A Method for Determining the Realizability of Tropical Curves in $\mathbb{R}^3$

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To my parents, Jerry and Lee Lee Dusing. Your support and encouragement have kept me going on this long journey.

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Abstract

A tropical variety is a weighted polyhedral complex whose maximal dimensional cells are pure dimensional, rational, connected, and balanced weighted around every vertex. It is known that every irreducible algebraic variety can be tropicalized, that is, there is a way one can derive a tropical variety from an algebraic one. However, there exist tropical varieties that are not the tropicalization of algebraic varieties. The goal of this work is to answer whether a given tropical curve (a $1-$dimensional tropical variety) in $\mathbb{R}^3$ is the tropicalization of an algebraic curve in $\mathbb{R}^3$.

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Chapter 1

Introduction

Suppose we have a field K. The collection of points taking values from K that are the solutions to some collection of polynomial equations form an algebraic variety. The polynomial equations they satisfy generate an ideal in a polynomial ring. Algebraic varieties that are irreducible and $1-$dimensional are called algebraic curves. Examples of algebraic curves include familiar objects like circles, lines, and parabola, and, perhaps, less familiar ones like elliptic curves. The study of ideal, varieties, and curves fall under the areas of algebraic geometry and commutative algebra with the ideal-variety correspondence giving a tight connection between the two.

Algebraic varieties can be tropicalized, that is, there is a map taking an algebraic variety and returning a weighted polyhedral complex called a tropical variety. As an example, we focus on algebraic curves in $\mathbb{Q}_2$, the rational numbers with the $2-$adic valuation. Say, $f = \frac{1}{2}xy+\frac{1}{4}x+\frac{1}{4}y+\frac{1}{2}$. The curve defined by the equation $f = 0$ is a hyperbola. The tropicalization of f is the weighted polyhedral complex consisting of the following five rays/line segments:

1. The ray beginning at $(1, 1)$ in the direction of $(1, 2)$
2. The ray beginning at $(1, 1)$ in the direction of $(2, 1)$
3. The line segment connecting $(1, 1)$ to $(-1, -1)$ through the origin
4. The ray beginning at $(-1, -1)$ in the direction of $(-1, -2)$
5. The ray beginning at $(-1, -1)$ in the direction of $(-2, -1)$

The weight on each polyhedra is 1. The union of these polyhedra is the set of points in $\mathbb{R}^2$ for which the minimum in $\min\{-1+x+y, -2+x, -2+y, -1\}$ is achieved twice. For general polynomials, say, $f = \sum a_{(i,j)}x^iy^j \in \mathbb{Q}[x,y]$, the tropicalization of f is the set of points in $(s,t) \in \mathbb{R}^2$ where the minimum $\min\{\mathrm{val}(a_{(i,j)}) + is + jt : a_{(i,j)} \neq 0\}$ is achieved at least twice. At this point, it is important to note that not all tropical varieties arise as tropicalizations of algebraic varieties, for example, see Exercise 4.7.14 in [10].

Tropicalization, in a sense, reduces an algebraic variety to its combinatorial information. An example of information that is preserved by tropicalization is the dimension of an algebraic variety which is found by computing a Gröbner basis for its defining ideal. The dimension of a polyhedral complex can be easily read off the dimension of the affine span of its largest (with respect to containment) cell. The dimension of an algebraic variety is the same as the dimension of the largest (with respect to containment) cell in the tropical variety.

1.1 More Examples of Tropical Curves and Realizability

Tropical curves are the combinatorial analogue to algebraic curves. They are weighted polyhedral complexes whose maximal (with respect to containment) polyhedra are $1-$dimensional – they consist of rays, lines, and line segments. Some examples of tropical curves are seen in Figures 1.1 and 1.2.

In both Figures 1.1 and 1.2, the polyhedra making up the tropical curves have dimension at most 1, i.e. they consist of points, rays, and line segments. The line segments and rays in Figure 1.1 all have weight 1. In 1.2, the horizontal line in the direction from the origin to $(1,0)$ has weight 2, while the other rays have weight equal to 1.

Tropical curves satisfy a 'balancing' condition around every vertex, that is, if a vertex and all adjacent $1-$dimensional cells are translated to the origin, then the weighted sum of the first lattice points of each cell will be equal to zero. For example, in Figure 1.2, taking the

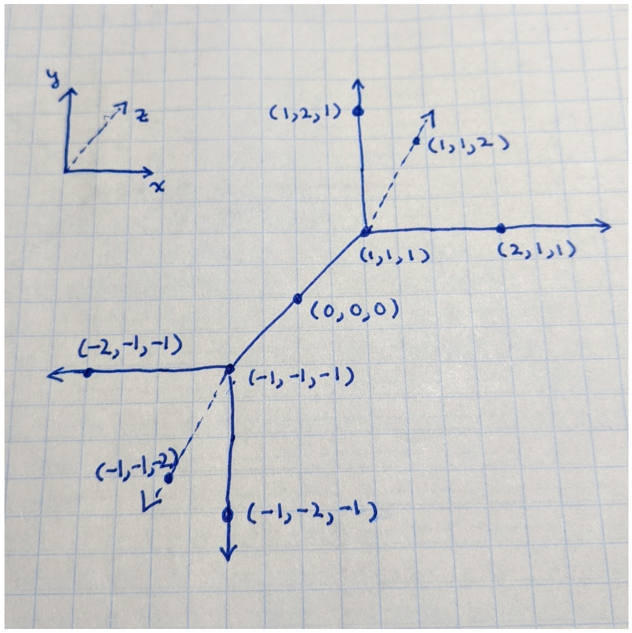

Figure 1.1: A Tropical Curve in $\mathbb{R}^3$

weighted sum of the first lattice points around the origin, we get $(-1,1)+(-1,-1)+2(1,0) = (0,0)$.

The tropical curve in Figure 1.2 is realizable. It is dual to a Newton polygon of the polynomial $f = xy + y^2 + x + 1$. More generally, every tropical curve in $\mathbb{R}^2$ is realizable [10]. This is a consequence of Proposition 3.3.10 in [10], noting that tropical curves and tropical hypersurfaces are the same thing in $\mathbb{R}^2$.

A more complicated problem is as follows: Given a tropical variety Σ in $\mathbb{R}^n$, does there exist an algebraic variety that tropicalizes to Σ? We will answer this question for a class of tropical curves in $\mathbb{R}^3$.

1.2 Rough Sketch of Procedure and Road Map

The procedure we present will answer (in most cases) the question of the realizability of a tropical curve Σ in $\mathbb{R}^3$ that satisfies the following conditions:

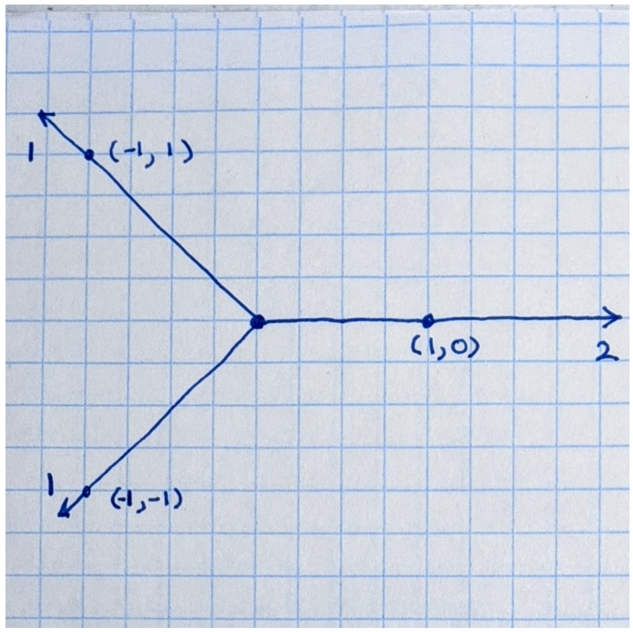

Figure 1.2: A Tropical Curve in $\mathbb{R}^2$

1. Σ is minimally balanced at each vertex
2. $\Sigma = \pi_{xy}^{-1}(\Sigma_{xy}) \cap \pi_{xz}^{-1}(\Sigma_{xz}) \cap \pi_{yz}^{-1}(\Sigma_{yz})$ where the $\Sigma_{xy}, \Sigma_{xz}, \Sigma_{yz}$ denote the (refined) projections of Σ to each of the $xy-$, $xz-$, and $yz-$planes
3. There exists a 1-dimensional cell $\sigma \in \Sigma$ that is parallel to the vector $(1, *, *) \in \mathbb{R}^3$ such that $\text{mult}_{\Sigma_{xy}}(\pi_{xy}(\sigma)) = \text{mult}_{\Sigma_{xz}}(\pi_{xz}(\sigma)) = 1$
4. The Σ_{xy}, Σ_{xz}, Σ_{yz} are tropically irreducible

The procedure that is described in this thesis can be summarized in the following way: Project the tropical curve to each coordinate plane, that is, the $xy-, xz-, yz-$planes. Find polynomials in $K[x, y], K[x, z], K[y, z]$ that realize the projected curve, call them f_{xy}, f_{xz}, f_{yz}. Our procedure determines if the coefficients of the polynomials found in the previous step satisfy certain compatibility conditions so that either the ideal $\langle f_{xy}, f_{yz}, f_{yz}\rangle$ or one of its associated primes is realizes the tropical curve we began with.

The first of the compatibility conditions are a collection of lattice polygons in $\mathbb{R}^2$ corresponding to the subdivided Newton polygons of f_{xy}, f_{xz}, f_{yz}. Each of these polygons must satisfy the hypotheses of Theorem 3.18 and Theorem 3.21. If none of these polytopes can be found, other methods will have to be used to determine if our starting curve is realizable, but for the purposes of this work, our procedure will output 'unknown.' On the other hand, once these polytopes are found, the procedure is guaranteed to answer whether the starting tropical curve is realizable or not. The validity of this procedure is proven in Theorem 3.26 and Theorem 3.27.

Chapter 2 reviews some of the basic concepts used in this work including commutative algebra, Gröbner basis theory, polyhedral geometry, and, of course, a very brief introduction to tropical geometry. In Chapter 3, we define the procedure in detail, as well as its accompanying assumptions and conditions for the procedure to work. The correctness of the procedure is also proven in Chapter 3. Finally, in Chapter 4, we will show that the tropical curve represented by Figure 1.1 is realizable.

1.3 Other Approaches to Determining Realizability

In this section, we provide a brief survey of other work done for determining the realizability of a tropical curve. The work in [15] and [2] are more computational in that they are based on explicitly constructing the variety whose tropicalization is the given tropical curve. In contrast, [8], [3], and [11] are more concerned with finding conditions that are easier to check that are equivalent to realizablility.

In [15], the author considers the realizability problem in the following setting: Given a tropical curve C in $\mathbb{R}^n$ and ideal $I \subseteq \mathbb{C}[c_1, ..., c_k, x_1, ..., x_n]$, for what, if any, $a = (a_1, ..., a_k) \in \mathbb{C}^k$ do we have $\mathrm{Trop}(I_a) = C$? In this context, I_a is the image of I under the map $\iota_a : \mathbb{C}[c_1, ..., c_k, x_1, ..., x_n] \to \mathbb{C}[x_1, ..., x_n]$ where $\iota_a(c_i) = a_i$ and $\iota_a(x_j) = x_j$. The author presents an algorithm for doing so based on comprehensive Gröbner bases [1]. This approach treats the coefficients of the generators for I as parameters and finds compatibility conditions for these parameters so that the specialization I_a of I tropicalizes to C. The approach taken in this work has some similarities based on Newton polytopes. More specifically, the subdivided Newton polygons obtained from projecting our original Tropical curve gives us the compatibility conditions needed to determine realizability. The work done by [15] in that the prescribed method works for tropical curves in any arbitrary dimension but where the field underlying the polynomial ring is $\mathbb{C}$ having trivial valuation. The method presented here is different in three main ways: first, the field K underlying polynomial ring can have non-trivial valuation, second, we work over the polynomial $K[x, y, z]$, and finally, our method avoids having to find the tropicalized family of algebraic curves that our tropical curve is contained in.

The work done in [2] considers the problem for tropical curves C contained in $\Sigma = \text{Trop}(E)$ where $E \subset (K^*)^n$ for some hyperplane E and for some algebraically closed field. Essentially, the tropical curves are projected to the $2-$dimensional coordinate planes, their Newton polygons are found, and then compatibility conditions based on combinatorial data is found which fully answers whether C is realizable or not. The approach of projecting to coordinate planes provided the inspiration for the procedure presented in this work. However, like [15], the method presented in [2] is defined for tropical curves in arbitrary dimension, but where the underlying field has trivial valuation.

Another approach to relative realizability is seen in [8] using rigid geometry and the combinatorics of Chow complexes to determine if a tropical curve is realized by some curve contained in a toric variety.

In [3], the authors show that tropical curves that are non-superabundant, smooth, and $3-$colorable are realizable. Smooth tropical curves are those for which the weights on the maximal dimensional cells are all equal to 1 and the edges around every vertex lie in the directions of $\{b_1, ..., b_n, -\sum_{i=1}^{n} b_i\}$ for some basis $\{b_i\}_{i=1}^{n}$ for $\mathbb{Z}^n$. The $3-$colorability of a tropical curves comes from viewing the tropical curve as a metric graph with unbounded edges. Our method just requires the tropical curve to be minimally balance, Definition 3.22, and the projections to be tropically irreducible, Definition 2.61. Non-superabundance is a combinatorial condition that places a bound on the expected dimension of a cone of tropical maps. For a precise definition we refer the reader to [3], Section 4 or [11], Section 2.2.

The perspective taken by [13] is to show that for certain genus-1 tropical curves, 'well-spacedness' is a sufficient condition for being realizable. In [11], the author uses Artin fans to study families of tropical curves. Using this approach, they show that there exist genus-1 tropical curves that are not well-spaced but are realizable. The method presented here places no restriction on the genus of the given tropical curve.

Chapter 2

Background

We begin this chapter with brief review of commutative algebra. In the first section, we spend some time on Gröbner basis theory. Gröbner bases provides us with computational tools for working with ideals, and hence, varieties. In the section following our foray into Gröbner bases, some concepts from polyhedral geometry and tropical geometry are introduced, and we will see how tropical geometry ties all these concepts together.

2.1 Commutative Algebra

The terminology used in this section is consistent with most introductory commutative algebra texts, namely, "Steps in Commutative Algebra" by R. Y. Sharp [12], "A Course in Commutative Algebra" by Gregor Kemper [9], and "Commutative Algebra" by David Eisenbud [6].

By a ring in this text, we mean a commutative Noetherian ring with unity. We will primarily be working over the ring of polynomials $K[x_1, ..., x_n]$ over some field K, as well as its quotient rings. We assume that K is algebraically closed and has valuation $\text{val} : K^* \to \mathbb{R}$.

The first basic quantity we need to define is the dimension of an ideal.

Definition 2.1 ((Saturated) Chain of Prime Ideals, Length of a Chain)**.** *Let $P_0, P_1, ..., P_m$ be a sequence of prime ideals in some ring R. This sequence is called a chain if $P_0 \subset P_1 \subset .... \subset P_m$, and we say that the chain has length m. A chain is saturated if for each $1 \leq i \leq m$, there does not exist a prime ideal P such that $P_{i-1} \subset P \subset P_i$.*

Definition 2.2 ((Krull) Dimension of a Ring)**.** *The (Krull) dimension of a ring R,* $\dim(R)$*, is the supremum over the lengths of all saturated chains in R.*

By convention, we take the dimension of the zero ring, that is, $R = 0$ to be -1.

Definition 2.3 (Dimension of an Ideal)**.** *Suppose $I \subseteq R$ is an ideal of a ring R. The dimension of I, written* $\dim(I)$*, is the dimension of the quotient ring R/I.*

Definition 2.4 (Algebraic Independence)**.** *Suppose A is a $K-$algebra.*

Let $a_1, ..., a_m \in A$ with $k < \infty$.

We say $a_1, ..., a_m$ is algebraically independent if there does not exist a nonzero polynomial $f \in K[x_1, ..., x_m]$ such that $f(a_1, ..., a_m) = 0$.

Definition 2.5 (Transcendence Degree)**.** *Suppose A is a $K-$algebra.*

We define the transendence degree of A to be the quantity

$$trdeg(A) := \sup\{|T| : T \subseteq A \text{ is finite and algebraically independent}\}$$

When A is a ring that is also a finitely generated $K-$algebra, as in the case that A is a polynomial ring over a field, $\dim(A) = trdeg(A)$ [9, Theorem 5.9]. In the next section, we will see that Gröbner bases gives us an algorithmic way of finding the transcendence degree, and thus the dimension, of an ideal over a polynomial ring.

The codimension of an ideal is another important related concept. Intuitively, the codimension of an ideal I is the dimension of the ring minus the dimension of the ideal.

Definition 2.6 (Codimension of a Prime Ideal)**.** *Let P be a prime ideal in a Noetherian ring R. The codimension of P, written $ht(P)$, is the supremum over the lengths of all saturated chains of prime ideals $P_0 \subset P_1 \subset ... \subset P_m$ with $P = P_m$*

Definition 2.7 (Minimal Primes of a Ring)**.** *Let R be a ring, and P a prime ideal of R. Then, P is a minimal prime of R if it has the property that if Q is a prime ideal of R with $Q \subseteq P$, then $Q = P$.*

Definition 2.8 (Minimal Primes over an Ideal)**.** *Let R be a ring, and I an ideal of R.*

The minimal primes over I are those prime ideals P in R whose image in the quotient ring R/I is a minimal prime of the quotient ring R/I.

Every Noetherian ring has at least one minimal prime, and has at most finitely many minimal primes [9, Corollary 3.14]. Definition 2.8 is equivalent to the following: P is a minimal prime over the ideal I if P satisfies the condition if Q is a prime ideal such that $I \subseteq Q \subseteq P$, then $P = Q$.

Definition 2.9 (Codimension of an Ideal)**.** *Let I be an ideal of the ring R. The codimension of I is the number*

$$ht(I) = \min\{ht(P) : P \textit{ is a minimal prime over } I\}$$

By convention, we take $ht(I) = n + 1$ when $I = R$ and $\dim(R) = n < \infty$.

Observe the following:

1. If P is a minimal prime of R, then $ht(P) = 0$
2. If R is an integral domain, then $ht(\langle 0 \rangle) = 0$
3. When I is prime, Definitions 2.6 and 2.9 coincide.
4. If $I \subseteq J$, then $ht(I) \leq ht(J)$
5. If R is a Noetherian $K-$algebra and $I \subseteq R$ is an ideal, $ht(I) + \dim(R/I) = \dim(R)$

The codimension of non-unit monomial ideals are (relatively) easy to compute. The next theorem describes in detail how this is done. In essence, codimension of a monomial ideal is the smallest collection of indeterminates $x_1, ..., x_n$ on which every generator of the ideal is 'dependent on.'

Theorem 2.10. *Let I be a non-unit monomial ideal in $K[x_1, ..., x_n]$. Say $I = \langle m_1, ..., m_t \rangle$ where m_i is a monomial in $K[x_1, ..., x_n]$ for $1 \leq i \leq t$. Define the following:*

$$\begin{aligned} M_i &= \{l \in \{1, ..., n\} : m_i \in \langle x_l \rangle\} \\ \mathbb{M} &= \{L \subseteq \{1, ..., n\} : L \cap M_i \neq \emptyset \textit{ for all } 1 \leq i \leq t\} \end{aligned} \tag{2.1}$$

Then, the codimension of I is equal to $\min\{|L| : L \in \mathbb{M}\}$

Proof. Let $k := \min\{|L| : L \in \mathbb{M}\}$. We will show that $k = \min\{ht(P) : P$ is a minimal prime over $I\}$.

Fix some $J \in \mathbb{M}$ such that $|J| = k$. Say, $J = \{i_1, ..., i_k\}$, then set $P := \langle x_{i_1}, ..., x_{i_k}\rangle$.

Given $1 \leq j \leq t$, $m_j \in \langle x_{i_l}\rangle$ for some $1 \leq l \leq k$ since $J \cap M_j \neq \emptyset$ for all $1 \leq j \leq t$. Hence, $I \subseteq P$. Suppose Q is some prime such that $I \subsetneq Q \subsetneq P$. Then Q is generated by some proper subset L of J, but this implies that $|J| = \min\{|J| : J \in \mathbb{M}\} > |L|$ which is a contradiction, so P is a minimal prime of I. The chain

$$\langle 0\rangle \subset \langle x_{i_1}\rangle \subset \langle x_{i_1}, x_{i_2}\rangle \subset ... \subset \langle x_{i_1}, x_{i_2}, ..., x_{i_k}\rangle$$

is saturated and is of maximal length with P as the terminal prime ideal. Hence, $ht(I) \leq ht(P) = k$.

As $J = \{i_1, ..., i_k\}$ is the element of $\mathbb{M}$ having smallest cardinality, there does not exist a prime monomomial ideal Q satisfying $I \subsetneq Q \subsetneq \langle x_{i_1}, x_{i_2}, ..., x_{i_k}\rangle$ so $ht(I) = ht(P)$ as desired. □

Theorem 2.10 gives us an easy way of determining if a monomial ideal in $K[x_1, ..., x_n]$ has codimension n. A set of polynomial equations that generate a codimension$-n$ ideal has finitely many solutions. Our next proposition gives a characterisation for a codimension$-n$ ideal.

Proposition 2.11. *Let I be a non-unit monomial ideal in $K[x_1, ..., x_n]$. Say $I = \langle m_1, ..., m_t\rangle$ where m_i is a monomial in $K[x_1, ..., x_n]$ for $1 \leq i \leq t$. Then, I has codimension n if and only if for each $i \in [n]$, there exists $j \in [t]$ and $d \in \mathbb{Z}_{>0}$ such that $m_j = x_i^d$*

Proof. Recall from Theorem 2.10, that

$$M_i = \{l \in \{1, ..., n\} : m_i \in \langle x_l\rangle\} \text{ for each } i \in [t]$$
$$\mathbb{M} = \{L \subseteq \{1, ..., n\} : L \cap M_i \neq \emptyset \text{ for all } 1 \leq i \leq t\}$$

That is, $\mathbb{M}$ is the collection of subsets L of $\{x_1, ..., x_n\}$ such that every monomial generating I, that is $m_1, ..., m_t$, involves every indeterminate in L. The codimension of a monomial ideal is the size of $L \in \mathbb{M}$ with smallest cardinality.

Suppose that, for each $i \in [n]$, there exists $j \in [t]$ and positive integer d such that $m_j = x_i^d$. Then, the smallest subset L of $\{x_1, ..., x_n\}$ for which every monomial generating I involves each indeterminate in L must have size n. Otherwise, if L_0 is strictly smaller than $\{x_1, ...x_n\}$ and $x_i \in \{x_1, ...x_n\} - L_0$, then no monomial $m_1, ..., m_t$ would be a power of x_i which is a contradiction of our hypothesis.

Now, assume to the contrary that none of the monomials generating I is a power of any of the indeterminates, that is, none of $m_1, ..., m_t$ can be written purely in terms of any one of $x_1, ..., x_n$, but this implies that $m_1, ..., m_t$ can be written in terms of all the indeterminates in $\{x_1, ..., x_n\} - \{x_j\}$ since none of the monomials depend on any single indeterminate. So, $\{x_1, ..., x_n\} - \{x_j\}$ has size strictly less then n, and $\min\{|L| : L \in \mathbb{M}\} < n$. So, I has codimension strictly less then n.

Thus, we've proven the contrapositve of the statement: if $I = \langle m_1, ..., m_t \rangle$ is a non-unit (codimension strictly less than $n + 1$) monomial ideal having codimension n, then for each indeterminate $x_1, ..., x_n$, there is a monomial $m_1, ..., m_t$ that depends purely on that indeterminate. □

2.2 Gröbner Bases

Gröbner bases are used heavily in this work. The terminology used here is consistent with the terminology used in the texts Ideals, Varieties, and Algorithms by Cox, Little, and O'Shea [5] and Using Algebraic Geometry [4] by the same authors. Before we can define a Gröbner basis, we will first need some way of ordering the monomials of R. This is equivalent to defining an ordering on the elements of $\mathbb{Z}^n_{\geq 0}$ which are the exponent vectors of the monomials of R.

2.2.1 Monomial Orderings on $K[x_1, ..., x_n]$

Definition 2.12 (Monomial Ordering, [5]). *By a monomial (or term) ordering on $K[x_1, ..., x_n]$, we mean a relation $<$ on $\mathbb{Z}^n_{\geq 0}$ that satisfies the following conditions:*

1. *The relation $<$ is a total ordering on $\mathbb{Z}^n_{\geq 0}$*

2. *If $\alpha < \beta$, then $\alpha + \gamma < \beta + \gamma$ for all $\gamma \in \mathbb{Z}^n_{\geq 0}$*

3. *Every nonempty subset $S \subseteq \mathbb{Z}^n_{\geq 0}$ has a least element, i.e. the relation $<$ satisfies the well-ordering principle.*

An example of a monomial ordering on a polynomial ring, say $R = K[x, y, z]$, is the lexicographic order with the ordering $x > y > z$. In this order, $x^{r_1}y^{s_1}z^{t_1} < x^{r_2}y^{s_2}z^{t_2}$ if either $r_1 < r_2$, or $r_1 = r_2$ and $s_1 < s_2$, or $r_1 = r_2$ and $s_1 = s_2$ and $t_1 < t_2$.

Another example of a monomial ordering on R is the degree lexicographic order with the ordering $x > y > z$. In this order, $x^{r_1}y^{s_1}z^{t_1} < x^{r_2}y^{s_2}z^{t_2}$ if either $r_1 + s_1 + t_1 < r_2 + s_2 + t_2$, or $r_1 + s_1 + t_1 = r_2 + s_2 + t_2$ and $r_1 < r_2$, and so on. The degree lexicographic order can be thought of as first ordering monomials by total degree (the sum of the exponents of the monomials), and then breaking ties using the lexicographic order.

Monomial orders on $K[x_1, ..., x_n]$ can also be defined in terms of an **independent weight vector**, that is, a vector $\mathbf{w} = (w_1, ..., w_n) \in \mathbb{R}^n$ such that the w_i's are positive and $\mathbb{Q}$-linearly independent.

Definition 2.13. *[5, p. 75] The **weight order** $<_{\boldsymbol{w}}$ determined by an independent weight vector $\boldsymbol{w} \in \mathbb{R}^n$ is defined by:*

$$\alpha <_{\boldsymbol{w}} \beta \iff \alpha \cdot \boldsymbol{w} < \beta \cdot \boldsymbol{w}$$

for $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$, and "$\cdot$" denotes the usual dot product on $\mathbb{R}^n$.

We show that Definition 2.13 is indeed a monomial ordering on $K[x_1, ..., x_n]$.

Proposition 2.14. *[5, p. 75] The weight order defined by an independent weight vector $\boldsymbol{w}$ is a monomial order.*

Proof. Let $\alpha, \beta, \gamma \in \mathbb{Z}^n_{\geq 0}$.

First, we show that $<_{\mathbf{w}}$ is a total ordering on $\mathbb{Z}^n_{\geq 0}$.

Suppose $\alpha \cdot \mathbf{w} \leq \beta \cdot \mathbf{w}$ and $\beta \cdot \mathbf{w} \leq \alpha \cdot \mathbf{w}$. Then $(\alpha - \beta) \cdot \mathbf{w} \leq 0$ and $(\alpha - \beta) \cdot \mathbf{w} \geq 0$, so that $(\alpha - \beta) \cdot \mathbf{w} = 0$. But, $\mathbf{w}$ is an independent weight vector, so $\alpha = \beta$.

Suppose $\alpha <_{\mathbf{w}} \beta$ and $\beta <_{\mathbf{w}} \gamma$. Then $\alpha \cdot \mathbf{w} < \beta \cdot \mathbf{w}$ and $\beta \cdot \mathbf{w} < \gamma \cdot \mathbf{w}$. But this implies that $\alpha \cdot \mathbf{w} < \gamma \cdot \mathbf{w}$ so that $\alpha <_{\mathbf{w}} \gamma$.

Since $\alpha \cdot \mathbf{w} \in \mathbb{R}$ for all $\alpha \in \mathbb{Z}^n$, either $\alpha \leq_{\mathbf{w}} \beta$ or $\beta \leq_{\mathbf{w}} \alpha$ for all $\alpha, \beta \in \mathbb{Z}^n$. So, all $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$ are comparable.

Thus, $<_{\mathbf{w}}$ is a total ordering on $\mathbb{Z}^n_{\geq 0}$.

Suppose $\alpha \cdot \mathbf{w} < \beta \cdot \mathbf{w}$. Then, since $\gamma \cdot \mathbf{w} \geq 0$, it follows that $\alpha \cdot \mathbf{w} + \gamma \cdot \mathbf{w} < \beta \cdot \mathbf{w} + \gamma \cdot \mathbf{w}$. It follows then that $\alpha <_{\mathbf{w}} \beta$ implies $\alpha + \gamma <_{\mathbf{w}} \beta + \gamma$.

Finally, since $\alpha \cdot \mathbf{w} \geq 0$ follows from our choice of $\mathbf{w} \in \mathbb{Z}^n_{\geq 0}$, $<_{\mathbf{w}}$ is a well-ordering on $\mathbb{Z}^n_{\geq 0}$. $\square$

Suppose $n \geq 2$ and $\mathbf{w} = (w_1, ..., w_n) \in \mathbb{R}^n$. If $\mathbf{w}$ were not an independent weight vector, then $<_{\mathbf{w}}$ fails total ordering as the following example illustrates:

Example. Let $n = 2$, $\mathbf{w} = (1, 1)$, $\alpha = (2, 0)$ and $\beta = (0, 2)$. Clearly, $\alpha \neq \beta$ even though $\alpha \cdot \mathbf{w} = 2 = \beta \cdot \mathbf{w}$. So the relation $\leq_{(1,1)}$ fails to be anti-symmetric and, thus, fails to be a total ordering on $\mathbb{Z}^2$.

In this case, we say that α, β are 'tied.' In fact, ties will always occur whenever $\mathbf{w} \in \mathbb{Q}^n$. More formally, we have the following:

Proposition 2.15. *If $\boldsymbol{w} = (w_1, ..., w_n) \in \mathbb{Q}^n$ and $w_i > 0$ for all i, then $\exists \alpha \neq \beta \in \mathbb{Z}^n_{\geq 0}$ such that $\alpha \cdot \boldsymbol{w} = \beta \cdot \boldsymbol{w}$.*

Proof. Since $0 < w_i \in \mathbb{Q}$ for each i, $1/w_i \in \mathbb{Q}$. Let k denote the least common denominator of the $1/w_i$. Now let $\alpha = (2k/w_1, ..., 2k/w_{n-1}, 2k/w_n)$ and $\beta = (k/w_1, ..., k/w_{n-1}, 2kn/w_n)$. By our choice of k, it follows that each $2k/w_j$ is an integer. Hence, $\alpha \neq \beta \in \mathbb{Z}^n_{\geq 0}$.

Now, $\alpha - \beta = (k/w_1, ..., k/w_{n-1}, k(1-n)/w_n)$ and hence, $\mathbf{w} \cdot (\alpha - \beta) = kw_1/w_1 + ... + kw_{n-1}/w_{n-1} + k(1-n)w_n/w_n = (n-1)k + k(1-n) = 0$ $\square$

A more general way of specifying a monomial ordering on $K[x_1, ..., x_n]$ is by way of a real-valued $m \times n$ matrix M and we write $<_M$ for the monomial order derived from M. We can view M as matrix with m vectors in $\mathbb{R}^n$, say

$$M = \begin{bmatrix} \mathbf{w_1} \\ \mathbf{w_2} \\ \dots \\ \mathbf{w_m} \end{bmatrix}$$

If $u, v \in \mathbb{Z}^n_{\geq 0}$ are exponent vectors of monomials, then $x^u <_M x^v$ iff $\mathbf{w_1} \cdot u < \mathbf{w_1} \cdot v$ or $\mathbf{w_1} \cdot u = \mathbf{w_1} \cdot v$ and $\mathbf{w_2} \cdot u < \mathbf{w_2} \cdot v$, and so on. If $\ker(M) \cap \mathbb{Z}^n = 0$, then the matrix M specifies a monomial order, and, in fact, monomial orders on $K[x_1, ..., x_n]$ arise in this way.

The lex order on $K[x, y, z]$ comes from the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the grlex order comes from

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Once a monomial order on R has been specified, it makes sense to talk about the 'largest' term in a polynomial.

Definition 2.16 (Leading Term, Leading Monomial, Leading Coefficient)**.** *Let $<$ be a monomial order on $K[x_1, ..., x_n]$, and let $f \in K[x_1, ..., x_n]$.*

Suppose $f = c_{\alpha_d} x_d^{\alpha} + ... + c_{\alpha_1} x_1^{\alpha}$. Here, $\alpha_i < \alpha_{i+1} \in \mathbb{Z}^n_{\geq 0}$, for $i = 1...n$, and $x^{\alpha_i} = x_1^{\alpha_{i,1}} x_2^{\alpha_{i,2}} .... x_n^{\alpha_{i,n}}$.

*Here, x^{α_d} is called the **leading monomial (with respect to $<$)**, denoted by $LM_<(f)$, c_{α_d} is the **leading coefficient**, denoted by $LC_<(f)$, and $c_{\alpha_d} x_d^{\alpha}$ is the **leading term (with respect to $<$)** of f, denoted by $LT_<(f)$.*

Definition 2.17. *[5] Let $<$ be a monomial ordering on $K[x_1, ..., x_n]$ and let I be a nonzero ideal in $K[x_1, ..., x_n]$. Then,*

$$LT_<(I) = \{LT_<(f) | f \in I - \{0\}\}$$

*The monomial ideal $\langle LT_<(I) \rangle$ is called the **initial ideal of I (with respect to $<$)**, which we will also refer to as the **ideal of leading terms of I (with respect to $<$)**.*

A monomial order gives a consistent way of ordering the terms of a polynomial. In turn, this gives us a way of generalizing the division algorithm on univariate polynomials to polynomials with finitely many variables. This generalized division algorithm is defined in [5, Theorem 2.3.3].

It takes as input polynomials $f, f_1, ..., f_s \in K[x_1, ..., x_n]$ (with some monomial order $<$ specified on $K[x_1, ..., x_n]$) and returns polynomials $r, q_1, ..., q_s$ that satisfy

$$f = r + q_1 g_1 + ... + q_s g_s$$

where $\mathrm{LT}_<(\mathrm{r})$ is not divided by $\mathrm{LT}_<(\mathrm{g_i})$ for any $i \in [s]$

Finally, we are ready to define a Gröbner basis of an ideal.

Definition 2.18. *Let $I \subseteq K[x_1, ..., x_n]$ be an ideal, $<$ a monomial order on $K[x_1, ..., x_n]$, and $G = \{g_1, ..., g_s\} \subseteq I$.*

We say G is a Gröbner basis for I with respect to $<$ if $\langle LT_<(I)\rangle = \langle LT_<(g_1), ..., LT_<(g_s)\rangle$

Theorem 2.19. *Let $I \subset K[x_1, ..., x_n]$ be a nonzero ideal.*

Suppose $G = \{g_1, ..., g_s\} \subset I$ is a Gröbner basis for I. Then, $\langle G\rangle = I$

Proof. By hypothesis, $G \subseteq I$. So, $\langle G\rangle \subseteq I$.

Assume to the contrary that the containment is strict, i.e. there is a polynomial $f \in I$, but $f \notin \langle G\rangle$. Since $f \in I$, $\mathrm{LT}_<(\mathrm{f}) \in \langle \mathrm{LT}_<(\mathrm{I})\rangle = \langle \mathrm{LT}_<(\mathrm{g_1}), ..., \mathrm{LT}_<(\mathrm{g_s})\rangle$. By the division algorithm on $K[x_1, ..., x_n]$ [5, Theorem 2.3.3], there exists $q_1, ..., q_s, r \in K[x_1, ..., x_n]$ such that $f = r + q_1 g_1 + ... + q_s g_s$ so that $r = f - (q_1 g_1 + ... + q_s g_s) \in I$ and $\mathrm{LT}_<(\mathrm{r}) \in \langle \mathrm{LT}_<(\mathrm{I})\rangle = \langle \mathrm{LT}_<(\mathrm{g_1}), ..., \mathrm{LT}_<(\mathrm{g_s})\rangle$. This implies that $\mathrm{LT}_<(\mathrm{r})$ is divided by $\mathrm{LT}_<(\mathrm{g_i})$ for some $i \in [s]$, which is an impossibility as this is a contradiction of what the division algorithm on $K[x_1, ..., x_n]$ returns as output. □

Gröbner bases give us an algorithmic way of performing computations on ideals. For example, we see in [9, Corollary 11.14], that if $I \subset K[x_1, ..., x_n]$ is an ideal and $G \subset I$ is a Gröbner basis with respect to a fixed monomial ordering $<$ on $K[x_1, ..., x_n]$, then the Krull dimension of I is the same as $\langle \mathrm{LT}_<(\mathrm{G}) \rangle$. That is,

$$\dim(I) = \dim(\langle \mathrm{LT}_<(\mathrm{G}) \rangle) \tag{2.2}$$

Given a finite collection $S \subset K[x_1, ..., x_n]$ and a polynomial $f \in K[x_1, ..., x_n]$, some books refer to the property that no monomial of f is divisible by $\mathrm{LT}_<(\mathrm{g})$ for each $g \in S$ as f being in **normal form** with respect to S.

Theorem 2.20. *Let $I \subset K[x_1, ..., x_n]$ be an ideal.*

Fix a monomial ordering $<$ on $K[x_1, ..., x_n]$, and let $G = \{g_1, ..., g_s\}$ be a Gröbner basis for I with respect to $<$.

Then, the set of monomials not in $\langle \mathrm{LT}_<(\mathrm{I}) \rangle$ forms a $K-$basis for the quotient ring $\dfrac{K[x_1, ..., x_n]}{I}$.

Proof. By Proposition 1 [5][p. 248], $M = \{x^\alpha = x_1^{\alpha_1} .... x_n^{\alpha_n} : x^\alpha \notin \langle \mathrm{LT}_<(\mathrm{I}) \rangle\}$ are $K-$linearly independent in $\dfrac{K[x_1, ..., x_n]}{I}$. Consider the map $K[x_1, ..., x_n] \to \dfrac{K[x_1, ..., x_n]}{I}$ sending each element of $K[x_1, ..., x_n]$ to its equivalence class in $\dfrac{K[x_1, ..., x_n]}{I}$. According to [5, Theorem 2.3.3], given any $f \in K[x_1, ..., x_n]$ there are $r, q_1, ..., q_s \in K[x_1, ..., x_n]$ such that

$$f = r + q_1 g_1 + ... + q_s g_s$$

with $\mathrm{LT}_<(\mathrm{r})$ not divisible by $\mathrm{LT}_<(\mathrm{g_i})$ for each $i \in [s]$. Hence, no term of r lies in $\langle \mathrm{LT}_<(\mathrm{I}) \rangle$. So, every element in $\dfrac{K[x_1, ..., x_n]}{I}$ is a $K-$linear combination of the elements of M. □

2.2.2 Elimination Theory

The intersection of an ideal with a subring of R, for example $I \cap K[x, y]$, is an ideal in the subring. Once a Gröbner basis for I has been found it is easy to find a subset of the Gröbner basis that generates the ideal in the subring.

Definition 2.21. *Let I be an ideal in $K[x_1, .., x_l, x_{l+1}, ..., x_n]$. The $l^{th}-$**elimination ideal** is the ideal $I \cap K[x_{l+1}, ..., x_n]$ which is an ideal in $K[x_{l+1}, ..., x_n]$.*

Different orderings on the indeterminates will produce different monomial orderings.

Definition 2.22. *Let $S \subseteq \{x_1, ..., x_n\}$.*

A monomial order $<$ on $K[x_1, ..., x_n]$ is of S-elimination type if any monomial in S is greater than all other monomials.

An example of an elimination order on $R = K[x, y, z]$ is a z-elimination order, i.e. any monomial in z is greater than all monomials in $K[x, y]$. This order can be induced by the matrix

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Then, clearly, $z^2 >_M x^2y^3z >_M x^4y^4$.

Proposition 2.23. *Suppose $l := \{x_1, ..., x_l\} \subsetneq \{x_1, ..., x_n\}$*

Let I be an ideal in $K[x_1, ..., x_n]$, and let $<$ be a monomial order on I that is of $l-$elimination type.

If G is a Gröbner basis for I with respect to $<$, then $G \cap K[x_{l+1}, ..., x_n]$ is a Gröbner basis for $I \cap K[x_{l+1}, ..., x_n]$.

Proof. Let $G_l := G \cap K[x_{l+1}, ..., x_n] = \{g_1, ..., g_s\}$ and $I_l := I \cap K[x_{l+1}, ..., x_n]$.

Obviously, $\langle \mathrm{LT}_<(\mathrm{G_l}) \rangle \subseteq \langle LT_<(I_l) \rangle$ since $G_l \subseteq I_l$.

Now, suppose $f \in I_l$. Since $f \in I$, there exists $g \in G$ such that $\mathrm{LT}_<(\mathrm{f}) = \mathrm{LT}_<(\mathrm{g})$. As f depends only on the indeterminates $x_{l+1}, ..., x_n$, it follows that LT(f) does as well. Since $<$ is of l-elimination type and $\mathrm{LT}(\mathrm{g}) = \mathrm{LT}(\mathrm{f}) \in K[x_{l+1}, ..., x_n]$, no term of g depends on any of $x_1, ..., x_l$ so that $g \in K[x_{l+1}, ..., x_n]$. Thus, if $\mathrm{LT}_<(\mathrm{f}) \in I_l$, then $\exists g \in G_l$ such that $\mathrm{LT}_<(\mathrm{g}) = \mathrm{LT}_<(\mathrm{f})$. □

Finding a Gröbner basis with respect to an elimination order allows us to find the primary decomposition (not necessarily minimal or irredundant) of an elimination ideal:

Proposition 2.24. *Suppose $Q \subset K[x_1, ..., x_l, x_{l+1}, ..., x_n]$ is a primary ideal with prime radical $P := \sqrt{Q}$.*

If $Q \cap K[x_{l+1}, ..., x_n] \neq 0$, then $Q \cap K[x_{l+1}, ..., x_n]$ is primary and $\sqrt{Q \cap K[x_{l+1}, ..., x_n]}$ is equal to $P \cap K[x_{l+1}, ..., x_n]$

Proof. Let $<$ be some monomial order on $K[x_1, ..., x_n]$ that is of l-elimination type (Definition 2.22).

Suppose $fg \in Q \cap K[x_{l+1}, .., x_n]$. Then $fg \in Q$ and has no monomial divisible by any $x_1, ..., x_l$.

Since $fg \in Q$, either $f \in Q$ or $g^m \in Q$ for some $m \geq 1$.

Suppose $f \in Q$ and assume $f \notin K[x_{l+1}, .., x_n]$. Then some term of f divisible by at least one of $x_1, ..., x_l$. As $<$ is of l-elimination type, $LT_<(f)$ is divisible by at least one of $x_1, ..., x_l$, say $LT_<(f) = kx_1^{\alpha_1}...x_l^{\alpha_l}x_{l+1}^{\alpha_{l+1}}...x_n^{\alpha_n}$ for some $\gamma = (\alpha_1, .., \alpha_l) \neq 0$. But $fg \in K[x_{l+1}, .., x_n]$ so $LT_<(fg) = k_1 x_{l+1}^{\alpha'_{l+1}}...x_n^{\alpha'_n}$, i.e. there exists some $\gamma' = (\alpha'_1, ..., \alpha'_l) \geq 0$ such that $\gamma' + \gamma = 0$. As this is an impossibility, it follows that $f \in K[x_{l+1}, ..., x_n]$.

Since $g^m \in K[x_{l+1}, .., x_n]$ if and only if $g \in K[x_{l+1}, .., x_n]$, it follows that if $g^m \in Q$, then $g^m \in K[x_{l+1}, .., x_n]$ as well, reasoning similarly as before.

Hence, if $fg \in Q \cap K[x_{l+1}, .., x_n]$, then either $f \in Q \cap K[x_{l+1}, .., x_n]$ or $g^m \in Q \cap K[x_{l+1}, .., x_n]$ for some $m \geq 1$ so that $Q \cap K[x_{l+1}, .., x_n]$ is primary.

Obviously, $P \cap K[x_{l+1}, .., x_n]$ is a prime ideal in $K[x_{l+1}, .., x_n]$. Then we have the following equivalences:

$$\begin{aligned} f \in \sqrt{Q \cap K[x_{l+1}, .., x_n]} &\iff \exists m \geq 1 \text{ such that } f^m \in Q \cap K[x_{l+1}, .., x_n] \\ &\iff \exists m \geq 1 \text{ such that } f^m \in Q \text{ and } f \in K[x_{l+1}, .., x_n] \\ &\iff f \in P \text{ and } f \in K[x_{l+1}, .., x_n] \\ &\iff f \in P \cap K[x_{l+1}, .., x_n] \end{aligned} \tag{2.3}$$

Hence, $\sqrt{Q \cap K[x_{l+1}, .., x_n]} = P \cap K[x_{l+1}, .., x_n]$ so each $Q \cap K[x_{l+1}, .., x_n]$ is $P \cap K[x_{l+1}, .., x_n]-$primary. □

Proposition 2.25. *Suppose $I \subset K[x_1, ..., x_l, x_{l+1}, ..., x_n]$ is an ideal such that $I \cap K[x_{l+1}, .., x_n] \neq 0$, and that I has primary (not necessarily minimal or irredundant) decomposition $I = \cap_{i=1}^k Q_i$ for some $k \in \mathbb{Z}_{\geq 1}$. Suppose further that for each $i \in [k]$, there exists a prime ideal P_i such that $\sqrt{Q_i} = P_i$.*

Then, $I \cap K[x_{l+1}, .., x_n]$ has primary decomposition $\cap_{i=1}^k (Q_i \cap K[x_{l+1}, .., x_n])$ and each $Q_i \cap K[x_{l+1}, .., x_n]$ is $P_i \cap K[x_{l+1}, .., x_n]-$primary.

Proof. Since $I \cap K[x_{l+1}, .., x_n] \neq 0$ and $I \subseteq Q_i$ for each $i \in [k]$, $Q_i \cap K[x_{l+1}, .., x_n] \neq 0$ for each i as well since $I \cap K[x_{l+1}, .., x_n] \subseteq Q_i \cap K[x_{l+1}, .., x_n]$.

Obviously, $K[x_{l+1}, .., x_n] \cap I = K[x_{l+1}, .., x_n] \cap (\cap_{i=1}^{k} Q_i) = \cap_{i=1}^{k}(Q_i \cap K[x_{l+1}, .., x_n])$. By Proposition 2.24, for each $i \in [k]$, $Q_i \cap K[x_{l+1}, ..., x_n]$ is primary and $\sqrt{Q_i \cap K[x_{l+1}, .., x_n]}$ is equal to $P_i \cap K[x_{l+1}, .., x_n]$.

Thus, I has primary decomposition $\cap_{i=1}^{k}(Q \cap K[x_{l+1}, .., x_n])$. $\square$

2.3 Polytopes

In this section, we define what is meant by a polytope in $\mathbb{R}^n$ which is a combinatorial object defined by vertices. We then derive some results about polytopes and define the Newton polytope of a polynomial. Polytopes, particularly Newton polytopes, are a fundamental object used in this text.

Definition 2.26. *A set $S \subseteq \mathbb{R}^n$ is **convex** if given $x, y \in S$, $sx + ty \in S$ for all $s, t \in [0, 1]$ such that $s + t = 1$.*

Clearly, $\mathbb{R}^n$ is a convex set, and the empty set vacuously satisfies the definition of convexity. By convention, we consider a set consisting of a single element to be convex.

Proposition 2.27. *Suppose $P \subset \mathbb{R}^n$.*

Then, P is convex if and only if $t_1u_1 + ... + t_ku_k \in P$ for all $u_i \in P$ and $t_1, ..., t_k \in [0, 1]$ such that $t_1 + ... + t_k = 1$.

Proof. Suppose that P satisfies the condition that $t_1u_1 + ... + t_ku_k \in P$ for all $u_i \in P$ and $t_1, ..., t_k \in [0, 1]$ such that $t_1 + ... + t_k = 1$. Obviously this implies tha P is convex since we can choose $k = 2$.

Suppose P is convex. Our proof is by induction on $k \geq 2$.

Suppose $k = 2$. Suppose $u_1, u_2 \in P$ and $t_1, t_2 \in [0, 1]$ such that $t_1 + t_2 = 1$, then since P is convex, $t_1u_1 + t_2u_2 \in P$ by definition.

Suppose $k > 2$ and that the result holds for $s \leq k$. Let $t_1, ..., t_s, t_{s+1} \in [0,1]$ be such that $t_1 + ... + t_s + t_{s+1} = 1$ and let $u_1, ..., u_s, u_{s+1} \in P$. Then we have the following:

$$t_1u_1 + ... + t_su_s + t_{s+1}u_{s+1} = (1 - t_{s+1})(\frac{t_1}{1 - t_{s+1}}u_1 + ... + \frac{t_s}{1 - t_{s+1}}u_s) + t_{s+1}u_{s+1} \qquad (2.4)$$

But,

$$\frac{t_1}{1 - t_{s+1}} + ... + \frac{t_s}{1 - t_{s+1}} = \frac{t_1 + ...t_s}{1 - t_{s+1}} \qquad (2.5)$$

Since $t_1 + ... + t_s + t_{s+1} = 1$ it follows that $1 - t_{s+1} = t_1 + ...t_s$. Hence,

$$\frac{t_1}{1 - t_{s+1}}u_1 + ... + \frac{t_s}{1 - t_{s+1}}u_s \in P$$

and since $1 = (1 - t_{s+1}) + t_{s+1}$ with $t_{s+1}, 1 - t_{s+1} \in [0,1]$, it follows that

$$(1 - t_{s+1})(\frac{t_1}{1 - t_{s+1}}u_1 + ... + \frac{t_s}{1 - t_{s+1}}u_s) + t_{s+1}u_{s+1} \in P$$

□

Proposition 2.28. *Suppose J is some nonempty set and P_α are convex subsets of $\mathbb{R}^n$ for all $\alpha \in J$.*

Then, $\bigcap_{\alpha \in J} P_\alpha$ is convex.

Proof. If $\bigcap_{\alpha \in J} P_\alpha = \emptyset$, then $\bigcap_{\alpha \in J} P_\alpha$ is vacuously convex.

Suppose $x, y \in \bigcap_{\alpha \in J} P_\alpha$. Since the P_α are convex by hypothesis, $sx + ty \in P_\alpha$ for all $s, t \in [0,1]$ with $s + t = 1$ and $\alpha \in J$. So, $sx + ty \in \bigcap_{\alpha \in J} P_\alpha$ for all $s, t \in [0,1]$ such that $s + t = 1$. Thus, $\bigcap_{\alpha \in J} P_\alpha$ is convex. □

Definition 2.29. *Given a set $S \subset \mathbb{R}^n$, the* ***convex hull of*** *S is defined as:*

$$\text{conv}(S) = \cap\{C : S \subseteq C \subseteq^{convex} \mathbb{R}^n\}$$

Observe the following:

1. The convex hull of S is the smallest convex set in $\mathbb{R}^n$ containing S

2. S is convex if and only if $\text{conv}(S) = S$.

Proposition 2.30. *Let $S \subseteq \mathbb{R}^n$.*

Then, $\text{conv}(S) = \{t_1v_1 + ...t_sv_s : t_1, ..., t_s \in [0,1], t_1 + ...t_s = 1, \text{ and } v_1, ..., v_s \in S\}$

Proof. Let R denote the set in right-hand side in the proposition statement. Then, R is obviously convex containing S, so $\text{conv}(S) \subseteq R$.

Suppose C is convex containing S. Then, for all $t_1, t_2 \in [0,1]$ with $t_1 + t_2 = 1$ and for all $u_1, u_2 \in S$, $t_1u_1 + t_2u_2 \in C$. But, $t_1u_1 + t_2u_2 \in R$ as well by construction. Hence, R is contained in all convex subsets of $\mathbb{R}^n$ containing S, so $R \subseteq \text{conv}(S)$. □

Observe that the convex hull of a convex set is itself.

Definition 2.31 (Polytope and Lattice Polytope)**.** *By a **polytope** P in $\mathbb{R}^n$, we mean the convex hull of some finite set of points (called **vertices**) $v_1, ..., v_k \in \mathbb{R}^n$. If $n = 2$, then we say P is a **polygon**. By a **lattice polytope**, we mean the convex hull of a set of vertices in $\mathbb{Z}^n_{\geq 0}$.*

Definition 2.32. *If $P \subseteq \mathbb{R}^n$ is a polytope, we define the **dimension of a polytope** P as*

$$\dim(P) = \inf\{\dim(A) : P \subseteq A \subseteq^{\textit{affine}} \mathbb{R}^n\}$$

That is, the dimension of P is the dimension of the smallest affine subset of $\mathbb{R}^n$ containing P.

Recall that an affine set A of $\mathbb{R}^n$ is a translate of a linear subspace L, i.e. $A = L + a$ for some $a \in \mathbb{R}^n$.

Example. Let $P \subset \mathbb{R}^2$ be the polytope generated by the points $(0,1)$ and $(1,1)$. This is the same as the line segment joining the points $(0,1)$ and $(1,1)$. The smallest affine set containing P is the line $\{(x,y) : y = 1\}$.

Definition 2.33. *Let $U, V \subseteq \mathbb{R}^n$. The **Minkowski sum** of U and V, written $U + V$, is the set*

$$U + V := \{u + v : u \in U, v \in V\}$$

Proposition 2.34. *If P, Q are convex sets, then their Minkowski sum $P+Q$ is again convex.*

Proof. Let $u, v \in P+Q$, and $s, t \in [0,1]$ be such that $s+t=1$. Then $u = p_u + q_u$ and $v = p_v + q_v$ for some $p_u, p_v \in P$ and $q_u, q_v \in Q$. So,

$$\begin{aligned} su + tv &= s(p_u + q_u) + t(p_v + q_v) \\ &= sp_u + sq_u + tp_v + tq_v \\ &= (sp_u + tp_v) + (sq_u + tq_v) \end{aligned} \tag{2.6}$$

Since $sp_u + tp_v \in P$ and $sq_u + tq_v \in Q$ by convexity, $su + tv \in P + Q$ so that $P+Q$ is convex. $\square$

Definition 2.35. *Suppose $f \in K[x_1, ..., x_n]$ say $f = \sum c_{k_1,...,k_n} x_1^{k_1} ... x_n^{k_n}$.*

*The **Newton polytope** of f is the lattice polytope that is the convex hull in $\mathbb{R}^n$ of the set $\{(k_1, ..., k_n) : c_{k_1,...,k_n} \neq 0\}$.*

Proposition 2.36. *Suppose $\Delta := \operatorname{conv}(v_1, .., v_k) \subset \mathbb{R}^n$ for some $k \in \mathbb{Z}_{>0}$ and $v_i \neq v_j$ for all $i \neq j$. Let $S(\Delta) := \{v_1, ..., v_k\}$ and $I := \{S \subseteq S(\Delta) : \operatorname{conv}(S) = \Delta\}$. Then, I has a unique minimal element.*

Proof. Obviously, I is nonempty since, at least, $S(\Delta) \in I$. The collection $(I, \leq)$, where $S_1, S_2 \in I$ satisfies $S_1 \leq S_2$ if and only if $S_2 \subseteq S_1$, is a partially ordered set.

Let $\{S_i\}_{i=1}^n$ be a nonempty subcollection of I satisfying $S_1 \leq S_2 \leq ... \leq S_n$. Then, $S_n \subseteq S_{n-1} \subseteq ... \subseteq S_1$. Then, S_n is the maximal element of $\{S_i\}_{i=1}^n$ with respect to $\leq$. Hence, every chain in I has a maximal element with respect to $\leq$, and by Zorn's lemma, I has a maximal element with respect to $\leq$.

Suppose $S, S' \in I$ are both maximal with respect to $\leq$. Then, say $S = \{u_i\}_{i=1}^r$ and $S' = \{w_i\}_{i=1}^s$ with $[r], [s] \subseteq [k]$. Since $\operatorname{conv}(S) = \operatorname{conv}(S') = \Delta$, there exists $t_{1i}, ..., t_{ri} \in [0,1]$ such that $t_{1i} + ... + t_{ri} = 1$ and $t_{1i}u_1 + ... + t_{ri}u_r = w_i$ for each $i \in [s]$. Similarly, there exists $\gamma_{1j}, ..., \gamma_{sj} \in [0,1]$ such that $\gamma_{1j}w_1 + ... + \gamma_{sj}w_s = u_j$ for each $j \in [r]$. So,

$$\begin{aligned} u_1 &= \gamma_{11}w_1 + ... + \gamma_{s1}w_s \\ &= \gamma_{11}(t_{11}u_1 + ... + t_{r1}u_r) + ... + \gamma_{1s}(t_{1s}u_1 + ... + t_{rs}u_r) \\ &= \alpha_1 u_1 + ... + \alpha_r u_r \end{aligned}$$

where $\alpha_i = \sum_j \gamma_{ij} t_{ij}$ So, $(1-\alpha_1)u_1 = \alpha_2 u_2 + ... + \alpha_r u_r$. But, $\alpha_1 = \sum_j \gamma_{1j} t_{1j} < \sum_j t_{1j} = 1$, so, $1-\alpha_1 > 0$. Hence, u_1 can be written as a convex sum of $u_2, ..., u_r$. So, the set $\{u_2, ..., u_r\} \subsetneq S$ generates Δ which contradicts the minimality of S. $\square$

Definition 2.37 (Vertices of a Polytope)**.** *Suppose* $\Delta := \mathrm{conv}(v_1, .., v_k) \subset \mathbb{R}^n$ *for some* $k \in \mathbb{Z}_{>0}$ *and* $v_i \neq v_j$ *for all* $i \neq j$*. Let* $S(\Delta) := \{v_1, ..., v_k\}$ *and* $I := \{S \subseteq S(\Delta) : \mathrm{conv}(S) = \Delta\}$*.*

The vertices of Δ*,* $V(\Delta)$*, is the minimal over all* $S \in I$*. When* Δ *is the Newton polytope of a polynomial* f*, we write* $V(f)$*.*

Lemma 2.38. *Suppose* $\Delta := \mathrm{conv}(v_1, .., v_k) \subset \mathbb{R}^n$ *for some* $k \in \mathbb{Z}_{>0}$ *and* $v_i \neq v_j$ *for all* $i \neq j$*.*

Then, v_1 *is a vertex of* Δ *if and only if* $\Delta - \{v_1\}$ *is convex.*

Proof. Suppose v_1 is a vertex of Δ, but $\Delta - \{v_1\}$ is nonconvex. Then, there exists $t_2, ..., t_k \in [0, 1]$ such that $t_2 + ... + t_k = 1$ and $t_2 v_2 + ... + t_k v_k \notin \Delta - \{v_1\}$. Since Δ is convex, it follows that $t_2 v_2 + ... + t_k v_k \in \Delta$. Hence, $t_2 v_2 + ... + t_k v_k = v_1$. Since $\Delta = \mathrm{conv}(v_1, v_2, ..., v_k)$ and $t_2 v_2 + ... + t_k v_k = v_1$, it follows that $\Delta = \mathrm{conv}(v_2, ..., v_k)$. So there is a subset of $\{v_1, ..., v_k\}$ not containing v_1 that generates Δ. This contradicts our hypothesis that v_1 is a vertex of Δ. So, $\Delta - \{v_1\}$ is convex.

Now, suppose that $\Delta - \{v_1\}$ is convex, but v_1 is not a vertex of Δ. Then, $\mathrm{conv}(v_2, ..., v_k) = \Delta$. Since $v_1 \in \Delta$, there exists $t_2, ..., t_k \in [0, 1]$ such that $t_2 + ... + t_k = 1$ and $t_2 v_2 + ... + t_k v_k = v_1 \notin \Delta - \{v_1\}$. So, we have contradicted our hypothesis that $\Delta - \{v_1\}$ is convex. $\square$

Proposition 2.39. *Suppose* $\Delta := \mathrm{conv}(v_1, .., v_k) \subset \mathbb{R}^n$ *for some* $k \in \mathbb{Z}_{>0}$ *and* $v_i \neq v_j$ *for all* $i \neq j$*.*

If $v_1 = ra + sb$ *for some distinct* $a, b \in \Delta$ *and positive* r, s *such that* $r + s = 1$*, then* v_1 *is not a vertex.*

Proof. First, $a, b \neq v_1$. Otherwise, if, say $a = v_1$, then $v_1 = ra + sb = rv_1 + sb$ so that $v_1 = b$ which is a contradiction of our hypothesis that a, b are distinct. Then, $a, b \in \Delta - \{v_1\}$ since $a, b \neq v_1$ but $a, b \in \Delta$.

Assume to the contrary that v_1 is a vertex. Then, $\Delta - \{v_1\}$ is convex, but then, $ra + sb = v_1 \in \Delta - \{v_1\}$ which is a contradiction. $\square$

Lemma 2.40. *Suppose $f, g \in K[x_1, ..., x_n]$. Let $P(f), P(g)$ denote the Newton polytopes of f, g respectively.*

Then,

$$P(f \cdot g) = P(f) + P(g)$$

Proof. Let $S(f)$ denote the support of $f = \sum c_u x_1^{u_1} ... x_n^{u_n}$, that is

$$S(f) = \{u = (u_1, ..., u_n) \in \mathbb{Z}^n_{\geq 0} : c_u \neq 0\}$$

and $S(g)$ is defined analogously for $g = \sum d_{v_1,...,v_n} x_1^{v_1} ... x_n^{v_n}$. Then, $P(f) = \text{conv}(S(f))$ and $P(g) = \text{conv}(S(g))$. Without loss, we may assume that $S(f) = \{a_1, ..., a_l\}$ and $S(g) = \{b_1, ..., b_m\}$ where each a_i and b_j is a $n-$tuple of nonnegative integers.

First, we show that $\text{conv}(S(f) + S(g)) = P(f) + P(g)$. Obviously, since $P(f) = \text{conv}(S(f))$ by definition, $\text{conv}(S(f) + S(g)) \subseteq P(f) + P(g)$. Now, suppose $u \in P(f) + P(g)$. Then, there exists nonnegative $s_1, ..., s_l$ and $t_1, ..., t_m$ such that $\sum s_i = \sum t_j = 1$ and

$$u = \sum_{i=1}^{l} s_i a_i + \sum_{j=1}^{m} t_j b_j$$

Since $\sum s_i = \sum t_j = 1$, we have

$$\begin{aligned} \sum_{i=1}^{l} s_i a_i + \sum_{i=1}^{m} t_i b_i &= \sum_{i=1}^{l} s_i (\sum_{j=1}^{m} t_j) a_i + \sum_{j=1}^{m} t_j (\sum_{i=1}^{l} s_i) b_j \\ &= \sum_{i=1}^{l} \sum_{j=1}^{m} s_i t_j (a_i + b_j) \end{aligned}$$

so that $u \in \text{conv}(S(f) + S(g))$ and $P(f) + P(g) = \text{conv}(S(f) + S(g))$.

We can write $f \cdot g = \sum_w \alpha_w x_1^{w_1} ... x_n^{w_n}$, where $\alpha_w = \sum\limits_{u+v=w} c_u d_v$ and each u, v lies in $S(f)$ and $S(g)$ respectively. Suppose $w \in S(f \cdot g)$. Then, there exists some $c_u \neq 0$ and $d_v \neq 0$ so that $\sum_{k+e=u} c_u d_v = \alpha_w \neq 0$ so $w = u + v$ for some $u \in S(f)$ and $v \in S(g)$. Hence, $S(f \cdot g) \subseteq S(f) + S(g)$ which further implies that $P(f \cdot g) \subseteq \text{conv}(S(f) + S(g)) = P(f) + P(g)$.

We note that $V(P(f) + P(g)) \subseteq V(f) + V(g)$. To show this, we argue as before but replace $S(f), S(g)$ with $V(f), V(g)$ respectively. This gives $\text{conv}(V(P(f) + P(g)) = P(f) + P(g) =$

conv($V(f) + V(g)$). By Definition 2.37, $V(P(f) + P(g))$ is minimal over generators of $P(f) + P(g)$, so the vertices of $P(f) + P(g)$ are contained in $V(f) + V(g)$.

It remains to show that the vertices of $P(f) + P(g)$ lie in the support of $f \cdot g$. Let w be a vertex of $P(f) + P(g)$. Then, there exists i, j vertices of $P(f)$ and $P(g)$ respectively such that $w = i + j$. Since i, j are vertices $c_i d_j \neq 0$.

Assume to the contrary that $w \notin S(f \cdot g)$, then $\alpha_w = 0$. Since $0 = \alpha_w = \sum_{u+v=w} c_u d_v$, $c_i d_j \neq 0$ and $i + j = w$, we have:

$$\begin{aligned} 0 &= \alpha_w \\ &= c_i d_j + \sum_{\substack{u+v=w \\ u\neq i, v\neq j}} c_u d_v \end{aligned}$$

so that $0 \neq c_i d_j = - \sum_{\substack{u+v=w \\ u\neq i, v\neq j}} c_u d_v$. This further implies that $c_u d_v \neq 0$ for at least on pair of u, v not equal to i, j respectively. For each u, v with $c_u v_v \neq 0$ in the sum $\sum_{\substack{u+v=w \\ u\neq i, v\neq j}} c_u d_v$, we have $u \neq i$ and $v \neq j$ but $u + v = i + j$.

To obtain our contradiction, we will show that w can be written as $ra + sb$ for some positive r, s with $r + s = 1$ and distinct $a, b \in P(f) + P(g)$ with neither a, b equal to w. Since $u + v = i + j$, we have

$$\begin{aligned} w = i + j &= \frac{1}{2}(2i + 2j) \\ &= \frac{1}{2}(i + j + u + v) \\ &= \frac{1}{2}(i + v + j + u) \\ &= \frac{1}{2}(i + v) + \frac{1}{2}(j + u) \end{aligned}$$

Then, $i + v \neq j + u$, otherwise we have,

$$u + v = i + j$$
$$u + v + i + v = i + j + j + u$$
$$2v = 2j$$
$$v = j$$

which contradicts the fact that $v \neq j$. Furthermore, since $u \neq i$ and $v \neq j$, it follows that neither $i + v = i + j$ nor $j + u = i + j$. Thus, by Proposition 2.39, $w = i + j$ is not a vertex of $P(f) + P(g)$ which is a contradiction of our assumption that w is a vertex. So, $\alpha_w \neq 0$, and $w \in S(f \cdot g)$ as desired. Thus, $P(f) + P(g) = P(f \cdot g)$. □

Definition 2.41 (Newton Subdivision)**.** *Let $P(f)$ denote the Newton polygon of the polynomial $f \in K[x, y]$, where K is a field with nontrivial valuation* $\mathrm{val} : K^* \to \mathbb{R}$*. Suppose $P(f) = \mathrm{conv}\{(u_1, v_1), ..., (u_m, v_m)\}$ where the coefficient of the monomial $x^{u_i}y^{v_i}$ is nonzero for each $i \in [m]$. Define $P^e(f) \subset \mathbb{R}^2 \times \mathbb{R}$ as the set $\mathrm{conv}\{(u_i, v_i, z_i) : i \in [m]$ and $z_i \geq \mathrm{val}(c_i)\}$ where c_i is the coefficient of the monomial $x^{u_i}y^{v_i}$.*

The Newton subdivision on $P(f)$ is the image of $P^e(f)$ under the map $\pi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ by $(r, s, t) \mapsto (r, s)$.

Example. Let $f = \frac{1}{2}xy + \frac{1}{4}x + \frac{1}{4}y + \frac{1}{2} \in \mathbb{Q}[x, y]$ where $\mathbb{Q}$ has the $2-$adic valuation. In this example, $P(f) = \mathrm{conv}\{(1, 1), (1, 0), (0, 1), (0, 0)\}$, and the Newton subdivision on $P(f)$ is the same as $P(f)$ with a line segment connecting $(1, 0)$ and $(0, 1)$. In this work, by subdivided Newton polygons, we mean the Newton polygon with its Newton subdivision.

2.4 Tropical Geometry

2.4.1 Polyhedral Complexes and Fans

Definition 2.42 (Polyhedron, Face of a Polyhedron)**.** *By a polyhedron P in $\mathbb{R}^n$, we mean the intersection of finitely many closed half-spaces, that is, a set having the form*

$$\{x \in \mathbb{R}^n : Ax \leq b\}$$

Here, for some $\mathbb{R}-$valued $m \times n$ matrix A and $b \in \mathbb{R}^m$

By a face of P, we mean a set determined by some $w \in (\mathbb{R}^n)^\vee$ defined by $\{x \in P : w \cdot x \leq w \cdot y$ for all $y \in P\}$

Notice that polytopes are bounded polyhedra.

Definition 2.43 (Polyhedral Cone)**.** *By a (polyhedral) cone in $\mathbb{R}^n$, we mean a polyhedron C, satisfying the condition that if $v_1, ..., v_k \in C$ and $\lambda_1, ..., \lambda_k \geq 0$, then $\lambda_1 v_1 + ... + \lambda_k v_k \in C$.*

The simplest nonempty cone in $\mathbb{R}^n$ is $\{\mathbf{0}\}$. For every cone in $\mathbb{R}^n$, there exists an $m \times n$ matrix A such that $C = \{x \in \mathbb{R}^n : Ax \leq \mathbf{0}\}$

Definition 2.44 (Polyhedral Complex)**.** *By a polyhderal complex in $\mathbb{R}^n$, we mean a collection Σ of polyhedra, called the cells of Σ, satisfying the following conditions:*

1. *If $P \in \Sigma$, then any face of P also lies in Σ*

2. *If $P, Q \in \Sigma$, then either $P \cap Q = \emptyset$ or $P \cap Q$ is a face of both P and Q*

Definition 2.45 (Support of a Polyhedral Complex)**.** *Suppose Σ is a polyhedral complex in $\mathbb{R}^n$. The support of Σ, written $|\Sigma|$ is the union of the cells in Σ.*

Definition 2.46 (Subdivision of a Polytope)**.** *The subdivision of a polytope P is a polyhedral complex C consisting of polytopes such that $|C| = P$.*

We are particularly interested in 'regular' subdivisions of polytopes as defined in Definition 5.3 of [16] and Definition 2.3.8 of [10].

Definition 2.47 (Polyhedral Fan)**.** *Let $\mathfrak{F} = \{C_1, ..., C_k\}$ be a nonempty finite collection of cones in $\mathbb{R}^n$.*

We call $\mathfrak{F}$ a (polyhedral) fan if it satisfies the following conditions:

1. *Every nonempty face of a cone is again a cone in $\mathfrak{F}$*

2. *The intersection of any two cones in $\mathfrak{F}$ is a face of both*

Definition 2.48 (Recession Cone, Recession Fan). *Let σ is a polyhedron in $\mathbb{R}^n$, then there exists some $m \times n$ matrix A with entries in $\mathbb{R}^n$ and $b \in \mathbb{R}^m$ such that $\sigma = \{x \in \mathbb{R}^n : A \cdot x \leq b\}$*

We define the recession cone of σ to be the set

$$rec(\sigma) = \{x \in \mathbb{R}^n : A \cdot x \leq \mathbf{0}\}$$

For a polyhedral complex Σ in $\mathbb{R}^n$, the recession fan of Σ is the collection

$$rec(\Sigma) = \{rec(\sigma) : \sigma \in \Sigma\}$$

Definition 2.49. *A polyhedral complex Σ is pure of dimension d if every face that is maximal with respect to containment has dimension d.*

Definition 2.50 (Refinement of a Polyhedral Complex). *Let Σ and Σ' be polyhedral complexes.*

We say that Σ' is a refinement of Σ if $|\Sigma| = |\Sigma'|$, and, for every $\sigma \in \Sigma$, there exists a $\sigma' \in \Sigma'$ such that $\sigma' \subseteq \sigma$

Definition 2.51 (Common Refinement). *Let Σ_1 and Σ_2 be pure $d-$dimensional polyhedral complexes in $\mathbb{R}^n$. The polyhedral complex $\Sigma_1 \wedge \Sigma_2 := \{\sigma_1 \cap \sigma_2 : \sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2\}$ is called the common refinement of Σ_1 and Σ_2 .*

Proposition 2.52. *Suppose $|\Sigma_1| = |\Sigma_2|$. Then, $|\Sigma_1 \wedge \Sigma_2| = |\Sigma_1| = |\Sigma_2|$.*

Proof. First, we note that if P is a polyhedral complex in $\mathbb{R}^n$, then

$$|P| = \{x \in \mathbb{R}^n : x \in \sigma \text{ for some } \sigma \in P\}$$

So, $x \in |\Sigma_1 \wedge \Sigma_2|$ if and only if $x \in \sigma_1 \cap \sigma_2$ for some $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$, if and only if $x \in |\Sigma_1| \cap |\Sigma_2| = |\Sigma_1| = |\Sigma_2|$. □

Definition 2.53. *A polyhedral complex Σ is connected through codimension l if given $P, Q \in \Sigma$, there exists a finite sequence of cells, say $P = P_1, ..., P_k = Q$ such that $P_i \cap P_{i+1}$ has codimension l for $i = 1, ..., k-1$*

Definition 2.54 (Equivalent Polyhedral Complexes). *Let $\Sigma = \{\sigma_i\}_{i \in I}$ and $\Sigma' = \{\sigma'_j\}_{j \in J}$ be pure dimensional polyhedral complexes. We say that Σ' and Σ are equivalent if $|\Sigma| = |\Sigma'|$*

We discuss the effect of projection on a polyhedral complex. Naively projecting a polyhedral complex to one of the coordinate hyperplanes does not always preserve the structure of a polyhedral complex.

Take, for example, Σ to collection in $\mathbb{R}^3$ consisting of the polyhedra $\sigma_1 = \mathbb{V}(y, z+1)$ and $\sigma_2 = \mathbb{V}(x, z-1)$. Clearly, $\sigma_1 \cap \sigma_2 = \emptyset \in \Sigma$. Also, the only proper faces of σ_1 and σ_2 are the empty set. Hence, Σ is a polyhedral complex.

Projecting Σ to the $xy-$plane gives the collection $P = \{\pi(\sigma_1) = \mathbb{V}(y), \pi(\sigma_2) = \mathbb{V}(x)\}$. While the faces of each element of P are again in P, $\pi(\sigma_1) \cap \pi(\sigma_2) = (0,0) \notin P$. So, P is not a polyhedral complex.

However, if define $\Sigma_i = \{s_1, s_2, s_3, s_4, (0,0)\}$ where $s_1 = \pi(\sigma_1) \cap \{x \geq 0\}$, $s_3 = \pi(\sigma_1) \cap \{x \leq 0\}$, $s_2 = \pi(\sigma_2) \cap \{y \geq 0\}$, $s_4 = \pi(\sigma_2) \cap \{y \leq 0\}$. We get a refinement of P that is a polyhedral complex. Also, notice that $|P| = |\Sigma_i|$.

Below, we provide a definition for the projection of a polyhedral complex to a coordinate hyperplane.

Definition 2.55 (Projection of a Polyhedral Complex). *Let Σ be a $1-$dimensional polyhedral complex in $\mathbb{R}^n$ that is not contained in a hyperplane. Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ denote the projection to the i^{th}-coordinate hyperplane.*

We define the projection of Σ to be the complex Σ_i containing the following:

1. *The zero dimensional cells are:*

   (a) $\pi(\sigma)$ *where* $\sigma \in \Sigma$ *with* $\dim(\sigma) = 0$

   (b) $\pi(\sigma) \cap \pi(\tau)$ *where* $\sigma, \tau \in \Sigma$ *with* $\dim(\pi(\sigma)) = \dim(\pi(\tau)) = 1$ *and* $\dim(\pi(\sigma) \cap \pi(\tau)) = 0$

   (c) $\pi(\sigma)$ *where* $\sigma \in \Sigma$ *has* $\dim(\sigma) = 1$, *but* $\dim(\pi(\sigma)) = 0$

2. *The $1-$dimensional cells are finite intersection of some maximal (with respect to containment) collection* $\{\pi(\sigma) : \sigma \in \Sigma \text{ and } \dim(\pi(\sigma)) = 1\}$

2.4.2 Weighted Polyhedral Complexes

Definition 2.56. *A weighted polyhedral complex is a pure dimensional polyhedral complex* Σ *together with a function* $m : \Sigma \to \mathbb{Z}_{>0}$ *called the multiplicities of the cells of* Σ*.*

Definition 2.57. *Let* (Σ, m) *be a weighted pure* $1-$*dimensional polyhedral complex, and let* $\sigma_1, ..., \sigma_k \in \Sigma$ *be a collection of* $1-$*dimensional cells sharing a single common* $0-$*dimensional face,* x*. Let* $v_1, ..., v_k$ *be the first lattice points of* $\sigma_1, ..., \sigma_k$ *respectively.*

We say that Σ *is balanced weighted at* x *if*

$$\sum_{i=1}^{k} m(\sigma_i) v_i = 0$$

If we say that Σ *is balanced weighted without specifying the* $0-$*dimensional cell, then we mean that* Σ *is balanced weighted at each of its* $0-$*dimensional cells.*

Definition 2.58. *Suppose* (Σ, m) *and* (Σ', m') *are balanced weighted, pure* $1-$*dimensional polyhedral complexes.*

We say that (Σ, m) *and* (Σ', m') *are equivalent if we have the following:*

1. $|\Sigma| = |\Sigma'|$
2. *If* $\sigma \in \Sigma$ *and* $\sigma' \in \Sigma'$ *have* $\dim(\sigma \cap \sigma') > 0$*, then* $m(\sigma) = m'(\sigma')$

Definition 2.59 (Multiplicities of Projected Cells)**.** *Let* (Σ, m) *be a balanced-weighted, pure* $1-$*dimensional polyhedral complex in* $\mathbb{R}^n$*, and* Σ_i *denote the projection of* Σ *as defined in Definition 2.55.*

Let $\sigma \in \Sigma_i$ *be a* $1-$*dimensional cell. We define the multiplicity of* σ *to be:*

$$m'(\sigma) = \sum_{\substack{\pi(\tau)=\sigma \\ \tau \in \Sigma}} m(\tau)[(\mathbb{Z}^{n-1})_\sigma : \pi((\mathbb{Z}^n)_\tau)]$$

If σ *is a* $1-$*dimensional polyhedron in* $\mathbb{R}^n$*, then* $(\mathbb{Z}^n)_\sigma$ *denotes the lattice points in the linear subspace of* $\mathbb{R}^n$ *parallel to* σ*. If* σ *is* $0-$*dimensional, then* $(\mathbb{Z}^n)_\sigma = \mathbb{Z} \cdot \mathbf{0}$*. The index* $[(\mathbb{Z}^{n-1})_\sigma : \pi((\mathbb{Z}^n)_\tau)]$ *denotes the index of* $\pi((\mathbb{Z}^n)_\tau)$ *as a subgroup of* $(\mathbb{Z}^{n-1})_\sigma$

Definition 2.60 (Tropical Curve). *By a tropical curve, we mean a balanced weighted, pure $1-$dimensional polyhedral complex (Σ, m) that is connected.*

Definition 2.61 (Tropically Irreducible). *Let (Σ, m) be a tropical curve in $\mathbb{R}^n$.*

We say (Σ, m) is tropically irreducible if the following hold:

1. *If Σ_1, Σ_2 are polyhedral complexes such that $\Sigma = \Sigma_1 \cup \Sigma_2$, then either $\Sigma = \Sigma_1$ or $\Sigma = \Sigma_2$*
2. *If $m' : \Sigma \to \mathbb{Z}_{>0}$ is such that (Σ, m') is balanced and $m'(\sigma) \leq m(\sigma)$ for all $\sigma \in \Sigma$, then either $m' = 0$ or $m' = m$.*

2.4.3 Tropicalizations of Ideals and Polynomials

Definition 2.62 (Tropicalization of a Polynomial). *Suppose $f \in K[x_1, ..., x_n]$ and $f = \sum_{u \in \mathbb{Z}^n_{\geq 0}, c_u \neq 0} c_u x^u$ and $w \in \mathbb{R}^n$. The tropicalization of f with respect to w, written $\operatorname{trop}(f)$ is given by*

$$\operatorname{trop}(f)(w) = \min\{\operatorname{val}(c_u) + w \cdot u : c_u \neq 0\}$$

Definition 2.63 (Tropical Hypersurface). *Let $f \in K[x_1, ..., x_n]$. The tropical hypersurface $\operatorname{Trop}(f)$ is the set $\{w \in \mathbb{R}^n :$ the minimum in $\operatorname{trop}(f)(w)$ is achieved more than once $\}$*

Definition 2.64 (Initial Form of a Polynomial). *Suppose $f \in K[x_1, ..., x_n]$ and $f = \sum_{u \in \mathbb{Z}^n_{\geq 0}, c_u \neq 0} c_u x^u$ and $w \in \mathbb{R}^n$. The initial form of f with respect to w, written $in_w(f)$, is the sum of those terms of f that attain the minimum in $\operatorname{trop}(f)(w)$.*

In the trivial valuation case, $in_w(f)$ is the sum of the terms of f with minimum w-weight.

These definitions lead to Kapranov's Theorem [10, Theorem 3.1.3]:

Theorem 2.65. *(Kapranov's Theorem) Suppose K is an algebraically closed field with nontrivial valuation. Fix $f \in K[x_1, ..., x_n]$. Then the following three sets are the same:*

1. *the tropical hypersurface $\operatorname{Trop}(f)$ in $\mathbb{R}^n$*
2. *the set $\{w \in \mathbb{R}^n : in_w(f)$ is not monomial$\}$*
3. *the closure in $\mathbb{R}^n$ of the set $\{(\operatorname{val}(y_1), ..., \operatorname{val}(y_n)) : (y_1, ..., y_n) \in \mathbb{V}(f) \subset K^n\}$*

Definition 2.66 (Initial Ideal). *Suppose $I \subseteq K[x_1, ..., x_n]$ is an ideal and $w \in \mathbb{R}^n$. The initial ideal of I with respect to w is given by*

$$in_w(I) = \langle in_w(f) : f \in I \rangle$$

Definition 2.67 (Tropicalization of an Ideal). *Suppose $I \subseteq K[x_1, ..., x_n]$ is an ideal and $w \in \mathbb{R}^n$. The tropicalization of I is given by*

$$\operatorname{Trop}(I) = \{w \in \mathbb{R}^n : in_w(I) \textit{ does not contain a monomial } \}$$

If I is a principal ideal, say $I = \langle f \rangle$, then the tropicalization of I coincides with the tropical hypersurface $\operatorname{Trop}(f)$.

2.4.4 The Weights on a Tropical Variety

Definition 2.68 (Multiplicity of a Minimal Associated Prime). *Let $I \subset K[x_1, ..., x_n]$ be an ideal, and P a minimal associated prime of I. We define the multiplicity of P, written* $\operatorname{mult}(P, I)$, *is defined as the largest possible length of $K[x_1, ..., x_n]_P-$modules in $((I : P^\infty)/I)_P$*

Definition 2.69. *Suppose Σ is supported on the ideal I and $\sigma \in \Sigma$ is a cell. The weight on the cell σ is called the multiplicty of σ, written* $\operatorname{mult}_\Sigma(\sigma) = \operatorname{mult}_{\operatorname{Trop}(I)}(\sigma)$, *and is given by the formula*

$$\operatorname{mult}_{\operatorname{Trop}(I)}(\sigma) = \sum_{P \in A} \operatorname{mult}(P, in_w(I))$$

where A is the collection of all minimal associated primes P of $in_w(I)$ for any $w \in$ $\operatorname{relint}(\sigma)$.

2.4.5 Tropicalization of an Ideal Produces a Weighted Polyhedral Complex

Let $I \subseteq K[x_1, ..., x_n]$ and $w \in \mathbb{R}^n$. We define $C_I[w] := \{w' \in \mathbb{R}^n : in_w(I) = in_{w'}(I)\}$. The closure in $\mathbb{R}^n$, $\overline{C_I[w]}$, is a polyhedron, and the collection $\{(\overline{C_I[w]}, \operatorname{mult}(\overline{C_I[w]})\}$ is a pure dimensional polyhedral complex that is connected in codimension 1.

Theorem 2.70. *Let $I \subseteq K[x_1, ..., x_n]$ be an ideal. Then,* $\mathrm{Trop}(I)$ *is the support of the Gröbner complex of I.*

Proof. Let $\Sigma(I) := \{(\overline{C_I[w]}, \mathrm{mult}(\sigma_w))\}$. We wish to show that $|\Sigma(I)| = \mathrm{Trop}(I)$. To this end, we will let $\square$

Theorem 2.71. *Suppose I is an ideal of $K[x_1, ..., x_n]$. Then,* $\mathrm{Trop}(I)$ *has dimension d if and only if* $\dim(I) = d$.

Definition 2.72. *Suppose Σ is a tropical variety in $\mathbb{R}^n$ and I is an ideal in $K[x_1, ..., x_n]$. We say that I realizes Σ (equivalently, Σ is supported on I) if* $\mathrm{Trop}(I) = |\Sigma|$*, and that a tropical variety Σ is realizable if such an ideal exists.*

We complete this section by showing that every Tropical curve in $\mathbb{R}^2$ is realizable by some ideal.

Theorem 2.73. *Every tropical curve Σ in $\mathbb{R}^2$ is realizable.*

Proof. By Proposition 3.3.10 [10], there exists a Laurent polynomial $f \in K[x^{\pm 1}, y^{\pm 1}]$ such that $|\Sigma| = \mathrm{Trop}(f)$. Furthermore, by the same proposition, if Δ denotes the subdivided Newton polytope dual to Σ, then the lattice lengths of each edge of Δ corresponds to the multiplicity of its dual. That is, if $e \in \Delta$ is a $1-$dimensional cell with length $l(e)$, then the cell $\sigma \in \Sigma$ corresponding to σ has multiplicity $\mathrm{mult}_{\Sigma}(\sigma) = l(e)$ $\square$

Chapter 3

Procedure: Definition and Proof of Correctness

3.1 Statement of Procedure and Necessary Hypotheses

Recall that we are trying to answer the following question: Is a tropical curve Σ in $\mathbb{R}^3$ realizeable by some $1-$dimensional prime ideal $I \subset K[x, y, z]$? In this chapter, we define the procedure that will answer this question and prove the correctness of this procedure.

Essentially, the procedure is to project the tropical curve to each of the $xy-, xz-,$ and $yz-$planes, and since tropical curves in $\mathbb{R}^2$ are realizable, find subdivided newton polytopes that realize each of the projected tropical curves, and they satisfy certain compatiblity conditions that we define later. In order for our procedure to work, we will need the following hypotheses to be true for our tropical curve Σ:

(i) $\Sigma = \pi_{xy}^{-1}(\Sigma_{xy}) \cap \pi_{xz}^{-1}(\Sigma_{xz}) \cap \pi_{yz}^{-1}(\Sigma_{yz})$ where the $\Sigma_{xy}, \Sigma_{xz}, \Sigma_{yz}$ denote the polyhedral complexes as defined in Theorem 2.55

(ii) Σ is minimally balanced at each vertex (this is defined in section 3.22)

(iii) There exists a 1-dimensional cell $\sigma \in \Sigma$ that is parallel to the vector $(1, *, *) \in \mathbb{R}^3$ such that $\mathrm{mult}_{\Sigma_{xy}}(\pi_{xy}(\sigma)) = \mathrm{mult}_{\Sigma_{xz}}(\pi_{xz}(\sigma)) = 1$

(iv) The projections of Σ, that is Σ_{xy}, Σ_{xz}, Σ_{yz}, are tropically irreducible as in Definition 2.61.

Once Σ has been verified to satisfy the preceding hypotheses, we can follow the procedure below to determine if our tropical curve is realizable.

1. Determine if Σ is contained in a plane. If it is, then the procedure outputs 'Yes.' Otherwise, proceed to the next step.

2. Project Σ to the $xy-$, $xz-$, $yz-$ planes and refine as prescribed in Definition 2.55 to obtain polyhedral complexes in $\mathbb{R}^2$

3. Find subdivided Newton polytopes P_{xy}, P_{xz}, and P_{yz} dual to Σ_{xy}, Σ_{xz}, Σ_{yz} respectively. We may do so by Theorem 2.73.

4. Find polytopes P'_{xy}, P'_{xz}, and P'_{yz} corresponding to P_{xy}, P_{xz}, and P_{yz} respectively, that satisfy the conditions of Theorem 3.18 and Theorem 3.21. If none can be found, then output 'Unknown.' Otherwise, we proceed to the next step.

5. Construct a matrix M of coefficients whose rows are indexed by the polynomials contained in the set $(P'_{xy} + P_{xy}) \cup (P'_{xz} + P_{xz}) \cup (P'_{yz} + P_{yz})$ and whose columns are indexed by the monomials contained in each $P'_i + P_i$. We describe this process and some necessary conditions on the P'_i's in greater detail in Subsection 3.3.1.

6. Compute a set of generators G for the maximal minors of the matrix M.

7. Let α denote the vector of valuations that determine the subdivision on the Newton polytopes P_{xy}, P_{xz}, P_{yz} and compute the initial ideal $in_\alpha(G)$.

8. If $in_\alpha(G)$ contains no monomials, then the procedure returns 'Yes.' Return 'No' otherwise.

As seen above, our procedure necessarily produces one of the following answers regarding the tropical curve Σ:

(i) Yes – There is a prime ideal I such that $\mathrm{Trop}(I) = \Sigma$.

(ii) No – There does not exist a prime ideal I such that $\text{Trop}(I) = \Sigma$.

(iii) Unknown – It is unknown if Σ is realizable.

An implication of the Fundamental Theorem of Tropical Algebraic Geometry [10, Theorem 3.2.3] is that if $in_\alpha(G)$ contains no monomials, then there exists $p \in \mathbb{V}(G)$ such that the pointwise valuations, $\text{val}(p) = \alpha$. Hence, if f_{xy}, f_{xz}, f_{yz} have the same Newton polytopes as the duals of $\Sigma_{xy}, \Sigma_{xz}, \Sigma_{yz}$ respectively, and have coefficients specialized at $p \in \mathbb{V}(G)$, then M has strictly less than full rank. We'll show that this condition on the rank of M implies that $\langle f_{xy}, f_{xz}, f_{yz}\rangle$ has an associated prime I that satisfies the elimination conditions.

3.2 Multiplicities of the One-Dimensional Cells of Tropical Curves

In this section, we develop some additional machinery that powers our procedure. We begin by proving some results about the behavior of tropical curves under projection to the coordinate planes. Following this, we define the polytopes and coefficient matrix referenced in the statement of the procedure, and then we develop some results from commutative algebra that allow for their use. For these upcoming technical results, we do not assume the hypotheses on Σ that we defined at the beginning of this chapter unless otherwise explicitly noted.

3.2.1 Projections of Tropical Curves

The projection of tropical varieties is studied in [14]. In this section, we derive some results regarding the projection of tropical curves.

Suppose Σ is a tropical curve in $\mathbb{R}^n$ such that $\text{Trop}(I) = \Sigma$ for some ideal in $K[x_1, ..., x_n]$. In Definition 2.55, we defined a polyhedral complex Σ_i that contains the projection of Σ to the $i^{th}-$coordinate hyperplane. While $\text{Trop}(I \cap K[x_1, ..., x_{i-1}, x_{i+1}, ..., x_n]$ may not equal to Σ_i, they are equivalent as we will show in the following theorem:

Theorem 3.1 (Tropical Closure Theorem)**.** *Suppose $I \subset K[x_1, ..., x_n]$ is an ideal.*

Define the following:

$$\begin{aligned}
&\pi : \mathbb{R}^n \to \mathbb{R}^{n-1} \textit{ by} \\
&\pi : (s_1, ..., s_n) \mapsto (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n) \\
&I_i := I \cap K[x_1, ..., x_{i-1}, x_{i+1}, ..., x_n] \qquad (3.1)\\
&\overline{\pi(\mathbb{V}(I))} \textit{ is the smallest variety containing } \pi(\mathbb{V}(I)) \\
&\Sigma_i \textit{ is the polyhedral complex as defined in Definition 2.55}
\end{aligned}$$

Then, $|\Sigma_i| = |\operatorname{Trop}(I_i)|$.

Proof. By [5, Theorem 3.2.3], $\overline{\pi(\mathbb{V}(I))} = \mathbb{V}(I_i)$ so that $\operatorname{Trop}(\overline{\pi(\mathbb{V}(I))}) = \operatorname{Trop}(I_i)$.

Next we note that the projection map is a monomial map, so by [10, Corollary 3.2.13], $|\Sigma_i| = |\operatorname{Trop}(I_i)|$. □

Theorem 3.1 shows that the polyhedral complex Σ_i is equivalent to the tropicalization of the elimination ideal. The theorem says nothing about the resulting multiplicities, however. The last two theorems of this section shows how the multiplicities of the $1-$dimensional cells of Σ_i and $\operatorname{Trop}(I_i)$ are related. The proof of these theorems rely on intersecting a $1-$dimensional cell of $\Sigma_i \wedge \operatorname{Trop}(I_i)$ (we showed in Proposition 2.52 that $|\Sigma_i| = |\Sigma_i \wedge \operatorname{Trop}(I_i)| = |\operatorname{Trop}(I_i)|$) with a classical line that satisfies certain properties. The following theorem shows that such a line does indeed exist.

Another concept that plays in important role in the next few results is that of the stable intersection between polyhedral complexes.

Definition 3.2 (Stable Intersection of Tropical Curves)**.** *Let Σ_1, Σ_2 be rational balanced weighted pure polyhedral complexes in $\mathbb{R}^n$. The stable intersection of Σ_1 and Σ_2 is defined by*

$$\Sigma_1 \cap_{st} \Sigma_2 = \bigcup_{\substack{\sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2 \\ \dim(\sigma_1 + \sigma_2) = n}} \sigma_1 \cap \sigma_2$$

If $\dim(\sigma_1 \cap \sigma_2) = n$*, then it has multiplicity*

$$\operatorname{mult}_{\Sigma_1 \cap_{st} \Sigma_2}(\sigma_1 \cap \sigma_2) = \sum_{\tau_1, \tau_2} \operatorname{mult}_{\Sigma_1}(\tau_1) \operatorname{mult}_{\Sigma_2}(\tau_2)[\mathbb{Z}^n : (\mathbb{Z}^n)_{\tau_1} + (\mathbb{Z}^n)_{\tau_1}]$$

Here, the sum is over each $\tau_1 \in star_{\Sigma_1}(\sigma_1 \cap \sigma_2)$ and $\tau_2 \in star_{\Sigma_2}(\sigma_2 \cap \sigma_2)$ with $\tau_1 \cap (\tau_2 + v) \neq \emptyset$ for some fixed generic v. Further, letting L_τ denote the linear subspace of $\mathbb{R}^n$ parallel to τ, $(\mathbb{Z}^n)_\tau = L_\tau \cap \mathbb{Z}^n$ which is nonempty since τ has rational slope.

Notice that the stable intersection of Σ_1 and Σ_2 is contained in their set-theoretic intersection. The containment is possibly strict.

Example. Let $\Sigma_1 = \{(\sigma_1, m_1)\}$ and $\Sigma_2 = \{(\sigma_2, m_2)\}$ where $\sigma_1 = \mathbb{R} \cdot (1,1) \subset \mathbb{R}^2$ and $\sigma_2 = \mathbb{R} \cdot (1,0)$ with $m_1 = m_2 = 1$. Observe that Σ_1 and Σ_2 are tropical curves, and their stable intersection is the zero-dimensional polyhedral complex $\Sigma_1 \cap_{st} \Sigma_2 = \{(\mathbf{0}, m)\}$ where $m = [\mathbb{Z}^2 : (\mathbb{Z}^2)_{\sigma_1} + (\mathbb{Z}^2)_{\sigma_2}] = |\det \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}| = 1$

Lemma 3.3. *Let $I \subset K[x,y,z]$ be an ideal and $\Sigma = \mathrm{Trop}(I)$ be a tropical curve in $\mathbb{R}^3$ that is not contained in any plane, and let $\sigma \in \Sigma$ be a $1-$dimensional cell whose projection $\pi_{xy}(\sigma)$ is also $1-$dimensional.*

Then, there exists $w = (w_1, w_2, w_3) \in \mathrm{relint}(\sigma)$ such that for at least one of $X = \mathbb{V}(x - w_1)$ or $Y = \mathbb{V}(y - w_2)$, the set theoretic intersection of X (or Y) with Σ is equal to their stable intersection.

Proof. $\mathrm{Trop}(I)$ is the union of finitely many $1-$dimensional cells (lines or line segments), say $\mathrm{Trop}(I) = \sigma \cup \tau_1 \cup ... \cup \tau_k$ for some $k \in \mathbb{Z}_{>0}$.

For each $i = 1...k$, choose $H_i \subseteq \mathbb{R}^3$ in the following manner:

If $\pi_{xy}(\tau_i) \subseteq \mathbb{R}^2$ is $1-$dimensional, then let L_i be the affine span in $\mathbb{R}^2$ of $\pi_{xy}(\tau_i)$. Then $H_i := \pi_{xy}^{-1}(L_i)$ is a plane in $\mathbb{R}^3$ containing τ_i. If $\pi_{xy}(\tau_i)$ is zero-dimensional, then $\pi_{xy}(\tau_i) = (v_1, v_2)$. Let $L_i := \mathbb{V}(x - v_1, y - v_2)$, then $H_i := \pi_{xy}^{-1}(L_i)$ is a line in $\mathbb{R}^3$ containing τ_i. Let $S \subseteq \{1, ..., k\}$ be such that if $s \in S$, then $\sigma \not\subset H_s$. If $S = \emptyset$, then $\sigma \subset H_i$ for each $i = 1...k$. Then, $\pi_{xy}(\sigma) \subset L_i$ for each $i = 1...k$, so that each L_i is $1-$dimensional since $\pi_{xy}(\sigma)$ is $1-$dimensional by hypothesis, but this implies that $L_i = L_j$ for each $i, j = 1...k$ which further implies that $\mathrm{Trop}(I)$ is contained in a plane. As this contradicts our assumption that $\mathrm{Trop}(I)$ is not contained in a plane, $S \neq \emptyset$. Let $P = \bigcup_{s \in S} H_s$. Then $P \cap \sigma$ has at most finitely many points, so we may choose $w = (w_1, w_2, w_3) \in \mathrm{relint}(\sigma) \cap (\mathbb{R}^3 - P)$.

At least one of $\mathbb{V}(x - w_1)$ or $\mathbb{V}(y - w_2)$ intersects σ only at w. If σ intersects both $\mathbb{V}(x - w_1)$ and $\mathbb{V}(y - w_2)$ at a point different from w, then $\sigma \subseteq \mathbb{V}(x - w_1, y - w_2)$ which implies that $\pi_{xy}(\sigma)$ is $0-$dimensional which is a contradiction of our hypothesis.

Say $\mathbb{V}(x - w_1)$ intersects σ only at w. Then, as a plane in $\mathbb{R}^3$, $\mathbb{V}(x - w_1)$ intersects $\mathrm{Trop}(I)$ at only finitely many points, otherwise $\mathbb{V}(x - w_1)$ contains a $1-$dimensional cell of $\mathrm{Trop}(I)$ which implies that $\mathbb{V}(x - w_1) \in \mathbb{P}$ which in turn contradicts our choice of w, and as a line in $\mathbb{R}^2$, $\mathbb{V}(x - w_1)$ intersects $\mathrm{Trop}(I \cap K[x, y])$ at only finitely many points, and in particular, $\mathbb{V}(x - w_1)$ intersects $\mathrm{Trop}(I \cap K[x, y])$ transversely at (w_1, w_2). Then, any $g = cx \in K[x] \subseteq K[x, y]$ with $\mathrm{val}(c) = w_1$ will satisfy $\mathrm{Trop}(g) = x - w_1$. $\square$

Proposition 3.4. *Let Σ be a tropical curve in $\mathbb{R}^3$ that is not contained in a plane, let $g \in K[x, y]$ a polynomial such that $\mathrm{Trop}(g)$ intersects Σ at finitely many places and Σ_{xy} transversely everywhere, and let $(\Sigma \cap \mathrm{Trop}(g))_{xy}$ denote the polyhedral complex that is the refinement of the $\pi_{xy}(\Sigma \cap \mathrm{Trop}(g))$ as in Definition 2.55. Then,*

$$\Sigma_{xy} \cap \mathrm{Trop}(g) = (\Sigma \cap \mathrm{Trop}(g))_{xy}$$

Proof. Suppose $\Sigma \cap \mathrm{Trop}(g) = \{u_1, ..., u_k\}$ where $u_j = (u_{j,1}, u_{j,2}, u_{j,3}) \in \mathbb{R}^3$ for each $1 \leq j \leq k$. Then, $\pi_{xy}(\Sigma \cap \mathrm{Trop}(g)) = \{\pi_{xy}(u_j) : 1 \leq j \leq k\}$, but without loss, we may suppose that $\pi_{xy}(\Sigma \cap \mathrm{Trop}(g)) = \{\pi_{xy}(u_1), ..., \pi_{xy}(u_l)\}$ where $l \leq k$. Refining $\pi_{xy}(\Sigma \cap \mathrm{Trop}(g))$ leaves the set unchanged by Definition 2.55, so $\pi_{xy}(\Sigma \cap \mathrm{Trop}(g)) = (\Sigma \cap \mathrm{Trop}(g))_{xy}$.

Next, we show that $\Sigma_{xy} \cap \mathrm{Trop}(g)$ is the image of $\Sigma \cap \mathrm{Trop}(g)$ under π_{xy}. Since $\Sigma_{xy} \cap \mathrm{Trop}(g)$ is $0-$dimensional, it consists of finitely many points. Let u be one such point, then $u = \sigma \cap \mathrm{Trop}(g)$ for some $1-$dimensional cell $\sigma \in \Sigma_{xy}$. By Definition 2.55, $\sigma = \bigcap_{\substack{\tau \in \Sigma \\ \dim(\pi(\tau))=1}} \pi(\tau)$. Since $\sigma \cap \mathrm{Trop}(g) \neq \emptyset$, $\tau \cap \mathrm{Trop}(g) \neq \emptyset$ for each $\tau \in \Sigma$ with $\dim(\pi(\tau)) = 1$ and $\pi(\tau) \cap \sigma \neq \emptyset$ since $\mathrm{Trop}(g)$ is a plane in $\mathbb{R}^3$. So, $u = \pi_{xy}(\tau \cap \mathrm{Trop}(g)) \in \pi_{xy}(\Sigma \cap \mathrm{Trop}(g))$. Hence, $\Sigma_{xy} \cap \mathrm{Trop}(g) \subseteq (\Sigma \cap \mathrm{Trop}(g))_{xy}$.

Next, we show $(\Sigma \cap \mathrm{Trop}(g))_{xy} \subseteq \Sigma_{xy} \cap \mathrm{Trop}(g)$.

Suppose $u \in (\Sigma \cap \mathrm{Trop}(g))_{xy}$. Then, there is a $w \in \Sigma \cap \mathrm{Trop}(g)$ such that $\pi_{xy}(w) = u$. But, $w \in \Sigma \cap \mathrm{Trop}(g)$ if and only if there is a $\sigma \in \Sigma$ such that $w = \sigma \cap \mathrm{Trop}(g)$. Since

$\pi_{xy}(\sigma)$ is $1-$ dimensional, there exists a 'subdivision' of $\pi_{xy}(\sigma)$ that is contained in Σ_{xy} and contains u. Hence, $w \in \Sigma_{xy} \cap \text{Trop}(g)$, and $\Sigma_{xy} \cap \text{Trop}(g) = (\Sigma \cap \text{Trop}(g))_{xy}$ as desired. $\square$

Proposition 3.5. *Let $N \subseteq \mathbb{Z}^n$ be a sublattice having rank n. Say $N := \{a_1v_1 + ... + a_nv_n : a_i \in \mathbb{Z}\}$ for some set of integer vectors v_i. If V is the $n \times n$ matrix whose columns are the v_i's, then,*

$$[\mathbb{Z}^n : N] = |\det(V)|$$

Proof. There exist $n \times n$ matrices U and W with integer entries and $\det(U) = \det(W) = \pm 1$, and diagonal matrix T with integer entries such that $T = UVW$, here, T is called the Smith Normal Form of V. The entries of T correspond are the invariant factors of N in the decomposition

$$N = \mathbb{Z}^n \oplus \frac{\mathbb{Z}}{f_1\mathbb{Z}} \oplus ... \oplus \frac{\mathbb{Z}}{f_k\mathbb{Z}}$$

where f_i divides f_{i+1} for each $i \in [k-1]$

Since $\frac{\mathbb{Z}^n}{N} \cong \frac{\mathbb{Z}}{f_1\mathbb{Z}} \oplus ... \oplus \frac{\mathbb{Z}}{f_k\mathbb{Z}}$ as abelian groups,

$[\mathbb{Z}^n : N] = |\frac{\mathbb{Z}^n}{N}| = f_1 \cdot f_2 \cdot ... \cdot f_k = |\det(U)| \cdot |\det(T)| \cdot |\det(W)| = |\pm 1| \cdot f_1 \cdot ... \cdot f_k \cdot |\pm 1| = |\det(V)|$ $\square$

Proposition 3.6. *Let Σ be a tropical curve in $\mathbb{R}^3$ consisting of the (classical) line through the origin $\sigma = \mathbb{R} \cdot (s_1, s_2, s_3)$ such that $s_1, s_2 \neq 0$. Fix some $(r_1, r_2, r_3) \in \sigma$ and let $g = cy$ where $c \in K$ has valuation equal to $-r_2$ so that* $\text{Trop}(g) = \mathbb{V}(y - r_2)$. *Then,*

$$\text{mult}_{\Sigma \cap \text{Trop}(g)}(r_1, r_2, r_3) = \text{mult}_{\Sigma_{xy} \cap \text{Trop}(g)}(r_1, r_2)$$

Proof. Since σ is not contained in the plane $\mathbb{V}(y - r_2)$, so $\text{Trop}(g) + \sigma = \mathbb{R}^3$. Hence, $\dim(\sigma + \text{Trop}(g)) = 3$ as linear subspaces of $\mathbb{R}^3$. So, $\Sigma \cap \text{Trop}(g)$ satisfies [10, Definition 3.6.5], that is, the set-theoretic intersection of Σ and $\text{Trop}(g)$ is equal to their stable intersection. Likewise, since $\pi_{xy}(\sigma) = \mathbb{R} \cdot (s_1, s_2)$ is not parallel to the line $\text{Trop}(g) = \mathbb{V}(y - r_2)$, $\dim(\pi_{xy}(\sigma) + \text{Trop}(g)) = 2$ as linear subspaces of $\mathbb{R}^2$. So, the set-theoretic intersection of $\Sigma_{xy} = \{\pi_{xy}(\sigma)\}$ and $\text{Trop}(g)$ is equal to their stable intersection.

Using the formula given in [10, Definition 3.6.5],

$$\mathrm{mult}_{\Sigma\cap\mathrm{Trop}(g)}(r_1,r_2,r_3) = \mathrm{mult}_{\Sigma}(\sigma)\,\mathrm{mult}_{\mathrm{Trop}(g)}(\mathrm{Trop}(g))[\mathbb{Z}^3 : (\mathbb{Z}^3)_\sigma + (\mathbb{Z}^3)_{\mathrm{Trop}(g)}]$$

since $\sigma \cap \mathrm{Trop}(g) = (r_1,r_2,r_3)$ and, here, $(\mathbb{Z}^3)_S = L_S \cap \mathbb{Z}^3$ where S is either σ or $\mathrm{Trop}(g)$ and L_S is the linear subspace in $\mathbb{R}^3$ parallel to S. Similarly,

$$\mathrm{mult}_{\Sigma_{xy}\cap\mathrm{Trop}(g)}(r_1,r_2) = \mathrm{mult}_{\Sigma_{xy}}(\pi_{xy}(\sigma))\,\mathrm{mult}_{\mathrm{Trop}(g)}(\mathrm{Trop}(g))[\mathbb{Z}^2 : (\mathbb{Z}^2)_{(\pi_{xy}(\sigma))} + (\mathbb{Z}^2)_{\mathrm{Trop}(g)}]$$

The lattice $(\mathbb{Z}^3)_\sigma$ is given by $\mathbb{Z}\cdot(z_1,z_2,z_3)$ where (z_1,z_2,z_3) is the first lattice point on $\mathbb{R}\cdot(r_1,r_2,r_3)$. The index $[\mathbb{Z}^3 : (\mathbb{Z}^3)_\sigma + (\mathbb{Z}^3)_{\mathrm{Trop}(g)}]$ is given by the absolute value of the determinant of the matrix

$$\begin{bmatrix} z_1 & z_2 & z_3 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so $[\mathbb{Z}^3 : (\mathbb{Z}^3)_\sigma + (\mathbb{Z}^3)_{\mathrm{Trop}(g)}] = |-z_2| = |z_2|$ so that

$$\mathrm{mult}_{\Sigma\cap\mathrm{Trop}(g)}(r_1,r_2,r_3) = \mathrm{mult}_{\Sigma}(\sigma)\,\mathrm{mult}_{\mathrm{Trop}(g)}(\mathrm{Trop}(g))|z_2|$$

Since $(\mathbb{Z}^2)_{\pi_{xy}(\sigma)} = \mathbb{Z}\cdot(\frac{z_1}{g},\frac{z_2}{g})$ where $g = \gcd(z_1,z_2)$, $[\mathbb{Z}^2 : (\mathbb{Z}^2)_{\pi_{xy}(\sigma)} + (\mathbb{Z}^2)_{\mathrm{Trop}(g)}]$ is given by the absolute value of the determinant of the matrix

$$\begin{bmatrix} \frac{z_1}{g} & \frac{z_2}{g} \\ 1 & 0 \end{bmatrix}$$

and so, $[\mathbb{Z}^2 : (\mathbb{Z}^2)_{(\pi_{xy}(\sigma))} + (\mathbb{Z}^2)_{\mathrm{Trop}(g)}] = \dfrac{|z_2|}{g}$ giving

$$\mathrm{mult}_{\Sigma_{xy}\cap\mathrm{Trop}(g)}(r_1,r_2) = \mathrm{mult}_{\Sigma_{xy}}(\pi_{xy}(\sigma))\,\mathrm{mult}_{\mathrm{Trop}(g)}(\mathrm{Trop}(g))\frac{|z_2|}{g}$$

By Definition 2.55,

$$\text{mult}_{\Sigma_{xy}}(\pi_{xy}(\sigma)) = \text{mult}_{\Sigma}(\sigma)[(\mathbb{Z}^2)_{\pi_{xy}(\sigma)} : \pi_{xy}((\mathbb{Z}^3)_\sigma)]$$

Since $(\mathbb{Z}^3)_\sigma = \mathbb{Z}(z_1, z_2, z_3)$, $\pi_{xy}((\mathbb{Z}^3)_\sigma) = \mathbb{Z} \cdot (z_1, z_2)$ which gives us

$$[(\mathbb{Z}^2)_{\pi_{xy}(\sigma)} : \pi_{xy}((\mathbb{Z}^3)_\sigma)] = g$$

in turn

$$\text{mult}_{\Sigma_{xy} \cap \text{Trop}(g)}(r_1, r_2) = \text{mult}_{\Sigma}(\sigma)\, \text{mult}_{\text{Trop}(g)}(\text{Trop}(g))|z_2|$$

So, $\text{mult}_{\Sigma \cap \text{Trop}(g)}(r_1, r_2, r_3) = \text{mult}_{\Sigma_{xy} \cap \text{Trop}(g)}(r_1, r_2)$ as desired $\square$

Proposition 3.7. *Let Σ be a tropical curve in $\mathbb{R}^3$ that is not contained in a plane, and $I \subseteq K[x, y, z]$ such that $\text{Trop}(I) = \Sigma$. Fix $\sigma \in \Sigma$, and let $w \in \text{relint}(\sigma)$ and $g \in K[x, y]$ be as in Lemma 3.3.*

If $u = \pi_{xy}(w) \in \Sigma_{xy} \cap \text{Trop}(g) = (\Sigma \cap \text{Trop}(g))_{xy}$, then

$$\text{mult}_{\Sigma_{xy} \cap \text{Trop}(g)}(u) = \text{mult}_{(\Sigma \cap \text{Trop}(g))_{xy}}(u)$$

Proof. Without loss (by Lemma 3.3), we may assume that $\text{Trop}(g)$ is parallel to the $x-$axis. Then, $(\mathbb{Z}^3)_{\text{Trop}(g)} = span_{\mathbb{Z}}\{(1, 0, 0), (0, 0, 1)\}$ and $(\mathbb{Z}^2)_{\text{Trop}(g)} = span_{\mathbb{Z}}\{(1, 0)\}$.

By Definition 2.55,

$$\text{mult}_{(\Sigma \cap \text{Trop}(g))_{xy}}(u) = \sum_{\substack{w \in \Sigma \cap \text{Trop}(g) \\ \pi_{xy}(w) = u}} \text{mult}_{\Sigma \cap \text{Trop}(g)}(w)[(\mathbb{Z}^2)_u : \pi_{xy}((\mathbb{Z}^3)_w)]$$

but, u and w are $0-$dimensional, so $(\mathbb{Z}^2)_u = \mathbb{Z} \cdot (0, 0)$ and $\pi_{xy}((\mathbb{Z}^3)_w) = \pi_{xy}(\mathbb{Z} \cdot (0, 0, 0)) = \mathbb{Z} \cdot (0, 0)$. So, $[(\mathbb{Z}^2)_u : \pi_{xy}((\mathbb{Z}^3)_w)] = 1$ and

$$\text{mult}_{(\Sigma \cap \text{Trop}(g))_{xy}}(u) = \sum_{\substack{w \in \Sigma \cap \text{Trop}(g) \\ \pi_{xy}(w) = u}} \text{mult}_{\Sigma \cap \text{Trop}(g)}(w)$$

By Definition [10, Definition 3.6.5] and Lemma 3.3,

$$\mathrm{mult}_{\Sigma\cap\mathrm{Trop}(g)}(w) = \mathrm{mult}_{\Sigma}(\sigma_w)\,\mathrm{mult}_{\mathrm{Trop}(g)}(\mathrm{Trop}(g))[\mathbb{Z}^3 : (\mathbb{Z}^3)_{\sigma_w} + (\mathbb{Z}^3)_{\mathrm{Trop}(g)}]$$

where $\sigma_w \in \Sigma$ satisfies $\sigma_w \cap \mathrm{Trop}(g) = w$. Since $\dim(\mathrm{Trop}(g) + \sigma_w) = 3$ by Lemma 3.3, $(\mathbb{Z}^3)_{\sigma_w} + (\mathbb{Z}^3)_{\mathrm{Trop}(g)}$ has rank 3, and if $(\mathbb{Z}^3)_{\sigma_w} = \mathbb{Z}\cdot(z_1(w), z_2(w), z_3(w))$, then

$$\begin{aligned}[\mathbb{Z}^3 : (\mathbb{Z}^3)_{\sigma_w} + (\mathbb{Z}^3)_{\mathrm{Trop}(g)}] &= \det\begin{bmatrix} z_1(w) & z_2(w) & z_3(w) \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= |z_2(w)|\end{aligned}$$

Hence,

$$\mathrm{mult}_{\Sigma\cap\mathrm{Trop}(g)}(w) = \mathrm{mult}_{\Sigma}(\sigma_w)\,\mathrm{mult}_{\mathrm{Trop}(g)}(\mathrm{Trop}(g))|z_2(w)|$$

which implies that

$$\mathrm{mult}_{(\Sigma\cap\mathrm{Trop}(g))_{xy}}(u) = \mathrm{mult}_{\mathrm{Trop}(g)}(\mathrm{Trop}(g)) \sum_{\substack{w\in\Sigma\cap\mathrm{Trop}(g)\\ \pi_{xy}(w)=u}} \mathrm{mult}_{\Sigma}(\sigma_w)|z_2(w)|$$

Next, we note that by [10, Definition 3.6.5] and Lemma 3.3

$$\mathrm{mult}_{\Sigma_{xy}\cap\mathrm{Trop}(g)}(u) = \mathrm{mult}_{\Sigma_{xy}}(\tau)\,\mathrm{mult}_{\mathrm{Trop}(g)}(\mathrm{Trop}(g))[\mathbb{Z}^2 : (\mathbb{Z}^2)_{\tau} + (\mathbb{Z}^2)_{\mathrm{Trop}(g)}]$$

where $\tau \in \Sigma_{xy}$ is the $1-$dimensional cell such that $\tau \cap \mathrm{Trop}(g) = u$.

Since τ and $\mathrm{Trop}(g)$ are both $1-$dimensional such that $\dim(\tau + \mathrm{Trop}(g)) = 2$, $(\mathbb{Z}^2)_\tau + (\mathbb{Z}^2)_{\mathrm{Trop}(g)}$ is a lattice with rank 2. Since $\mathrm{Trop}(g)$ is parallel to the $x-$axis, $(\mathbb{Z}^2)_{\mathrm{Trop}(g)}$ is generated by $(1,0)$. Say, $(\mathbb{Z}^2)_\tau$ is generated by the integer vector (s_1, s_2) then, $(\mathbb{Z}^2)_\tau + (\mathbb{Z}^2)_{\mathrm{Trop}(g)} = span_{\mathbb{Z}}\{(s_1, s_2), (1,0)\}$. The matrix whose rows are (s_1, s_2) and $(1,0)$ has nonzero determinant since the lattice $(\mathbb{Z}^2)_\tau + (\mathbb{Z}^2)_{\mathrm{Trop}(g)}$ has rank 2. In fact, its determinant is s_2 so

$[\mathbb{Z}^2 : (\mathbb{Z}^2)_\tau + (\mathbb{Z}^2)_{\mathrm{Trop}(g)}] = |s_2|$ and

$$\mathrm{mult}_{\Sigma_{xy} \cap \mathrm{Trop}(g)}(u) = \mathrm{mult}_{\Sigma_{xy}}(\tau)\,\mathrm{mult}_{\mathrm{Trop}(g)}(\mathrm{Trop}(g))|s_2|$$

By Definition 2.55,

$$\mathrm{mult}_{\Sigma_{xy}}(\tau) = \sum_{\substack{\tau' \in \Sigma \\ \pi_{xy}(\tau') = \tau}} \mathrm{mult}_{\Sigma}(\tau')[(\mathbb{Z}^2)_\tau : \pi_{xy}((\mathbb{Z}^3)_{\tau'})]$$

If $\tau' \in \Sigma$ satisfies $\pi_{xy}(\tau') = \tau$, then $\tau' \cap \mathrm{Trop}(g) = w$ for some $w \in \mathrm{relint}(\tau')$ and $\pi_{xy}(w) = \pi_{xy}(\tau' \cap \mathrm{Trop}(g)) = \pi_{xy}(\tau') \cap \mathrm{Trop}(g) = \tau \cap \mathrm{Trop}(g) = u$. So, we may rewrite $\mathrm{mult}_{\Sigma_{xy}}(\tau)$ as

$$\mathrm{mult}_{\Sigma_{xy}}(\tau) = \sum_{\substack{w \in \Sigma \cap \mathrm{Trop}(g) \\ \pi_{xy}(w) = u \\ \pi_{xy}(\sigma_w) = \tau}} \mathrm{mult}_{\Sigma}(\sigma_w)[(\mathbb{Z}^2)_\tau : \pi_{xy}((\mathbb{Z}^3)_{\sigma_w})]$$

Since $\pi_{xy}(\tau') = \tau$, $\pi_{xy}((\mathbb{Z}^3)_{\tau'})$ is a sublattice of $(\mathbb{Z}^2)_\tau$ If $(\mathbb{Z}^3)_{\sigma(w)}$ is generated by the integer vector $(z_1(w), z_2(w), z_3(w))$, then, $\pi_{xy}((\mathbb{Z}^3)_{\tau'})$ is generated by the integer vector $(z_1(w), z_2(w))$. Since $\pi_{xy}((\mathbb{Z}^3)_{\tau'}) = \mathbb{Z} \cdot (z_1(w), z_2(w)) \subseteq \mathbb{Z} \cdot (s_1, s_2) = (\mathbb{Z}^2)_\tau$, there exists $k_w \in \mathbb{Z}_{\geq 1}$ such that $k_w s_i = z_i(w)$ for $i = 1, 2$, so

$$[(\mathbb{Z}^2)_\tau : \pi_{xy}((\mathbb{Z}^3)_{\sigma_w})] = k_w$$

Hence,

$$\mathrm{mult}_{\Sigma_{xy} \cap \mathrm{Trop}(g)}(u) = \mathrm{mult}_{\mathrm{Trop}(g)}(\mathrm{Trop}(g))|s_2| \sum_{\substack{w \in \Sigma \cap \mathrm{Trop}(g) \\ \pi_{xy}(w) = u \\ \pi_{xy}(\sigma_w) = \tau}} \mathrm{mult}_{\Sigma}(\sigma_w) k_w$$

Hence, dividing, we get

$$\frac{\operatorname{mult}_{\Sigma_{xy}\cap \operatorname{Trop}(g)}(u)}{\operatorname{mult}_{(\Sigma\cap\operatorname{Trop}(g))_{xy}}(u)} = \frac{\operatorname{mult}_{\operatorname{Trop}(g)}(\operatorname{Trop}(g))|s_2| \sum_{\substack{w\in\Sigma\cap\operatorname{Trop}(g)\\ \pi_{xy}(w)=u\\ \pi_{xy}(\sigma_w)=\tau}} \operatorname{mult}_\Sigma(\sigma_w)k_w}{\operatorname{mult}_{\operatorname{Trop}(g)}(\operatorname{Trop}(g)) \sum_{\substack{w\in\Sigma\cap\operatorname{Trop}(g)\\ \pi_{xy}(w)=u}} \operatorname{mult}_\Sigma(\sigma_w)|z_2(w)|}$$

$$= \frac{|s_2| \sum_{\substack{w\in\Sigma\cap\operatorname{Trop}(g)\\ \pi_{xy}(w)=u\\ \pi_{xy}(\sigma_w)=\tau}} \operatorname{mult}_\Sigma(\sigma_w)k_w}{\sum_{\substack{w\in\Sigma\cap\operatorname{Trop}(g)\\ \pi_{xy}(w)=u}} \operatorname{mult}_\Sigma(\sigma_w)|k_w s_2|}$$

$$= \frac{\sum_{\substack{w\in\Sigma\cap\operatorname{Trop}(g)\\ \pi_{xy}(w)=u\\ \pi_{xy}(\sigma_w)=\tau}} \operatorname{mult}_\Sigma(\sigma_w)k_w}{\sum_{\substack{w\in\Sigma\cap\operatorname{Trop}(g)\\ \pi_{xy}(w)=u}} \operatorname{mult}_\Sigma(\sigma_w)k_w}$$

The $w \in \operatorname{relint}(\sigma)$ in the hypothesis can be a priori chosen so that if $\pi_{xy}(\sigma_w \cap \operatorname{Trop}(g)) = u$, then $\pi_{xy}(\sigma_w)$ can be refined to be equal to τ. Thus,

$$\operatorname{mult}_{\Sigma_{xy}\cap\operatorname{Trop}(g)}(u) = \operatorname{mult}_{(\Sigma\cap\operatorname{Trop}(g))_{xy}}(u)$$

$\square$

Theorem 3.8. *Let Σ_{xy} denote the projection of* $\operatorname{Trop}(I) = \Sigma$ *to the $xy-$plane. Then,*

$$\operatorname{mult}_{\Sigma_{xy}}(\sigma) \geq \operatorname{mult}_{\operatorname{Trop}(I\cap K[x,y])}(\sigma)$$

for each 1-dimensional cell $\sigma \in \Sigma_{xy} \wedge \operatorname{Trop}(I \cap K[x, y])$.

Proof. By Theorem 3.1, Σ_{xy} and $\operatorname{Trop}(I\cap K[x, y])$ are equivalent polyhedral complexes with $|\Sigma_{xy}| = |\operatorname{Trop}(I \cap K[x, y])| = |\Sigma_{xy} \wedge \operatorname{Trop}(I \cap K[x, y])|$ by Proposition 2.52. Let σ denote a $1-$dimensional cell in $\mathbb{R}^2$ in $\Sigma_{xy}\wedge\operatorname{Trop}(I\cap K[x, y])$. Then, $\sigma = \tau_1\cap\tau_2$ for some $1-$dimensional cells $\tau_1 \in \Sigma_{xy}$ and $\tau_2 \in \operatorname{Trop}(I \cap K[x, y])$. By $\operatorname{mult}_{\Sigma_{xy}}(\sigma)$, we mean $\operatorname{mult}_{\Sigma_{xy}}(\tau_1)$. Likewise, $\operatorname{mult}_{\operatorname{Trop}(I\cap K[x,y])}(\sigma) = \operatorname{mult}_{\operatorname{Trop}(I\cap K[x,y])}(\tau_2)$

Let $g \in K[x,y]$ be chosen as in Lemma 3.3 so that $\mathrm{Trop}(g)$ intersects Σ_{xy} transversely at some fixed $w \in \mathrm{relint}(\sigma)$, and $\Sigma \cap \mathrm{Trop}(g) = \mathrm{Trop}(I + \langle g \rangle)$ is $0-$dimensional– this means that all nonempty $\tau \in \mathrm{Trop}(I + \langle g \rangle)$ are points.

By Definition 2.59 and Proposition 3.7,

$$\begin{aligned}\mathrm{mult}_{\Sigma_{xy}\cap\mathrm{Trop}(g)}(w) &= \mathrm{mult}_{(\Sigma\cap\mathrm{Trop}(g))_{xy}}(w)\\ &= \sum_{\substack{\pi(\tau)=w\\ \tau\in\Sigma\cap\mathrm{Trop}(g)}} \mathrm{mult}_{\Sigma\cap\mathrm{Trop}(g)}(\tau)[(\mathbb{Z}^2)_w : \pi((\mathbb{Z}^3)_\tau)]\end{aligned}$$

where $(\Sigma \cap \mathrm{Trop}(g))_{xy}$ denotes the polyhedral complex that is the refinement of the $\pi_{xy}(\Sigma \cap \mathrm{Trop}(g))$ as in Definition 2.55.

Since w and τ are $0-$dimensional, $(\mathbb{Z}^2)_w = \mathbb{Z}\cdot(0,0)$ and $(\mathbb{Z}^3)_\tau = \mathbb{Z}\cdot(0,0,0)$ so that $\pi((\mathbb{Z}^3)_\tau) = \mathbb{Z}\cdot(0,0)$, and the index $[(\mathbb{Z}^2)_w : \pi((\mathbb{Z}^3)_\tau)] = 1$ for each $\tau \in \mathrm{Trop}(I + \langle g \rangle)$.

Since $I + \langle g \rangle$ is $0-$dimensional, $\mathrm{mult}_{\mathrm{Trop}(I+\langle g\rangle)}(\tau) = \dim_K(\frac{K[x,y,z]}{(I+\langle g\rangle)_\tau})$ for each $\tau \in \mathrm{Trop}(I + \langle g \rangle)$ by [10, Proposition 3.4.8].

Hence,

$$\mathrm{mult}_{\Sigma_{xy}\cap\mathrm{Trop}(g)}(\tau) = \sum_{\substack{u\in\mathrm{Trop}(I+\langle g\rangle)\\ \pi(\tau)=w}} \dim_K(\frac{K[x,y,z]}{(I+\langle g\rangle)_\tau})$$

Similarly, as $I \cap K[x,y] + \langle g \rangle$ is 0-dimensional, we have,

$$\mathrm{mult}_{\mathrm{Trop}(I\cap K[x,y]+\langle g\rangle)}(w) = \dim_K(\frac{K[x,y]}{(I\cap K[x,y]+\langle g\rangle)_w})$$

Since $(I + \langle g \rangle)_\tau$ is zero-dimensional, $(I + \langle g \rangle)_\tau = \bigcap_{\substack{v\in K^3\\ \mathrm{val}(v)=\tau}} Q_v$ where each $v = (v_1, v_2, v_3)$ and $\sqrt{Q_v} = \langle x - v_1, y - v_2, z - v_3 \rangle$ since zero-dimensionality of an ideal implies that its associated primes are zero dimensional, but zero-dimensional primes are maximal. Likewise, $(I \cap K[x,y] + \langle g \rangle)_w = \bigcap_{\substack{v\in K^3\\ \mathrm{val}(v_1,v_2)=w}} (Q_v \cap K[x,y])$ because the radical of each primary component of $(I \cap K[x,y] + \langle g \rangle)_w$ is equal to $\langle x - v_1, y - v_2 \rangle$. Since $(I + \langle g \rangle)_\tau \cap K[x,y] = \bigcap_{\substack{v\in K^3\\ \mathrm{val}(v)=\tau}} (Q_v \cap K[x,y])$

and $\pi_{xy}(\tau) = w$,

$$(I + \langle g\rangle)_\tau \cap K[x,y] \subseteq (I \cap K[x,y] + \langle g\rangle)_w$$

So, $\displaystyle\bigcap_{\substack{\tau\in \mathrm{Trop}(I+\langle g\rangle)\\ \pi_{xy}(\tau)=w}} (I + \langle g\rangle)_\tau \cap K[x,y] \subseteq (I \cap K[x,y] + \langle g\rangle)_w$ which implies that

$$\frac{K[x,y]}{(I \cap K[x,y] + \langle g\rangle)_w} \subseteq \frac{K[x,y]}{\bigcap_{\substack{\tau\in \mathrm{Trop}(I+\langle g\rangle)\\ \pi_{xy}(\tau)=w}} (I + \langle g\rangle)_\tau \cap K[x,y]}$$

But, $\displaystyle\frac{K[x,y]}{\bigcap_{\substack{\tau\in \mathrm{Trop}(I+\langle g\rangle)\\ \pi_{xy}(\tau)=w}} (I + \langle g\rangle)_\tau \cap K[x,y]}$ injects into $\displaystyle\bigoplus_{\substack{\tau\in \mathrm{Trop}(I+\langle g\rangle)\\ \pi_{xy}(\tau)=w}} \frac{K[x,y]}{(I + \langle g\rangle)_\tau \cap K[x,y]}$ so,

$$\sum_{\substack{\tau\in \mathrm{Trop}(I+\langle g\rangle)\\ \pi_{xy}(\tau)=w}} \dim_K\Big(\frac{K[x,y]}{(I + \langle g\rangle)_\tau \cap K[x,y]}\Big) \geq \dim_K\Big(\frac{K[x,y]}{(I \cap K[x,y] + \langle g\rangle)_w}\Big)$$

But, for each $\tau \in \mathrm{Trop}(I + \langle g\rangle)$, $\dim_K\big(\frac{K[x,y,z]}{(I+\langle g\rangle)_\tau}\big) \geq \dim_K\big(\frac{K[x,y]}{(I + \langle g\rangle)_\tau \cap K[x,y]}\big)$ so,

$$\sum_{\substack{\tau\in \mathrm{Trop}(I+\langle g\rangle)\\ \pi(\tau)=w}} \dim_K\Big(\frac{K[x,y,z]}{(I + \langle g\rangle)_\tau}\Big) \geq \dim_K\Big(\frac{K[x,y]}{(I \cap K[x,y] + \langle g\rangle)_w}\Big)$$

Hence, $\mathrm{mult}_{\Sigma_{xy}\cap \mathrm{Trop}(g)}(w) \geq \mathrm{mult}_{\mathrm{Trop}(I\cap K[x,y]+\langle g\rangle)}(w)$ so that, by Definition 3.6.5 [10, p. 137]

$$\begin{gathered}\mathrm{mult}_{\Sigma_{xy}}(\sigma)\,\mathrm{mult}_{\mathrm{Trop}(g)}(\mathrm{Trop}(g))[\mathbb{Z}^2 : (\mathbb{Z}^2)_\sigma + (\mathbb{Z}^2)_{\mathrm{Trop}(g)}] \\ \geq \\ \mathrm{mult}_{\mathrm{Trop}(I\cap K[x,y])}(\sigma)\,\mathrm{mult}_{\mathrm{Trop}(g)}(\mathrm{Trop}(g))[\mathbb{Z}^2 : (\mathbb{Z}^2)_\sigma + (\mathbb{Z}^2)_{\mathrm{Trop}(g)}]\end{gathered} \tag{3.2}$$

and thus,

$$\mathrm{mult}_{\Sigma_{xy}}(\sigma) \geq \mathrm{mult}_{\mathrm{Trop}(I\cap K[x,y])}(\sigma)$$

as desired. □

Proposition 3.9. *Suppose σ is a $1-$dimensional cell of the tropical curve* $\mathrm{Trop}(I) = \Sigma$ *that is parallel to the vector* $(1, z_1, z_2)$ *for some* $z_1, z_2 \in \mathbb{Z}$. *If* $\mathrm{mult}_{\Sigma_{xy}}(\pi_{xy}(\sigma)) = 1$, *then*

$$\mathrm{mult}_{\Sigma}(\sigma) = 1$$

Proof. Since σ is parallel to $(1, z_1, z_2) \in \mathbb{Z}^3$, the projection of σ to the $xy-$ plane is parallel $(1, z_1)$. In particular, $\dim(\pi_{xy}(\sigma)) = 1$.

By Theorem 3.8, $1 = \mathrm{mult}_{\Sigma_{xy}}(\pi_{xy}(\sigma)) \geq \mathrm{mult}_{\mathrm{Trop}(I \cap K[x,y])}(\pi_{xy}(\sigma))$. So, $\mathrm{mult}_{\Sigma_{xy}}(\pi_{xy}(\sigma)) = \mathrm{mult}_{\mathrm{Trop}(I \cap K[x,y])}(\pi_{xy}(\sigma)) = 1$. Furthermore, by Lemma 3.3, there exists $w = (w_1, w_2, w_3) \in \mathrm{relint}(\sigma)$ so that the intersection of $\mathrm{Trop}(g) = \mathbb{V}(x - w_1)$ (here $g = cx$ with $\mathrm{val}(c) = -w_1$) with Σ is equal to their stable intersection.

Since σ is parallel to $(1, z_1, z_2)$, $(\mathbb{Z}^3)_\sigma = \mathbb{Z} \cdot (1, z_1, z_2)$. Then, $\pi_{xy}((\mathbb{Z}^3)_\sigma) = \mathbb{Z} \cdot (1, z_1) = (\mathbb{Z}^2)_{\pi_{xy}(\sigma)}$ since $\gcd(1, z_1) = 1$.

This gives

$$[\mathbb{Z}^3 : (\mathbb{Z}^3)_\sigma + (\mathbb{Z}^3)_{\mathrm{Trop}(g)}] = \det \begin{bmatrix} 1 & z_1 & z_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$$

Similarly, $[\mathbb{Z}^2 : (\mathbb{Z}^2)_{\pi_{xy}(\sigma)} + (\mathbb{Z}^2)_{\mathrm{Trop}(g)}] = 1$

By [10, Definition 3.6.5], we have the following:

$$\begin{aligned}
&\mathrm{mult}_{\Sigma_{xy} \cap \mathrm{Trop}(g)}(w_1, w_2) = \\
&\qquad \mathrm{mult}_{\Sigma_{xy}}(\pi_{xy}(\sigma))\, \mathrm{mult}_{\mathrm{Trop}(g)}(\mathrm{Trop}(g))[\mathbb{Z}^2 : (\mathbb{Z}^2)_{\pi_{xy}(\sigma)} + (\mathbb{Z}^2)_{\mathrm{Trop}(g)}] \\
&\qquad\qquad = \mathrm{mult}_{\Sigma_{xy}}(\pi_{xy}(\sigma))\, \mathrm{mult}_{\mathrm{Trop}(g)}(\mathrm{Trop}(g)) \\
&\qquad\qquad = 1
\end{aligned}$$

On the other hand, Lemma 3.3 gives:

$$\begin{aligned}
\mathrm{mult}_{\Sigma_{xy} \cap \mathrm{Trop}(g)}(w_1, w_2) &= \sum_{\substack{u \in \Sigma \cap \mathrm{Trop}(g) \\ \pi_{xy}(u) = (w_1, w_2)}} \mathrm{mult}_{\Sigma \cap \mathrm{Trop}(g)}(u)[(\mathbb{Z}^2)_{(w_1,w_2)} : \pi_{xy}((\mathbb{Z}^3)_u)] \\
&= \sum_{\substack{u \in \Sigma \cap \mathrm{Trop}(g) \\ \pi_{xy}(u) = (w_1, w_2)}} \mathrm{mult}_{\Sigma \cap \mathrm{Trop}(g)}(u)
\end{aligned}$$

But, by Definition 2.59, we have

$$\mathrm{mult}_{\Sigma\cap\mathrm{Trop}(g)}(u) = \mathrm{mult}_{\Sigma}(\tau_u)\,\mathrm{mult}_{\mathrm{Trop}(g)}(\mathrm{Trop}(g))[\mathbb{Z}^3 : (\mathbb{Z}^3)_{\tau_u} + (\mathbb{Z}^3)_{\mathrm{Trop}(g)}]$$

where $\tau_u \in \Sigma$ satisfies $\tau_u \cap \mathrm{Trop}(g) = u$ and $\pi_{xy}(u) = (w_1, w_2)$. However, it must be that

$$\sum_{\substack{\tau_u\in\Sigma-\{\sigma\}\\ \tau_u\cap\mathrm{Trop}(g)=u\\ \pi_{xy}(u)=(w_1,w_2)}} \mathrm{mult}_{\Sigma}(\tau_u)[\mathbb{Z}^3 : (\mathbb{Z}^3)_{\tau_u} + (\mathbb{Z}^3)_{\mathrm{Trop}(g)}] = 0$$

since $\mathrm{mult}_{\Sigma_{xy}\cap\mathrm{Trop}(g)}(w_1, w_2) = 1$. So, we have

$$1 = \mathrm{mult}_{\Sigma_{xy}\cap\mathrm{Trop}(g)}(w_1, w_2) = \mathrm{mult}_{\Sigma}(\sigma)[\mathbb{Z}^3 : (\mathbb{Z}^3)_{\sigma} + (\mathbb{Z}^3)_{\mathrm{Trop}(g)}]$$

Since we showed that $[\mathbb{Z}^3 : (\mathbb{Z}^3)_{\sigma} + (\mathbb{Z}^3)_{\mathrm{Trop}(g)}] = 1$, it follows that $\mathrm{mult}_{\Sigma}(\sigma) = 1$. □

This shows that if $\mathrm{mult}_{\Sigma_{xy}}(\pi_{xy}(\sigma)) = 1$ and σ is parallel to the $x-$axis, then there can only be a single $1-$dimensional cell in Σ mapped to $\pi_{xy}(\sigma)$.

Proposition 3.10. *If $J \subset K[x, y, z]$ is a $0-$dimensional ideal, then*

$$\dim_K\left(\frac{K[x, y, z]}{J}\right) \leq \dim_K\left(\frac{K[x, y]}{J \cap K[x, y]}\right) \cdot \dim_K\left(\frac{K[x, z]}{J \cap K[x, z]}\right)$$

Proof. Let $<$ be some monomial ordering on $K[x, y, z]$. By Theorem 2.20, the collection M of monomials not in $\langle \mathrm{LT}_<(\mathrm{J})\rangle$ forms a $K-$basis for $\frac{K[x, y, z]}{J}$, and since J has Krull dimension equal to zero, $\frac{K[x, y, z]}{J}$ is a finite dimensional $K-$vector space.

Let V denote the $K-$span of $M \cap K[x, y]$ in $\frac{K[x, y]}{J \cap K[x, y]}$, and W the $K-$span of $M \cap K[x, z]$ in $\frac{K[x, z]}{J \cap K[x, z]}$. We note that $V \bigotimes_K W$ has dimension equal to $\dim(V) \cdot \dim(W)$. We show that $V \bigotimes_K W$ surjects onto $\frac{K[x, y, z]}{J}$ by constructing a surjective bilinear map from $V \times W$ onto $\frac{K[x, y, z]}{J}$.

Define $\phi : V \times W \rightarrow \dfrac{K[x,y,z]}{J}$ by $\phi(f,g) = fg$. If $f_1, f_2 \in V$ and $k \in K$, then $\phi(f_1 + kf_2, g) = f_1g + kf_2g = \phi(f_1, g) + k\phi(f_2, g)$. Similarly, if $g_1, g_2 \in V$ and $k \in K$, then $\phi(f, g_1 + kg_2) = \phi(f, g_1) + k\phi(f, g_2)$. So, ϕ is $k-$bilinear.

To show that ϕ is surjective, it suffices to show that if $m \in M$ and $m = x^\alpha y^\beta z^\gamma$ with $\beta, \gamma > 0$, then there are monomials in $M \cap K[x,y]$ and $M \cap K[x,z]$ mapped to m by ϕ. Observe that $1 \in M$ since $1 = x^0y^0z^0$ and $1 \notin \langle \mathrm{LT}_<(\mathrm{J}) \rangle$ by zero-dimensionality of J. By Proposition 2.11, there exist $c, d, e \in \mathbb{Z}_{\geq 1}$ such that $x^c, y^d, z^e \in \langle \mathrm{LT}_<(\mathrm{J}) \rangle$. Hence, $1, x, ..., x^{c-1}, y, ...y^{d-1}, z, ..., z^{e-1} \in M$ which implies that, $1, x, ..., x^{c-1}, y, ..., y^{d-1} \in M \cap K[x,y]$ and $1, x, ..., x^{c-1}, z, ..., z^{e-1} \in M \cap K[x,z]$. Each of α, β, γ must be less than c, d, e respectively, for if, say $\alpha \geq c$, then, since $x^c \in \langle \mathrm{LT}_<(\mathrm{J}) \rangle$, $x^\alpha \in \langle \mathrm{LT}_<(\mathrm{J}) \rangle$ so that $m \in \langle \mathrm{LT}_<(\mathrm{J}) \rangle$ which is a contradiction. Next, $x^\alpha y^\beta \in M \cap K[x,y]$, otherwise, $x^\alpha y^\beta \notin M$ which implies that $x^\alpha y^\beta \in \langle \mathrm{LT}_<(\mathrm{J}) \rangle$ which again implies that $m \in \langle \mathrm{LT}_<(\mathrm{J}) \rangle$. Together, these imply that $\alpha \in \{0, 1, ..., c-1\}$, $\beta \in [e-1]$, and $\gamma \in [d-1]$ and $\phi(x^\alpha y^\beta, z^\gamma) = m$. So, $\phi : V \times W \rightarrow \dfrac{K[x,y,z]}{J}$ is a surjective, $K-$bilinear map which lifts to a surjective $K-$module homomorphism $\Phi : V \bigotimes_K W \rightarrow \dfrac{K[x,y,z]}{J}$ and $\dim(V) \cdot \dim(W) \geq \dim(\dfrac{K[x,y,z]}{J})$.

Since V and W are $K-$vector subspaces of $\dfrac{K[x,y]}{J \cap K[x,y]}$ and $\dfrac{K[x,z]}{J \cap K[x,z]}$ respectively, we have $\dim(\dfrac{K[x,y]}{J \cap K[x,y]}) \geq \dim(V)$ and $\dim(\dfrac{K[x,z]}{J \cap K[x,z]}) \geq \dim(W)$. Thus,

$$\dim(\frac{K[x,y]}{J \cap K[x,y]}) \cdot \dim(\frac{K[x,z]}{J \cap K[x,z]}) \geq \dim(\frac{K[x,y,z]}{J})$$

as desired. $\square$

Theorem 3.11. *Suppose σ is a $1-$dimensional cell of the tropical curve $\mathrm{Trop}(I) = \Sigma$ parallel to line $(1, z_1, z_2)$ for some $z_1, z_2 \in \mathbb{Z}$. Suppose further that:*

$$\mathrm{mult}_{\mathrm{Trop}(I \cap K[x,y]}(\pi_{xy}(\sigma)) = \mathrm{mult}_{\mathrm{Trop}(I \cap K[x,z]}(\pi_{xz}(\sigma)) = 1$$

Then, $\mathrm{mult}_{\mathrm{Trop}(I)}(\sigma) = 1$

Proof. Since σ is parallel to $(1, z_1, z_2) \in \mathbb{Z}^3$, the projections of σ to the $xy-$ and $xz-$planes are respectively parallel $(1, z_1)$ and $(1, z_2)$. In particular, $\dim(\pi_{xy}(\sigma)) = \dim(\pi_{xz}(\sigma)) = 1$.

By Theorem 3.8, $1 = \text{mult}_{\Sigma_{xy}}(\pi_{xy}(\sigma)) \geq \text{mult}_{\text{Trop}(I\cap K[x,y])}(\pi_{xy}(\sigma))$. So, $\text{mult}_{\Sigma_{xy}}(\pi_{xy}(\sigma)) = \text{mult}_{\text{Trop}(I\cap K[x,y])}(\pi_{xy}(\sigma)) = 1$. Likewise, $\text{mult}_{\text{Trop}(I\cap K[x,z])}(\pi_{xz}(\sigma)) = 1$. Furthermore, by Lemma 3.3, there exists $w = (w_1, w_2, w_3) \in \text{relint}(\sigma)$ so that the intersection of $\text{Trop}(g) = \mathbb{V}(x - w_1)$ (here $g = cx$ with $\text{val}(c) = -w_1$) with Σ is equal to their stable intersection.

Suppose $(\mathbb{Z}^3)_\sigma = \mathbb{Z}\cdot(1, z_1, z_2)$. Then, $\pi_{xy}((\mathbb{Z}^3)_\sigma) = \mathbb{Z}\cdot(1, z_1) = (\mathbb{Z}^2)_{\pi_{xy}(\sigma)}$ and $\pi_{xz}((\mathbb{Z}^3)_\sigma) = \mathbb{Z}\cdot(1, z_2) = (\mathbb{Z}^2)_{\pi_{xz}(\sigma)}$ since $\gcd(1, z_1) = \gcd(1, z_2) = 1$.

This gives

$$[\mathbb{Z}^3 : (\mathbb{Z}^3)_\sigma + (\mathbb{Z}^3)_{\text{Trop}(g)}] = \det \begin{bmatrix} 1 & z_1 & z_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$$

Similarly, $[\mathbb{Z}^2 : (\mathbb{Z}^2)_{\pi_{xy}(\sigma)} + (\mathbb{Z}^2)_{\text{Trop}(g)}] = [\mathbb{Z}^2 : (\mathbb{Z}^2)_{\pi_{xy}(\sigma)} + (\mathbb{Z}^2)_{\text{Trop}(g)}] = 1$

By [10, Definition 3.6.5] and Lemma 3.3, we have the following:

$$\begin{aligned}
&\text{mult}_{\text{Trop}(I\cap K[x,y]+g)}(w_1, w_2) = \\
&\quad \text{mult}_{\text{Trop}(I\cap K[x,y])}(\pi_{xy}(\sigma)) \,\text{mult}_{\text{Trop}(g)}(\text{Trop}(g))[\mathbb{Z}^2 : (\mathbb{Z}^2)_{\pi_{xy}(\sigma)} + (\mathbb{Z}^2)_{\text{Trop}(g)}] \\
&\qquad = \text{mult}_{\text{Trop}(I\cap K[x,y])}(\pi_{xy}(\sigma)) \,\text{mult}_{\text{Trop}(g)}(\text{Trop}(g)) \\
&\qquad = 1
\end{aligned}$$

Arguing similarly, we have $\text{mult}_{\text{Trop}(I+g)}(w_1, w_2, w_3)$ and $\text{mult}_{\text{Trop}(I+g)}(w_1, w_2, w_3) = 1$.

Since $I + \langle g\rangle$ is $0-$dimensional, all of its associated primes are $0-$dimensional as well. So, $(I + \langle g\rangle)_w$ is $0-$dimensional. Similarly, $(I \cap K[x, y] + \langle g\rangle)_{(w_1,w_2)}$ and $(I \cap K[x, z] + \langle g\rangle)_{(w_1,w_3)}$ are $0-$dimensional, and we have the following:

$$\begin{aligned}
\text{mult}_{\text{Trop}(I+\langle g\rangle)}(w) &= \dim_K\Big(\frac{K[x, y, z]}{(I + \langle g\rangle)_w}\Big) \\
\text{mult}_{\text{Trop}(I\cap K[x,y]+\langle g\rangle)}(w_1, w_2) &= \dim_K\Big(\frac{K[x, y]}{(I \cap K[x, y] + \langle g\rangle)_{(w_1,w_2)}}\Big) \\
\text{mult}_{\text{Trop}(I\cap K[x,z]+\langle g\rangle)}(w_1, w_3) &= \dim_K\Big(\frac{K[x, z]}{(I \cap K[x, z] + \langle g\rangle)_{(w_1,w_3)}}\Big)
\end{aligned} \tag{3.3}$$

Proposition 3.10 gives

$$\begin{aligned}\dim_K\left(\frac{K[x,y,z]}{(I+\langle g\rangle)_w}\right) &\leq \dim_K\left(\frac{K[x,y]}{(I\cap K[x,y]+\langle g\rangle)_{(w_1,w_2)}}\right)\cdot\dim_K\left(\frac{K[x,z]}{(I\cap K[x,z]+\langle g\rangle)_{(w_1,w_3)}}\right)\\ &\leq 1\end{aligned}$$

So, $\dim_K(\frac{K[x,y,z]}{(I+\langle g\rangle)_w}) = 1$ which implies

$$\mathrm{mult}_{\mathrm{Trop}(I)}(\sigma) = \mathrm{mult}_{\mathrm{Trop}(I+g)}(w_1,w_2,w_3) = 1$$

as desired. □

3.3 A Coefficient Matrix and Some Necessary Bounds on Dimension

3.3.1 Some Polytopes and a Coefficient Matrix

We will denote by P_{xy} the Newton polygon (in the xy-plane) of the polynomial f_{xy}. Similarly, P_{xz} and P_{yz} are the Newton polygons of f_{xz} and f_{yz} in the xz- and yz-planes respectively.

Throughout this section, we will construct lattice polytopes $P'_{xy}, P'_{xz}, P'_{yz}$ whose Minkowski sums $P'_{xy}+P_{xy}$, $P'_{xz}+P_{xz}$, and $P'_{yz}+P_{yz}$ we will study in detail. In particular, we are interested in the monomials whose exponent vectors lie in the set

$$(P'_{xy}+P_{xy})\cup(P'_{xz}+P_{xz})\cup(P'_{yz}+P_{yz})$$

We denote this set of monomials by S, and by T we mean the (ordered) set of polynomials $\{x^\alpha y^\beta z^\gamma f_i : (\alpha,\beta,\gamma)\in P'_i\cap\mathbb{Z}^3_{\geq 0}$ and $i\in\{xy,xz,yz\}\}$.

The rows of the coefficient matrix M are indexed by the polynomials in T, and its columns by the monomials in S. That is, given a monomial $s\in S$ and polynomial $t\in T$, the (t,s)-entry will be the coefficient of the monomial s in the polynomial t. If the polynomial t does not have the monomial s, then that (t,s)-entry is zero.

3.3.2 The Rank of the Coefficient Matrix and Dimension of the Three Generator Ideal

Proposition 3.12. *Suppose f_{xy}, f_{xz}, f_{yz} are specialized so that they are all nonconstant. Then,* $\dim(\langle f_{xy}, f_{xz}, f_{yz}\rangle)$ *is either equal to* -1, 0, *or* 1.

Proof. Fix a monomial order $<$ on $K[x_1, ..., x_n]$. By 2.2, $\dim(I) = \dim(\langle \mathrm{LT}_<(\mathrm{I})\rangle)$. It suffices to consider the ideal $L := \langle \mathrm{LT}_<(\mathrm{f_{xy}}), \mathrm{LT}_<(\mathrm{f_{xz}}), \mathrm{LT}_<(\mathrm{f_{yz}})\rangle \subseteq \langle \mathrm{LT}_<(\mathrm{I})\rangle$ as $L \subseteq \langle \mathrm{LT}_<(\mathrm{I})\rangle$ implies that $\dim(L) \geq \dim(I)$.

Say $LT_<(f_{xy}) = Ax^\alpha y^\beta$. Since f_{xy} is nonconstant, at least one of α, β is positive. Similarly,

$$LT_<(f_{xz}) = Bx^{\alpha'} z^\gamma$$
$$LT_<(f_{yz}) = Cy^{\beta'} z^{\gamma'}$$

with at least one of α', γ positive and at least one of β', γ' positive.

Let $M_{xy} := \{s \in \{x, y, z\} : s | x^\alpha y^\beta\}$, i.e. the set of variables that divide the leading monomial of f_{xy}. Since f_{xy} is nonconstant, $(\alpha, \beta) \neq 0$. If both α and β are nonzero, then both x and y divide $x^\alpha y^\beta$. So, $M_{xy} = \{x, y\}$. Now, if $\alpha = 0$, then $\beta \neq 0$ so that y divides $x^0 y^\beta$. Then, $M_{xy} = \{y\}$. Likewise, if $\beta = 0$, then $\alpha \neq 0$, so that x divides $x^\alpha y^0$. Hence, $M_{xy} = \{x\}$. Thus, $M_{xy} = \{x, y\}$, $\{x\}$, or $\{y\}$. Similarly, $M_{xz} = \{x, z\}, \{x\}$ or, $\{z\}$ and $M_{yz} = \{y, z\}, \{y\}$ or, $\{z\}$.

Let $\mathbb{M} := \{J \subseteq \{x, y, z\} : J \cap M_i \neq \emptyset$ for $i = xy, xz,$ and $yz\}$. Certainly, $\{x, y, z\} \cap M_i \neq \emptyset$ so that $\{x, y, z\} \in \mathbb{M}$. Say, $\{x\} \in \mathbb{M}$, then $\{x\} \cap M_{yz} \neq \emptyset$, but this is a contradiction since x cannot divide any term of f_{yz}, in particular, $LM_<(f_{yz}) = y^{\beta'} z^{\gamma'}$ is not divided by x. Likewise, $\mathbb{M}$ cannot contain the singletons $\{y\}$ or $\{z\}$ since the terms of f_{xz} and f_{xy} have no terms divided by y or z respectively.

Hence, $\min(|J| : J \in \mathbb{M}) = 2$ or 3 and since $\dim(L) = 3 - \min(|L| : L \in \mathbb{M})$, $\dim(L) = 0$ or 1. Thus, $\dim(\langle f_{xy}, f_{xz}, f_{yz}\rangle)$ is either -1, 0, or 1. □

Next, we show that $\dim(I) = 1$ is a necessary condition for I to satisfy the elimination conditions:

$$I \cap K[x,y] = \langle f_{xy} \rangle$$
$$I \cap K[x,z] = \langle f_{xz} \rangle$$
$$I \cap K[y,z] = \langle f_{yz} \rangle$$

That is, I has nonzero and principal elimination ideals in each of $K[x,y]$, $K[x,z]$, and $K[y,z]$.

Proposition 3.13. *If a proper ideal $I \subset K[x,y,z]$ has nonzero and principal elimination ideals in each of $K[x,y]$, $K[x,z]$, and $K[y,z]$, then* $\dim I = 1$.

Proof. We assume that I has nonzero and principal elimination ideals but $\dim(I) = 0$, and derive a contradiction.

Say $I \cap K[x,y] = \langle f_{xy} \rangle$, $I \cap K[x,z] = \langle f_{xz} \rangle$, and $I \cap K[y,z] = \langle f_{yz} \rangle$ for some nonconstant polynomials f_{xy}, f_{xz}, f_{yz} in $K[x,y], K[x,z], K[y,z]$ respectively. Since $\dim I = 0$, $I \cap K[x], I \cap K[y], I \cap K[z]$ all nonzero ideals (in fact, they are nonzero, principal ideals by Theorem 2.10 and [9, Corollary 11.14]). Say,

$$\begin{aligned} I \cap K[x] &= \langle f \rangle \\ I \cap K[y] &= \langle g \rangle \\ I \cap K[z] &= \langle h \rangle \end{aligned} \tag{3.4}$$

and $f = x^d + c_1 x^{d-1} + ... + c_{d-1}x + c_d$ and $g = y^e + b_1 y^{e-1} + ... + b_{e-1}y + b_e$.

Then, $f, g \in I$ implies that $f, g \in I \cap K[x,y] = \langle f_{xy} \rangle$. Under any term order, the leading terms of f and g are x^d and y^e respectively. So, there must exist some term of f_{xy}, say $kx^\alpha y^\beta$, such that $x^d = (\frac{x^{s_1}y^{t_1}}{k}) \cdot kx^\alpha y^\beta$ and $y^e = (\frac{x^{s_2}y^{t_2}}{k}) \cdot kx^\alpha y^\beta$ for some $s_i, t_j \in \mathbb{Z}_{\geq 0}$. This is equivalent to finding $s_i, t_j \in \mathbb{Z}_{\geq 0}$ satisfying the system of equations

$$\begin{aligned} d &= \alpha + s_1 \\ 0 &= \beta + t_1 \\ 0 &= \alpha + s_2 \\ e &= \beta + t_2 \end{aligned} \tag{3.5}$$

As this is an impossibility, it follows that $I \cap K[x, y]$ is not principal, and, in particular, I fails to satisfy the elimination conditions in xy.

Hence, we conclude that if I satisfies the elimination conditions, then $\dim(I) \geq 1$, and since Proposition 3.12 implies that $\dim(I) \leq 1$, it follows that $\dim(I) = 1$. □

Definition 3.14. *Let R be a ring.*

*A sequence of elements $f_1, ..., f_n \in R$ is a **regular sequence** if*

1. *$\langle f_1, ..., f_n\rangle \neq R$, and*

2. *For $i = 1, ..., n$, f_i is a non-zerodivisor in $R/\langle f_1, ..., f_{i-1}\rangle$*

Proposition 3.15. *Suppose f_{xy}, f_{xz}, f_{yz} are specialized so that they are all nonconstant.*

If there exists a distinct pair of polynomials chosen from f_{xy}, f_{xz}, f_{yz} that is not relatively prime, then $\dim\langle f_{xy}, f_{xz}, f_{yz}\rangle = 1$.

Proof. Assume that f_{xy} and f_{xz} are not relatively prime, then let $g := \gcd(f_{xy}, f_{yz})$. Since $f_{xy} \in K[x, y]$ and $f_{xz} \in K[x, z]$ and are not relatively prime by hypothesis, $g \in K[x]$ and is nonconstant. Since $f_{yz} \in K[y, z]$, every nonconstant term of f_{yz} and g are relatively prime so that $\{g, f_{yz}\}$ is a Gröbner basis for $\langle g, f_{yz}\rangle$ irrespective of term order.

Certainly, $I \subseteq \langle g, f_{yz}\rangle$ since g divides both f_{xy}, f_{xz} by assumption. Fix some monomial ordering $<$ on $K[x, y, z]$, and consider $\langle LT_<(g), LT_<(f_{yz})\rangle$. We will use the computation in the proof of Theorem 2.10 to find the codimension.

Let $M_g := \{t \in \{x, y, z\} : t | LM_<(g)\}$ and $M_{yz} := \{t \in \{x, y, z\} : t | LM_<(f_{yz})\}$. Since $g \in K[x]$ is nonconstant, $M_g := \{x\}$. Since $f_{yz} \in K[y, z]$, $M_{yz} = \{y\}, \{z\}$, or $\{y, z\}$.

Let $\mathbb{M} := \{L \subseteq \{x, y, z\} : L \cap M_i \neq \emptyset \text{ for } i = g, yz\}$. Then $\mathbb{M}$ contains $\{x, y, z\}$, and either $\{x, z\}$, $\{x, y\}$, or both. In either case, $\min(|L| : L \in \mathbb{M}) = 2$, so that $\dim(\langle g, f_{yz}\rangle) = 1$. Since $\dim(I) \geq \dim(\langle g, f_{yz}\rangle) = 1$, $\dim(I) = 1$ by Proposition 3.12. □

The contrapositive to this proposition is that if $\dim\langle f_{xy}, f_{xz}, f_{yz}\rangle = 0$, then f_{xy}, f_{xz}, f_{yz} are pairwise relatively prime. We will exploit this fact in the next theorem.

Theorem 3.16. *Suppose the coefficients of f_{xy}, f_{xz}, f_{yz} are specialized so that they are nonconstant.*

If $\langle f_{xy}, f_{xz}, f_{yz}\rangle \subsetneq R$ is zero-dimensional, then the generators f_{xy}, f_{xz}, f_{yz} form a regular sequence.

Proof. Since $I \neq R$ by hypothesis, f_{xy}, f_{xz}, f_{yz} satisfies the first condition for being a regular sequence on R. Next, we show that f_i is a non-zerodivisor in $R/\langle f_1, ..., f_{i-1}\rangle$ for $i = 1, 2, 3$.

Since R is a domain and f_{xy} is nonzero by hypothesis, f_{xy} is a non-zerodivisor in R. We show that f_{xz} is a non-zerodivisor in $R/\langle f_{xy}\rangle$. Since $K[x, y, z]$ is a unique factorization domain, we can find a prime factorization for f_{xy} and f_{xz}. Say $f_{xy} = up_1^{\alpha_1}...p_k^{\alpha_k}$ and $f_{xz} = vq_1^{\beta_1}...q_l^{\beta_l}$ where p_i, q_j are primes in $K[x, y, z]$ and $\alpha_i, \beta_j \in \mathbb{Z}_{\geq 0}$. Since $\dim(I) = 0$, f_{xy}, f_{xz} are relatively prime by Proposition 3.15. In particular, $p_i \neq q_j$ for all i, j. So, $af_{xy} = bf_{xz}$ only if $a \in \langle f_{xz}\rangle$. Hence, f_{xy} is a non-zerodivisor in $R/\langle f_{xz}\rangle$.

Finally, we show f_{yz} is a non-zerodivisor in $R/\langle f_{xy}, f_{xz}\rangle$. Again, we assume to the contrary that f_{yz} is a zerodivisor in $R/\langle f_{xy}, f_{xz}\rangle$. Then $f_{yz} \in P$ for some minimal associated prime P of $\langle f_{xy}, f_{xz}\rangle$, so $\langle f_{xy}, f_{xz}, f_{yz}\rangle \subseteq P$. By a generalization of the Principal Ideal Theorem, $3 - \dim(P) \leq 2$ so that $\dim(P) \geq 1$, but $\langle f_{xy}, f_{xz}, f_{yz}\rangle \subseteq P$ implies that $1 \leq \dim(P) \leq \dim\langle f_{xy}, f_{xz}, f_{yz}\rangle = 0$ which is an impossibility. So, f_{yz} is contained in no minimal associated prime of $\langle f_1, f_2\rangle$. □

Let $R = K[x, y, z]$, and let $f_{xy}, f_{xz}, f_{yz} \in R$ be such that the ideal $\langle f_{xy}, f_{xz}, f_{yz}\rangle$ is zero-dimensional and $I \neq R$.

Consider the **Koszul complex** below

$$0 \longrightarrow R \xrightarrow{\mu} R^3 \xrightarrow{\psi} R^3 \xrightarrow{\phi} I \longrightarrow 0 \tag{3.6}$$

here μ is the map corresponds to scalar multiplication of $r \in R$ with

$$\begin{bmatrix} f_{xy} \\ f_{xz} \\ f_{yz} \end{bmatrix}$$

and ψ is the R-linear map corresponding to multiplication by the matrix

$$\begin{bmatrix} 0 & -f_{yz} & f_{xz} \\ f_{yz} & 0 & -f_{xy} \\ -f_{xz} & f_{xy} & 0 \end{bmatrix}$$

and ϕ maps $(h, g, k) \in R^3$ to $hf_{xy} + gf_{xz} + kf_{yz} \in I$

We will show that this complex is exact.

Theorem 3.17. *Suppose the coefficients of f_{xy}, f_{xz}, f_{yz} are specialized so that they are nonconstant.*

If $\langle f_{xy}, f_{xz}, f_{yz}\rangle \subsetneq R$ is zero-dimensional, then the Koszul complex (3.6) is exact.

Proof. It was shown in Theorem 3.16 that the generators f_{xy}, f_{xz}, f_{yz} form a regular sequence, assuming the given hypotheses. Exactness of the Koszul complex (3.6) is a direct application of Corollary 17.5 in [6]. □

Next, we show that if the coefficient matrix M is strictly rank deficient and the polytopes $P'_{xy}, P'_{xz}, P'_{yz}$ are constructed in a 'certain' way, then the ideal $\langle f_{xy}, f_{xz}, f_{yz}\rangle$ is a $1-$dimensional ideal in $K[x, y, z]$.

Theorem 3.18. *Suppose P_{xy}, P_{xz}, P_{yz} are Newton polytopes of some generic polynomials f_{xy}, f_{xz}, f_{yz}.*

Let $d_x(f_*)$ denote the largest degree of x appearing in f_*. The integers $d_y(f_*), d_z(f_*)$ are defined analogously.

Let P'_{xy} be a polytope contained in the convex hull of the lattice points (α, β, γ) with

$$\begin{aligned} 0 &\leq \alpha < d_x(f_{xz}) \\ 0 &\leq \beta < d_y(f_{yz}) \end{aligned} \tag{3.7}$$

Let P'_{xz} be a polytope contained in the convex hull of the lattice points (α, β, γ) with

$$\begin{aligned} 0 \leq \alpha < d_x(f_{xy}) \\ 0 \leq \gamma < d_z(f_{yz}) \end{aligned} \tag{3.8}$$

Let P'_{yz} be a polytope contained in the convex hull of the lattice points (α, β, γ) with

$$\begin{aligned} 0 \leq \beta < d_y(f_{xy}) \\ 0 \leq \gamma < d_z(f_{xz}) \end{aligned} \tag{3.9}$$

If the coefficients of f_{xy}, f_{xz}, f_{yz} are specialized so that they are non-constant and M is rank deficient, then $\langle f_{xy}, f_{xz}, f_{yz}\rangle$ is a $1-$dimensional ideal in $K[x, y, z]$.

Proof. We may assume that f_{xy}, f_{xz}, f_{yz} are pairwise relatively prime. Otherwise, by, Proposition 3.15, $\langle f_{xy}, f_{xz}, f_{yz}\rangle$ is $1-$dimensional, and we have our result.

Let $M_* = \{m_{*,1}, ..., m_{*,k_*}\}$ be the set of monomials whose exponent vectors lie in the polytope P'_* where $* = xy, xz$, or yz. Since M is rank deficient, there exists scalars $a_i, b_j, c_l \in K$ not all zero such that

$$\sum(a_i m_{xy,i} f_{xy}) + \sum(b_j m_{xz,j} f_{xz}) + \sum(c_l m_{yz,l} f_{yz}) = 0$$

Letting $g_1 = \sum a_i m_{xy,i}$, $g_2 = \sum b_j m_{xz,j}$, and $g_3 = \sum c_l m_{yz,l}$, we see that $g_1, g_2, g_3 \in K[x, y, z]$ are polynomials such that $g_1 f_{xy} + g_2 f_{xz} + g_3 f_{yz} = 0$.

First, we show that $g_1 \neq 0$. To this end, we assume that to the contrary $g_1 = 0$. Then, $g_2 f_{xz} = -g_3 f_{yz}$. So, f_{xz} divides $-g_3 f_{yz}$. Since f_{xz} and f_{yz} are relatively prime by assumption, it follows that f_{xz} divides g_3 since factorization is unique in $K[x, y, z]$, but this is an impossibility since the $z-$degree of g_3 is strictly less than the $z-$degree of f_{xz} by construction. Likewise, g_2 has maximum $y-$degree strictly less than the maximum $y-$degree of f_{yz} so it is not possible for f_{yz} to divide g_2. Hence, $g_1 \neq 0$. Arguing similarly, we get that g_2 and g_3 are nonzero as well.

Let $h \in K[z]$ denote the coefficient of x having the maximum $x-$degree in $f_{xz} \in K[z][x]$. Let $z = \alpha$ where $h(\alpha), g_1(x, y, \alpha) \neq 0$, and $\overline{f}$ denote the specialization of the polynomial $f \in K[x, y, z]$ at $z = \alpha$.

Since $g_1 f_{xy} + g_2 f_{xz} + g_3 f_{yz} = 0$ and $g_1, g_2, g_3 \neq 0$, it follows that $\overline{g_1}\overline{f_{xy}} + \overline{g_2}\overline{f_{xz}} + \overline{g_3}\overline{f_{yz}} = 0$ so $\overline{g_1}\overline{f_{xy}} \in \langle \overline{f_{xz}}, \overline{f_{yz}} \rangle \subseteq K[x, y]$. We note that $\overline{f_{xz}} \in K[x]$ and $\overline{f_{yz}} \in K[y]$ forms a Gröbner basis for $\langle \overline{f_{xz}}, \overline{f_{yz}} \rangle$ irrespective of monomial ordering on $K[x, y]$ since $\gcd(\mu, \nu) = 1$ for each nonconstant monomial μ, ν of $\overline{f_{xz}}$ and $\overline{f_{yz}}$ respectively. So, $\langle \overline{f_{xz}}, \overline{f_{yz}} \rangle$ is a $0-$dimensional ideal in $K[x, y]$ following Theorem 2.10.

Assume to the contrary that $\overline{g_1} \in \langle \overline{f_{xz}}, \overline{f_{yz}} \rangle$. Let $<$ denote the degree lexicographic order on $K[x, y]$, then $\overline{f_{xz}}, \overline{f_{yz}}$ form a Gröbner basis with respect to $<$ and $\mathrm{LT}_<(\overline{\mathrm{g_1}}) \in \langle \mathrm{LT}_<(\overline{\mathrm{f_{xz}}}), \mathrm{LT}_<(\overline{\mathrm{f_{xz}}}) \rangle$, so that at least one of $\mathrm{LT}_<(\overline{\mathrm{f_{xz}}})$ or $\mathrm{LT}_<(\overline{\mathrm{f_{xz}}})$ divides $\mathrm{LT}_<(\overline{\mathrm{g_1}})$, but this is an impossibility as the maximum $x-$degree of g_1 is strictly less than the maximum $x-$degree of f_{xz} and the maximum $y-$degree of g_1 is strictly less than the maximum $y-$degree of f_{yz}. Hence, $\overline{g_1} \notin \langle \overline{f_{xz}}, \overline{f_{yz}} \rangle$

However, as $\overline{g_1}\overline{f_{xy}} \in \langle \overline{f_{xz}}, \overline{f_{yz}} \rangle$, f_{xy} is a zero-divisor, and so cannot be a unit, in the quotient ring $\dfrac{K[x, y]}{\langle \overline{f_{xz}}, \overline{f_{yz}} \rangle}$. Hence, $\langle \overline{f_{xy}}, \overline{f_{xz}}, \overline{f_{yz}} \rangle \neq K[x, y]$.

Thus, we've shown that for generic $z = \alpha$, the variety $\mathbb{V}(f_{xy}, f_{xz}, f_{yz}, z - \alpha)$ is nonempty, consisting of finitely many points, from whence it follows that the variety $\mathbb{V}(f_{xy}, f_{xz}, f_{yz}) \subset K^3$ is $1-$dimensional. Thus, $\langle f_{xy}, f_{xz}, f_{yz} \rangle$ is a $1-$dimensional ideal in $K[x, y, z]$. $\square$

It is a necessary (but not sufficient) condition for $\langle f_{xy}, f_{xz}, f_{yz} \rangle$ to be $1-$dimensional in order for it to satisfy the elimination conditions.

However, while $\langle f_{xy}, f_{xz}, f_{yz} \rangle$ itself may not necessarily satisfy the elimination conditions, it will at least have an associated prime that does assuming it is $1-$dimensional.

Theorem 3.19. *Let f_{xy}, f_{xz}, f_{yz} be the polynomials whose tropicalizations are Σ_{xy}, Σ_{xz}, and Σ_{yz} respectively.*

Suppose Σ_{xy}, Σ_{xz}, and Σ_{yz} are all nontrivial, and the polytopes $P'_{xy}, P'_{xz}, P'_{yz}$ are constructed to satisfy the conditions of Theorem 3.18,

If f_{xy}, f_{xz}, f_{yz} have been specialized so that M is rank deficient, then $\langle f_{xy}, f_{xz}, f_{yz} \rangle$ has an associated prime ideal I that satisfies the elimination conditions, and in particular, *the elimination ideals in each of the coordinate planes are each respectively generated by* f_{xy}, f_{xz}, f_{yz}.

Proof. By Theorem 3.18, $\langle f_{xy}, f_{xz}, f_{yz}\rangle$ is 1−dimensional since M has been specialized to have strictly less than full rank. By hypothesis, $\langle f_{xy}, f_{xz}, f_{yz}\rangle \neq K[x, y, z]$ so it is 1−dimensional and has an associated prime I that is 1−dimensional. This associated prime can be chosen so that $I \cap K[x, y]$ is 1−dimensional.

Since $I \cap K[x, y]$ is 1−dimensional and prime, it is principal. Say $\langle g\rangle = I \cap K[x, y]$. Since $f_{xy} \in I \cap K[x, y]$, $f_{xy} = gh$ for some $h \in K[x, y]$. So, $\text{Trop}(f_{xy}) = \text{Trop}(g) \cup \text{Trop}(h)$. But, by assumption, Σ_{xy} is not the union of any proper tropical curves, so $\text{Trop}(h)$ is 0−dimensional, which implies that h is equal to a constant. Thus, $I \cap K[x, y] = \langle f_{xy}\rangle$.

Next, we show that both $I \cap K[x, z]$ and $I \cap K[y, z]$ are 1−dimensional as well. Assume to the contrary that $I \cap K[x, z]$ is 0−dimensional. Since it is a 0−dimensional prime, $I \cap K[x, y] = \langle x - a, z - b\rangle$ for some $a, b \in K$. As I is 1−dimensional, it then follows that $I = \langle x - a, z - b\rangle$.

Since $f_{xy} \in I$ and f_{xy} was shown to be irreducible, it must be that $f_{xy} = x - a$. Hence, Σ_{xy} is a classical line. This further implies that Σ is contained in a plane which is a contradiction of our hypothesis on Σ.

Thus, as both $I \cap K[x, z]$ and $I \cap K[y, z]$ must also be 1−dimensional, I satisfies the elimination conditions as desired. $\square$

Next, we prove a converse to Theorem 3.18. Recall that $S = (P'_{xy} + P_{xy}) \cup (P'_{yz} + P_{yz}) \cup (P'_{xz} + P_{xz})$, $T = \{x^\alpha y^\beta z^\gamma f_i : (\alpha, \beta, \gamma) \in P'_i \cap \mathbb{Z}^3_{\geq 0} \text{ and } i \in \{xy, xz, yz\}\}$, and that our matrix of coefficients M has size $|T| \times |S|$.

For the next theorem, we impose the condition that $|S| \geq |T|$, i.e. there are at most as many monomials than there are Newton polytopes, we will also need the following proposition:

Proposition 3.20. *Let U, V be vector spaces of K^n for some $n < \infty$. Then,*

$$\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V)$$

where $U + V$ denotes the intersection of the subspaces of K^n containing both U and V.

Proof. Let $\{b_1, ..., b_p\}$ denote a basis for $U \cap V$. Then, $\dim(U \cap V) = p$, and $\{b_1, ..., b_p\}$ can be extended to a basis of U by, say, $\{u_1, ..., u_k\}$. So, $\dim(U) = p + k$. Likewise, $\{b_1, ..., b_p\}$ can be extended to a basis of V by $\{v_1, ..., v_l\}$ so that $\dim(V) = p + l$.

Let $B := \{b_1, ..., b_p, u_1, ..., u_k, v_1, ...v_l\}$. Every subspace containing both U and V contains B as well, so B spans $U + V$. We will show that B is a linearly independent set. Assume that there exists $\alpha_h, \beta_i, \gamma_j \in K$ such that

$$\alpha_1 b_1 + ... + \alpha_p b_p + \beta_1 u_1 + ... + \beta_k u_k + \gamma_1 v_1 + ... + \gamma_l v_l = 0$$

This gives:

$$\gamma_1 v_1 + ... + \gamma_l v_l = -(\alpha_1 b_1 + ... + \alpha_p b_p + \beta_1 u_1 + ... + \beta_k u_k) \in U$$

since $u_1, ..., u_k \in U$ and $b_1, ...b_p \in U \cap V$, it follows that the $\gamma_1 v_1 + ... + \gamma_l v_l \in U \cap V$. Hence, we may write $\gamma_1 v_1 + ... + \gamma_l v_l = d_1 b_1 + ... + d_p b_b$ which gives $\gamma_1 v_1 + ... + \gamma_l v_l - d_1 b_1 - ... - d_p b_p = 0$. Since $v_1, ..., v_l, b_1, ...b_p$ is a basis for V, it follows that γ_j and d_h are all zero. Arguing similarly, we get that the β_j are all zero as well. Thus, B is a basis for $U + V$ and

$$\begin{aligned}\dim(U + V) &= p + k + l \\ &= p + k + p + l - p \\ &= \dim(U) + \dim(V) - \dim(U \cap V)\end{aligned}$$

□

Theorem 3.21. *Suppose the polytopes $P'_{xy}, P'_{xz}, P'_{yz}$ satisfy*

$$\begin{aligned}&|P'_{xy} \cap \mathbb{Z}^2_{\geq 0}| + |P'_{xz} \cap \mathbb{Z}^2_{\geq 0}| + |P'_{yz} \cap \mathbb{Z}^2_{\geq 0}| + |S \cap K[x, z]| \\ &\qquad > |\{u \in \mathbb{Z}^2_{\geq 0} : u + P_{xz} \subset \mathrm{conv}(S \cap K[x, z])\}| + |S|\end{aligned}$$

If the coefficients of f_{xy}, f_{xz}, f_{yz} are specialized so that the ideal $\langle f_{xy}, f_{xz}, f_{yz}\rangle$ has an associated prime I satisfying $I \cap K[x, z] = \langle f_{xz}\rangle$, then M has strictly less than $|T|$.

Proof. Suppose f_{xy}, f_{xz}, f_{yz} have been specialized so that $\langle f_{xy}, f_{xz}, f_{yz}\rangle$ has an associated prime I so that $I \cap K[x,z] = \langle f_{xz}\rangle$. Assume to the contrary that M has rank equal to $|T|$.

Define W_{xy} to be the K-span of those monomials whose exponent vectors lie in $P'_{xy} \cap \mathbb{Z}^2_{\geq 0}$. Define W_{xz} and W_{yz} analogously. Then, W_{xy} (analogously W_{xz} and W_{yz}) is a K-vector space in $K[x,y,z]$ with $\dim_K(W_{xy}) = |P'_{xy} \cap \mathbb{Z}^2_{\geq 0}|$. Then the vector space $W := W_{xy} \oplus W_{xz} \oplus W_{yz}$ has $K-$dimension equal to $|T|$

Let V denote the $K-$span in $K[x,y,z]$ of the monomials in S. Then M is the matrix representation of the map $W \to V$ by $(p,q,r) \mapsto pf_{xy} + qf_{xz} + rf_{yz}$. Hence, the image of W in V under M is contained in $\langle f_{xy}, f_{xz}, f_{yz}\rangle \subseteq I$.

Since the rank of M is equal to $|T|$ by assumption, $I \cap V$ is at least $|T|$-dimensional, and $V \cap K[x,z]$ has the monomials $S \cap K[x,z]$ as basis, so $V \cap K[x,z]$ is $|S \cap K[x,z]|$-dimensional. This gives:

$$\dim(V \cap K[x,z]) + \dim(V \cap I) \geq |S \cap K[x,z]| + |T|$$

By hypothesis, $I \cap K[x,z] = \langle f_{xz}\rangle$. So, $V \cap K[x,z] \cap I = (V \cap K[x,z]) \cap (I \cap K[x,z]) = (V \cap K[x,z]) \cap \langle f_{xz}\rangle$. Now, $h \in (V \cap K[x,z]) \cap \langle f_{xz}\rangle$ if and only if, there exists some $\alpha_m \neq 0$ and $g \in K[x,z]$ such that

$$\begin{aligned} h &= \sum_{m \in S \cap K[x,z]} \alpha_m m \\ &= g f_{xz} \end{aligned}$$

Let $P(f)$ denote the Newton polytope of f, then

$$\begin{aligned} P(h) &= \operatorname{conv}\{m \in S \cap K[x,z] : \alpha_m \neq 0\} \\ &= P(g f_{xz}) \\ &= P(g) + P_{xz} \end{aligned}$$

the last line follows from Lemma 2.40. This implies that $V \cap K[x,z] \cap I = \{g \in K[x,z] : P(g) + P_{xz} \subseteq \operatorname{conv}(S \cap K[x,z])\}$. Since $g \in K[x,z]$ satisfies $P(g) + P_{xz} \subseteq \operatorname{conv}(S \cap K[x,z])$ if each monomial, $u \in \mathbb{Z}^2_{\geq 0}$, of g satisfies $u + P_{xz} \subset \text{conv}(S \cap K[x,z])$, we have $V \cap K[x,z] \cap I \subseteq \{u \in \mathbb{Z}^2_{\geq 0} : u + P_{xz} \subset \text{conv}(S \cap K[x,z])\}$. Hence,

$$\dim(V \cap K[x,z] \cap I) \leq |\{u \in \mathbb{Z}^2_{\geq 0} : u + P_{xz} \subset \text{conv}(S \cap K[x,z])\}|$$

Hence, we have

$$\dim((V \cap K[x,z]) + (V \cap I)) + \dim(V \cap K[x,z] \cap I) \leq |S| + |\{u \in \mathbb{Z}^2_{\geq 0} : u + P_{xz} \subset \text{conv}(S \cap K[x,z])\}|$$

Let $(V \cap K[x,z]) + (V \cap I)$ denote the intersection of all vector subspaces in V containing both $(V \cap K[x,z])$ and $(V \cap I)$. Hence, $|S| = \dim(V) \geq \dim((V \cap K[x,z]) + (V \cap I))$.

From Proposition 3.20, we have the following formula:

$$\dim(V \cap K[x,z]) + \dim(V \cap I) = \dim((V \cap K[x,z]) + (V \cap I)) + \dim(V \cap K[x,z] \cap I)$$

So,

$$\begin{aligned} &|S \cap K[x,z]| + |T| \leq \\ &|\{u \in \mathbb{Z}^2_{\geq 0} : u + P_{xz} \subset \text{conv}(S \cap K[x,z])\}| + |S| \end{aligned}$$

but, since $|T| = |P'_{xy} \cap \mathbb{Z}^2_{\geq 0}| + |P'_{xz} \cap \mathbb{Z}^2_{\geq 0}| + |P'_{yz} \cap \mathbb{Z}^2_{\geq 0}|$, we have:

$$\begin{aligned} &|P'_{xy} \cap \mathbb{Z}^2_{\geq 0}| + |P'_{xz} \cap \mathbb{Z}^2_{\geq 0}| + |P'_{yz} \cap \mathbb{Z}^2_{\geq 0}| + |S \cap K[x,z]| \\ &\leq |\{u \in \mathbb{Z}^2_{\geq 0} : u + P_{xz} \subset \text{conv}(S \cap K[x,z])\}| + |S| \end{aligned}$$

However, this contradicts our hypothesis that

$$\begin{aligned} &|P'_{xy} \cap \mathbb{Z}^2_{\geq 0}| + |P'_{xz} \cap \mathbb{Z}^2_{\geq 0}| + |P'_{yz} \cap \mathbb{Z}^2_{\geq 0}| + |S \cap K[x,z]| > \\ &|\{u \in \mathbb{Z}^2_{\geq 0} : u + P_{xz} \subset \text{conv}(S \cap K[x,z])\}| + |S| \end{aligned}$$

Thus, the coefficient matrix M cannot have rank greater than or equal to $|T|$. $\square$

A consequence of Theorem 3.21 is that if $\langle f_{xy}, f_{xz}, f_{yz}\rangle$ satisfies the elimination conditions, then M is strictly rank deficient.

3.4 The Correctness of the Procedure

In this section, we prove that the procedure specified at the beginning of this chapter is indeed correct. We begin with some preliminaries.

Definition 3.22. *Let $\{(\sigma_i, m_i)\}_{i=1}^{k} \subset \Sigma$, where σ_i is a $1-$dimensional cell of Σ, and $m_i > 0$ is the multiplicity of σ_i. Suppose these cells meet at a vertex x, and let $v_1, ..., v_m$ be the first lattice points on each cell.*

*We say Σ is **minimally balanced** at x if the following conditions hold:*

(i) $\gcd(m_1, ..., m_k) = 1$

(ii) The subspace $\{(c_1, ..., c_k) : c_1 v_1 + ... + c_k v_k = 0\} \subset \mathbb{R}^k$ is $1-$dimensional.

The polyhedral complex Σ is minimally balanced if Σ is minimally balanced at each of its vertices.

Proposition 3.23. *Suppose Σ is a tropical curve that is minimally balanced, and $\Sigma' = \{(\sigma_i, m_i)\}$ is a weighted polyhedral complex contained in Σ.*

If Σ' is balanced, then $\Sigma' = \Sigma$. Furthermore, there exists some $\alpha \in \mathbb{Z}_{\geq 0}$, we have

$$\text{mult}_{\Sigma'}(\sigma) = \alpha \cdot \text{mult}_{\Sigma}(\sigma)$$

for every $\sigma \in \Sigma = \Sigma'$

Proof. Assume to the contrary that $\Sigma' \subsetneq \Sigma$.

Let x be a vertex at which the 1-dimensional cells $\sigma_1, ..., \sigma_n \in \Sigma$ meet. We can choose x so that at least one of the cells, say σ_1, lies in Σ but not in Σ'.

Let $v_1, ..., v_k$ be the first lattice points on $\sigma_1, ..., \sigma_k$ respectively, $m_i = \text{mult}_{\Sigma}(\sigma_i) > 0$, and $m_i' = \text{mult}_{\Sigma'}(\sigma_i) \geq 0$.

By the balancing condition on Σ and Σ', $m_1v_1+m_2v_2+...+m_kv_k = 0$ and $m_2'v_2+...m_k'v_k = 0$.

Let V denote the set $\{(c_1, ..., c_k) : c_1v_1 + ... + c_kv_k = 0\}$, $u = (m_1, ..., m_k)$, and $u' = (0, m_2, ..., m_k)$. Since Σ is minimally balanced, V is a 1-dimensional subspace of $\mathbb{R}^k$ and $\gcd(m_1, .., .m_k) = 1$. Hence, $u' = \alpha u$ for some $\alpha \in \mathbb{Z}_{\geq 0}$ so that $\alpha m_1 = 0$, $\alpha m_2 = m_2'$, ... , $\alpha m_k = m_k'$, but as $m_1 > 0$, we conclude that $\alpha = 0$ so that Σ' is the empty complex.

Next, we show that there exists $\alpha \in \mathbb{Z}_{\geq 0}$ such that $\alpha m_i = m_i'$ for all $i \in [k]$. We showed that $m_i', m_i > 0$ for all i, and that $u = (m_1, ..., m_k)$ and $u' = (m_1', ..., m_k')$ are both in V. As V is 1-dimensional by minimal balancing and the multiplicities are all in $\mathbb{Z}_{>0}$, there exists $\alpha \in \mathbb{Z}_{\geq 0}$ such that $\alpha u = u'$ or $\alpha u' = u$.

Assume that $\alpha u' = u$. Then, $\alpha m_i' = m_i$ for all $i \in [k]$. But, $1 = \gcd(m_1, ...m_k) = \gcd(\alpha m_1', ..., \alpha m_k') = \alpha \gcd(m_1', ..., m_k')$, which implies that α and $\gcd(m_1', ..., m_k')$ are both 1. Hence, $u = u'$.

Thus, $\text{mult}_{\Sigma'}(\sigma) = \alpha \cdot \text{mult}_{\Sigma}(\sigma)$ $\square$

Then we have the following results:

Theorem 3.24. *Suppose f_{xy}, f_{xz}, f_{yz} are polynomials whose tropicalizations are $\Sigma_{xy}, \Sigma_{xz}, \Sigma_{yz}$ respectively.*

If I is an associated prime ideal of $\langle f_{xy}, f_{xz}, f_{yz}\rangle$ satisfying

(i) $I \cap K[x, y] = \langle f_{xy}\rangle$

(ii) $I \cap K[x, z] = \langle f_{xz}\rangle$

(iii) $I \cap K[y, z] = \langle f_{yz}\rangle$

then $\text{Trop}(I) = \Sigma$.

Proof. By Theorem 3.19, $\langle f_{xy}, f_{xz}, f_{yz}\rangle$ has an associated prime I that has principal elimination ideals. Theorem 3.19 further gives us that $I \cap K[x, y] = \langle f_{xy}\rangle$, $I \cap K[x, z] = \langle f_{xz}\rangle$, and $I \cap K[y, z] = \langle f_{yz}\rangle$. By Theorem 3.1, $\Sigma_{xy} = \text{Trop}(I \cap K[x, y])$. Hence, $\Sigma_{xy} = \text{Trop}(f_{xy})$.

Since $\text{Trop}(I) \subset \pi_{xy}^{-1}(\Sigma_{xy})$ and, by hypothesis, $\pi_{xy}^{-1}(\Sigma_{xy}) \cap \pi_{xz}^{-1}(\Sigma_{xz}) \cap \pi_{yz}^{-1}(\Sigma_{yz}) = \Sigma$, it follows that $\text{Trop}(I) \subseteq \Sigma$.

By hypothesis, Σ is minimally balanced, and since $\text{Trop}(I)$ is a balanced complex contained in Σ, $\text{Trop}(I)$ and Σ have the same underlying set of polyhedra with $\text{mult}_{\text{Trop}(I)}(\sigma) \geq \text{mult}_{\Sigma}(\sigma)$ for all cells in $\text{Trop}(I)$ and Σ by Proposition 3.23.

We wish to show that, in fact, $\text{mult}_{\text{Trop}(I)}(\sigma) = \text{mult}_{\Sigma}(\sigma)$ for all cells σ in $\text{Trop}(I)$ and Σ.

By our hypothesis on Σ, there exists a cell $\sigma \in \Sigma$ such that $\text{mult}_{\Sigma_{xy}}(\pi_{xy}(\sigma)) = \text{mult}_{\Sigma_{xz}}(\pi_{xz}(\sigma)) = 1$. By Theorem 3.11, $\text{mult}_{\Sigma}(\sigma) = 1$. As $\Sigma_{xy} = \text{Trop}(f_{xy})$ and $\Sigma_{xz} = \text{Trop}(f_{xz})$, it follows that $\text{mult}_{\text{Trop}(I)}(\sigma) = 1$ as well.

Let x be the vertex at which the cells $\sigma_1 = \sigma, \sigma_2, ..., \sigma_k$ meet, let $v_1, ..., v_k$ be their first lattice points, and let $m_1 = 1, m_2, ..., m_k$ and $m_1' = 1, m_2, ..., m_k$ be the multiplicities of the σ_i as cells in Σ and $\text{Trop}(I)$ respectively. Since $\text{Trop}(I)$ and Σ are balanced,

$$v_1 + m_2 v_2 + ... + m_k v_k = v_1 + m_2' v_2 + ... + m_k' v_k = 0$$

Subtracting, we get $(m_2' - m_2)v_2 + ... + (m_k' - m_k)v_k = 0$ which implies that $m_i' - m_i \in V$ where $V \subset \mathbb{R}^3$ is the set $\{(c_1, ..., c_k) : c_1 v_1 + ... + c_k v_k = 0\}$. As V is $1-$dimensional by minimal balancing, there exists $\alpha \in \mathbb{Z}_{\geq 0}$ such that $\alpha \cdot m_i = m_i' - m_i$ for all $i \in [k]$. But, $m_1 = 1 > 0$. So, $\alpha m_1 = m_1' - m_1 = 0$ is possible only if $\alpha = 0$. Thus, $m_i = m_i'$ for all $i \in [k]$. $\square$

Theorem 3.25. *Suppose Σ is a tropical curve in $\mathbb{R}^3$ that is not contained in a plane.*

If $I \subseteq K[x, y, z]$ is a prime ideal such that $\text{Trop}(I) = \Sigma$, *then I has principal elimination ideals.*

Furthermore, if $\langle f_{xy} \rangle = I \cap K[x, y]$, $\langle f_{xz} \rangle = I \cap K[x, z]$, and $\langle f_{yz} \rangle = I \cap K[y, z]$, then $|\Sigma_i| = |\text{Trop}(f_i)|$ and $\text{mult}_{\Sigma_i}(\sigma) = \text{mult}_{\text{Trop}(f_i)}(\sigma)$ for each $1-$dimensional cell σ and for $i = xy, xz, yz$.

Proof. Since $\Sigma = \text{Trop}(I)$ is 1-dimensional and is not contained in a plane, it follows that I is $1-$dimensional and the projection Σ_{xy} to the $xy-$plane is a $1-$dimensional polyhedral complex in $\mathbb{R}^2$. By Theorem 3.1, $|\text{Trop}(I \cap K[x, y])| = |\Sigma_{xy}|$, so $I \cap K[x, y]$ is a $1-$dimensional ideal. Since $\dfrac{K[x, y]}{I \cap K[x, y]}$ is a $1-$dimensional integral domain ($I \cap K[x, y]$ inherits prime-ness from I), it follows that $I \cap K[x, y]$ is principal. Reasoning similarly, we get that $I \cap K[x, z]$ and $I \cap K[y, z]$ are principal as well.

Let f_{xy}, f_{xz}, f_{yz} be polynomials such that generate $I \cap K[x, y]$, $I \cap K[x, z]$, and $I \cap K[y, z]$ respectively. Then, $|\Sigma_i| = |\operatorname{Trop}(f_i)|$ for $i = xy, xz, yz$.

Let $X_i = \operatorname{Trop}(f_i) \wedge \Sigma_i$ denote the common refinement of $\operatorname{Trop}(f_i)$ and Σ_i. For $\sigma \in X_i$, we have $\operatorname{mult}_{\Sigma_i}(\sigma) \geq 1$ and $\operatorname{mult}_{\operatorname{Trop}(f_i)}(\sigma) \geq 0$. By Theorem 3.8, we also have that $\operatorname{mult}_{\Sigma_i}(\sigma) \geq \operatorname{mult}_{\operatorname{Trop}(f_i)}(\sigma)$. Hence, $\operatorname{mult}_{\Sigma_i}(\sigma) - \operatorname{mult}_{\operatorname{Trop}(f_i)}(\sigma) \geq 0$ forms a collection of weights for each $1-$dimensional cell σ in Σ_i.

By Definition 2.61, since $\operatorname{mult}_{\Sigma_i}(\sigma) \geq \operatorname{mult}_{\Sigma_i}(\sigma) - \operatorname{mult}_{\operatorname{Trop}(f_i)}(\sigma)$ and Σ_i is tropically irreducible, either $\operatorname{mult}_{\Sigma_i}(\sigma) - \operatorname{mult}_{\operatorname{Trop}(f_i)}(\sigma) = \operatorname{mult}_{\Sigma_i}(\sigma)$ or $\operatorname{mult}_{\Sigma_i}(\sigma) - \operatorname{mult}_{\operatorname{Trop}(f_i)}(\sigma) = 0$. Since $\operatorname{Trop}(f_i) \neq 0$, it follows that $\operatorname{mult}_{\Sigma_i}(\sigma) = \operatorname{mult}_{\operatorname{Trop}(f_i)}(\sigma)$. □

We now have the preliminary results needed to prove correctness. We start first with the theorem that if our procedure returns 'Yes,' then the Tropical curve Σ is realizable.

Theorem 3.26. *Let α denote the vector of valuations determined by the subdivision on P_{xy}, P_{xz}, P_{yz}.*

Suppose there exists polytope $P'_{xy}, P'_{xz}, P'_{yz}$ satisfying the conditions of Theorems 3.18.

If $\alpha \in \operatorname{Trop}(G)$, then there exists a $1-$dimensional prime ideal $I \subseteq K[x, y, z]$ such that $\operatorname{Trop}(I) = \Sigma$.

Proof. If $\alpha \in \operatorname{Trop}(G)$, then $\exists p \in \mathbb{V}(G)$ such that $\operatorname{val}(p) = \alpha$.

Specializing f_{xy}, f_{xz}, f_{yz} at p means the ideal $\langle f_{xy}, f_{xz}, f_{yz}\rangle$ is $1-$dimensional by Theorem 3.18. Then $\langle f_{xy}, f_{xz}, f_{yz}\rangle$ has a $1-$dimensional associated prime ideal I that has principal elimination ideals by Theorem 3.19 and, $I \cap K[x, y] = \langle f_{xy}\rangle$, $I \cap K[x, z] = \langle f_{xz}\rangle$, and $I \cap K[y, z] = \langle f_{yz}\rangle$.

Hence, by Theorem 3.24, $\operatorname{Trop}(I) = \Sigma$. □

Next, we show that if our procedure returns 'No,' then our Tropical curve Σ is not realizable. We do this by proving the contrapositive.

Theorem 3.27. *Let α denote the vector of valuations determined by the subdivision on P_{xy}, P_{xz}, P_{yz}*

If there exists a $1-$dimensional prime ideal $I \subseteq K[x, y, z]$ such that $\operatorname{Trop}(I) = \Sigma$, then $\alpha \in \operatorname{Trop}(G)$.

Proof. Let I be a 1−dimensional prime ideal such that $\text{Trop}(I) = \Sigma$. By Theorem 3.25, I satisfies the elimination conditions, that is, $I \cap K[x, y]$, $I \cap K[x, z]$, $I \cap K[y, z]$ are respectively generated by f_{xy}, f_{xz}, f_{yz}. Since I satisfies the elimination conditions, Theorem 3.21 implies that the matrix of coefficients generated by f_{xy}, f_{xz}, f_{yz} has strictly less than full rank, so if p denotes that vector of coefficients of f_{xy}, f_{xz}, f_{yz}, then $p \in \mathbb{V}(G)$.

As $\text{val}(p) = \alpha$, it follows then that $\alpha \in \text{Trop}(G)$. □

Chapter 4

Example

We conclude by applying the procedure developed in the previous chapter. Let Σ in $\mathbb{R}^3$ denote the tropical curve consisting of the straight line segment through the origin connecting $(1,1,1)$ and $(-1,-1,-1)$. The curve branches off at $(1,1,1)$ in the the directions of $(2,1,1)$, $(1,2,1)$, and $(1,1,2,)$. Similarly, the curve branches at $(-1,-1,-1)$ in the the directions of $(-2,-1,-1)$, $(-1,-2,-1)$, and $(-1,-1,-2)$. This is the same tropical curve seen in Figure 1.1.

In the language of [13], Σ is a zero tension curve of genus zero. Hence, Theorem 3.2 in the same paper gives that Σ is realizable. We will verify this using the methods developed in this work.

The projections of this tropical curve to each of the coordinate planes all have the same form. They consist of a line segment connecting the coordinates $(-1,-1)$ to $(1,1)$ passing through the origin. At $(-1,-1)$ the curve branches into two rays in the direction of $(-2,-1)$ and $(-1,-2)$. Similarly, at $(1,1)$ the curve branches into two rays in the direction of $(2,1)$ and $(1,2)$.

The subdivided newton polytopes that are dual to the projections in each coordinate plane also all have the same form. They consist of a square with vertices at $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$. The subdivision is a line segment connecting the vertices $(1,0)$ and $(0,1)$. In order to obtain this subdivision, we will need the valuation of the coefficient of the monomials corresponding to $(1,0)$ and $(0,1)$ to be both less than that of the monomials corresponding to $(0,0)$ and $(1,1)$.

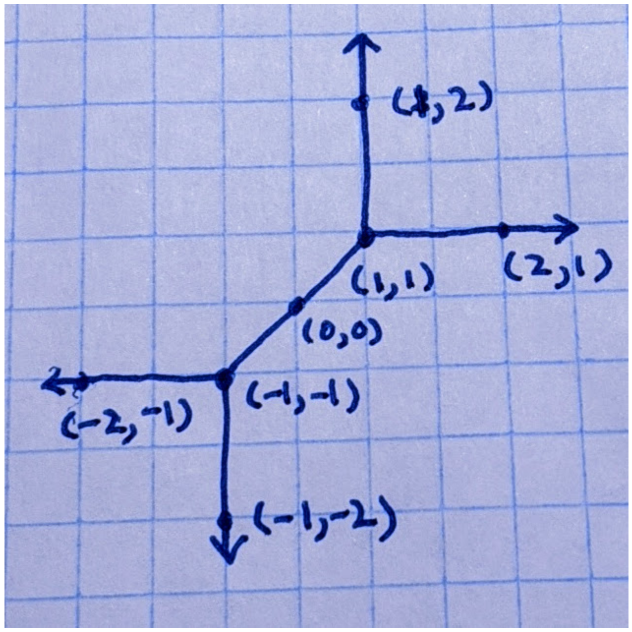

Figure 4.1: Projection of Σ to coordinate planes

The polynomials that have the same subdivided Newton polygons are:

$$f_{xy} = axy + bx + cy + d$$
$$f_{xz} = exz + fx + gz + h$$
$$f_{yz} = iyz + jy + kz + l$$

where the valuations of b, c, f, g, j, k are all equal to -2 and the valuations of a, d, e, h, i, l are equal to -1. For these polynomials, the polytopes $P'_{xy}, P'_{xz}, P'_{yz}$ that satisfy that conditions of Theorem 3.18 are given by:

1. $P'_{xy} = \text{conv}\{1, z\}$

2. $P'_{xz} = \text{conv}\{1, y\}$

3. $P'_{yz} = \text{conv}\{1, x\}$

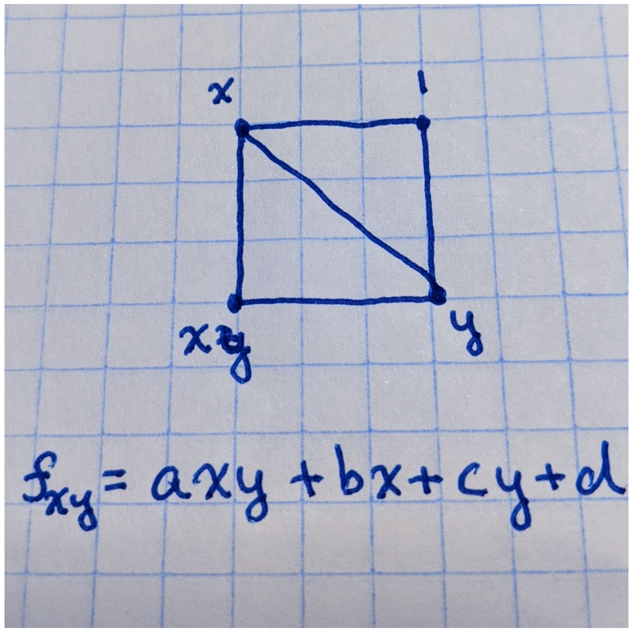

Figure 4.2: Subdivided Newton Polygon Dual to Σ_{xy}

Hence, the entries of the rows of M are the coefficients of the polynomials $f_{xy}, zf_{xy}, f_{xz}, yf_{xz}, f_{yz}, xf_{yz}$, and the columns of M correspond to the monomials $1, z, y, x, yz, xz, xy, xyz$. So, the coefficient matrix M as defined in 3.3.1 is exactly:

$$M = \begin{bmatrix} d & b & c & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d & b & c & a \\ h & f & 0 & 0 & g & e & 0 & 0 \\ 0 & 0 & h & f & 0 & 0 & g & e \\ l & 0 & j & 0 & k & 0 & i & 0 \\ 0 & l & 0 & j & 0 & k & 0 & i \end{bmatrix}$$

Next, we verify that $P'_{xy}, P'_{xz}, P'_{yz}$ satisfy the hypotheses of Theorem 3.21. First, $|P'_{xy} \cap \mathbb{Z}^2_{\geq 0}| = |P'_{xz} \cap \mathbb{Z}^2_{\geq 0}| = |P'_{yz} \cap \mathbb{Z}^2_{\geq 0}| = 2$. The set S is the collection of monomials in $(P_{xy} + P'_{xy}) \cup (P_{xz} + P'_{xz}) \cup (P_{yz} + P'_{yz})$. For these polynomials, S equal to $\{1, x, y, z, xy, xz, yz, xyz\}$.

So, $|S| = 8$ and $|S \cap K[x,z]| = 4$. Next, $|\{u \in \mathbb{Z}^2_{\geq 0} : u + P_{xz} \subset \operatorname{conv}(S \cap K[x,z])\}| = 0$ since both $\operatorname{conv}(S \cap K[x,z])$ and P_{xz} are both a square generated by the vertices $1, x, z, xz$. Hence, there is no $u \in \mathbb{Z}^n_{\geq 0}$ such that $u + P_{xz}$ is properly contained in $\operatorname{conv}(S \cap K[x,z])$. Finally, for the formula

$$\begin{aligned}|P'_{xy} \cap \mathbb{Z}^2_{\geq 0}| + |P'_{xz} \cap \mathbb{Z}^2_{\geq 0}| + |P'_{yz} \cap \mathbb{Z}^2_{\geq 0}| + |S \cap K[x,z]|&\\ > |\{u \in \mathbb{Z}^2_{\geq 0} : u + P_{xz} \subset \operatorname{conv}(S \cap K[x,z])\}| + |S|&\end{aligned}$$

we have $10 > 8$. So, Theorem 3.21 assures us that if the coefficients of f_{xy}, f_{xz}, f_{yz} are specialized so that $\langle f_{xy}, f_{xz}, f_{yz} \rangle$ has an associated prime that satisfies the elimination conditions, then M will be rank deficient.

We use the computational commutative algebra software Macaulay2 [7], for the remaining computations. Using the following input in Macaulay2,

```
w = {1,2,2,1,1,2,2,1,1,2,2,1}
R = QQ[a,b,c,d,e,f,g,h,i,j,k,l, Weights => w];
M = matrix{{d,b,c,a,0,0,0,0},
           {0,0,0,0,d,b,c,a},
           {h,f,0,0,g,e,0,0},
           {0,0,h,f,0,0,g,e},
           {l,0,j,0,k,0,i,0},
           {0,l,0,j,0,k,0,i}};
I = minors(6,M);
all(flatten entries groebnerBasis I,
    f -> ((u,wt) = weightRange(w,f);
    (u, nxt) = weightRange(w, f - leadTerm f); wt == nxt))
```

we find that the initial ideal with weight $w = (-1, -2, -2, -1, -1, -2, -2, -1, -1, -2, -2, -1)$ contains no monomial, so this tropical curve is realizable.

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Vita

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