To the Graduate Council:

I am submitting herewith a thesis written by Michael J. Gallagher entitled "A Comparative Study of Optimal Control Techniques." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science in Electrical Engineering.

Dr. James Hung, Major Professor

We have read this thesis and

recommend its acceptance:

Accepted for the Council:

Associate Vice Chancellor

and Dean of the Graduate School

STATEMENT OF PERMISSION TO USE

In presenting this thesis in partial fulfillment of the requirements for a Master's degree at The University of Tennessee, Knoxville, I agree that the Library shall make it available to borrowers under the rules of the Library. Brief quotations from this thesis are allowable without special permission, provided that accurate acknowledgment of the source is made.

Permission for extensive quotation from or reproduction of this thesis may be granted by my major professor, or in his absence, the Head of Interlibrary Services when, in the opinion of either, the proposed use of the material is for scholarly purposes. Any copying or use of the material in this thesis for financial gain shall not be allowed without my written permission.

Signature Michael J. Dallaire

Date 23 November 1994

A COMPARATIVE STUDY OF OPTIMAL CONTROL TECHNIQUES

A Thesis

Presented for the

Master of Science

Degree

The University of Tennessee, Knoxville

Michael J. Gallagher

December 1994

DEDICATION

This thesis is dedicated to every member of my family,

whose support made everything I've done possible.

ACKNOWLEDGMENTS

I would like to thank my major professor, Dr. James Hung, for his cooperation and assistance. Additionally, I would like to thank Dr. James Rochelle of the Electrical Engineering department, and Dr. Vasilios Alexiades of the Mathematics department for being part of the committee.

ABSTRACT

Two least-integral-square based optimal control techniques are presented and compared here for single-input-single-output processes. The foundation of both techniques is the selection of a performance index based upon sound engineering principles. It is the selection of this performance index that, in fact, will guarantee the evolution of an optimal controller for the process at hand.

This study first develops the least-square based mathematical model proposed by Youla-Bongiorno-Jabr, 1976. The study then continues by examining the method developed by S.S.L Chang, 1961, and derives results for both single, and multiple controller system configurations. The method proposed by Youla, et. al., starts by selecting a specific control system configuration. Then, using a performance based index, the derivation of a single optimal compensator which may account for external noise and disturbance is made possible. Chang's method, also least-square based, begins with no specific preassigned control configuration. This is an important difference that sets Chang's and the Youla, et. al. method apart. Chang's method may be used to obtain one, or more, optimal closed-loop transfer functions for the system. After a specific control configuration is chosen, one or more optimal controllers may be found using the results from Chang's general control configuration. These controllers also may account for noise and disturbance effects which may be present external to the system.

These results are then applied to a DC motor and load control situation, with an external disturbance. Single optimal controllers, which account for this disturbance in

their derivation are found using both methods mentioned above, and are compared by simulation to compensators which account for disturbance in their evolution. These results are compared to a dual controller configuration, in which the two optimal compensators were derived by Chang's method. Results show that the two controller configuration has superior disturbance suppression characteristics than the single compensator configuration using both Youla, et. al and Chang's single controllers. These results, therefore, support the use of Chang's method due to its ability to achieve multiple optimal controllers and its ease of use.

TABLE OF CONTENTS

CHAPTER		PAGE
1.	INTRODUCTION	1
2.	METHOD OF LIS CONTROL	5
	2.1 Introduction to LIS control	5
	2.2 Derivation of the YBJ technique	6
	2.3 Chang's Method: Single controller case	12
	2.4 Chang's Method: Two controller case	17
	2.5 Root-Square Locus	20
3.	APPLICATION TO DC MOTOR AND LOAD CONTROL	25
	3.1 Application of the YBJ method: Part I	25
	3.2 Application of YBJ method: Part II	34
	3.3 Application of Chang's method: Part I	37
	3.4 Application of Chang's method: Part II	40
	3.5 Application of Chang's method: Part III	44
4.	DISCUSSION AND CONCLUSIONS	52
LIST OF REFERENCES		55
APPENDICES		54
	Appendix A. Details of Chang's Method	58
	Appendix B. List of abbreviations	60
VITA		61

LIST OF FIGURES

FIGURE PAGE

1. General control configuration used for the YBJ method. 7
2. General control configuration used for Chang's method. 14
3. DC Motor and Load. 26
4. Root-Square Locus. 29
5. Disturbance input to DC motor and load system. 32
6. Response of DC motor and load system to unit step input and disturbance occurring at five seconds, using a zero-disturbance optimal controller derived by the YBJ method. 33
7. Response of DC motor and load system to unit step input and disturbance occurring at five seconds, using an optimal controller accounting for disturbance derived by the YBJ method. 36
8. Response of DC motor and load system to unit step input and disturbance occurring at five seconds, using a one optimal controller zero-disturbance design derived by Chang's method. 39
9. Response of DC motor and load system to unit step input and disturbance occurring at five seconds, using one optimal controller accounting for disturbance derived by Chang's method. 43
10. DC Motor and load, two controller configuration. 45

11. Response of DC motor and load system to unit step input and disturbance occurring at five seconds, using two optimal controllers derived by Chang's method for k = 1, 2, 5, 10, 40, 300, and 3000. 49

12. Plant input as a function of time for k = 1 and 3000. 50

CHAPTER 1

INTRODUCTION

During the past 30 years, there has been a rapidly growing interest in control theory. This interest coupled with the ever growing call for the effective use of natural resources, material, etc., directly leads to the implementation of optimal control. Optimal control theory was first developed in the mid-Fifties. After the Soviet mathematician L.S. Pontryagin and his colleagues [1] developed the so called maximum principle, along with the advent of dynamic programming by Bellman [2], [3], it was discovered that optimal control theory could be treated in the domain of calculus of variations. Both time and frequency domain optimal control theory have been developed, yet time domain techniques have remained the choice of most engineers and mathematicians.

There are two major areas of optimal control, linear quadratic and least-integral-square. Linear quadratic (LQ) control [4], [5], [6], [7], [8] which appeared in the early 1960's has been a popular theoretical technique of "modern control". LQ control is an inherently time-domain technique, and uses state space formulation. One special case of LQ control is known as *linear quadratic regulator* (LQR) control. In LQR control a plant is modeled in standard state space notation

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

$$\mathbf{y}(t) = C\mathbf{x}(t)$$

where $\mathbf{x}(t)$ is the state vector, $\mathbf{u}(t)$ is the control input vector, $\mathbf{y}(t)$ is the output vector, A, B, and C are constant system matrices. The control vector, $\mathbf{u}(t)$, is designed such that the cost function

$$J = \int_0^T [\mathbf{x}(t)^T Q\mathbf{x}(t) + \mathbf{u}(t)^T R\mathbf{u}(t)]\, dt$$

is minimized. Note the integrand consists of quadratic forms in $\mathbf{x}$ and $\mathbf{u}$, and the control is to bring the state to the origin of the state space, hence this is known as LQR control. The control input vector $\mathbf{u}$ is given by

$$\mathbf{u}(t) = K_c\mathbf{x}(t)$$

where

$$K_c = -R^{-1}B^T P.$$

P is found by solving the matrix Riccati differential equation

$$-\dot{P} = A^T P + PA + Q - PBR^{-1}B^T P, \quad P(T) = 0.$$

This method does offer attractive characteristics such as good stability margins, namely, a phase margin in the neighborhood of 60° and infinite gain margin. This method does require the entire state be accessible for measurement, which especially in the case of large scale systems, is often not possible.

There are other extensions of LQ control, such as linear quadratic Gaussian (LQG), and LQG/LTR (linear transfer recovery), which are treated in [9], and [10]. There are, however, several pitfalls to LQ control. First, there is no easy way to incorporate practical design specifications such as risetime, overshoot, bandwidth, etc. This,

however, has been partially solved by pole-placement techniques [4], [5] which is a hybrid time and frequency domain approach. Second, the effects of plant zeros are not accounted for in the design. This is due to the fact that the constant system matrix, C, is not contained in the Riccati equation used to solve for the control input vector. Third, most of the LQ family of control techniques are formulated for regulator control, and the implementation of LQ for tracking control is quite complicated.

During the same time that LQ control was being developed, frequency-domain techniques were also being developed. Although most control professionals have devoted much of their time to time-domain techniques, frequency-domain techniques offer some advantages over time-domain techniques. For this reason, frequency domain techniques deserve to be revisited. Two notable techniques are those proposed by Chang [11], [12], and Youla-Bongiorno-Jabr (YBJ) [13]. The purpose of this paper is to examine these two techniques in detail. While these two methods are similar in principle, there are important differences as far as performance, and ease of design are concerned.

Chang's approach uses integral-square cost functions, instead of quadratic, and is known as a *least-integral-square* (LIS) method. This technique starts with time-domain quantities but exploits Parseval's theorem and transforms the time-domain terms to the frequency-domain. This method also easily accommodates the system design specifications mentioned above by a method known as root-square-locus (RSL), also developed by Chang. Using Chang's approach, optimum closed-loop transfer functions are synthesized, then after a suitable control-system configuration is chosen, the

controllers may be determined. Chang's method, as will be shown, is extremely flexible and can be used to derive one, or multiple controller configurations.

A similar technique to Chang's has been proposed by Youla-Bongiorno-Jabr [13]. In this approach, an optimum closed-loop sensitivity function is found for a system, then from this, a single optimal controller may be found. As in LQ control, this method alone does not allow for the implementation of practical design specifications. However, combining this method with Chang's RSL technique, these specifications may be accommodated. Although the techniques proposed by Chang and YBJ are both LIS in nature, they do differ in formulation and in detail. Therefore, they may result in different control implementations with different control performances.

This paper consists of two main parts. Chapter 2 first reviews LS optimal control theory, and develops results using the method proposed by YBJ. Chapter 2 then continues by examining the method proposed by S.S.L Chang and derives results for two different control scenarios. Chapter 2 concludes with the concept of RSL, a convenient tool in carrying out Chang's method. Chapter 3 then applies the results developed in Chapter 2 to a practical system and compares each system's transient response to a command input and disturbance input.

CHAPTER 2

METHOD OF LIS CONTROL

2.1 INTRODUCTION TO LIS CONTROL

One goal of LIS control theory is to minimize error, e, which is defined as the difference between system input and output, both positive and negative and for all time. One way to achieve this is to minimize the cost function

$$J_e = \int_0^\infty e^2(t)\, dt \tag{2.1}$$

which is the well known integral square error (ISE). Using the ISE cost function is a practical and theoretically sound method. Minimizing ISE will in turn minimize both positive and negative error. It also represents a form of total energy, leading to linear compensators. Another goal of this LIS control is to constrain plant input, i, to avoid saturation. It is convenient to achieve this is by constraining the integral square input of the plant, which is represented by

$$J_m = \int_0^\infty i^2(t)\, dt \le L \tag{2.2}$$

where L is the constraining upper bound on J_m. Letting $s = j\omega$, and by using Parseval's theorem, J_e and J_m may be translated from their time domain representations to their respective frequency domain representations, and are given to be

$$J_e = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} E(s)E(-s)\, ds \tag{2.3}$$

$$J_m = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} I(s)I(-s)\,ds. \tag{2.4}$$

Where E(s) and I(s) are the Laplace transformations of e(t) and i(t), respectively.

For convenience, the shorthand notation

$$\left.\begin{aligned} E &\equiv E(s) \\ \bar{E} &\equiv E(-s) \\ I &\equiv I(s) \\ \bar{I} &\equiv I(-s) \end{aligned}\right\}$$

will be used throughout. More generally, any function $X \equiv X(s)$, and $\bar{X} \equiv X(-s)$. Minimizing ISE, subject to the constraint that the integral square plant input is bounded, is equivalent to minimizing the cost function

$$J = J_e + \lambda J_m = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty}\left\{E\bar{E} + \lambda I\bar{I}\right\} ds \tag{2.5}$$

where λ is the Lagrange multiplier, and is so selected to satisfy the constraint equation (2.2). This procedure is commonly referred to as the Lagrange multiplier method [6].

2.2 DERIVATION OF THE YBJ TECHNIQUE

Having shown the general concepts of the LIS optimal control technique, the method proposed by YBJ [13] is first examined. In order to begin, a specific control system configuration must be selected in advance, and is shown in Figure 1. In Figure 1, R is the command input, I is the plant input, D is external disturbance, C is system output, G_C is the optimal controller, and G_P is the plant and may be written as N_P/D_P.

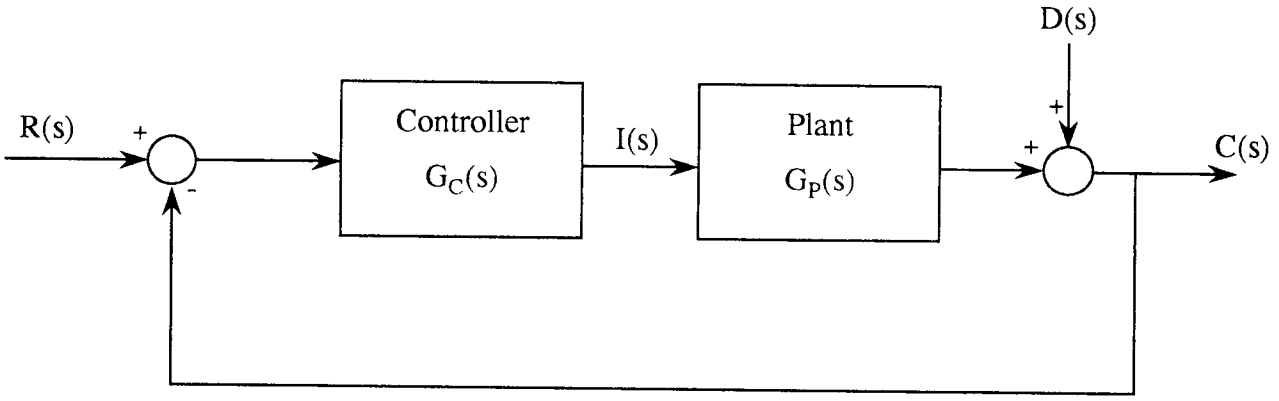


Figure 1. General control configuration used for the YBJ method.

N_P and D_P are polynomials which represent the numerator and denominator of the plant, respectively. The idea behind this method is to achieve an optimal *sensitivity function*, and from that an optimal controller. This method will allow for the use of other configurations, however it must be noted that only one optimal controller may be present. Starting with the analysis of the system in Figure 1, define

$$\text{Error: } E = R - C \tag{2.6}$$

$$\text{Output: } C = IG_P + D \tag{2.7}$$

where I is plant input, R is command-input, and D is external disturbance.

Define S, as

$$S = \frac{1}{1 + G_P G_C}. \tag{2.8}$$

In fact, S is the sensitivity function of the system. Using (2.8), E, and I may be expressed as

$$C = G_C G_P SR + SD = 1 - S(R) + SD \tag{2.9}$$

$$E = SR - SD + (1 - S) \tag{2.10}$$

and plant input, I is

$$I = G_C S(R + D) = \frac{1 - S}{G_P}(R + D). \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.5), and assuming all processes independent and zero mean, the total cost function may be written as

$$J = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} \left\{ S\bar{S}A\bar{A} + (1 - S)(1 - \bar{S})B\bar{B} \right\} ds. \tag{2.12}$$

Let $X\overline{X}$ represent the spectral density of a general function X. Therefore the spectral density of R may be written $R\overline{R}$. In (2.12), $B\overline{B}$, and $A\overline{A}$ are

$$B\overline{B} = \lambda \frac{1}{G_P \overline{G_P}} \left(D\overline{D} + R\overline{R} \right) \tag{2.13}$$

$$A\overline{A} = R\overline{R} + D\overline{D} \tag{2.14}$$

Let

$$S = S_o + \varepsilon \chi_r \delta \tag{2.15}$$

where S_o is the optimal sensitivity function , and $\varepsilon\chi_r\delta$ is the perturbation. ε is any real number, δ is any function of s which is free of right-half plane (RHP) poles and zeros, and χ_r is a function of s which possess all RHP zeros of $N_P D_P$. Substituting (2.15) into (2.12), gives

$$J = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} \left\{ \left(\bar{S}_o + \varepsilon \bar{\chi}_r \bar{\delta} \right) R\overline{R} + \left(S_o + \varepsilon \chi_r \delta \right) \left(\bar{S}_o + \varepsilon \bar{\chi}_r \bar{\delta} \right) A\overline{A} \right.$$
$$\left. + \left(1 - \left(S_o + \varepsilon \chi_r \delta \right) \right) \left(1 - \left(\bar{S}_o + \varepsilon \bar{\chi}_r \bar{\delta} \right) \right) D\overline{D} \right\} ds$$

and multiplying through yields

$$J = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} \left\{ \left[S_o \bar{S}_o + \varepsilon \left(\chi_r \delta \bar{S}_o + \bar{\chi}_r \bar{\delta} S_o \right) + \varepsilon^2 \chi_r \bar{\chi}_r \delta \bar{\delta} \right] A\overline{A} \right.$$
$$\left. + \left[(1 - S_o)(1 - \bar{S}_o) - \varepsilon \left[(1 - \bar{S}_o) \chi_r \delta + (1 - S_o) \bar{\chi}_r \bar{\delta} \right] B\overline{B} \right] + \varepsilon^2 \chi_r \bar{\chi}_r \delta \bar{\delta} B\overline{B} \right\} ds. \tag{2.16}$$

At the optimal condition

$$\left. \frac{\partial J}{\partial \varepsilon} \right|_{\varepsilon = 0} = 0. \tag{2.17}$$

Carrying out (2.17) on (2.16) yields

$$\left.\frac{\partial J}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty}\left\{\chi_r \delta \bar{S}_o A\bar{A} + \bar{\chi}_r \bar{\delta} S_o A\bar{A}\right.$$
$$\left. -\left(1-\bar{S}_o\right)\chi_r \delta B\bar{B} - \left(1-S_o\right)\bar{\chi}_r \bar{\delta} B\bar{B}\right\} ds = 0$$

Rearranging terms and simplifying give

$$\left.\frac{\partial J}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty}\left\{S_o \bar{\chi}_r\left(A\bar{A}+B\bar{B}\right) - \bar{\chi}_r B\bar{B}\right\}\bar{\delta}\, ds = 0 \qquad (2.18)$$

where $\bar{\delta}$ contains RHP poles and zeros only. Making the substitution

$$\left\{S_o \bar{\chi}_r\left(A\bar{A}+B\bar{B}\right) - \bar{\chi}_r B\bar{B}\right\} = \Lambda \qquad (2.19)$$

into (2.18) gives

$$\left.\frac{\partial J}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty}\Lambda\bar{\delta}\, ds = 0. \qquad (2.20)$$

The theory of residue may be used to evaluate (2.20). In order to use the theory of residues, the integration path in (2.20) must be closed. This may be achieved by closing the integral to the right or to the left of the jω axis. Since $\bar{\delta}$ contains only RHP poles and zeros, it would not be wise to close the integral to the right of the jω axis. Therefore, closing the integration path to the left of the jω axis gives

$$\oint \Lambda\bar{\delta}\, ds = 0 \qquad (2.21)$$

where $\oint$ represents complex integration by enclosing the left-half of the complex plane (LHP). In order for (2.21) to hold, Λ must also only posses RHP poles.

Letting

$$Y\bar{Y} = \left(A\bar{A} + B\bar{B}\right)$$

and with a little rearranging, it is easy to write (2.19) as

$$\bar{\chi}_r Y\bar{Y}S_o = \chi_r\left(B\bar{B}\right) + \Lambda. \tag{2.22}$$

Dividing through by terms that contain RHP poles only in (2.22), leaves

$$\bar{\chi}_r YS_o = \frac{\bar{\chi}_r}{\bar{Y}}\left(B\bar{B}\right) + \frac{\Lambda}{\bar{Y}} \tag{2.23}$$

and it is clear, $\bar{\chi}_r YS_o$ has LHP poles and zeros only, $\frac{\bar{\chi}_r}{\bar{Y}}\left(B\bar{B}\right)$ has both LHP and RHP poles and zeros, and $\frac{\Lambda}{\bar{Y}}$ has RHP poles and zeros only. By performing partial fraction decomposition on

$$\frac{\bar{\chi}_r}{\bar{Y}}\left(B\bar{B}\right)$$

will express the LHP and RHP terms as a sum, which may be written as

$$\frac{\bar{\chi}_r}{\bar{Y}}\left(B\bar{B}\right) = \left[\frac{\bar{\chi}_r}{\bar{Y}}\left(B\bar{B}\right)\right]_L + \left[\frac{\bar{\chi}_r}{\bar{Y}}\left(B\bar{B}\right)\right]_R.$$

Where $[\ ]_L$ and $[\ ]_R$ represent the LHP and RHP terms resulting from partial fraction decomposition. Therefore (2.22) may be written as

$$\bar{\chi}_r YS_o = \left[\frac{\bar{\chi}_r}{\bar{Y}}\left(B\bar{B}\right)\right]_L + \left[\frac{\bar{\chi}_r}{\bar{Y}}\left(B\bar{B}\right)\right]_R + \frac{\Lambda}{\bar{Y}}$$

and RHP terms cancel each other, giving

$$S_o = \frac{1}{\bar{\chi}_r Y}\left\{\left[\frac{\bar{\chi}_r}{\bar{Y}}\left(B\bar{B}\right)\right]_L + f\right\}. \tag{2.24}$$

Here f is a real polynomial and is known as the *Lagrange interpolation polynomial* [14], [15], and is uniquely determined by the interpolation conditions

$$\left.\begin{aligned} &S_o(p_r) = 0, \text{ all r} \\ &S_o(z_r) = 1, \text{ all r} \\ &1 - S_o(s) = O\left(\frac{1}{s^2}\right), \ |s| \text{ large} \end{aligned}\right\}$$

The third condition states that $1-S_o(s)$ approaches zero at the rate of $\frac{1}{s^2}$ as $|s| \to \infty$, p_r and z_r are the RHP poles and zeros of G_p, respectively. It is then easy to find the optimal controller, G_C, by substituting (2.24) into (2.8) which yields

$$G_C = \frac{\frac{1}{S_o} - 1}{G_P} = \frac{1 - S_o}{G_P S_o}. \tag{2.25}$$

2.3 CHANG'S METHOD: SINGLE CONTROLLER CASE

There are three pertinent points concerning Chang's method. First, the method begins with the determination of the optimal closed-loop response(s) of the system which are in the form of transfer functions, and may be one, or both of the following:

-Optimal input-to-output response, C/R.

-Optimal disturbance-to-output response, C/D.

Second, the solution of the optimal frequency domain problem is obtained by using a method developed by Chang. The details are given in the Appendix A. Third, obtaining the optimum closed loop responses are made easy by using the RSL method, also developed by Chang.

It must be stressed that in Chang's method, unlike YBJ's, the specific control system configuration is not specified at the outset. Rather, the configuration is selected after the closed-loop transfer functions have been obtained and is based on the convenience of hardware realization. The general configuration is shown in Figure 2, where in the single controller case of Chang's method, external disturbance, D, is ignored.

Here the same cost functions given in (2.3) and (2.4) are used. Applying the concepts above to the general system illustrated in Figure 2, the input to output transfer function is defined as

$$H_1 \equiv \frac{C}{R} \tag{2.26}$$

Plant input and error are also easily found and are given as

$$I = R\frac{H_1}{G_P} \tag{2.27}$$

and

$$E = R - C = R - RH_1 = (1 - H_1)R. \tag{2.28}$$

Multiplying I and E by their respective complex conjugates, gives

$$I\bar{I} = \frac{1}{G_P \bar{G}_P}\left[H_1 \bar{H}_1 R\bar{R}\right] \tag{2.29}$$

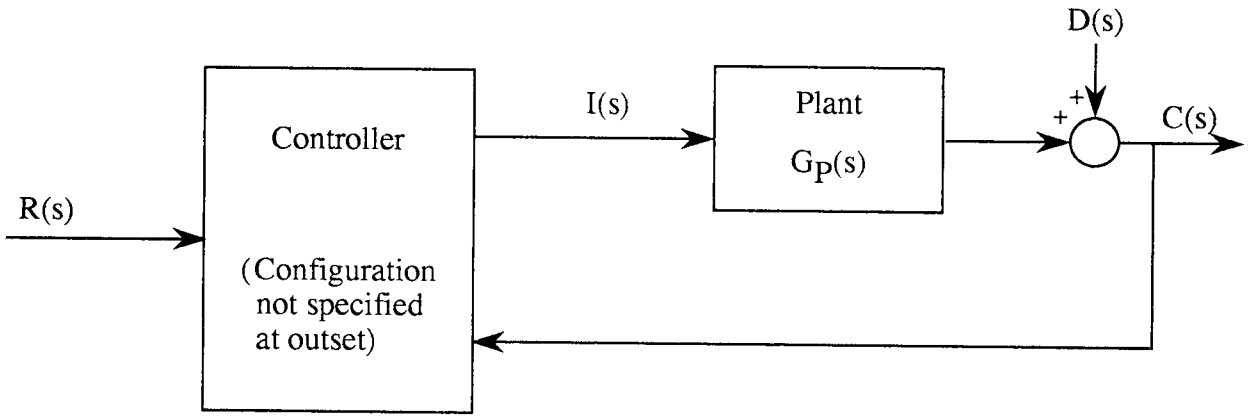


Figure 2. General control configuration used for Chang's method.

$$E\bar{E} = (1 - H_1)(1 - \bar{H}_1)R\bar{R} \tag{2.30}$$

Using equations (2.29) and (2.30), let

$$\Sigma = E\bar{E} + \lambda I\bar{I} \tag{2.31}$$

then

$$\Sigma = (1 - H_1)(1 - \bar{H}_1)R\bar{R} + \frac{\lambda}{G_P\bar{G}_P}\left[H_1\bar{H}_1R\bar{R}\right] \tag{2.32}$$

and the total cost function, as given in (2.5), can be written as

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty}\Sigma\, ds$$

As shown in Appendix A, for an optimal H_1 the following condition must hold

$$\frac{1}{2\pi j}\oint\frac{\partial\Sigma}{\partial\bar{H}_1}\Delta\bar{H}_1 = 0. \tag{2.33}$$

Using the same arguments as in Section 2.2, residue theory is used and the integration path is closed to the left of the jω axis. In order for (2.33) to be satisfied, $\frac{\partial\Sigma}{\partial\bar{H}_1}$ must contain all RHP poles.

Taking the partial derivative of Σ with respect to $\bar{H}_1$ yields

$$\frac{\partial\Sigma}{\partial\bar{H}_1} = -(1 - H_1)R\bar{R} + \frac{\lambda}{G_P\bar{G}_P}\left[H_1R\bar{R}\right]$$

collecting terms and simplifying

$$\frac{\partial\Sigma}{\partial\bar{H}_1} = \left(1 + \frac{\lambda}{G_P\bar{G}_P}\right)R\bar{R}H_1 - R\bar{R} = \Omega. \tag{2.34}$$

The optimal condition states that Ω must only possess RHP poles only. Making a change in variables, let

$$\left.\begin{aligned} Y\bar{Y} &= \left(1+\frac{\lambda}{G_P\bar{G}_P}\right) \\ Z\bar{Z} &= R\bar{R} \end{aligned}\right\} \tag{2.35}$$

where Y and Z have only LHP poles and zeros, $\bar{Y}$ and $\bar{Z}$ have RHP poles and zeros. Backsubstituting (2.35) into (2.34), gives

$$Y\bar{Y}Z\bar{Z}H_1 = Z\bar{Z} + \Omega.$$

Dividing through by terms that contain only LHP poles and zeros gives

$$YZH_1 = \frac{Z}{\bar{\bar{Y}}} + \frac{\Omega}{\bar{\bar{Y}}\bar{Z}} \tag{2.36}$$

where the left-hand side of (2.36) has only LHP terms, but the right-hand side is mixed. The term $\frac{Z}{\bar{\bar{Y}}}$ has both LHP and RHP terms, and $\frac{\Omega}{\bar{\bar{Y}}\bar{Z}}$ contains RHP only. Decomposing terms with contain both LHP and RHP poles and zeros, (2.36) may be written as

$$YZH_1 = \left[\frac{Z}{\bar{\bar{Y}}}\right]_L + \left[\frac{Z}{\bar{\bar{Y}}}\right]_R + \frac{\Omega}{\bar{\bar{Y}}\bar{Z}}.$$

As before, terms which contain RHP poles and or zeros only will cancel each other. This leaves the optimal input-to-output closed loop transfer function

$$H_1 = \frac{1}{YZ}\left[\frac{Z}{\bar{\bar{Y}}}\right]_L. \tag{2.37}$$

Now, all that is left to be determined is the optimal controller, which may be done after the system configuration is selected. The reader many have noticed since there is

only one degree of freedom, H_1, only a single controller configuration is possible in this case.

It should be noted that this procedure is known as a "no-disturbance" design, since disturbance was neglected from the outset. It may be possible, however, to design a single optimal H_1 with regard to disturbance, however in this case the system configuration must be known prior to the determination of H_1.

2.4 CHANG'S METHOD: TWO CONTROLLER CASE

As stated earlier, because of the flexibility of Chang's method, one or more closed-loop transfer function may be determined, namely C/R and C/D, where C/D is defined as

$$H_2 \equiv \frac{C}{D}. \tag{2.38}$$

It is then natural to attempt to exploit the flexibility of Chang's method. For example an optimal H_1 *and* H_2 may be derived which by equations (2.26) and (2.27) would yield an optimal command input to output, and an optimal disturbance to output transfer function. The intent of deriving an optimal H_2 is to aid H_1 in the suppression of any external disturbance. The derivation is straightforward and uses the same concepts as used to find (2.37).

Since disturbance is now being accounted for I and E are written as

$$I = R\frac{H_1}{G_P} + D\frac{(H_2 - 1)}{G_P}$$

and with a little manipulation, I is given to be

$$I = \frac{1}{G_P}\left[H_1 R + D(H_2 - 1)\right]$$

and error E, is

$$E = R - C = R - RH_1 - DH_2 = (1 - H_1)R - DH_2.$$

Again, multiplying I and E by their respective complex conjugate and assuming R, and D are independent and zero mean, yields

$$I\bar{I} = \frac{1}{G_P \bar{G}_P}\left[H_1 \bar{H}_1 R\bar{R} + (H_2 - 1)(\bar{H}_2 - 1)D\bar{D}\right] \tag{2.39}$$

and

$$E\bar{E} = (1 - H_1)(1 - \bar{H}_1)R\bar{R} + D\bar{D}H_2 \bar{H}_2. \tag{2.40}$$

Making the substitution defined in (2.31), let

$$\begin{aligned}\Sigma = {} & (1 - H_1)(1 - \bar{H}_1)R\bar{R} + H_2 \bar{H}_2 D\bar{D} \\ & + \frac{\lambda}{G_P \bar{G}_P}\left[H_1 \bar{H}_1 R\bar{R} + (H_2 - 1)(\bar{H}_2 - 1)D\bar{D}\right]\end{aligned} \tag{2.41}$$

The same procedure in obtaining (2.34) to (2.37) is followed, and again

$$H_1 = \frac{1}{YZ}\left[\frac{Z}{\bar{Y}}\right]_L$$

which is identical to (2.37). In order to achieve an optimal H_2, the exact arguments used to achieve an optimal H_1 are used here, which results in $\frac{\partial \Sigma}{\partial \bar{H}_2}$ containing RHP poles only. Taking this partial derivative gives

$$\frac{\partial \Sigma}{\partial \bar{H}_2} = H_2 D\bar{D} + \frac{\lambda}{G_P \bar{G}_P}(H_2 - 1)D\bar{D} = \Omega_2. \tag{2.42}$$

Unlike the single transfer function case, disturbance terms have been accounted for in this derivation. Rearranging terms it can be seen

$$\left[D\overline{D}+\frac{\lambda}{G_P\overline{G}_P}\left(D\overline{D}\right)\right]H_2=\frac{\lambda}{G_P\overline{G}_P}D\overline{D}+\Omega_2 \tag{2.43}$$

making the substitutions

$$U\overline{U}=\left[D\overline{D}+\frac{\lambda}{G_P\overline{G}_P}D\overline{D}\right] \tag{2.44}$$

and

$$V\overline{V}=\frac{\lambda}{G_P\overline{G}_P}D\overline{D} \tag{2.45}$$

leads to a more compact form of (2.43), which is

$$U\overline{U}H_2=V\overline{V}+\Omega_2.$$

Where U and V contain LHP terms, $\overline{U}$ and $\overline{V}$ contain RHP terms only. Dividing through by terms which contain only RHP terms yields

$$UH_2=\frac{V\overline{V}}{\overline{U}}+\frac{\Omega_2}{\overline{U}}. \tag{2.46}$$

Again by arguments used previously, performing partial fraction decomposition on mixed terms leaves LHP only, and since RHP terms cancel, the optimal H_2 is found to be

$$H_2=\frac{1}{U}\left[\frac{V\overline{V}}{\overline{U}}\right]_L. \tag{2.47}$$

The addition of transfer function H_2 provides an extra degree of freedom to the designer, and hence will allow multiple controller configurations. The hope is the

multiple controller configuration will yield better transient responses to the command input and external disturbance.

2.5 ROOT-SQUARE LOCUS (RSL)

So far nothing has been said concerning the determination of the Lagrange multiplier, λ. One brute force method, is to substitute (2.29) into

$$J_m = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} I\bar{I}\, ds \le L$$

which would yield

$$J_m = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} \frac{H_1\bar{H}_1}{G_P\bar{G}_P} R\bar{R}\, ds \le L$$

and then carry out the complex integration to obtain the range of λ. This can be done with the help of integral tables, the theory of residues, or any other applicable means. However, the integration can easily become unruly, and would probably not be a favorite choice of the typical engineer. Also, there does not seem to be any relation between λ and practical design specifications such as, risetime, overshoot, etc. There is still hope, however. By using RSL, λ may be indirectly determined. As will be demonstrated, however, it is not necessary to find the actual value of λ. Anyone familiar with classic root-locus techniques should feel perfectly at ease with this method, and RSL also provides the designer with a clear relation to design specifications.

First let

$$Y\overline{Y} = 1 + \frac{\lambda}{G_P \overline{G}_P} \tag{2.48}$$

where G_P takes the general form of

$$G_P = k \frac{\prod_i (s - z_i)}{\prod_j (s - p_j)}. \tag{2.49}$$

Substituting (2.49) into (2.48) gives

$$\frac{1}{Y\overline{Y}} = \frac{\frac{1}{\lambda} G_P \overline{G}_P}{1 + \frac{1}{\lambda} G_P \overline{G}_P}, \tag{2.50}$$

and in expanded form

$$\frac{1}{Y\overline{Y}} = \frac{\frac{k^2}{\lambda} \frac{\prod_i (s - z_i)(-s - z_i)}{\prod_j (s - p_j)(-s - p_j)}}{1 + \frac{k^2}{\lambda} \frac{\prod_i (s - z_i)(-s - z_i)}{\prod_j (s - p_j)(-s - p_j)}} = \frac{\frac{k^2}{\lambda} \frac{\prod_i (\Omega - Z_i)}{\prod_j (\Omega - P_j)}}{1 + \frac{k^2}{\lambda} \frac{\prod_i (\Omega - Z_i)}{\prod_j (\Omega - P_j)}},$$

which can be simplified to the more compact form of

$$\frac{k^2}{\lambda} \frac{\prod_i (\Omega - Z_i)}{\prod_j (\Omega - P_j)}, \tag{2.51}$$

where

$$\Omega = -s^2,\ Z_i = -z_i^2,\ P_j = -p_j^2,\ Q_j = -q_j^2\ .$$

Using the root-locus analogy, $\frac{k^2}{\lambda}$ is open-loop gain, Z_i is open-loop zero, P_j is open-loop pole, Q_j is closed-loop pole, Ω is complex frequency, and $\frac{1}{YY}$ is the closed-loop transfer function. The procedure for RSL is as follows. Given G_P, its zeros, z_i and poles p_j, calculate

$$\left.\begin{aligned} Z_i &= -z_i^2 \\ P_j &= -p_j^2 \end{aligned}\right\}. \tag{2.52}$$

Then these are used as the open-loop zeros and poles to plot the loci of Q_j's as functions of gain on the Ω-plane. From the plot of the locus, the Q_j's may be determined. Then Y is determined by

$$\frac{1}{Y} = \frac{\prod_{i=1}^{m}\left(1-\frac{s}{z_i}\right)}{\prod_{j=1}^{n}\left(1-\frac{s}{q_j}\right)} \tag{2.53}$$

where q_j's are determined from

$$q_j = \pm\sqrt{-Q_j}$$

and the sign of the radical is chosen so $\mathrm{Re}\{q_j\}$ is negative.

It is now straightforward to find H. For a plant with integration, and a step reference input, the following relation holds

$$H = \frac{1}{Y}.$$

The proof of this is straightforward. Starting with (2.37) it is known that

$$Z\bar{Z} = \left(\frac{A}{s}\right)\left(-\frac{A}{s}\right) \tag{2.54}$$

and then from (2.54) it can be seen that

$$Z = \left(\frac{A}{s}\right) \tag{2.55}$$

$$\bar{Z} = \left(-\frac{A}{s}\right) \tag{2.56}$$

then substituting (2.55) into (2.37), H becomes

$$H = \frac{s}{YA}\left[\frac{A}{s\bar{Y}}\right]_L$$

which is readily simplified to

$$H = \frac{s}{Y}\left[\frac{1}{sY(-s)}\right]_L$$

then taking partial fraction decomposition yields the optimal H, which is

$$H = \frac{1}{Y(0)Y(s)}. \tag{2.57}$$

Finally, since it was assumed G_P has integration, $Y(0) = 1$. This is easily shown by

$$\left[Y\bar{Y}\right]_{s=0} = \left[1 + \frac{\lambda}{G_P \bar{G}_P}\right]_{s=0}$$

and

$$G_P(s=0) = \infty$$

therefore

$$Y(0) = 1.$$

Hence, the optimal H was found, and the issue of λ was completely avoided. However, if one wished to determine λ, this can be accomplished by comparing Y found from (2.35) to that found in (2.54), and backing out λ.

CHAPTER 3

APPLICATION TO DC MOTOR AND LOAD CONTROL

3.1 APPLICATION OF THE YBJ METHOD: PART I

The theory developed in preceding sections will now be used to derive an optimal sensitivity function and transfer function(s) H_1 (and H_2). The system to be used is a DC motor and load (Figure 3), which is a common and practical system. Referring to Figure 3, let the system parameters be

$$\left.\begin{aligned} K_T &= 2.1 \frac{lb * ft}{ampere} \\ K_e &= 0.3 \frac{Volt}{rpm} \\ R_a &= 2.7 \text{ ohms} \\ B &= 0.3343 \text{ lb} * \text{ft/rad / sec} \\ J &= 0.11 \text{ slug} * \text{ft}^2 \end{aligned}\right\}$$

where K_T is the torque constant of the motor, K_e is the back-emf constant, R_a is the armature resistance, and J and B are the moment of inertia and friction constant of the motor and load, respectively. Note that the entire motor and load is not G_P in this case. First define G_P as

$$G_P = \left(\frac{K_T}{Js^2 + Bs}\right)\left(\frac{1}{200}\right)$$

$$G_P = \frac{0.0955}{s(s+3.04)} \tag{3.1}$$

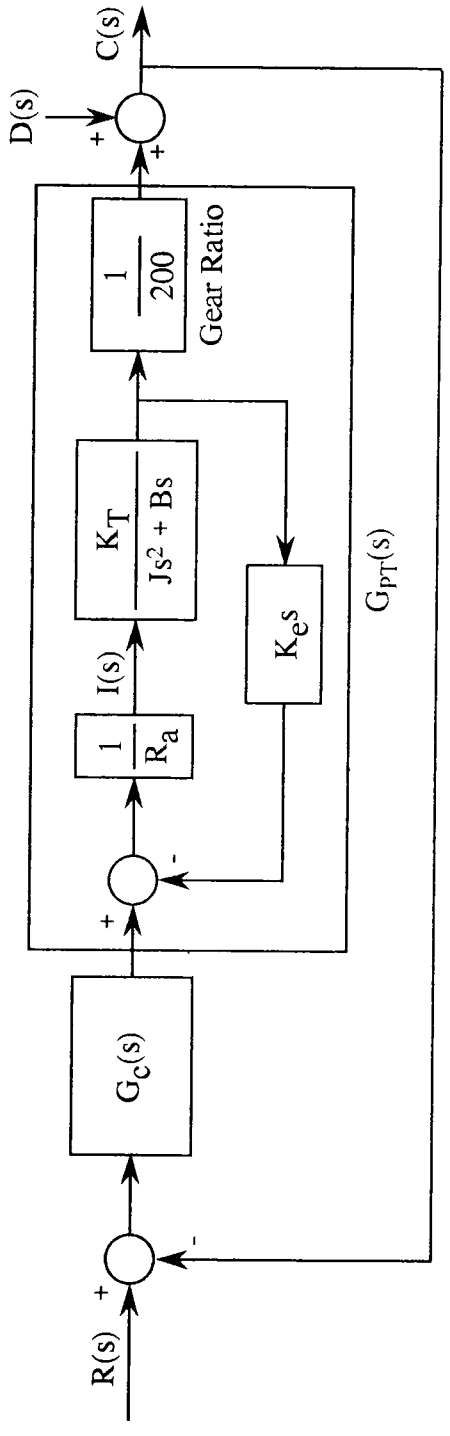


Figure 3. DC Motor and Load

and let the entire motor and load be defined as G_{PT}, which can be written

$$G_{PT} = \frac{0.0354}{s(s+23.33)}. \tag{3.2}$$

First, an optimal controller will be designed without considering external disturbance ($D\overline{D} = 0$). Since disturbance is being neglected, this design procedure will be known as a zero-disturbance design. Then the system response due to a disturbance, D, where D is chosen to be

$$D(s) = \frac{1}{s+10}, \tag{3.3}$$

will be evaluated in order to justify the design of an optimal controller which will account for the disturbance. Since the configuration of the DC motor and load is identical to the configuration used to derive the YBJ optimal controller in Chapter 2, those results may be used directly.

The first step in obtaining the optimum controller is the determination of the Lagrangian multiplier. If the method in [13] was followed exactly, λ would be determined by evaluating

$$J = J_t + \lambda J_s$$

over several decades of λ. Then a suitable value for λ could be chosen. However, Chang's RSL method is far easier to carry out, and will not effect the integrity of the YBJ process.

The first step is to determine the P_j's and Z_i's from the p_j's and z_i's of G_P, which are found to be

$$\left.\begin{array}{ll} p_1 = 0 & P_1 = -p_1{}^2 = 0 \\ p_2 = -3.04 & P_2 = -p_2{}^2 = -9.24 \end{array}\right\}.$$

Now the RSL may be plotted using P_1 and P_2 as the poles, as shown in Figure 4.

Second, the closed-loop poles Q_j are determined from the RSL plot, and are chosen to be

$$Q_{1,2} = 9.12 \angle 120.4°$$

and hence

$$-Q_{1,2} = 9.12 \angle 59.6°.$$

As previously stated the q_j's in (2.53) are determined by taking the square root of the Q_j's and keeping the LHP poles. This results in

$$\left.\begin{array}{l} q_3 \\ q_2 \end{array}\right\} = \pm\sqrt{-Q_1} = \pm\sqrt{4.62 + j7.9} = \begin{cases} 2.62 + j1.515 \\ -2.62 - j1.515 \end{cases}$$

$$\left.\begin{array}{l} q_4 \\ q_1 \end{array}\right\} = \pm\sqrt{-Q_2} = \pm\sqrt{4.62 - j7.9} = \begin{cases} 2.62 + j1.515 \\ -2.62 + j1.515 \end{cases}$$

and the LHP poles are

$$\begin{aligned} q_1 &= -2.62 + j1.515 \\ q_2 &= -2.62 - j1.515 \end{aligned} \tag{3.4}$$

Substituting the values in (3.4) into (2.53)

$$\frac{1}{Y} = \frac{q_1 q_2}{(s - q_1)(s - q_2)}$$

and

$$\frac{1}{Y} = \frac{9.16}{s^2 + 5.32s + 9.16}. \tag{3.5}$$

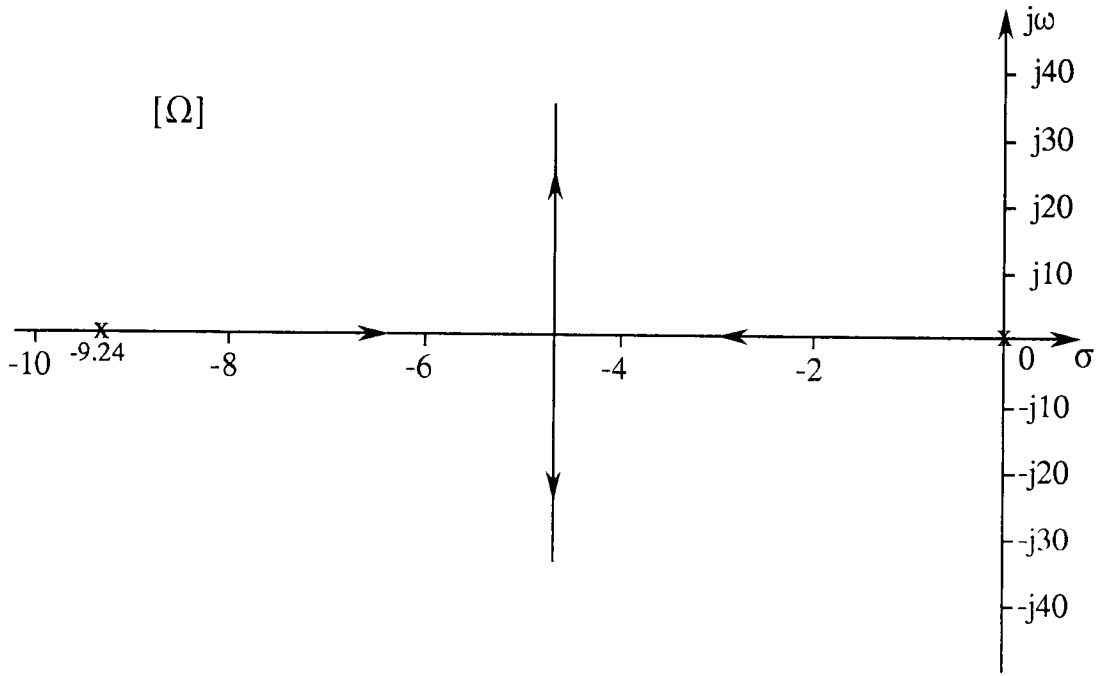


Figure 4. Root-Square Locus

Now that (3.5) has been determined via RSL, λ may be found by comparing Y found by (2.35) to that found by (3.5). Equation (2.35) yields

$$Y\overline{Y} = (109.64\lambda)s^4 - (1013.3\lambda)s^2 + 1$$

and by trial and error, $\lambda = 0.0001$. This value of λ gives poles very close to those in (3.4).

Using a unit step as the system's command input, which takes the form of $\frac{1}{s}$, and this being a zero-disturbance design, let

$$\left.\begin{aligned} R\overline{R} &= \frac{-1}{s^2} \\ D\overline{D} &= 0 \end{aligned}\right\}.$$

Then from (2.13) and (2.14)

$$A\overline{A} = \frac{-1}{s^2}$$

$$B\overline{B} = 109.65\lambda(s+3.04)(-s+3.04)$$

where $\lambda = 0.0001$, then $Y\overline{Y}$ is

$$Y\overline{Y} = \frac{-0.0109\left(s^4 - 9.24s^2 + 91.2\right)}{s^2}.$$

The next unknown to be determined is χ. This is found by multiplying the numerator and of Gp by its denominator. Hence χ for this system is found to be

$$\chi = 0.0955(s)(s+3.04)$$

and from χ it is easy to find

$$\bar{\chi}_r = \sqrt{0.0955}.$$

Performing partial fraction decomposition on

$$\frac{\bar{\chi}_r}{\bar{Y}}\left(B\bar{B}\right)$$

reveals there are no LHP terms to incorporate into (2.24). Hence the optimal sensitivity function reduces to

$$S_o = \frac{1}{\bar{\chi}_r Y}(f)$$

where, since G_P has no LHP poles or zeros, f is uniquely determined by the interpolation condition

$$1 - S_0(s) = O\left(\frac{1}{s^2}\right), \ |s| \text{ large}$$

which gives

$$f = (1/30.995)s$$

and then

$$S_o = \frac{s^2}{s^2 + 5.32s + 9.55}.$$

Now, from (2.25), the zero-disturbance optimal controller is found to be

$$G_C = \frac{150.39\left(s^2 + 25.12s + 41.85\right)}{s}.$$

Figure 5 shows the external disturbance, D, as a function of time. Figure 6 illustrates the motor and load system response to a unit step input, and to disturbance

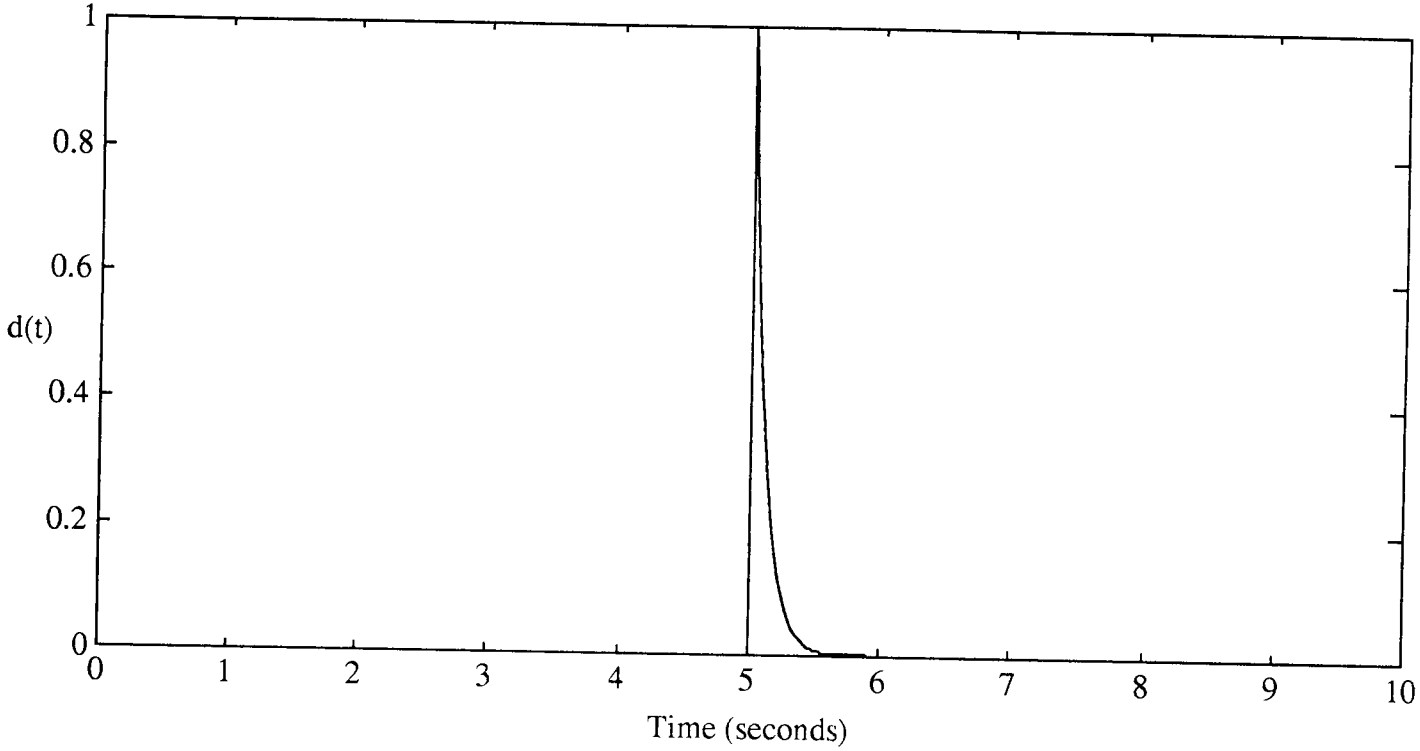


Figure 5. Disturbance input to DC motor and load system.

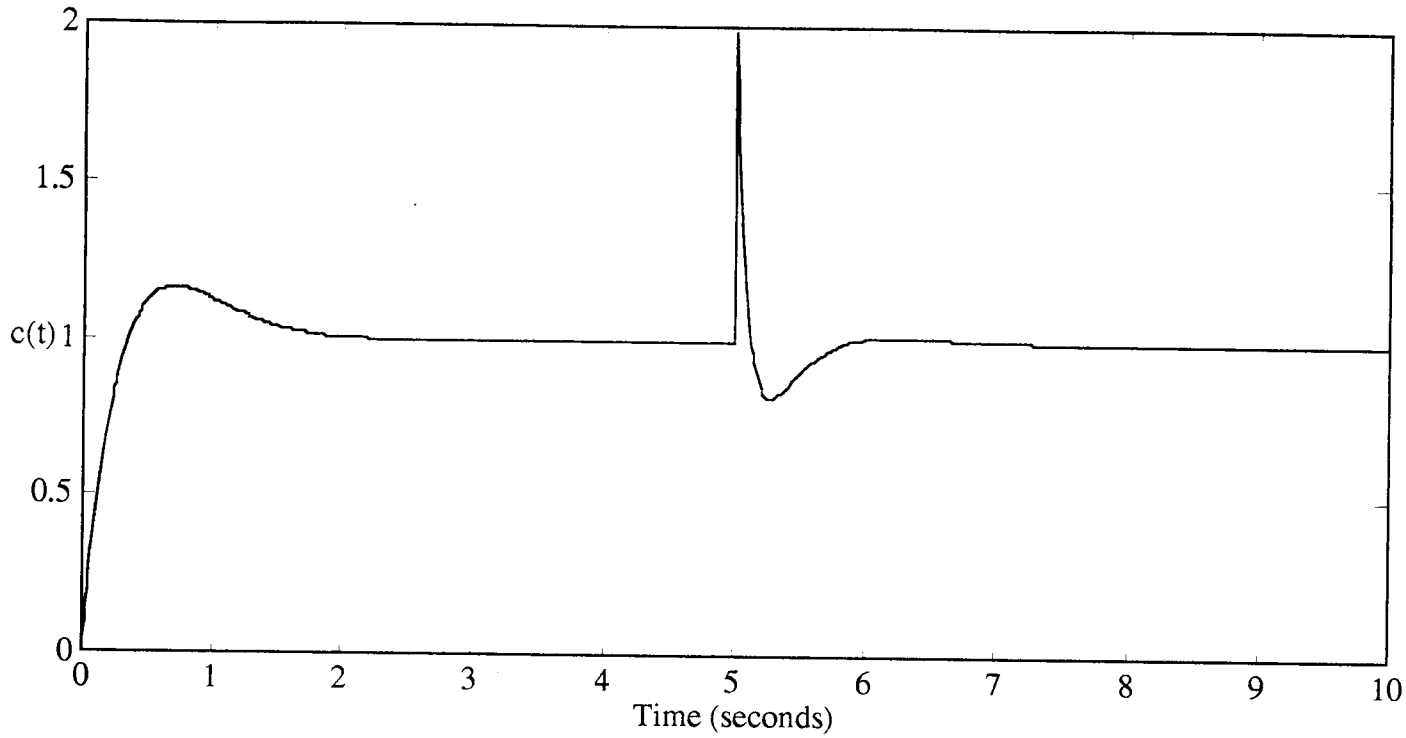


Figure 6. Response of DC motor and load system to unit step input and disturbance occurring at five seconds, using a zero-disturbance optimal controller derived by the YBJ method.

injection occurring at a steady-state time of five seconds. The results shown in Figure 6 clearly show the optimal controller gives the system a good response to the unit step input. However, since disturbance peaks at 2, the zero-disturbance design is not capable, to any degree, of disturbance suppression. This result then clearly obviates the need to evaluate performance when disturbance is included in the determination of the optimal controller.

3.2 APPLICATION OF THE YBJ METHOD: PART II

In this section, an optimal controller will be derived with regard to disturbance in order to evaluate the system's response to that disturbance. The same command input, and disturbance model used in section **3.1** will be used here. Including the disturbance terms in the derivation, gives

$$\left.\begin{aligned} R\bar{R} &= \frac{-1}{s^2} \\ D\bar{D} &= \frac{1}{\left(100 - s^2\right)} \end{aligned}\right\}.$$

Again from (2.13) and (2.14)

$$A\bar{A} = 2\frac{s^2 - 50}{s^2\left(100 - s^2\right)}$$

$$B\bar{B} = -218.75\lambda\frac{(s+3.04)(s-3.04)\left(s^2 - 50\right)}{(s+10)(s-10)}$$

since the same plant is being used as in **3.1**, $\lambda = 0.0001$. Carrying out the same calculations as in section **3.1** yields the following quantities

$$Y\overline{Y} = -0.0219\frac{s^6 - 59.24s^4 + 553.28s^2 - 4560.12}{s^4 - 100s^2}$$

$$\overline{\chi}_r = \sqrt{0.0955}$$

and (2.24) yields

$$S_o = \frac{1}{\overline{\chi}_r Y}\left\{\frac{0.747}{s+10} + f\right\}.$$

Again, since there are no RHP poles or zeros in G_P, f is uniquely determined by the single interpolation condition

$$1 - S_0(s) = O\left(\frac{1}{s^2}\right), \ |s| \text{ large}.$$

Using this condition, f, is found to be

$$f = (1/21.866)s.$$

The optimal sensitivity function S_o, may now be found from (2.24) and is

$$S_o = \frac{s\left(s^2 + 10s + 16.33\right)}{(s+7.07)\left(s^2 + 5.32s + 9.51\right)}$$

from (2.25) the optimal controller is given by

$$G_C = 67.6\frac{s^3 + 36.22s^2 + 328.93s + 658.1}{s^2 + 10s + 16.33}.$$

Figure 7 shows even when the optimal controller is derived with regard to disturbance, the system's disturbance response has not improved. Whether this poor

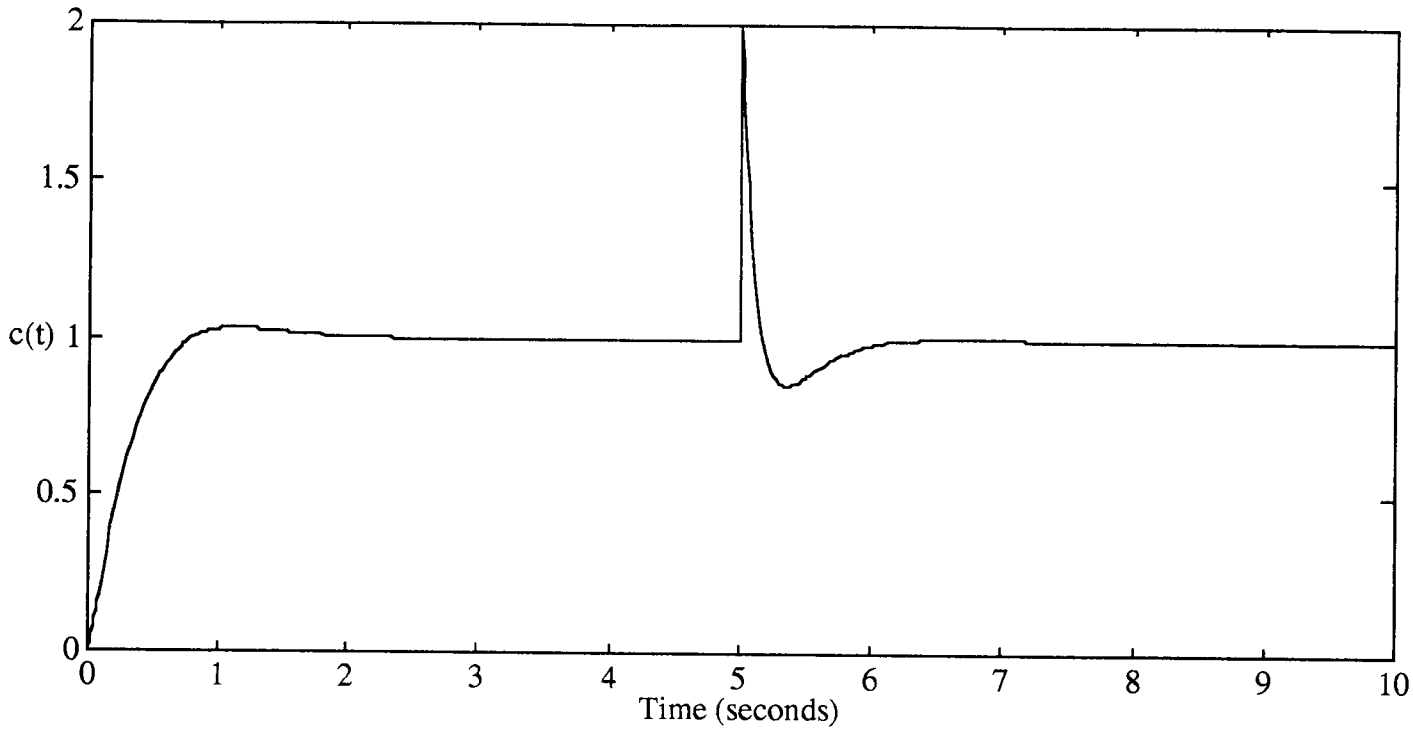


Figure 7. Response of DC motor and load system to unit step input and disturbance occurring at five seconds, using an optimal controller accounting for disturbance derived by the YBJ method.

disturbance response occurs due to a flaw in the original derivation, or if the single controller configuration is to blame, will remain to be seen in the following sections.

3.3 APPLICATION OF CHANG'S METHOD: PART I

Having shown results using the YBJ technique, it is now time to derive both single and multiple controllers for the DC motor and load, using the method derived by Chang. First, Chang's method for one controller will be implemented and will use the DC motor and load configuration in Figure 3. Since it has not been shown thus far that a single controller configuration using Chang's method can account for disturbance in its derivation, this will be also called a zero-disturbance design. All system parameters used in sections **3.1** and **3.2** will also be used here, namely a unit step command input, and, since the plant has not changed, $\lambda = 0.0001$. Using these parameters, (2.35) yields

$$\left.\begin{aligned} Z\overline{Z} = R\overline{R} &= \frac{-1}{s^2} \\ Y\overline{Y} &= 0.0109\left(s^4 - 9.29s^2 + 91.74\right) \end{aligned}\right\} \tag{3.6}$$

then (2.37) gives

$$\frac{Z}{\overline{Y}} = \frac{1}{s\left(\sqrt{0.0109}\left(s - 2.66 \pm j1.57\right)\right)}$$

performing partial fraction decomposition and keeping LHP terms results in

$$\left[\frac{Z}{\overline{Y}}\right]_L = \frac{1}{s}.$$

The optimal control ratio can then be determined, and is given as

$$H_1 = \frac{9.55}{s^2 + 5.32s + 9.55}. \tag{3.7}$$

Now that the optimal command-input-to-output transfer function has been found, the optimal controller, G_C, may determined by analysis of the DC motor load configuration. By straightforward analysis the input to output is transfer function can be found, and is

$$\frac{C}{R} = H_1 = \frac{G_C G_{PT}}{1 + G_C G_{PT}} \tag{3.8}$$

Solving for G_C in (3.8) gives

$$G_C = \frac{H_1}{G_{PT} - G_{PT} H_1} \tag{3.9}$$

where G_{PT} is known from Figure 3, and H_1 is given in (3.7). Making substitutions gives the optimal controller

$$G_C = 269.77 \frac{(s + 23.33)}{(s + 5.32)}.$$

Figure 8 shows the system response due to step input and disturbance injected at t = 5 seconds. It is clear from Figure 8, that G_C gives the system a very good transient response to a step input. However, from Figure 8, it can be seen that G_C leaves the system virtually helpless to disturbance D. Comparing Figures 6, and 8 it can be seen that both the Chang and YBJ methods for a zero-disturbance design, produce nearly identical results.

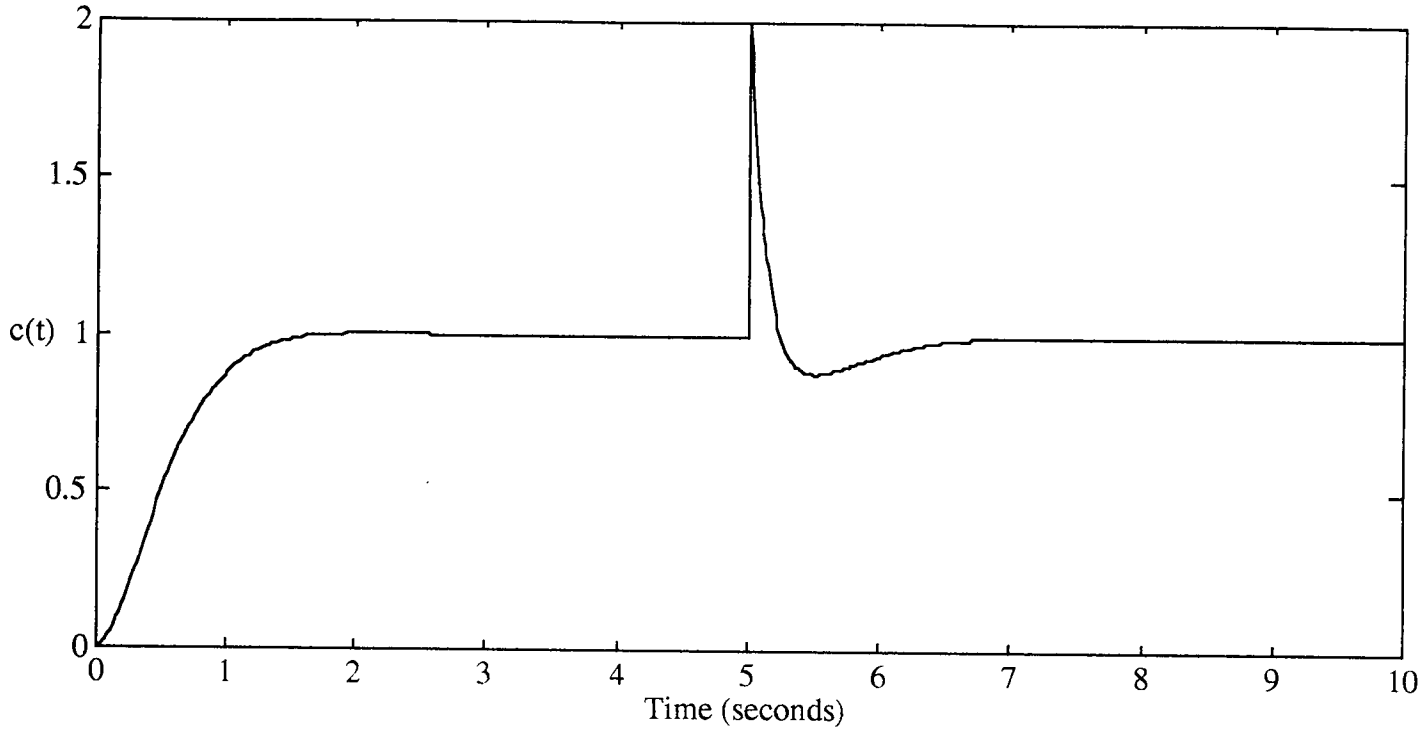


Figure 8. Response of DC motor and load system to unit step input and disturbance occurring at five seconds, using a one optimal controller zero-disturbance design derived by Chang's method.

3.4 APPLICATION OF CHANG'S METHOD: PART II

Since both a zero-disturbance design and a design with regard to disturbance was shown using the YBJ method, the same will be done using Chang's method. As stated before, since the system configuration is not set at the outset of Chang's method, using a single controller configuration does not account for disturbance in its derivation. This is due to the fact that only H_1, the command input to output transfer function, may be used in the determination of a single controller. However, disturbance may be implemented if and only if the disturbance to output transfer function, H_2, can be put in terms of H_1. Analysis of the DC motor load configuration gives H_2 as

$$\frac{C}{D} = H_2 = \frac{1}{1 + G_C G_{PT}} \tag{3.10}$$

Notice from (3.8) and (3.10), that

$$H_2 = 1 - H_1 \tag{3.11}$$

Therefore H_2 may be put in terms of H_1 and a single optimal controller may be derived with regard to disturbance. Substituting (3.11) into (2.41) yields

$$\Sigma = R\bar{R}(1-H_1)(1-\bar{H}_1) + D\bar{D}(1-H_1)(1-\bar{H}_1) + \frac{\lambda H_1 \bar{H}_1}{G_P \bar{G}_P} R\bar{R} + \frac{\lambda H_1 \bar{H}_1}{G_P \bar{G}_P} D\bar{D}$$

collecting terms gives

$$\begin{aligned} \Sigma = R\bar{R}\left[(1-H_1)(1-\bar{H}_1) + \frac{\lambda H_1 \bar{H}_1}{G_P \bar{G}_P}\right] \\ + D\bar{D}\left[(1-H_1)(1-\bar{H}_1) + \frac{\lambda H_1 \bar{H}_1}{G_P \bar{G}_P}\right] \end{aligned} \tag{3.12}$$

and it is assumed that D and R are independent and zero mean. Finding the optimal H_1 in this case is the same as before, and taking the partial derivative of (3.12) with respect to $\overline{H}_1$ gives

$$\frac{\partial \Sigma}{\partial \overline{H}_1} = R\overline{R}\left(1+\frac{\lambda}{G_P\overline{G}_P}\right)H_1 - R\overline{R} + D\overline{D}\left(1+\frac{\lambda}{G_P\overline{G}_P}\right)H_1 - D\overline{D} = \Omega_2 . \qquad (3.13)$$

Collecting terms and simplifying (3.13) leads to

$$\left(R\overline{R}+D\overline{D}\right)\left(1+\frac{\lambda}{G_P\overline{G}_P}\right)H_1 = R\overline{R}+D\overline{D}+\Omega_2 \qquad (3.14)$$

and as before, the optimal condition requires Ω_2 to contain RHP poles only. Making the substitutions

$$\left.\begin{aligned} X\overline{X} &= \left(R\overline{R}+D\overline{D}\right) \\ Y\overline{Y} &= \left(1+\frac{\lambda}{G_P\overline{G}_P}\right) \end{aligned}\right\} \qquad (3.15)$$

leads to a more compact form of (3.14), that being

$$X\overline{X}Y\overline{Y}H_1 = R\overline{R}+D\overline{D}-\Omega_2 .$$

Dividing through by terms which contain RHP poles and zeros only, namely $\overline{X}$ and $\overline{Y}$, then as before , H_1 is given to be

$$H_1 = \frac{1}{XY}\left[\frac{X}{\overline{Y}}\right]_L . \qquad (3.16)$$

No changes were made to the system configuration, therefore G_C may be found by (3.9).

Since G_P has not changed, the value of λ remains as found previously. Using a unit step reference input, and disturbance as given in (3.3), (3.15) reduces to

$$X\overline{X} = \frac{-2\left(s^2 - 50\right)}{s^2(s+10)(s-10)} \tag{3.17}$$

and $Y\overline{Y}$ given in (3.6) still holds for this derivation. Using (3.6) and (3.17) in (3.16), gives

$$\begin{bmatrix} X \\ \overline{\overline{Y}} \end{bmatrix} = \frac{13.51\left(s^2 - 50\right)}{(s-2.66 \pm j1.57)(s)(s+10)(s-7.07)}$$

and performing the partial fraction decomposition

$$\left[\frac{13.51\left(s^2 - 50\right)}{(s-2.66 \pm j1.57)(s)(s+10)(s-7.07)}\right]_L = \frac{1}{s} + \frac{0.0243}{(s+10)}$$

and hence the optimal control ratio is

$$H_1 = \frac{6.92(s+9.76)}{(s+2.66 \pm j1.57)(s+7.07)}. \tag{3.18}$$

Substituting (3.18) into (3.9), the optimal controller for this case is

$$G_C = \frac{H_1}{G_{PT} - G_{PT}H_1} = \frac{195.4s(s+23.33)(s+9.76)}{(s-.00016)(s+6.197 \pm j1.37)}. \tag{3.19}$$

Figure 9 shows system response due to step input and disturbance injected at t = 5 seconds, respectively. Comparing Figs. 7 and 9, illustrates the two system responses are nearly identical for both Chang and YBJ disturbance designs. The conclusion for this system is then clear. Even if a single controller is designed to compensate for an injected disturbance, the system will yield a poor disturbance response. This may be due to the

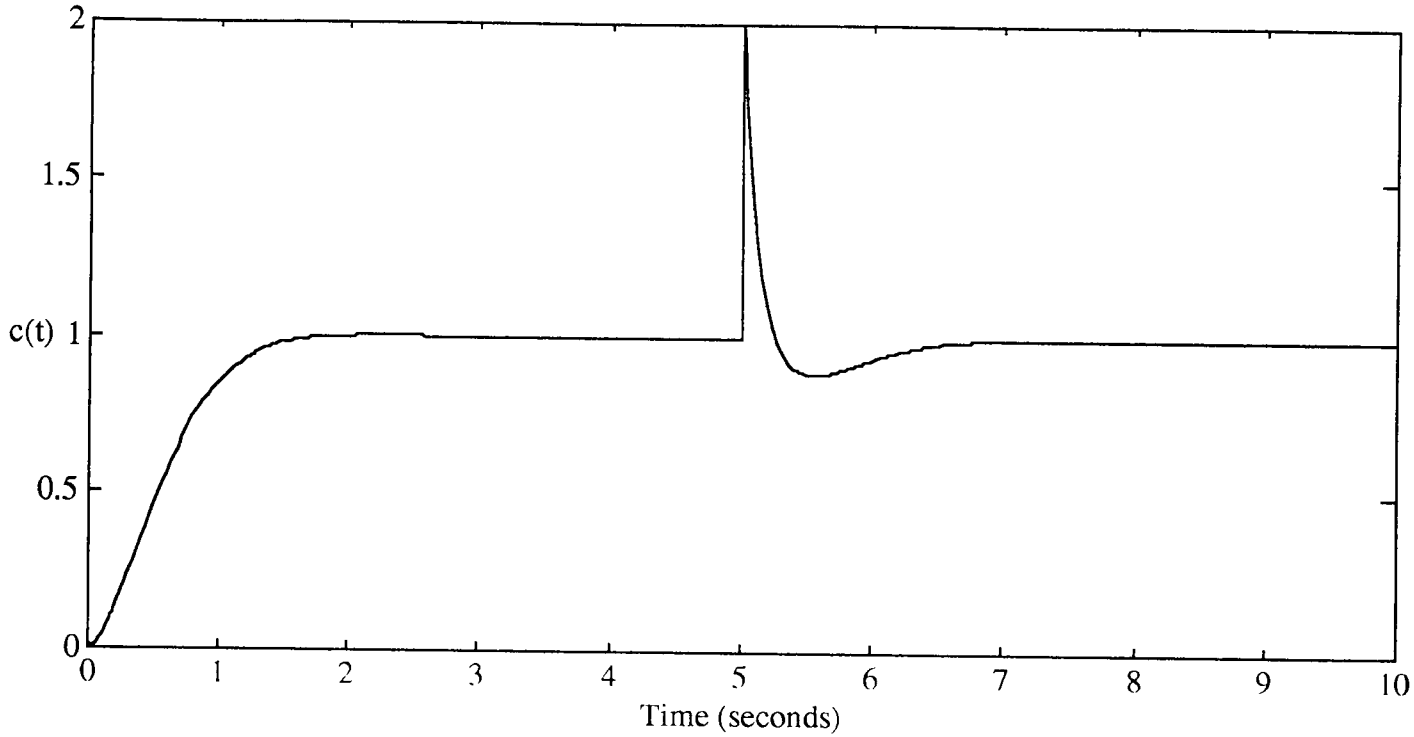


Figure 9. Response of DC motor and load system to unit step input and disturbance occurring at five seconds, using one optimal controller accounting for disturbance derived by Chang's method.

fact that, although H_2 was put in terms of H_1, only one optimal closed-loop transfer function was used to obtain G_C. Therefore, only one system input can be handled.

3.3 APPLICATION OF CHANG'S METHOD: PART III

The last test for Chang's methodology, is to design two optimal controllers. Here a modified version of Figure 3 is given in Figure 10. By inspection of Figure 10, G_P is the plant to be controlled, and is obviously a fixed entity. G_C is a controller to be designed, and it is obvious that it is unknown at the outset, which leaves G_F as a second unknown. In Chang's technique, it has been shown that more than one closed-loop transfer functions may be employed. Therefore, if two optimal transfer functions are found, then two unknowns may be determined. It is natural then, to employ G_F as a second controller.

Comparing (2.8) and (3.10) shows that H_2 is identical to system sensitivity as defined in [13]. Therefore, system sensitivity is directly related to the D to C transfer function for the DC motor and load configuration being used here. This is an extremely important finding. If disturbance could be somehow weighted such that its contribution to the system would decrease, then system sensitivity would decrease and would, in turn, make for a more robust system. The addition of a disturbance weighting term can be implemented into (2.40), which gives

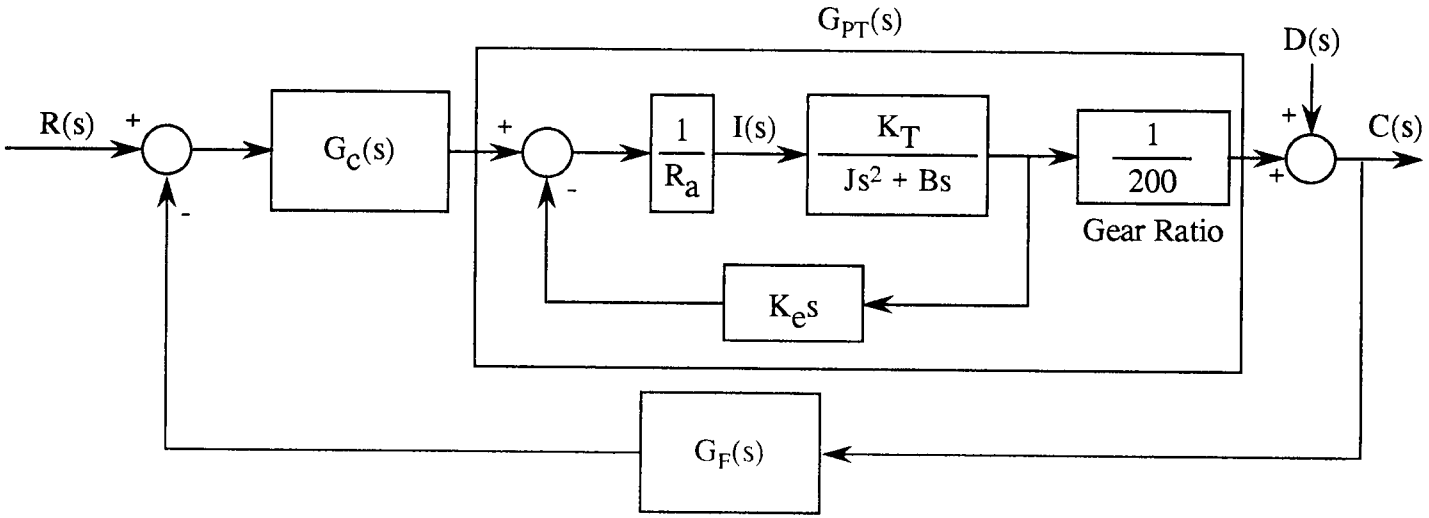


Figure 10. DC Motor and Load, two controller configuration

$$E\overline{E} = (1 - H_1)(1 - \overline{H}_1)R\overline{R} + kD\overline{D}H_2\overline{H}_2 \tag{3.20}$$

where k is the weighting coefficient. As k is increased, disturbance effects reduce, making the system more robust. Derivations for obtaining H_1, and H_2, are identical to the derivation in section **2.4**. H_1 and H_2 are identical in form to (2.37) and (2.47), except $U\overline{U}$ is now

$$U\overline{U} = \left[k + \frac{\lambda}{G_P \overline{G}_P} \right] D\overline{D} \tag{3.21}$$

Using the same reference input and disturbance given in (3.3), H_1 is

$$H_1 = \frac{9.55}{s^2 + 5.32s + 9.55} \tag{3.22}$$

which is the same as the single controller case without regard to disturbance.

Substituting

$$\left.\begin{aligned} k &= 3000 \\ \lambda &= 0.0001 \end{aligned}\right\}$$

in (3.21) gives

$$U\overline{U} = \frac{-.0109s^4 + .0103s^2 - 3000}{(s+10)(s-10)} \tag{3.23}$$

Making the appropriate substitutions in (2.45), $V\overline{V}$ is

$$V\overline{V} = \frac{-.0109s^2(s+3.04)(-s+3.04)}{(s+10)(-s+10)}$$

and therefore

$$\left[\frac{V\overline{V}}{\overline{U}}\right] = \frac{-.104s^2(s+3.04)(-s+3.04)}{(s+10)(-s+16.24 \pm j16.1)}$$

and

$$\left[\frac{V\overline{V}}{\overline{U}}\right]_L = \frac{1.0025}{s+10}. \tag{3.24}$$

Substituting (3.24) and U found from (3.23) into (2.47), the optimal H_2 can be found, and is given as

$$H_2 = \frac{9.57}{\left(s^2 + 32.48s + 523.1\right)}. \tag{3.25}$$

To get controllers G_C and G_F, the form of H_1 and H_2 must be determined from Figure 10. By analysis of Figure 10,

$$\frac{C}{R} = H_1 = \frac{G_C G_{PT}}{1 + G_C G_F G_{PT}} \tag{3.26}$$

$$\frac{C}{D} = H_2 = \frac{1}{1 + G_C G_F G_{PT}} \tag{3.27}$$

From (3.26)

$$G_F = \frac{1}{H_1} - \frac{1}{G_C G_P} \tag{3.28}$$

substitute (3.28) into (3.27)

$$\frac{1}{H_2} = 1 + G_C G_P \left(\frac{1}{H_1} - \frac{1}{G_C G_P}\right)$$

$$= \frac{G_C G_P}{H_1}$$

solving for G_C

$$G_C = \frac{1}{G_P}\left(\frac{H_1}{H_2}\right) \tag{3.29}$$

and using (3.29) in (3.28)

$$G_F = \frac{1}{H_1} - \frac{G_P H_2}{H_1}\frac{1}{G_P}$$
$$= \frac{1}{H_1} - \frac{H_2}{H_1}$$

which leaves

$$G_F = \frac{1 - H_2}{H_1}. \tag{3.30}$$

It is now possible to obtain the two optimal controllers,

$$G_C = \frac{28.2(s)(s+23.33)\left(s^2 + 32.48s + 523.1\right)}{\left(s^2 + 5.32s + 9.55\right)} \tag{3.31}$$

$$G_F = \frac{0.105\left(s^2 + 32.48s + 513.53\right)\left(s^2 + 5.32s + 9.55\right)}{\left(s^2 + 32.48s + 523.1\right)}. \tag{3.32}$$

Figure 11 shows unit step response with disturbance injected at t = 5 seconds for various values of ***k***. For ***k*** = 1, the responses to command input and disturbance are similar to the single controller cases, but gives slightly improved disturbance suppression. Figure 11 also illustrates that the contribution of D to the system decreases as ***k*** is increased. However, the practicality of such values of ***k*** would be a concern in a real life situation. This is illustrated in Figure 12. Although disturbance suppression increases, the magnitude of the plant input increases along with an increase in ***k***. Therefore the

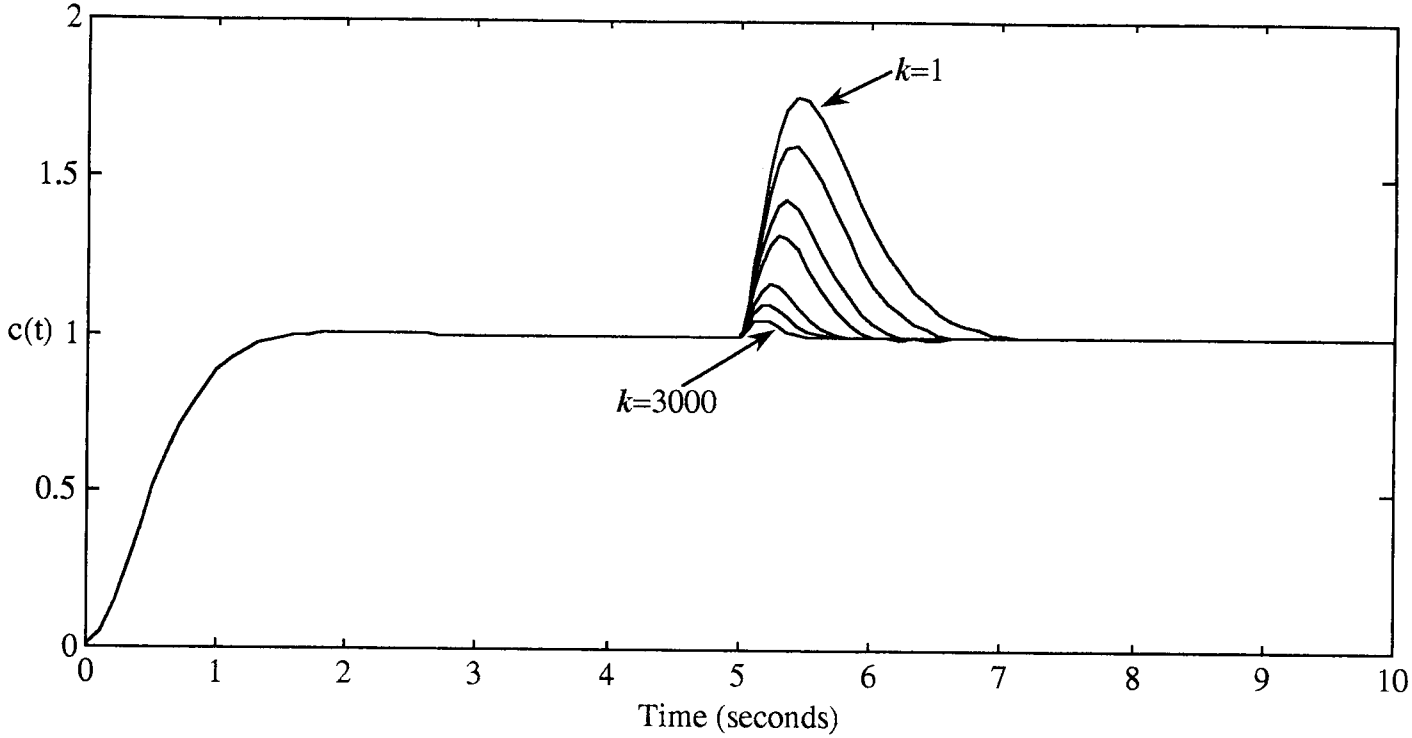


Figure 11. Response of DC motor and load system to unit step input and disturbance occurring at five seconds, using two optimal controllers derived by Chang's method for k =1, 2, 5, 10, 40, 100, 300, and 3000.

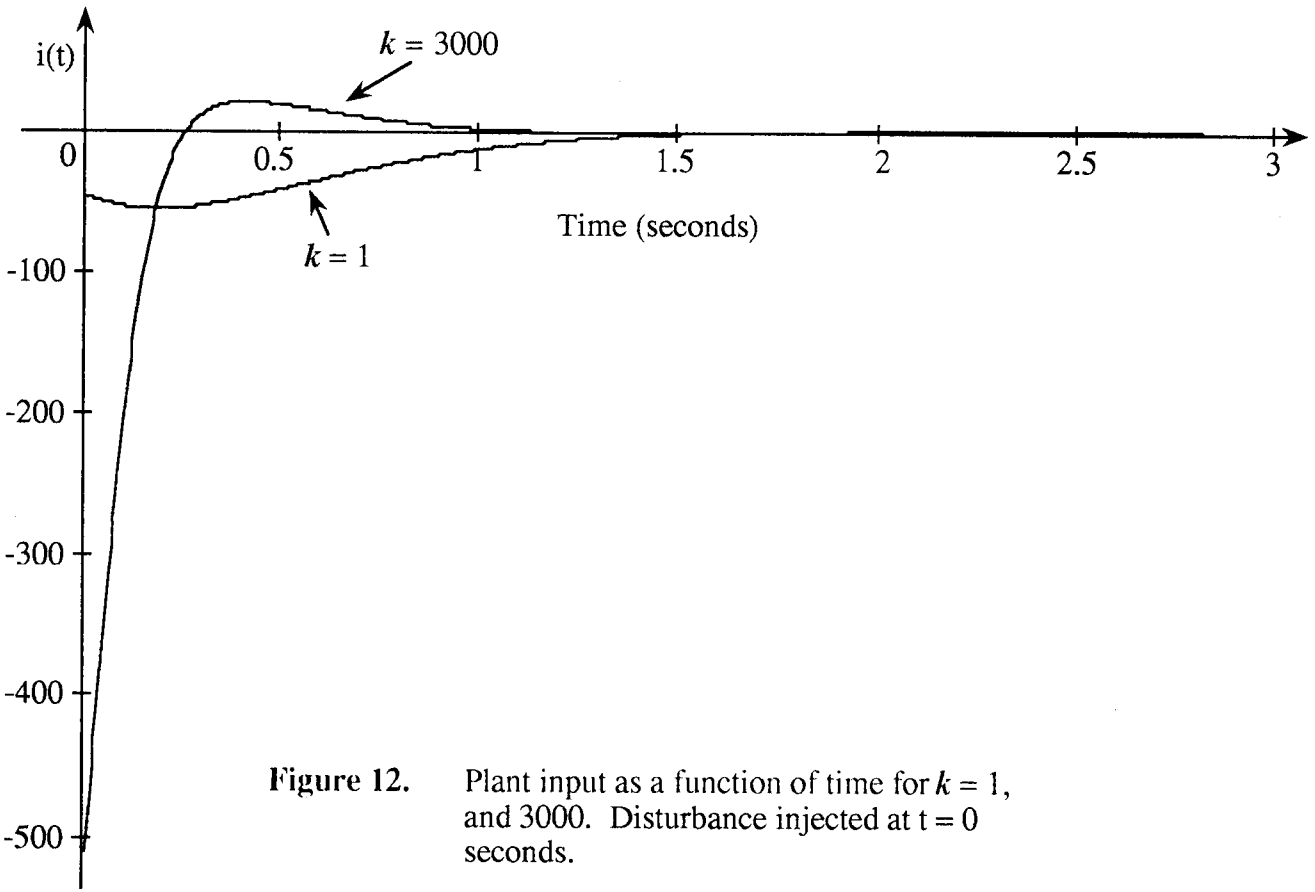


Figure 12. Plant input as a function of time for $k = 1$, and 3000. Disturbance injected at t = 0 seconds.

designer must select the value of k to achieve both suitable disturbance suppression and to constrain plant input to a suitable range.

CHAPTER 4

DISCUSSION AND CONCLUSIONS

This study evaluated the performance of two LS control techniques, one developed by Youla-Bongiorno-Jabr, the other by S.S.L Chang. LS control in general was shown to have several advantages to LQ control. First, the ability to incorporate practical design specifications is made quite straightforward via Chang's RSL technique. Second, the effects of plant zeros are inherently accommodated in all LS control techniques. Third, since LS control is frequency domain and does not use state space formulation, the LQ control requirement that the entire state be accessible for measurement, is not required.

The methods proposed by YBJ and Chang both start with the goals of system error minimization and plant input constraint. However, the equations used to represent system error and plant input differs greatly between the two techniques. Also, Chang's method was shown to be much more flexible then the method proposed by YBJ due to the fact that no requirements need be made on the specific control system being used. In the YBJ technique, one *must* start by specifying the exact control system to be used, where, in Chang's technique, there is no need for this specification. Chang's method also was shown to have the distinct advantage of being able to accommodate multiple controller configurations.

The YBJ and Chang technique were used to synthesize an optimal controller for a DC motor and load which did not account for external disturbance in its derivations, and

an optimal controller which did account for this disturbance. Chang's technique was then used to design two optimal controllers for the same DC motor and load, which were used in a multiple controller configuration. Simulations were then performed using each of these controllers, and performance of the DC motor and load was then evaluated for each controller. None of the four single controllers designed were able to satisfactorily suppress the external disturbance injected into the motor and load system. However, the goals of error minimization and plant input constraint were met by the controllers since both goals are guaranteed in the derivation of said controllers. Extremely encouraging results were obtained, however, when two optimal controllers derived by Chang's technique were implemented into the motor and load system. The two controller configuration, to some extent, was able to suppress the external disturbance even with the disturbance weighting coefficient, k, set to unity. Results were even more encouraging when the disturbance weighting coefficient was increased. This increase lead to even further disturbance suppression and, at the greatest value of k used, almost completely negated disturbance effects.

Continued studies are recommended which should include the evaluation of both YBJ and Chang's methods for a non-minimum phase plant (a plant which does not contain all LHP poles and zeros). Also, since the synthesis of the controllers both methods depend on the form of disturbance selected, an extensive study should be done to evaluate performance, when disturbances other than that selected, are injected into the system.

LIST OF REFERENCES

REFERENCES

[1] Pontryagin, L.S., Boltyanskii, V. G., Gamkrelidze, R. V., and Mishchenko, E. F., "The Mathematical Theory of Optimal Processes," Interscience, 1962.

[2] Bellman, R., "Dynamic Programming," Princeton University Press, NJ, pp. 81-115, 1957.

[3] Bellman, R. and Dreyfus, S, "Applied Dynamic Programming," Princeton University Press, NJ, pp. 180-204, 1962.

[4] Kwakernaak, H. and Sivan, R., "Linear Optimal Control Systems," Wiley-Interscience, pp. 193-327, 1972.

[5] Anderson, B. D. O and Moore, J. B., "Linear Optimal Control," Prentice-Hall, 1971.

[6] Naslin, P., "Essentials of Optimal Control," Boston Technical Publishers, Inc., pp. 229-232, 1969.

[7] Lee, E. B. and Markus, L., "Foundations of Optimal Control Theory," Wiley, pp. 169-238, 1967.

[8] Kirk, D. E., "Optimal Control Theory," Prentice-Hall, pp. 29-46, 1970.

[9] Doyle, J. C. and Stein, G., "Robustness with Observer," IEEE Transaction on Automatic Control, vol. 21, pp. 607-611, 1979.

[10] Maciekowski, J. M., "Multivariable Feedback Design," Addison-Wesley, pp. 222-264, 1989.

[11] Chang, S. S. L., "Synthesis of Optimum Control Systems," McGraw-Hill, 1961.

[12] Chang, S. S. L., "Two Network Theorems for Analytical Determination of Optimum-Response Physically Realizable Network Characteristics," Proceeding of IRE, vol. 43, pp.1128-1135, 1957.

[13] Youla, D. C., Bongiorno, J. J., and Jabr, H. A., "Modern Wiener-Hopf Design of Optimal Controllers Part I: The Single-Input-Output Case," IEEE Transaction on Automatic Control, vol. AC-21, pp. 3-13, Feb. 1976.

[14] Gantmacher, F. R., "The Theory of Matrices," Chelsea, vol. I, pp.101-103, 1960.

[15] Schwarz, H. R., "Numerical Analysis, A Comprehensive Introduction," Wiley, pp. 85-95, 1989.

APPENDICES

APPENDIX A

DETAILS OF CHANG'S METHOD

The goal is to minimize the total cost function, J, with respect to H (and automatically to $\overline{H}$). Using the s-domain optimization theorem developed by Chang [1], and a general cost function

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} \Phi\left(s, H, \overline{H}\right)\, ds.$$

The variation of J with respect to the variation of H is given by

$$\Delta J = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} \left\{\frac{\partial \Phi}{\partial H}\Delta H + \frac{\partial \Phi}{\partial \overline{H}}\Delta \overline{H}\right\} ds$$

$$\Delta J = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} \frac{\partial \Phi}{\partial H}\Delta H\, ds + \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} \frac{\partial \Phi}{\partial \overline{H}}\Delta \overline{H}\, ds$$

since Φ is an even function in s, we get

$$\Delta J = \frac{2}{2\pi j}\int_{-j\infty}^{+j\infty} \frac{\partial \Phi}{\partial \overline{H}}\Delta \overline{H}\, ds. \qquad (A.1)$$

For a physical system the integrant goes to zero as $|s|$ goes to infinity, at least fast as $\frac{1}{s^2}$, taking the contour integral of (A.1) yields

$$\Delta J = \frac{2}{2\pi j}\oint \frac{\partial \Phi}{\partial \overline{H}}\Delta \overline{H}\, ds$$

For an optimal $\overline{H}$, the variation ΔJ disappears, which then shows the relationship

$$\frac{1}{2\pi j}\oint \frac{\partial \Phi}{\partial \overline{H}} \Delta \overline{H}\, ds = 0 \tag{A.2}$$

which can only be satisfied if $\frac{\partial \Phi}{\partial \overline{H}}$ only possesses RHP poles. Therefore let

$$\frac{\partial \Phi}{\partial \overline{H}} = \Omega \tag{A.3}$$

where Ω has only RHP poles. Equation (A.3) is the necessary and sufficient condition for optimality and is one of a class of equations which should be called Chang's equations.

APPENDIX B

LIST OF ABBREVIATIONS

Integral Square Error	ISE
Least Integral Square	LIS
Left-half Complex Plane	LHP
Linear Quadratic	LQ
Linear Quadratic Gaussian	LQG
Linear Quadratic Regulator	LQR
Linear Transfer Recovery	LTR
Right-half Complex Plane	RHP
Root-Square Locus	RSL
Youla-Bongiorno-Jabr	YBJ

VITA

Michael James Gallagher was born in Springfield, Delaware County, Pennsylvania on June 25, 1970. He attended elementary schools in the Springfield Area School District as well as Harrison Township School District (Mullica Hill, NJ) and graduated from Clearview Region High School (Mullica Hill) in June 1988. The following September he entered Drexel University in Philadelphia, PA, and in June, 1993 received a Bachelor of Science in Electrical Engineering. He then entered The University of Tennessee, Knoxville in August, 1993 and received a Master of Science in Electrical Engineering.