August 1, 1973

To the Graduate Council:

I am submitting herewith a thesis written by William Thomas Ransone entitled "Green's Relations on Semigroups of Nonnegative Matrices." I recommend that it be accepted for nine quarter hours of credit in partial fulfillment of the requirements for the degree of Master of Arts, with a major in Mathematics.

Robert Plemmons
Major Professor

We have read this thesis
and recommend its acceptance:

Stephen Wolfram

N. Harold Row Jr.

Accepted for the Council:

Hilton A. Smith
Vice Chancellor for
Graduate Studies and Research

GREEN'S RELATIONS ON SEMIGROUPS OF NONNEGATIVE MATRICES

A Thesis

Presented to

the Graduate Council of

The University of Tennessee

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

---

by

William Thomas Ransone

August 1973

ACKNOWLEDGMENTS

The author wishes to express his appreciation to Dr. Robert J. Plemmons for his guidance and assistance in the preparation of this thesis, and to Dr. J. S. Montague for his guidance in the theory of vector cones.

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ABSTRACT

The collection of all $n \times n$ matrices with real nonnegative entries, denoted by N_n, forms a semigroup with respect to the usual matrix multiplication. Various matrix equations in N_n are investigated in this thesis. For example, necessary and sufficient conditions for the equations $AX = B$, $BY = A$ to be solvable for X and Y in N_n are derived. These equations lead to characterizations of the well-known Green's relations L, R, H, $\mathcal{D}$, and J in the semigroup N_n for regular matrices of N_n, that is, matrices A satisfying $A = AXA$ for some X in N_n. These results are given in Chapter II where it is also shown that $J = \mathcal{D}$ on the regular elements. In Chapter III, the relations L, R, and H are characterized for arbitrary elements in N_n. It is not known whether $J = \mathcal{D}$ on the entire semigroup.

TABLE OF CONTENTS

CHAPTER	PAGE
I. INTRODUCTION	1
II. THE REGULAR CASE	8
III. THE GENERAL CASE	19
BIBLIOGRAPHY	30
VITA	32

CHAPTER I

INTRODUCTION

A semigroup is a system consisting of a nonempty set S and an associative binary operation on S, usually written multiplicatively. A semigroup may or may not have an identity element with respect to the operation. The following example is the semigroup with which this thesis is concerned.

Denote by N_n the set of all $n \times n$ matrices with real nonnegative entries. The operation on N_n is the usual matrix multiplication. This operation is known to be associative in general and clearly the product of two nonnegative matrices is again nonnegative. The semigroup N_n also has an identity element since the usual $n \times n$ identity matrix is nonnegative.

An element a in a semigroup S is said to be regular in S if there exists an element x in S such that $a = axa$, that is, if $a \in aSa$. If in addition $x = xax$, then a and x are said to be inverses in S. Note that if S is a group, a and x are the usual group inverses. If every element of a semigroup S is regular, then S is called a regular semigroup. As an example, consider N_2. If

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

then

$$X = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

is a solution in N_2 to the matrix equation $A = AXA$. Thus A is regular in N_2. However, N_2 is not a regular semigroup since for

$$B = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} ,$$

the only solution to $B = BXB$ is

$$X = \begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix} ,$$

which is not an element of N_2 for $a > 0$.

From the algebraic theory of semigroups, the following relations will be considered. The definitions and notation follow that in [1]. Let a and b be elements of a semigroup S. The relation $R[L]$ is defined on S by the rule $aRb[aLb]$ if and only if a and b generate the same principal right [left] ideal in S, that is, $aS^1 = bS^1$ $[S^1a = S^1b]$. S^1 is used to denote S if S has an identity element, and otherwise to denote S with an identity element 1 adjoined. The relation H is defined to be $L \cap R$. The relation $\mathcal{D}$ is defined to be the intersection of all equivalence relations on S which contain $L \cup R$. In [1] it is shown that $\mathcal{D} = L \circ R = R \circ L$, where $(a,b) \in L \circ R$ if and only if there

exists $c \in S$ such that $a\mathcal{L}c$ and $c\mathcal{R}b$. Finally the relation $\mathcal{J}$ is defined by $a\mathcal{J}b$ if and only if a and b generate the same two-sided principal ideal in S, that is, $S^1aS^1 = S^1bS^1$. Since $\mathcal{L} \subseteq \mathcal{J}$ and $\mathcal{R} \subseteq \mathcal{J}$ then $\mathcal{D} \subseteq \mathcal{J}$ also. However, in general $\mathcal{D} \neq \mathcal{J}$. This is the case for the semigroup of binary relations on a set [4]. These five relations are known as Green's relations and are important in the study of semigroup structure. It should be clear that all five are equivalence relations on the set S.

In order to investigate these relations in the matrix semigroup N_n, it will be convenient to use the following equivalent definitions of $\mathcal{R}$, $\mathcal{L}$, and $\mathcal{J}$ for elements a and b from any semigroup S.

$$a\mathcal{L}b \text{ iff } \exists x,y \in S \ni a = xb \text{ and } b = ya,$$

$$a\mathcal{R}b \text{ iff } \exists x,y \in S \ni a = bx \text{ and } b = ay,$$

$$a\mathcal{J}b \text{ iff } \exists x,x',y,y' \in S \ni a = xby \text{ and } b = x'ay'.$$

These are the definitions that will be used throughout the theorems and proofs in the following chapters.

The following theorem concerning the relation $\mathcal{D}$ for regular elements is proved in [1].

THEOREM 1.1. If a $\mathcal{D}$-class D of a semigroup contains a regular element, then every element of D is regular.

The proof of this theorem first establishes the same result for $\mathcal{L}$- and $\mathcal{R}$-classes. Thus we can investigate Green's relations for regular elements separately. This will be done in Chapter II. When this is the case, we speak of the equivalence classes as regular $\mathcal{L}$-classes, etc.

When working in the semigroup N_n of nonnegative matrices, it should be noted that only one of the relations $\mathcal{L}$ and $\mathcal{R}$ need be described, since the other follows immediately. This is shown in the following theorem.

THEOREM 1.2. Let A and B be elements of N_n. Then $A\mathcal{L}B$ if and only if $A^T\mathcal{R}B^T$.

PROOF. Assume $A\mathcal{L}B$. Then there exist matrices X and Y in N_n such that

$$A = XB , \quad B = YA$$

which implies

$$A^T = (XB)^T , \quad B^T = (YA)^T$$

or

$$A^T = B^T X^T \ , \ B^T = A^T Y^T \ .$$

Since X^T and Y^T must also belong to N_n , we have that $A^T R B^T$. The converse follows in the same manner.

It should also be noted that in the semigroup N_n , the definitions of Green's relations in terms of matrix equations immediately implies that two elements of N_n of unequal rank cannot belong to the same equivalence class under any of these relations.

Some of the work in Chapter III involves the idea of a cone of vectors [2]. Let V be an n-dimensional vector space over an ordered field R . Our work here is done exclusively over the field of real numbers although, in general, R may be arbitrary. Let $P = \{v_1 , v_2 , \ldots , v_k\}$ be a finite collection of vectors from V . Then the <u>polyhedral convex cone</u> generated by P , denoted Cone P , is defined by

$$\text{Cone } P = \left\{ a_1 v_1 + a_2 v_2 + \ldots + a_k v_k \mid \begin{array}{r} \text{for each } i \ , \\ a_i \geq 0 \ \text{ in } \ R \end{array} \right\} \ .$$

Cone P is simply the collection of all vectors in V which can be expressed as a linear combination of vectors in P with nonnegative coefficients. If the vectors in P are nonnegative, then Cone P is simply a convex pyramid in the first orthant

of V with vertex at the origin and half-lines as edges.

A collection of vectors P is said to <u>span</u> a convex cone C if every vector of C can be written as a nonnegative linear combination of vectors in P. Alternately, a convex cone C is generated by a collection of vectors P if C is the smallest cone containing each vector in P. The <u>dimension</u> of a convex cone is the cardinality of the minimal collection of vectors that span the cone. For polyhedral cones, this corresponds to the number of edges. If we denote by S the entire subspace of V generated by the vectors of P, then dim $S \leq$ dim Cone P. This is seen by letting P_0 denote a minimal subcollection of P that spans Cone P. Then each vector of $P - P_0$ lies in Cone P_0 and thus is a linear combination of vectors in P_0. So S is generated by some subset of P_0.

A closely related idea is that of the <u>cone of a matrix</u>. Here we simply consider the usual convex cone generated by the column vectors of the matrix. The convex cone generated by a matrix A will be denoted AR_+^n. Thus we can write

$$AR_+^n = \{a_1A^1 + a_2A^2 + \ldots + a_nA^n \mid \text{for each } i\text{ , } a_i \geq 0 \text{ in } R\}$$

where A^i denotes the i^{th} column vector of A. In order to consider the cone generated by the row vectors of A, we simply consider $A^TR_+^n$.

The theory of vector cones is used in studying Green's relations for arbitrary elements of the semigroup N_n in Chapter III.

However, we will first complete the investigation begun by Plemmons in [3] of these relations for regular elements of N_n. This is the subject of the next chapter.

CHAPTER II

THE REGULAR CASE

As pointed out earlier, not every matrix in the semigroup N_n of nonnegative matrices is regular. However, R. J. Plemmons [3] has characterized the regular elements of N_n and also the regular $\mathcal{D}$-classes. We shall state his results and then provide characterizations for the remaining four Green's relations for regular elements.

An $n \times n$ matrix A is called r-monomial if A has rank r and if each row and each column of A contains at most one nonzero entry. If $r = n$, then A is simply called monomial. Nonnegative monomial matrices are the only nonsingular matrices in N_n that have nonnegative inverses. Hence they are the only regular nonsingular matrices in N_n.

In [3] the following description of regular matrices in N_n was given.

THEOREM 2.1. Let A be an $n \times n$ nonnegative matrix of rank r. Then the following statements are equivalent.

(i) A is regular in N_n.

(ii) A has an inverse in N_n of the form $D_1 A^T D_2$ where D_1 and D_2 are nonnegative diagonal matrices.

(iii) A has an inverse in N_n of the r-monomial type.

(iv) A has a monomial submatrix of order r.

Statement (iv) will prove to be the most useful description of regular matrices of rank r in N_n, since it provides a criterion by which regular matrices can be easily identified.

Also in [3], the following characterization of the regular $\mathcal{D}$-classes in N_n was established.

THEOREM 2.2. Let $A \in N_n$ be regular of rank r and let D denote the $\mathcal{D}$-class containing A. Then D contains the canonical idempotent

$$E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} .$$

Moreover, D consists of all members of N_n of rank r containing a monomial submatrix of order r.

So there are exactly $n + 1$ regular $\mathcal{D}$-classes in N_n, each consisting of all regular matrices of equal rank.

In order to extend the theory of Green's relations in the semigroup N_n, nonsingular elements will be briefly examined and then complete characterizations will be provided for Green's $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$, and $\mathcal{J}$ relations for regular elements. We begin with the following description of the $\mathcal{R}$-classes for nonsingular matrices.

THEOREM 2.3. Let A and B be nonsingular in N_n. Then ARB if and only if there exists a monomial matrix $M \in N_n$ such that $A = BM$.

PROOF. Assume ARB. Then there exist matrices X and Y in N_n such that $A = BX$ and $B = AY$. Since rank A = rank $B = n$, then rank X = rank $Y = n$ also. Thus X is nonsingular and $AX^{-1} = B = AY$. Since A is nonsingular, the first and last expressions yield $X^{-1} = Y$. Hence we have that X is nonsingular and nonnegative and has a nonnegative inverse. By earlier remarks, X must be monomial.

Now assume there exists a monomial matrix $M \in N_n$ such that $A = BM$. If M is monomial in N_n, then M^{-1} is also nonnegative. So we have $B = AM^{-1}$ and $A = BM$ where $M, M^{-1} \in N_n$. Thus ARB.

Note that the condition requires that the column vectors of A are nonnegative multiples of the column vectors of B in a one-to-one fashion. Consider the following nonsingular matrices from N_3.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 3 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 4 & 2 \\ 2 & 4 & 0 \\ 0 & 2 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 6 & 1 \\ 1 & 6 & 0 \\ 0 & 3 & 2 \end{bmatrix}.$$

ARB by inspection, but A and C are not R-related since the first column of A has no column multiple in C.

Dually, L-equivalence for nonsingular elements A and B in N_n depends on the existence of a monomial matrix $M \in N_n$ such that $A = MB$.

In order to simplify the characterization of the R relation for regular elements, a convenient column permutation of a regular matrix of N_n will be discussed and two preliminary lemmas will be given.

If A is any regular matrix of rank r in N_n, then A has a monomial submatrix of order r (Theorem 2.1). So there exist r columns of A which can be used in forming this submatrix. Furthermore, there exists a permutation matrix P that will permute these r columns into the first r column positions. This yields

$$AP = [M , U] \tag{2.1}$$

where M is $n \times r$ and contains a monomial submatrix of order r. The following lemma will also be most useful.

LEMMA 2.4. If A and B are elements of N_n and P_1 and P_2 are permutation matrices such that $A_1 = AP_1$ and $B_1 = BP_2$, then ARB if and only if A_1RB_1.

PROOF. Assume ARB. Then there exist X and Y in N_n such that $A = BX$ and $B = AY$. This implies $AP_1 = BXP_1 = B_1(P_2^TXP_1)$ and $BP_2 = AYP_2 = A_1(P_1^TYP_2)$ or simply that $A_1 = B_1R$ and $B_1 = A_1S$ where $R , S \in N_n$. Hence A_1RB_1. The converse is proved in the same manner.

The following lemma was given in [3] and will be stated here for use in the proof of Theorem 2.6.

LEMMA 2.5. Let A be an $n \times r$ nonnegative matrix of rank r. Then A has a nonnegative left inverse if and only if A has a monomial submatrix of order r.

A left inverse for such a matrix A is simply an $r \times n$ matrix X such that $XA = I_r$, the identity matrix of order r.

We are now ready to prove the major result concerning R-related regular elements of N_n.

THEOREM 2.6. Let A and B be regular of rank r in N_n. Let P_1 and P_2 be permutation matrices such that $A_1 = AP_1$ and $B_1 = BP_2$ where A_1 and B_1 have the form (2.1). Then ARB if and only if there exists a matrix $Q \in N_n$ of the form

$$Q = \begin{bmatrix} C & K \\ 0 & 0 \end{bmatrix} \tag{2.2}$$

where C is $r \times r$ and monomial, such that $A_1 = B_1 Q$.

PROOF. Assume ARB. Then $A_1 R B_1$ by Lemma 2.4. Since A_1 and B_1 have the form (2.1),

$$A_1 = [M_1 , U_1] \quad , \quad B_1 = [M_2 , U_2]$$

where M_1 and M_2 are $n \times r$ and contain monomial submatrices of order r. Since $A_1 R B_1$, there exists a matrix $X \in N_n$ such that $A_1 = B_1 X$. This simply says that the columns of A_1 are nonnegative linear combinations of the columns of B_1. Since M_2 is $n \times r$ and has a monomial submatrix of order r, the columns of M_2 must be linearly independent. But B_1 has rank r, so the columns of U_2 are linear combinations of the r columns of M_2. Moreover, these linear combinations must all have nonnegative coefficients since there are r rows of M_2, each having exactly one nonzero entry with each entry appearing in a different column. Therefore, each column of M_1 can be expressed as a nonnegative linear combination of the columns of M_2. So there exists an $r \times r$ nonnegative matrix C such that $M_1 = M_2 C$. But since M_1 and M_2 have rank r, C must also have rank r and is therefore nonsingular. This implies $M_1 C^{-1} = M_2$.

$A_1 R B_1$ also implies there exists a matrix $Y \in N_n$ such that $B_1 = A_1 Y$. So, by argument similar to that given above, there exists an $r \times r$ nonnegative matrix H such that $M_2 = M_1 H$. We therefore have $M_1 C^{-1} = M_1 H$. But M_1 has a left inverse M_L by Lemma 2.5. Thus $M_L M_1 C^{-1} = M_L M_1 H$ which implies $I_r C^{-1} = I_r H$ or simply $C^{-1} = H$. But if C is nonnegative and nonsingular with a nonnegative inverse, then C must be monomial.

As pointed out earlier, the columns of U_1 are also nonnegative linear combinations of the columns of M_2. So there exists an $r \times (n-r)$ nonnegative matrix K such that $U_1 = M_2 K$. We have now

established the equality

$$[M_1 \ , \ U_1] = [M_2 \ , \ U_2] \begin{bmatrix} C & K \\ 0 & 0 \end{bmatrix}$$

or simply $A_1 = B_1 Q$ where Q has the desired form (2.2).

For the converse, assume there exists a matrix Q in N_n of the form (2.2) such that $A_1 = B_1 Q$. This says

$$A_1 = [M_2 \ , \ U_2] \begin{bmatrix} C & K \\ 0 & 0 \end{bmatrix}$$

so

$$A_1 \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} = [M_2 \ , \ U_2] \begin{bmatrix} C & K \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

or

$$A_1 \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} = [M_2 \ , \ U_2] \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = [M_2 \ , \ 0] \ .$$

But, as noted earlier, the columns of U_2 are nonnegative linear combinations of the columns of M_2. Hence, there exists an $r \times (n-r)$ nonnegative matrix W such that $U_2 = M_2 W$. So

$$A_1 \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_r & W \\ 0 & 0 \end{bmatrix} = [M_2 \ , \ 0] \begin{bmatrix} I_r & W \\ 0 & 0 \end{bmatrix}$$

or

$$A_1 \begin{bmatrix} C^{-1} & C^{-1}W \\ 0 & 0 \end{bmatrix} = [M_2 , U_2] = B_1 .$$

Since C^{-1} and W are both nonnegative, we have a matrix X in N_n such that $B_1 = A_1X$. By assumption, $A_1 = B_1Q$ with $Q \in N_n$. Thus A_1RB_1 and by Lemma 2.4, ARB.

A restatement of the above result in words will make it possible to easily identify R-related, nonnegative, regular matrices by inspection. Basically Theorem 2.6 says that ARB if and only if we can choose a set of r columns from A that are multiples of r columns from B in a one-to-one fashion, where each set of columns can be used in forming a monomial submatrix of order r in each matrix.

The Green's L relation in N_n can now be characterized using Theorem 1.2. However, we shall state a theorem analogous to Theorem 2.6 that applies directly. The proof is essentially the same as the previous one, replacing all column operations by row operations. First note that for a regular matrix A of rank r in N_n, we can choose r rows which can be used in forming a monomial submatrix of order r in A. These rows can then be permuted into the first r row positions by a permutation matrix P, yielding

$$PA = \begin{bmatrix} M \\ U \end{bmatrix} \tag{2.3}$$

where M is $r \times n$ and contains a monomial submatrix of order r.

THEOREM 2.7. Let A and B be regular of rank r in N_n. Let P_1 and P_2 be permutation matrices such that $A_1 = P_1A$ and $B_1 = P_2B$ where A_1 and B_1 have the form (2.3). Then $A \mathcal{L} B$ if and only if there exists a matrix Q in N_n of the form

$$Q = \begin{bmatrix} C & 0 \\ K & 0 \end{bmatrix}$$

where C is $r \times r$ and monomial, such that $A_1 = QB_1$.

Thus $A \mathcal{L} B$ if and only if there exist r rows of A that are multiples of r rows of B in a one-to-one fashion, where each set of rows can be used in forming a monomial submatrix of order r in each matrix.

Consider now the following examples from N_3.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 3 \\ 0 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 & 5 \\ 2 & 0 & 4 \\ 0 & 6 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}.$$

Each of these rank two matrices is regular since each contains a monomial submatrix of order two. $A \mathcal{L} B$ since the first and third

rows of A are multiples of the second and third rows of B. BRC since the first and second columns of B are multiples of the second and third columns of C. The rows and columns considered here are all used in forming a monomial submatrix of order two in each matrix. Note also that A and C are neither R-related nor L-related.

Theorems 2.6 and 2.7 also give us a characterization of Green's H relation for regular elements of N_n since $H = R \cap L$. As an example, consider

$$X = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 2 & 0 \\ 4 & 1 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 1 & 0 \\ 5 & \frac{1}{2} & 2 \end{bmatrix}.$$

X and Y are regular of rank two in N_3. XHY since the first and second rows of X are multiples of the first and second rows of Y, and the second and third columns of X are multiples of the second and third columns of Y. Again these are the rows and columns that can be used in forming 2×2 monomial submatrices.

In order to complete the theory of Green's relations for regular elements of N_n, we consider the J relation, which is the subject of the following theorem.

THEOREM 2.8. If A and B are regular in N_n, then AJB if and only if $A\mathcal{D}B$.

PROOF. Since $\mathcal{D} \subseteq J$ in any semigroup, we immediately have $A\mathcal{D}B$ only if AJB.

For the converse, assume AJB. Then there exist matrices X_1, X_2, Y_1, and Y_2 in N_n such that $A = X_1BY_1$ and $B = X_2AY_2$. The first of these equalities implies rank $A \leq$ rank B. Similarly, the second equality gives rank $B \leq$ rank A. Hence rank A = rank B. Since all regular elements of N_n of the same rank are contained in a single $\mathcal{D}$-class (Theorem 2.2), then $A\mathcal{D}B$.

Thus, in the semigroup N_n, the regular $\mathcal{D}$-classes and regular J-classes are the same. It is not known, however, if this result also holds for arbitrary elements of N_n. Since the discussion of the regular case is now completed, we move to the case of arbitrary elements where the L, R, and H relations will be characterized.

CHAPTER III

THE GENERAL CASE

We will now consider the Green's relations $\mathcal{R}$, $\mathcal{L}$, and $\mathcal{H}$ for arbitrary elements of N_n . In order to accomplish this, the following preliminary theorem concerning nonnegative vectors will be needed.

THEOREM 3.1. Let $P = \{v_1 , v_2 , \ldots , v_k\}$ be a collection of nonnegative vectors in n-space that generate a subspace S of dimension r . Then the cone generated by P is generated by a minimal subcollection P' of t vectors of P , with $t \geq r$. Furthermore, P' contains r vectors that generate S .

PROOF. First consider v_1 . If Cone P = Cone $(P - \{v_1\})$ then omit v_1 . Otherwise retain v_1 . In either case, denote the collection of vectors that remain by P_1 . Now consider v_2 . If Cone P_1 = Cone $(P_1 - \{v_2\})$ then omit v_2 . Otherwise retain v_2 . Denote the collection of vectors that remain by P_2 . Continue this procedure through v_k , finally obtaining P_k , a minimal subcollection of P that generates the same cone as P .

Now card $P_k = t \geq r$ since the dimension of Cone P cannot be less than the dimension of the subspace S . Furthermore, P_k contains a set of r independent vectors since S has dimension r and each vector in $P - P_k$ lies in Cone P_k , that is, S is

generated by vectors from P_k .

We are now ready to prove the main result which provides a characterization of the R relation for arbitrary elements of N_n . The theorem presents three equivalent statements and will be proved by showing that statements (i) and (ii) are equivalent and that statements (i) and (iii) are equivalent. This is done because the first equivalence is rather easy to show and the proof of the second equivalence provides some interesting additional information about elements of N_n .

THEOREM 3.2. Let A and B be matrices of rank r in N_n . Then the following statements are equivalent.

(i) ARB

(ii) $AR_+^n = BR_+^n$

(iii) (a) There exist t columns of A that are multiples of t columns of B in a one-to-one fashion with $t \geq r$, and each such collection of t columns contains a sub-collection of r independent columns.

(b) In each matrix, each of the remaining $n - t$ columns can be expressed as a nonnegative linear combination of the t columns in (a).

PROOF. If ARB in N_n , then there exist X and Y in N_n such that $A = BX$ and $B = AY$. The first equation says that the

columns of A are nonnegative linear combinations of the columns of B. This means each column of A belongs to BR_+^n. So $AR_+^n \subseteq BR_+^n$. Similarly, the second equation will yield $BR_+^n \subseteq AR_+^n$. Hence we have (ii).

Now assume $AR_+^n = BR_+^n$. Each column of A clearly belongs to AR_+^n and thus belongs to BR_+^n. So each column of A can be expressed as a nonnegative linear combination of the columns of B. This implies there exists a nonnegative matrix X such that $A = BX$. Similarly, each column of B belongs to AR_+^n, so $B = AY$ with $Y \in N_n$. Hence ARB and (i) is equivalent to (ii).

In order to show that (i) implies (iii), let A be of rank r in N_n. We will construct a general form for any matrix B that is R-related to A. First permute the columns of A so that A can be partitioned into the minimum number of rank one column blocks, say k of these. Then select one nonzero column from each block. Theorem 3.1 says that there exist t of these column vectors ($t \geq r$), each of which cannot be expressed as a nonnegative linear combination of the other $k - 1$ column vectors. So permute the column blocks of these t columns to the far right and at the same time permute all zero columns to the far left in the first column block. Thus we obtain the following column block form.

$$AP = [A_1 \ \cdots \ A_{k-t} \mid A_{k-t+1} \ \cdots \ A_k] \ ,$$

where P is a permutation matrix that produces the special form described above.

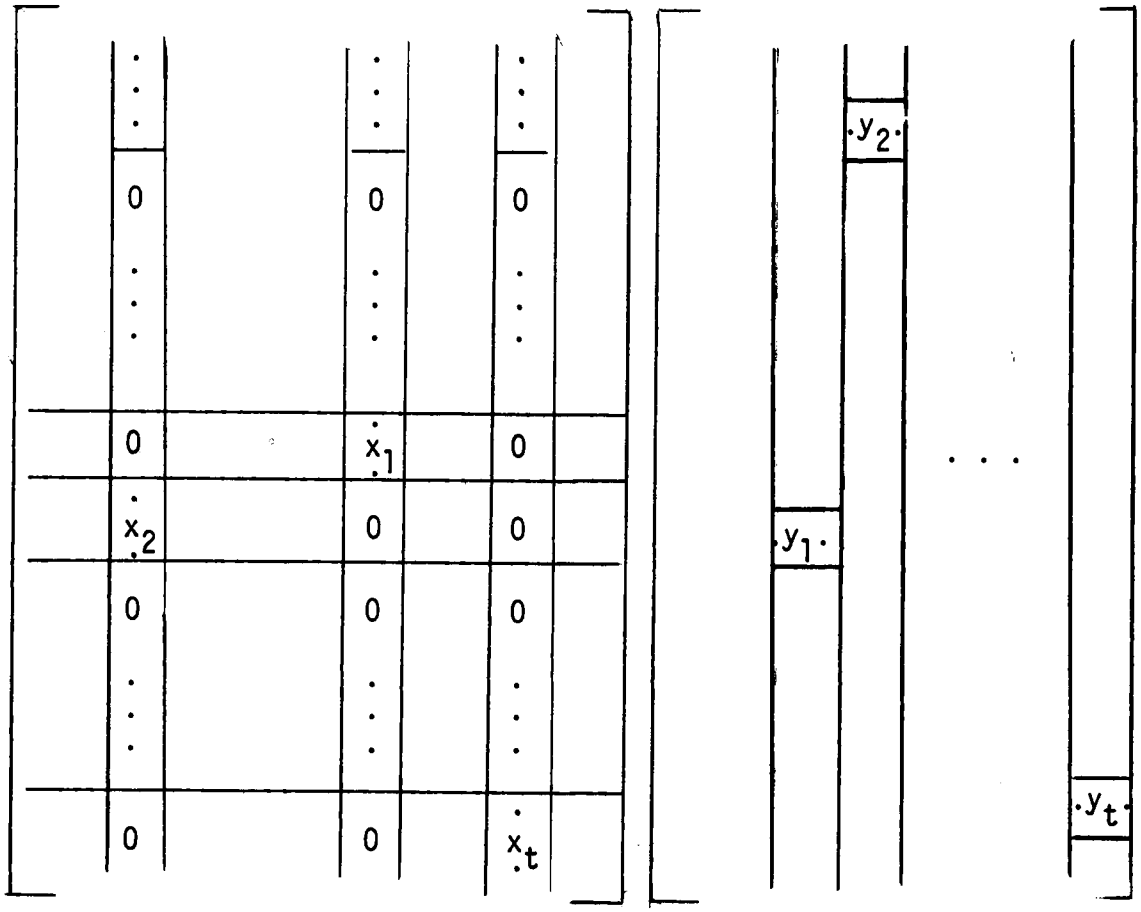
Next, consider the general form of any nonnegative matrix Z such that $AP = APZ$. By properties already established for the columns of the last t blocks and the fact that each block has rank one, it follows that

$$[A_1 \cdots A_{k-t} \mid A_{k-t+1}A_{k-t+2} \cdots A_k] =$$

$$[A_1 \cdots A_{k-t} \mid A_{k-t+1}A_{k-t+2} \cdots A_k] \begin{bmatrix} & Z_1 & Z_2 & \cdots & Z_t \\ & 0 & 0 & & 0 \\ & \vdots & \vdots & & \vdots \\ & M_1 & \cdot & & \cdot \\ & 0 & M_2 & & \cdot \\ & \cdot & 0 & & \cdot \\ & \cdot & \cdot & & \cdot \\ & \cdot & \cdot & & 0 \\ & 0 & 0 & & M_t \end{bmatrix}$$

where each M_i is square and each Z_i has as many rows as there are zero columns in A_1. Note that the position of the first row in M_i corresponds to the position of the first column in A_{k-t+i}. Next, consider any factorization of the above matrix Z into two nonnegative matrices X and Y :

$Z = XY =$



where x_i , $y_i > 0$ for $i = 1, \ldots, t$. Each x_i will be multiplied by the corresponding y_i when performing the matrix multiplication and the result contributes to an entry in M_i . These correspondences are guaranteed since each M_i must contain at least one nonzero entry. By denoting APX by B' and noting that $B'Y = APXY = APZ = AP$, we have constructed a general form for any matrix B' such that $(AP)RB'$.

By examining closely the forms of AP and X, we can determine certain properties of B'. Since A_{k-t+i} has rank one, the column in X containing x_i will yield a column in B' which is simply a positive multiple of any column in A_{k-t+i}. Since this is true for $i = 1, \ldots, t$, we have t pairs of column multiples from AP and B'. Since these t columns from AP are one each from the last t column blocks, r of these columns are independent by Theorem 3.1. Hence their multiples in B' are independent and we have (iii) - (a).

Any other column from the last t blocks of AP is a multiple of one of the t columns chosen above. Also, any column from the first k-t blocks of AP can be expressed as a nonnegative linear combination of the chosen t columns by Theorem 3.1.

Now any column of B' is a nonnegative linear combination of all columns of AP, and by remarks above, a nonnegative linear combination of the chosen t columns of AP. But these are only multiples of the chosen t columns of B', so (iii) - (b) holds for both AP and B'.

This portion of the proof is completed by simply noting that any matrix which is R-related to A is also R-related to AP, and that A and AP have the same collection of column vectors.

Now assume the conditions of (iii) for A and B. Denote by $A^1, \ldots, A^t$ and $B^1, \ldots, B^t$ the column vectors from A and B that are multiples. Assume that the correspondence is such that $A^i = c_i B^i$ for $i = 1, \ldots, t$ where each c_i is a positive real number. Therefore we can write

$$[A^1A^2 \ldots A^t] = [B^1B^2 \ldots B^t]C$$

where C is a nonnegative $t \times t$ diagonal matrix.

Condition (iii) - (b) concerning the remaining $n - t$ columns of A guarantees that there exists a $t \times (n - t)$ nonnegative matrix K such that

$$[A^{t+1}A^{t+2} \ldots A^n] = [A^1A^2 \ldots A^t]K \quad .$$

It follows that

$$[A^1 \ldots A^tA^{t+1} \ldots A^n] = [A^1 \ldots A^t, 0 \ldots 0] \begin{bmatrix} I_t & K \\ 0 & 0 \end{bmatrix}$$

$$= [B^1 \ldots B^t, 0 \ldots 0] \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_t & K \\ 0 & 0 \end{bmatrix}$$

$$= [B^1 \ldots B^t, 0 \ldots 0] \begin{bmatrix} C & CK \\ 0 & 0 \end{bmatrix} \quad .$$

However, the zero column vectors in the left-hand matrix of this last product may be replaced by any column vectors we choose, since the entries there are multiplied by zeros from the right-hand matrix and therefore do not contribute to the product. So we will replace these columns by the remaining $n - t$ columns of B, obtaining

$$[A^1 \ldots A^tA^{t+1} \ldots A^n] = [B^1 \ldots B^tB^{t+1} \ldots B^n] \begin{bmatrix} C & CK \\ 0 & 0 \end{bmatrix} \quad .$$

But there exist permutation matrices P_1 and P_2 for which the last equality reduces to

$$AP_1 = BP_2 \begin{bmatrix} C & CK \\ 0 & 0 \end{bmatrix}$$

or

$$A = BP_2 \begin{bmatrix} C & CK \\ 0 & 0 \end{bmatrix} P_1^T \quad .$$

Since C and K are nonnegative, we have $A = BX$ with $X \in N_n$.

By reversing the roles of A and B (beginning with $c_i^{-1}A^i = B^i$ for $i = 1, \ldots, t$) and repeating the argument, $Y \in N_n$ is obtained such that $B = AY$ where

$$Y = P_3 \begin{bmatrix} C^{-1} & C^{-1}M \\ 0 & 0 \end{bmatrix} P_4^T \quad .$$

P_3 and P_4 are permutation matrices and M is a $t \times (n - t)$ nonnegative matrix. Thus ARB and the proof is complete.

The conditions given in (iii) are the most useful since they will allow us to identify R-related matrices in N_n by examination. Consider the following two matrices of rank two in N_4.

$$A = \begin{bmatrix} 0 & 2 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 3 & 2 & 4 & \frac{1}{2} \\ 5 & 2 & 5 & 1 \\ 2 & 0 & 1 & \frac{1}{2} \end{bmatrix} .$$

Again denote the columns of A and B by A^i and B^i for $i = 1, 2, 3, 4$. A^1 and A^2 are linearly independent and are multiples of B^2 and B^4. In addition,

$$A^3 = A^1 + A^2 \quad , \quad B^1 = \tfrac{1}{2}B^2 + 4B^4$$

$$A^4 = 2A^1 + A^2 \quad , \quad B^3 = \tfrac{3}{2}B^2 + 2B^4 \quad .$$

Thus, in Theorem 3.2, both conditions of (iii) are satisfied, so ARB.

The L relation for arbitrary elements of N_n can now be characterized in a manner similar to that for the R relation, simply by considering rows instead of columns. Therefore the following theorem will only be stated since its proof is analogous to that of Theorem 3.2.

THEOREM 3.3. Let A and B be matrices of rank r in N_n. Then the following statements are equivalent.

(i) ALB

(ii) $A^T R_+^n = B^T R_+^n$

(iii) (a) There exist t rows of A that are multiples of t rows of B in a one-to-one fashion with $t \geq r$, and each collection of t rows contains a subcollection of r independent rows.

(b) In each matrix, each of the remaining n - t rows can be expressed as a nonnegative linear combination of the t rows in (a).

The proof of this theorem will establish the fact that if $A \mathcal{L} B$, then there exist matrices X and Y in N_n such that $A = XB$ and $B = YA$ where

$$X = P_1^T \begin{bmatrix} C & 0 \\ KC & 0 \end{bmatrix} P_2 \quad , \quad Y = P_3^T \begin{bmatrix} C^{-1} & 0 \\ MC^{-1} & 0 \end{bmatrix} P_4 \quad .$$

Each P_i is a permutation matrix, C is $t \times t$ and diagonal, and K and M are each $(n - t) \times t$ and nonnegative.

Since $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, we have also characterized the $\mathcal{H}$ relation for arbitrary elements of N_n. $A \mathcal{H} B$ if and only if A and B satisfy the conditions of Theorem 3.2 (iii) and Theorem 3.3 (iii). We will conclude with an example of two $\mathcal{H}$-related matrices of rank two in N_3.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 3 & 5 \\ 3 & 3 & 6 \end{bmatrix} \quad .$$

Denote the row vectors of A and B by A_i and B_i, $i = 1, 2, 3$ and denote the column vectors by A^j and B^j, $j = 1, 2, 3$.

A_1 and A_3 are linearly independent with $A_1 = \frac{1}{3}B_3$ and $A_3 = B_1$. Also A^1 and A^2 are independent with $A^1 = \frac{1}{3}B^2$ and $A^2 = B^1$. So the first condition is satisfied in Theorem 3.2 (iii) and in Theorem 3.3 (iii).

Examining the remaining rows and columns, we see that

$$A_2 = \tfrac{1}{2}A_1 + \tfrac{1}{2}A_3 \qquad B_2 = \tfrac{1}{2}B_1 + \tfrac{1}{2}B_3$$

$$A^3 = A^1 + A^2 \qquad B^3 = B^1 + B^2 \quad .$$

Since all coefficients are nonnegative, the second condition is also satisfied in (iii) of each theorem. Thus ARB, ALB, and so AHB. After some computation, we find that

$$A = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{6} \\ 1 & 0 & 0 \end{bmatrix} B = B \begin{bmatrix} 0 & 1 & 1 \\ \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 0 & 1 \\ \frac{3}{2} & 0 & \frac{1}{2} \\ 3 & 0 & 0 \end{bmatrix} A = A \begin{bmatrix} 0 & 3 & 3 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} .$$

The problem of characterizing the Green's relations $\mathcal{D}$ and $\mathcal{J}$ for arbitrary elements of N_n essentially remains unsolved.

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VITA

William Thomas Ransone was born in Washington, North Carolina on August 19, 1948. He attended the public schools of that city and graduated from Washington High School in 1966. He then attended East Carolina University on academic and athletic scholarships and received a Bachelor of Science degree in Mathematics in May, 1970. He was employed as an instructor of mathematics at Lake Taylor High School in Norfolk, Virginia during the 1970-71 school year. In September, 1971, he entered the University of Tennessee and has been a graduate student in mathematics since that time.