6-1963

Stochastic Income Model Using Optimal Inventory Rules, A

Daniel Orr

A Stochastic Income Model Using Optimal Inventory Rules
Author(s): Daniel Orr
Source: *The Review of Economic Studies*, Vol. 30, No. 2 (Jun., 1963), pp. 84-92
Published by: Oxford University Press
Stable URL: http://www.jstor.org/stable/2295805
Accessed: 09/09/2011 15:13

A Stochastic Income Model Using Optimal Inventory Rules[1]

I. Introduction

The optimality of various inventory rules has been intensively investigated from the standpoint of the individual firm; however, no comparable analytic study has been devoted to the question of what happens when such rules interact in a complex, multi-stage economy. This question is considered here in the context of a highly simplified model, but one in which aggregation between some of the stages within the economy (e.g. production and retailing) is eliminated. Thus, in its objectives, this paper is in the tradition pioneered by Lundberg [7] and Metzler [8]. The characteristic procedure of that tradition is to investigate, in the context of non-stochastic, linear, and highly aggregative models, the implications of various patterns of desired inventory investment for the dynamic behaviour of income. In method of analysis, however, the present study will be seen to fall more naturally into a second tradition: that of Slutzky, [14], Yule [15], Frisch [4], and more recently Muth [11], which holds that random shocks may be a principal cause of economic fluctuation.

The linear difference equation approach of the first tradition has undoubtedly yielded insight into economic mechanisms, but it has scored no outstanding triumphs in confrontation with data. The lack of success may be due to any of these obvious drawbacks, acting singly or in combination: (1) The absence of stochastic shock leaves only two explicit explanations of oscillatory behavior, i.e., complex roots (or a dominant negative real root) of unit modulus in the characteristic equation of the (second order) model; or some type of relaxation mechanism, such as was suggested by Hicks in [5], in combination with roots of greater than unit modulus. (2) Linear, delay-free models cannot be expected to represent a non-linear, delay-prone " real world ". (3) The inventory holding objectives of a retailer are likely to differ from those of a manufacturer: it is impossible to capture in a single inventory-sales relation a representation of both sets of motives.

The present study attempts to answer, at least in part, the first and third of these objections. The objective is to investigate the impact of stochastic shocks on output stability when the shocks are transmitted through a " multi-echelon " supply system, with inventories at each echelon controlled by rules that are appropriate for use at the echelon in question.

The " optimality " of (or returns to the individual firm from) the s,S (or two-bin) type of inventory control rule has been thoroughly explored;[2] one outcome of this discussion has been the realization that policies of this type are useful for requisitioning stock only in

[1] I am indebted to members of the University of Chicago Workshop in Econometrics—particularly Zvi Griliches, Merton Miller and Lester Telser—for thorough and incisive criticism of an earlier draft of this paper. Remaining errors or imperfections are the fault—or whim—of the author alone. Research was initially undertaken under the auspices of the Econometric Research Program, Princeton University: this support is gratefully acknowledged.

[2] C.f. [2, 16]. The s,S policy is: identify two inventory levels s and S, $s < S$. When inventory on hand falls below s, order up to S; when inventory on hand is above s, order nothing. The rationale is to balance costs of holding inventory (a linear function of the inventory level) against costs of placing an order (which are a lump sum, i.e., invariant with the order size).

situations where smoothness in the time series of quantities requisitioned is unimportant. (For example, a retailer may not be disturbed by the presence of substantial fluctuations in his orders for a good he sells; but the manufacturer of that good will take pains to smooth out the pattern of ordering into finished goods inventory, since these orders determine the level of production and employment in his factory, and hence affect his operating costs. To the retailer, the s,S policy may be extremely useful; to the manufacturer, not so.)

Where smoothing of the requisition load is desirable, other classes of policy can be expected to perform satisfactorily; an example is the policy that requisitions an amount equal to a demand forecast obtained from an exponentially weighted moving average of past demand.[1]

In the following, I will deal with a basic model involving a population of consumers, who generate a time series of demands (d_t) which are placed with a set of retailers. These retailers requisition goods to fill consumer demands: the requisition series (r_t) constitutes the demand load on the manufacturer. He, in turn, fills these requisitions out of current production (p_t) and/or his finished goods inventory. Two variations on this model will be treated: the first will assume there is no linkage between production and demand; the second will consider a case in which there is a linkage via which output partially determines current customer demand (i.e., a sort of consumption function is introduced). I will investigate the stability and variability-transmitting properties of such systems. The stability criterion applies to the variance of the time path of production (which is equivalent to income in our analysis). Let Vp_t and Vd_t represent the variances of production and demand in period t. Then, if there is a finite number z which satisfies the condition

$$\lim_{k=\infty} Vp_{t+k} \leqslant z$$

the system is said to be stable. If

$$\lim_{k=\infty} Vp_{t+k} \leqslant Vd_t$$

the system is said to be variance-attenuating; if the last inequality is reversed, it is variance-amplifying.

II. Initial Assumptions

The structure of retail operations is initially characterized in these assumptions: (1) There are m retailers, who sell one good to customers; this good is purchased from a single manufacturer. (2) The retailers are identical in all salient respects; the same probability distribution of customer demands confronts each retailer, and the inventory-associated costs are the same for each retailer. The nature of these costs is such that the retailers all utilize the familiar economic order quantity (e. o. q.) analysis[2] to determine how much to order from the manufacturer. Each retailer's e. o. q. is Q units, where Q is an integer. (3) When an order is placed by a retailer, the manufacturer fills it in the

[1] This type of policy is investigated in [9]; in [12] the optimality of exponentially weighted forecasts is investigated, and other economic problems for which these forecasts have been found useful are listed.

[2] [16], chapter 3. The usual e.o.q. formula is $Q = (2R\lambda/H)^{1/2}$, where R is a lump-sum cost of ordering, H is the cost of holding one unit of stock for one period, and λ is the mean demand rate.

same period;[1] consequently, the reorder point[2] is zero, and since all retailers are " always " able to satisfy customer demands in the period in which they are placed, there is independence between the demand distribution of retailer i and the inventory positions of the other m—1 retailers. (4) The demand sequence confronting each retailer is generated by a Poisson process,[3] with parameter λ: that is, over a time interval of length h $(t, t+h)$, the probability that the firm i receives demands for n units of goods is:

$$(1) \qquad P(d^i = n) = e^{-\lambda h}(\lambda h)^n/n! \qquad (i = 1, \ldots, m)$$

Difficulties are avoided at a later stage of the analysis if we select a sufficiently small h so that the probability that one retailer places more than one order in the interval $(t, t+h)$ is effectively zero, i.e.,

$$\sum_{n=Q}^{\infty} e^{-\lambda h} (\lambda h)^n/n! \sim 0,$$

and if we let h be our unit of time measure, i.e., $h = 1$. The variable r_t represents the total quantity requisitioned from the manufacturer by all the retailers during time interval t. The manufacturer, anxious to smooth fluctuations in his finished goods inventory and production sequences, uses the output rule[4]

$$(2) \qquad p_{t+1} = \alpha p_t + (1-\alpha) r_t \qquad 0 \leqslant \alpha < 1$$

where p_t is the quantity produced during time period t.

III. The Distribution of requisitions on the Manufacturer

Because the demands placed upon the individual retailers are mutually independent, the distribution of orders upon the producer during a unit time interval would be

$$(3) \qquad P(\sum_{i=1}^{m} d^i \leqslant d) = \sum_{n=0}^{d} e^{-m\lambda} (m\lambda)^n/n!,$$

if the retailers passively relayed demands to the producer as they occurred, rather than transferring them only in lots of size Q. The distribution (3) has mean and variance $m\lambda$.

When the retailers each order Q units at a time, the total orders placed upon the producer in the time interval of unit length can range from 0 to mQ. The probability that an individual retailer orders during an arbitrary unit interval $(t, t+1)$, denoted $\phi(1)$, is equal to the probability of the compound event (inventory at $t = k$ units)· [sales during $(t, t+1) \geqslant k$ units] where k ranges from 1 to Q. The probability of the first part of this

[1] By the Poisson assumption there is a finite (but small) probability that the retailers place an "infinite" number of requisitions on the manufacturer in a single period. If all requisitions are to be filled in the same period as they are placed, the Poisson assumption must mean that finished goods inventories at the manufacturing level are " infinitely " high. This difficulty is not too hard to live with; selection of an appropriately short time period assures that there is always a finite level of manufacturer's inventory which will enable him to meet requisitions " immediately " with any probability we care to specify. In terms of the influence on the results obtained from this model, an extremely rare stockout is equivalent to no stockouts.

[2] [16], chapter 3.

[3] [3], pp. 400-402. Implicit in this assumption is the condition that customers order only one unit at a time. The assumption also requires that the demand process has no memory, i.e., demand in the interval $(t, t+h)$ is independent of demand in the interval $(0, t)$.

[4] H. D. Mills shows in [9] that the rule (2) can, by choice of α, yield a minimum value to any weighted sum of inventory variance and production variance. Properly speaking, the producer is concerned with mean square deviations of inventory and production when he uses the rule (2), not merely with fluctuations in these two variables through time. It should be noted that (2) is formally equivalent to the rule

$$p_{t+1} = p_t - \beta(f_t - i^*)$$

where f_t is the final finished goods stock level of period t, i^* is a constant, and $\beta = (1-\alpha)$, i.e., the complement of the exponential weight in the rule (2).

compound event is $1/Q$; the inventory distribution at a randomly selected future time t is uniform.[1] Hence, we have:

$$\phi(1) = 1/Q \sum_{k=1}^{Q} \sum_{n=k}^{\infty} e^{-\lambda} \lambda^n/n! \tag{4}$$

The part of (4) under the double summation sign can be rewritten

$$e^{-\lambda}\left(\sum_{n=1}^{Q} n\lambda^n/n! + \sum_{n=Q+1}^{\infty} Q\lambda^n/n! \right)$$

and if Q is sufficiently large,[2] this is equivalent to

$$e^{-\lambda} \sum_{n=1}^{\infty} n\lambda^n/n! = \lambda \sum_{n=1}^{\infty} e^{-\lambda}\lambda^{n-1}/(n-1)! = \lambda,$$

or

$$\phi(1) = \lambda/Q.$$

The probability distribution of orders confronting the manufacturer in a unit interval is the binomial expression

$$P(\sum_i d^i = r = jQ) = \binom{m}{j} (\lambda/Q)^j (1 - \lambda/Q)^{m-j} \tag{5}$$

which has mean $m\lambda$ and variance $m\lambda\ (Q - \lambda)$.[3]

In §II, the time unit was stipulated in such a way that λ is quite small vis-a-vis Q. Thus, it is permissible to approximate (5) by the Poisson expression

$$P(r = jQ) = e^{-\lambda m/Q}\ (m\lambda/Q)^j/j!$$

which has mean $m\lambda$ and variance $m\lambda Q$. Strictly speaking, if $Q - \lambda > 1$, there will be amplification of variance as the system transforms demands into requisitions upon the producer: in the Poisson approximation, $Q > 1$ is the condition which yields amplification from d to r.

[1] Proof: The Poisson process (reference [3]) is described by $P(i_t = k) = \lambda\Delta t \cdot P(i_{t-\Delta t} = k+1) + (1-\lambda\Delta t) \cdot P(i_{t-\Delta t} = k) + o(\Delta t)$ for $k = 1, 2, \ldots, Q-1$; or in more convenient notation $P_k(t) = \lambda\Delta t\, P_{k+1}\ (t-\Delta t) + (1-\lambda\Delta t)\, P_k(t-\Delta t) + o(\Delta t)$ where the last term on the right side is of an order of magnitude smaller than Δt.

$$\frac{P_k(t) - P_k(t-\Delta t)}{\Delta t} = \lambda[P_{k+1}(t-\Delta t) - P_k(t-\Delta t)] - \frac{o(\Delta t)}{\Delta t}$$

and as $\Delta t \to 0$, the last term $\to 0$, and hence the limit of the left side exists. This is equivalent to the differential equation

$$P'_k(t) = -\lambda P_k(t) + \lambda P_{k+1}(t).$$

The statement we wish to prove implies that the steady-state distribution of the inventory process is uniform. Setting $P'_k(t) = 0$ to investigate the steady-state conditions yields

$$\lambda P_{k+1}(t) = \lambda P_k(t). \tag{*}$$

The case for $k = Q$ is equally easy to work out; hence QED by (*).

[2] The requisite " largeness " of Q of course depends upon the size of λ. From the Poisson tables [10], we see that the cumulative term

$$\sum_{n=Q}^{\infty} e^{-\lambda}(\lambda)^n/n\,!$$

is less than 10^{-8} for $Q = 5$ if $\lambda = .01$, $Q = 6$ if $\lambda = .1$, $Q = 10$ if $\lambda = 1$, and $Q = 30$ if $\lambda = 10$. Notice that even in the most demanding of the cases here enumerated, i.e., when $\lambda = .01$, the condition to be satisfied is $Q = (2R\lambda/H)^{1/2} > 500\ \lambda$ or $2R\lambda > 25H$, which is perfectly plausible for $\lambda \geqslant .01$.

[3] If (5) referred to the probability that j units were ordered, the mean would be $m\lambda/Q$ and the variance $m\lambda(Q-\lambda)/Q^2$, but since each order is for Q units, the mean and variance are increased by factors of Q and Q^2, respectively. Note that the mean quantity ordered from the producer during $(t,\ t+1)$ is the same, regardless of whether the producers use *e. o. q.*'s, which we would expect, since the ordering rules do not allow for any inventory accumulation or decumulation, and hence average orders must be the same as average sales.

IV. System Behaviour with Stationary, Random Demands

The production scheduling rule (2) is equivalent to

$$p_t = \alpha^t p_0 + (1 - \alpha) \sum_{j=0}^{t-1} \alpha^j r_{t-1-j}$$

Thus, for large t (after the system has been in operation for many periods) the first two moments of p are

$$(6) \qquad \begin{cases} Ep = Er = m\lambda, \\ Vp = (1 - \alpha)^2 \sum_{j=0}^{t-1} \alpha^{2j} \; Vr = \left(\dfrac{1 - \alpha}{1 + \alpha}\right) m\lambda Q. \end{cases}$$

The manufacturer can, by manipulation of α in (2), obtain a production variance that is arbitrarily small. Thus, it cannot be argued that the use of economic order quantities by firms in the retail echelon necessarily leads to a greater variance of production than would result if demands were placed directly on the manufacturer. However, either higher production variance *or* higher inventory variance can be anticipated with certainty. The variance of finished goods period-ending inventory levels implied by use of the rule (2) is (again for large t)

$$Vf = \sum_{j=0}^{\infty} \alpha^{2j} \; Vr = m\lambda Q/(1 - \alpha^2).$$

To illustrate the impact of the presence of retailers' *e. o. q.*'s upon the cost position of a manufacturer, suppose that demands had been placed directly on the manufacturer, and his production response had been governed by the rule (2) with $\alpha = 0$, i.e.,

$$p_{t+1} = \sum_i d_t^i.$$

It follows that

$$Vp = Vf = V_{\Sigma d} = m\lambda.$$

Next, suppose that retailers began ordering in batches, as described in § III, but the manufacturer wished to keep his production variance at the level $m\lambda$. What is the concomitant amplification in inventory variance? The desired condition is (from equation (6))

$$\left(\frac{1 - \alpha}{1 + \alpha}\right) m\lambda Q \leqslant m\lambda,$$

or

$$\frac{1 + \alpha}{1 - \alpha} \geqslant Q.$$

The smallest α satisfying this condition is

$$\alpha = (Q - 1) / (Q + 1)$$

which, when substituted into Vf, gives

$$Vf = m\lambda Q \Big/ \left[1 - \left(\frac{Q - 1}{Q + 1}\right)^2\right] = m\lambda(Q + 1)^2/4.$$

Thus, the variance of inventory is amplified by a factor of $(Q + 1)^2/4$ if the objective of maintaining $Vp = m\lambda$ is carried out: this implies an increase in inventory holding cost to the manufacturers by a factor of $(Q + 1)/2$ if these costs are linear and the effective protection against stockouts is not to change.

This type of system will always be stable: for any admissible value of α, both Vf and Vp converge to finite values, regardless of the number of retailers or their order quantity sizes.

V. A Multiplier Mechanism: Production-to-Consumption Feedback

There are certain interesting questions concerning decision making by the firm that are raised by the foregoing analysis. The stationary, independent random demand formulation is less unappealing in a microeconomic study than in a macroeconomic model like the present one. In particular, the model of §§ III-IV would appear useful for the study of the question: What recourses (such as price differentials to retailers who order uniform amounts at regular time intervals, or alternative production policies) are available to the manufacturer who would like to reduce the variability of his requisition load. If the foregoing model is to be treated as a macroeconomic representation, however, the absence of any relation between consumption (demands) and income (production) is a serious drawback: despite the fact that we are dealing with short-run responses, a recognizable sort of consumption function mechanism is necessary.

Accordingly, the following change is proposed: the quantity demanded by all customers is a Poisson-distributed random variable with parameter that is a linear function of the present rate of output. The process is constrained to have stable mean behaviour. The production control rule (2) is used. Thus, the system is described by the equations

$$p_{t+1} = \alpha p_t + (1 - \alpha)\, r_t \tag{2}$$

$$\sum_i d_t^i = \text{Pois}\,(j, (\beta + \gamma p_t)) \tag{7}$$

$$P(r_t = kQ) = e^{-(\beta+\gamma p_t)/Q}\left(\frac{\beta + \gamma p_t}{Q}\right)^k \Big/ k! \tag{8}$$

$$\beta = (1 - \gamma)\, Ep. \tag{9}$$

Evidently, r_t is a random variable with value contingent on the random variable p_t. Taking expectations on (8), we see that

$$Er_t \mid p_t = \beta + \gamma p_t$$

and the unconditional expectation is given by[1]

$$Er_t = \beta + \gamma Ep = Ep:$$

the equality between the second and third terms is assured by the condition (9).

Similarly, from (8) we see that

$$Vr_t \mid p_t = Q(\beta + \gamma p_t)$$

which has the unconditional value[2]

$$Vr_t = QEp + \gamma^2 Vp_t$$

[1] Omission of the subscript from the expected value of production in period t is justified on the ground that the mean of production is stationary through time: this is assured by the condition (9), as can be seen by recursive calculation:

$$Er_0 = \beta + Ep_0 = Ep_0$$
$$Ep_1 = \alpha Ep_0 + (1-\alpha)\, Er_0 = Ep_0 = Ep \text{ etc.}$$

[2] This unconditional variance is given in a theorem of Raiffa and Schlaifer [13], p.107: $Vr = E^p Vr \mid p + V^p Er \mid p$, where the superscript indicates the variable over which expectations are taken.

Since r_t and p_t are not independent, to trace the behaviour of the variance of production through time, we need the covariance of r_t and p_t, which is[1]

$$Cr_t,\ p_t = \gamma Vp_t$$

The variance of p in successive time periods is related by

$$(10) \qquad \begin{aligned} Cp_{t+1} &= \alpha^2 Vp_t + (1-\alpha)^2\,[\gamma^2 Vp_t + QEp] + 2\alpha(1-\alpha)\gamma Vp_t \\ &= [\alpha^2 + 2\alpha(1-\alpha)\gamma + (1-\alpha)^2\gamma^2]Vp_t + (1-\alpha)^2\,QEp \end{aligned}$$

which is of the form $Vp_{t+1} = AVp_t + B$. If $A < 1$, the system converges to the value

$$\lim_{k=\infty} Vp_{t+k} = B/(1-A);$$

for γ less than 1, the coefficient $[\alpha^2 + 2\alpha(1-\alpha)\gamma + (1-\alpha)^2\,\gamma^2]$ is less than one for all admissible values of α; consequently, for any reasonable marginal and average propensity to consume, the system must be stable by the variance criterion. If Ep in (10) is taken as an equivalent to $m\lambda$ in the first model, however, it is apparent that the system with feedback, though stable, does lead to a substantial amplification of the variance of production.

VI. The Variance as a Measure of Shock-Induced Oscillations

Because the models of this paper are stochastic, the standard stability criteria, involving the roots of the characteristic equation, do not adequately characterize the oscillatory behaviour that can be expected to obtain. The stability and amplification of variances through time has been adopted as an attractive alternative criterion because of the ease with which they can be traced, and some readily apparent connections between the variances and time paths of important variables. The variance measure may, under a wide variety of circumstances, afford a convenient key to the classification and analysis of cycles that are caused by the interactions of successive random shocks. The suggestion that cycles are indeed produced by such a mechanism was advanced and rationalized in the work of Slutzky [14] and Yule [15], and later by Frisch [4]. More recently, Adelman and Adelman [1] have conjectured that random shocks are a necessary ingredient to the production of realistic fluctuation in a highly structured macroeconomic model. To date, however, little is known about the process by which the economic system transforms the amplitude (variance) of the shock (or shocks) into the observable oscillations in economic series: the results presented in this paper suggest that the mechanism may produce cycles of substantial amplitude from small shocks. We expect from [14, 15] that the period of these shock induced cycles will be fairly regular.

[1] Let p_t be a random variable with arbitrary distribution function $F(p_t)$, and let the joint distribution of p_t and r_t be $G(p_t\, r_t)$.

$$\begin{aligned} Cp_t, r_t &= \int_{p_t}\int_{r_t} (r_t - Er_t)\,(P_t - Ep_t)\; dG(p_t\, r_t) \\ &= \int_{p_t} \left\{ \sum_{k=0}^{\infty} (kQ - Ep)\,(p_t - Ep)\, e^{-(\beta+\gamma p_t)/Q} \left(\frac{\beta+\gamma p_t}{Q}\right)^k \Big/ k! \right\} d\,F(p_t) \\ &= E[p_t(\beta+\gamma p_t) - Ep(\beta+\gamma p_t)] \\ &= \gamma Vp_t. \end{aligned}$$

Knowledge of the first two moments of our initial variables is sufficient to characterize the time series in which these variables will be observed. Throughout, discussion has been confined to systems that produce stable means. Thus, monotonically divergent time series will not occur. In neither case is there a possibility of variances which converge to zero; consequently, both models produce irregular oscillations around fixed levels.

Standard non-stochastic dynamic models generate various classes of time series: if the roots of a second order system are complex with modulus greater than 1, for example, the time path will diverge and oscillate. The variance of the time series composed of regular observations on this time path, will increase without bound through time. A logical concomitant of this implication is that an unbounded variance (such as would be observed with $\gamma = 1$ in equation (10)) is a sufficient condition for an explosive oscillatory time path. Similarly, a stable variance is sufficient for regular oscillation with constant amplitude (the unit modulus time path). There is, however, a much richer array of time path possibilities in the stochastic models: there is no limitation to cycles with regular period and constant (or steadily expanding) amplitude, and no dependence upon fortuitous properties of characteristic roots to obtain realistic fluctuations. In a paper which also rejects the standard deterministic difference equation approach in favor of a random shock mechanism [11], Muth has shown that " cobweb " behaviour can be obtained without the assumption of irrational price expectations on the part of farmers, a necessary ingredient in cobwebs spun in the absence of random shocks.

VII Conclusions

The principal results of this study are (*a*) " multiplier-accelerator " mechanisms can be modified to accommodate random shocks; the modified mechanism generates series which afford a better representation of observable series than do the more familiar non-stochastic difference equation models. (*b*) The rules for inventory investment prescribed for retailers in the new normative literature of managerial economics contain characteristics which may lead to substantial amplification in the inherent variability of the system; so long as the smoothing motive remains attractive to the manufacturer, however, this amplification may in part be offset before it is transmitted to the employment and output decision level. (*c*) Under all conditions that can be foreseen in models which relate consumption to *current* income, systems are stable, i.e., the variance of the income variable converges to some finite level.

Chicago. DANIEL ORR.

REFERENCES

[1] Irma Adelman and Frank L. Adelman, " The Dynamic Properties of the Klein-Goldberger Model," *Econometrica*, 27, 1959.

[2] Kenneth J. Arrow, Theodore Harris and Jacob Marschak, " Optimal Inventory Policy," *Econometrica*, 19, 1951.

[3] William Feller, *An Introduction to Probability Theory and Its Applications*, Vol. I, 2nd edition, New York, 1957.

[4] Ragnar Frisch, " Propogation Problems and Impulse Problems in Dynamic Economics," *Economic Essays in Honor of Gustav Cassel*, London, 1933.

[5] J. R. Hicks, *A Contribution to the Theory of the Trade Cycle*, London, 1950.

[6] Charles C. Holt, Franco Modigliani, John F. Muth, and Herbert Simon, *Planning Production, Inventories, and Work Force*, Englewood Cliffs, 1960.

[7] Erik Lundberg, *Studies in the Theory of Economic Expansion*, London, 1937.
[8] Lloyd A. Metzler, " The Nature and Stability of Inventory Cycles," *Review of Economic Statistics*, 23, 1941.
[9] Harlan D. Mills, " Smoothing in Inventory Processes," *Navy Supply System Research*, Vol. 1, Mathematica, Princeton, New Jersey, 1960.
[10] E. C. Molina, *Poisson's Exponential Binomial Limit*, Bell System, New York, 1942.
[11] John F. Muth, " Rational Expectations and the Theory of Price Movements," *Econometrica*, 29, 1961.
[12] John F. Muth, " Optimal Properties of Exponentially Weighted Forecasts," *Journal of the American Statistical Association*, 55, 1960.
[13] Howard Raiffa and Robert Schlaifer, *Applied Statistical Decision Theory*, Boston, 1961.
[14] E. Slutzky, " The Summation of Random Causes as the Source of Cyclic Processes," *Econometrica*, 5, 1937.
[15] G. U. Yule, " On a Method of Investigating Periodicity in Disturbed Series," *Transactions of the Royal Society*, London, A, 1927.
[16] Thomson M. Whitin, *The Theory of Inventory Management*. 2nd edition, Princeton 1957.