

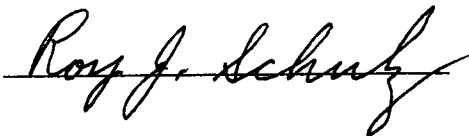
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I am submitting herewith a thesis written by David Marcelo Garza entitled "Investigation of the Composite Amplitude in Solid Rocket Motor Combustion Chamber Pressure Oscillations." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Aerospace Engineering.

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Associate Vice Chancellor and
Dean of the Graduate School

**INVESTIGATION OF THE COMPOSITE AMPLITUDE IN SOLID
ROCKET MOTOR COMBUSTION CHAMBER PRESSURE
OSCILLATIONS**

A Thesis
Presented for the
Master of Science
Degree
The University of Tennessee, Knoxville

David Marcelo Garza

August, 1996

Dedication

I would like to dedicate this thesis to my family, whose unwavering support has kept me sane through this experience.

Acknowledgments

I would like to thank my advisor, Dr. Gary Flandro, for his guidance and support in completing this work. Also, I would like to thank the other members of my thesis committee, Dr. Robert Roach and Dr. Roy Schulz, for their valuable comments and suggestions on this thesis. I would like to express my appreciation to Linda Engels whose scanner was an indispensable tool in performing the work described here.

Abstract

To date, a well developed nonlinear combustion instability theory remains elusive. Flandro developed a nonlinear theory using the concept of an acoustic energy balance. The theory expresses the pressure and velocity oscillations in a rocket motor chamber as a superposition of acoustic modes and models growth and decay responses using a differential equation to describe the composite wave amplitude. This work attempted to apply the theory developed by Flandro to some data from a recent test program done by the Naval Warfare Center (NWC). Two different data reduction methods were used. Both methods were only partially successful and a parameter study was carried out to investigate the equations and solution behavior which were being fit to the data. It was concluded that multiple pulse responses must be combined into a single data set to determine a set of triggering and limit-cycle oscillations, and this data set must include both growth and decay responses to reliably estimate triggering and limit-cycle amplitudes in a third-order or higher system. This work recommends that future test programs should be more focused on a particular motor geometry and propellant combination with greater repetition and less variation in the testing than the NWC test program.

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1 Introduction

Combustion instability in rocket motor combustion chambers has plagued motor development for many decades. When a motor begins to experience severe oscillations in the chamber pressure and thrust output it can severely impact performance of the vehicle. At its mildest the changes in motor performance can be compensated by adjustments in the guidance system. At worst, severe oscillations can damage the motor, vehicle, or payload. Oscillations are typically induced by pressure disturbances in the combustion chamber. In solid propellant motors these disturbances can be caused by a sizable chunk of unburned propellant passing through the nozzle and momentarily blocking the throat, or by the uncovering of a crack in the propellant grain which increases the burning area of the propellant and increases the mass flow into the chamber. Both linear and nonlinear theories to explain this phenomenon are the subject of continuing research.

While a linear theory has been developed to the point that a standard predictive code and data reduction method exist to implement it, a nonlinear theory remains elusive. The present work attempts to apply a theory developed by Flandro, and summarized later in this chapter, to some recent test data.^{1,2} The present work attempts to apply the theory to some recent test data from the Naval Weapons Center (NWC) and to establish some guidelines for future test programs which try to verify this and other nonlinear combustion instability theories.^{3,4}

1.1 Nonlinear Chamber Oscillations

Rocket motor combustion oscillations have exhibited several nonlinear features including steep wave fronts, mean pressure shifts, limit-cycles, and triggering phenomenon.

Figures 1 and 2 show typical results from a motor test.⁴ Figure 1 shows a pressure-time history for a representative solid rocket motor. In this figure the phenomenon of

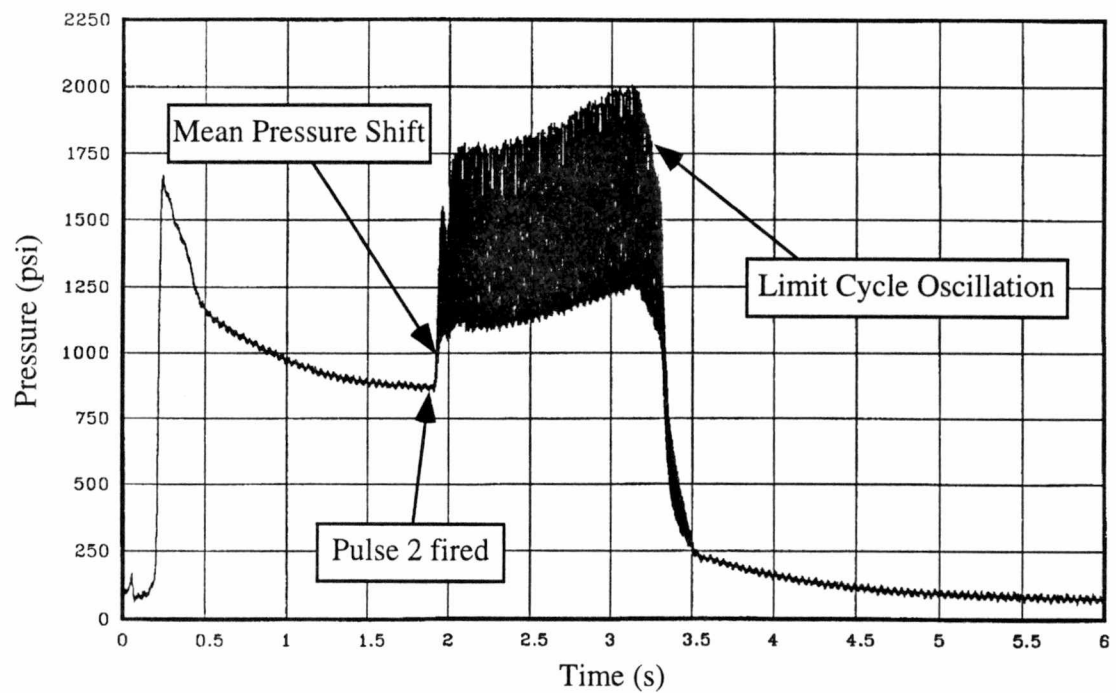


Figure 1. NWC Motor 2 Chamber Pressure History.

Source: Blomshield, F.S., H.B. Mathes, J.E. Crump, and M.W. Beckstead, *Stability Testing and Pulsing of Full Scale Tactical Motors, Part 2. Appendix*, NAWCWPNS TP 8060, Naval Air Warfare Center Weapons Division, China Lake, CA, Feb. 1996

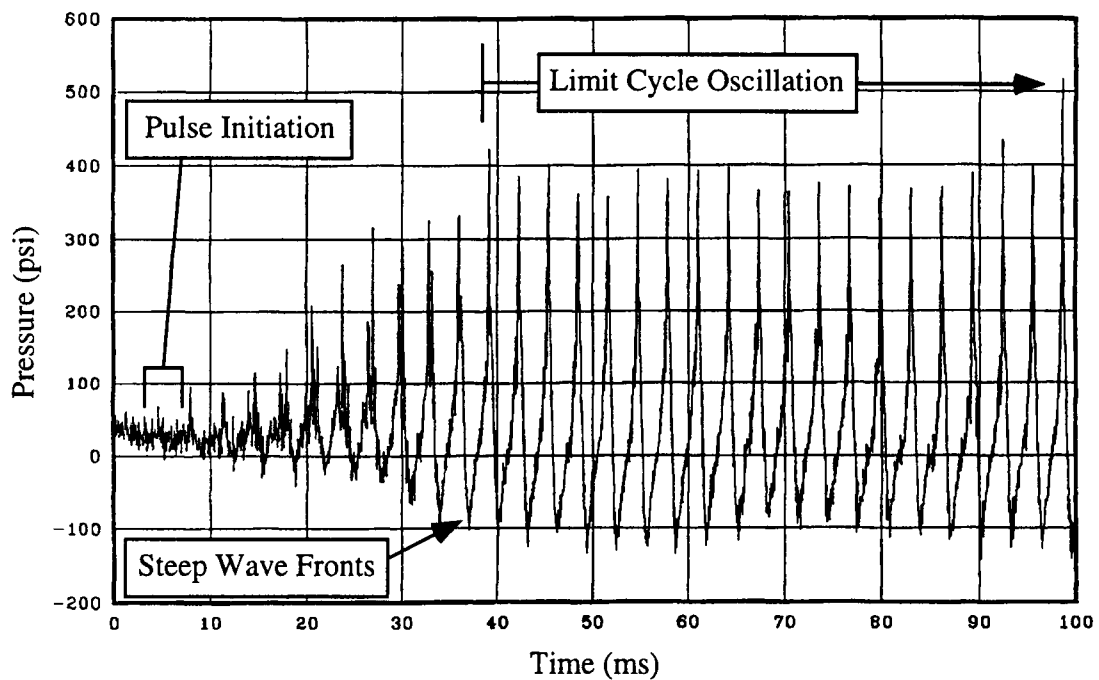


Figure 2. Expanded View of Motor 2 Response to Pulse 2.

Source: Blomshield, F.S., H.B. Mathes, J.E. Crump, and M.W. Beckstead, *Stability Testing and Pulsing of Full Scale Tactical Motors, Part 2. Appendix*, NAWCWPNS TP 8060, Naval Air Warfare Center Weapons Division, China Lake, CA, Feb. 1996

triggering and limit-cycle oscillations are apparent. A pulse was fired at about 1.7 seconds into the burn. The pulse exceeded a critical triggering amplitude and the motor transitioned into a high-amplitude limit-cycle oscillation. Also evident in the figure is another nonlinear feature of motor combustion chamber oscillations—the mean pressure shift. This is a shift in the mean chamber pressure from its normal value to a higher pressure. Figure 2 is an expanded view of the growth to the limit-cycle. In this figure, which only plots the oscillation about the mean, the steep wave fronts characteristic of many rocket motor pressure oscillations are quite noticeable.

Limit-cycle behavior is characterized by an initial, linearly unstable departure from the steady-state pressure. As the oscillation amplitude increases nonlinear losses begin to check the growth of the oscillations, and finally stabilize the oscillations at a new amplitude. Triggering describes a motor's behavior when it resists growth to large-amplitude oscillations until a pressure disturbance larger than a certain level is reached. When a disturbance is stronger than this "trigger" value, the motor will begin to experience large-amplitude oscillations.

1.2 Nonlinear Combustion Instability Theory

Flandro developed a nonlinear theory based on observations of motor test results and application of a nonlinear acoustic energy balance. The method begins by expressing the unsteady components of gas velocity and pressure using Fourier series:

$$\frac{p'}{P_0} = \varepsilon(t) \sum_{n=1}^{\infty} C_n \cos(k_n x) \sin(\omega_n t - \phi_n) \quad (1)$$

$$\frac{u'}{a_0} = \varepsilon(t) \sum_{n=1}^{\infty} C_n \sin(k_n x) \cos(\omega_n t - \phi_n) \quad (2)$$

where n is the mode number, C_n is the relative amplitude of mode n , and ϕ_n is the relative phase angle. k_n is the wave number and ω_n is the frequency for mode n . These quantities describe the wave form of the composite wave and account for wave form-related nonlinear features such as steep wave fronts. The form of equations 1 and 2 are based on observations that steep-fronted waves which occurred in rocket motor combustion chambers could be reduced to a superposition of acoustic modes. Values for C_n and ϕ_n which closely match the waveforms observed in tactical rockets is

$$\begin{aligned} C_n &= 2\sqrt{1+(n\pi)^2} / (n\pi)^2 \\ \phi_n &= -\tan^{-1}[1/(n\pi)] \end{aligned} \tag{3}$$

Of note is the fact that the relative amplitude and relative phase of the different modes remain fixed in time. This assumption that the waveform is fixed has been the subject of some debate.⁵ For the motors studied in this work, this assumption seems to hold. The spectra of all of the triggered pulse responses are shown in appendix A. Examination of these figures shows little mode coupling during the growth of the disturbance to the large-amplitude limit-cycle.

Alone this provides a wave with the proper wave form but does not provide a means for modeling the time-varying amplitude. The composite wave amplitude, ϵ , determines the time-dependant amplitude, and is assumed to vary over a time scale much longer than the oscillation period. The expression describing the behavior of ϵ over time, and thus the growth or decay of the waves given by the sums in equations 1 and 2, is found using an acoustic energy balance. The unsteady enthalpy, H' , mass flow, m' , and density, ρ' are expanded using ϵ as a perturbation parameter

$$\begin{aligned}
H' &= \varepsilon H_1 + \varepsilon^2 H_2 + \varepsilon^3 H_3 + O(\varepsilon^4) \\
m' &= \varepsilon m_1 + \varepsilon^2 m_2 + \varepsilon^3 m_3 + O(\varepsilon^4) \\
\rho' &= \varepsilon \frac{p'}{a_0^2} + \varepsilon^2 \rho_0 \frac{(1-\gamma)}{2\gamma^2} \left(\frac{p'}{P_0} \right)^2 + \\
&\quad \varepsilon^3 \rho_0 \frac{(1-\gamma)(1-2\gamma)}{6\gamma^3} \left(\frac{p'}{P_0} \right)^3 + O(\varepsilon^4)
\end{aligned} \tag{4}$$

The H_i and m_i coefficients are found by substituting these results into the enthalpy function

$$\nabla H = \nabla P / \rho + \mathbf{u} \cdot \nabla \mathbf{u} \tag{5}$$

and solving for them in terms of the unsteady pressure and velocity. Next, the unsteady acoustic energy density, E' , is expanded to 4th-order in ε in a manner similar to that carried out above on H' and m' :

$$E' = \varepsilon^2 E_1 + \varepsilon^3 E_2 + \varepsilon^4 E_3 + O(\varepsilon^5) \tag{6}$$

This expansion and equations 4 are inserted into the energy balance equation

$$\frac{\partial E'}{\partial t} = -\nabla \cdot (H' m') \tag{7}$$

which is then solved to give expressions for the E_i in terms of the unsteady pressure and velocity. The time average of equation 7 is taken to provide a differential equation for ε :

$$\frac{d\varepsilon}{dt} = \frac{-\int_V \langle \nabla \cdot (H' m') \rangle dV}{\varepsilon (2E_1 + 3\varepsilon E_2 + 4\varepsilon^2 E_3 + \dots)} \tag{8}$$

After additional algebra and a development of a shock loss model accurate to ε^4 , equation 8 is reduced to an equation of the form

$$\frac{d\varepsilon}{dt} = A\varepsilon + B\varepsilon^2 + C\varepsilon^3 \quad (9)$$

The coefficients A , B , and C involve combinations of the steady and unsteady pressure and velocities, as well as the motor geometry and propellant response function.

The development of equation 9 can be done to any order accuracy in ε , making a more general form of equation 9

$$\frac{d\varepsilon}{dt} = \sum_{i=1}^N \alpha_i \varepsilon^i \quad (10)$$

where N is the order of the system and the α_i are coefficients determined by motor geometry, flow properties, etc. Design information such as grain geometry, motor dimensions, propellant, and chamber pressure influence the growth and decay of oscillations in the combustion chamber through the α_i . If only one term is used, the equation reduces to

$$\frac{d\varepsilon}{dt} = \alpha\varepsilon(t) \quad (11)$$

which is the linear equation used extensively in motor analysis and design today, and which is the basis for the Standard Stability Program (SSP), the standard stability prediction program used in government and industry.⁶

1.3 Limit Cycles and Triggering

Equation 10 can be rewritten using roots of the polynomial on the right-hand side.

This gives

$$\frac{d\varepsilon}{dt} = \alpha\varepsilon \prod_{i=1}^{N-1} (\varepsilon - r_i) \quad (12)$$

where the r_i are the roots of the right-hand side of equation 10 when $d\varepsilon/dt=0$.

These roots are called the fixed-points of the nonlinear system and represent the limit-cycle amplitudes and triggering amplitudes as shown in figure 3. If the system is operating at one of the fixed-points, then it will remain at that point, however, if it is shifted slightly from the fixed point, it may or may not depart the vicinity of the fixed-point. The left diagram shows an unstable fixed point. If the system is perturbed slightly below r_1 then $d\varepsilon/dt$ is negative and the system begins to move away from r_1 as time progresses. Conversely, if the system is forced slightly above r_1 , $d\varepsilon/dt$ becomes positive and the system again departs the vicinity of r_1 without further outside influence. The diagram on the right shows a stable fixed point. If the system is perturbed from r_1 to a point slightly below r_1 , then $d\varepsilon/dt$ is positive and the system moves back toward r_1 . On the other hand, if the system is perturbed to a point slightly above r_1 , $d\varepsilon/dt$ becomes negative and ε drops back to r_1 . Thus, stability of a fixed point describes the behavior of the system in the vicinity of the fixed point and relative to the fixed point. In the context of rocket motor stability, the term "linear stability" describes the behavior of the motor when perturbed slightly from its normal operating mode, which corresponds to the origin, and relative to that operating mode. So a linearly unstable motor is one which does not damp out small perturbations introduced into the combustion chamber, but rather transitions to

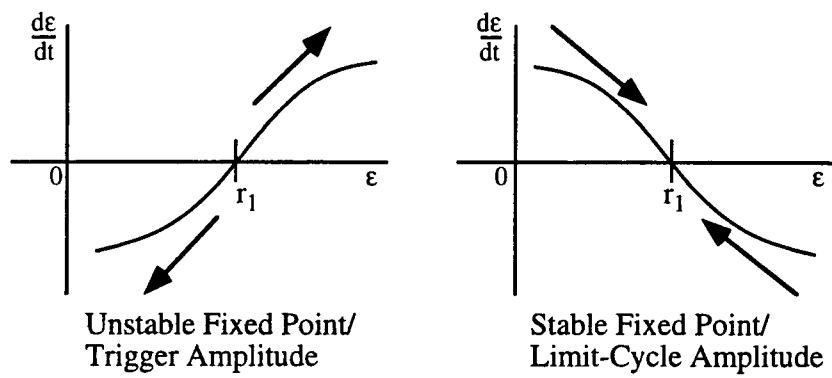


Figure 3. Stable and Unstable Fixed Points.

a new mode of operation normally characterized by the large-amplitude oscillations and other features described in the previous section.

With equation 10 the origin is forced to be a root of the system and it can be a stable or unstable fixed-point. The system would spontaneously move to either a positive or negative amplitude oscillation if there was a non-zero y-intercept. Negative amplitudes have no physical meaning because it is only the unsteady behavior about the mean chamber conditions that is being modeled. A system which spontaneously moves to a positive amplitude without any initial perturbation does not match the observed behavior of these motors since they all had to be forced into oscillations by a pressure pulse introduced into the combustion chamber.

Using a fourth-order model, equation 10 becomes

$$\frac{d\varepsilon}{dt} = \alpha_1\varepsilon + \alpha_2\varepsilon^2 + \alpha_3\varepsilon^3 + \alpha_4\varepsilon^4 \quad (13)$$

while equation 12 becomes

$$\frac{d\varepsilon}{dt} = \alpha\varepsilon(\varepsilon - r_1)(\varepsilon - r_2)(\varepsilon - r_3) \quad (14)$$

Multiplying out the right-hand side of equation 14, and setting the coefficients of like powers of ε in the right-hand sides of equations 13 and 14 equal to each other gives expressions relating the α_i to r_i :

$$\begin{aligned} \alpha_1 &= -\alpha r_1 r_2 r_3 \\ \alpha_2 &= \alpha(r_1 r_2 + r_1 r_3 + r_2 r_3) \\ \alpha_3 &= -\alpha(r_1 + r_2 + r_3) \\ \alpha_4 &= \alpha \end{aligned} \quad (15)$$

These equations relate the parameters determined by the motor design, the α_i , to the parameters which describe the key features of the system behavior, α and the r_i . These two sets of parameters are not independent, but instead two different set of parameters which describe the same behavior. If the α_i are known, equations 15 can be solved for α and the r_i and vice versa. A note will be made here on notation in the remainder of this paper. The parameter α , without a subscript, refers to the α in equation 14. This α will always be either α_N or $-\alpha_N$. Throughout this work α without a subscript will mean the α in equation 14, while a subscript will indicate a coefficient from equation 10.

Figure 4 shows the phase space of a fourth-order nonlinear system with two stable fixed points and one unstable non-zero fixed point. Perturbations to a point between the origin and r_1 cause the system to shift to operation at r_1 over time. If the system is perturbed to a point between r_1 and r_2 it will shift back towards r_1 . If it is perturbed to a point between r_2 and r_3 , the system grows to its new level at r_3 . Solutions to this system given initial conditions between r_1 and r_3 are shown in figure 5. As the system initial conditions progress past the value of r_2 , the steady-state solution changes from r_1 to r_3 . In a solid rocket motor a small pressure perturbation could come from a piece of propellant breaking off and passing through the nozzle, burning on the surface of a newly exposed crack in the propellant grain, pyrotechnic device, or other mechanism. This perturbation would be represented in figure 5 by a set of initial conditions at the appropriate value of ϵ . If this disturbance is strong enough, it will force a linearly stable motor into a limit-cycle oscillation, or it will force a linearly-unstable motor from one limit-cycle to a new limit-cycle at higher amplitude.

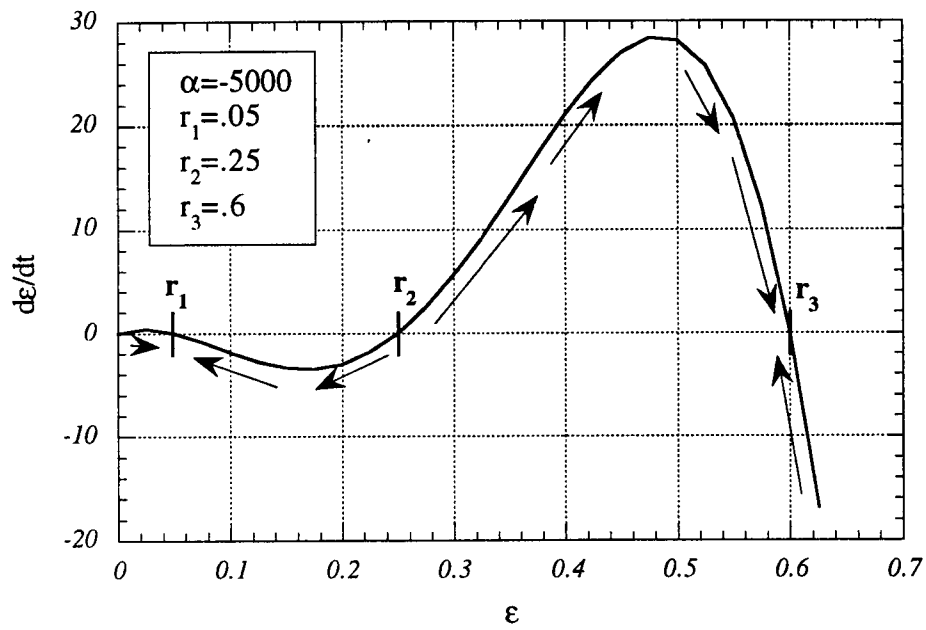


Figure 4. Phase Space of a Typical Fourth-Order System.

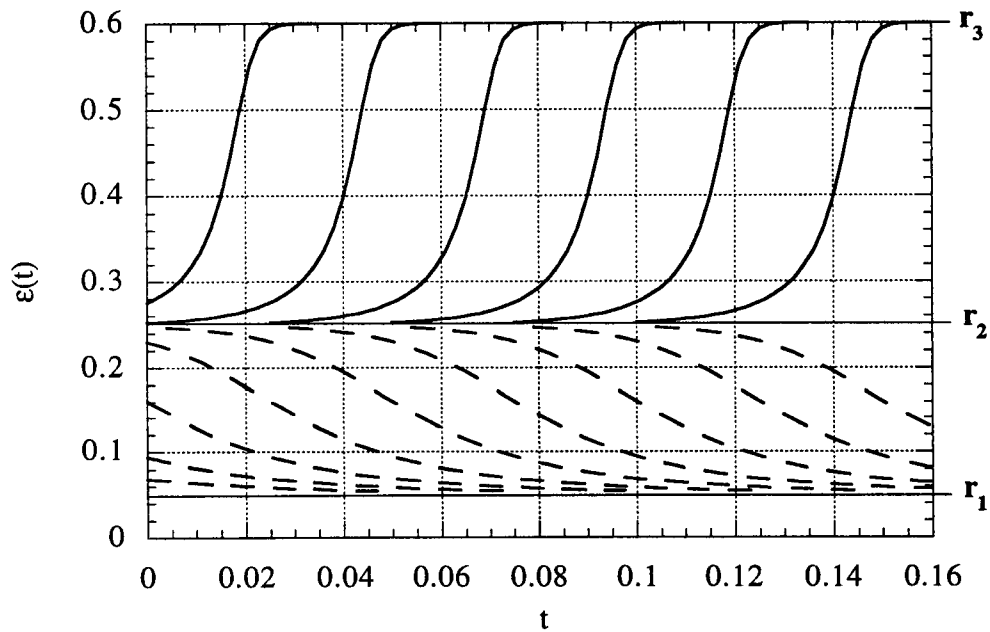


Figure 5. Solutions to the System in Figure 4.

1.4 NWC Test Program and Data

The experimental data used for this research comes from a multi-year project conducted by Blomshield, Crump, Mathes, and Beckstead.³ They fired 24 motors with different combinations of propellant, geometry, and chamber pressure to obtain data for nonlinear analysis of combustion stability. Appendix B lists the 24 motors fired. Most motors were pulsed three times during their burn by pyrotechnic devices to induce oscillations in the combustion chamber pressure. The pulses were planned to be fired at specific times, with specific pulse pressures. Due to inconsistencies in igniter performance, pulser performance, and miscellaneous technical problems, few of the actual pulse pressures and times met those specified by the test matrix. Of the 24 motors fired, 10 went into violent, large-amplitude (triggered) limit-cycle oscillations during firing. Table 1 lists those motors, while figure 6 shows the different geometries which were used in the testing. All these geometries were pulsed unstable at least once.

A subset of the motors in table 1 was used in this study. Only the star-aft, full-length motor configuration was tested enough to provide the quantity of useful data necessary for this study. It will be shown later that the data reduction methods require multiple pulse responses, both growth and decay responses, to accurately determine the α , which describe the motors' nonlinear behavior. Also, the limited availability of a complete set of data for some motors, like motor 8a, restricted the motors used to a smaller group: motor 2, 3, and 5. Motor 5 had the same configuration as motors 2 and 3 but was only pulsed once, at approximately 1 second into the burn. This motor provided an additional decay response which was combined with those from motors 2 and 3. These motors were all operated at a mean chamber pressure of 800 psi and used the NWR-11 propellant.

Table 1. NWC Motors Pulsed Unstable.

Motor	Configuration	Triggering Pulse	Time of Pulse
		Strength (psi)	Initiation (sec)
1	SAFT3	36	2.016
2	SAFT3	12	1.708
3	SAFT3	50	1.790
4	SFWD3	22	1.962
7	SFULL3	38	0.748
8a	SAFT3	10	3.020
8b	SAFT3	32	3.149
8c	SAFT3	20	0.617
9	CYL3	20	0.745
10	CYL3	36	0.887

Source: Blomshield, F.S., H.B. Mathes, J.E. Crump, and M.W. Beckstead, *Stability Testing and Pulsing of Full Scale Tactical Motors, Part 1. Final Report*, NAWCWPNS TP 8060, Naval Air Warfare Center Weapons Division, China Lake, CA, Feb. 1996

Notes: SAFT3—Full length, star aft motor; SFWD3—Full length, star forward motor; SFULL3—Full star, full length motor; CYL3—Full cylinder, full length motor; see figure 6 for more configuration details; Pulse times measured from initiation of firing

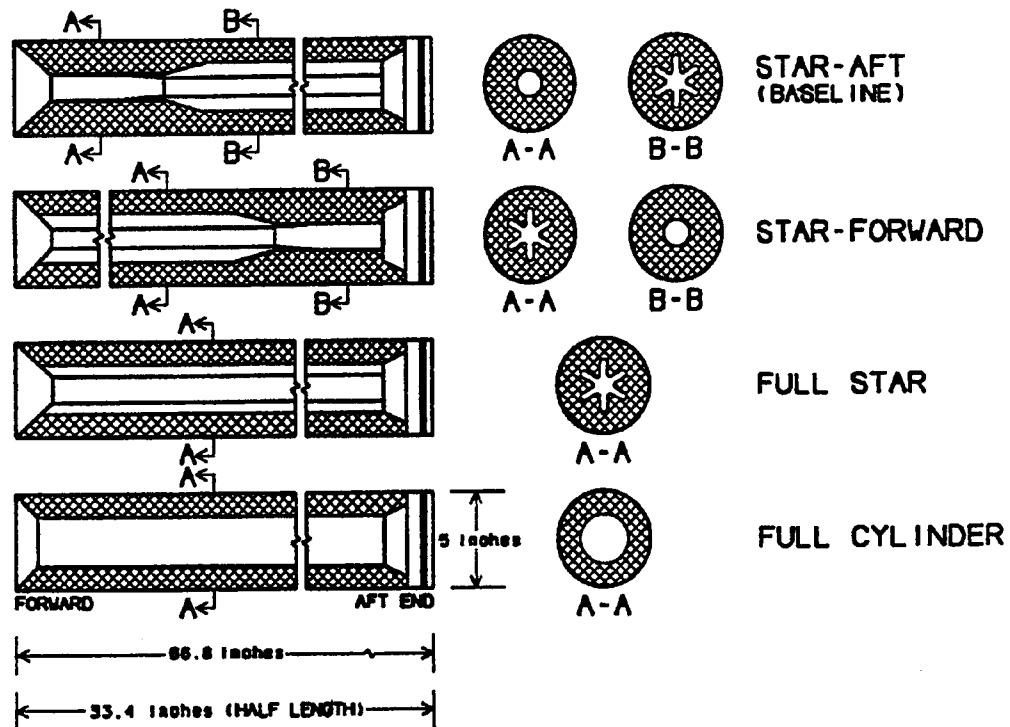


Figure 6. Motors Configurations Tested in NWC Study.

Source: Blomshield, F.S., H.B. Mathes, J.E. Crump, and M.W. Beckstead, *Stability Testing and Pulsing of Full Scale Tactical Motors, Part 1. Final Report*, NAWCWPNS TP 8060, Naval Air Warfare Center Weapons Division, China Lake, CA, Feb. 1996

2 Data Reduction Approaches

This chapter describes the data analysis that was done, how it was done, and why it was done. The first section provides an overview of the process. The following sections describe the two data reduction methods in more detail.

2.1 Analysis Approach

This work attempted to reduce motor data by applying the nonlinear theory described in Chapter 1 to data from a recent motor test program, and to investigate the possibility that a fourth-order, not a third-order, form of equation 10 describes the behavior of those motors. The data reduction was performed using two methods. The first approach, called the direct approach, attempts to fit the analytical solution of equation 10 to the data. The second method, called the integral method, uses integrals of the terms in equation 10 to find the values of the α_i . Determining whether or not the motors were operating as fourth-order systems was done using the results of the data reduction.

Although one goal of this project was application of the fourth-order model to reduction of real data, second- and third-order curve fits were also made. They provided a means for analyzing the procedures and allowed comparisons to check the results from the fourth-order analysis. Also, while a second-order system does not accurately represent the behavior of the motors, a third-order system can represent the behavior if one neglects low-amplitude oscillations. This is the view taken in the NWC report which stated that low level acoustic oscillations at frequencies in the vicinity of the first acoustic mode did not merit intense study.³

In an effort to investigate the poor results from these two data reduction methods, which are described in chapter 3, and to assess the requirements these methods place on

the types and amount of data collected, three somewhat different approaches were undertaken in this study to assess the sensitivity of the models to the data. First, a parameter study was carried out to investigate the effects of small variations in α and the r_i on the solution to equation 10. Second, a sensitivity analysis was performed using the integral method, and third, some noisy test data was generated and passed through the two routines to examine the impact of noise and the way data sets were grouped on the values of α and the r_i returned from the data reduction methods.

2.2 Direct Solution Method

The direct solution method coupled the analytical solution to equation 10 with a Levenberg-Marquardt nonlinear curve-fitting algorithm. An overview of that curve-fitting technique is given in Appendix C and more detail, as well as sample code, is given by Press, Teukolsky, Vetterling, and Flannery.⁷ Data was taken and incorporated into an input file. This file included all the data sets for a particular attempt. A program called Lmfit performed the curve-fit on the data in the data file. The program provided estimates for each iteration on the parameters being determined, as well as the final values and covariances. Parameter values could be fixed in the input file to reduce the number of degrees-of-freedom in a problem. The covariances are a measure of the uncertainty in a parameter and should be small for a reliable value.

Equation 10 is a first-order, separable, nonlinear equation, and can be solved analytically. To demonstrate the process a fourth-order system will be used, but the solution process is the same for any N in the right-hand side in equation 10. Beginning with equation 13, the solution can be carried out by dividing equation 13 by the polynomial right-hand side and then factoring the polynomial. Multiplying both sides by αdt gives

$$\left[\frac{I}{(\varepsilon - r_1)(\varepsilon - r_2)(\varepsilon - r_3)} \right] \frac{d\varepsilon}{\varepsilon} = \alpha dt \quad (16)$$

The left-hand side can be broken into a sum using partial fractions

$$\left(\frac{A}{\varepsilon - r_1} + \frac{B}{\varepsilon - r_2} + \frac{C}{\varepsilon - r_3} \right) \frac{d\varepsilon}{\varepsilon} = \alpha dt \quad (17)$$

The coefficients A , B , and C can be found by combining the terms in the parentheses and setting the numerator of the result equal to the numerator in the brackets in equation 16, giving

$$\begin{aligned} A + B + C &= 0 \\ -A(r_2 + r_3) - B(r_1 + r_3) - C(r_1 + r_2) &= 0 \\ Ar_2r_3 + Br_1r_3 + Cr_1r_2 &= I \end{aligned} \quad (18)$$

The solution to this set of equations is

$$\begin{aligned} A &= \frac{I}{(r_1 - r_2)(r_1 - r_3)} \\ B &= \frac{I}{(r_2 - r_1)(r_2 - r_3)} \\ C &= \frac{I}{(r_3 - r_1)(r_3 - r_2)} \end{aligned} \quad (19)$$

Each term can then be integrated individually. The solution for the integration of one term is given by

$$A \int_{\varepsilon_0}^{\varepsilon} \frac{d\varepsilon}{\varepsilon(\varepsilon - r_1)} = \frac{A}{r_1} \ln \left| \frac{\varepsilon_0(\varepsilon - r_1)}{\varepsilon(\varepsilon_0 - r_1)} \right| \quad (20)$$

This solution is singular at $\varepsilon=r_1$. Examination of equation 10 shows when the system is operating at one of its roots the derivative will be zero and the motor will remain at that root. Once perturbed from the root, by a pressure pulse for example, the system begins behaving by equation 20. The resulting solution for a fourth-order system is

$$\frac{\ln \left| \frac{\varepsilon_0(\varepsilon - r_1)}{\varepsilon(\varepsilon_0 - r_1)} \right|}{r_1(r_1 - r_2)(r_1 - r_3)} + \frac{\ln \left| \frac{\varepsilon_0(\varepsilon - r_2)}{\varepsilon(\varepsilon_0 - r_2)} \right|}{r_2(r_2 - r_1)(r_2 - r_3)} + \frac{\ln \left| \frac{\varepsilon_0(\varepsilon - r_3)}{\varepsilon(\varepsilon_0 - r_3)} \right|}{r_3(r_3 - r_1)(r_3 - r_2)} = \alpha(t - t_0) \quad (21)$$

where ε_0 and t_0 are the initial perturbation pressure amplitude and initial time. The solution for an N^{th} -order polynomial right-hand side for equation 10 is given by

$$\sum_{i=1}^N \frac{\ln \left| \frac{\varepsilon_0(\varepsilon - r_i)}{\varepsilon(\varepsilon_0 - r_i)} \right|}{r_i \prod_{\substack{j=1 \\ j \neq i}}^N (r_i - r_j)} = \alpha(t - t_0) \quad (22)$$

These nonlinear algebraic equations cannot be solved explicitly for ε , but can easily be solved to express time as a function of amplitude. In using this equation in the program Lmfit a combination of Regula-Falsi and bisection root solving algorithms were used to find ε at specific times.

From a data reduction perspective, the direct method offers a simple procedure for reducing test data. While this method does not place any explicit limits on the number of test required, it was found in the reduction that this method requires one approach to each

limit-cycle in the system be contained in the data set. The behavior of the solution in the vicinity of a fixed point r_i is governed primarily by the leading coefficient α , the fixed point r_i . For a fourth-order system this means at least two tests are required to determine the r_i and α , with the two tests bracketing the triggering amplitude. That is, the two tests should have different pulse strengths, one which induces a large-amplitude limit-cycle oscillation, and another which decays to the low-amplitude limit-cycle.

At this point it should be noted that in a solid rocket motor the geometry, mean chamber pressure, and thus α_i are changing throughout the burn time. This means α and the r_i are not constants, but vary with time. Therefore an additional restriction on the testing is that for data sets from different pulses to be combined to find a particular set of α and r_i , the pulses should be at the same or similar time, and thus, must come from two separate motor firings. By finding an estimate of the r_i and α_i using data at one point in the motor burn, and then finding separate estimates using pulse responses from a different point in the motor burn, a picture of the time dependence of the r_i and α_i can be found.

2.3 Integral Method

An alternative to using data with the analytical solution directly is to integrate each term in equation 10 separately.⁸ The integration can remedy noise in the data, but potentially requires more testing than the direct method because it reduces a series of data points to a single value. Integrating equation 10 term by term gives

$$\ln \left| \frac{\varepsilon}{\varepsilon_0} \right| = \sum_{i=1}^n \alpha_i I_i \quad (23)$$

$$I_i = \int_{t_0}^t \varepsilon^{i-1} dt$$

which becomes the following for a fourth-order system:

$$\begin{aligned}
\ln \left| \frac{\varepsilon_f}{\varepsilon_0} \right| &= \alpha_1 I_1 + \alpha_2 I_2 + \alpha_3 I_3 + \alpha_4 I_4 \\
I_1 &= (t_f - t_0) \\
I_2 &= \int_{t_0}^{t_f} \varepsilon dt \\
I_3 &= \int_{t_0}^{t_f} \varepsilon^2 dt \\
I_4 &= \int_{t_0}^{t_f} \varepsilon^3 dt
\end{aligned} \tag{24}$$

where ε_f and t_f are the oscillation amplitude and time, respectively, at the end of the pulse response.

Each pulse response is reduced to one equation, and several responses are grouped together to form a system of equations

$$\begin{bmatrix} I_{11} & I_{21} & I_{31} & I_{41} \\ I_{12} & I_{22} & I_{32} & I_{42} \\ I_{13} & I_{23} & I_{33} & I_{43} \\ I_{14} & I_{24} & I_{34} & I_{44} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} \ln |\varepsilon_f / \varepsilon_0|_1 \\ \ln |\varepsilon_f / \varepsilon_0|_2 \\ \ln |\varepsilon_f / \varepsilon_0|_3 \\ \ln |\varepsilon_f / \varepsilon_0|_4 \end{bmatrix} \tag{25}$$

where I_{ij} is the integral of the i^{th} term for pulse j , and $\ln(\varepsilon_f / \varepsilon_0)_j$ is $\ln(\varepsilon_f / \varepsilon_0)$ from pulse response j . The number of variables can be reduced by determining the limit-cycle amplitudes separately, for instance by measuring the amplitude directly from a plot like figure 1 or 2. The α_i can be replaced with expressions involving r_i using equations 15 and the limit-cycle amplitudes can then be substituted into the new equation. This does, however, require accurate measurement of the limit-cycle amplitudes. This technique to reduce the number of free parameters was used in this work.

The integral method as implemented solved the system of equations formed by application of equation 23 to the data. The method was implemented as a *Mathematica* "notebook" and typically required at least one root to be set because of a lack of sufficient

repetition in the test performed in the NWC program. The data sets were combined in the notebook, which then performed the integration and solution of the equations for the parameters α and r_i .

One advantage of the integral method is that overlap of data from growth and decay pulses does not cause numerical problems as it does in the direct method. This is because each set of data is transformed into integrals and then used to solve for the parameters, rather than trying to fit a model simultaneously to the original data. When data from different pulses which had overlapping initial amplitudes are combined, this method provides an advantage in flexibility.

3 Motor Data Analysis

This chapter describes the analysis performed for this work. The chapter begins with descriptions of the results of the data reduction that was performed, followed by the parameter study, sensitivity analysis, and results of testing with artificial data. It concludes with analysis of the potential that the behavior of the motors could be modeled by a fourth-order form of equation 10.

3.1 Direct Results

Table 2 summarizes the results obtained by the author from second-, and third-order analysis using the integral method when attempting to fit a new curve to each pulse. From the table it is readily apparent that second-order curve fit with this data did not work. Convergence of the curve-fit routine did not occur when trying to fit a curve to a decay response, and the first-order term of the fits to growing pulses has the wrong sign. This is not surprising since the motor behavior and the qualitative behavior of a second-order system do not match. A second-order system will only have a single non-zero root, giving the system either a trigger point or limit-cycle. For the growth data the system has already been perturbed to a point from which it will enter a limit-cycle oscillation. Since the information which is provided by the pulse response data is clustered around the limit-cycle, that becomes the feature which determines the value of the fixed-point. The result is a linearly-unstable system with a limit-cycle oscillation. Since the system had to be perturbed to induce the limit-cycle, the linearly-unstable model is not correct.

In the case of the decaying pulses, the data decays to either a low limit-cycle or to the root at zero amplitude. A low-amplitude limit-cycle would require the same inconsistency as a high limit-cycle, namely a linearly-unstable system. A decay to the

Table 2. Results of Direct Method Applied to Individual Pulse Response Data Sets.

Motor and Pulse	Order	α	r_1	r_2	α_1	α_2	α_3
Motor 2, Pulse 1	2			No convergence			
Motor 2, Pulse 2	2	-320	0.46		147.2	-320	
Motor 3, Pulse 1	2			No convergence			
Motor 3, Pulse 2	2	-192.9	0.574		110.725	-192.9	
Motor 5, Pulse 1	2			No convergence			
Motor 2, Pulse 1	3			No convergence			
Motor 2, Pulse 2	3	-816	-0.1643	0.458	61.4035	239.659	-816
Motor 3, Pulse 1	3	2.11×10^6	0.00499	0.0221	233	-57160	2.11×10^6
Motor 3, Pulse 2	3	-676	0.0733	0.570	-28.244	434.871	-676
Motor 5, Pulse 1	3			No convergence			

root at the origin is representative of a first-order system, and lacking any data which provides a broader picture of the system behavior, the curve fit algorithm tries to force a second-order model to the first-order behavior.

A third-order system offers the chance to model both behaviors of the motor, limit-cycles and triggering, but it also requires more data. This is shown in the unacceptable results of third-order systems in table 2. When a fit was attempted with the response to pulse 2 in motor 2, a negative root was returned. While this by itself is not a problem, it does not fit the behavior of the physical system because non-zero roots must be positive to exhibit both triggering and limit-cycle behavior like the NWC motors. The resulting linear stability is positive, not negative, which is also inconsistent with the behavior of the motors. Using the data from pulse 1 motor 3 resulted in a curve for which both roots are positive, but the resulting system is linearly unstable because it tries to model the apparent fixed point at $\epsilon=.005$. The trigger value is higher than the limit-cycle, the reverse of what the data showed. Data from pulse 2 motor 3 gave the most plausible solution in this set of attempts. The curve fit produced a fixed point at $\epsilon=.57$, a triggering value which is positive, and a linearly stable solution. While this fit is accurate qualitatively, it cannot be reliably determined if the fit is a valid quantitative model of the motor nonlinear stability at this point. The linear stability term computed by the curve fit is -28.2 sec^{-1} . The SSP-predicted linear stability at this time in the burn was -84.1 sec^{-1} . While these results are the same order of magnitude, the SSP linear stability is still three times that obtained by the curve fit. Table 3 lists the covariances of the different parameters. The covariances provide a measure of the uncertainty of the parameters returned by the Levenberg-Marquardt algorithm. Smaller values, normally less than one, are better, and indicate a reliable estimate. The size of the covariances for α and r_I indicate unreliable estimates, very unreliable in the case of α . Some of the difference can be attributed to the lack of data in the vicinity of the trigger point. This leads to the high

uncertainty in the trigger value and α , which in turn can help explain the large difference between the SSP linear stability and the third-order fit's linear stability. Another factor which may or may not invalidate this third-order fit is that the computed trigger value is lower than the highest amplitude in a decay response from this motor, motor 3. That means the fit predicts that the initial point in the decay response should cause growth to a limit-cycle, rather than decay towards zero, which is what actually happened. This does not necessarily mean the fit is bad because the trigger points and limit-cycle oscillations will vary with time during the burn just as the linear stability varies during the burn. It would indicate, however that the fit is a view of the stability at the time that pulse 2 was fired and not a reliable guide for determining if an arbitrary pulse strength will decay or trigger growth to a limit-cycle. Further data would have to be obtained to make firm conclusions about the reliability of the fit.

Table 3. Covariances for 3rd-Order Curve Fit Using the Direct Method.

Variable	Value	Covariance
α	-676	29200
r_1	.0733	11.60
r_2	.570	0.377

Curve fits were also made using combinations of data sets from different pulses and motors, though the motors were all of the same grain type; for example, data from a star-forward, full-length motor should not be combined with data from a star-aft motor. Table 4 lists the results for third- and fourth-order models applied to composite data sets obtained by combining data from different pulses in the same motor; for example, combining the pulse 1 decay data and pulse 2 growth data to obtain a single curve fit. The motivation for using combined data was to provide data which would describe the behavior of the system more completely, not just in the vicinity of a single fixed-point.

Table 4. Direct Method Results Using Combinations of Pulse Responses From Each Motor.

Motor	Order	α	r_1	r_2	r_3	α_1	α_2	α_3	α_4
Motor 2	3	-1110	-0.0664	0.458	N/A	33.7564	434.676	-1110	N/A
Motor 3	3	-880	0.134	0.57	N/A	-67.2144	619.52	-880	N/A
Motor 2	3				No convergence				
Motor 3	3	-875	0.1334	0.57*	N/A	-66.5332	615.475	-875	N/A
Motor 2	3	-1110	-0.0663	0.458*	N/A	33.7056	434.787	-1110	N/A
Motor 3	3				No convergence				
Motor 2	3	-455	-0.29	0.514*	N/A	67.8223	101.92	-455	N/A
Motor 3	3	-4658	0.255	0.514*	N/A	-610.524	3582	-4658	N/A
Motor 2	4				No convergence				
Motor 3	4				No convergence				
Motor 2	4				No convergence				
Motor 3	4				No convergence				
Motor 2	4	-13190	-1.035	0.0809	0.514*	-567.671	7572.89	-5804.92	-13190
Motor 3	4	-18210	-0.0232	0.26	0.514*	-56.4592	-2106.59	13672.1	-18210
Motor 2	4				No convergence				
Motor 3	4				No convergence				

Note: All fits are to combinations of pulse 1 and 2 responses from the indicated motor; * indicates root which was preset when the fit was attempted

Fits were made where all parameters were determined, as well as where some values of were preset in an effort to improve the chances of a reliable fit. This type of combination data set suffers from a potential flaw. It was mentioned before that the values of the parameters are assumed to vary with time. This means that the data from two pulses in the same motor describe two different states of the motor. Combining data sets from two pulses in the same motor will give an approximation which combines the behavior of the motor at two different times into one set of parameters. While useful, a fit made in this way will still not provide a reliable, quantitative means of predicting a motor's response to a particular pulse strength. It will provide a qualitative guide to the behavior of the motor.

Results from combining data from each motor into a single set for each motor provided mixed results. Third-order models using motor 2 data generally failed. When no roots were fixed, the resulting fit has a negative root which does not agree with the behavior of the motor, as explained above. When a root was set to .570, the value obtained earlier for the limit-cycle in motor 3, the nonlinear curve fit routine did not converge at all. When a value of .514 was used instead of .570, the resulting fit again had a negative root. Both fits that converged were linearly unstable, a condition which is contradicted by the motor behavior. A value of .514 was used because that is the average of the limit-cycle amplitudes from motors 2 and 3. The third-order motor 3 data shows somewhat better results. While the fit did not converge when the data limit-cycle amplitude was set to .458, the value in motor 2, both of the other attempts converged to systems with two positive root and negative linear stability terms. When all the parameters were allowed to vary the resulting linear stability term was -67.2 sec^{-1} , closer to the SSP predictions reported in the NWC report.³ When the limit-cycle amplitude is set to .514, the linear stability term grows by almost a factor of 10, to -611 sec^{-1} . This is

much more stable than predicted by the linear analysis done by the NWC and by the motor tests.

Fourth-order fits were less successful using the combined motor data. Of all the attempts made to generate a fourth-order curve fit only two attempts were successful, and both had one root forced to a value of .514. The fit using data from motor 2 produced a positive root and a negative root. Because the fourth-order system includes three non-zero fixed-points it is possible to model a system which has triggering and a high limit-cycle as well as a system which possesses a low limit-cycle, triggering, and high limit-cycle. The results from the successful fourth-order motor 2 fit demonstrate behavior which can be modeled by a third-order system. Only two non-zero roots are positive, which is a situation which can be modeled by a third-order system as well as a fourth-order system. The presence of a negative root could indicate that a fourth-order representation has too many unknown parameters compared to the third-order system. The computed trigger value of .0809 is low compared to the initial amplitudes of the decay responses. However, since this fit is based on what is likely to be a combination of two different systems this is not too surprising. The linear stability term is again much larger than that computed in the NWC linear analysis, which also tends to indicate this is not a reliable fit. When a fourth-order fit was applied to data from motor 3 the results were more encouraging. While the curve again has a negative root, which means this system has only one limit-cycle, this root is smaller by two orders of magnitude than the negative root from the motor 2 fit. The system has a trigger point at $\epsilon=.260$ and the linear stability term is the same order of magnitude as those in the NWC analysis. Comparing this curve to the best third-order fit made to this data shows fair agreement in the linear stability term, but the trigger points are quite different. Comparing the higher order terms shows the coefficients of ϵ^2 in the fourth-order fit is about 3.4 times the size of the same coefficient in the third-order fit and the coefficient of ϵ^3 is over 15 times the value of the

same coefficient in the third-order fit. It seems the data combined in this fashion also tends to be insufficient to produce a reliable fourth-order fit.

A problem which made using these multiple data sets more difficult was the general lack of triggering dependence on pulse strength, as reported by NWC. The motor 2 decay data began at a higher value of ϵ than that of the motor 2 growth data. To get around this problem, the first data point from the decay data set was removed, artificially creating a region for a trigger point. The triggering amplitudes in the fits from motor 2 data all lie in this region which is between the highest point in a decay data set and the lowest point in a growth data set. The motor 3 decay and growth data do not overlap, and so do not need any modification to accommodate a trigger point.

Another attempt at combining multiple data sets to find a curve was made by combining all data from pulse 1 and all data from pulse 2 into two data sets, and the results are shown in table 5. This is a better method of combining data in the sense that the different data sets which are used together are all from approximately the same time, however, it suffers from a problem similar to that encountered when using the single data sets--the data describes only one portion of the system's total behavior and high-order curve fits have too many free parameters to make an accurate fit to such limited information. This problem likely caused the failure of three of the attempts. The only attempt which succeeded was a third-order fit to the data from the second pulse. Unfortunately the resulting fit has a single positive root and is linearly unstable, both conditions contrary to the actual motor behavior.

3.2 Integral Results

A method by Flandro which is less susceptible to problems arising from noise in the data is to use integrals of the individual terms in equation 10 as described in section 2.3.⁸ Table 6 lists the results using this method. Solution attempts were first made combining

Table 5. Direct Method Results Using Combinations of Pulse Responses from Different Motors.

Pulse	Motors	Order	α	r_1	r_2	r_3	α_1	α_2	α_3	α_4
Pulse 2	2 & 3	3	-209	-0.927	0.541003	N/A	105.0	-80.8	-209	N/A
Pulse 1	2, 3, & 5	3				No convergence				
Pulse 2	2 & 3	4				No convergence				
Pulse 1	2, 3, & 5	4				No convergence				

Table 6. Integral Method Results.

Motor	Pulse	Order	α	r_1	r_2	r_3	α_1	α_2	α_3	α_4
Motor 2	1 & 2	2	221	0.343	N/A	N/A	-75.9	221	N/A	N/A
Motor 3	1 & 2	2	100.4	0.457	N/A	N/A	-45.9	100.4	N/A	N/A
Motor 2	1 & 2	3	-2340	0.0756	0.458*	N/A	-80.9	1247	-2340	N/A
Motor 3	1 & 2	3	-772	0.1111	0.57*	N/A	-48.9	526	-772	N/A
Motor 2	1 & 2	3	-927	0.1633	0.514*	N/A	-77.9	628	-927	N/A
Motor 3	1 & 2	3	377	-0.229	0.514*	N/A	-44.4	-107.6	377	N/A
Motor 2	1 & 2	4	-1.11x10 ⁶	0.00425*	0.216	0.458*	465	-112700	751000	-1.11x10 ⁶
Motor 3	1 & 2	4	-73000	0.00425*	0.379	0.570*	67	-16060	69600	-73000
Motor 2	1 & 2	4	-564000	0.00425*	0.368	0.514*	453	-108800	500000	-564000
Motor 3	1 & 2	4	-43700	0.00425*	0.693	0.514*	66.2	-15800	53000	-43700
Motors 2, 3, & 5	1	3	-448000	0.01051	0.0343	N/A	-161.4	20000	-448000	N/A
Motors 2, 3, & 5	1	4	1.55x10 ⁹	0.00425*	0.01083	0.0656	-4680	1.60x10 ⁶	-1.25x10 ⁸	1.55x10 ⁹
Motors 2, 3, & 5	1	4	939000	0.0105	0.0324	0.541*	-164.2	21000	-523000	939000

Note: * indicates root which was preset when the fit was attempted

different pulse responses from the same motor. When determining the values for a second-order system, the method returned reasonable results, although the suitability of a second-order system in this case is questionable. The second-order fits both give positive roots and negative linear stability terms. The linear stabilities are of the same order of magnitude as the NWC computations and lie close to the linear stabilities in the more useful fits in the direct analysis. Third-order fits required one parameter to be set since each motor used in this analysis had only two pulses, giving only two equations. Four attempts were made: one with motor 2 data and a root set to .458, the limit-cycle amplitude determined from one of the direct solutions; one with motor 3 data and a root set to .570; and two with a root set to .514, one each with the motor 2 and 3 data. Three of the curves provided two positive roots with each having a trigger value between .075 and .16. The attempts using motor 3 data with a root set to .514 gave a negative value for the second root, something which, as stated before, is not fully consistent with the observed motor behavior. All four methods have negative linear stability terms of the same order of magnitude as the linear analysis done by NWC.

The fourth-order fits generally fell into the pattern of a linearly unstable motor with a low-amplitude limit-cycle. All fits are linearly unstable with a low limit-cycle set to .00425. This value is between the motor 2 and 3 final decay amplitudes and served as a reasonable estimate of a possible low-amplitude limit-cycle. At 800 psi, this is a 3.4 psi deviation in the combustion chamber pressure. A pressure disturbance this small should not be difficult to induce, and the motor could send itself into this low limit-cycle just from variations in propellant burning, propellant casting, or ignition transients. Three of the four curve fits have reasonable trigger values. Attempts with a fixed-point set to .514 and using data from motor 3 produced a root above .514, making .514 a trigger value and .693 the new high stable fixed point. This is not consistent with the motor behavior. Examination of the higher-order coefficients revealed large absolute values, between 10^4

and 10^6 . The source of these large quantities is the coefficient of the fourth-order term, which has values from -5×10^5 and -10^6 . This term is likely this high as a result of the rapid decay of the pulse response in the motors; the decays were much faster than in the growth pulses.

Attempts at combining data from the same pulse were also marginally successful. All these fits had negative linear stability terms which made the first and third positive roots trigger points rather than limit-cycle oscillations. Also, the higher-order coefficients were all very high with absolute values ranging from 2×10^4 to 1.5×10^9 .

3.3 Parameter Study

A parameter study provided some guidance on the problem of determining reasons for the lack of reliable results from the curve fits, and provided some insight into the requirements for future experimental programs. The following data were used as a baseline model from which to study the impact of different parameters:

$$\begin{array}{ll} \alpha = -5000 & \alpha_1 = 37.5 \\ r_1 = .05 & \alpha_2 = -962.5 \\ r_2 = .25 & \alpha_3 = 4500 \\ r_3 = .6 & \alpha_4 = -5000 \end{array}$$

The effects of varying α , r_1 , and r_2 individually on a growth to the high fixed-point are shown in figures 7, 8, and 9, respectively. Each was varied by 10% from its baseline value. Increasing values of r_1 cause the growth to become less rapid, but the change is barely noticeable in figure 8. This is a reflection of the increasing influence r_1 has on the solution around r_3 as r_1 approaches r_3 . Increasing values of α also served to make growth more rapid, but with the changes much more pronounced than those caused by varying r_1 . Increasing values of r_2 have the effect of decreasing the pace of the solution's

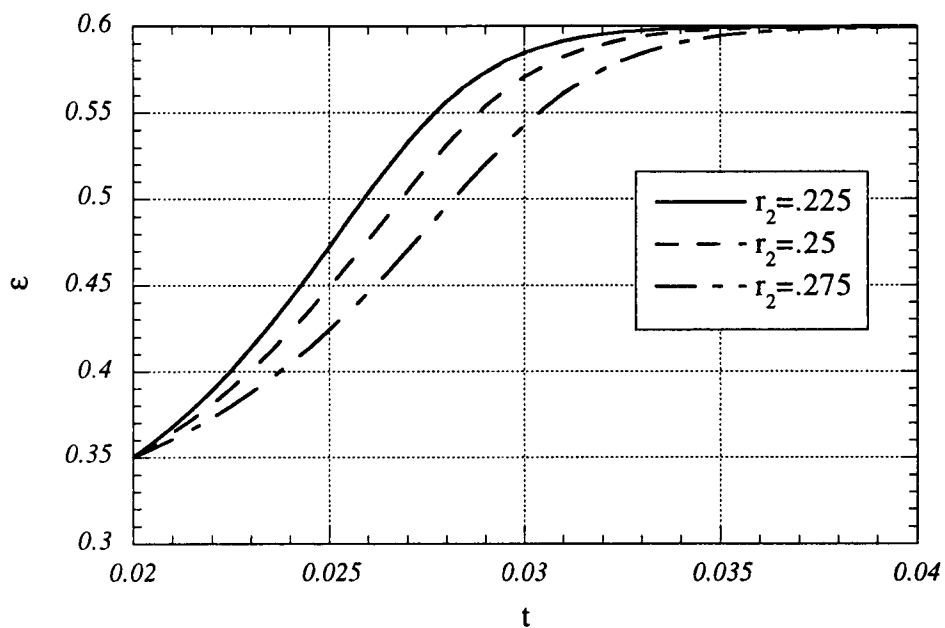


Figure 7. $\varepsilon(t)$ for Values of r_2 between .225 and .275. $\varepsilon_0=.35$, $t_0=.02$

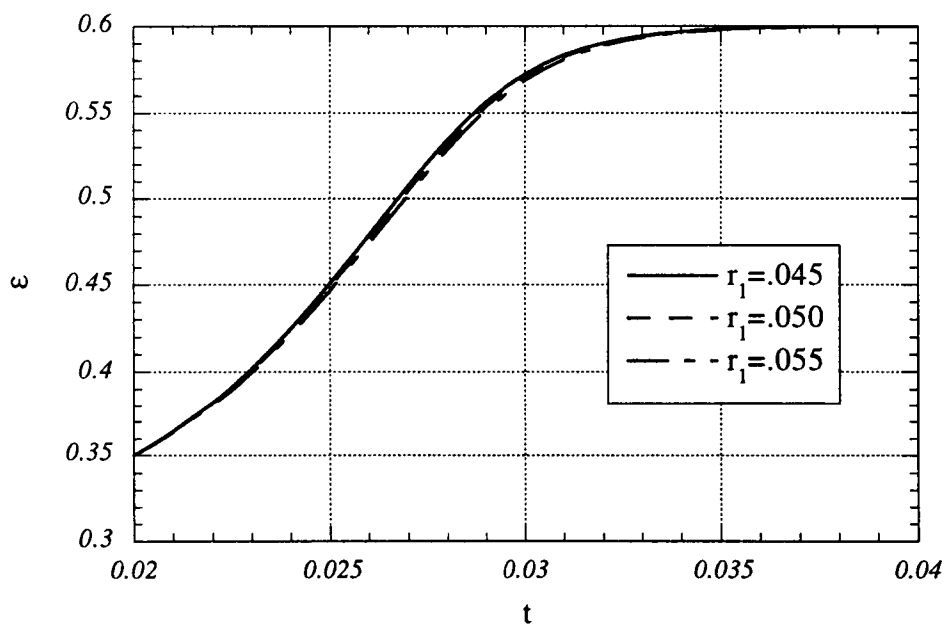


Figure 8. $\varepsilon(t)$ for Values of r_1 between .045 and .055. $\varepsilon_0=.35$, $t_0=.02$

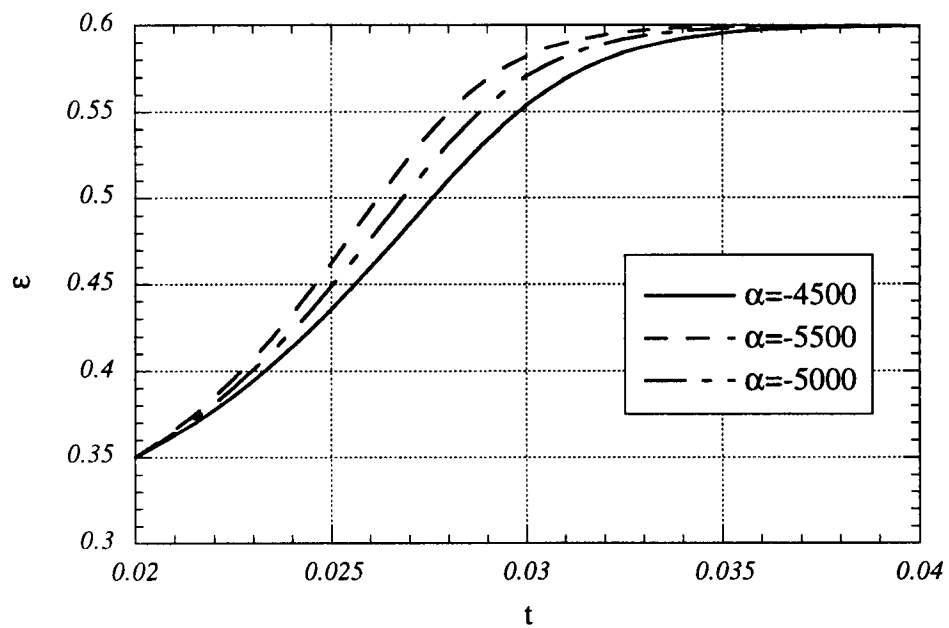


Figure 9. $\varepsilon(t)$ for Values of α between -4500 and -5500. $\varepsilon_0=.35$, $t_0=.02$

growth to r_3 . Comparing figure 7 to figure 9 one notices a similarity in the changes to the solutions—both increasing and decreasing r_2 cause the solution to progress more rapidly to r_3 from its initial condition. A curve-fitting method must deal with this issue and not having sufficient data will lead to poor solutions with a fourth-order system. The algorithm must determine if change should be made by adjusting α or r_2 . The closer the initial point is to r_2 , the easier this will be. This is because changes to the solution shape caused by changes in r_2 and α become different as the initial condition approaches r_2 .

Similar effects can be found in solutions decaying to the low fixed-point. Figures 10, 11, and 12 show solutions when α , r_2 , and r_3 are changed. Increasing values of all three parameters cause the solution to approach r_1 more slowly. The change from r_2 and r_3 is caused by the decreasing influence of those roots on the solution about r_1 as r_2 and r_3 move away. Since α is a coefficient of the whole system it affects all solutions the same way, causing the transients to happen over a shorter period of time as the value gets larger. The problem of discerning differences in how the parameters affect the solution occurs here, with the added complication that a 10% change in r_3 has a noticeable effect, similar to a slightly smaller percentage change in α .

Combining these observations allows the different effects of changes in the parameters to be seen. Table 7 summarizes the patterns. Examination of the table allows much better choices to be made about which parameters should be changed. This is the motivation behind combining growth and decay data sets. The greater variety of data allows the curve fitting algorithm to more precisely decide which parameter to change.

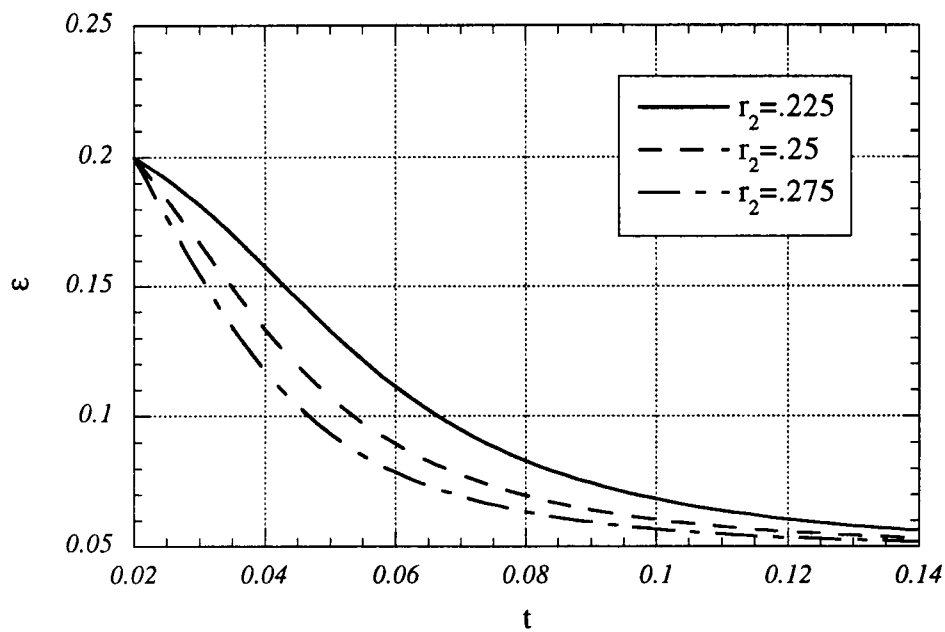


Figure 10. $\varepsilon(t)$ for Values of r_2 between .225 and .275. $\varepsilon_0=.2$, $t_0=.02$

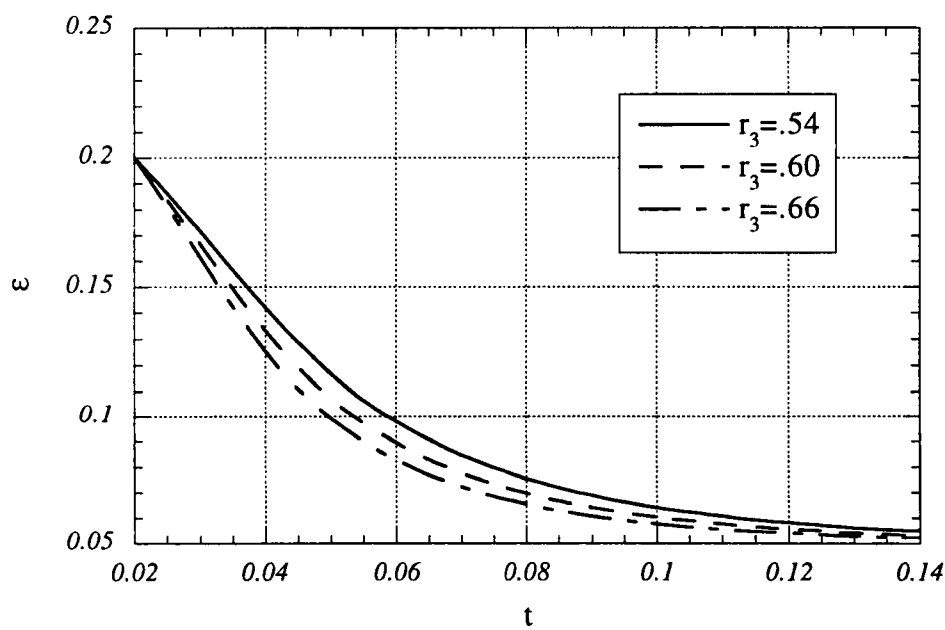


Figure 11. $\varepsilon(t)$ for Values of r_3 between .045 and .055. $\varepsilon_0=.2$, $t_0=.02$

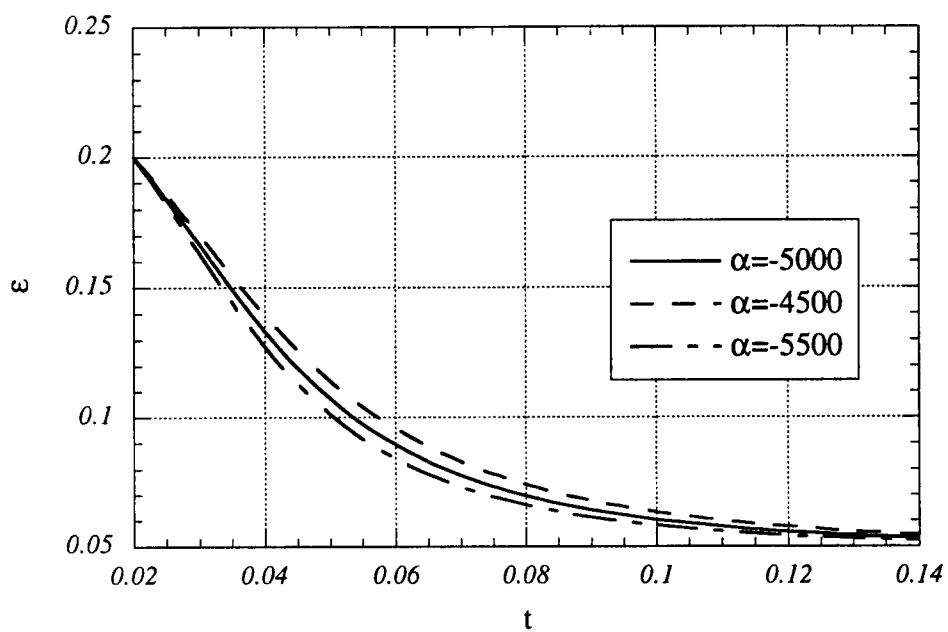


Figure 12. $\varepsilon(t)$ for Values of α between -4500 and -5500. $\varepsilon_0=.2$, $t_0=.02$

Table 7. Patterns in Fourth-Order Solution Behavior.

Parameter Changing	ε in the vicinity of r_1	ε in the vicinity of r_3
r_1 Increasing	Increasing	Decreasing
r_2 Increasing	Decreasing	Decreasing
r_3 Increasing	Decreasing	Increasing
α Increasing	Increasing	Decreasing

3.4 Sensitivity Analysis

Given a function $\varepsilon(t)$ and the equations

$$I_i = \int_{t_0}^{t_f} \varepsilon^i(t) dt$$

$$\ln \left| \frac{\varepsilon_f}{\varepsilon_0} \right| = \sum_{i=1}^n \alpha_i I_i \quad (26)$$

a single α_i can be isolated, yielding

$$\alpha_i = \frac{1}{I_i} \left(\ln \left| \frac{\varepsilon_f}{\varepsilon_0} \right| - \sum_{\substack{j=1 \\ j \neq i}}^n \alpha_j I_j \right) \quad (27)$$

Taking the derivative with respect to each of the integrals $I_i, i=1, \dots, n$, gives the sensitivities

$$\frac{d\alpha_i}{dI_j} = \begin{cases} \frac{1}{I_i^2} \left(\sum_{\substack{k=1 \\ k \neq i}}^n \alpha_k I_k - \ln \left| \frac{\varepsilon_f}{\varepsilon_0} \right| \right) & i = j \\ -\alpha_i / I_i & i \neq j \end{cases} \quad (28)$$

To find the sensitivities of the system fixed points to changes in the I_i use the chain rule to get

$$\frac{dr_i}{dI_j} = \sum_{k=1}^n \frac{dr_i}{d\alpha_k} \frac{d\alpha_k}{dI_j} \quad (29)$$

Where the individual $dr_i/d\alpha_k$ are determined from equations 15.

3.5 Results of Noisy Test Data

Four sets of noisy test data were generated to determine the impact noise would have when using multiple pulse responses of the same type. In other words, can noise render combinations of multiple growths or decays near useless? The test data was generated using the following system parameters:

$$\begin{array}{ll} \alpha = -5000 & \alpha_1 = 37.5 \\ r_1 = .05 & \alpha_2 = -962.5 \\ r_2 = .25 & \alpha_3 = 4500 \\ r_3 = .6 & \alpha_4 = -5000 \end{array}$$

Table 8 lists the results using the integral and direct method. Because of the difficulty in determining if a system is a third- or fourth-order system, attempts were made to fit both types to the data. It is evident from the data that the noise has a dramatic effect on the solution. Attempts at using the direct method failed for all but one attempt. While this method only converged for one attempt, that one solution is quite reasonable, and hints that perhaps the direct method is a somewhat better choice for the data reduction. Overall, however, both methods show high sensitivity to noise in the data. While some

Table 8. Results of Testing with Synthetic Data.

Method	Growth/	Order	α	r_1	r_2	r_3	α_1	α_2	α_3	α_4
Decay										
Integral	Growth	3	3390	.460	.6	N/A	936	-3590	3390	N/A
Integral	Decay	3	40000	.01068	.6	N/A	256	-24400	40000	N/A
Integral	Growth	4	8200	.05	.480	.6	-118.1	2810	-9270	8200
Integral	Decay	4	5.74×10^6	.05	-.00530	.6	912	152400	-3.70×10^6	5.74×10^6
Direct	Growth	3	-992	0.249	.6	N/A	-148.2	842	-992	N/A
Direct	Decay	3				No Convergence				
Direct	Growth	4				No Convergence				
Direct	Decay	4				No Convergence				

of the difficulties could be due to the method used to generate the noise, it is certainly true that any error can have a noticeable effect on the success of both methods.

3.6 Fourth-Order Verification

Fourth-order fits were generally consistent with the results as long as one realizes that for a fourth-order system to model two limit-cycles and a trigger point, the fourth-order system must be linearly unstable. Given the very low amplitude of the suspected low limit-cycle oscillation, this is possible. Figure 13 shows a fourth- and third-order system with identical high limit-cycles and trigger points. The fourth-order system has a low limit-cycle at $\epsilon=.005$. In the vicinity of the trigger point and high limit-cycle the curves look quite similar, and to the right of the low limit-cycle the basic shapes are identical. It is possible that most motors are generally operating at the low limit-cycle, and all experimental data will come from the portion to the right of this low limit-cycle. In this situation it is difficult to tell the correct order since both systems behave the same way. Since the motor is already operating at a low-amplitude limit-cycle all linear data will be modeling the decay back to the low limit cycle, not to a non-oscillatory condition. This means it will remain up to the numerical and analytical models to determine if the motor is truly linearly stable or unstable, and whether it should be modeled as a third- or fourth-order system. Any test firings will have sufficient random pressure fluctuations to trip the motor immediately into low limit-cycle operation.

One goal of this work was to determine if a fourth-order model was an appropriate model for the motors used in the NWC tests. At this point the decision cannot be made reliably. While it is evident from plots such as figure 2 that a low-level limit-cycle could be present, the data used in this study were not sufficient to determine a reliable fourth-order model. As was stated earlier, one consequence of having both low and high limit-cycles in a fourth-order model is that the linear stability term becomes positive. While it

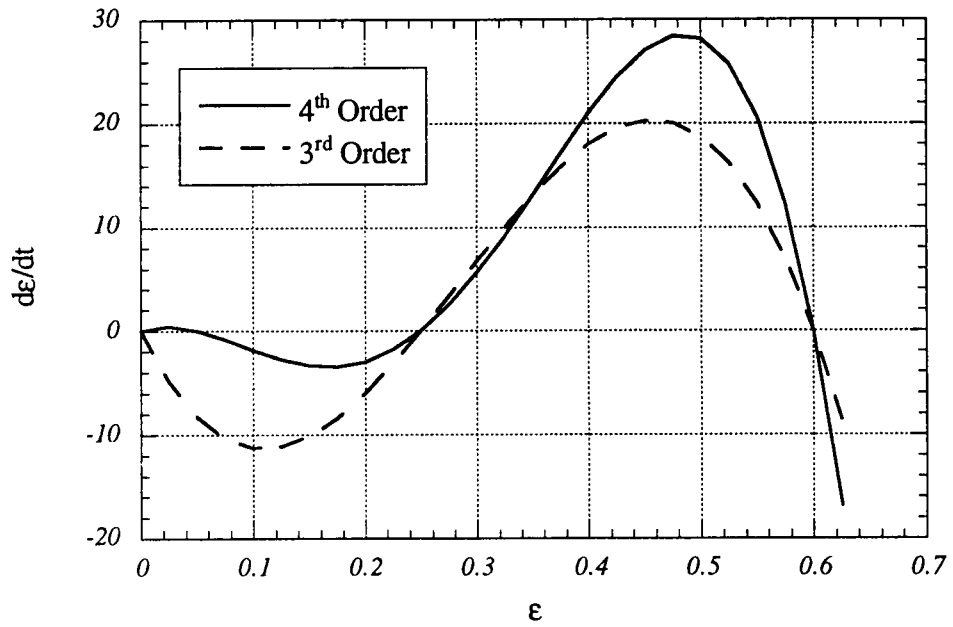


Figure 13. Comparison of the Phase Space of Third- and Fourth-Order Systems.

is entirely possible to have low limit-cycle operation induced from the variability in the motor burn, linear stability models will have to be reexamined to properly estimate the true linear stability and some work is already underway in this direction.⁹ Using the NWC motors as an example, all the motors were predicted using the SSP to have negative linear stabilities, that is they were linearly unstable. Assuming they are fourth-order systems with two limit cycles and a trigger point requires that the linear stability be positive, the opposite of what was computed. One possible way to reconcile this is to recognize that the low limit-cycle can be very small—almost zero relative to the trigger value and high limit-cycle amplitude. In this case, it can become difficult to tell the difference between the low limit-cycle and the origin. Response data will indicate a decay which will look sufficiently close to a decay to zero that it seems the motor is linearly stable when the true situation is that it is decaying back to a low-amplitude limit-cycle. Certainly some of the results for third-order models done here fit in a qualitative manner with the linear analysis and the observed motor behavior. In the end, more motor pulsing should be conducted to finally determine the applicability of a fourth-order model.

4 Conclusions and Recommendations

4.1 Conclusions

Attempts to apply the nonlinear theory developed by Flandro were attempted, but were generally unsuccessful or unreliable. Determination of whether or not the motors represent a third- or fourth-order nonlinear system could not be made. In light of these problems some studies were made to determine requirements for future experimental programs. The following conclusions were drawn from this investigation:

1. Multiple pulse responses are required to effectively analyze nonlinear motor behavior.
2. If a motor is never pulsed unstable at a certain time in the burn, then only low-order nonlinear models can provide reliable estimates of the α_i coefficients. In other words one cannot estimate what cannot be observed.
5. Combining data from pulse responses which occurred at different points in the motor burn is not a reliable technique for estimating the values of the nonlinear parameters because of the time dependence of the parameters.

4.2 Recommendations

The following recommendations are made for future motor test programs:

1. Propellant and geometry variations should be kept to a minimum, or even eliminated in a given test program. Verification of nonlinear theories relating to triggering and transient response to pulses will require a significant amount of repetition and the parameters are all influenced by changes in propellant and motor geometry.
2. Pulse strengths should be varied for each time the motor is pulsed and these strengths should be distributed over a wide range.

3. Pretest evaluation of the order of the nonlinear model should be made to provide guidance on the kind of data that will be needed. For example, using a third- or fourth-order model will demand that a motor be pulsed unstable by a strong pulse and resist triggering from a weaker pulse at the same time in the motor burn.
4. Further investigations of the flowfield produced as the shock from a pulser enters the combustion chamber should be performed to develop a more precise method to determine the actual pulse characteristics. The current method has a significant amount of uncertainty involved.
5. Development of a standard marginally stable motor configuration would be helpful. This configuration should be mildly stable with as little time variation in stability as practical. Having a standard motor would simplify the job of comparing theories by forcing them to prove the same behavior. Having little time variation in the motor stability would make the analysis job simpler by relaxing the constraint of having all pulses which are being combined occur at the same time.
6. A general recommendation for future experimental programs is do not be overly ambitious about what is being investigated. It is difficult enough to do quantitative analysis for a single geometry-propellant combination with time variation. Trying to investigate the impact of many different motor geometries and propellants adds further complication to the program, and cannot provide much additional quantitative information about a theory without significant additional expense in the form of more motor firings.

4.3 Reflections

Given the generally poor correlation between the actual pulse response and the predictions of either the SSP or the nonlinear analysis attempted here, it has occurred to the author that perhaps better understanding of the initial pulse growth from existing the

pulser to full development as a flat traveling shock should be the subject of further investigation. The Aerojet and Hercules pulser theories, the accepted linear theory, and the nonlinear theory presented here reduce the pulse severity, and pulse-related phenomenon of triggering, to a single variable, ε , and certain key values of it.³ It has occurred to the author that perhaps there is not enough consideration being given to the complex process of pulse initiation and initial evolution. The nonlinear theory presented could potentially handle this situation by incorporating pulse mechanisms in the formulation of the α_i . The α_i would be determined on both a slow and fast time scale. A slow time scale to express the growth to a high limit-cycle, and a fast time scale, which essentially couples them to the oscillation amplitude. It should be noted that once this initial transient has occurred, in these motors at least, the spectral content of the waves is fixed. However, during the initial pulse evolution—the time between emergence of a shock from a pulser to the coalescence of a traveling normal shock—the interactions are more complex and this complexity is where a more quantitative description of triggering is found.

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Appendices

Appendix A

Spectra of Unstable Pulse Responses

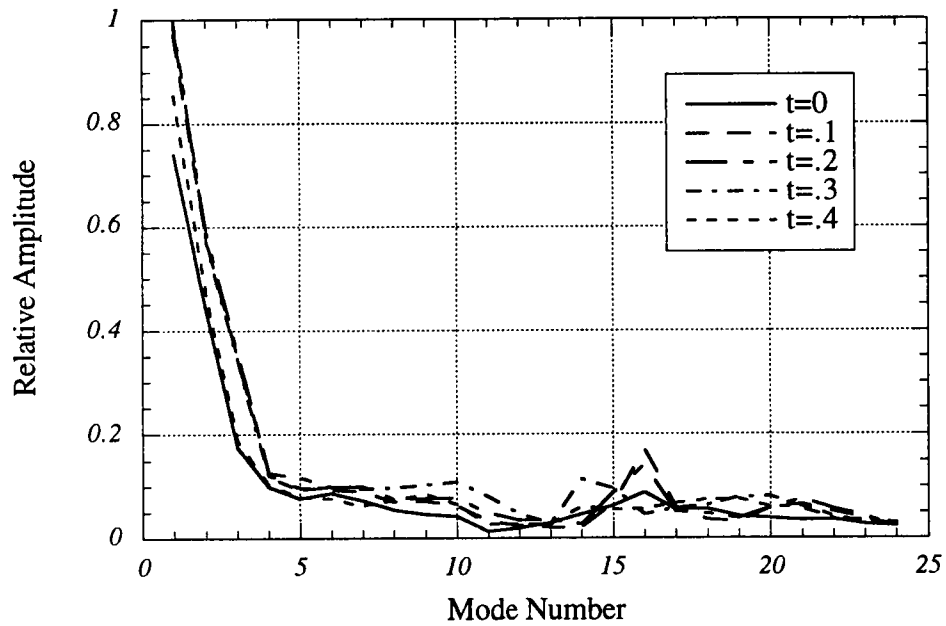


Figure A1. Motor 1 Spectrum.

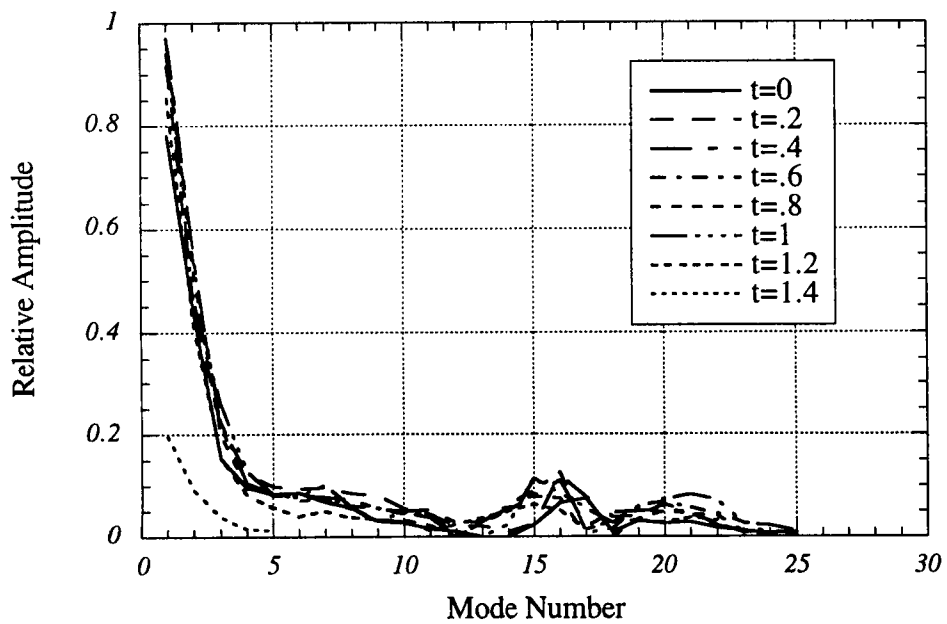


Figure A2. Motor 2 Spectrum.

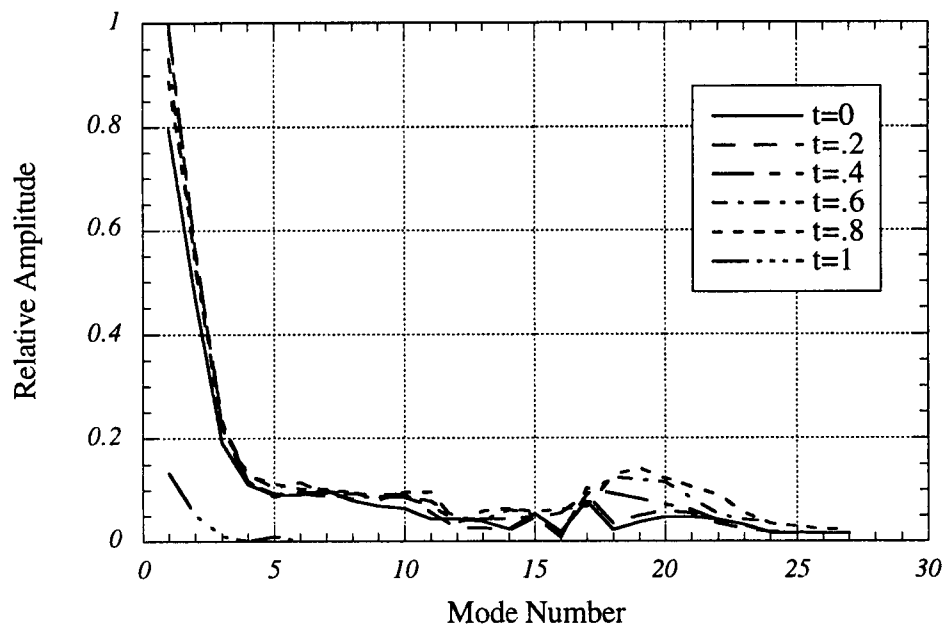


Figure A3. Motor 3 Spectrum.

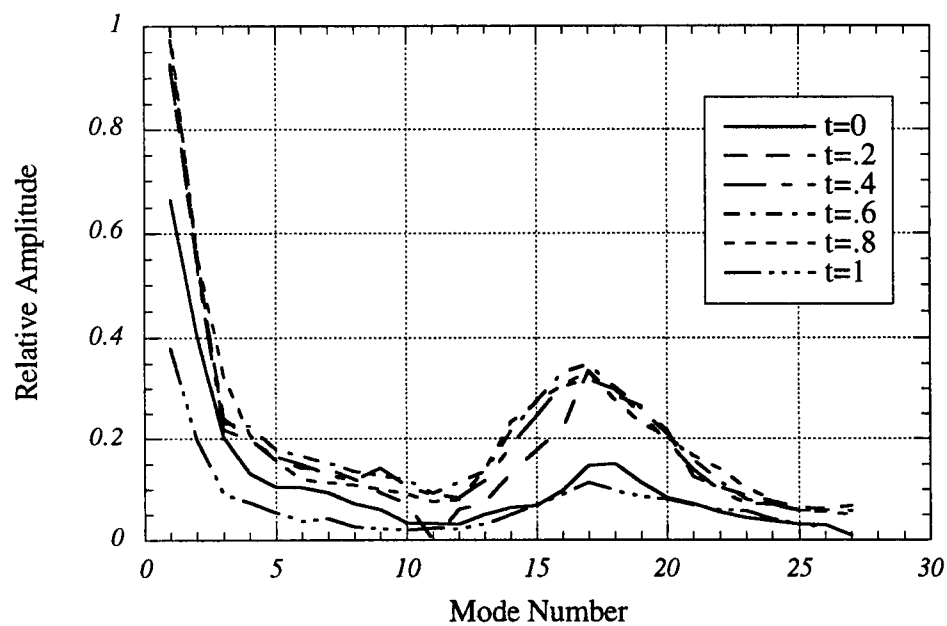


Figure A4. Motor 4 Spectrum.

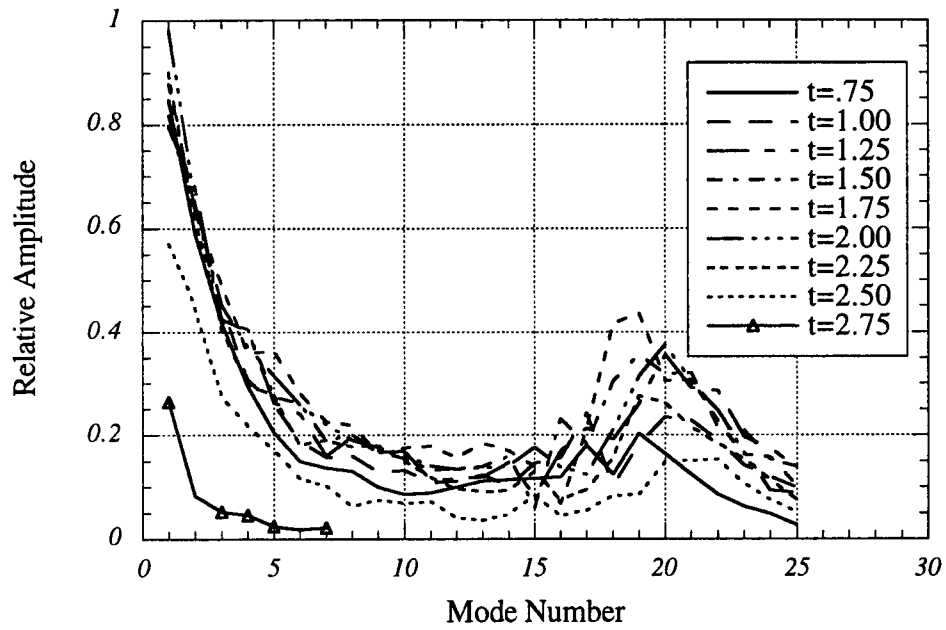


Figure A5. Motor 7 Spectrum.

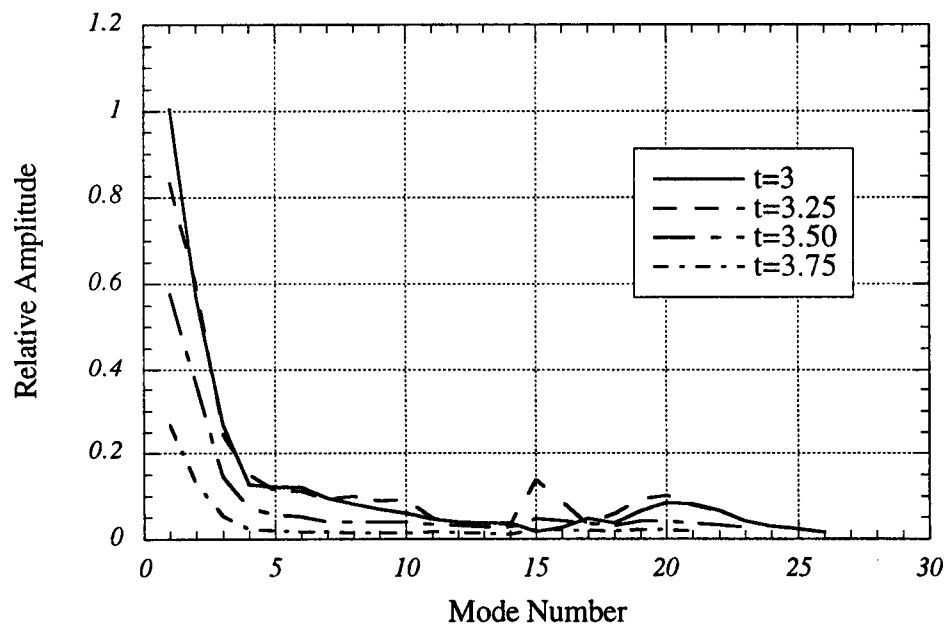


Figure A6. Motor 8a Spectrum.

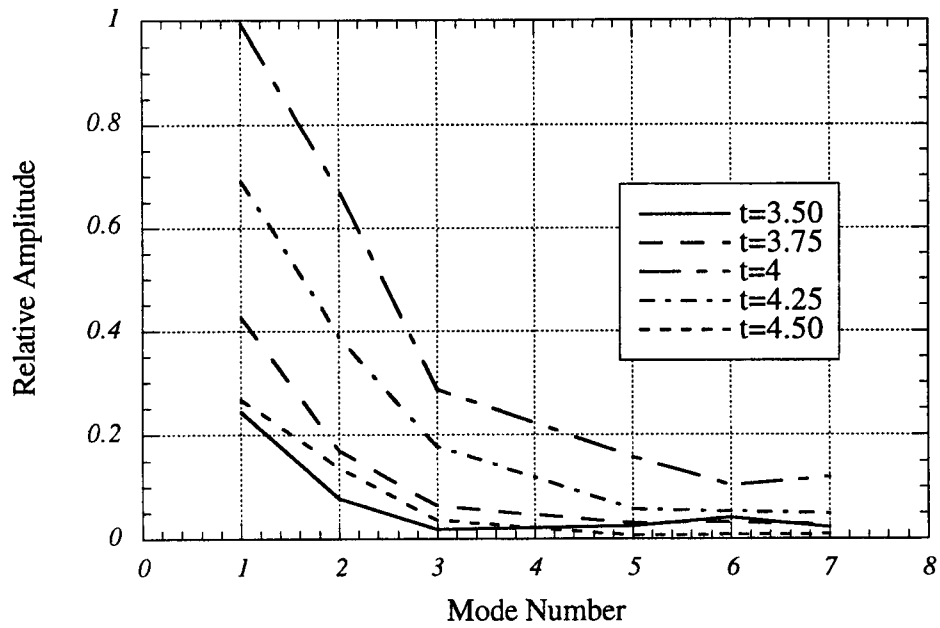


Figure A7. Motor 8b Spectrum.

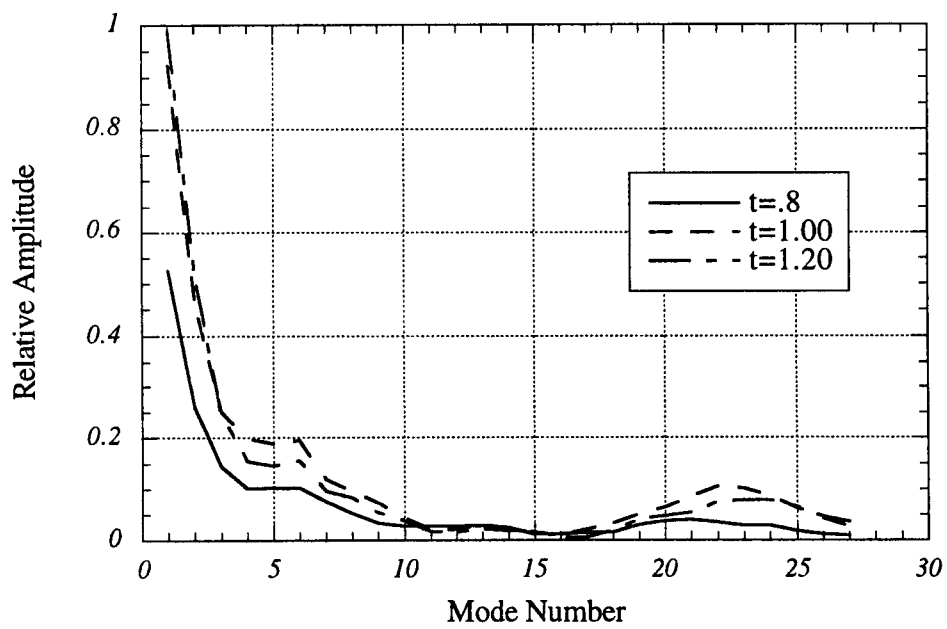


Figure A8. Motor 8c Spectrum.

Appendix B

Naval Warfare Center Test Program Motors

Motor #	Grain Configuration	Full/Half Length	Propellant	Chamber Pressure (psi)
1	Star Aft	Full Length	NWR-11	800
2	Star Aft	Full Length	NWR-11	800
3	Star Aft	Full Length	NWR-11	800
4	Star Forward	Full Length	NWR-11	900
5	Star Aft	Full Length	NWR-11	800
6	Star Aft	Full Length	NWR-11	540
7	Full Star	Full Length	NWR-11	900
8a	Star Aft	Full Length	NWR-11	800
8b	Star Aft	Full Length	NWR-11	540
8c	Star Aft	Full Length	NWR-11	1330
8d*	Star Aft	Full Length	NWR-11	800
9	Full Cylinder	Full Length	NWR-11	830
10	Full Cylinder	Full Length	NWR-11	610
11	Star Aft	Half Length	NWR-11B	480
12	Star Aft	Half Length	NWR-11B	800
13	Full Cylinder	Half Length	NWR-11B	580
14	Star Forward	Half Length	NWR-11B	660
15	Star Aft	Full Length	NWR-12 with 8 μ m Al ₂ O ₃	940
16	Star Aft	Full Length	NWR-13 with ZrC	720
17	Star Aft	Full Length	NWR-17 with 8 μ m Al ₂ O ₃	830
19	Star Aft	Full Length	NWR-11B	1460
20	Star Forward	Full Length	NWR-11B	730
21	Star Forward	Full Length	NWR-11B	730

* No pulses fired in this test

Appendix C

Levenberg-Marquardt Nonlinear Curve Fitting Algorithm

The description here is taken from reference 6. That reference provides more complete discussion of some of the assumptions. The notation used is essentially the same as that used in reference 6.

First, define the merit function for the model error

$$\chi^2(\mathbf{a}) = \sum_{j=1}^N \left[\frac{y_j - y(x_j, \mathbf{a})}{\sigma_j} \right]^2 \quad (30)$$

where $\mathbf{a}$ is the length M vector of parameters being fit, M is the number of fit parameters, N is the total number of data points, (x_j, y_j) form data point j , and $y(x, \mathbf{a})$ is the model function. When χ is sufficiently close to a minimum, it can be assumed to behave according to the quadratic form

$$\chi^2(\mathbf{a}) \approx \gamma - \mathbf{d} \cdot \mathbf{a} + \frac{1}{2} \mathbf{a} \cdot \mathbf{D} \cdot \mathbf{a} \quad (31)$$

where $\mathbf{d}$ is a vector of length M and $\mathbf{D}$ is an $M \times M$ matrix. The step from a given point $\mathbf{a}_i$ to $\mathbf{a}_{\min}$, the point which minimizes χ , is

$$\mathbf{a}_{\min} = \mathbf{a}_i + \mathbf{D}^{-1} \cdot [-\nabla \chi^2(\mathbf{a}_i)] \quad (32)$$

If $\mathbf{a}$ is not sufficiently close to the minimum to be modeled by equation 31, then the best improvement that can be made is to take a small step in the direction of the local gradient

$$\mathbf{a}_{i+1} = \mathbf{a}_i - c \nabla \chi^2(\mathbf{a}_i) \quad (33)$$

where c is a small constant.

The gradient of χ^2 is

$$\frac{\partial \chi^2}{\partial a_k} = -2 \sum_{j=1}^N \frac{[y_j - y(x_j, \mathbf{a})]}{\sigma_j} \frac{\partial y(x_j, \mathbf{a})}{\partial a_k} \quad (34)$$

where $k=1 \dots M$. The components of the Hessian, D , are given by

$$\frac{\partial^2 \chi^2}{\partial a_k \partial a_l} = 2 \sum_{j=1}^N \frac{1}{\sigma_j^2} \left[\frac{\partial y(x_j, \mathbf{a})}{\partial a_k} \frac{\partial y(x_j, \mathbf{a})}{\partial a_l} - [y_j - y(x_j, \mathbf{a})] \frac{\partial^2 y(x_j, \mathbf{a})}{\partial a_k \partial a_l} \right] \quad (35)$$

By convention the factors of two are removed using the definitions

$$\beta_k \equiv -\frac{1}{2} \frac{\partial \chi^2}{\partial a_k} \quad \alpha_{kl} \equiv \frac{1}{2} \frac{\partial^2 \chi^2}{\partial a_k \partial a_l} \quad (36)$$

This makes equations 32 and 33

$$\sum_{l=1}^M \alpha_{kl} \delta a_l = \beta_k \quad (37)$$

$$\delta a_l = c \beta_l \quad (38)$$

where $\delta \mathbf{a} = \mathbf{a}_{\min} - \mathbf{a}_i$. Typically the second derivative in equation 35 is dropped. For an explanation of the reasoning see Press.⁶ This makes the definition of α_{kl}

$$\alpha_{kl} \equiv \sum_{j=1}^N \frac{1}{\sigma_j^2} \left[\frac{\partial y(x_j, \mathbf{a})}{\partial a_k} \frac{\partial y(x_j, \mathbf{a})}{\partial a_l} \right] \quad (39)$$

Equations 37 and 38 can be combined using the definitions

$$c \equiv \frac{1}{\lambda \alpha_{ll}} \quad (40)$$

$$\begin{aligned} \alpha'_{jj} &\equiv \alpha_{jj}(1 + \lambda) \\ \alpha'_{jk} &\equiv \alpha_{jk} \quad (j \neq k) \end{aligned} \quad (41)$$

where λ is a small constant which can be varied during the fitting process. This gives

$$\sum_{l=1}^M \alpha'_{kl} \delta a_l = \beta_k \quad (42)$$

When λ is small equation 42 approaches equation 37. Conversely, when λ is large the matrix α' becomes diagonally dominant and equation 42 approaches equation 38.

Control of λ allows the algorithm to determine whether it will use a new estimate of a based on a linear or quadratic approximation to the local shape of χ^2 . The algorithm recommended by Marquardt is

1. Compute $\chi^2(a)$.
2. Pick a small value of λ , for example $\lambda=0.0001$.
3. Solve the system of equations 42 for δa and compute $\chi^2(a + \delta a)$.
4. If $\chi^2(a + \delta a) \geq \chi^2(a)$, increase λ by a factor of 10, or another substantial factor, and go to step 3.
5. If $\chi^2(a + \delta a) < \chi^2(a)$, decrease λ by a factor of 10, update the solution $a_{\text{new}} = a + \delta a$, and go back to step 3.

Stopping criteria generally are small absolute or relative changes in χ^2 , perhaps .01 absolute or .001 relative. An increase in χ^2 is not a good stopping criteria because it normally only indicates that λ has not reached an optimal value.

Vita

David Garza was born on April 6, 1971 in McAllen, Texas. He grew up in Texas, mostly in Austin, and graduated from Westwood High School in May, 1989. He entered The University of Texas at Austin the following August and graduated in December, 1993 with a Bachelor of Science in Aerospace Engineering. In August, 1994 he entered The University of Tennessee Space Institute in Tullahoma, Tennessee and received the degree of Master of Science in Aerospace Engineering in August, 1996.