machinery, and a parameter denoting the effect of the rate of use of machinery on the cost of complementary inputs. The following variables are controlled by the firm in each period: the rate of use and the replacement age of machinery and the rate at which equipment is added to the firm's stock of durable production goods.

The procedure used to solve the problem is to set out the present value of the total

[3] Prepared at Princton University under Office of Naval Research Contract.

costs of services in terms of the parameters and the variables and to minimize the total costs with respect to the replacement age and the rate of use of machinery while subject to the restriction that sufficient output must be produced to satisfy the output projection.

The solution gives the number of machines the firm should purchase in the present period and a plan for the future purchase of machines. The solution only holds for the present period, however, since over time the parameters may change. Thus, at the end of each period a new solution is obtained which again gives the number of machines to be purchased for the coming period and a plan for future periods.

OPERATIONS RESEARCH FOR BUSINESS AND MANAGEMENT

Tuesday afternoon, August 30. Merrill M. Flood, Columbia University, chairman. Papers were discussed by John M. Danskin, Massachusetts Institute of Technology, and Robert V. Hidgon, Haller, Raymond, and Brown, Inc.

Implicit Mathematical Decision Criteria, Harlan D. Mills, General Electric Company.

Problems in physical sciences and social sciences may differ significantly in aims and intuitive background. Physical science problems are generally existential in character with an intuitive background which can be highly formalized. Social science problems are not usually existential, and the intuitive background is often extremely hard to formalize.

This raises the question, in social sciences, whether mathematics can be used to obtain implicit criteria for solving problems on an intuitional basis, rather than requiring explicit mathematical solutions. In short, can mathematics be brought to the problems rather than vice versa?

Mathematical programs which are presentations of decision situations furnish several examples of such possibilities. Without finding a solution to a program, one may be able to use a mathematical analysis to: (1) estimate how a feasible solution should be altered; (2) estimate the effects of boundary conditions on the optimal value of the program which may, themselves, be decision variables in more strategic problems; and (3) find necessary conditions for an optimal solution.

For example, consider the program:

$$\max f_0(x) = \max f_0(x_1, x_2, \cdots, x_n) = F(r_1, \cdots, r_m),$$

when

$$f_i(x) = f_i(x_1, x_2, \cdots, x_n) = r_i, \qquad i = 1, 2, \cdots, m,$$

where $f_0(x), f_1(x), \cdots, f_m(x)$ are twice differentiable, and $m < n$. If Lagrange Multipliers, $(y_1, y_2, \cdots, y_m)$, are introduced by the relations

$$\sum_{i=1}^{m} y_i \frac{\partial f_i(x)}{\partial x_j} = \frac{\partial f_0(x)}{\partial x_j}, \qquad j = 1, 2, \cdots, n$$

they are over-determined. If S is the set of maximal square subsystems of (1), denote solutions to them by

$$y^s(x) = (y_1^s, y_2^s, \cdots, y_m^s), \qquad s \in S.$$

A necessary condition that x^0 be optimal in the program is that $y^s(x^0)$ is invariant over S. Further, it is easy to verify, under certain conditions, that

$$\frac{\partial F(r)}{\partial r_i} = y_i \quad \text{where} \quad y_i = y_i^s(x^0).$$

Hence the functions $y^s(x)$ can be used for all three examples, by (1′) estimating how to alter a feasible solution, x, to reduce variance in the set $\{y^s(x)\}$; (2′) estimating $y(x^0)$ from the functions $y^s(x)$; and (3′) using the necessary condition of invariance of $\{y^s(x)\}$ as a criterion for optimality.

The Application of Linear Programming to Valuation Problems, J. O. HARRISON, JR., Johns Hopkins University.

THE LINEAR PROGRAMMING problem of combining limited stocks of input resources so as to produce an optimum mix of output products has as its dual, the problem of valuing each input resource. The implications of the valuation process are obtained by interpreting the principal mathematical properties connecting a pair of dual linear programming problems. These implications are:

1. An optimum valuation of input resources exists if and only if production is feasible and the production process is limited to a finite total return.
2. If valuation is accomplished by any feasible solution of the valuation problem, and if production is accomplished by any feasible solution of the production problem, then the value of the total return cannot exceed the total value of the input resources.
3. If production is scheduled in the optimum manner and if resources are valued in the optimum manner, then the total value of the input resources is conserved.
4. Any resource which is not completely used is assigned zero unit value, and any resource which is assigned positive value is completely used.
5. Any product which requires a mix of resources whose unit value is in excess of its unit return will not be produced, and any product that is produced has values assigned to its inputs in such a way that the unit value of its input mix is equal to its unit return.
6. Subject to certain conditions of existence and continuity, the value of any resource represents the change in the maximum value of the total return per unit change in the stock of that resource.

DECISIONS UNDER UNCERTAINTY

Wednesday morning, August 31. Jacob Wolfowitz, Cornell University, chairman. Papers were discussed by M. A. Woodbury, Logistics Research Project, George Washington University; and Stanley Reiter, Purdue University.

Foundations of a Stochastic Theory of Organizations, DAVID ROSENBLATT, The American University.

THE STUDY OF organizations focuses in general on the investigation of hierarchies of relationships between parts and wholes. In organization theory, one is led quite naturally to seek formalisms which may be expressed in experimental language (in terms of observables) and which provide some reasonable representation of the concepts: environment, process, responsiveness, history, behavior (activities, actions, decisions), and uncertainty.

This paper presents a formal postulational system OFK, Organization of the First Kind, which provides a calculus in which to formulate models of behavioral interaction of "entities" in organizations. The system OFK is phrased in the language of operator theory and of stochastic processes. The approach adopted leads to the apparently novel representation of an organization as a complex whose structure and behavior are governed by interdependent discrete time parameter stochastic processes; the component processes are in general non-stationary and non-Markovian. The stochastic processes are postulated to proceed in accord with designated rules which are permitted to vary in time, thus leading to the treatment of multiple (relativized) time series of sequentially related activities or decisions of "entities." Results on quantized models stemming from OFK have been reported earlier in *Annals of Mathematical Statistics*, September, 1955. Theorems of the following character are now reported for systems qualifying as OFK: (1) conditions for treating an entire system as a composite "entity" in the sense of OFK; (2) conditions permitting a complete system with "recursive interaction structure" to be viewed as a single "conjoint" discrete time parameter stochastic process defined over a continuous state space; (3) conditions for constructively characterizing processes of the latter type by stationary Markov transition functions. These results have been applied in models in the following areas: administrative structure and behavior, group instruction-training, accounting control and system design schemata, and system stability *vs.* flexibility. In general, the system OFK leads to the study of controlled interdependent stochastic processes from a normative standpoint. Although large OFK models have already been studied, it appears fruitful to enlist the aid of modern large-scale computers for more general OFK realizations.

The present approach may be related to the conception of treating a complex of interdependent probabilistic "Turing machines," wherein interest centers on the behavioral evolution of the complex in time and on the detailed properties of cross-section and time averages. This view has been elaborated in the light of the work of Pitts-McCulloch on neural networks, the logical theory of automata of von Neumann, and the experimental studies of W. R. Ashby and D. M. MacKay.

The Use of Probability Forecasts in Making a Sequence of Decisions under a Quadratic Criterion, Herbert A. Simon, Carnegie Institute of Technology.

Published as an article in the January, 1956 issue under the title "Dynamic Programming under Uncertainty with a Quadratic Criterion Function."

CAPITAL RE-EQUIPMENT

Wednesday afternoon, August 31. John Meyer, Harvard University, chairman. Papers were discussed by George Terborgh, Machinery and Allied Products Institute.

Economic Equipment Policies: An Evaluation, Vernon L. Smith, Purdue University and The Harvard Economic Research Project.

The classical Preinreich–Lutz–Alchian theory of economic replacement, modified in accordance with Terborgh's concept of linear obsolescence, is applied to the replacement of line-haul truck-tractor equipment and compared with the firm's actual replacement behavior. The theory involves the determination of L such that the following fundamental expression is maximized.

$$v = \sum_{k=0}^{\infty} e^{-\rho kL} \left\{ \int_0^L Q(kL, t)e^{-\rho t}\, dt - p + S(L)e^{-\rho L} \right\},$$

where v is the prevent value of the net earnings profile per unit of equipment, ρ is the continuous rate of discount, L is the period of equipment service, p is the initial cost per unit of the equipment, and $S(L)$ is its market resale value after L years of service. $Q(kL, t)$ is the earnings of a unit of equipment net only of operating costs (and not capital cost), as a function of age t and the point in calendar time, kL, when the asset was purchased new. It is technological change, affecting the performance of successive new models, that causes earnings to depend upon kL.

Application has been made to four replacement situations arising in two trucking firms. In each case the replacement problem involved a large number of equipment units (from 20 to 250), on which maintenance and operating cost data were available over the life of each equipment unit. Only the results of one application were discussed, but the results were similar in all four of the replacement problems studied.

Average repair cost per mile, r, is found to rise at a decreasing rate with cumulative mileage, M, as follows:

$$r = A_r(1 - e^{-b_r M}), \qquad (A_r, b_r > 0).$$

Fuel consumption per mile, f, rises at a linear rate with cumulative mileage as follows:

$$f = A_f + B_f M, \qquad (A_f, B_f > 0).$$

The average rate of equipment utilization is approximately constant in line-haul service over the life of the equipment, i.e., cumulative mileage is simply proportional to length of service.

Truck-tractor resale value is found to fall rather abruptly after initial purchase, and then to decline at a constant linear rate, at least for the first 5 to 6 years, i.e.,

$$S(L) = B - \gamma L, \qquad (B, \gamma > 0).$$

The major component of technological change is increases in equipment load capacity, within the legal restrictions on overall dimensions and maximum axle loading. The obsolescence of a unit of equipment is estimated to develop at a rate between 100 and 200 dollars/year.

The empirical results indicate that the firms in question followed replacement policies which did not vary greatly from the optimal policies predicted from the above model. At the same time it was found that the age variable costs which bear upon the replacement decision are such a small component of the total equipment operating costs that delays of a year or two beyond the optimal replacement point do not greatly reduce earnings. A delay of 1.2 years lowers total earnings by less than 2 per cent.

The theory omits important aspects of the replacement problem, e.g., limitations on the capital budget and the interdependence of replacement and expansion decisions. Further research is being carried out in this area with the intention of modifying and extending the theory of investment of the firm, and applying it to actual investment problems in specific firms or industries.

The Derivation and Solution of Replacement Decision Equations, MARTIN H. GREENBERGER, Harvard Computation Laboratory.

WORKING WITHIN the framework of Terborgh's capital equipment model, numerical techniques are developed for the determination of a machine's optimum service life and its corresponding annual cost. This determination is the crucial prerequisite to making the replacement decision formulated in the MAPI study *Dynamic Equipment Policy*.[4]

[4] *Dynamic Equipment Policy*, George Terborgh, McGraw-Hill (1949). Machinery and Allied Products Institute (MAPI).

Assuming a zero terminal salvage value, a series of equations are first derived for the equipment's annual time-adjusted capital cost, operating inferiority (deterioration plus obsolescence), and adverse average (of capital cost and operating inferiority). These are all functions of constant problem parameters and a variable length of equipment service. A smallest adverse average, called the adverse minimum, is obtainable when the equipment is held for the optimum number of years, its service life. An implicit equation for the mathematical service life is derived and an approximation method described for solving it. An equation giving the adverse minimum in terms of this service life completes one approach to the determination of the adverse minimum. For large volume work this approach is particularly suited to adaptation on an automatic computer.

A second approach, more convenient perhaps for small volume hand calculation, is also described. The two approaches are compared on a sample problem. The method of successive approximations used by Approach I is illustrated with a graph, and the table used by Approach II is also provided.

In the remaining portion of the paper the assumption of no terminal salvage value is dropped and three different salvage value models are discussed. These are: (1) straight line decline, (2) exponential decay, and (3) declining balance. Under the assumption of a declining balance (which appears most in line with Terborgh's over-all model) it is shown in what direction and to what extent salvage value affects the earlier determinations of service life and adverse minimum.

EXTREMUM PROBLEMS I

Thursday morning, September 1. E. G. Begle, Yale University, chairman. Papers were discussed by Richard Bellman, RAND Corporation, and Tjalling C. Koopmans, Yale University.

On Distance Methods in Statistical Inference, JACOB WOLFOWITZ, Cornell University.

No abstract available.

Some Aspects of Dynamic Programming, SAMUEL KARLIN, California Institute of Technology.

No abstract available.

EXTREMUM PROBLEMS II

Thursday afternoon, September 1. A. W. Tucker, Princeton University, chairman. Papers were discussed by Donald Bratton, New York City, and H. W. Kuhn, Bryn Mawr College.

Some Problems of Linear Programming, PAUL ROSENBLOOM, University of Minnesota.

No abstract available.

Programming in Linear Spaces, LEONID HURWICZ, University of Minnesota.

IN MANY IMPORTANT problems of economics, in particular those of extremization under constraints, it is unnatural to confine oneself to a finite number of variables and/or constraints. (Examples: programming over time without an arbitrarily chosen finite horizon; activity analysis with a continuum of processes; certain problems involving decision-making under uncertainty; "continuous commodity spectra.") The method of Lagrange multipliers has long been used for finite-dimensional spaces with equality constraints; it was extended, on the one hand, to Banach spaces but still with equality constraints (H. H. Goldstine, *Bull. A.M.S.*, 1940), and, on the other hand, to cases involving inequality constraints but in a finite-dimensional Euclidean space (Fritz John, *Studies and Essays Presented to R. Courant*, 1948, with an infinity of constraints permitted; H. W. Kuhn and A. W. Tucker, *Second Berkeley Symposium*, 1950; M. Slater, Cowles Commission Discussion Paper, Math. 403, 1950, with relaxed differentiability requirements). The present paper extends the domain of validity of the Lagrangean method to a wide class of linear topological spaces (including, but broader than, Banach) with inequalities as well as equalities among the constraints, permitting infinity of constraints and non-differentiable functions; many, but not all, of the results of the authors cited above are included as special cases. (Detailed formulation and proofs are given in the writer's Cowles Commission Discussion Papers Econ. 2109 (with Addenda) and 2110.)

Let P_x, P_y, P_z be closed convex cones in the topological linear spaces denoted respectively by X, Y, Z, and write $x' \geqslant x''$ to mean $x' - x'' \in P_x$, so that $x \geqslant 0_x$ (where 0_x is the null element of X) means $x \in P_x$, with analogous notation for Y and Z. [$x \geqslant 0_x$ may represent a set of equalities (when $P_x = \{0_x\}$) or inequalities (when P_x is the nonnegative "orthant") or a variety of mixtures; the restriction that $x \geqslant 0_x$ becomes vacuous when $P_x = X$.] Given the functional relations f and g with a common domain in X and the respective ranges in Y and Z, x' is said (vectorially) to maximize $f(x)$ subject to $g(x) \geqslant 0_z$, $x \in X_0$, if $f(x'') \geqslant f(x')$ implies $f(x') \geqslant f(x'')$ whenever $g(x'') \geqslant 0_z$, $x'' \in X_0$. (X_0 is a given subset of the domain, often taken as P_x.)

With the Lagrangean expression defined as $\phi(x, z^*; y^*) = y^*(f(x)) + z^*(g(x))$ where y^* and z^* are linear continuous functionals on Y and Z respectively, it is shown that (under appropriate restrictions on the nature of the above spaces, cones, and functions) (a) $\bar{x}$ is maximal if ϕ has a "nonnegative" saddlepoint at $\bar{x}$, and (b) if $\bar{x}$ is "properly" (see Kuhn and Tucker, *op. cit.*) maximal, ϕ will have a nonnegative saddlepoint at $\bar{x}$ if f and g are concave, or satisfy certain first order differential conditions if they are not. The latter result is obtained (following Kuhn and Tucker) thanks to a generalization of the Minkowski-Farkas lemma to the case where the range of the transformation is in a locally convex Hausdorff space.

OPTIMIZATION PROBLEMS

Thursday afternoon, September 1. C. G. Hildreth, North Carolina State College, chairman. Papers were discussed by Victor Smith, Michigan State College.

Optimal Coupling of a Factory Warehouse System, CHARLES C. HOLT, Carnegie Institute of Technology.

THE PROBLEM CONSIDERED is that of integrating the decision-making in a factory warehouse system in order to optimize its over-all performance. The decisions considered are: aggregate production rate, allocation of this production over individual products, shipments from the factory to the warehouses, and allocation of these shipments over products. Optimal (cost minimizing) solutions for each of these decision problems *taken in isolation* can be found in the literature. The purpose of the paper is to seek a method for obtaining a *simultaneous* solution of these problems that will be simple enough to apply even in situations which involve dozens of warehouses and thousands of products. A procedure based on approximations and systematic suboptimization is proposed to satisfy these requirements.

Each of the decisions (with the exception of aggregate production rate) is considered statically and in isolation, and suboptimal decisions determined. The ordering of these analyses is important. The dynamic interactions between the decisions are then considered by examining the costs of departing from the suboptimal decisions. Taylor series approximations (through quadratic terms) of these costs yield a cost function that may be minimized while taking into account the constraints between the decision variables by means of a LaGrange multiplier analysis. A set of linear equations is obtained which constitutes the first order conditions for a minimum. It appears that the solutions of these equations for the optimal decisions can be extremely simple. From the decision rules for individual products, a cost function may be obtained through aggregation that can be used to make the aggregate decisions.

The general procedure is illustrated by its application to the problem of determining the optimal lot sizes for many products when total inventory is determined by the aggregate production rate decision. In making the latter decision, its influence on the costs that are dependent upon the lot size decisions is considered.

The method which is presented is designed to be compatible with the recent work on linear decision rules for production and inventory control which has been done by F. Modigliani, J. F. Muth, H. A. Simon, and the author.

Indirect Taxes in a Leontief System, LIONEL W. MCKENZIE, Duke University.

WE INVESTIGATE the impact on economic efficiency of indirect taxes in a Leontief system with substitution. There is no ultimate factor except labor, and markets are purely competitive. For incipient taxation the "competitive" prices, reflecting costs of production, suffer second order changes. For this reason the actual price structure is influenced only by forward tax shifting. If p is the initial price vector and dt is a vector of incipient taxes, $dp = (I - A)^{-1\prime}\, dt$. This is the source of special simplicity in the present case.

If some of the taxes fall on intermediate products, there will be a loss in productive efficiency, since firms choose wrong production coefficients. The loss is measured by the labor which could be released by efficient production.[5] This equals $(\frac{1}{2})dp \cdot da$, where da is the gross substitution in the input of intermediate products resulting from dp when levels of activity are unchanged.

If the loss of economic efficiency is measured by the equivalent variation,[6] the total loss from dt is $(\frac{1}{2})dp'Sdp$ where S is a matrix summing the substitution effects for all producing and consuming units in the economy. For a given tax revenue, the equivalent variation is minimized when $dt = -\theta S^{*-1}x$, where θ is an appropriate positive constant, $S^* = (I - A)^{-1}S\,(I - A)^{-1\prime}$, and x is the vector of initial gross outputs. It is an easy consequence of this that $(dx)_s = -\theta y$, where $(dx)_s$ is the gross substitution effect for the economy and y is the initial net output vector.[7]

Four implications are immediate. (1) For purely intermediate products gross substitution effects should be zero. (2) If sales to consumers can be separated from sales to firms, (1) implies that all taxes should fall on final sales.[8] (3) Then (2) implies $(dy)_s = -\theta y$, which is a version of Ramsey's result. (4) If the labor supply is fixed, and only consumer sales are taxed, (3) implies $\theta = 0$ and the taxes should be proportionate to prices. But, in general, (4) will be inapplicable.

[5] Compare Gerard Debreu, "A Classical Tax-Subsidy Problem," *Econometrica*, 1954.

[6] Marcel Boiteux, "Le 'Revenue Distribuable' et les Pertes Economiques," *Econometrica*, 1951.

[7] This is reminiscent of Frank Ramsey, "A Contribution to the Theory of Taxation," *Economic Journal*, 1927.

[8] A result anticipated in my "Ideal Output and the Interdependence of Firms," *Economic Journal*, 1951.