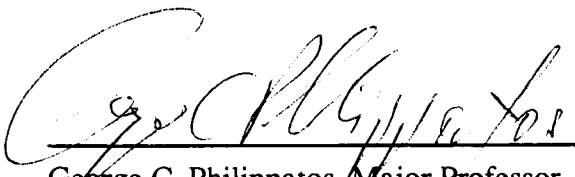
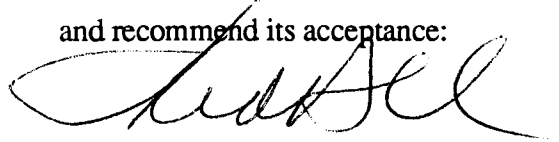


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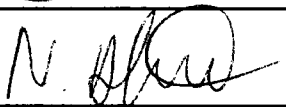
I am submitting herewith a dissertation written by Efstathia Pilarinu entitled "A Chaotic approach to the analysis of Financial data." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Business Administration.


George C. Philippatos, Major Professor

We have read this dissertation
and recommend its acceptance:





N. 

Accepted for the Council:


Associate Vice Chancellor
and Dean of The Graduate School

A CHAOTIC APPROACH TO THE ANALYSIS OF FINANCIAL DATA

A Dissertation

Presented for the

Doctor of Philosophy

Degree

The University of Tennessee, Knoxville

Efstathia Pilarinu

May, 1993

DEDICATION

To my first teacher, Nikos Kakoliris,
and to my parents
for their intellectual support.

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ABSTRACT

This dissertation presents an effort to implement non-linear dynamic tools adapted from chaos theory in Financial applications. Chaotic processes, although deterministic, generate time series indistinguishable from random ones due to sensitivity to initial conditions. Utilizing techniques capable of accounting for deterministically generated stochasticity offers the advantage of describing structurally unstable systems and non-equilibrium states of nature. Furthermore, dependencies pertaining to the entire probability distribution that cannot be quantified through standard statistical procedures can be detected with the proposed methodology. The present work is along the line of a recent trend towards emphasizing disorder, instability, diversity, disequilibrium, and non-linear relationships in the description of complex system behavior.

From a theoretical point of view special consideration is given in discriminating the non-linearities underlying chaotic processes from those modeled through non-linear stochastic specifications in Finance. Economic pathways precluding chaotic dynamics along with potential settings generating random and complex appearing behavior with a deterministic origin, are presented. Finally, the implications of a chaotic scenario for prediction, control, and asset pricing are thoroughly discussed.

A novel methodology is introduced in Financial applications that serves as a criterion for selecting the appropriate time delay in reconstructing the phase space from a single observable. This approach (entropic criterion) acts also as a measure of general dependence. The empirical part of this work focuses on the potential of the

aforementioned techniques in analyzing daily prices and volume series of the DJIA individual stocks, daily US index levels, as well as, weekly index levels and returns of International indices. The results suggest that the static and dynamic behavior of stock prices depend on the portfolio aggeragation level and the holding period, but not on idiosyncartic characteristis. Daily price and volume series of individual stocks may be modeled as the outcome of a low dimensional system.

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CHAPTER 1

INTRODUCTION

The main purpose of this doctoral dissertation is to provide a better understanding of the recent trend in Financial research towards non-linear and especially chaotic dynamic techniques. Within this framework, an extensive empirical analysis of Financial series that have not been examined before, is pursued. Furthermore, an improvement in the estimation of a system's correlation dimension from a single observable is presented.

Chaos theory is not a Financial theory, but merely a mathematical tool¹. This dissertation will focus, in part, on rationales for bringing chaos theory to the attention of Financial researchers. Some crucial assumptions of Financial theory will be presented which, if relaxed will generate the type of dynamic behavior implied by chaos. The emphasis will be on distinguishing the non-linearities implied by stochastic specifications versus those of deterministic (chaotic) specifications.

The empirical analysis employs a dimension measure (correlation dimension) that quantifies the average spatial correlations of a time series. A high estimate of the correlation dimension is consistent with a purely random process. On the other hand, a low value is in support of the existence of a low dimensional structure; that is, a simple non-linear underlying system with few variables. The observed randomness in this case is only due to the nature of the non-linearities (i.e. coupling of feedback effects).

The algorithm is improved by using an entropic criterion (redundancy) for selecting the time delay required for reconstructing the phase space. This technique is

¹ Current Financial theory (EMH, Capital Market theory, Portfolio theory, etc.) has been formed on the basis of another mathematical tool, namely stochastic processes.

used for the first time and serves a dual purpose. First, it addresses the issue of effectiveness of the time delay method and second, it quantifies general dependence instead of simple linear dependence.

The original contribution of this work is two-fold in scope. From a theoretical perspective it is focused on the adaptation of a mathematical tool, namely chaos theory, to Financial theory. From an empirical standpoint the innovation lies in improving the methodology for reconstructing a system's phase space from a single observable. In addition, a broad spectrum of Financial series that have not been examined with any of the new non-linear techniques is analyzed. The behavior of weekly index levels and their implied rates of change of international indices from eleven industrialized countries are examined. An extensive analysis of daily prices (instead of returns), as well as, volume series of individual stocks from the DJIA is also undertaken. Finally, the behavior of the individual stock price series is compared to that of two indices including them (DJIA, S&P500).

This dissertation is organized as follows. Chapter 2 motivates the use of techniques borrowed from chaos theory. The behavior of linear systems versus non-linear systems is contrasted. Their implicit assumptions and implications from a theoretical perspective are compared. Most importantly, chaotic dynamics which represent a very special type of non-linearity are differentiated from other non-linear statistical techniques employed recently in financial research.

Chapter 3 investigates the possibility that the type of behavior described in Chapter 2 as chaos in Finance is plausible. The basic assumptions that preclude chaotic dynamics, and have therefore led to the current stochastic framework are discussed in detail. Theoretical justification for relaxing such assumptions is presented. Possible scenarios that may lead a specific type of non-linear behavior exhibited by chaotic systems are discussed.

Formal definitions of the fundamental notions of chaos and its components are presented in Chapter 4 along with an example of a chaotic map. A number of diagnostic criteria in the form of Lyapunov exponents, as well as, dimension and entropy measures, are discussed.

The empirical methodology utilized for calculating the correlation dimension of a system from a single observable, is described in Chapter 5. The reconstruction of the phase space is emphasized and the shortcomings of the algorithm are discussed.

In Chapter 6 a functional literature review of the empirical issues pertaining to the present topic of non-linear dynamics, is presented. Three different methods of non-parametric non-linear analyses are found which lead to the detection of dependencies that cast reasonable doubt on the current techniques employed. The implications of the results are discussed.

A new method that serves as a selection criterion for the time delay in reconstructing the phase space is presented in Chapter 7. The importance of the criterion based on the mutual information values of the series at various lags is discussed. The algorithm for measuring general dependencies in time series, is described. Chapter 8 presents the time series analyzed with the new non-linear techniques. The issues addressed by the above analysis are discussed in detail.

In Chapter 9 the empirical results obtained from the applied methodologies are discussed. Additionally, a number of issues pertaining to further future research are addressed.

CHAPTER 2

CHAOS AS A SPECIAL TYPE OF NON-LINEARITY

Overview

There has recently been considerable interest in chaotic dynamics in a variety of disciplines. Extremely influential scientists emphasize disorder, instability, diversity, disequilibrium, non-linear relationships, as those aspects of reality that characterize today's dynamic environment¹. The objective of this Chapter is to motivate research in chaotic dynamics by explaining how this type of behavior is different from linear models and, most importantly, from other existing non-linear formulations. At this point no attempt is made to justify or reject any specific model. The implied assumptions and the direct implications of the different types of possible model specifications used in Finance are emphasized.

Nature and significance of a linear system in financial economics

Current Financial theory is based on a set of assumptions that characterize the so called linear paradigm. The existence of a rational investor who is risk averse is the building block of the linear view in Financial Economics. It is assumed that the individual's investment decisions are formed by maximizing expected utility across

¹ A detailed overview of the importance of this scientific change can be found in the two books of the 1977 Nobel Prize winner for his work in thermodynamics of non equilibrium systems (Prigogine (1980), *From Being to Becoming*, and Prigogine and Stengers (1984), *Order Out of Chaos*).

existing alternatives. The expected payoffs for each investment are determined by the expected future payouts of the securities (rational expectations hypothesis). Returns are normally distributed and less risk is preferred to more, for a given level of expected return.

The definition of *rational decision making* under uncertainty in Financial Economics is expressed through the *axioms of cardinal utility*, which provide the minimum set of conditions for rational behavior. The five axioms are: (i) completeness, (ii) transitivity, (iii) strong independence, (iv) measurability, and (v) ranking (Fama and Miller (1972), pp. 79). As a result, individuals can make choices among any large number of alternatives in a fashion that is rational, as defined by the above properties. Thus, there is always an optimum choice for each individual and his choices across time must be consistent² Preferences can be mapped into a measurable and order preserving utility function, as a result of the measurability and ranking axioms. Furthermore, utility functions are assumed to be increasing and strictly concave. More wealth is preferred to less and addictive behavior is ruled out³. Strict concavity guarantees *risk-averse* investors.

Subsequently, the choice between consumption and investment in capital markets is determined by the *Fisher Separation theorem* and the *unanimity principle* (Copeland and Weston (1988), pp. 12, 19). Separability between the investment and consumption decisions holds. The Fisher Separation theorem states that these decisions take place in two separate and distinct steps. The unanimity principle guarantees that differences in shareholder preferences are independent of the production and investment decision. As a result, the optimization of the investment sector can be studied independently from the

² A rational investor that likes Chevrolets more than Fords and Fords more than Toyotas, when having to choose between Toyotas and Chevrolets must choose Chevrolets (transitivity). Not being able to decide or choosing Toyotas, is considered irrational.

³ Addictive behavior would be consistent with a increasing marginal utility.

consumption decision. Furthermore, the existing differences of opinion and preferences among shareholders need not be taken into account when deciding the optimal production and investment mix for the firm.

The separable choice among the various investment alternatives is determined by maximizing a utility function (with the properties implied by rational behavior) and by forming expected payoffs according to the *rational expectations hypothesis*. In such a framework, markets are efficient because information is already reflected in market prices through the individual decision making process. *Market Efficiency* formalizes the price determination process on the basis of information and sets the utility value of the gain from information equal to zero (Fama (1976)). Since the flow of information between now and the next period cannot be anticipated in an efficient market, price changes are serially uncorrelated. Thus, tomorrow's expected price given the information set available today, will be in the long run equal to today's price. Therefore, it is only a change in the information set that results in the variability observed. A linear reaction to such changes (i.e. arrival of new information) guarantees that the system stabilizes. The crucial, implicit assumptions of the Market Efficiency framework - rational expectations and existence of a stable market equilibrium - are clearly stated in Fama's discussion (1976), (Chapter 5, pp. 135 and 137):

On the basis of the information ϕ_{t-1}^m , the market assesses a joint probability distribution of security prices for time t , $f_m(p_{1t}, \dots, p_{nt} | \phi_{t-1}^m)$. From this assessment of the distribution of prices at t , the market then determines appropriate current prices, $p_{1,t-1}, \dots, p_{n,t-1}$, for individual securities. The appropriate current prices are determined by some model of a market equilibrium - that is, by a model that determines what equilibrium current prices should be on the basis of the characteristics of the joint distribution of prices at t Any test is simultaneously a test of efficiency and of the

assumptions about the characteristics of market equilibrium. If the tests are unsuccessful, we face the problem of deciding whether this reflects a true violation of market efficiency (the simple proposition that prices fully reflect available information) or poor assumptions about the nature of market equilibrium. The point, of course, is that any test of market efficiency is simultaneously a test of assumptions about market equilibrium.

An even more restrictive form of the Market Efficiency hypothesis - namely, the random walk hypothesis - is obtained under the assumption of *normally distributed returns*, where security prices exhibit changes that are independent and identically normally distributed (IID). The nature of equilibrium implied in this setting guarantees instantaneous return to equilibrium after the imposition of an external shock. No types of correlations (short or long) and no type of memory fits into this framework⁴.

The significance of the independence and normality assumption lies in that it allows linear aggregation⁵ and reduces the complexity of decision making⁶. In this case, the consumption-investment decision is reduced to a *three-dimensional problem* involving the choice of consumption, expected return and variance. With normally distributed returns and risk-averse investors, dispersion is completely summarized by variance. Thus, risk is measured by the standard deviation and investors prefer less standard deviation (for a given level of return) to more. As a result, only efficient portfolios will be held and by assuming complete agreement of all market participants as

⁴ The submartingale form of efficient markets allows for a short term memory, however, a linear reaction is still implied and stabilization is guaranteed although not instantaneous.

⁵ The sum of normal distributions is again a normal distribution.

⁶ The jointly normal probability distribution of returns can be completely described by the mean and the standard deviation.

of the joint probability distribution of securities, all individuals will hold the market portfolio. The latter, is the basis of Portfolio theory and the Capital Asset Pricing Model.

Stochastic non-linear models and the curious behavior of volatility

Empirical evidence against the normality and independence assumptions of the linear scenario and contrary to the rational expectations hypothesis, has led to the development of a variety of non-linear stochastic models in Finance. The presence of skewness, kurtosis, and fat tails in the probability distribution of returns, are one source (Copeland and Weston (1989)). Second, is the curious behavior of volatility. It appears that actual volatility is much higher than that expected in a rational market's framework, where the valuation of stocks is based on expected dividends (Shiller (1989)). Furthermore, volatility does not scale over time according to the $T^{1/2}$ Rule⁷ and its behavior is disturbingly unstable through time (Turner and Weigel (1990)).

On the basis of stylized facts that are inconsistent with a linear view, a number of non-linear econometric models have been developed in Finance. Some of the most sophisticated specifications are presented here as representative techniques:

- (1) The Autoregressive Conditionally Heteroskedastic (ARCH) model put forth by Engle (1982) has proved very useful in describing heteroskedasticity in time series:

$$x_t = \sigma_t * u_t, \text{ where } u_t \sim N(0,1) \text{ and IID} \quad (2.1a)$$

$$\sigma_t = \sqrt{a + \phi * x_{t-1}^2} \quad (2.1b)$$

⁷ For normal distributions, the standard deviation of 5-day returns should be equal to the square root of five times the standard deviation of daily returns.

(2) The econometric model termed General Autoregressive Conditionally Heteroskedastic (GARCH) model in its various forms captures even more detailed stylized facts than ARCH models. For example, the GARCH-M (in the Mean) is intended to capture the fact that returns have persistent volatility and, ex ante, the higher the conditional variance, the higher the expected return. This model allows the mean to change with the variance (Bollerslev et al. (1988)):

$$x_t \sim N(\mu_t, \sigma_t) \quad (2.2a)$$

$$\sigma_t = a + \beta * \sigma_{t-1} + \gamma * x_{t-1}^2 \text{ and } \mu_t = \phi * \sigma_t \quad (2.2b)$$

(3) Another variation is the GARCH-EAR (Exponential AutoRegression) captures the fact that stock returns are more locally autocorrelated following periods when the local variance is smaller (LeBaron (1992)). Finally, the multivariate extensions of GARCH models allow for interaction effects within the conditional mean and conditional variance of two or more time series (Bollerslev (1990)). Such specifications not only capture departures from normality and independence within a time series, but also attempt to model dependencies across sectors that are considered separable under the linear view.

Some of the above sophisticated non-linear econometric models appear to fit the data and at the same time explain certain dependencies (especially in volatility) that were not modeled under the linear paradigm. Overall, these specifications do yield substantial improvement compared to linear ones, by taking into account higher order dependencies in the moments of the underlying distribution. However, all the specifications are equilibrium models and ascribe unpredictability and randomness to unforeseeable exogenous events as in the linear case. The distinction between linear and non-linear

stochastic models is the differential response of each system to an exogenous force. Although in both cases, equilibrium is assumed and guaranteed the route towards stability is different.

Chaotic models

Another class of non-linear models generating complex and random appearing behavior, has received lately attention in Economics and Finance. Informally⁸, one can define chaos as a non-linear dynamic system that displays *sensitivity to initial conditions* (i.e. small changes are magnified as time evolves). As Poincare⁹ (1952) has pointed out, randomness may arise if a small imperceptible cause at one time determines a large effect at a later time. This property results in time series that appear indistinguishable from random ones (at least through standard statistical tests). However, the randomness is bounded within a specific region, the so called *attractor set*. The trajectories of the system do not wander aimlessly as in a random walk, but are confined within this region.

A number of non-linear deterministic models that can produce time paths observationally indistinguishable from stochastic processes traditionally used in Finance, are described below:

- (1) *The tent map*: This is a piece-wise linear map that resembles a Bernoulli process:

$$x_{t+1} = \begin{cases} 2x_t, & 0 \leq x_t \leq 0.5 \\ 2 - 2x_t, & 0.5 \leq x_t \leq 1 \end{cases} \quad (2.3)$$

⁸ Mathematical definitions are presented in Chapter 4.

⁹ "A very slight cause which escapes our notice determines a considerable effect which we cannot fail to see, and then we say that this effect is due to chance".

Suppose the above two segments of the unit interval are labeled "H" and "T", respectively. If one selects an arbitrary point from the unit interval and carries out the iteration specified from the tent map by labeling each successive iterate by the half of the interval into which it falls, then he will obtain a sequence of Heads and Tails. It can be shown that the density function of the variable x describing the fraction of the time when more Heads than Tails exist, is a Bernoulli process with probability 0.5. Thus, the iteration of the deterministic map is mathematically and observationally indistinguishable from the sample path followed by flipping a fair coin (Casti (1989)).

(2) *The logistic map*: Malliaris and Philippatos (1992) have generated a time series of price changes from the logistic equation:

$$dp(t+1) = 3.89 * dp(t) * [1 - dp(t)] \text{ with } dp(0) = 0.52 \quad (2.4)$$

The hypothesis that $b=1$ could not be rejected at the 5% significance level for the regression model:

$$dp(t+1) = a + b * dp(t) + u(t+1) \quad (2.5)$$

(3) *The triangular recursion map*: Sakai and Tokumaru (1980), have shown that the triangular recursion specification:

$$x_{t+1} = 1 - 2 * \left| x_t - \frac{1}{2} \right| \quad (2.6)$$

produces trajectories that appear to be white noise.

(4) *A modified version of the triangular recursion map* (Sakai and Tokumaru (1980)): Most trajectories of the difference equation $x_{t+1} = F(x_t)$, where:

$$F(x) = \begin{cases} \frac{x}{a} & 0 \leq x \leq a \\ \frac{(1-x)}{(1-a)} & a \leq x \leq 1 \end{cases} \quad (2.7)$$

generate the same autocorrelation coefficients as the first-order autoregressive (AR) process:

$$x_{t+1} = (2a - 1)x_t + u_{t+1} \quad (2.8)$$

All the above non-linear models, appear to "deceive" many statistical tests that aim in revealing the underlying structure. In effect, the dependencies of chaotic models seem similar, if not identical, to those modeled by stochastic specifications.

Stochastic versus chaotic non-linearities

Despite the apparent similarities between random behavior produced by chaotic models and stochastic specifications, the two broad groups have fundamental differences. One of the main points that dichotomize stochastic non-linear models from deterministic chaos is whether the fluctuations observed in Financial markets are due to **external** forces or mostly due to **internal** dynamics. Stochastic models (linear and non-linear) imply that if the system is left to its own devices it will tend to equilibrium. In other words, the underlying assumption is that of an inherently stable system. The only difference between linear and non-linear models, in that respect, is the route followed until the system settles to a new equilibrium. On the other hand, chaotic dynamics are consistent with the idea of fluctuations produced internally. The variability is not due to

the stochasticity of some error term, but rather to the compounded effect of non-linear, feedback relations. As a result, even if no new information arrives the market will not equilibrate.

But why should one need to make the distinction that exists between the two types of models and stress the difference between internally and externally produced variability? Unfortunately, the difference is neither subtle, nor merely philosophical. The implications of the two views (external forces versus internal dynamics) are drastically different. **Prediction and control** in Financial markets are two areas in which one obtains drastically different policy recommendations from the two views. The internally generated dynamics' view implies that short-term prediction can be improved and supervisory stabilization policy may be proven effective (Brock et al. (1991)). The exogenous shock view would lead to the idea that stabilization policy is ineffective and probably useless. Prediction improvement may be hard to achieve, as a consequence of the unpredictability of future shocks.

The real cause that results in opposing policy and control recommendations, between stochastic non-linear models and deterministic chaos, is their different stability properties. For example, the **Structural Stability Assumption** states that a model is realistic or useful only if its qualitative properties do not change with perturbations of the parameters. All stochastic models employed carry this assumption as an apriori restriction to allow the argument that, within a large range of parameters and for almost any initial condition qualitative results hold. In effect, economic theory has accepted the structural stability assumption as an axiom (Malliaris and Stein (1992)). Chaotic dynamics, however, are consistent with the presence of structural instability. A chaotic system is inherently unstable resulting in a stochastic appearance but with a deterministic origin. The qualitative behavior of the system can change with a very small perturbation

in the parameter values or the initial conditions (i.e. there exists sensitivity to initial conditions).

Both chaotic systems and non-linear stochastic models are capable of accounting for departures from normality, independence, rationality, and separability assumptions. However, non-linearities of chaotic models result in disequilibrium market clearing points. From another point of view, chaotic dependencies imply **a more general notion of equilibrium** consistent with feedback effects and non-linearities that do not stabilize, but are confined within the attractor set. Randomness is due to the complexity of compounding feedback effects and to the alternating qualitative behavior of the system. A chaotic system offers **a more restrictive notion of randomness** (i.e. the attractor) and although it seems an oxymoron, randomness co-exists with order.

Furthermore, the differentiating features mentioned above (i.e. internally versus externally produced variability and structural stability versus structural instability), lead to another indirect distinction between the two types of systems. **The sources of uncertainty** and therefore its behavior, control, and measurement, are not the same. Uncertainty arises in stochastic Financial models from the unpredictability of future shocks. It is not due to the structure of the system itself that is unstable and sensitive to changes in initial conditions. The importance of this difference is well reflected in the above discussion about the effectiveness of government intervention. In addition, given that uncertainty arises from different sources, risk measurement will also differ not only quantitatively but also qualitatively.

The major distinctions discussed above motivate research in the area of non-linear chaotic dynamics by simply articulating important differences among the various types of non-linear models.

CHAPTER 3

CHAOS IN FINANCE

Overview

As was outlined in Chapter 2 chaotic systems, despite strong similarities, are fundamentally different¹ from stochastic non-linear specifications. In this Chapter interest is focused on explaining how financial theory could take advantage of these differences. Some stylized facts that lead to recognizing the advantages of non-linear and chaotic modeling, are discussed. Simple economic settings that have been found to generate such dynamic behavior, are also presented. The potential of applying chaos theory in Finance is investigated first from a theoretical perspective.

Economic pathways that preclude chaotic dynamics

(1) The existence of equilibrium.

The notion of equilibrium describes the state of the system that remains unchanged provided no external force is applied. Stable equilibrium is the equilibrium state to which the system will return after any external force dissipates. Unstable equilibrium is the equilibrium state from which the system deviates even after the external force applied has dissipated. The essential distinction between a sequence of stable equilibrium points versus a sequence of unstable equilibrium points lies in the

¹ Later it will be shown that chaotic randomness also requires different techniques for analysis.

causal factors of the observed states of the system. In the "stable" case only the external forces can change the state of the system, whereas in the "unstable" case the movements can also be produced endogenously. In economics, the external force view has dominated. Specific assumptions have been employed that guarantee a unique solution to decision making and optimization procedures. Several stylized models, however, have been developed that provide a different perspective consistent with the general notion of equilibrium in chaos theory and with internally generated movements².

Proof of non-unique equilibrium situations is frequently found in game theory (Simonsen (1987)). Non-cooperative games that describe how a market works in the absence of a Walrasian auctioneer are one area. It has been shown that equilibrium exists only if each player immediately locates a Nash equilibrium³. In the absence, however, of such a foresight assumption there is no guarantee of a unique solution to the decision-making process.

Additional evidence of the existence of multiple equilibria is found in modeling consumer choice (constrained maximization of a utility function) and producer decision making (constrained maximization of profits). A tâtonnement process with three goods can result in a stable limit cycle⁴ (Scarf (1960)). But, with four or more goods, the process can also generate chaotic dynamics (Kehoe (1987)). Production economies with four goods and four consumers may result in three equilibria, although the utility functions are still convex and the excess demand functions remain homogeneous (Kehoe (1985)).

Chaotic models can generate the type of instabilities present in real economic behavior. The existence of an attractor guarantees multiple equilibria. Most importantly,

² A more detailed account of various models that generate chaotic time paths will be considered later in this Chapter.

³ A Nash equilibrium is defined as a set of strategies, one for each player, such that no player can improve his expected utility by unilaterally changing his strategy.

⁴ A stable limit cycle is an unstable equilibrium situation, which oscillates with a fixed periodicity.

chaotic models can produce endogenously this type of "erratic" behavior. Non-linear stochastic models that attempt to determine the statistical distribution function (i.e. normal, leptokurtic, etc.) are ad hoc specifications. The strange attractor specification provides an explanation for the inconclusive results from time series analysis (Malliaris and Stein (1992)). The non-linear dynamics provide structure to complex and random appearing behavior.

Given the theoretical evidence supporting internally generated instabilities and non-unique equilibria, the real issue is what assumptions (i.e. what economic forces) can result in such complex dynamics? In other words, the issue is "what assumptions actually lead to stability"? Rational behavior of individuals and in the aggregate, complete markets and common knowledge beliefs suffice. All these assumptions lead toward theories of stability and imply amplitude reducing mechanisms. That is, economic forces guarantying that any disturbance will dissipate and the system will stabilize.

(2) Common knowledge beliefs among all investors.

Common knowledge beliefs⁵ have played the role of intertemporal smoothing mechanisms. All the major paradigms, Capital Asset Pricing model, Arbitrage Pricing Theory, Option Pricing Theory, and the Miller-Modiglianni framework, are based on these assumptions. It has been demonstrated, however, that the usual assumption of homogeneous beliefs⁶ is not strong enough to guarantee the aggregation required for CAPM, APT, OPM, MM to hold (Guth and Philippatos (1989)). In simple terms, it has been shown that even if homogeneous beliefs hold (but not common knowledge) the

⁵ Common knowledge beliefs about an event, hold between individual i and j if and only if both agents know the event has occurred, i knows that j knows the event has occurred, j knows that i knows that j knows the event has occurred, *ad infinitum*.

⁶ Homogeneous beliefs about an event, hold between individual i and j if and only if both agents know the event has occurred.

above equilibrium relationships cannot be obtained. The potential for multiple equilibria exists within the above settings.

As Arrow (1987) has pointed out, even though each first-order reaction to a disturbance may be stabilizing, the multiple connections within the economy might cause instability. Thus, even though individuals may act rationally ex-ante it is not at all clear that the result of the interactions of such individuals can be defined uniquely. If the dynamic process of such interaction does not result in a fully revealing state of nature, then the system will not equilibrate in the traditional point-equilibrium way. Some degree of uncertainty will remain unresolved at the market clearing point. Thus, is it not reasonable to expect complex dynamics evolving from an economy with decentralized knowledge? At the risk of being repetitive, it must be stressed that such dynamic behavior cannot be modeled from non-linear statistical models as explained in Chapter 2.

Settings under which chaotic regimes can be sustained by partially revealing market clearing points (i.e. non-common knowledge), are the following:

- (a) For a given set of information, the remaining (intrinsic) uncertainty at every market clearing point is perpetuated long enough. This may be due to low level of sophistication of investors, high costs of information, high degree of non-common knowledge.
- (b) As new information arrives (which creates extrinsic uncertainty in the traditional way), the intrinsic uncertainty may have not yet been resolved. The coupling of these sources complicates the inference procedure (disentangling of information) for market participants.
- (b) The inference procedure, is complex enough that the deterministic and chaotic behavior of prices is not common knowledge. In such a setting, although price behavior at the aggregate results in chaotic prices, individuals cannot reverse the dynamics and

"learn" the mechanisms at work. An example of such behavior in bond markets has been presented by Ploeg (1985).

(3) Market participants act ex-ante rationally both individually and in the aggregate.

Investors are assumed to form subjective probabilities in a rational and unbiased manner. Their decisions are based on a probability weighted average of the possible outcomes by utilizing all available information. Research into human decision making presents evidence against such behavior. Tversky (1990) suggests that investors are not always risk-averse. Evidence of individual behavior that is not based on an unbiased probability weighted average is clear in cases that involve losses. People that play in state lotteries (a typical negative expected return situation) feel that the possibility of a large return offsets the risk of a small loss, even if the probabilities are against them. Furthermore, experiments present evidence of "irrational" behavior by finding no differences in the decisions of groups that are segregated into specialists and non-specialist as well as informed and uninformed (Philippatos and Moscato (1969)). But even if individuals do act ex-ante rationally, what is the reason to believe that the aggregate outcome is rational?

At the individual level, rationality would preclude non-linear reactions to information by market participants. In the aggregate, it would rule out non-linear feedback effects and social behavior based on fads and fashions⁷. Linear feedback effects in price movements would imply fluctuations in prices from their "correct" value, that can be described as noise. At best, a mean-reverting behavior of prices should be expected under linear feedback effects. However, there exists supporting empirical evidence of trend-reinforcing behavior (Peters (1991)). Social experiments support systematically

⁷The ample empirical evidence supporting the presence of feedback mechanisms and persistence in Financial series will be discussed in detail in Chapter 6.

suboptimal behavior as the outcome of managing capital investment in a simple multiplier-accelerator economy (Sterman (1989)). Furthermore, Savit (1988) argues that individuals overreact to bad news. Thus, non-linear feedback mechanisms are more appropriate even at this level. In addition, such complexities at the individual level can not average out in the aggregate, and thus, simple linear feedback mechanisms are inappropriate at this level also.

Along the above lines of thought, theories based on social behavior that take into account various trends have been developed⁸. A social imitation model that is non-linear and accounts for transitions from random walk markets to periods of orderly price trends and periods of chaotic fluctuations, has been developed by Vaga (1990).

(4) The econometric treatment of interdependencies.

The notions of equilibrium, common knowledge, and rational behavior discussed above simplify Financial theories, but simultaneously preclude certain types of behavior. In effect, equilibrium guarantees unique solutions to decision making problems, common knowledge allows for aggregation, and rationality provides a formal behavioral framework individually and in the aggregate. On the other hand, multiple solutions to optimization procedures, structurally unstable relationships at the aggregate, and irrational behavior, are ruled out. All theories based on the above ideas are tested by using standard econometric techniques, which essentially are associated with a stochastic framework. However, the potential problems⁹ arising in such standard parametric methods are another important aspect of the classical Financial framework.

When using econometric models, the *ceteris paribus* assumption is implicit. In fact, the main purpose of econometric models is to isolate the influence of factors on one

⁸Shiller (1984), has done extensive work in relating observed social trends with economic behavior.

⁹ Parametric techniques are in disadvantage, especially if one has no clear idea about the data generating process.

another. This is the statistical analog of comparative static analysis that quantifies the impact of one variable on another, all other factors held constant. For example, when examining the influence of the market index (be it mean-variance efficient or not) on average individual stock returns, a regression model with independent and identically distributed (IID) error terms is estimated. This procedure assumes that the stated relationship is independent of any other factors. Investor attitudes towards risk, business cycle considerations, degree of completeness of the markets, etc., are considered "exogenous." All the qualitative aspects of decision making are ignored. Assuming independence among variables does simplify the problem, but unfortunately it also changes its nature.

The above analysis, for example, implies that the composition of the market index and specifically the proportion by which the individual security is represented in the market, is irrelevant to the relationship. In simple terms, a market that is composed of 10,000 companies with a 0.001% weighting of each company, is viewed the same way as a market consisting of 2,000 companies with a 0.005% weighting of each company¹⁰. If for a given change in the weight of the security the influence of the market on the individual average returns changes monotonically, then one should not worry¹¹. Assume that two investors use different efficient market indices to estimate their influence on the same security, which has a 10% representation in the first index and a 20% in the second. The betas are found to be 1.5 and 0.5, respectively. If both investors triple the representations (30% and 60%), then the betas do not change monotonically. From 1.5 the first beta drops to 1.1 and from 0.5 the other increases to 0.8. Thus, the problem is not a "bias" in the quantitative behavior of the estimate, but in its' qualitative behavior.

¹⁰ Both markets being mean-variance efficient.

¹¹ Unfortunately, this is not true as in many other cases of economic relationships. The problem can be seen through a very simple example, based on the efficient set mathematics of Roll (1977).

Separability theorems in Economics have guided us towards modeling techniques and testable hypotheses that abstract from any kind of interaction. Separability in utility functions, over goods-securities, and across individuals, have enabled us to disaggregate. Effects of substitutability and of the complexity of the multiple degrees of conditioning inherent in the process of communicating information, are avoided.

Assuming that dependencies can be treated just as friction is handled in physics can be misleading. Chaos theory promises a more general framework that does not allow for decomposition. It offers a way of visualizing the results of interactions among different interacting layers of a system. First, it describes the behavior of a system at the exact aggregation level needed to incorporate feedback and forward effects without assuming any separability within the system. Distinctions of exogenous versus endogenous factors have no place in chaos theory, since the basic idea is to produce the observed behavior endogenously. Second, the inability of traditional statistical techniques to reveal causal factors can be overcome by studying the nature of the endogenous interactions and, furthermore, by examining self-similar structures at various scale levels (time scales or group scales)¹².

The fact is that econometric methods have been exhausted according to scientific observations of distinguished scientists¹³. Further justification for analyzing financial data through chaos theory is discussed by Casti (1990, pp. 251):

At the present, the most interesting possibility for a genuine explanatory mechanism underlying price fluctuation seems to be the work involving deterministic chaos..... But

¹² Emphasis on the non-parametric nature of the chaotic inference techniques and its advantages have also been discussed in detail in a survey paper by Frank and Stengos (1988). Since we have no clear idea about the data generating process, non-parametric techniques may be able to capture characteristics that otherwise would have been missed by their parametric counterparts.

¹³ Details of the grounds of such observations from a historical perspective are well documented by Mirowski (1990).

the methods and ideas are only in their infancy, and it's much too early to speak with any authority as to whether or not they will banish the EMH to that scientific grave yard where lie the skeletons of other explanatory mechanisms that don't explain.

(5) The Efficient Market hypothesis and its implications.

Market Efficiency postulates that prices fully reflect all information, in the sense that any change in the information set will be reflected in prices. This hypothesis is a formal representation of the external force view. In essence, it is not only that any change in the information set will be reflected in prices, but it is only such a change that can cause prices to move. A chaotic approach may well be compatible with the first part of Efficiency (i.e. prices fully reflect information). Sensitivity to initial conditions is an ideal property that can guaranty the velocity of responsiveness required by Market Efficiency. However, changes in prices can be generated internally without the imposition of an external shock.

The martingale hypothesis, which is consistent with "instantaneously unpredictable" processes, is incompatible with a chaotic scenario that allows short-term predictability. The methods for empirical testing of the EMH by examining the significance of the coefficients of various lagged models, will not be valid under a chaotic regime.

In summary, chaos theory as an approach to randomness appears to be a promising alternative to the stochastic approach. It attempts to describe and model disorder, discontinuities, erratic behavior and sudden ups and downs. Simple mathematical equations are used to model complex or random appearing behavior. Rather than accounting for the observed variability with stochastic exogenous shocks, variations can occur even without any exogenous disturbance within the equilibrium set of the attractor. The appeal to exogenous variables can be avoided, since chaos theory

postulates that simple causes may result in complex phenomena. Ad hoc non-stationarity assumptions can be abandoned, since chaos theory accommodates for endogenously produced qualitative changes.

Under a chaotic scenario researchers should replace equilibrium asset pricing models with dynamic models that describe transitions from point equilibrium, to limit cycles, and finally to chaos. Prices can "wander" under a chaotic regime, but the difference between this and the stochastic regime is that the attractor bounds "randomness" and order does exist.

Economic Models of Chaotic Dynamics

There exist a variety of economic models by which chaos can arise. These models are more or less of pedagogical nature¹⁴. They do not claim to be approximations of the actual data observed, but simply serve as examples of potential complex behavior the same way the prisoner's dilemma has served as a conceptual example of the likelihood of a suboptimal solution in the case of a non-cooperative game. The interesting feature is the simplicity of the assumptions concerning the dependencies within the system, that lead to erratic dynamics:

- (1) Day (1983), considers a classical economic growth model, where the level of output determines the rate of population growth. Output in turn depends on the size of the labor force and the author shows that the interaction of these two effects can produce cycles or chaos.
- (2) Benhabib and Day (1981), analyze consumer choice. In their model actual consumption can influence tastes in future periods (by addiction). The conclusion is that

¹⁴ The only model that has been subjected to numerical investigation is that concerning business cycle behavior (Lorenz (1987)).

the dynamics of consumer choice depend on the level of wealth. The choices of poor consumers may be fairly constant, while the choices of richer consumers can be cyclical or chaotic.

(3) Gaertner (1988), analyzes a model in which individual tastes are affected by the choices of others in society. The last two models overcome the assumptions of stable preferences and assume preferences that change in a stable way (completely deterministic) resulting in erratically fluctuating behavior.

(4) Baumol and Benhabib (1989), consider the relationship of firm profits and advertising budget decision. Since an increase in advertising outlay always raises total revenue, after a point its marginal net profit yield becomes negative, thus the relation is hill-shaped and any kind of behavior can emerge by fine tuning of the parameters.

(5) The business cycle literature has heavily invested in this area. Kaldor (1940) type models (nonlinearities in the savings and investment functions), have been reformulated. Lorenz (1987), has shown the existence of a strange attractor for a Kaldor type business cycle model for specific parameter values.

(6) Shaffer (1991), models firm behavior by assuming that a fixed proportion of realized profits is paid out as dividends and the remainder is reinvested according to a marginal efficiency of investment curve that declines linearly down to some minimum rate of return. These simple conditions suffice to produce chaos.

(7) Dana and Montrucchio (1986), analyze the complexity of time paths generated by a model of dynamic oligopoly. For small discount factors, essentially anything is possible as a trajectory for the market prices and outputs.

Possible pathways to chaotic dynamics have also been proposed by various researchers as areas that need further investigation. Introduction of a group of households that heavily discount the future, may result in sensitive dependence on initial conditions (Woodford (1989)). Abandoning the equilibrium assumption in favor of learning

dynamics (Anderson, Arrow and Pines (1988)) can generate inherently unstable settings. Abandoning the assumption of complete markets by allowing for liquidity constraints can generate instabilities (Bewley (1983)). Finally, allowing agents to be price setters instead of price takers, imposing complex dynamics directly on tastes and technology, and many other settings surveyed by Boldrin and Woodford (1990).

In summary, all proposed settings that may lead to chaotic dynamics imply some type of non-linear market mechanism. If market participants either use nonlinear models and/or believe that others use such type of models, they will maximize return and reduce risk based on this knowledge. As a result one can expect that the nature of the price movements will thereby be altered. Market prices in a non-linear system can be very sensitive to small changes in the environment and it is not at all clear how such systems would respond to this additional kind of feedback. This is analogous to the interference by an observer on the phenomenon observed, as emphasized in quantum mechanics. In fact, the market as a system under examination, offers a laboratory superior to that of the natural sciences in that respect. In any physical experiment there is a natural need to abstract from the effect of the observer on the experiment in order to obtain the true (measurement-free, interference-free) law governing the phenomenon. However, in the market the observer is not simply a part of the system, but the system itself. In fact, it is this exact effect of the observer (market participant) on the measurement obtained (whether it is the price of a security, the trading volume, the demand for a new instrument, etc.) that we need to study.

A scaling issue

According to Einstein, "God does not play dice with the universe". This strong stance reflects a view that ultimately all systems are deterministic. One may model a

system as if it were the outcome of a random process but that is merely a simplification for purposes of modeling. Lack of information and complexity within a system may result in a random appearance. We do not wish to discuss the philosophical debate between a stochastic and a deterministic perspective, but we solely intend to focus on the aspects of the recent advances in chaos theory and their potential to change Financial theories.

Once one admits the Einstein conjecture, the real question is whether the scale at which structure exists is too fine for our practical purposes, thus proving empirically useless. Casti (1990), comments on the importance of detecting chaos in Economics in a section titled "Clarity of Chaos"(p. 248) articulating the above ideas:

But why should we care about knowing whether the economic data are generated by random or deterministic means? After all, you might well argue that even if the underlying rule is deterministic, it's too deeply hidden for us to have even a ghost of a chance of ever discovering what it really is....

First of all, there would be definite implications for government policy....

The second response is.... we can cling to the hope of being able to internalize (endogenize) the error terms in our models.

Randomness is a function of scale. For example, the output of a random number generator is the result of deterministic operations. If statistical tests can reveal the dependencies, then randomness ceases to exist. Therefore, processes that are random as measured by a Tsay test (1986), may well appear orderly when measured by the BDS test (Brock et al. (1991)).

Even if it is true in Finance that structure exists at low aggregation levels and not at higher aggregation levels, the new conceptual tool of chaos (from its theoretical

implications) will have offered the profession a valuable piece of knowledge. In summary, the complete investigation of a chaotic approach to randomness and complexity will address the following issues¹⁵:

- (a) *Locate the exact level of aggregation above or below which random appearing behavior can be modeled with simple mathematical equations.* At such levels a chaotic approach, that describes complex phenomena as results of simple causes with non-linear effects, will be appropriate. Levels of aggregation, either with respect to holding periods or with respect to portfolio compositions, may be decomposed in categories according to the complexity and "source" of randomness that they exhibit.
- (b) *Improve our understanding of the interaction effects (feedback mechanisms, complexity levels) that do result in linear relationships at certain aggregation levels.* At these levels a stochastic, equilibrium approach suffices to describe the outcome of the individual interactions. Thus, although it is evident that individual decision making is deterministic, the aggregation process can be approximated to a satisfactory degree by a "smooth" process, as if the mechanisms at work are indeed amplitude reducing.
- (c) *Eliminate an alternative modeling device from the broad alternatives among which we search to describe the system.* Even if a chaotic approach proves to be completely useless (from an empirical point of view), the rejection of this alternative modeling device is essential in our understanding of the mechanisms at work.

¹⁵ This Thesis deals only with issues pertaining to (a) and (b).

CHAPTER 4

CHAOS: A MATHEMATICAL PERSPECTIVE

Overview

Chaos is an "abused" term in dynamical systems. The word has a cataclysmic ring to it, reminiscent of catastrophe theory and to call your field "chaos" may seem as an admission of defeat at the outset. However, chaotic dynamics have become important, not only in mathematics, but also in physics, chemistry, engineering, etc.. Since there is no uniformly accepted definition of chaos in the literature a topological approach is presented here because of its simplicity¹. A simple non-linear map illustrates chaotic characteristics. Various measures that make feasible the definition of system parameters resulting in chaotic behavior, are discussed. Furthermore, techniques that distinguish random from deterministic systems in the case that the system equations are unknown, are defined.

Chaotic Maps

A dynamical system is described by the following evolution equation:

$$\frac{dx}{dt} = F(x) \quad (4.1a)$$

¹ The definitions are from Devaney (1990).

$$x(t+1) = f(x(t)) \quad (4.1b)$$

where F and f are differentiable functions and the variables x vary over a phase space M , which can be in $\mathbb{R}^m$, a compact manifold or an infinite dimensional space. The nonlinear time evolution operator f^t is a group on M :

$$f^0 x = x \text{ and } f^{t_1+t_2}(x) = (f^{t_1} \circ f^{t_2})(x) \quad (4.2)$$

The map f is said to be chaotic if:

1. f has sensitive dependence to initial conditions.
2. f is topologically transitive.
3. Periodic points of f are dense in M .

If f exhibits *sensitivity to initial conditions*, then there exists an arbitrarily small distance δ such that for every initial state x and for every neighborhood of this state $V(x, \delta)^2$, there exists a state y initially in this neighborhood but within a larger distance later in time.

$$\exists \delta > 0 \text{ such that } \forall x \in M \wedge \forall V(x) \Rightarrow \exists (y \in V, n > 0) \text{ such that } |f^n(x) - f^n(y)| > \delta \quad (4.3)$$

In other words, given any initial state x there exists a nearby state y (as close as δ) whose trajectory diverges from x . The distance between x and y although initially arbitrarily small (i.e. equal to δ), becomes very large as time passes. An infinitely small order inaccuracy in measuring the initial condition of the system, may grow exponentially

² $V(x, \delta)$ is a neighborhood of x with radius δ .

as the system evolves in time. This property renders the system unpredictable even if the law of motion is known. The difference with a purely random process (which is also unpredictable) lies in what causes it. In the latter case it is due to statistical independence, but for chaotic systems it is due to the lack of infinitely accurate information coupled with sensitivity to initial conditions. In practical terms, this property allows for major effects that are caused from small, imperceptible changes. It is a manifestation of the accumulated effect of the feedback nonlinear mechanisms at work, since sensitivity to initial conditions cannot be present in a linear world.

A mapping is *topologically transitive* if there is a trajectory that eventually enters any preassigned region, no matter how small. This property guarantees that the system cannot be separated into subsystems that behave independently. In other words, the trajectories visit the entire space, which is equivalent to the existence of a dense orbit. Chaotic prices may wander anywhere. They can soar up and plunge to zero. A formal definition of topological transitivity and thus, non-decomposability of a system is as follows:

$$\forall U, V \text{ open sets, } \exists k > 0 \text{ such that } F^k(U) \cap V \neq \emptyset \quad (4.4)$$

Finally, chaotic maps exhibit an element of regularity and structure, namely the *periodic points are dense*. Not only periodicity co-exists with unpredictability, but for any initial condition x there exists another y arbitrary close to x that is periodic (i.e. the property of density).

In summary, in a chaotic system as time evolves the trajectories tend toward a set, namely the *attractor*, due to the topological transitivity property. This set is compact and almost all initial conditions of the set will converge to it. Unpredictability and structure

co-exist within the system due to sensitivity to initial conditions and to the density of the periodic points.

Very simple difference equations exhibit the complex behavior described above. A theorem developed by Li and Yorke (1975) has facilitated the understanding of the existence of chaotic behavior in a one dimensional setting. A very simple condition that guarantees erratic and complex behavior of chaotic nature is obtained for any first order difference equation that is hill-shaped³.

To illustrate, imagine that stock price movements are described by a simple nonlinear function, the exponential logistic map (May (1974)):

$$P_{t+1} = P_t * \exp\{\mu * (1 - P_t)\} \quad (4.5)$$

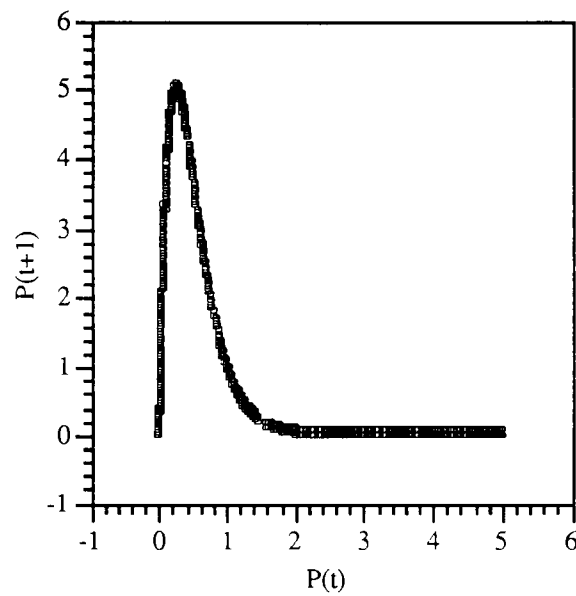
This map is used, since it exhibits only positive values, which make economic sense for price series. The exponent of this map is essentially a negative feedback term that under certain circumstances helps to stabilize price fluctuations. The pedagogical value of the metaphor used above as a model for describing stock price behavior lies in the simplicity of this map and its rich behavioral variety. Traditionally, the discrete logistic map $P_{t+1} = \mu * P_t * (1 - P_t)$ has been used, since changes in prices are the variables examined. Table 4.1 shows the dynamic behavior of the exponential logistic map for various parameter values. The time series of this map for parameter values in the chaotic region is indistinguishable from a random series. Even frequency domain analysis is not able to identify the underlying dependencies (Bunov and Weiss (1979)). However, the hidden structure is clearly detected even in a two dimensional phase space (Figure 4.1)⁴.

³ Hill-shaped are functions that exhibit one minimum or maximum point. In Chapter 3 we discussed how such dynamic relationships can arise in economics.

⁴ The phase plot uses 1,000 observations from time 4,000 to 5,000. The initial price value is 20 and the parameter value μ is equal to 4.

Table 4.1 Dynamic behavior of the exponential logistic map

Dynamic Behavior	Parameter Values
Stable Equilibrium Point	$2 > \mu > 0$
2 - point cycle	$2.526 > \mu > 2$
4 - point cycle	$2.656 > \mu > 2.526$
8 - point cycle	$2.685 > \mu > 2.656$
Cycles of period 2^n , $n > 3$	$2.692 > \mu > 2.685$
Chaotic Behavior	$\mu > 2.692$

**Figure 4.1** Two-dimensional phase plot of the exponential logistic map

Static Invariants

Any type of rule that can determine the control parameters of a system leading to chaos, is called a **predictive rule**. In order to apply such criteria, knowledge of the mathematical model describing the system or at least some property (e.g. behavior of derivatives), is required.

Diagnostic criteria test whether a system was or is in a state of chaotic dynamics. Measurement of a single observable from the time evolution of the system is required. These criteria include techniques that enable us to distinguish random from deterministic systems. In essence, they are non-parametric methods that provide information as of the stability properties and the dimensionality of the system.

Initially, we will present diagnostic criteria that fall in the category of static invariants⁵. These measures have different values (or ranges of values) for the deterministic case than for the non deterministic, thus enabling us (at least theoretically) to distinguish random from deterministic.

Dimension is probably the most basic attribute of an object. The traditional concept of dimension defines an object as n -dimensional if and only if the smallest number of parameters needed to describe its points is n . Thus, any object in the Euclidean n -space (R^n) is said to be n -dimensional⁶. New definitions of dimension measures have been developed that are not always integer values (Procaccia (1983)).

Let f be a map on M describing the law of motion of the dynamic system under consideration, equipped with an invariant probability measure μ . Consider a partition Φ

⁵ Quantities that remain invariant under any smooth change of coordinates.

⁶ This approach became questionable for its interpretation in the early part of this century, when Cantor discovered a one-to-one correspondence between points on a line and the points on a plane and Peano constructed a continuous mapping from an interval on the whole square. It appeared that there was no distinction between Euclidean n -space and Euclidean m -space.

$= \{\phi_1, \phi_2, \dots, \phi_m\}$ of M and denote $p_i = \mu(\phi_i)$. Given the probabilities of the partitioned sets, we define the information of order q , H_q , as:

$$H_1 = -\sum_i p_i \log p_i \quad (4.6a)$$

$$H_q = -\frac{\log \left(\sum_i p_i^q \right)}{(1-q)} \quad (4.6b)$$

If the dynamical system has an attractor A , and $\Psi(\varepsilon)$ is a finite partition of the basin of attraction of A of diameter ε , then we define:

$$H_q(\varepsilon) = \inf_{\Psi(\varepsilon)} H_q(\Psi(\varepsilon)) \quad (4.7)$$

The dimension of order q is defined as:

$$D_q(A) = -\lim_{\varepsilon \rightarrow 0} \frac{\log N(\varepsilon)}{\log \varepsilon} \quad (4.8)$$

For $q = 0$, we have the *fractal dimension* D_0 (or *Hausdorff dimension*, or *limit capacity*), which is independent of the invariant probability measure and is a characterization of the geometrical nature of the attractor. It can be shown easily that it is equal to:

$$D_0(A) = -\lim_{\varepsilon \rightarrow 0} \frac{\log N(\varepsilon)}{\log \varepsilon} \quad (4.9)$$

where $N(\varepsilon)$ is the minimal number of boxes of size ε needed to cover the attractor A . The intuition of this measure of dimension is that it quantifies the new information gained by increasing the resolution of measurement (i.e. decreasing ε), under the assumption that all elements of the partition are equally likely. For example, it is an integer (equal to the topological dimension) for a limit cycle or a torus or any closed curve. The Cantor set is the classical example of a non-integer fractal dimension.

For $q = 1$, we have the *information dimension* D_1 , which provides a measure for the exponential rate of increase in information gained by measuring the state of the system on the attractor with increased resolution:

$$D_1(A) = -\lim_{\varepsilon \rightarrow 0} \sum_i^{n(\varepsilon)} \frac{p_i \log p_i}{\log \varepsilon} \quad (4.10)$$

The idea is to measure on the average the information gained with increased resolution, based on the probability distribution of the partition elements. Thus, the fractal dimension will be an upper bound for the information dimension. In fact for any $q > q'$ it holds that the q -order dimension is less than or equal to the q' -order dimension (the equal case occurs when the invariant probability measure is constant).

For $q = 2$, we have the *correlation dimension* $D_2(A)$, which again measures the average information gained by increased resolution as the information dimension, but takes into account the conditional probabilities of the partitioned sets (i.e. the correlation of points along an orbit):

$$D_2(A) = \lim_{\varepsilon \rightarrow 0} \lim_{N \rightarrow \infty} \frac{\log C(\varepsilon)}{\log \varepsilon} \quad (4.11a)$$

where

$$C(\varepsilon) = N^{-2} \{\text{number of pairs } (x(i), x(j)), \text{ such that } d[x(i), x(j)] < \varepsilon\} \quad (4.11b)$$

d is a norm.

The correlation integral $C(\varepsilon)$ can also be adjusted to eliminate pairs $(x(i), x(j))$, with $i = j$ by dividing by $(N^2 - N)$. By construction, the correlation dimension is a lower bound to information dimension. Since the correlation dimension is the most conservative of all three above invariants, it is the one employed for experimental purposes. Higher order invariants (dimensions) have not been used in applications, since the above have shown to provide enough information to solve empirical problems.

Dynamic Invariants

The intuition behind the various dynamic invariants (entropies, characteristic exponents) stems from the exact nature of the attractor. It is the stretching and folding that holds the trajectories within the attractor and it is the frequency that these phenomena occur that can be studied through dynamic invariants.

Lyapunov exponents are the average rate of separation of various trajectories over the entire attractor. Thus, chaos requires at least one positive Lyapunov exponent λ_i . Notice that there are as many Lyapunov exponents as there are degrees of freedom (i.e. dimension of the true phase space). The idea is to follow the evolution through time of an infinitesimal hypersphere of radius ε . At time t the sphere transforms to an ellipsoid with semi-axes:

$$\varepsilon_i(T) = \varepsilon_i(0)e^{\lambda_i t} \quad (4.12)$$

Entropy is a measure of disorder and thus can potentially disentangle the dynamics enclosed in the attractor (where the disorder is created through sensitive dependence on initial conditions). We define the q -order entropies, which are dynamic invariants (i.e. depend on the properties of the trajectories).

Suppose that the phase space is partitioned to boxes of size ε . The state of the system is measured by $X(t)$ at intervals τ . The system is embedded in a probability space in the following fashion. Let $p(i_1, i_2, \dots, i_d)$ be the joint probability that $X(t = \tau)$ is in box i_1 , $X(t = 2\tau)$ is in box i_2 , $\dots$ and $X(t = d\tau)$ is in box i_d . Then the q -order entropy is:

$$K_q = -\lim_{\tau \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \lim_{d \rightarrow \infty} \frac{\sum p^{q(i_1, i_2, \dots, i_d)}}{d\tau (q-1)} \quad (4.13)$$

Topological entropy, is defined for $q = 0$. Kolmogorov (K) or metric entropy, is defined for $q = 1$. Regular dynamical systems (fully deterministic and thus, not chaotic) exhibit $K = 0$. Kolmogorov entropy is infinite for a random signal and finite for a chaotic system. It quantifies how chaotic the signal is. As time passes, two starting points that are empirically indistinguishable produce trajectories which are distinguishable. Kolmogorov entropy measures the mean rate at which they become distinguishable. In essence, it measures an aggregate of the stretching behavior, whereas Lyapunov exponents quantify the stretching and contracting in various directions. In empirical applications, the entropy of order $q = 2$ has been used as a lower bound on the metric entropy (Kolmogorov entropy), which could be easily extracted from a signal.

The dynamic invariants (Lyapunov exponents, entropy) are related to the static (dimension measures) ones. These relations are very powerful tools. From a theoretical perspective they are relating the stability properties of an attractor to the information needed to specify an initial condition or the degrees of freedom of the attractor. From an

empirical perspective it is possible to extract information without actually calculating the invariant.

(1) Kaplan-Yorke (1979), conjecture that the information dimension is related to the spectrum of the Lyapunov exponents as follows:

$$D_1 = j + \sum_{i=1}^j \frac{\lambda_i}{|\lambda_{j+1}|} \quad (4.14)$$

where j is such that the sum of the first j characteristic exponents is non-negative.

(2) Mori (1980), has conjectured that the fractal dimension is related to the spectrum of the Lyapunov exponents as follows:

$$D_0 = j + \frac{\sum_{i=1}^j \lambda_i}{\sum_{i=j+1}^n |\lambda_i|} \quad (4.15)$$

The two formulas give the same results for low dimension systems (two or three), but for large dimensional systems the results differ. Studies of infinite dimensional systems are the only ones that have managed to distinguish between the conjectures, in favor of the Kaplan-Yorke one.

(3) Kolmogorov entropy is equal to the sum of all positive Lyapunov exponents.

CHAPTER 5

CORRELATION DIMENSION

Overview

A detailed account of the algorithm for calculating the correlation dimension from a single time series of a multi-dimensional system, is presented in this Chapter. The reconstruction of the phase space by the time delay method and the correlation dimension estimation from a finite series of the single observable, is discussed. The shortcomings of the empirical procedure of extracting the static invariant (correlation dimension) are also discussed.

When applying the techniques that distinguish random from deterministic processes (diagnostic criteria) the researcher has to decide which measure is more appropriate for the analysis. If one is mainly interested in establishing or rejecting the presence of chaos, then the empirical attractiveness of the entropic measure over the calculation of Lyapunov exponents is obvious due to its relative simplicity of calculation. Based on the relations among the various invariants, the entropic measures suffice for detection purposes. Since entropy is equal to the sum of the positive Lyapunov exponents, one can reject or fail to reject the hypothesis of a chaotic structure.

On the other hand, the entropic measures and the dimension measures can be readily calculated using the same correlation integral information (as will be discussed later). Thus, obtaining both has only interpretational advantages concerning the meaning of the estimated dimension values and entropic measures obtained.

Reconstruction of the phase space

Suppose a time series of one variable X_i is the only measurement that is available for studying the behavior of the underlying system. Assume that dimension of the true system generating the time series (through an observer function) is n . The **time-delay method** is a technique for reconstructing the phase space from the observation of a single coordinate. This "picture" preserves the dynamic and topological properties of the true, multidimensional system. Takens (1981) has shown that there exists a diffeomorphism¹ between the true data generating process and d -histories, provided that $d \geq 2n+1$. The diffeomorphism property guarantees that the dynamic and static characteristics of the reconstructed phase space (in an embedding dimension d) are the same as the true phase space, which is unknown (state variables and functional form). In simple terms, one can construct "artificial" observables from a single time series, by displaying the time value. The paths followed by the reconstructed system are a topological approximation of the shape of the attractor. Takens has shown that if enough variables are used, typically about twice the Hausdorff dimension of the system, one should generally obtain a good projection. In practice the quality of the projection depends on the choice of variables and in particular on the time delay (Eckmann and Ruelle (1985)).

An illustration of the time delay method for a two-dimensional system can be studied for the Henon map:

$$\begin{cases} x_{t+1} = y_t + 1 - 1.4 * x_t^2 \\ y_{t+1} = 0.3 * x_t \end{cases} \quad (5.1)$$

¹ A diffeomorphism is a differentiable map with a differentiable inverse.

These equations exhibit a chaotic attractor with fractal dimension equal to 1.21. A two dimensional reconstruction of the phase space from delayed values of the x variable (Figure 5.1) and from delayed values of the y variable (Figure 5.2), clearly show graphically the similarity of geometric characteristics. The intuitive idea is that for a nonlinear system of simultaneous equations, the inherent interdependencies will be captured appropriately by examining only one variable. The time delay method "fills" the other dimensions (variables) with lagged values of the one observable. Thus, the reconstructed phase space can be calculated with just one observable and no equations of motion.

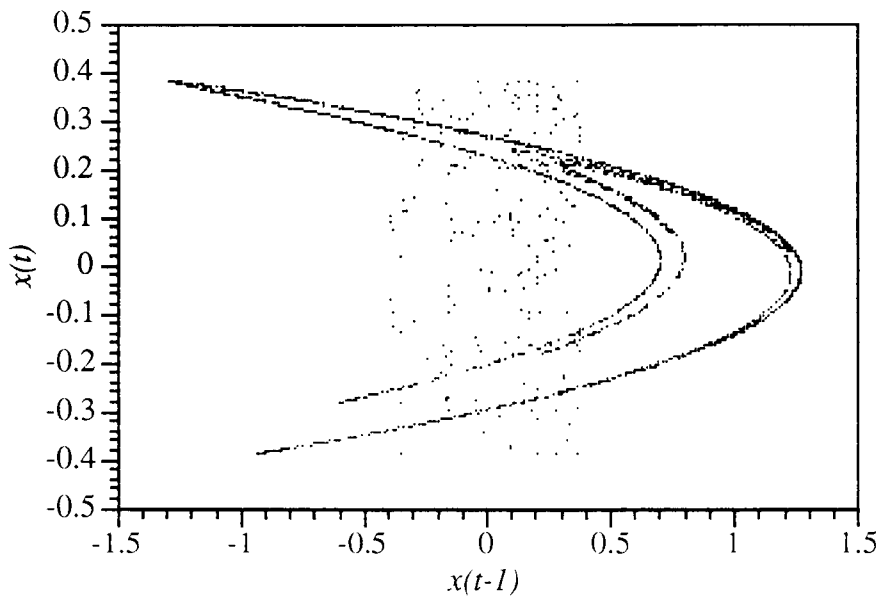


Figure 5.1 The Henon attractor reconstructed from the x variable

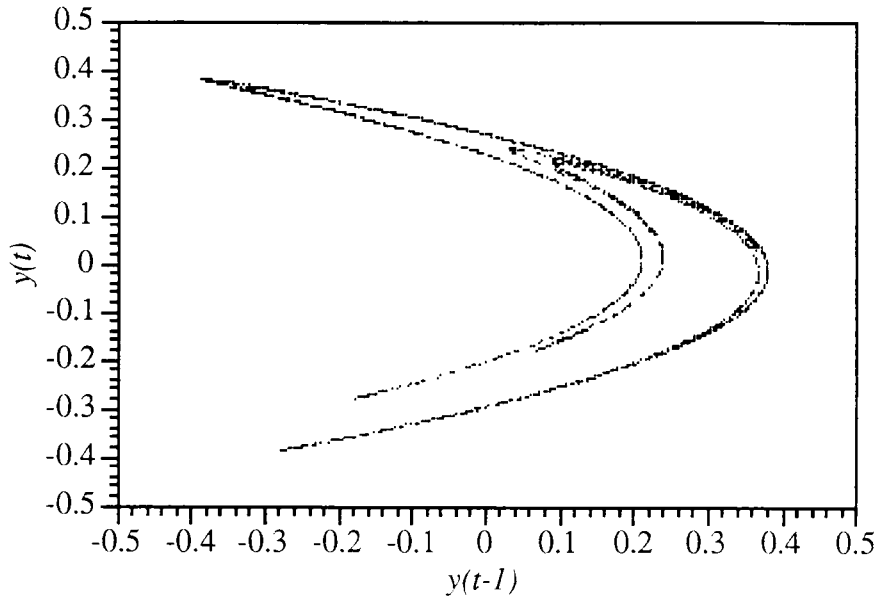


Figure 5.2 The Henon attractor reconstructed from the y variable

In general, if the true system is described by the following set of differential equations (Packard, et al. (1980)):

$$\begin{cases} x = f(x^{(k)}, y^{(k)}, z^{(k)}) \\ y = g(x^{(k)}, y^{(k)}, z^{(k)}) \\ z = h(x^{(k)}, y^{(k)}, z^{(k)}) \end{cases} \quad k=0, 1, 2, \dots \quad (5.2)$$

where $x^{(k)}$, $y^{(k)}$, $z^{(k)}$, are the k -order derivatives, describing the behavior of a system. The attractor formulated in the space (x, y, z) can be reconstructed by embedding a time series of observations of only one variable in time delays:

$$\vec{x}(t) = (x(t), x(t + \tau), x(t + 2\tau), \dots, x(t + (d-1)\tau)) \quad (5.3)$$

Correlation Dimension Estimation

In practice, d -dimensional vectors are constructed by the time delay method. For each point $z = (X(t), X(t+\tau), \dots, X(t+(d-1)\tau))$ in a d -dimensional space the number of points of the time series (the reconstructed one) whose distance from z is less than ε , are counted. The correlation integral $C(\varepsilon)$ is the sum of all points that lie in neighborhoods of radius ε of each point (See definition of $D_2(A)$). Subsequently, for a every embedding dimension d , the scaling behavior of the correlation integral $C^d(\varepsilon)$ with respect to ε , is examined. Grassberger and Procaccia (1983a) have shown that there exists a power scaling relationship of the form:

$$C^d(\varepsilon) \sim \varepsilon^D \quad (5.4)$$

where, D is the dimension. The intuition of this algorithm, stems from the fact that for a completely random series as the resolution ε changes, the number of neighborhoods included in the correlation integral changes (i.e. increases). If however, structure exists and not all the degrees of freedom are exploited by the system, then proportionally fewer new neighborhoods are included as we increase ε .

Since the correct dimension for reconstructing the phase space is unknown, the above procedure is repeated for higher embedding dimensions, thus calculating D^d for each d . If the behavior of D^d settles down at some level D_2 , this is the estimated dimension of the system. If as d increases, D^d also increases, the system is regarded as high dimensional or stochastic.

Therefore, for structure to exist the plot of $\log C(\varepsilon)$ versus $\log \varepsilon$ should be a straight line with slope D_2 by definition (this is the so called Grassberger-Procaccia (GP) plot). White noise has an infinite correlation dimension. A formal proof of this assertion has

been provided by Brock and Dechert (1986). The difference between the GP plots of random series and a completely deterministic (chaotic) is shown in Figure 5.3.

The process should be repeated (as noted above) for increasingly higher embedding dimensions d , up to the point that the slopes do not grow with increasing embedding dimension. The constant slope in the scaling region is the estimated correlation dimension.

If the procedure is repeated by calculating first the $\log$ of the number of points in each neighborhood of each point and then averaging over all points, we can obtain the information dimension D_I from the slope of $\langle \log C(\epsilon) \rangle^2$ versus $\log \epsilon$.

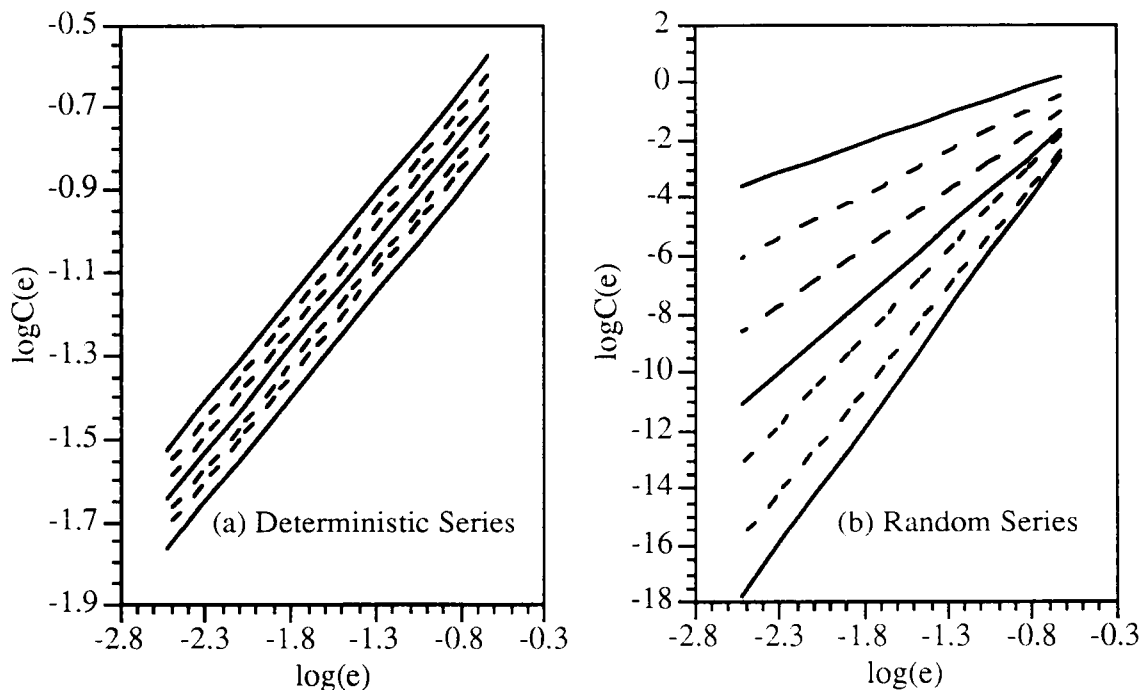


Figure 5.3 GP plots for Embedding dimensions $d = 2$ to $d = 8$.

² $\langle \cdot \rangle$ denotes the average over all points.

Using the same information, one can calculate the quantity $S^d = C^d(\epsilon)/C^{d-1}(\epsilon)$. The interpretation of S^d , using the supremum norm, is that it gives the conditional probability that two points will be close together, given that their past histories have been close together. Grassberger and Procaccia (1983b) proposed that the slope of $\log S^d$ over $\log \epsilon$ is a good approximation for the Kolmogorov entropy K_2 .

Shortcomings of the Correlation dimension procedure

The algorithm proposed by Grassberger and Procaccia for the estimation of the correlation dimension and of the Kolmogorov entropy appears to be a clear-cut procedure that could be effectively used to test for the presence of chaos. However, given that these techniques have basically been developed by physicists in fluid dynamics, their applications to other disciplines require some caution. This is due basically to the different nature of phenomena and to the specific assumptions underlying the system mechanism under examination. The issues that need to be examined in order to tailor the diagnostic to each specific application are :

- (1) The choice of the ranges of the embedding dimension d and the sample size.
- (2) The choice of the appropriate scaling region ϵ .
- (3) The meaningful statistical tests for the estimated values of the dimension measures.
- (4) The choice of the time delay τ in reconstructing the phase space.

Issue 1: The needed sample size for estimating the correlation dimension of an attractor has been "calculated" (a semi-empirical formula), as:

$$N_{min} = 21.5^D \quad (5.5)$$

where, N_{min} is the minimum sample size and D is the dimension of the attractor (Wolf et al. (1985)). Thus, for example, we need at least 460 observations to detect an attractor of dimension 2 and 10,000 points for a 3-dimensional attractor. With less than 1,000 observations (which is usually the case in economic series), the largest dimension that can be measured is between 2 and 3.

It is evident that since we do not know the dimension of the attractor (or an upper bound to its value), the length of the data set used will a priori determine the possible dimension obtainable. Furthermore, it will determine the possible maximum embedding dimension d . It is known (Farmer (1982)), that the embedding dimension M of an attractor³ contained in a smooth manifold is less than $2m+1$, where m is the attractor dimension. Thus, if the attractor is actually contained in a smooth manifold (which is not usually the case), we need an embedding dimension of 7 to be able to detect an attractor of dimension 3.

The effect on the estimated value of the correlation dimension, of the sample size and of the embedding dimension has been examined and quantified as an upward bias. Such bias depends on the size of the data and the embedding dimension. Ramsey and Yuan (1989) proposed a random coefficient regression model for the estimation of the correlation dimension. The idea was used to reevaluate previous economic studies that utilized the standard correlation dimension calculation procedure for small data sets to demonstrate their conjecture (Ramsey, Sayers and Rothman (1990)):

$$\log C(\epsilon) = a(\epsilon|N, d) + \beta(\epsilon|N, d) * \log(\epsilon) + u \quad (5.4)$$

³ The embedding dimension M of an attractor is the minimum dimension of a subset of Euclidean space into which a smooth manifold containing the attractor can be mapped through a diffeomorphism.

Ramsey and Yuan demonstrate empirically that the conditional mean of the estimated dimension D depends on the sample size N and the embedding dimension d in the following manner ($g_i, i=1, \dots, 6$, are constants):

$$\log D = g_1 + g_2 * N^{g_3} + g_4 * N^{g_5} * \left[\exp\left(\frac{g_1}{d^{g_6}}\right) - 1 \right] \quad (5.5)$$

In practice, for Financial applications that do not allow for resampling the procedure has little practical use. It requires a range of correlation dimension estimates for various sizes of subsamples of the original data set. Given the already small sample sizes traditionally used (1,000), the bias is evident in examining the behavior of the estimates for samples from 200 to 800 observations.

Issue 2: The choice of the scaling region (ϵ), has also received attention from the literature. For very low values of ϵ , the GP plots⁴ will be saturated (i.e. coincide for all embedding dimensions) because of the finite sample size. For large levels of ϵ , the graphs will also be saturated since the attractor has a finite size. Most studies in Finance⁵, chose a range of multiples of the standard deviation of the data to avoid the above bias.

Another important problem present in empirical applications and related to the resolution parameter, is the effect of noise on the estimated values of the correlation dimension. The effect may result in obtaining different ranges of $(\log \epsilon, \log \epsilon')$ associated with different slopes $\log C(\epsilon)$ versus $\log \epsilon$. Depending on the level of noise the obtained curves can appear as straight lines with "knees". In practice, the problem has been

⁴ The GP (Grassberger, Procaccia) plots are the logarithmic plots of the correlation integral against the resolution parameter.

⁵ See Brock et al. (1991) for a summary of all studies.

circumvented through a modification of the above procedure proposed by Ben-Mizrachi et al. (1984), which simultaneously supplies information on the noise level of the system.

In fact the relationship of $\log C(\epsilon)$ and $\log \epsilon$ is highly nonlinear and the traditional estimation procedure for the correlation dimension:

$$\log C(\epsilon) = a_0 + a_1 * \log(\epsilon) + u \quad (5.6)$$

creates a bias in the estimate of the dimension of the attractor. The existence of noise will not ruin the structure, but will cause fuzziness on length scales that are smaller or equal to the noise strength. Thus, given a noise level and for embedding dimensions higher than the true correlation dimension, one should observe fan-shaped versions of the GP plots. Above the length scales characterizing the noise strength all curves should be linear with a slope equal to the correlation dimension. All curves should break at the same value of ϵ , below which a slope equal to the embedding dimension should be seen (i.e. below the noise level the GP plots have slope equal to the embedding dimension d).

Issue 3: The most important aspect of correlation dimension analysis is the interpretation of the results obtained. As mentioned above, the effects of noise and the limitations of small data sets have more or less been addressed in this Chapter. The need for a statistical test for the estimated value is obvious, especially in disciplines (as Finance) where the possible modeling alternatives to a purely random case are multiple.

Brock (1988) has contributed significantly towards the direction of the interpretation of correlation dimension estimates, by noting that near unit root processes can generate low dimension estimates. In order to avoid this, he proposed the *Residual Test*, which concludes that the residuals from a linear time series (AR(q)) model have the same dimension as the original series.

Furthermore, Brock, Dechert, and Scheinkman (1986) constructed a statistic for the correlation dimension under the IID null. The quantity $W_m(\varepsilon, T)$ follows a standard normal distribution as T goes to infinity (thus if $|W| > 1.96$, the null hypothesis is rejected at the 5% significance level):

$$W_m(\varepsilon, T) = T^{1/2} * \frac{D_m(\varepsilon, T)}{b_m(\varepsilon, T)} \quad (5.7a)$$

where

$$D_m(\varepsilon, T) = C_m(\varepsilon, T) - [C_1(\varepsilon, T)]^m \quad (5.7b)$$

$$b_m(\varepsilon, T) = 4 * \left[K^m + 2 \sum_{j=1}^{m-1} K^{m-j} * C^{2j} + (m-1)^2 * C^{2m} - m^2 * K * C^{2m-2} \right] \quad (5.7c)$$

$$\text{where } C = C_1(\varepsilon, T) \text{ and } K = \frac{6}{T(T-1)(T-2)} * \sum_{i < s < r} I_e(x_i, x_s) * I_e(x_s, x_r) \quad (5.7d)$$

$I_e(x_i, x_s)$ is the indicator function that equals 1 if $\|x_i - x_s\| < \varepsilon$ and zero otherwise, where $\|.\|$ is the sup-norm

Although this is the only statistical inference associated with the estimation of the correlation dimension, it has very low power in the case of a high order AR in the residual test. The reason being that the estimation of the residuals itself creates dependence in the residuals. In fact, as the authors themselves note, experiments on the logistic and tent maps, by fitting a high AR(q) result in large correlation dimension estimates and the BDS statistic is incapable of detecting the hidden structure⁶. However, the test appears superior to other non-linearity tests as the Tsay test and the Engle test

⁶ A detailed discussion of the power of the BDS statistic can be found in Brock et al. (1991).

(Brock et al. (1991)). The Tsay test looks for non-zero terms in the third order cumulants of a zero mean time series and thus, has very low power against ARCH and GARCH models⁷. The Engle test looks for linear dependence in the conditional variance of a series and thus, has low power against specifications with a variance depending on its own past in a non-linear manner.

Another benchmark of comparison, aiding in the interpretation of the correlation dimension estimates is due to Scheinkman and LeBaron (the shuffle diagnostic). They calculate and compare the GP plot of the original residual series with a series 'scrambled' randomly (sampling with replacement). The idea is that if there is a hidden structure in the original residuals obtained along the lines of the residual test, it will be destroyed after shuffling and will cause a spread between the GP plots. Clearly, such procedure is restrained to the shuffling process and to the dependencies created in the residuals through estimation per se.

In summary, there exists no statistical test under the null hypothesis of chaos. The BDS statistic is a major improvement for non linearity testing under the null of IID processes. It is essentially a formal representation of the difference between the GP plot of the data (i.e. behavior of $C_m(\epsilon)$) and the GP plot of scrambled counterpart (that scales as $C_I(\epsilon)^m$).

Issue 4: The choice of the time delay in reconstructing the phase space, plays a crucial role in empirical applications. For infinite amounts of data and uncorrelated series the choice can be arbitrary. However, this is not the case in Finance. A selection criterion at each embedding dimension is proposed in this dissertation and will be discussed in Chapter 7.

⁷ These models have third order cumulants identically equal to zero by construction.

CHAPTER 6

APPLICATIONS IN FINANCE

Overview

The recent interest in non-linear dynamics of scientists in various disciplines extends also to Financial applications. The techniques discussed in the previous Chapter have been applied to various Economic time series. Studies that use methods capable of detecting non linearities that may lead to chaotic dynamics, are discussed in this Chapter. These include, frequency domain analysis based on high order spectra of the time series, Rescaled Range (R/S) analysis, and estimation of dimension measures and Lyapunov exponents.

Frequency Domain Tests

The bispectrum is the double Fourier transform of the third order cumulants of a probability distribution. Non constancy of the bispectrum strongly rejects linearity¹. There exists considerable empirical evidence supporting the presence of nonlinearities in various Financial time series, based on statistical techniques that use the estimated bispectrum values.

¹ A process is defined as linear if:

$$x(t) = \sum_{i=0}^{\infty} a(i) * e(t - i) \quad (6.1)$$

where e are IID and $a(i)$ are a square summable sequence.

(1) Hinich and Patterson (1985) and Brockett, Hinich, and Patterson (1988) present evidence against linearity and Gaussianity in daily returns of individual stocks from the NYSE and the AMEX (1962-1977). Thus, even if daily stock returns appear to be uncorrelated they are not realizations of independent random variables. More importantly, the nonlinear (stationary) structure that appears, is not due to some kind of nonstationarity in the series (possibly due to the long sample size), because nonstationarity will bias the test employed towards linearity and Gaussianity.

(2) Evidence of significant non-linearities also exists in 15-minute return series of individual stocks from the Dow Jones Industrial Average (1978-1981), which are calculated from a trade-to-trade data base. The sampling scheme is designed to avoid the introduction of aliased frequency components into the sampled returns (a low-pass Tukey anti-aliasing digital filter² was used for the one-minute returns that produced the above time series). The test statistic is highly significant for all stocks. Subsequently, daily returns (i. e. calculated from the filtered trade-by-trade series) and aliased returns (i. e. returns calculated from daily closing prices) are compared. Linearity is rejected for all 15 unaliased daily return series at the 5% level, whereas it is rejected for the 12 of the aliased daily return series (Hinich and Patterson (1988)).

(3) The null hypothesis of a linear generating mechanism is rejected at the 0.003% level (Ashley and Patterson (1989)) for daily returns of the CRSP value-weighted common stock index (1981-1984).

² The term *aliasing* refers to the phenomenon whereby the high-frequency structure of the source time series is reflected back into the lower frequencies of the sampled series. This results in statistics of the sampled series that are no longer representative of the underlying source of structure. The low pass Tukey anti-aliasing digital filter takes care of the aliasing phenomenon and removes the high-frequency energy in the time series.

(4) Brockett, Hinich, and Patterson (1988), find nonlinearities in spot price series, forward price series for foreign exchange rates, and US dollar to Japanese yen exchange rates.

Rescaled / Range Analysis

Rescaled range analysis (R/S) searches for fractal structure and non periodic cycles in time series. Fractal structure is symptomatic of non linear feedback effects that result in repetitive structure at smaller and smaller scales. Stability is banded and the presence of long term correlations (long memory) implies structure. The type of hidden structure and restricted stability detected from Rescaled range analysis are consistent with a chaotic process.

The intention of Rescaled range analysis is to examine whether the future is unrelated to the past and the present. This assumption for capital market analysis (i.e. instantaneous reaction to information) is necessary for the Central Limit theorem to apply. Standard econometrics can only apply to time series that are invariant with respect to time. Instead Rescaled range analysis includes the time dimension by examining whether the range of fluctuations change depending on the length of time used for measurement³. In effect time becomes as important as for a chaotic process.

(1) The monthly S&P500 returns and the 30-year Treasury bonds are found to exhibit long memory (Peters (1989)). The S&P500 exhibits an average cycle length of 48 months. This is a measure of the average length of time that past events have an impact

³ The statistic used for Rescaled range analysis is the Hurst exponent. If a series is random then the range of fluctuations would increase with the square root of time. Otherwise observations carry a "memory" of events that proceeded it.

on future returns. In effect, returns of indices based on a monthly holding period are highly inefficient and such lengthy memory casts doubts to the validity of results from event study methodologies.

(2) An extension of this study (Peters (1991b)), examines monthly returns of individual stocks (IBM, Xerox, Coca-Cola) and International indices (Morgan Stanley Capital international Index for U.K., Japan, Germany)⁴. All the results strongly reject the independence assumption by presenting evidence of long term memory. The markets appear to follow *trend-reinforcing behavior* and not mean-reverting behavior. The system appears persistent, since long trends have been identified. This is in support of potential forecastable structure and most importantly, of information that is not fully reflected in prices.

Correlation Dimension and Lyapunov Exponents

Evidence of nonlinear behavior has also been documented by employing the linearity test developed by Brock, Dechert, and Scheinkman (1986) based on the Grassberger-Procaccia-Takens dimension estimate.

(1) Scheinkman and LeBaron (1989), examine daily and weekly returns from the Value Weighted (VW) CRSP index around the mid 60's. Weekly returns of the index seem to be nonlinear, whereas for daily returns the evidence is weaker. Weekly returns for an individual stock (Abott Laboratories) show no evidence of any hidden structure.

⁴ Bond market data, currencies and economic indicators (industrial production, new business formation, etc.), are also examined.

Furthermore, the nonlinearities detected from the weekly returns of the index do not seem to be captured by an ARCH specification.

(2) Brock (1987), examined the Scheinkman and LeBaron weekly data set after dividing it in two equal subsamples. Both halves exhibit similar behavior (in contrast to the Ramsey et. al (1990) findings). Furthermore, he examined monthly returns of the VW NYSE index from the mid 20's to the late 70's. Compared to the EW NYSE index for the same period, the former showed stronger evidence against independence. In contrast, monthly returns of an individual stock (Kodak) appeared to be IID. Finally, first differences in T-bill returns showed strong evidence of non-linear dependence.

(3) Genotte and Marsh (1986), examined the same monthly data as above for both indices (EW, VW). The nonlinear dependence remained even when taking out the January seasonality. However, certain sub samples (42-71) appeared linear.

(4) Hsieh (1991), examined various time series in which the BDS statistic strongly rejects the hypothesis that stock returns are IID (without calculating the GP plots or the correlation dimension):

(a) Weekly returns of a Value-weighted index for a twenty five year period (1963-1987), a sample size approximately equal to 1250.

(b) Weekly returns of an Equally-weighted index for a twenty five year period (1963-1987).

(c) Weekly returns from ten value-weighted decile portfolios in which firms are ranked by size every quarter.

In order to check if the results are due to non-stationarity, the author looks at higher frequency data and shorter time periods:

- (a) Weekly returns of the S&P500 from 1962 to 1989 (i. e. 27 years), a sample size equal to 1350.
- (b) Daily returns of the S&P500 from 1983 to 1989 (i. e. 7 years), a sample size equal to 1750.
- (c) 15-minute returns during 1988 divided into four approximately equal samples of the S&P500, a sample size of 2000 observations each.

These returns appear to be also not IID, thus making unlikely that infrequent structural changes are causing the rejection of IID in weekly returns.

(5) Peters (1991a), recovers an attractor in monthly price levels of the S&P500. This is the only study besides the present research that uses actual values instead of rates of change. The data spans the period from 1950 to 1989 and is detrended by using a Consumer Price Index. The estimated correlation dimension is approximately equal to 2.33. No type of statistic is used in this study (i.e. the BDS or any bootstrapping technique). The GP plots and the phase plots support clearly the presence of structure. The EMH is rejected and the potential of forecastability is supported.

(6) Willey (1992), tests for non linear dependencies in the daily prices of the Standard & Poor's 100 Stock index and the Nasdaq 100 Stock Index. The BDS test and the Residual Diagnostic theorem are used, but the results are mixed.

(7) Eckmann et al. (1988), calculate the Lyapunov exponents for the weekly returns of the Scheinkman and LeBaron (1989) study. They find two positive Lyapunov exponents thus, supporting a chaotic underlying structure. Recall, that Lyapunov exponents measure the rate of divergence or contraction of trajectories across various directions.

(8) Other studies have examined rates of return on silver and gold (Frank and Stengos (1989)), daily foreign exchange rates (Hsieh (1989)), and futures prices for commodities (soybeans) and for financial futures markets on S&P500 (Blank (1991)).

(9) The most widely cited work for its theoretical and technical precision is that of Barnett and Chen (1986). It deals primarily with detecting chaos (correlation dimension criterion) in Divisia Demand and Supply monetary aggregates. For comparison the authors look at 2000 observations of daily common stock returns of IBM (beginning in January 1962). But since the phase plot is uniform (Gaussian noise) and the autocorrelation function (of 800 of the observations) resembles a delta function, the authors do not pursue the analysis.

In effect, evidence of uncorrelated series (supported by a uniform phase plot and a delta autocorrelation function) led to the conclusion of non-linear independence. However, one of the realizations of intensive research of Financial data has shown that linear independence can co-exist with non-linear dependence.

The Evidence

The tests of nonlinearity using tools from chaos theory, fractals, and polyspectra analysis, look for dependencies of the entire conditional distribution of the process. These dynamic tests of non-linearity are of a "global" nature. On the other hand, traditional tests of non-linearity search for dependencies in specific moments of the underlying probability distribution. For example, GARCH and ARCH specifications can be refuted only if the third order cumulants are shown to be non-zero. The martingale

hypothesis is violated only if dependencies exist in the first moment. Dependencies in the higher conditional moments would violate the more restrictive random walk hypotheses. In Finance it is important to be able to predict the conditional moment generating function. For example, Akgiray (1989) discusses applications of the forecastability of conditional variance. Prediction of conditional skewness or kurtosis can be effectively used in the trading of derivative assets (options).

The important question, however, that arises from the above evidence is whether the stochastic non linear specifications (i.e. departures from the random-walk hypothesis) that have been employed up to now for modeling stock returns can capture these dependencies. Examples of such models are, sophisticated autoregressive models (ARMA, ARCH, GARCH), bilinear models and threshold autoregressive models. Brock, Dechert, and Scheinkman have shown that GARCH processes generate zero bispectra, which is inconsistent with the evidence of non-zero bispectra of stock returns cited above (See Brock and Malliaris (1989)). The assumption of time varying volatility is not sufficient to capture the existing dependencies. Bispectral analysis has shown that these models are not consistent with stock return behavior. The bispectral based tests reject linearity and Gaussianity, but indicate that linear non-stationarity can not account for the observed variability. In essence, these tests offer strong evidence in favor of non-linear dynamics by showing that linear non-stationary specifications are incapable of capturing the existing dependencies. In effect, a stochastic framework that models the second moment of the underlying probability distribution of returns in a non-stationary fashion is not sufficient⁵.

Let us assume, however, that the dependencies detected from all the above non parametric methods can be captured reasonably, at least for all practical purposes, from

⁵ The BDS statistic, on the other hand, has shown low power when applied on residuals of ARCH and GARCH specifications or any high order autoregressive process (Brock et al. (1991)).

some even more sophisticated stochastic specification (e.g. a multivariate GARCH model). If this is true, is all the research using non parametric techniques (frequency domain tests, correlation estimates, Lyapunov exponents, Rescaled range analysis) useless or meaningless? The answer is no because the above research has led to another series of issues raised from the empirical findings of the above studies.

Issue 1: Hinich and Patterson (1985,1988) and Hsieh (1991) find evidence of nonlinearities for 15-minute returns and daily returns, whereas Scheinkman-LeBaron (1989) and Barnett-Chen (1986) find no such evidence. In fact, Brock (1987) and Scheinkman-LeBaron (1989) find evidence only in the weekly and monthly returns. Whether the structure of assets depends on the holding period is a question to be answered. Thus, a time scaling issue exists of whether the dependencies detected are similar across various investment holding periods.

Can some type of GARCH specification model this kind of structural instability? Is the solution as simple as using different versions of the model, depending on the holding period examined? But then what are the economic forces that may lead to markets in which volatility - the second moment, for example - not only is time varying, but also depends on the holding period considered? The only premise that is true is that current econometric techniques cannot allow and cannot model such relationships. Since their validity relies on the data confirming an ad hoc specification, it is only some type of non parametric technique that will aid in improving our understanding.

Issue 2: Hinich and Patterson (1985, 1988), Hsieh (1991), and Peters (1989, 1991a,b) present evidence of non-linearities for individual stocks and indices. But Brock (1987), Scheinkman-LeBaron (1989) and Barnett-Chen (1986) find no such evidence for individual stocks. Whether aggregation across assets preserves the order (structure) remains a question to be answered. Thus, a portfolio scaling issue exists. If so, a variety of conventional statistical models that account for the structural differences among levels

of asset aggregation must be developed. Furthermore, these models must exhibit internal consistency. For example, the variance behavior of individual stocks must "add up" to the variance of the portfolio or the index. But, then again no theory explains these differences and no improvement is accomplished in understanding such ad hoc modeling procedures.

In summary, the improved non linear techniques that allow for internally generated instabilities and non equilibrium behavior appear promising. Even in the case that the idea of chaotic dynamics per se is abandoned, the new insights and methods will have contributed in various ways:

- (a) for detecting dependencies above and beyond the capabilities of the current statistical techniques,
- (b) for internally producing and thus, explaining ad hoc non linearities in high order moments of the underlying distribution,
- (c) for presenting issues to the researchers attention that have been overlooked (as the empirical question of additivity across assets).

Furthermore, a whole new "technology" is presented that is revolutionary in terms of forecasting complex and random appearing time series. Such nonlinear modeling techniques will likely alter our short-term predictive ability vastly. Research currently under progress deals with new techniques of forecasting and noise reduction, under a chaotic scenario. Lapedes and Farber (1988) have employed neural networks to predict the future values of a chaotic time series with exceptional accuracy. Farmer and Sidorowich (1987b, c) have used an alternative approach to adaptive networks. A local approximation approach is also employed by Farmer and Sidorowich (1987a) and applied to the Dow-Jones Industrial Average, IBM stock returns and to gold currency prices. The authors state that the technique did not indicate any advantage over conventional linear techniques. However, they claim that the sample size was very small (less than 1000

observations) and furthermore, they did not include the improvements of their methodology (Farmer and Sidorowich (1987c)) which deal with issues of nonstationary behavior.

CHAPTER 7

MUTUAL INFORMATION

Overview

The interest in non-linear analysis of time series has resulted in the development of new techniques. The search for chaotic dynamics underlying complex and random appearing systems has been the basis for establishing a whole new technology. Advanced methods that may detect hidden structure are being improved. In this work the role of selecting an optimal time delay in reconstructing the phase space, is emphasized for the first time in Financial applications. Furthermore, a new entropic criterion is employed. The redundancy criterion (Fraser and Swinney (1986)), will be used as an "input" in the correlation dimension analysis and will also provide interesting insights as a measure of general dependence.

The proper choice of the time delay

A crucial development responsible for much of the recent progress in understanding low-dimensional chaos in diverse fields (such as chemistry, epidemiology, hydrodynamics), was the realization that multidimensional phase portraits could be constructed from measurements of single scalar series (Packard et al. (1980), Takens (1981)). The so called time delay method has become a "classic" technique among

chaologists¹. For an infinite amount of noise free data, the time delay can in principle be chosen arbitrarily. However, experiments show that the quality of the phase portraits depends on the time delay chosen (Roux et al. (1983)).

In all empirical applications to Economic or Financial data, the issue of choosing an appropriate time delay has been almost completely overlooked. Only Barnett and Chen (1986) refer to the problem in their research, but the authors deal with it through a trial and error procedure (i.e. choose the time delay which gives the more satisfactory results in terms of correlation integral behavior). The correct choice of lagged values in reconstructing the phase space is especially important in Financial applications. The significance of a proper phase space reconstruction is equivalent to the importance of a correct model specification in econometrics. In both cases the results obtained depend on whether the stochastic model or the reconstructed phase space is a correct representation of the true underlying process.

The effect of the time delay choice, in reconstructing the phase space for any given embedding dimension can be articulated by examining a periodic system. Assume that the time delay chosen is equal to the period of the system. Then a two-dimensional reconstructed phase space would be a single point. It is thus evident that the time delay should be chosen to make the coupled delayed values as independent as possible. Such coupling will result in a "clearer" image of the true relations in the system. A naive choice of τ could be the value at which the autocorrelation function first passes through zero. This, however, would be equivalent to requiring linear independence among the coupled delayed values. In Finance there exists some credible evidence (Hinich and Patterson (1985, 1988) that Financial time series are not Gaussian and thus, a more general measure of dependence is required. The mutual information criterion (or

¹ The term "chaologists" was used in an article of the Business Week magazine (November 2, 1992), to address those who have come to the kingdom of disorder, Wall Street, to explain the disorderly forces of the financial markets.

redundancy criterion) is employed in this work, because it captures non linear dependencies in time series. This criterion is superior to minimizing the autocorrelation function if the series examined are not Gaussian. If they are Gaussian, both methods provide the same results.

Early suggestions for choosing the time delay and simultaneously reducing noise by Broomhead and King (1986) have proved unsuccessful in general. The authors propose a sophisticated linear analysis of time series called the *singular system approach*. It is based on the Karhunen-Loeve theorem in information theory in Gallager (1968). The basic idea is that the mean-squared distance between points on the reconstructed attractor should be maximized. The procedure proposed has a built-in filter for reducing noise and the Broomhead and King approach of reconstructing the phase space appears to have a striking advantage. However, Fraser (1989a) and Mees et al. (1987) present evidence against this approach, based on the fact that the notion of independence underlying the singular approach procedure is only that of linear independence. Thus, if the densities of the observable are non-Gaussian, the singular approach procedure is not appropriate.

Fraser and Swinney (1986) proposed an approach that is based on general dependence, by choosing the time delay that produces the first local minimum of mutual information. Mutual information in a two dimensional setting measures how many bits on average can be predicted about one dimension, given a measurement in the other direction. If one considers scalar measurements versus lagged values (e.g. $(x(t), x(t-\tau))$) then mutual information measures the redundancy of the second axis of delayed values. The minimization is required in order to ensure that subsequent coordinates are as independent as possible and spanning the m -dimensional space to its full extent.

Fraser (1989b) generalized the algorithm from the two dimensional case to an n -dimensional case (embedding), where mutual information measures the number of bits

that are redundant in a vector measurement. The comparison of this method with the singular system approach is based on a distortion function that measures how close the reconstructed orbit in the phase space is to the true one. The minimum redundancy criterion appears superior. If the underlying distribution is Gaussian, then the two methods give the same results.

Various information criteria such as the BIC (Bayesian Information Criterion) or the AIC (Akaike Information Criterion) have also been employed to estimate the autoregression order of the time series and thereafter transforming them into uncorrelated time series by using the residual theorem of Brock (1988). The econometric criticisms of these methods have been discussed extensively (Gallant (1990), Nychka et. al (1991)) and their application indicates that they are very conservative measures.

Overall, the calculation of the mutual information among the delayed values of a single series serves a dual purpose. First, it provides a selection criterion for the proper choice of the time delay in reconstructing the phase space. Second, it offers a test and a measure of non-linear dependence.

Liebert and Schuster (1989) have shown, however, that the time delays at which mutual information $I_d(\tau)$ is minimized are the same as the minima of the first-order correlation integral² $C_1^d(\varepsilon, \tau)$. Furthermore, the BDS statistic also offers a non parametric test of non-linearity. The "innovation" and the "purpose" of using mutual information values instead of the BDS statistic and the first-order correlation integral behavior, is two-fold. First, it provides in addition to the above a measure of general dependence the same way the autocorrelation function does for linear dependence. Second, both objectives (i.e. optimal time delay and test of dependence) are accomplished

² Recall that $C_1^d(\varepsilon, \tau)$ is obtained by calculating the logarithm of the number of points in each neighborhood of every point and then averaging. It is the basis for the calculation of the information dimension.

through the use of only one procedure and most importantly, this method is general enough to account for multivariate cases.

The Redundancy criterion

Shannon entropy has been used in applications of information theory in Finance and Economics. It has been shown to serve as a more dynamic and general measure of uncertainty than variance, in a Mean-Variance like framework (Philippatos and Wilson (1972)). Entropy has also been used in decomposing capital market risk into systematic and unsystematic components (Philippatos and Wilson (1974)), and in testing the martingale hypothesis with filter-type security-trading models (Martell and Philippatos (1974)).

Entropy is defined as the expected information gain by the occurrence of an event. If the set of phenomena under consideration can be partitioned into non-overlapping subsets and each subset can be assigned a probability, then entropy is defined as:

$$H(S) = -\sum_s p_s * \log p_s \quad (7.1)$$

Mutual information in a two dimensional case is defined as:

$$I(Q, S) = H(Q) + H(S) - H(Q, S) \quad (7.2a)$$

$$H(S) = -\sum_s p_s * \log p_s \quad (7.2b)$$

$$H(Q, S) = -\sum_{sq} p_{sq} * \log p_{sq} \quad (7.2c)$$

It quantifies the average bits of information that can be predicted about q , given a measurement of s . A generalization for an n -dimensional case, measures the number of bits that are redundant in a vector measurement:

$$I(X_0, X_1, \dots, X_n) = \sum_i H(X_i) - H(X_0, X_1, \dots, X_n) \quad (7.3)$$

The intuition of the selection criterion can be best understood if one evaluates the redundancy measure at two extreme cases. At a zero redundancy value, knowledge of some components of a vector cannot be used to predict anything about the remaining components therefore, implying total independence and Market Efficiency as reflected in the random walk hypothesis. Any positive value indicates some degree of dependence. In summary:

- (a) If x determines y exactly (i.e. $y = f(x)$), then $I(x, y) = \infty$.
- (b) If the joint density probability distribution of two variables is such that $P_{x,y} = P_x * P_y$ (i.e. independent), then $I(x, y) = 0$.

At each embedding dimension d and time delay τ the redundancy measure calculated $I_d(\tau)$, quantifies the degree of predictability. Thus, higher dimensions that obviously have higher informational content, should show greater redundancy values at each time delay. *Ceteris paribus*, variables that are more independent, show smaller values of mutual information.

Fraser (1989a) also showed that if the underlying distribution is Gaussian, then the two methods of selecting the time delay (singular approach, redundancy approach) are the same. Mutual information (I) has an exact relation with the autocorrelation function (C), the difference is that the former can detect nonlinear correlations:

$$I(X,Y) = -\frac{1}{2} \log(1 - C^2(X,Y)) \quad (7.4)$$

The algorithm

The empirical procedure for calculating the mutual information values requires the estimation of certain probability density functions (equation (7.2)). If a box in the (s,q) plane of size $(\Delta s, \Delta q)$ has N_{sq} points in it, the joint probability P_{sq} is estimated to be equal to $N_{sq}/N_{total} \Delta s \Delta q$ uniformly across the box.

The first step in the algorithm is to change the variables (s,q) to equiprobable coordinates as follows:

$$u(s) = \int_{-\infty}^s p_s(t) dt \quad (7.5a)$$

$$v(q) = \int_{-\infty}^q p_q(t) dt \quad (7.5b)$$

A sequence of partitions in the (u, v) space is defined such that each partition $G_0, G_1, G_2, \dots$ is a rectangular grid of 4^m elements generated by dividing each axis into 2^m equiprobable segments. In other words, to obtain G_1 from G_0 , the latter is divided into four subelements $R_1(0), R_1(1), R_1(2), R_1(3)$. To get G_2 , each element of G_1 is divided into another four parts $\{R_2(k_1, k_2): k_1, k_2 \in \{0, 1, 2, 3\}\}$. A χ -square test is used to check for substructure in each $R_m(K_m)$ element of partition which contains $N(R_m(K_m))$ number of elements:

$$\chi_3^2 = \left[\frac{16}{9} (1/N) \sum_{i=0}^3 (a_i - N/4)^2 \right] < 1.547 \quad (7.6a)$$

$$\chi_{15}^2 = \left[\frac{256}{225} (1/N) \sum_{i,j=0}^3 (b_{ij} - N/16)^2 \right] < 1.287 \quad (7.6b)$$

$$N \equiv N(R_m(K_m)) \text{ and } a_i \equiv N(R_{m+1}(K_m, i)) \text{ and } b_{ij} \equiv N(R_{m+2}(K_m, i, j)) \quad (7.6c)$$

The null hypothesis is that P_{sq} is flat over $R_m(K_m)$. This implies that the distributions of the number of points in the partition elements $\{R_{m+1}(K_m, i): 0 \leq i \leq 3\}$ and $\{R_{m+2}(K_m, i, j): 0 \leq i \leq 3 \text{ and } 0 \leq j \leq 3\}$ are flat multinomials. If the χ -square test inequalities (7.6a, b) fail to hold, further partitioning is warranted since substructure exists. When P_{sq} is found to be uniform over $R_m(K_m)$ the mutual information is set equal to:

$$I(S, Q) = (1/N_0) * F(R_m(K_m)) - \log(N_0) \quad (7.7a)$$

$$F(R_m(K_m)) = P_{sq}(R_m(K_m)) * \log[P_{sq}(R_m(K_m))] \quad (7.7b)$$

The generalization of the algorithm for more than two variables requires modifying the χ -square test for the flatness to the appropriate number of degrees of freedom.

CHAPTER 8

A NON-LINEAR ANALYSIS OF FINANCIAL TIME SERIES

Overview

The new non-linear analysis techniques discussed in the previous Chapters have only recently received the attention of researchers. Evidently, there are many areas in Finance that have not yet been scrutinized through these new methodologies, such as, fixed income securities and a variety of derivative assets created in Financial markets. Since the area is in its infancy it offers many potential topics for investigation. In this Chapter, however, interest is focused on the issues that can be addressed by analyzing some Financial series with the new techniques. One of the fundamental questions concerns the portfolio scale at which dependencies in prices, rather than of returns, appear more prominent. In addition, whether, volume and price series may be simultaneously modeled as the outcome of a low dimensional system, will be explored.

Data

The time series examined in this study are selected in order to address issues pertaining to non-linear analysis, but are constrained by the availability of data. Domestic and foreign stock markets are analyzed. The structure of returns is contrasted to that of prices. The dynamic and static behaviors of individual stock prices is

contrasted to that of indices. Finally, trading volume series for the individual stocks are analyzed.

(1) Domestic markets: Daily closing prices and trading volume series of the individual stocks composing the DJIA index, are chosen. The data is obtained from the CRSP 1991 tapes and spans the period from July 2, 1962 through December 31, 1991 (#7,421). The composition of the DJIA index has changed over time, resulting in a total of 38 companies that have been continuously listed during this period (See Table 8.1). Due to scarcity of data the following seven companies are not presently examined: Allied Signal, American Express, Anaconda, Swift Co., USX Corp., General Foods, McDonald's. In addition, daily levels for two indices - the S&P500 index and the DJIA index - including all the above stocks, are also analyzed¹.

(2) Foreign Markets: The foreign sector analysis is on weekly observations of ten International Indices of Industrialized Countries and of the World index (See Table 8.2). The data source is Barron's National Business and Financial Weekly Magazine and spans over a period from January 16, 1976 to December 27, 1991 (# 833). All indices are based on local currencies and are value weighted². The World Index is constructed from the ten national indices as a value weighted portfolio based on the dollar value of GNP for each country.

Prices versus returns

The actual (natural) observable of the Financial system is *prices*. All empirical studies examine stock returns which are calculated from daily closing prices, aggregating to daily, weekly and monthly returns. The only other study that uses actual values instead

¹ The DJIA data was obtained from Heizer Software Inc..

² The weights are proportional to their market capitalizations.

Table 8.1 Composition of Dow Jones Industrial Average Index

* Companies that are not analyzed due to the scarcity of data

General Motors	
Primamerica	
Navistar Int.	
Exxon	
Texaco	
Westinghouse	
Woolworth	
AT&T	
Merck	Replaced Swift Co.* on 6/29/1979
Philip Morris	
Alcoa	
Coca Cola	Replaced Owens Illinois on 3/12/1987
Sears Roebuck	
Procter & Gamble	
International Paper	
Goodyear	
Owens Illinois	Replaced by Coca cola on 3/12/1987
American Tobacco	Replaced by McDonald's* on 10/30/1985
General Electric	
Minnesota M&M	Replaced Anaconda* on 8/9/1976
Boeing	Replaced Int. Nickel on 3/12/1987
Bethlehem	
Chevron	
Chrysler	Replaced by IBM on 6/29/1979
International Nickel	Replaced by Boeing on 3/12/1987
Johns Manville	Replaced by Amer. Express* on 8/30/1982
United Aircraft	
Union Carbide	
Eastman Kodak	
DuPont	
IBM	Replaced Chrysler on 6/29/1979

Table 8.2 International Stock Indices

<u>Country</u>	<u>Index</u>
Australia	All Ordinaries
Belgium	Stock Index
Canada	TSE 300
France	Agefi
Germany	Commerzbank
Italy	M.I.N.
Japan	Topix
Switzerland	Swiss Bank Corporation
United Kingdom	Financial Times Stock Exchange 100
United States	Standard & Poor 500

Source: B. Solnik (1988), p. 113.

of rates of change is that of Peters (1991a), in which an attractor is recovered for monthly price levels of the S&P500³. The estimated correlation dimension of the attractor for the monthly index levels of the S&P500 is found approximately equal to 2.33. No type of statistic is used in this study (as the BDS or any bootstrapping technique). The GP plots and the phase plots support clearly the presence of non-linear structure. Furthermore, the empirical findings of Hinich and Patterson (1985, 1988) show that nonlinearities are masked when daily returns are calculated from filtered closing prices versus daily returns calculated from filtered trade-to-trade prices.

³ The data spans the period from 1950 to 1989 and is detrended by using a Consumer Price Index.

In Finance, returns have received all the attention in empirical applications. Returns have been used because they appear to “fit” known distributions (e.g. Gaussian) and thus can be analyzed easily through standard statistical theory⁴. Economic modelers “rediscovered” in the 60’s the Brownian model (the most common distributional assumption in Finance) in books of physics and developed the Efficient Market Hypothesis framework to accompany it. Ever since, academicians try to test empirically the above setting. The results are contradictory and the arguments are perpetuated.

A simple argument against that of Brownian motion was first developed by Mandelbrot (1967). The idea was based on the observation that prices are *not continuous*, as are the sample functions of Brownian motion. Furthermore, they differ from other physical observables in that price formation involves both the knowledge of the present and anticipation of the future. The *anticipatory nature* of the system results in prices that can wander anywhere. Mandelbrot (1982) states that: “without institutional constraint and inertia to complicate matters, a price determined on the basis of anticipation can crash to zero or soar out of sight; it can do anything.” These ideas led to the development of a family of “stable” distributions called the Levy distributions⁵. However, in the 1970’s Gaussian distribution assumptions were again favored over Levy distributions and over Mandelbrot’s (1967, 1971) principle of a scaling structure in price changes.

Returns, independent increments, finite variances and Efficient Markets, remain on the “empirical stage”. Since the framework is stochastic, any conclusions concerning the properties of the distribution of returns can be transformed to the implied price distribution. For example, the assumption of normal returns implies lognormal prices. In

⁴ In fact the first scholar to describe the Brownian motion process of independent Gaussian increments was not a physicist. Louis Bachelier, whose 1900 Ph.D., was granted reluctantly by a committee of mathematicians first postulated that price changes do follow a Brownian motion.

⁵ The logarithmic transformation of their characteristic functions is the well known “stable Paretian” distribution. The Gaussian distribution is only one member of this family.

a stochastic framework there exists a one-to-one correspondence between price and return distributional assumptions. The latter validates the theoretical examination of return distributions instead of price distributions, but may not be valid from an empirical perspective. Returns series may fit a specific distribution with a given degree of accuracy, but price series may fit the implied price distribution with a substantially lower degree of accuracy.

In a chaotic framework, the stochastic theoretical correspondence may not hold. If one assumes that prices do follow a specific chaotic process, then the transformation of prices to return series may not preserve the functional form of the chaotic map, and vice versa⁶. Prices are the actual outcome produced from the system. Prices seem to exhibit **the complexity, the irregularities, the discontinuities, the sudden ups and downs**, that chaos theory attempts to describe. Deceptively simple mathematical equations of non-linear dynamic systems exist in chaos theory that seem capable of accounting for the above. In a chaotic framework it appears more appropriate to examine prices first.

A preliminary check on the empirical validity of the above ideas is performed by analyzing the International market data based on the methodological approach proposed by Willey (1992). The BDS statistic - which measures non-linear departures from the IID assumption - of the foreign index levels is compared to that of their rates of changes. Furthermore, the correlation integral behavior of raw prices series and returns is compared to detrended counterparts. The Residual Test of Brock (1988) guarantees that

⁶ If for example we assume that stock prices can be described from the logistic equation:

$$P_t = \mu * P_{t-1} * (1 - P_{t-1}) \quad (7.8)$$

then returns $R_t = \frac{P_t - P_{t-1}}{P_{t-1}}$ are not described from the logistic equation.

chaotic series have the same correlation dimension as the residuals obtained from a linear regression.

As a second step of the empirical analysis, the raw prices of the individual stocks from the DJIA index, are analyzed. All previous studies have focused on rejecting specific stochastic return models by presenting evidence of "extra" structure. The interest in this Thesis is not focused on detecting additional forecastable structure in the residuals of stochastic models. The purpose is to model the observed complexity and randomness in stock prices as if chaos were the only appropriate tool at our disposal. For this purpose, the DJIA stocks and the two domestic indices are studied. Mutual information values and correlation dimension estimates are calculated.

The implications of finding high correlation dimension estimates for a given holding period and portfolio aggregation level are not fatal for a chaotic approach to randomness. One should reject the existence of low dimensional structure at all possible time scales, in order to conclude that from an empirical perspective a chaotic regime is not a "better" modeling approach and hence, a stochastic model is appropriate. For example, the findings of Scheinkman and LeBaron (1989) support the existence of chaotic structure for weekly holding periods of indices, but not for individual stocks. In effect, weekly individual stock returns may be modeled by linear stochastic specifications, but not indices.

Low correlation dimension estimates of the individual stock price series would imply that daily stock prices may be modeled as chaotic. The implications of the existence of low dimensional structure in stock prices (at this level) have been emphasized in Chapter 2 and 3. Short-term predictability can be substantially improved while long-term forecasting is impossible. Ad hoc non-stationary distributional assumptions can be circumvented. Variability is endogenously produced without any "exogenous" effects.

Prices may fully reflect all information, since sensitivity to initial conditions "transmits" even an imperceptible change in the system. In this fashion Market Efficiency may hold under a chaotic regime. Risk can not be solely measured through the sample variance of the (artificially) implied probability distribution of the chaotic model. In a stochastic setting, variability (measured through variance or any higher moments) is the only source of uncertainty. Variability of conditional distributions is considered as a measure of relative risk. Chaos offers a dynamic framework and thus, risk is associated with the dynamic properties of the system. Entropic measures can capture the uncertainty associated with the diffusion of information. These measures can quantify average systematic and unsystematic risk. Lyapunov exponents measure the rate of information flow in various directions. Since these directions are the dimensions of the system, Lyapunov exponents could be used as measures of the dynamic evolution of sources of risk through time.

Individual stocks versus indices

Traditional Financial theory employs the same distributional assumptions for individual stock and portfolio returns and therefore, for prices at various portfolio aggregation levels. Under this stochastic approach to randomness, *aggregation across assets conserves the structure (order)*. However, Hinich and Patterson (1985,1988) and Hsieh (1991), present evidence of non-linearities for individual stocks and indices, but Brock (1987), Scheinkman-LeBaron (1989) and Barnett-Chen (1986) find no such evidence for individual stocks. The issue that arises is whether aggregation actually changes the structure. In effect, the question is "at which portfolio aggregation level is it more likely to detect stronger non-linearities?".

In order to investigate this issue, the static and dynamic behavior of the individual securities in the DJIA and the index itself, are compared. The S&P500 index is also studied, since it also contains the above stocks and has been one the most frequently examined indices. Finally, the correlation dimension behavior of the international indices is compared to that of the domestic U.S.A. indices, cited above.

In summary, the correlation dimension estimates of individual components are compared to those of portfolios (indices) that include them. If they are found equal or monotonically related to those of portfolios, then the structure in daily stock prices is either identical across portfolios or scales according to the number of individual components. If the correlation dimension estimates of the individual stocks are not monotonically related to those of portfolios, then structure is not preserved. This would imply that *the number of stocks in a portfolio or the level of diversification are not linear factors in determining the structure of daily stock returns across portfolios*. Aggregation across assets does not preserve the structure in prices. Non-linearity is also present in portfolio formation. Dynamic asset pricing models should include the number of securities and the level of diversification. In stochastic terminology, different distributional assumptions should be employed depending on the level of diversification and the number of individual components.

Trading volume and prices

Another natural observable of our system is trading volume. The importance of volume has been emphasized more frequently in the real world (e.g. by analysts), than in the academic profession. At the micro-level it has been shown (Wood et al. (1985)) that trading volume and price volatility follow the same pattern near the opening and the close. Wood et al. (1985) have documented that returns tend to be higher at the opening

of the exchange, fall over the first 20-30 minutes, look unpredictable in the middle, and tend to rise in the last 20-30 minutes to the close. These patterns appear to be statistically significant. If the trading day is considered as the time length of the decision making process of investors, then the above findings translate to *mixing patterns of periodicity with chaos at the transactions level* (for prices and volume).

The trading volume of a security can be interpreted as a measure of the existing disagreement in the beliefs of investors concerning its' value. The higher the disagreement among investors, the higher the trading volume, *ceteris paribus*. Trading volume may also be viewed as a measure of the departure from true and fully revealing equilibrium and as a general measure of uncertainty. The higher the uncertainty within the economy, the higher the trading volume. Economically stable periods are associated with lower trading volume averages than unstable ones, where expectations are not subject to frequent, large changes in magnitude.

The informational content of volume and its value within an EMH framework is an issue that has not been resolved to date. The only conjecture that can be stated is that the absence of a volume reaction to the release of new information implies that all investors have interpreted the information in some homogeneous fashion, but not vice versa. Verrecchia (1981), has shown that the presence of a volume reaction does not necessarily imply that the information is interpreted differently by investors. Beaver (1968), views volume as reflecting changes in expectations of individual investors.

Low correlation dimension estimates for daily trading volume of individual stocks imply structure in the observable. Such property leads to potential exploitation of the informational content of volume data for forecasting. Furthermore, it implies that conditional decision making of investors should take into account available volume information.

High correlation dimension estimates for daily trading volume of individual stocks would imply that the complex structure of the observable cannot be modeled under a chaotic scenario. The rare use of volume series in empirical testing of the EMH appears to be in agreement with the purely random behavior implied from the high dimensional estimates.

The most important conclusions can be drawn by comparing the correlation dimension values of volume series and price series for each security. Although *prices and volume are the two most important contemporaneous outputs of trading activity*, traditional Financial theory does not offer a model of simultaneous stock price and volume determination. Such models have been developed for price and volume quantities of real assets (Mosekilde et al. (1991)) but have not been extended to financial markets. However, there exists considerable work investigating the relationship of the two observables in financial markets.

Most authors claim that the degree of volume reaction should be looked at in conjunction with the degree of price change, in order to access the extent to which investors revise their expectations, given a set of information. Beaver (1968), views changes in prices as reflecting changes in expectations of the market as a whole, while volume as reflecting changes in expectations of individual investors.

Various models have been developed in the existing literature that result in contradictory predictions concerning the relationship of volume and prices:

- (a) Tauchen and Pitts (1983) claim that no relationship exists between price-change magnitude and volume.
- (b) Copeland (1976) postulates that a correlation exists between volume and absolute price changes.
- (c) Harris (1984) suggests that a correlation between volume and price changes per se exists.

(d) Epps (1975) concludes that higher volume is asymmetrically associated with price increases rather than with price decreases .

Empirical evidence obtained through standard statistical procedures is conflicting with respect to the above hypotheses (Karpoff (1987)). The results vary depending on the differencing interval employed (i.e. transactions, hourly, daily, weekly, monthly) and on the aggregation level of data used (i.e. individual stocks, indices). The dynamic relationship between price changes and volume depending (non-linearly) on the aggregation level is in agreement with our quest to investigate scaling properties. Furthermore, the "asymmetric volume-price change hypothesis" proposed by Karpoff (1987) justifies the null hypothesis from a stochastic perspective.

Karpoff (1987), proposes a non-monotonic relationship of volume and price change, which is consistent with a relationship of volume and absolute price change that is not a one-to-one function (the "asymmetric volume-price change hypothesis"). The model predicts that all the theoretical hypotheses (b, c, d) about the relationship of the variables are true, except for the no relationship conjecture. The whole idea is based on a non-linear relationship among these variables, which appears to explain most of the inconsistencies in the empirical results. However, it might be the case that this non-linear relationship is not identical across different frequency levels (transactions, daily, etc.). Thus, a dynamic model that relates the two state variables (volume, price change) and their appropriate derivatives appears to be a plausible description of the endogenous evolution of the system.

Low correlation dimension estimates of both series would imply that a plausible modeling approach might be a low dimensional deterministic system with prices and volume as two of the variables of the system. In this case the results of this study can be used in non-linear forecasting techniques. Volume and prices should be used as inputs for local approximation and near neighborhood forecasting techniques.

High correlation dimension estimates of at least one of the series would imply that stock prices and volume series cannot be modeled from a low dimensional system. The importance of the comparison of the correlation dimension estimates of two different series is that this approach circumvents the univariate nature of the correlation dimension technique. The structural similarities or differences that will result from this comparison can be used in a multidimensional setting for non-linear forecasting.

CHAPTER 9

EMPIRICAL RESULTS

Overview

The discussion up to this point has concentrated on new non-linear techniques and their potential in Financial applications. As Freidman has argued, the merit of a theory is not in the realism of its assumptions but in the quality of its predictions, explanations and its simplicity. If one accepts this line of thought, chaos theory becomes a promising candidate since very simple causes may generate complex and random appearing phenomena. However, its ability to explain and forecast (in a qualitative manner) have not been fully explored. The empirical analysis of this dissertation focuses on the potential of the new methodologies to explain above and beyond the traditional statistical techniques, but most importantly to account consistently for the already explained. In this Chapter the extensive results that address the issues discussed in Chapter 8, are presented and interpreted. Finally, the dynamic non-parametric approach currently pursued is contrasted to the traditional econometric analysis and future research directions are proposed.

Mutual Information Calculations

The mutual information values for the daily price series of the individual stocks and of the DJIA index and the S&P500 index, are shown in Appendix A. Table A-1

through A-7, correspond to the mutual information values for embedding dimensions ranging from 2 to 8, respectively. Tables A-8 and A-9 in Appendix A show the values for the two indices examined (i.e. DJIA and S&P500). Shaded values show the first local minimum of the mutual information values of each series and the time delay at which they occur. Tables B-1 through B-7 in Appendix B contain the results for the daily volume series of the individual stocks. All time delays refer to daily intervals and the mutual information values are significant at a 20% level.

Although the purpose of the mutual information analysis was primarily to obtain the "optimal" time delay in reconstructing the phase space by the time delay method, a comparative analysis of these results provides additional information. Recall that mutual information quantifies general dependence in time series. It answers the question, "Given a measurement of $x(t)$, how many bits on the average can be predicted about $x(t')$?" The measure is coordinate independent and most importantly reflects non-linear dependencies¹.

As expected, the mutual information values for each series are an increasing function of the embedding dimension. For example, mutual information values of individual stock price series (Appendix A) increase from approximately 3 to 15 for embedding dimensions 2 to 8, respectively. If the probability densities of the series (prices or volume, individual stocks or indices) were Gaussian, then the mutual information values at all embedding dimensions and time delays should be zero, implying linear independence. Thus, the results generally *reject a Gaussian underlying distribution and in effect the random walk hypothesis*.

However, note that linear independence can coexist with non-linear dependence. It may well be the case that markets are linearly efficient, but not if viewed through a

¹ At the risk of being repetitive, recall that the difference between mutual information and autocorrelation becomes trivial only for Gaussian random variables. Furthermore, the smaller the mutual information value, the "more independent" the delayed vector examined.

non-linear "filter". This would be manifested by uncorrelated series that exhibit positive non-linear dependencies (as measured by the mutual information values). If on the other hand, linear and non-linear dependence coexist in Financial time series a "relative" EMH may still hold. In that case, fast decaying autocorrelation functions will exist consistent with a quick - although not instantaneous - adjustment process to information. The important feature of such a scenario is that the effects of the non-linear component will "distort" the characteristics of the system. Again the random walk and EMH frameworks will not hold in their current form. The non parametric analysis² used, in addition to rejecting independence and normality as expressed from the EMH and the random walk hypotheses, provides additional information.

The dependencies among individual stock price series are similar

For any given embedding dimension the individual stocks exhibit mutual information values of approximately of the same order. For example, for an embedding dimension equal to 2 all individual stock price series exhibit mutual information values around 3.5 and 5.0 for time delays equal to one day and two days, respectively (See detailed results in Appendix A)³. The apparent consistency supports a uniform underlying structure with respect to feedback effects across all individual securities. Table 9.1 shows the average mutual information values and their dispersion across the 31 securities, grouped by embedding dimension and time delay (measured in days), providing a summary of the lengthy results.

² Since the analysis does not detect the form of dependencies.

³ Recall that mutual information for an embedding dimension d and a time delay τ measures the general dependence among the stock price at time t , the stock price at time $t-\tau$, the stock price at time $t-2\tau$,..., the stock price at time $t-d\tau$.

The increase in the mutual information values by embedding dimension, saturates at an embedding dimension close to 8, again consistently across all individual securities. In other words, higher dimensional phase reconstruction will not provide substantially more information. The increase in the information generation rate of the time series is relatively small at embedding dimensions higher than 8. The latter dictates that the tradeoff between increasing the embedding dimension and the bias induced in the correlation dimension estimation, will be large if one examines embedding dimensions as high as 20^4 . The saturating embedding dimension also serves as a rough bound to the dimensionality of the system, which once again appears to be identical across all individual securities.

Table 9.1 Mean and Standard deviation of Mutual information values of individual stock price series at all embedding dimensions and time delays

	Mean	Standard deviation	Mean	Standard deviation
	$\tau = 1$		$\tau = 2$	
d = 2	3.76	0.54	5.20	0.42
d = 3	8.07	0.67	9.01	0.91
d = 4	10.84	0.86	11.86	1.12
d = 5	12.46	1.12	13.29	1.26
d = 6	13.86	1.26	14.19	1.30
d = 7	14.32	1.41	14.48	1.34
d = 8	14.89	1.55	14.56	1.41

⁴ This is a usual practice in some previous empirical studies (Frank and Stengos (1988,1989)).

The time delay at which the first local minimum occurs is fairly consistent across all securities. For example, for an embedding dimension equal to two, the first local minimum of the mutual information values occurs at a time delay of one day for 30 out of 31 companies. Furthermore, the consistency holds across all embedding dimensions. For example, all individual stock price series exhibit the first local minimum of mutual information values at a time delay equal to one for embedding dimensions ranging from two to eight. Table 9.2 summarizes the "frequency distribution" of the optimal time delay (corresponding to the first local minimum of mutual information) across the various individual securities. For all embedding dimensions less than eight the "prevailing" optimal time delay is one. For $d = 8$ it is two.

Table 9.2 Proportion of individual companies with first local minimum at time delay τ and embedding dimension d

	$\tau = 1$	$\tau = 2$	$\tau = 3$	$\tau = 4$
$d = 2$	30 / 31	1 / 31	0	0
$d = 3$	31 / 31	0	0	0
$d = 4$	28 / 31	1 / 31	0	2 / 31
$d = 5$	28 / 31	2 / 31	0	1 / 31
$d = 6$	26 / 31	0	1 / 31	4 / 31
$d = 7$	20 / 31	8 / 31	1 / 31	2 / 31
$d = 8$	3 / 31	25 / 31	0	3 / 31

The important implication of finding first local minima of the mutual information at consistently small time delays (i.e. one or two) across all securities, is related to the EMH and the reaction of rational investors to the arrival of information. The EMH states that information is immediately discounted in prices, thus ruling out any lagged effects. The existence of fat tails and leptokurtotic distributions has been explained as coming from the infrequent arrival of information (or the arrival of information at lumps). But another plausible explanation is that the fat tails are due to the reaction to information in a discontinuous fashion. As Peters (1991) explains, investors may ignore information until trends are well in place and then react in a lagged - cumulative fashion to information previously ignored. Fat tails would be justified and people would react to information in a non-linear fashion⁵. This type of explanation would be consistent with a nonlinear form of lagged dependencies and would allow for long term correlations. Under such a scenario, small delay values will appear less dependent (since reaction to information has not yet occurred) than subsequent lagged values (at which the cumulative reaction occurs). The above empirical findings are consistent with the presence of feedback effects. Most importantly, daily individual stock prices do not seem to exhibit a decaying memory (as in the submartingale form).

Finally, a summary of the similarities among the DJIA individual stocks and a comparison of the results to those of previous studies leads to the following conclusions:

- (1) A rough estimate of the dimensionality of the underlying process of individual daily stock prices, is equal to eight across all securities (the saturating embedding dimension). The importance of the uniformity can be understood if one views, the above analysis as the non-linear analog of an APT framework and of factor analysis. The conclusion is that the exact same number of factors are "priced" across all the price series of individual securities.

⁵ Substantial theoretical justification of the presence of feedback effects was also discussed in Chapter 3.

(2) The magnitude of the dependencies at various embedding dimensions and time delays are approximately of the same order and the form of the feedback effects (the uniform small time delay at which minimum occurs) are similar across securities. This leads to the conclusion that the non-linearities detected are not due to firm specific characteristics or that at least these effects do not dominate. If the source of the dependencies were "idiosyncratic" features, then the dependencies would be expected to vary across different industries, firm sizes, etc..

The above conclusions are consistent with a scenario in which non-systematic factors (i.e. unique to the particular company) average out in a linear fashion, but aggregation of systematic components is non-linear. A plausible explanation is that reaction to firm specific information is immediate, but for "global" news the reaction occurs in a discontinuous lagged fashion (i.e. investors await until the trends are well in place). The latter would be consistent with a restricted Efficient Market framework. Under such a scenario, markets would be efficient conditional on information sets that are firm specific, but would be inefficient with respect to market-wide news.

(3) Return series of individual stocks show no evidence of additional structure when examined through the BDS statistic and the bootstrapping methods of the correlation integral (Brock (1987), Scheinkman and LeBaron (1989)) at the weekly and monthly aggregation levels. This conclusion is based on studying only two individual stocks - namely, Abott Laboratories and Kodak. Barnett and Chen (1986) do not even examine the daily returns of IBM, simply because the reconstructed phase plot and the autocorrelation function indicate no structure. A plausible explanation for the Barnett and Chen findings is that the arbitrary time delay of 20 lags used in the two-dimensional phase plot, is the cause of the observed independence. In fact, our results suggest that the appropriate time delay is equal to one day for most embedding dimensions. Furthermore, although the autocorrelation function resembles a delta function (i.e. white noise) this

does not preclude the presence of non-linear dependencies in the series. Daily individual stocks may well be uncorrelated but the mutual information values calculated for the 31 stocks indicate that they are not realizations of independent random variables. These findings are consistent with the results from bispectral-based tests of Hinich and Patterson (1985, 1988) that present evidence of non-linear dependencies in the third order cumulants of the probability distribution⁶.

The dependencies among the individual stock price series and the two index levels series are different.

Individual stock price series and the two index level series (which include all the individual stocks) do not exhibit the profound similarities detected within the group of individual securities. The mutual information values of all individual daily stock price series are substantially higher than those of the S&P500 index and slightly higher than the DJIA index (See appendix A for details). Since the S&P500 shows the highest differences, it appears that it exhibits "more independence" than the blue chips.

The feedback effects (i.e. significant lagged dependencies) reflected in daily individual stock prices do not appear to be additive. Additivity is a property that is consistent with a linear world in which the sum of Gaussian distributions is also Gaussian. One can thus, conclude that the above analysis warrants for different distributional assumptions for individual stocks and indices. In empirical research, models that reflect higher dependencies for individual stocks than for indices should be employed. Table 9.3 summarizes the extensive calculations, by presenting the difference between the two indices and between each index and the average individual stock values:

⁶ ARCH and GARCH models imply by construction zero third order cumulants.

Table 9.3 Differences in the mutual information values of the two domestic indices and the average values of the individual stocks

	Average individual stocks vs. DJIA		Average individual stocks vs. S&P500		DJIA vs. S&P500	
	$\tau = 1$	$\tau = 2$	$\tau = 1$	$\tau = 2$	$\tau = 1$	$\tau = 2$
d = 2	0.86	1.34	1.57	3.55	0.71	2.21
d = 3	2.07	2.36	4.43	5.83	2.36	3.47
d = 4	2.66	3.58	6.31	7.45	3.65	3.87
d = 5	3.01	3.78	7.31	8.22	4.30	4.35
d = 6	2.58	3.09	8.46	8.99	5.88	5.90
d = 7	1.60	1.72	8.80	9.24	7.20	7.52
d = 8	0.75	0.56	9.23	8.95	8.48	8.39

Furthermore, the analysis supports another difference in the dynamic behavior of price series across various portfolio aggregation levels. The S&P500 exhibits a saturation of information production at a substantially lower embedding dimension (5) than the individual stocks (8). The latter supports a system of lower dimensionality for modeling indices. In a traditional APT framework⁷, the S&P500 dynamic behavior would support five priced "risk-factors" and eight for the individual securities. Overall, it appears that aggregation across securities does not preserve the order.

The differences in general dependencies of the two indices and the individual stocks along with the similarities among the individual stocks are best shown graphically (Figure 9.1).

⁷ Only a non-linear version of APT could accommodate such dependencies.

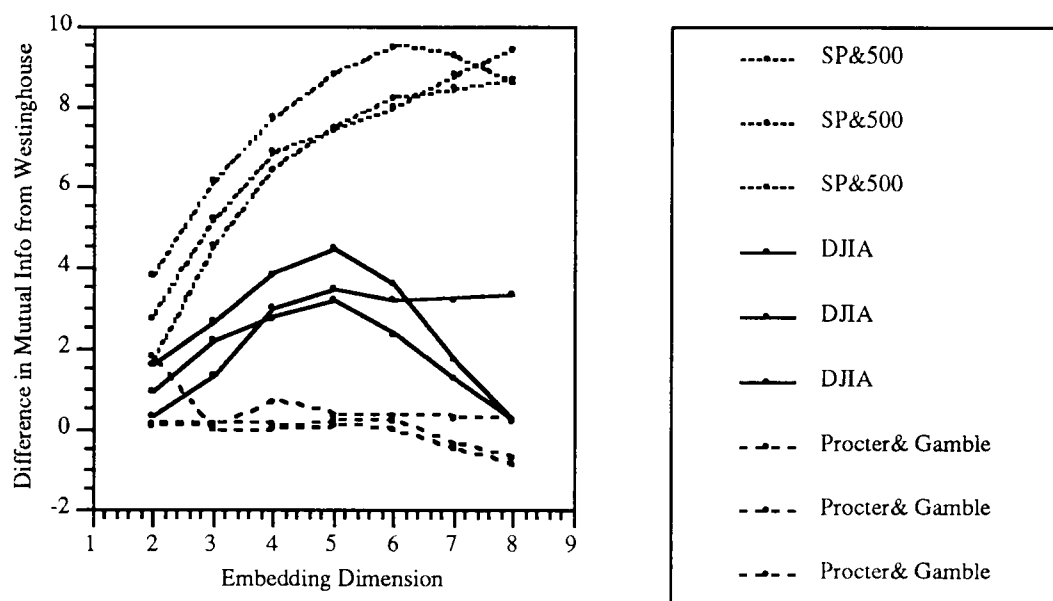


Figure 9.1 Differences in Mutual information values of Westinghouse from Procter & Gamble, DJIA, S&P500 by embedding dimension, for time delays from 1 to 3.

The differences are not as profound for the DJIA index as for the S&P500. In fact Table 9.3 indicates that the differences in mutual information values between the two indices are closer to those of the S&P500 with the individual stocks. In summary, the DJIA index appears to exhibit slightly more independence than its components, but the dimensionally (saturation of information production) is the same as the individual stocks. The results are consistent with the notion of diversification, which is more prominent in the S&P500 index than the DJIA.

Overall, the comparison of the two indices and the individual stock price behavior supports the following differences:

- (1) The dimensionality of the underlying system for the S&P500 is lower than that of the individual stocks, but not of the DJIA index.

- (2) The magnitude of the dependencies among indices and individual stocks differ, but more profoundly for the S&P500.

The dependencies among daily volume series are similar.

For any given embedding dimension, the trading volume series for the individual stocks exhibit mutual information values of approximately the same order (See Appendix B for details). For example, for an embedding dimension equal to two, all trading volume series of the individual stocks exhibit mutual information values approximately equal to 4. These values increase consistently by embedding dimension up to mutual information values approximately equal to 23 for an embedding dimension equal to eight. Table 9.4 presents a summary of the extensive results by embedding dimension and time delay averaged across all individual securities.

As was the case of individual stock price series, a reasonable saturation of the information generation occurs also for the trading volume series at an embedding dimension of 8. For example, average mutual information values double (from 4 to 8) for an increase of embedding dimension from 2 to 3 (See Table 9.4). However, the change in average mutual information values from an embedding dimension equal to 7 to an embedding dimension equal to 8 is only approximately 10% (from 21 to 23). The first local minimum of the mutual information values occurs in a fairly consistent fashion at a time delay of two days for all embedding dimensions higher than 2. Its frequency is summarized in the Table 9.5.

The volume series of four stocks did not exhibit a minimum at certain embedding dimensions. United Aircraft, showed decaying mutual information values for every embedding dimension higher than 5; Philip Morris, for embedding dimensions 7 and 8; Owens Illinois and International Nickel, for the embedding dimension equal to 8.

Table 9.4 Mean and Standard deviation of Mutual information values of daily trading volume series at all embedding dimensions and time delays

	Mean	Standard deviation	Mean	Standard deviation
	$\tau = 1$		$\tau = 2$	
d = 2	4.03	0.49	4.21	0.35
d = 3	8.04	0.49	7.80	0.53
d = 4	12.35	0.68	11.41	0.87
d = 5	15.65	0.94	14.43	1.09
d = 6	18.29	1.11	16.67	1.31
d = 7	21.11	1.30	18.87	1.65
d = 8	23.53	1.44	21.00	1.79

Table 9.5 Proportion of individual companies with first local minimum of the mutual information for the volume series at time delay τ and embedding dimension d

	$\tau = 1$	$\tau = 2$
d = 2	19 / 31	12 / 31
d = 3	1 / 31	30 / 31
d = 4	0	31 / 31
d = 5	0	30 / 31
d = 6	0	30 / 31
d = 7	0	29 / 31
d = 8	0	27 / 31

Similarities and differences of mutual information values among price and volume series of individual stocks

The analysis of both series indicates an information generation rate that saturates at an embedding dimension of 8. By comparing the magnitude of the dependencies between price series and volume series for each individual stock, it is apparent that volume series exhibit consistently higher mutual information values at all embedding dimensions (See Appendix A and B, for details). The difference between the mutual information values of individual stock price and volume series increases with embedding dimension. For example, the difference between the average values is equal to 0.27 for an embedding dimension equal to 2 and 8.64 for an embedding dimension equal to eight. Table 9.6 shows a summary of the differences between the average mutual information values of volume series and the average mutual information values of the price series.

Table 9.6 Differences in average mutual information values of individual stock volume series and price series

	$\tau = 1$	$\tau = 2$
d = 2	0.27	-0.99
d = 3	-0.03	-1.21
d = 4	1.51	-0.27
d = 5	3.19	1.14
d = 6	4.43	2.48
d = 7	6.79	4.39
d = 8	8.64	6.44

These average values are not very informative, since small deviations among securities exist. However, even on average the increase is apparent. A graphical illustration of the difference in general dependencies between price and volume (on an individual basis) for Westinghouse is shown in Figure 9.2, by embedding dimension for time delays from 1 to 3 days. On an individual basis, the differences increase by embedding dimension, but not by time delay simply because the minima for the two series occur at different delays. Prices, in fact appear to exhibit a different form of feedback effects, than volume series. The latter is manifested in the difference of the time delay, at which the first local minimum occurs for the two series. On average, it is one day for the price series and two days for volume series.

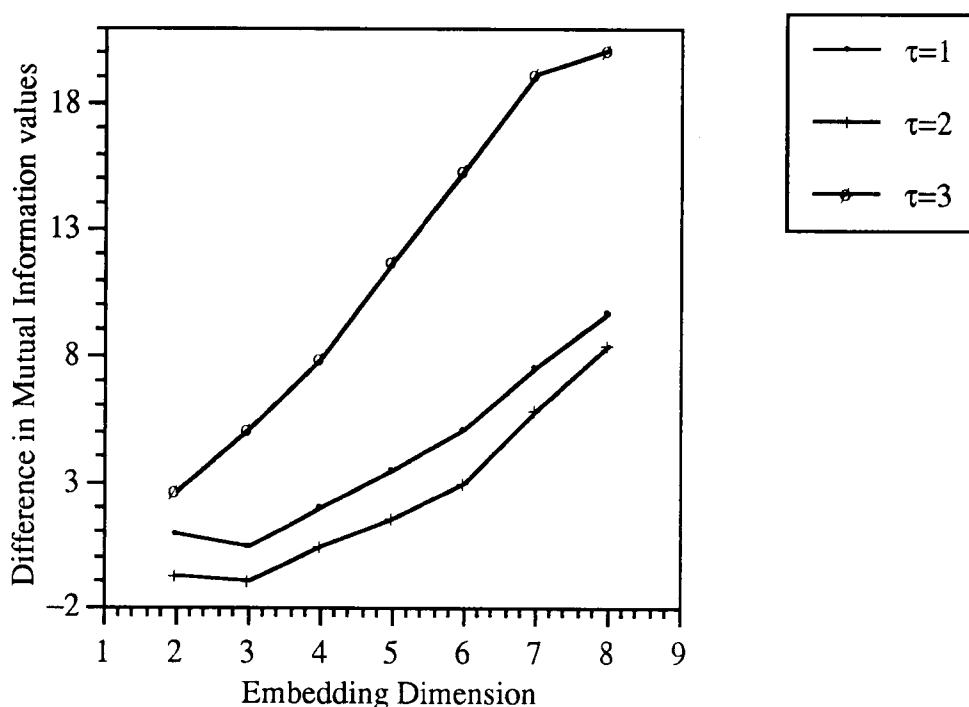


Figure 9.2 Difference in Mutual Information values among volume and price series of Westinghouse by embedding dimension, for three time delays

The latter translates to a "faster" adjustment to information inherent in volume series, relative to that in price series. This is consistent with prices that are more "sticky" relative to quantity. Furthermore, if volume reflects changes in expectations of individual investors and price movements reflect changes in the expectations of the market as a whole, then the former is more frequent than the latter. From this different point of view, again volume series would be expected to exhibit a faster adjustment than prices.

Correlation Dimension Calculations

The static behavior of the various series is examined, through the behavior of the correlation integral at various embedding dimensions and resolutions values. The procedure for the analysis of each time series is as follows:

- (a) The relevant ranges of the resolution parameter ε are determined by trial and error for each series. The ten different values⁸ selected are such that the correlation integral is different than one (i.e. all points included) and zero (i.e. no points included), thus avoiding the effects of noise and finite sample size.
- (b) For a given embedding dimension d the correlation integral is calculated for all ten resolution parameter values. The time delayed vectors at each embedding dimension are reconstructed based on the optimal time delay determined by the redundancy criterion (Appendix A and B). Thus, at different embedding dimensions the "coupling" of delayed values may well vary.
- (c) The embedding dimension values are increased two or three units higher than the saturation level indicated from the mutual information increase rate⁹. All previous studies arbitrarily increase the embedding dimension resulting in a substantial bias of the

⁸ Most studies use five resolution parameter values since the algorithm is very slow due to the large number of calculations.

⁹ The upper bound for all series is equal to eleven and only for the S&P500 it is eight.

estimated correlation dimension (Ramsey and Yuan (1989)). In some cases embedding dimension values as high as 20 for sample sizes of around 1000 observations have been used. The logarithmic plots of the correlation integral values versus the resolution parameter values are constructed for each embedding dimension. The GP plots (Grassberger-Procaccia plots) of all price and volume series analyzed, are in Appendix C and D (Figures C-1 through C-18 and D-1 through D-15), respectively.

(d) Following the procedure of Barnett and Chen (1986), the behavior of the slopes of the GP plots is examined as the resolution parameter changes in order to determine the appropriate linear scaling region (for each embedding dimension). This range of values is used in an OLS regression of the logarithm of the correlation integral values against the logarithm of the resolution parameter values. The slopes of the regressions, namely, correlation dimensions estimates, of all price and volume series analyzed, are in Appendix C and D (Table C-1 and D-1), respectively.

Recall that completely random series exhibit GP plots with curves for each embedding dimension that have slopes equal to that embedding dimension. Completely deterministic series exhibit GP plots with parallel straight lines¹⁰ that have a slope equal to the correlation dimension. Thus, by simply comparing the GP plots one concludes that structure does exist in all series.

Daily prices of individual stocks and daily index levels

Although any differences in the slopes (i.e. correlation integral convergence) between individual stocks and indices may not be evident from simply examining the graphs, one clearly detects a difference in the displacement of the curves among the two

¹⁰ Given that there is no noise present in the series (which would show as "knees" in the straight lines) and also no finite sample size or finite attractor size effects (which would show as saturated lines at very high and very low resolution parameter values).

groups (Compare Figures C-17,C-18 with C-1 through C-16). The GP plots of the individual stocks exhibit larger displacement from each other as the embedding dimension increases. The latter is a rough estimate of metric entropy (Procaccia (1985)) that quantifies predictability. Thus, indices show lower values of metric entropy than individual stock series. This is in complete agreement with their lower mutual information values. Overall, indices appear more predictable than individual stocks. All individual stocks exhibit similar nonlinear dependencies, which are different than those of indices.

Table C-1 displays the slopes at each embedding dimension. All individual stock prices exhibit similar behavior. The convergence of the correlation integral indicates weaker evidence of a low dimensional structure for individual stocks than for indices. The latter is consistent with the difference in the estimate of the degrees of freedom of the two series observed in the mutual information calculations¹¹. For individual stocks the estimates do not converge at a single value, as for the indices, and also the standard error of the estimate increases as the embedding dimension changes.

The S&P500 index has been analyzed through both the correlation integral technique and the Rescaled / Range analysis. Hsieh (1991), reports that weekly, daily and 15-minute returns of the S&P500 show structure (as measured by the BDS statistic) and thus, concludes that the dependencies detected are not due to non-stationarity. Although no correlation dimension estimates are obtained, the dependencies persist at higher frequency data and shorter time periods. Peters (1991a) reports a correlation dimension estimate of 2.33 for monthly levels of the S&P500. Also Peters (1989) finds an average cycle length of 48 months over which past events have an impact on future

¹¹ The geometry of the series described through the behavior of the correlation integral (Packard et al. (1980)) is consistent with the dynamic behavior examined by the mutual information invariant. In fact, the dynamic invariants of a system are related to the static invariants (Kaplan-Yorke (1979)), thus linking stability properties with information diffusion.

monthly levels of the S&P500. The analysis of the daily levels of the S&P500 in this Thesis shows a lower correlation dimension estimate approximately equal to 0.40.

The comparison of the behavior of individual stocks and indices (either through the mutual information values or the correlation integral behavior) suggests major differences. The underlying process for individual stocks, given a daily holding period, is of lower dimensionality than that of indices. Furthermore, the magnitude of the dependencies is attenuated as the portfolio aggregation level increases. Finally, based on the present results and all pervious studies examining indices it appears that neither the dependencies nor the dimensionality are uniform across different indices. For example, the daily levels of the S&P500 and the DJIA exhibit different behavior in this study and the daily returns of the S&P100 and NASDAQ100 also exhibit mixed dynamics (Willey (1992)).

Finally, in relation to the portfolio-scaling issue discussed in Chapter 8, the empirical results of this Thesis suggest that daily individual stock price levels exhibit similar static and dynamic behavior. Portfolios (indices) that include these stocks have two major characteristics: (a) they exhibit different dynamic and static behavior than the individual stocks, and (b) this behavior is not uniform across various indices. Furthermore, the types of dependencies detected from chaotic tools appear more prominent in daily index level series rather than daily individual stock price series.

Daily prices and volume series of the individual stocks

Table C-1 shows the convergence of the correlation integral by embedding dimension for price series. Table D-1 shows the convergence of the correlation integral by embedding dimension for volume series. No statistical test exists for comparing the correlation dimension estimates of two different series. However, if the estimates of two

variables are substantially different, one can reject the hypothesis that these variable can be modeled as the outcome of a low dimensional system. Assuming that it is not only reasonable but also desirable to model price and quantity from the same system, the correlation dimension estimates of the two variables are compared. The magnitude of the correlation integral at every embedding dimension is on average higher for price series than for the corresponding volume series. Furthermore, the convergence behavior of the price series is more consistent. The results cannot reject the possibility that price and volume can be modeled from the same low dimensional system. However, caution is required when using the estimates as a guideline in attempting to construct a price-volume model. Such a model needs information about whether past prices (volumes) help predict current and future volumes (prices) above and beyond the predictive content of past volumes (prices) for current and future volumes (prices). The univariate analysis of mutual information and correlation dimension does not quantify such relations.

Weekly foreign indices

For the analysis of these Financial series the redundancy criterion is not used because of the small sample size (#833). Furthermore, the objective is to examine whether price series are more appropriate than return series for analyzing with the new non-linear techniques (Philippatos and Pilarinu (1993)) and thus, compare the results to previous studies. Correlation integral behavior and BDS statistics are calculated for four different series: (a) The original return series, (b) residuals of an OLS regression on the return series, (c) transformed index level series, (d) residuals of an OLS regression on the index level series. Appendix E (Tables E-1 through E-11) shows the results (i.e. correlation integral values, BDS statistic values) for embedding dimension values form 2 to 10 and for five resolution parameter values ranging from 0.50 to 1.50 multiples of the

standard deviation of the data. The choice of the range of embedding dimension and resolution values is identical to that followed from all previous studies. Thus, the results can be compared to the findings of others for return series of indices.

BDS statistic

Recall that the BDS statistic measures departures from the IID hypothesis. As Brock et al. (1991) has noted: "So one can usefully think of the GP plot as a graphical representation of the BDS test." The BDS statistic is high for all series and all countries (critical value of 1.96 with a 5% significance). These values, however, are very different than those reported by Willey (1992) for two U.S. stock indices, the Standard & Poor's 100 and the NASDAQ 100. Willey finds significantly lower than the critical value of the BDS statistic for both raw and detrended series of the logged first difference values used. Thus the null of an IID process is retained. By comparing our results of the return series we can infer that the difference stems from the aggregation level examined (daily versus weekly). Overall, the BDS statistic leads to the conclusion that non-linearities exist at the weekly time-aggregation level and the index portfolio aggregation level and across all countries for all series examined, but this may not be the case for daily series (Willey (1992)).

Correlation integral values

All indices examined exhibit similar (approximately equal), correlation integral values at each embedding dimension and resolution parameter value. This consistency translates to identical dependencies across all countries and for all series examined. In

other words, the average local spatial correlations, as measured by the correlation integral), are of the same type for all countries.

Values for the raw price series range from 0.3 to 0.6 as the resolution parameter increases. These values remain essentially constant as the embedding dimension increases for each given resolution parameter value. The detrended counterpart exhibits similar behavior, although the values decrease as the embedding dimension increases. Hence we conclude that these series pass the Residual Brock Test¹². Since no formal statistical test exists that enables us to distinguish between two correlation dimension estimates, the GP plots of the raw series and the filtered counterpart can be compared in order to access any differences. Figures 9.3, 9.4 show an example of these plots for the World index¹³:

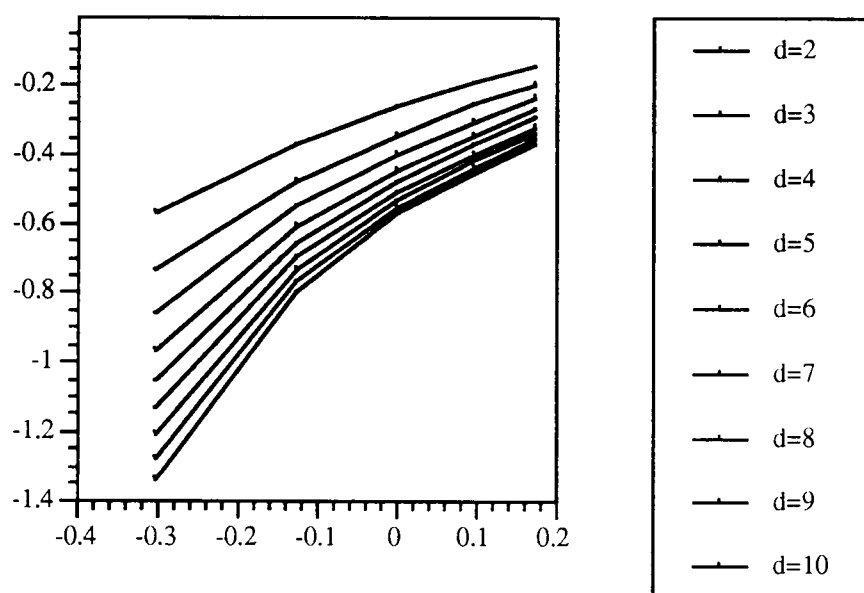


Figure 9.3 GP plots of filtered levels of the World index

¹² Recall that the Residual Test of Brock, states that for chaotic series the correlation dimension of raw series and linearly detrended counterparts are equal.

¹³ The GP plots across all countries are similar.

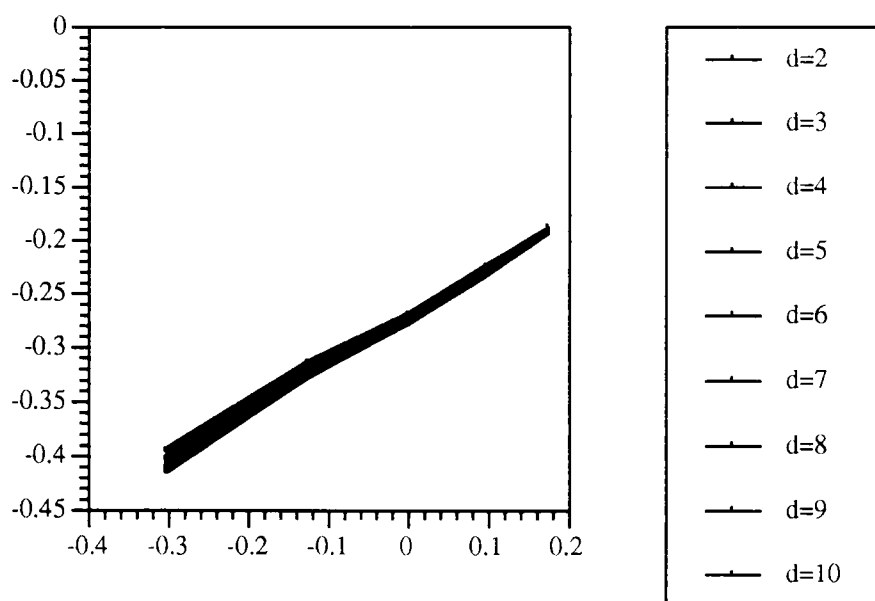


Figure 9.4 GP plots of raw World index levels.

The raw return series lack the convergence present in the raw price levels. However, the return series pass the Brock Residual Test, since their detrended counterpart shows similar behavior. Willey (1992) focuses on U.S. market data and finds that only the S&P100 Index passes this test. Figure 9.5 and 9.6 show the GP plots for the return series of the World Index. The correlation dimension estimates are of the order of 2. The same limiting value found by Willey (1992) for the daily S&P100 and smaller than that found in the Scheinkman and LeBaron (1989) study for the weekly returns on the VW CRSP index.

In summary, the dependencies at a weekly time aggregation level are very similar across all major stock indices. The spatial characteristics of all series examined (prices and returns) are similar. However, price series appear more prominent candidates for investigating further the existence of strange attractors.

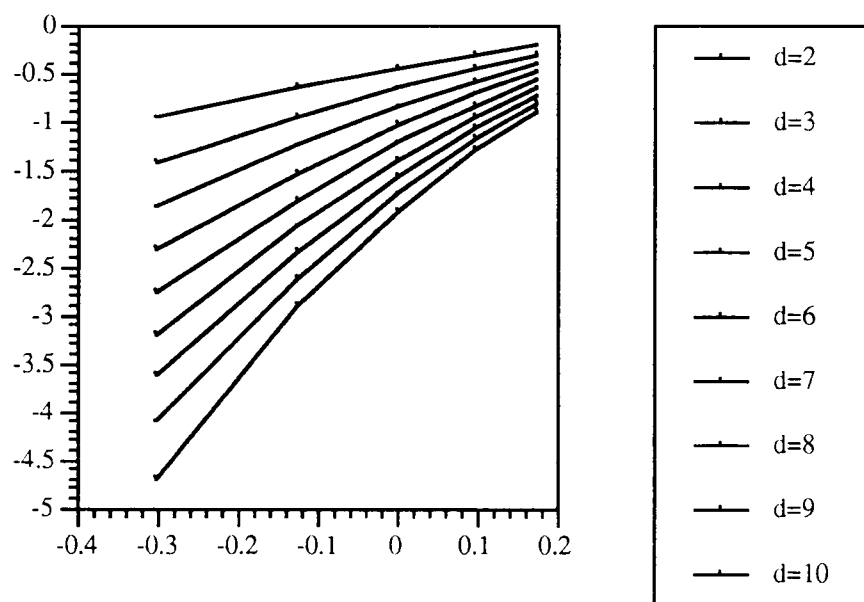


Figure 9.5 GP plots of filtered returns of the World index.

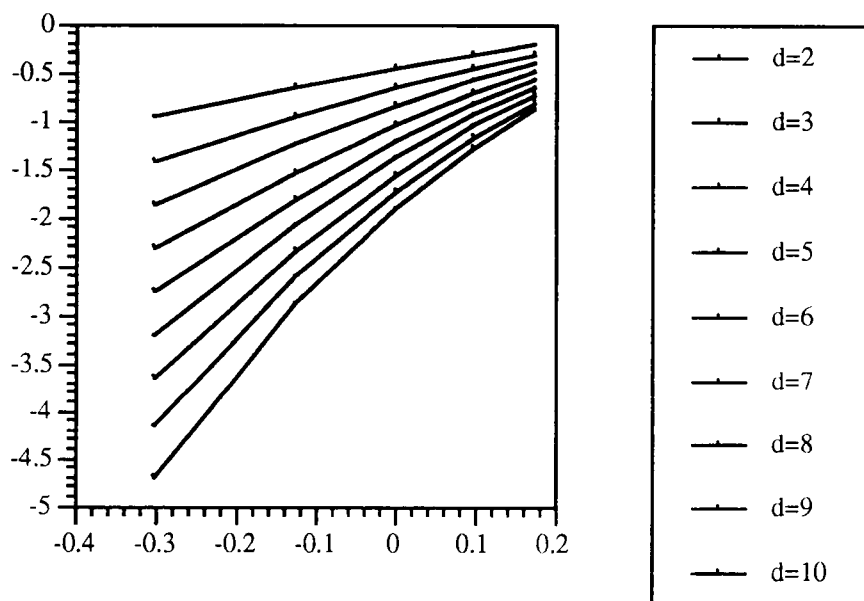


Figure 9.6 GP plots of raw returns of the World index.

Evidence of low dimensional structure as a plausible alternative in modeling complex and random appearing weekly index levels can not be inferred from the above analysis due to the scarcity of the data (which induces a bias in the estimates of the correlation dimensions).

Conclusions

The extensive investigation of Financial series using dynamic techniques borrowed from chaos theory has led to important implications even though there exists no statistical test under the null hypothesis of chaos. The static and dynamic measures based on the underlying structure of chaotic processes are capable of detecting dependencies above and beyond the capabilities of classical statistical techniques. Such kind of non-parametric analysis addresses issues that pertain to the dependencies of the entire probability distribution. Traditional non-linearity tests focus on dependencies in specific moments of the probability distribution (e.g. Tsay test looks for deviations from zero third order cumulants). The difference of the dynamic measures employed in this Thesis lies in their ability to capture dependencies due to structural instabilities and to endogenously produced variability. Recall, that the focus was on comparing individual stocks with indices, and price series with volume series. Note also that the new techniques employed not only avoid the contingency of the results upon the model selected, but also allow for non-equilibrium and structurally unstable underlying processes.

The present work falls in the category of a diagnostic type of analysis, however, it provides the scientist with a basis for future research in developing and testing pricing models consistent with chaotic non-linearities. The implications of the empirical analysis

are not only different from those obtained from a standard Financial framework but most importantly cannot be deduced through traditional econometric techniques.

This Thesis also presents an effort to improve the empirical techniques adapted from chaos theory. Based on the extensive discussion in Chapter 7 the significance of selecting the proper time delay in reconstructing the phase space from a single time series is evident from a theoretical perspective. After all it is only the realization that the properties (static and dynamic) of a multidimensional system can be obtained from measurements of a single observable that enable any application of diagnostic techniques adapted from chaos theory. The empirical results of this Thesis substantiate further the importance of employing an optimal time delay criterion in reconstructing the phase space from an empirical perspective. Evidence of a differential structure in price series that varies with the portfolio aggregation level is presented. All individual stock price series exhibit a uniform first minimum of general dependence at a time delay of one day. Index levels, however, do not follow the same structure. Finally, volume series show a two day minimum time delay suggesting a faster adjustment process for volume series rather than price series.

The novel methodology - namely, redundancy criterion - applied for the analysis of Financial time series should become standard in reconstructing the dynamics of the multidimensional Financial system from a single observable. Evidence of redundancy in the Financial system is not a novel idea but according to the present analysis it appears to vary with the portfolio aggregation level. If in fact these dependencies (redundancies) are a manifestation of non-equilibrium market clearing points and a structurally unstable system, then one may expect non-decaying feedback effects (i.e. memory) and trend-reinforcing behavior in Financial series that is not stable through time.

The empirical findings of this Thesis lead to important conclusions and suggestions for future research. The results suggest that given a daily holding period the

search for chaotic structure is more promising for indices rather than individual stocks. The new non-linear techniques employed also suggest that daily individual stock prices exhibit similar static and dynamic behavior which is different than that of daily index levels. Given the above, different asset pricing models should be developed for various portfolio aggregation levels. The empirical investigation of price versus return series of weekly index levels is inconclusive and further investigation is warranted. However, the analysis again suggests a similarity of non-linearities across all major stock indices for price and return series.

The importance of the implications of the extensive empirical analysis is reinforced since the above results cannot be deduced through standard econometric analysis.

Individual stocks versus indices: A "traditional" approach analogous to the analysis of this Thesis could be pursued by using some stochastic specification (e.g. GARCH) and thereafter comparing whether individual stock returns¹⁴ and indices exhibit a different fit. In effect, differences observed (if any) would relate only to the second moment of the distribution and to the specific functional form of dependencies assumed by GARCH models. The tests currently employed quantify non-parametrically the differences in the entire distribution, thus avoiding the confusion of joint hypotheses inherent in parametric econometric modeling. Even a non-parametric approach (e.g. factor analysis) would be inferior in that it detects only linearly related factors.

Prices versus volume series: A comparison of the dimensionality of price and volume series could be examined by using the traditional non-parametric tool of factor analysis. The disadvantage of such an approach lies in the linearity assumption of factor analysis. Thus, one would be able to detect only those factors that are linearly related to

¹⁴ In order to actually obtain an econometric analog of the present analysis, one should employ the stochastic model for prices implied by a GARCH specification for returns.

the time series examined. Furthermore, the mathematical theory of chaos provides the researcher with a powerful tool - namely, the time delay method - that guarantees replication of the dynamics of a multi-variable system from delayed values of a single observable.

In summary, the importance of general dependencies in the entire probability distribution is stressed in this Thesis through the methods employed. The implications do not only affect asset pricing models per se, but also the behavior of any type of derivative asset that directly depends on some moment of the distribution of the underlying asset (e.g. options).

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APPENDICES

APPENDIX A

**Mutual information values of stock price series
(shaded values are the first local minima)**

Table A-1 Mutual information values of DJIA individual stock price series for an embedding dimension equal to 2

	$\tau = 1$	$\tau = 2$
General Motors	3.66	5.12
Primamerica	3.61	5.66
Navistar Int.	4.07	5.64
Exxon	3.88	5.37
Texaco	3.56	5.09
Westinghouse	3.83	5.44
Woolworth	4.06	5.85
AT&T	4.62	4.40 [†]
Merck	3.67	5.15
Philip Morris	4.14	5.48
Alcoa	3.47	4.41
Coca Cola	4.57	5.53
Sears Roebuck	3.79	5.33
Procter & Gamble	2.01	5.35
International Paper	3.75	5.27
Goodyear	3.87	4.98
Owens Illinois	4.10	5.56
American Tobacco	3.44	5.41

[†] The mutual information value for a time delay equal to 3, is 4.72.

Table A-1 (continued)

	$\tau = 1$	$\tau = 2$
General Electric	3.51	4.81
Minnesota M&M	3.43	4.79
Boeing	3.37	5.09
Bethlehem	3.53	5.57
Chevron	3.37	5.39
Chrysler	3.31	4.68
International Nickel	3.90	5.27
Johns Manville	3.78	5.37
United Aircraft	5.38	6.05
Union Carbide	4.34	5.43
Eastman Kodak	3.68	4.90
DuPont	3.59	4.70
IBM	3.54	4.34

Table A-2 Mutual information values of DJIA individual stock price series for an embedding dimension equal to 3

	$\tau = 1$	$\tau = 2$
General Motors	7.80	5.12
Primamerica	8.01	9.66
Navistar Int.	8.76	9.84
Exxon	8.12	9.32
Texaco	7.57	8.99
Westinghouse	8.18	9.32
Woolworth	8.75	9.65
AT&T	8.68	9.45
Merck	7.79	8.96
Philip Morris	8.51	9.29
Alcoa	7.40	8.80
Coca Cola	9.01	9.55
Sears Roebuck	8.27	9.05
Procter & Gamble	8.20	9.15
International Paper	8.39	9.50
Goodyear	7.89	9.32
Owens Illinois	8.54	9.39
American Tobacco	7.91	9.34

Table A-2 (continued)

	$\tau = 1$	$\tau = 2$
General Electric	7.57	8.46
Minnesota M&M	7.65	8.83
Boeing	7.54	8.82
Bethlehem	7.93	9.58
Chevron	7.73	9.09
Chrysler	7.13	8.76
International Nickel	8.18	8.86
Johns Manville	8.08	9.14
United Aircraft	10.44	10.84
Union Carbide	8.61	9.23
Eastman Kodak	7.20	8.18
DuPont	7.04	7.81
IBM	7.30	8.20

Table A-3 Mutual information values of DJIA individual stock price series for an embedding dimension equal to 4

	$\tau = 1$	$\tau = 2$
General Motors	10.37	11.54
Primamerica	11.18	12.39
Navistar Int.	11.57	12.67
Exxon	10.85	11.84
Texaco	10.10	11.45
Westinghouse	10.94	11.94
Woolworth	11.45	12.53
AT&T	11.28	11.80
Merck	10.40	11.41
Philip Morris	11.30	12.11
Alcoa	10.00	11.19
Coca Cola	12.13	12.54
Sears Roebuck	10.86	11.63
Procter & Gamble	10.92	11.83
International Paper	11.18	12.18
Goodyear	10.69	12.03
Owens Illinois	11.42	12.10
American Tobacco	10.95	11.93

Table A-3 (continued)

	$\tau = 1$	$\tau = 2$
General Electric	10.30	10.89
Minnesota M&M	10.73	11.50
Boeing	9.98	11.12
Bethlehem	10.92	12.17
Chevron	10.41	11.55
Chrysler	9.55	10.94
International Nickel	11.13	11.63
Johns Manville	10.99	11.84
United Aircraft	13.99	14.51
Union Carbide	11.36	13.45
Eastman Kodak	9.61	8.18 [†]
DuPont	9.47	9.36 [‡]
IBM	10.01	9.85 [‡]

[†] Eastman Kodak shows a minimum at a time delay equal to 2. The mutual information value at $\tau = 3$ is 9.18.

[‡] DuPont and IBM show a minimum at a time delay equal to 4. The corresponding mutual information values at time delays 3, 4, and 5, respectively, are 9.34, 8.98, 13.04 for DuPont, and 9.78, 9.54, 13.79 for IBM.

Table A-4 Mutual information values of DJIA individual stock price series for an embedding dimension equal to 5

	$\tau = 1$	$\tau = 2$
General Motors	11.88	13.08
Primamerica	13.28	14.36
Navistar Int.	13.04	14.54
Exxon	12.45	13.53
Texaco	11.55	13.01
Westinghouse	12.65	13.89
Woolworth	12.96	14.23
AT&T	12.45	13.17
Merck	11.88	12.87
Philip Morris	12.97	14.03
Alcoa	11.67	12.81
Coca Cola	13.90	14.52
Sears Roebuck	12.01	13.01
Procter & Gamble	12.51	13.63
International Paper	13.06	13.98
Goodyear	12.36	13.62
Owens Illinois	13.26	13.97
American Tobacco	12.80	13.69

Table A-4 (continued)

	$\tau = 1$	$\tau = 2$
General Electric	11.95	12.47
Minnesota M&M	12.87	13.52
Boeing	11.20	12.38
Bethlehem	13.57	13.78
Chevron	12.12	13.25
Chrysler	10.94	12.36
International Nickel	12.51	13.09
Johns Manville	12.63	13.60
United Aircraft	16.45	16.87
Union Carbide	13.07	13.45
DuPont	10.51	10.15 [†]
IBM	11.37	10.97 [†]
Eastman Kodak	10.49	10.31 [‡]

[†] The mutual information values at a time delay equal to 3 for DuPont and IBM, respectively, are 10.16, 10.99.

[‡] Eastman Kodak shows a minimum at a time delay equal to 4. The mutual information values at time delays 3, 4, 5, respectively, are 10.12, 9.55, 14.82.

Table A-5 Mutual information values of DJIA individual stock price series for an embedding dimension equal to 6

	$\tau = 1$	$\tau = 2$
General Motors	13.19	14.08
Primamerica	14.88	15.57
Navistar Int.	13.95	15.21
Exxon	13.52	14.18
Texaco	12.58	13.71
Westinghouse	13.64	14.68
Woolworth	14.55	15.15
AT&T	13.37	13.77
Merck	12.95	13.67
Philip Morris	14.17	15.04
Alcoa	12.94	13.61
Coca Cola	15.35	15.85
Sears Roebuck	12.82	13.56
Procter & Gamble	13.66	14.45
International Paper	14.41	15.02
Goodyear	13.38	14.41
Owens Illinois	14.58	15.04
American Tobacco	14.10	14.65

Table A-5 (continued)

	$\tau = 1$	$\tau = 2$
General Electric	13.51	13.54
Minnesota M&M	14.47	14.71
Boeing	12.31	12.85
Bethlehem	14.43	14.68
Chevron	13.43	13.99
Chrysler	11.88	12.83
International Nickel	13.63	14.02
Johns Manville	13.87	14.53
United Aircraft	18.23	18.09 [†]
Union Carbide	14.39	14.37 [‡]
Eastman Kodak	11.45	11.13 [‡]
DuPont	11.67	11.34 [‡]
IBM	12.80	12.42 [‡]

[†] United Aircraft shows a minimum at a time delay equal to 3. The mutual information values at time delays 3, 4, respectively, are 15.74, 16.72.

[‡] The minimum occurs at a time delay equal to 4. The mutual information values at time delays 3, 4, 5 are 13.14, 13.,14, 15.88 for Union Carbide, 10.12, 9.55, 14.82 for Eastman Kodak, 11.19, 10.97, 18.53 for DuPont, and 12.24, 12.01, 20.01 for IBM.

Table A-6.1 Mutual information values of DJIA individual stock price series for an embedding dimension equal to 7

	$\tau = 1$	$\tau = 2$
General Motors	13.97	14.08
Primamerica	15.87	16.21
Navistar Int.	14.40	14.95
Exxon	13.94	14.16
Texaco	13.21	13.81
Westinghouse	13.98	14.49
Woolworth	14.55	15.15
AT&T	13.53	13.77
Merck	13.40	13.72
Philip Morris	14.88	15.19
Alcoa	13.71	13.83
Coca Cola	16.19	16.57
Sears Roebuck	12.86	13.34
Procter & Gamble	14.45	14.85
International Paper	15.37	15.66
Goodyear	13.99	14.32
Owens Illinois	15.33	15.50
American Tobacco	14.87	15.11

Table A-6.2 Mutual information values of DJIA individual stock price series for an embedding dimension equal to 7

	$\tau = 1$	$\tau = 2$	$\tau = 3$
General Electric	14.72	14.43	14.54
Minnesota M&M	15.69	15.48	15.55
Boeing	12.92	12.80	13.73
Bethlehem	15.28	15.11	15.29
Chevron	14.20	14.18	14.97
Chrysler	12.36	12.72	14.18
International Nickel	13.99	14.28	
Johns Manville	14.57	14.92	
United Aircraft	19.33	18.58	17.00 [†]
Union Carbide	14.86	14.61	15.39
Eastman Kodak	11.75	11.60	11.85
DuPont	12.31	12.18	12.05 [‡]
IBM	13.69	13.44	13.27 [‡]

[†] The mutual information value at a time delay equal to 4 is 17.35.

[‡] DuPont and IBM show a minimum at a time delay equal to 4. The mutual information values at time delays 4 and 5, respectively, are 11.62, 20.64 for DuPont and 12.37, 22.63 for IBM.

Table A-7 Mutual information values of DJIA individual stock price series for an embedding dimension equal to 8

	$\tau = 1$	$\tau = 2$	$\tau = 3$
General Motors	14.92	14.15	16.70
Primamerica	16.70	16.45	17.11
Navistar Int.	14.83	14.71	17.62
Exxon	14.35	14.13	16.83
Texaco	13.87	13.69	16.05
Westinghouse	14.35	14.21	16.97
Woolworth	15.08	14.90	16.09
AT&T	13.52	13.55	
Merck	13.72	13.39	15.34
Philip Morris	15.59	15.19	17.65
Alcoa	14.38	13.97	15.60
Coca Cola	16.86	16.99	17.03
Sears Roebuck	12.86	12.79	14.34
Procter & Gamble	15.23	14.93	16.74
International Paper	16.12	16.03	17.16
Goodyear	14.56	14.16	16.48
Owens Illinois	15.94	15.76	16.95
American Tobacco	15.57	15.19	15.56

Table A-7 (continued)

	$\tau = 1$	$\tau = 2$	$\tau = 3$
General Electric	15.80	15.42	15.90
Minnesota M&M	16.70	15.98	16.51
Boeing	13.37	12.83	14.96
Bethlehem	15.95	15.25	16.41
Chevron	14.98	14.32	16.29
Chrysler	12.55	12.59	15.47
International Nickel	14.55	14.41	15.97
Johns Manville	15.25	15.04	16.86
United Aircraft	19.95	18.52	17.95 [†]
Union Carbide	15.20	14.72	16.36
Eastman Kodak	11.95	11.59	12.27
DuPont	12.77	12.75	12.56 [‡]
IBM	14.39	14.01	13.95 [‡]

[†] United Aircraft shows a minimum at a time delay equal to 5. The mutual information values at time delays 4, 5, 6, respectively are 17.57, 16.21, 17.60.

[‡] DuPont and IBM show a minimum at a time delay equal to 4. The mutual information values at time delays 4, 5, respectively, are 12.13, 22.66 for DuPont and 13.44, 25.05 for IBM.

Table A-8 Mutual Information values of DJIA index

DJIA	$\tau = 1$	$\tau = 2$	$\tau = 3$	$\tau = 4$	$\tau = 5$
d = 2	2.90	3.86	3.85	2.86	5.30
d = 3	6.00	6.65	6.57	5.77	9.47
d = 4	8.18	8.10	7.98	7.87	13.36
d = 5	9.45	9.42	9.29	9.16	16.36
d = 6	11.28	11.10	11.00	10.83	19.70
d = 7	12.72	12.76	12.48	12.07	22.90
d = 8	14.14	14.00	13.64	13.35	26.02

Table A-9 Mutual Information values of S&P500 index

S&P500	$\tau = 1$	$\tau = 2$	$\tau = 3$	$\tau = 4$	$\tau = 5$
d = 2	2.19	1.65	1.41	1.42	1.54
d = 3	3.64	3.18	2.68	2.72	2.84
d = 4	4.53	4.23	4.11	3.85	3.98
d = 5	5.15	5.07	5.32	4.80	4.88
d = 6	5.40	5.20	6.21	5.11	5.30
d = 7	5.52	5.24	6.92	5.20	5.53
d = 8	5.66	5.61	7.54	5.74	5.91

APPENDIX B

Mutual information values of stock volume series
(shaded values are the first local minima)

Table B-1 Mutual information values of DJIA individual stock volume series for an embedding dimension equal to 2

	$\tau = 1$	$\tau = 2$	$\tau = 3$
General Motors	4.98	4.78	6.26
Primamerica	3.32	4.61	
Navistar Int.	4.02	4.17	
Exxon	4.59	4.61	
Texaco	4.38	4.35	5.81
Westinghouse	4.73	4.63	6.78
Woolworth	3.85	3.98	
AT&T	4.15	4.06	6.05
Merck	3.96	4.22	
Philip Morris	3.43	4.11	
Alcoa	3.69	3.98	
Coca Cola	3.86	4.22	
Sears Roebuck	4.24	4.16	5.75
Procter & Gamble	3.36	4.18	
International Paper	5.08	4.89	6.22
Goodyear	3.92	3.86	5.38
Owens Illinois	3.77	4.17	
American Tobacco	4.13	4.15	

Table B-1 (continued)

	$\tau = 1$	$\tau = 2$	$\tau = 3$
General Electric	4.68	4.61	6.91
Minnesota M&M	3.35	3.43	
Boeing	4.23	4.23	6.02
Bethlehem	4.27	4.19	5.70
Chevron	4.22	4.35	
Chrysler	3.63	3.59	5.95
International Nickel	3.21	3.50	
Johns Manville	3.21	3.71	
United Aircraft	4.23	4.61	
Union Carbide	4.39	4.60	
Eastman Kodak	4.18	4.35	
DuPont	3.97	4.23	
IBM	3.91	4.09	

Table B-2 Mutual information values of DJIA individual stock volume series for an embedding dimension equal to 3

	$\tau = 1$	$\tau = 2$	$\tau = 3$
General Motors	9.00	9.06	
Primamerica	7.69	7.36	10.37
Navistar Int.	8.12	7.79	10.73
Exxon	8.20	8.11	11.97
Texaco	8.61	8.14	11.65
Westinghouse	8.54	8.32	12.97
Woolworth	7.70	7.48	10.58
AT&T	7.45	7.18	11.19
Merck	7.97	7.67	10.80
Philip Morris	7.79	7.70	10.00
Alcoa	7.62	7.36	10.38
Coca Cola	7.95	7.94	10.52
Sears Roebuck	8.11	7.72	10.65
Procter & Gamble	7.79	7.79	10.58
International Paper	9.28	9.08	12.67
Goodyear	7.56	7.00	10.20
Owens Illinois	7.77	7.57	10.51
American Tobacco	7.91	7.49	11.82

Table B-2 (continued)

	$\tau = 1$	$\tau = 2$	$\tau = 3$
General Electric	8.92	8.78	13.31
Minnesota M&M	7.63	7.06	10.40
Boeing	8.09	7.78	11.45
Bethlehem	7.70	7.48	10.69
Chevron	8.44	7.96	11.83
Chrysler	7.11	6.92	11.37
International Nickel	7.57	7.47	10.42
Johns Manville	7.53	7.38	10.97
United Aircraft	8.55	8.20	9.22
Union Carbide	8.34	8.18	10.85
Eastman Kodak	8.32	8.06	10.97
DuPont	8.20	8.15	11.00
IBM	7.96	7.62	10.65

Table B-3 Mutual information values of DJIA individual stock volume series for an embedding dimension equal to 4

	$\tau = 1$	$\tau = 2$	$\tau = 3$
General Motors	13.81	13.65	18.79
Primamerica	11.97	10.79	14.22
Navistar Int.	12.55	11.24	15.20
Exxon	12.51	12.23	17.30
Texaco	13.02	12.14	16.49
Westinghouse	12.91	12.32	18.87
Woolworth	11.83	10.89	14.85
AT&T	11.39	10.93	15.75
Merck	12.39	11.04	14.88
Philip Morris	11.97	10.88	13.51
Alcoa	11.83	10.62	14.37
Coca Cola	12.05	11.20	14.03
Sears Roebuck	12.32	11.47	15.00
Procter & Gamble	12.09	11.26	14.54
International Paper	14.09	13.53	18.20
Goodyear	11.63	10.22	14.18
Owens Illinois	11.95	10.71	14.08
American Tobacco	12.19	11.15	16.59

Table B-3 (continued)

	$\tau = 1$	$\tau = 2$	$\tau = 3$
General Electric	13.77	13.21	19.23
Minnesota M&M	11.86	10.35	14.03
Boeing	12.46	11.63	16.30
Bethlehem	11.42	10.98	15.01
Chevron	12.99	11.77	16.28
Chrysler	11.31	10.53	15.51
International Nickel	12.00	10.80	14.16
Johns Manville	11.67	10.84	14.83
United Aircraft	12.36	10.61	11.31
Union Carbide	12.77	12.09	15.96
Eastman Kodak	12.75	11.76	15.69
DuPont	12.69	11.86	15.22
IBM	12.36	11.20	14.90

Table B-4 Mutual information values of DJIA individual stock volume series for an embedding dimension equal to 5

	$\tau = 1$	$\tau = 2$	$\tau = 3$
General Motors	17.57	17.18	24.37
Primamerica	15.24	13.84	16.81
Navistar Int.	15.82	14.48	18.58
Exxon	15.75	15.35	22.19
Texaco	16.55	15.18	20.42
Westinghouse	16.15	15.42	24.46
Woolworth	14.96	13.77	18.46
AT&T	14.20	13.41	19.69
Merck	15.84	14.09	17.63
Philip Morris	15.01	13.86	15.66
Alcoa	14.99	13.63	17.25
Coca Cola	15.22	14.38	16.78
Sears Roebuck	15.61	14.59	18.93
Procter & Gamble	15.28	14.50	17.22
International Paper	17.98	16.87	22.81
Goodyear	14.66	12.97	17.14
Owens Illinois	15.03	13.78	16.43
American Tobacco	15.43	13.91	20.59

Table B-4 (continued)

	$\tau = 1$	$\tau = 2$	$\tau = 3$
General Electric	17.54	16.42	24.85
Minnesota M&M	14.96	13.05	16.97
Boeing	15.48	14.52	20.41
Bethlehem	14.19	13.54	19.08
Chevron	16.40	14.84	19.24
Chrysler	16.69	13.08	19.92
International Nickel	14.99	13.93	16.96
Johns Manville	14.71	14.13	17.66
United Aircraft	14.64	12.69	12.42
Union Carbide	16.20	15.34	19.59
Eastman Kodak	16.14	14.94	19.07
DuPont	16.11	15.37	18.22
IBM	15.87	14.30	18.11

Table B-5 Mutual information values of DJIA individual stock volume series for an embedding dimension equal to 6

	$\tau = 1$	$\tau = 2$	$\tau = 3$
General Motors	20.63	20.14	28.95
Primamerica	17.99	15.97	18.54
Navistar Int.	18.78	16.86	20.76
Exxon	18.19	17.72	26.58
Texaco	19.38	17.31	24.03
Westinghouse	18.71	17.69	29.75
Woolworth	17.50	15.94	21.19
AT&T	16.26	15.31	23.04
Merck	18.69	16.39	19.52
Philip Morris	17.75	16.17	17.06
Alcoa	17.76	15.83	19.08
Coca Cola	18.02	16.87	18.63
Sears Roebuck	18.52	17.02	21.93
Procter & Gamble	18.00	16.90	19.17
International Paper	20.76	19.31	27.39
Goodyear	17.28	15.04	19.22
Owens Illinois	17.78	16.16	17.95
American Tobacco	18.04	15.79	23.68

Table B-5 (continued)

	$\tau = 1$	$\tau = 2$	$\tau = 2$
General Electric	20.44	18.92	30.11
Minnesota M&M	17.53	14.85	19.03
Boeing	18.58	16.52	23.49
Bethlehem	16.31	15.28	22.86
Chevron	19.17	17.01	22.27
Chrysler	16.69	15.12	23.69
International Nickel	17.67	16.30	18.65
Johns Manville	17.45	16.61	19.64
United Aircraft	17.21	14.20	13.31
Union Carbide	19.00	17.82	22.62
Eastman Kodak	18.99	17.23	21.55
DuPont	19.21	18.02	20.53
IBM	18.80	16.59	20.26

Table B-6 Mutual information values of DJIA individual stock volume series for an embedding dimension equal to 7

	$\tau = 1$	$\tau = 2$	$\tau = 3$
General Motors	23.86	23.33	33.95
Primamerica	20.79	17.89	19.67
Navistar Int.	21.61	19.21	22.10
Exxon	21.14	20.35	30.86
Texaco	22.29	19.64	27.00
Westinghouse	21.52	20.29	34.90
Woolworth	20.26	18.17	23.61
AT&T	18.84	17.47	26.36
Merck	21.46	18.49	20.78
Philip Morris	20.57	18.02	17.80
Alcoa	20.56	17.90	20.13
Coca Cola	20.80	18.85	19.92
Sears Roebuck	21.47	19.39	24.59
Procter & Gamble	20.69	19.00	20.43
International Paper	23.96	22.12	30.65
Goodyear	20.05	16.90	20.50
Owens Illinois	20.40	18.15	18.92
American Tobacco	20.79	17.57	25.83

Table B-6 (continued)

	$\tau = 1$	$\tau = 2$	$\tau = 3$
General Electric	23.58	21.70	34.93
Minnesota M&M	20.23	16.68	20.27
Boeing	21.59	18.56	26.45
Bethlehem	18.75	17.41	26.33
Chevron	22.14	19.13	24.73
Chrysler	19.32	17.31	27.11
International Nickel	20.41	18.37	19.70
Johns Manville	20.29	18.88	20.76
United Aircraft	19.31	15.41	13.70
Union Carbide	22.05	20.48	25.25
Eastman Kodak	21.86	19.29	23.84
DuPont	22.31	20.44	22.09
IBM	21.72	18.57	21.80

Table B-7 Mutual information values of DJIA individual stock volume series for an embedding dimension equal to 8

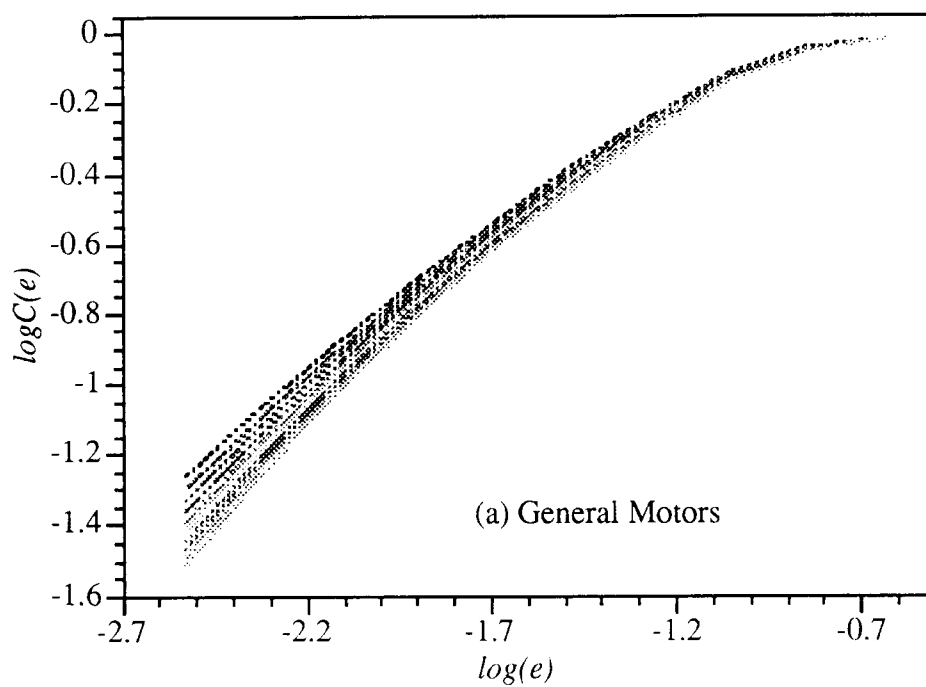
	$\tau = 1$	$\tau = 2$	$\tau = 3$
General Motors	26.73	25.98	38.12
Primamerica	23.00	19.63	20.33
Navistar Int.	23.99	21.34	23.23
Exxon	23.72	22.73	34.34
Texaco	24.71	21.51	29.75
Westinghouse	24.01	22.58	37.04
Woolworth	22.60	20.14	25.45
AT&T	20.92	19.19	29.44
Merck	23.71	20.57	21.70
Philip Morris	22.89	19.70	18.33
Alcoa	22.81	19.88	20.64
Coca Cola	23.03	20.98	20.62
Sears Roebuck	23.87	21.55	26.80
Procter & Gamble	22.91	21.17	21.20
International Paper	26.66	24.54	34.37
Goodyear	22.37	18.60	21.15
Owens Illinois	22.47	20.13	19.62
American Tobacco	23.04	19.13	26.79

Table B-7 (continued)

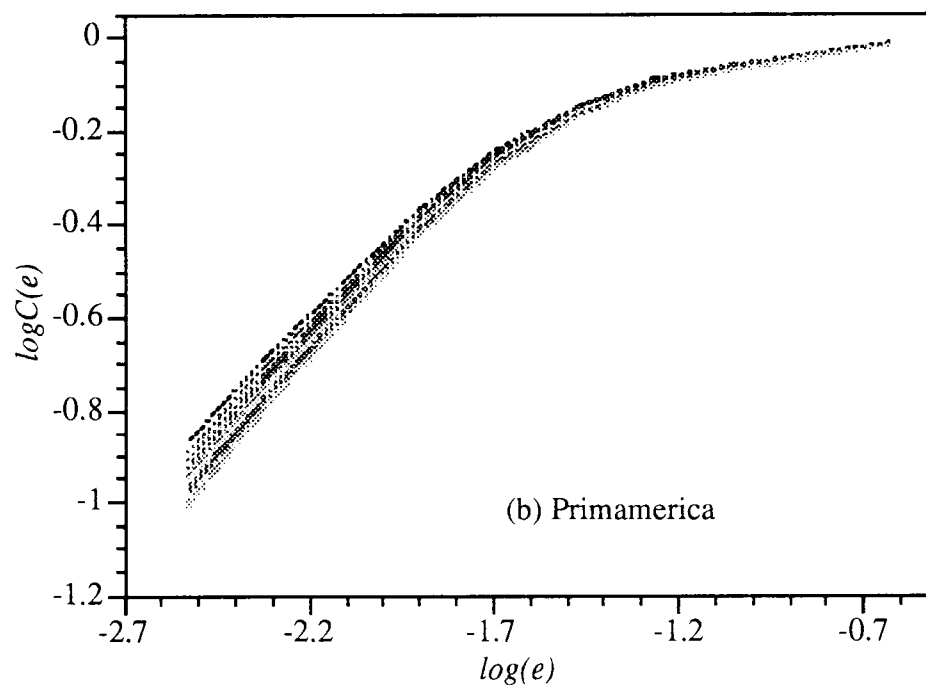
	$\tau = 1$	$\tau = 2$	$\tau = 3$
General Electric	26.20	24.00	39.00
Minnesota M&M	22.48	18.31	21.08
Boeing	24.04	20.36	29.28
Bethlehem	20.76	19.21	28.48
Chevron	24.60	20.98	26.69
Chrysler	21.56	19.01	30.17
International Nickel	22.57	20.47	20.46
Johns Manville	22.61	21.12	21.52
United Aircraft	20.98	18.30	17.20
Union Carbide	24.61	22.88	27.49
Eastman Kodak	24.18	21.27	25.62
DuPont	24.87	22.77	23.19
IBM	24.08	20.53	22.85

APPENDIX C

GRASSBERGER-PROCACCIA PLOTS (GP) OF DAILY PRICES AND CORRELATION DIMENSION ESTIMATES FOR INDIVIDUAL STOCKS AND TWO DOMESTIC INDICES

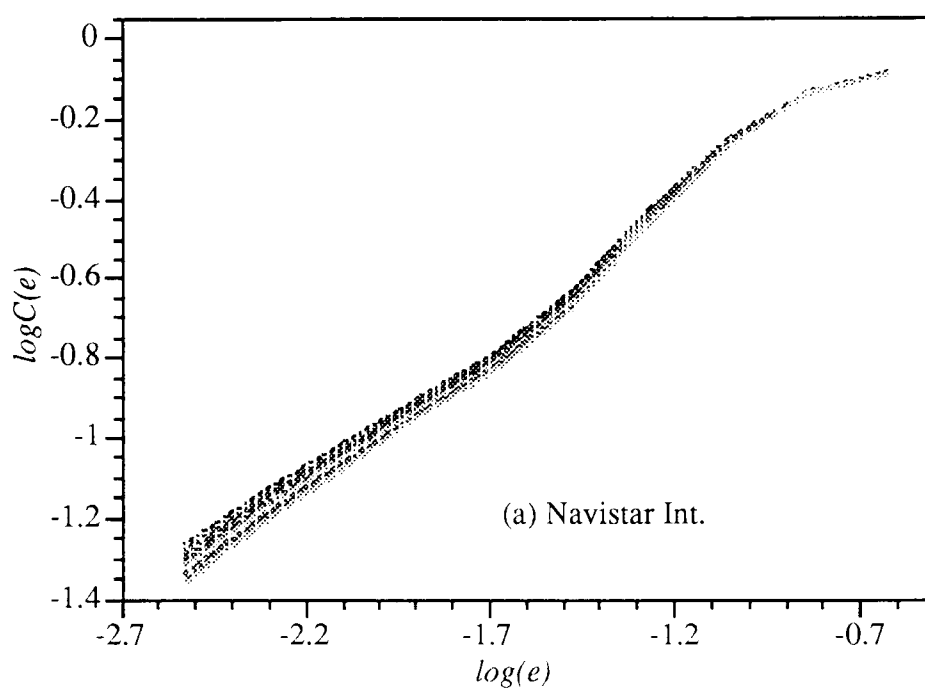


(a)

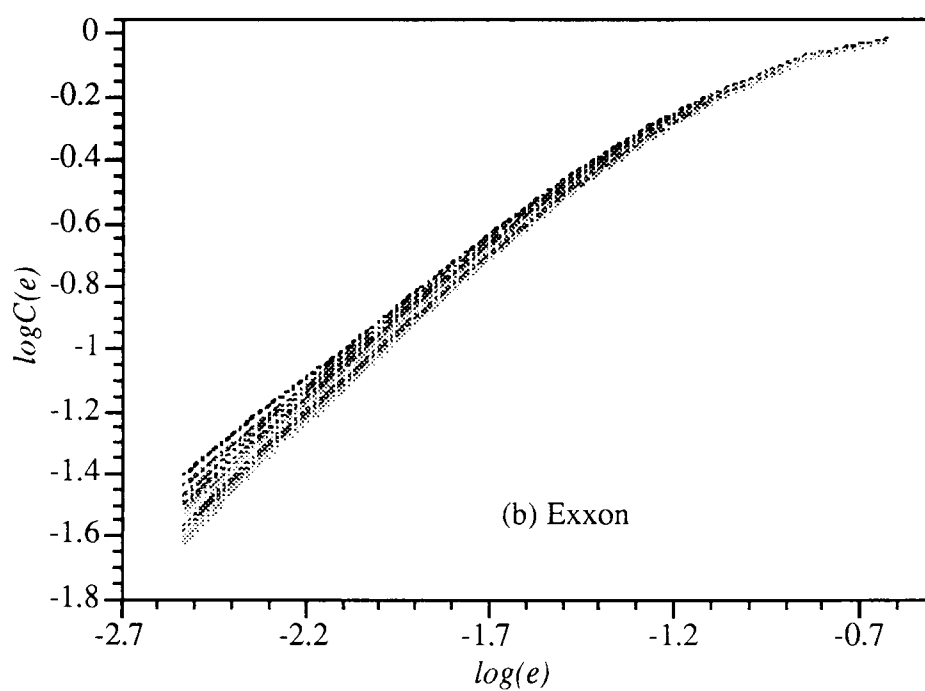


(b)

Figure C-1 GP plots of daily prices of (a) General Motors and (b) Primamerica

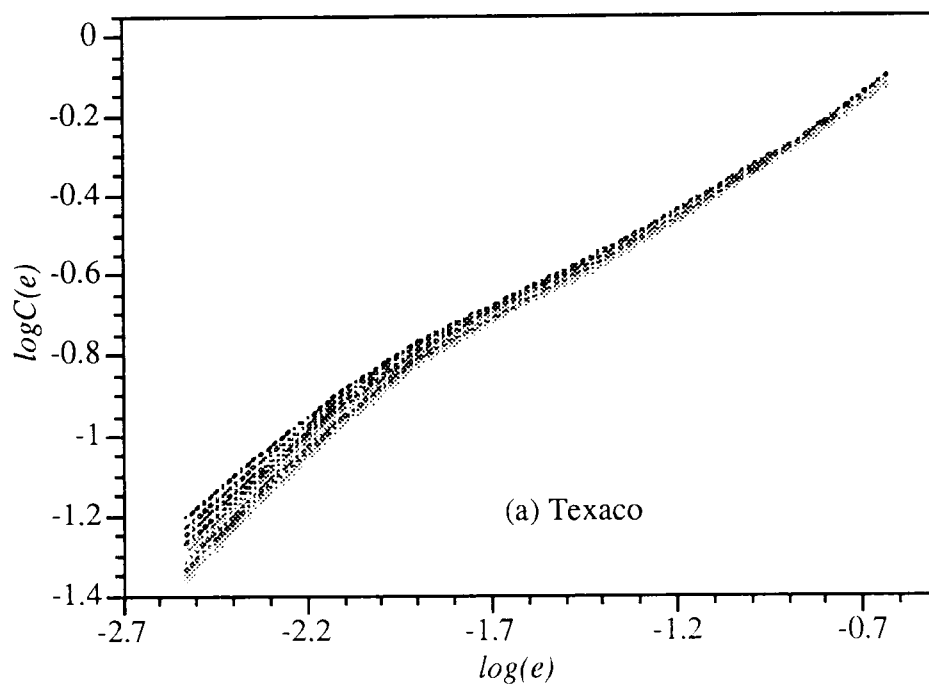


(a)

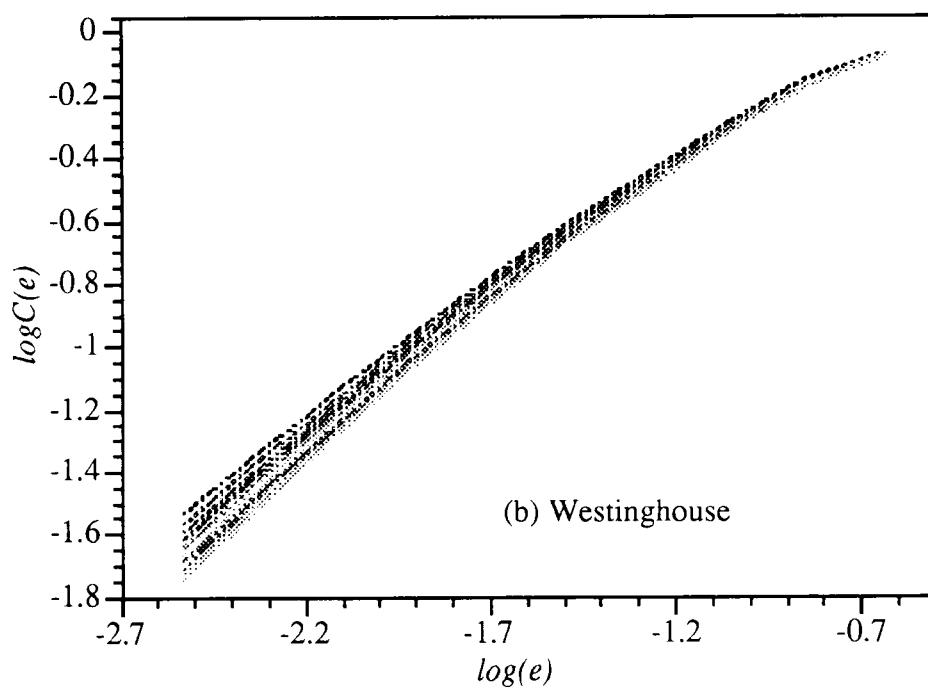


(b)

Figure C-2 GP plots of daily prices of (a) Navistar Int. and (b) Exxon

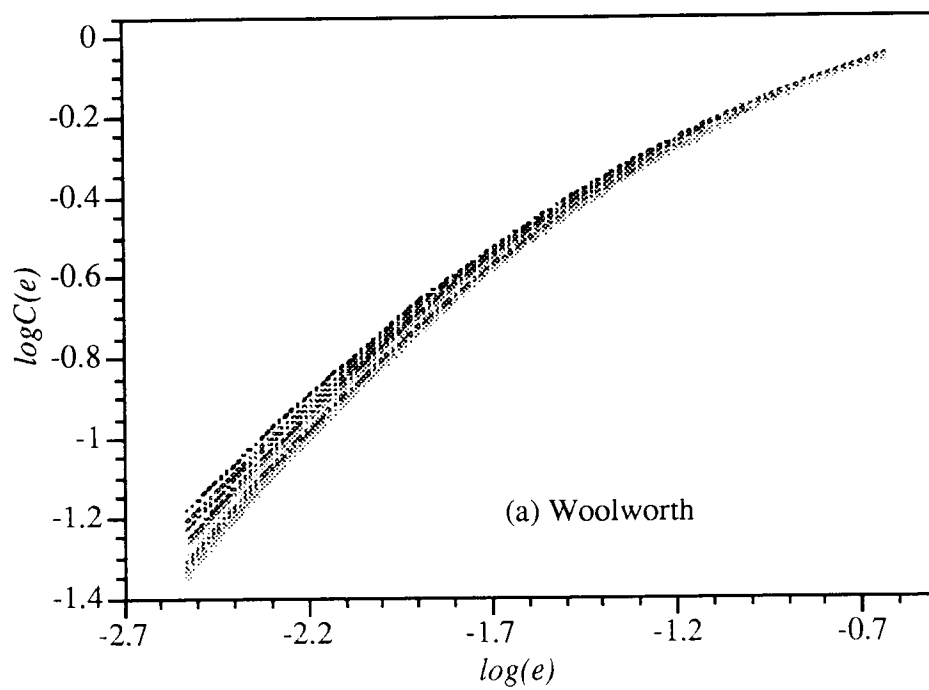


(a)

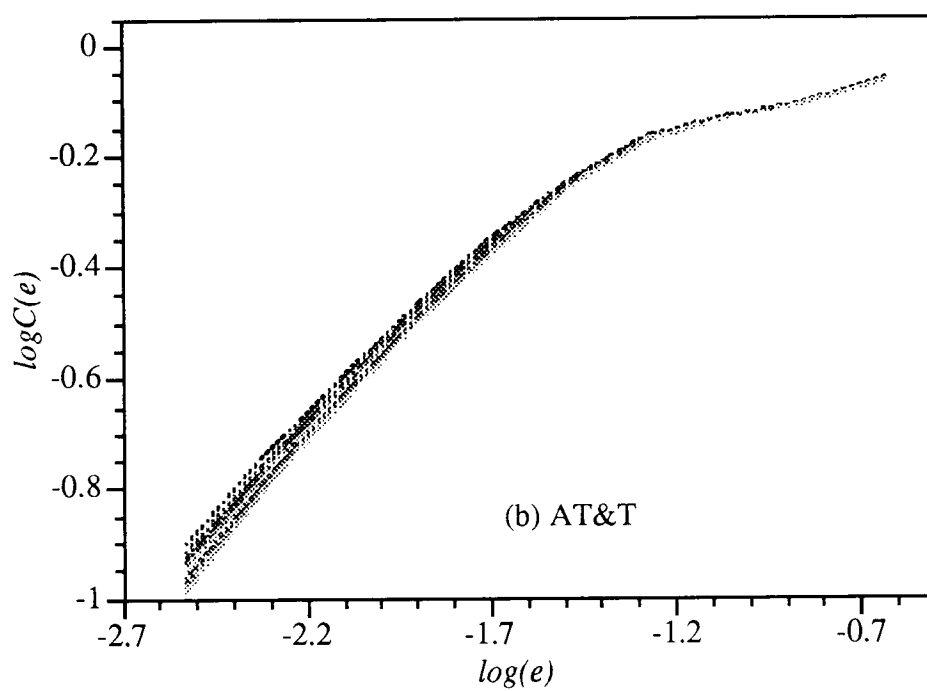


(b)

Figure C-3 GP plots of daily prices of (a) Texaco and (b) Westinghouse

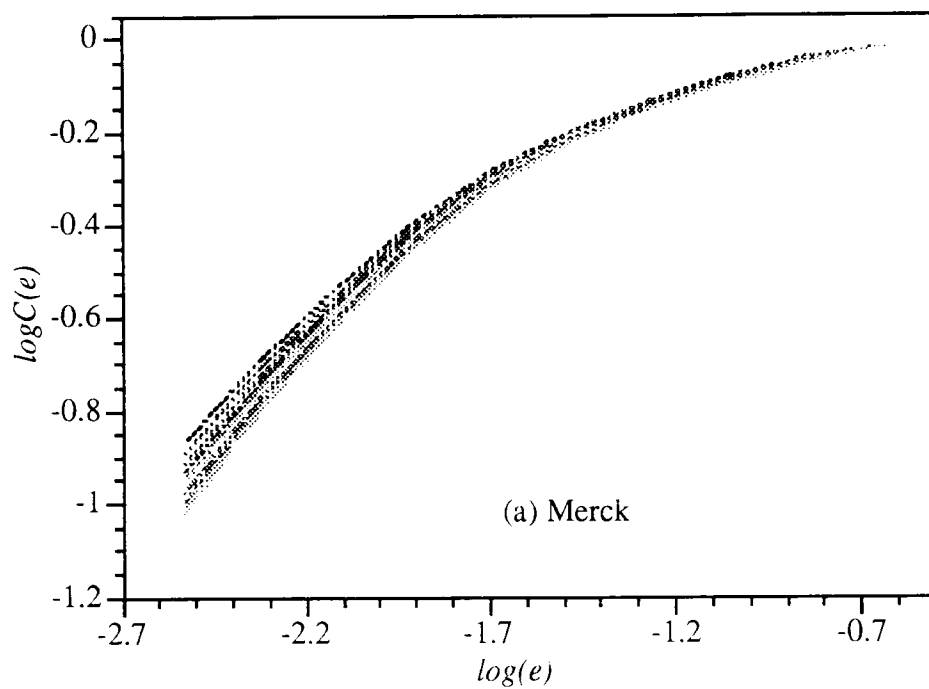


(a)

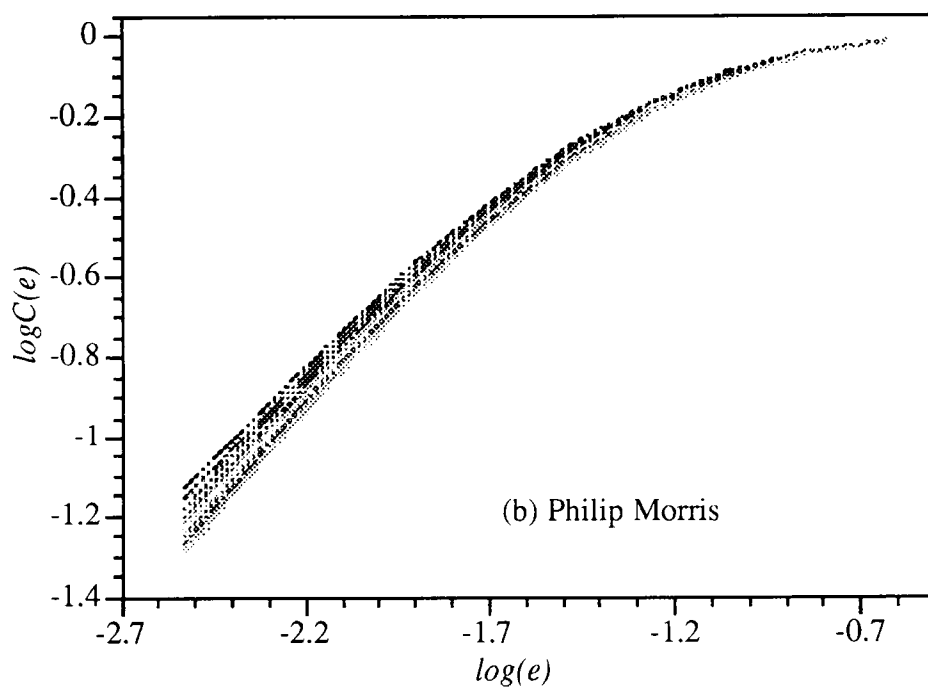


(b)

Figure C-4 GP plots of daily prices of (a) Woolworth and (b) AT&T

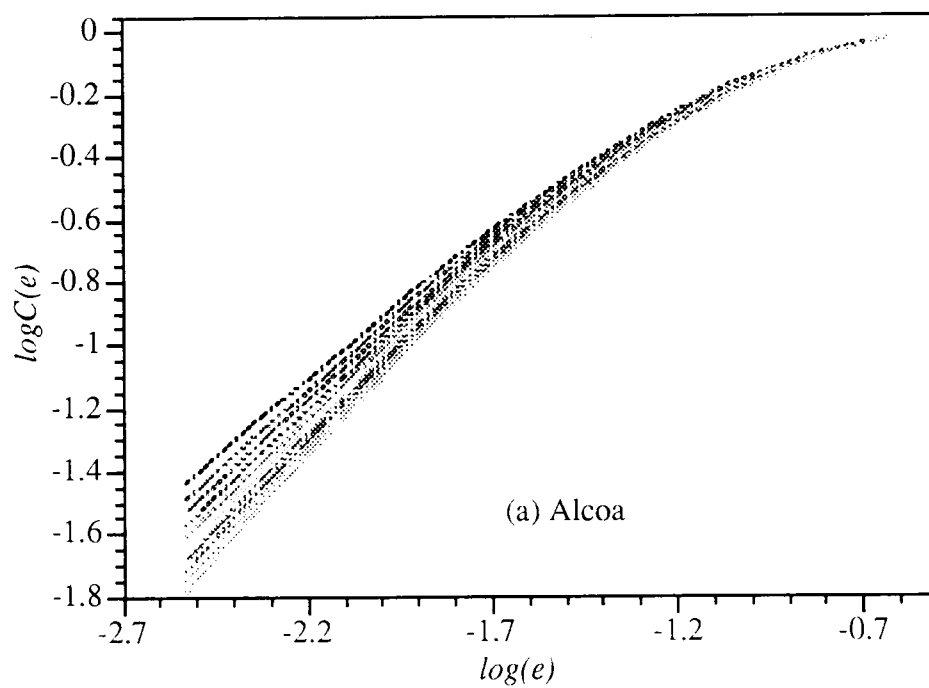


(a)

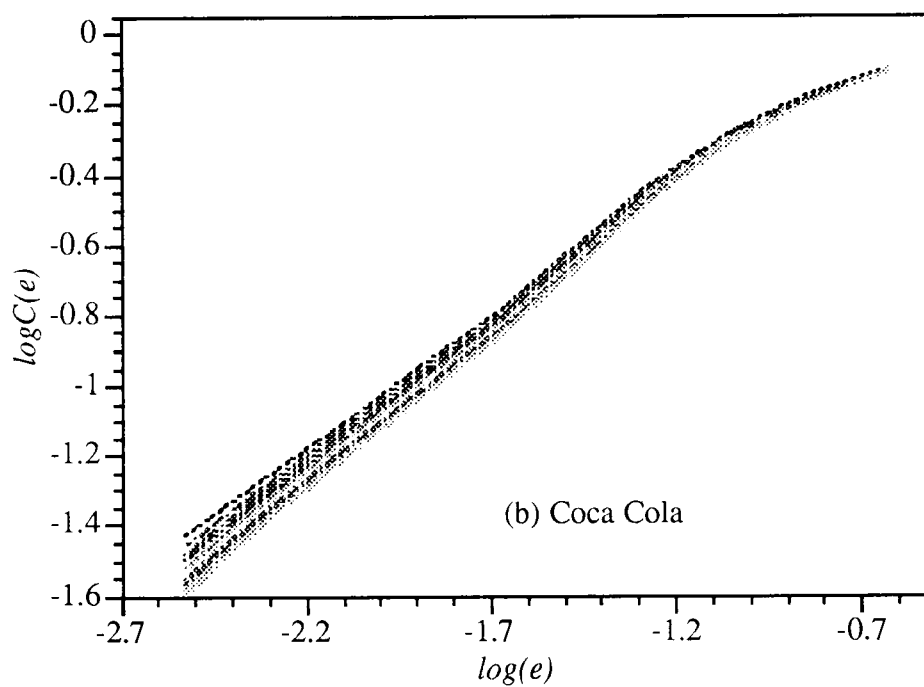


(b)

Figure C-5 GP plots of daily prices of (a) Merck and (b) Philip Morris



(a)



(b)

Figure C-6 GP plots of daily prices of (a) Alcoa and (b) Coca Cola

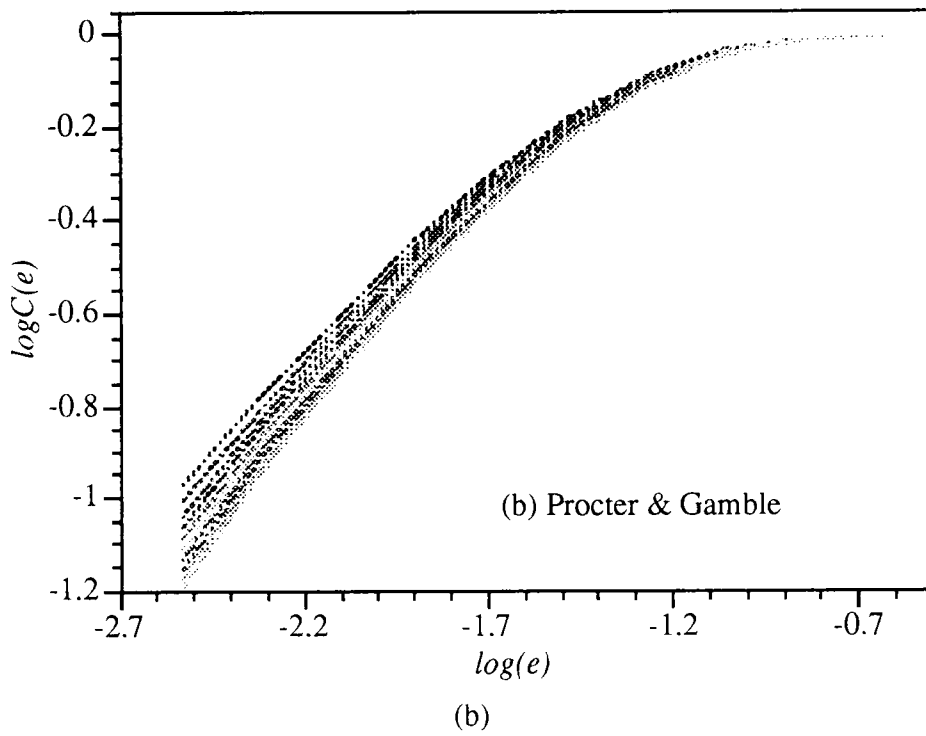
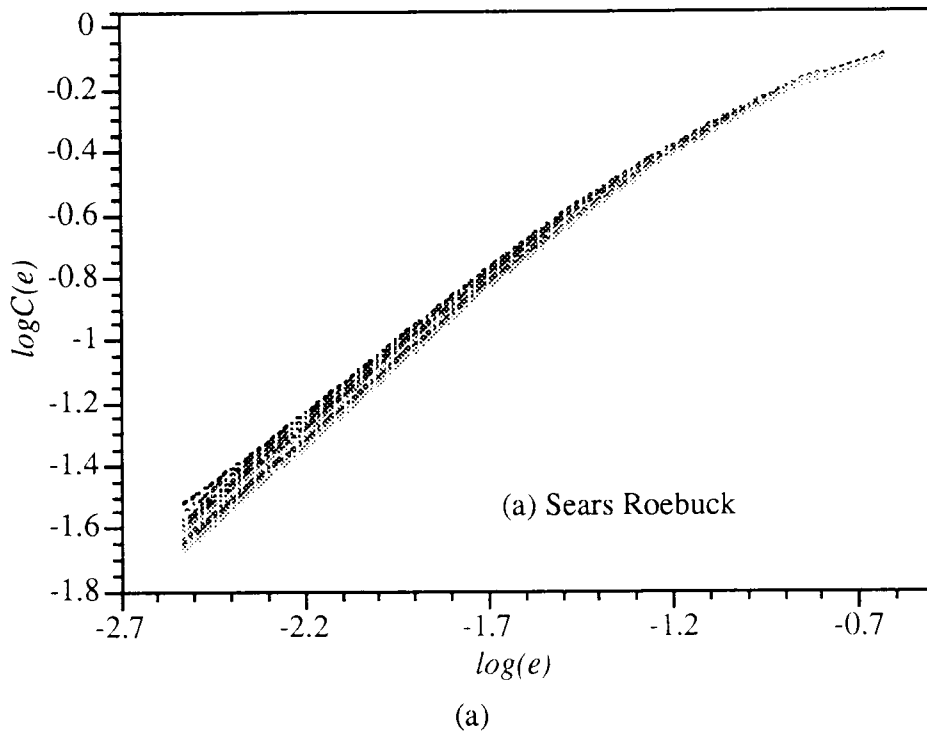
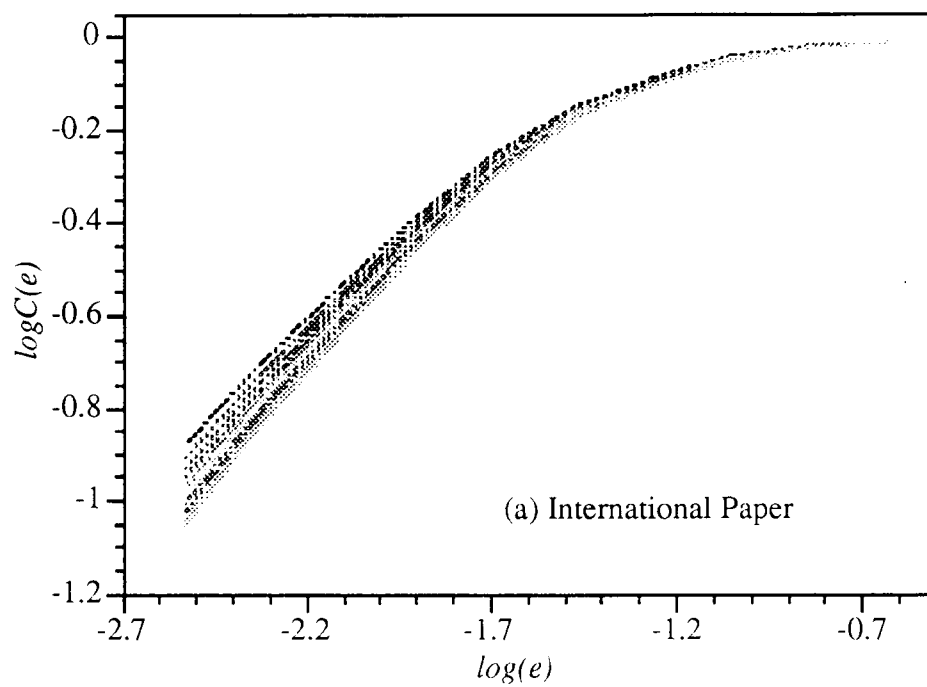
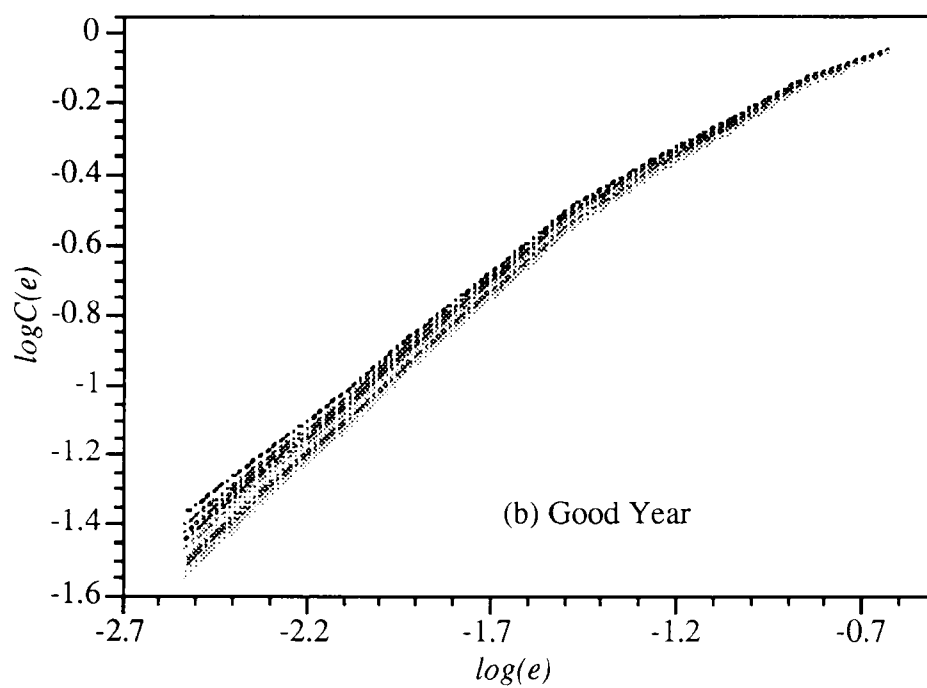


Figure C-7 GP plots of daily prices of (a) Sears Roebuck and (b) Procter & Gamble

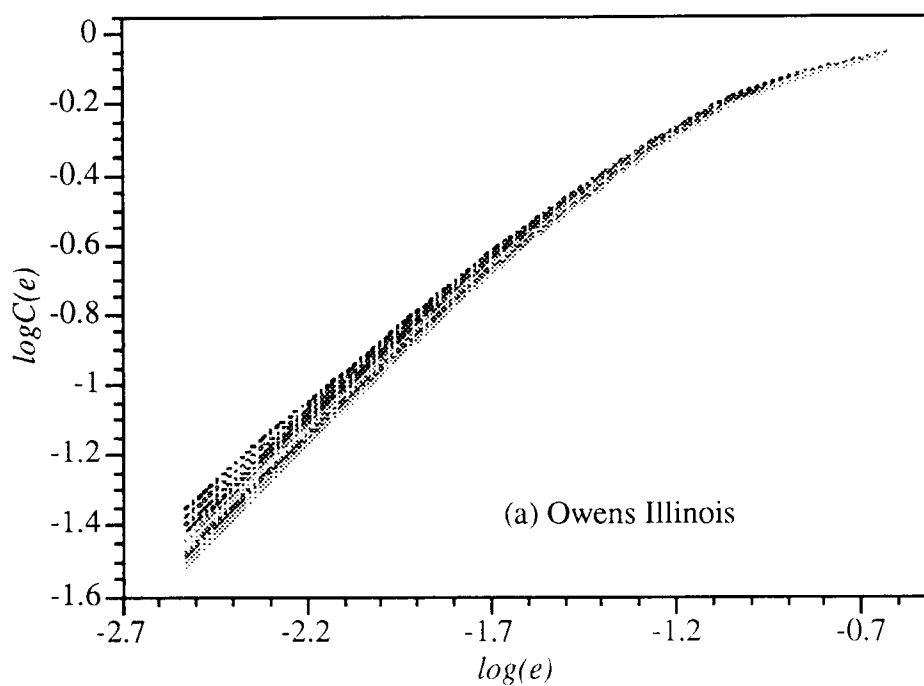


(a)

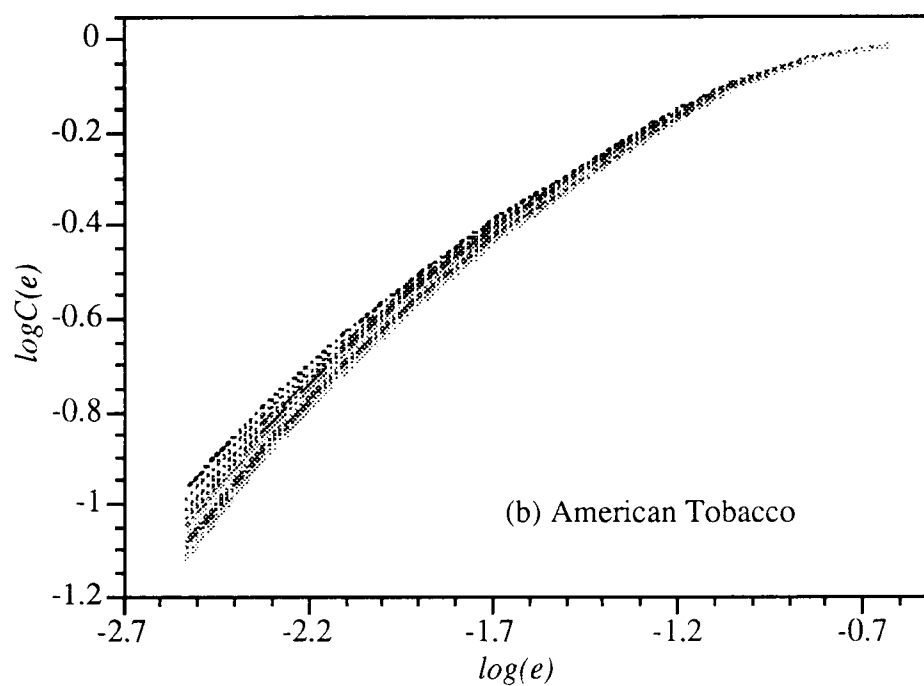


(b)

Figure C-8 GP plots of daily prices of (a) International Paper and (b) Goodyear

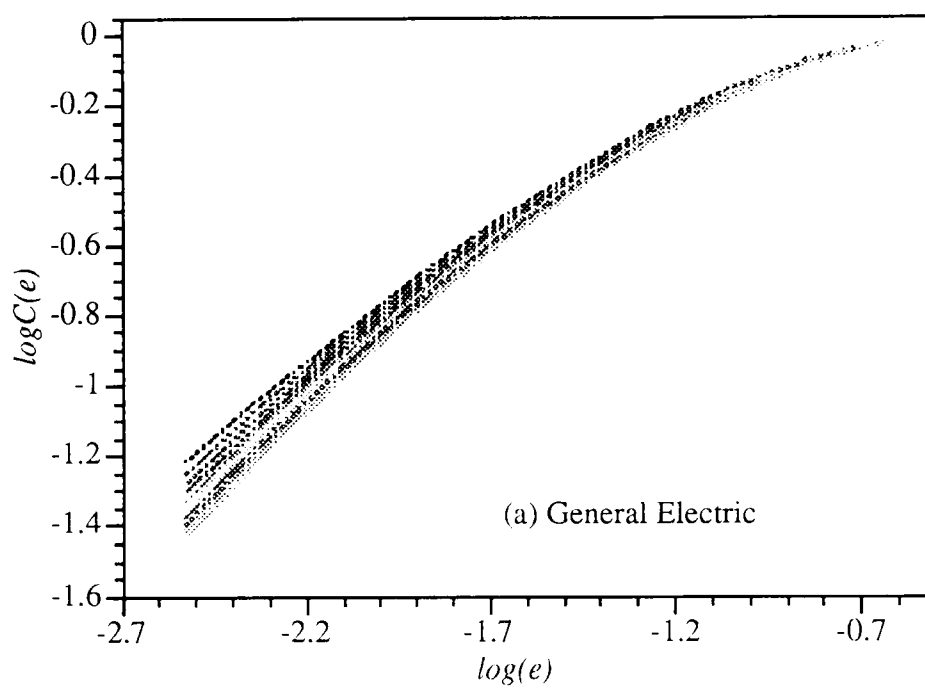


(a)

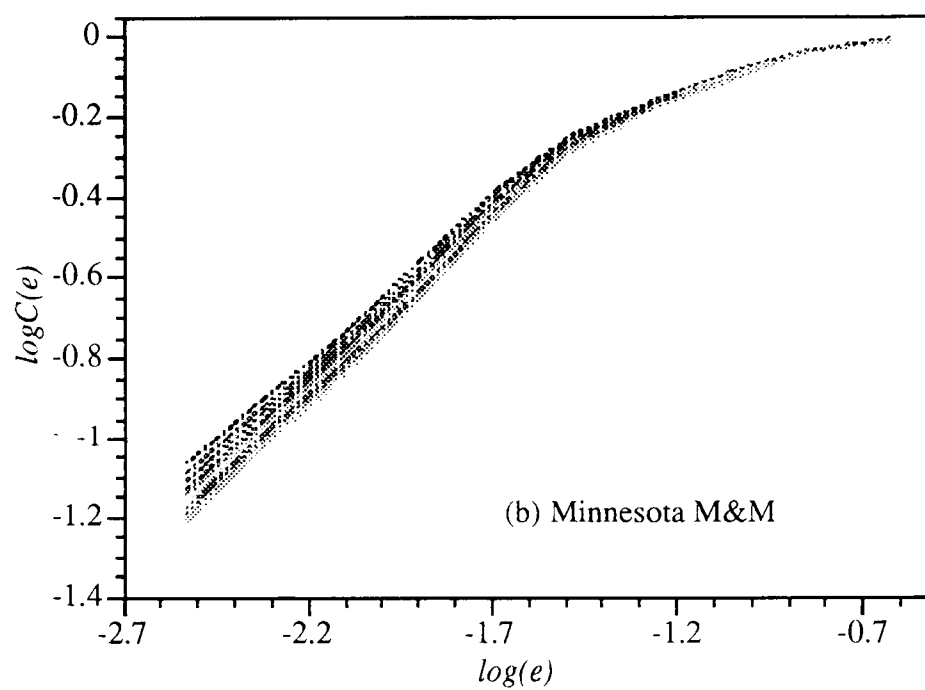


(b)

Figure C-9 GP plots of daily prices of (a) Owens Illinois and (b) American Tobacco

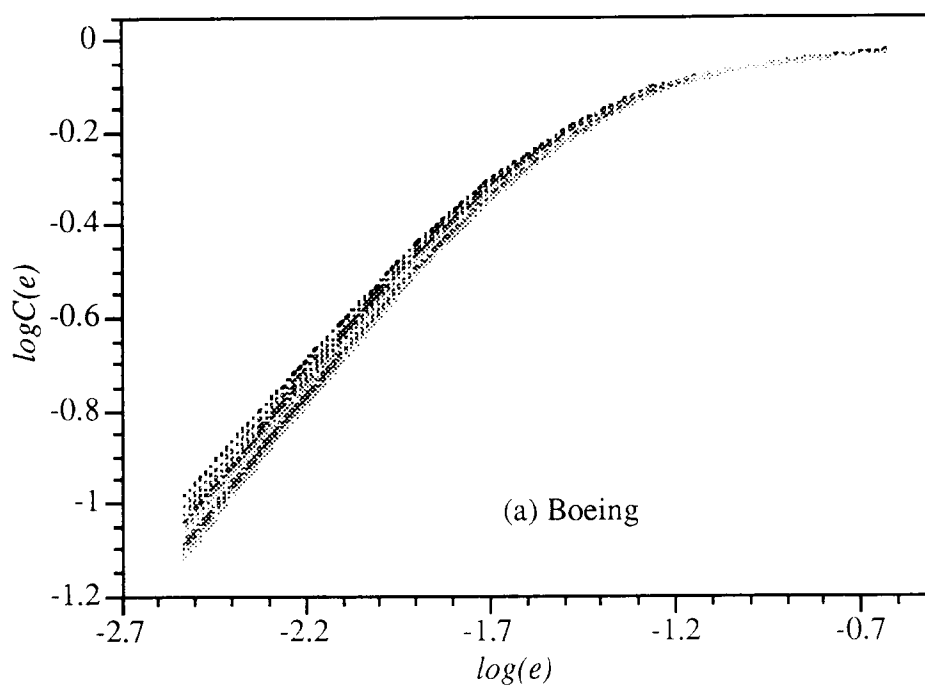


(a)

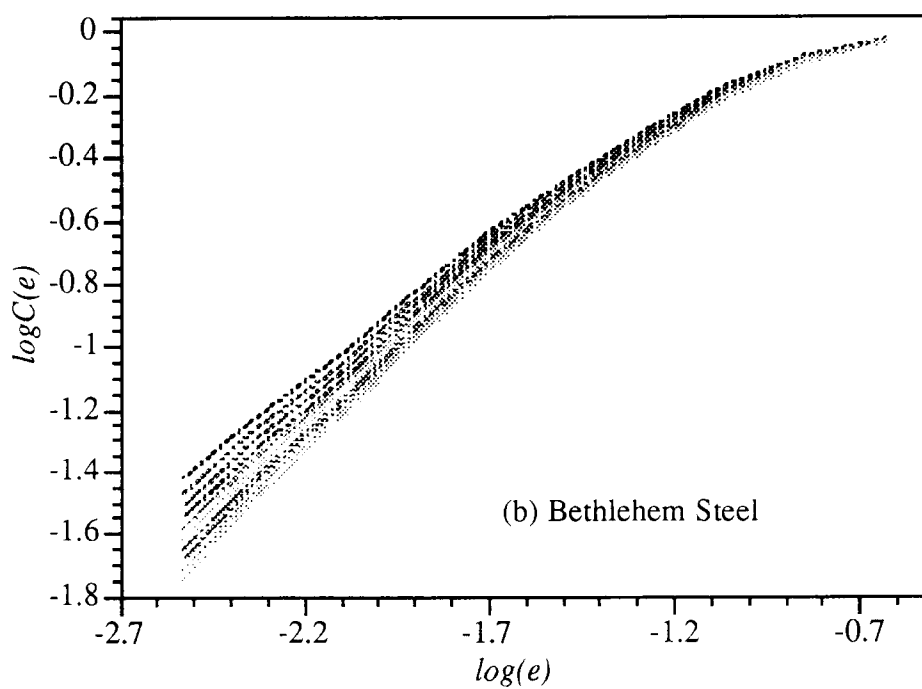


(b)

Figure C-10 GP plots of daily prices of (a) General Electric and (b) Minnesota M&M

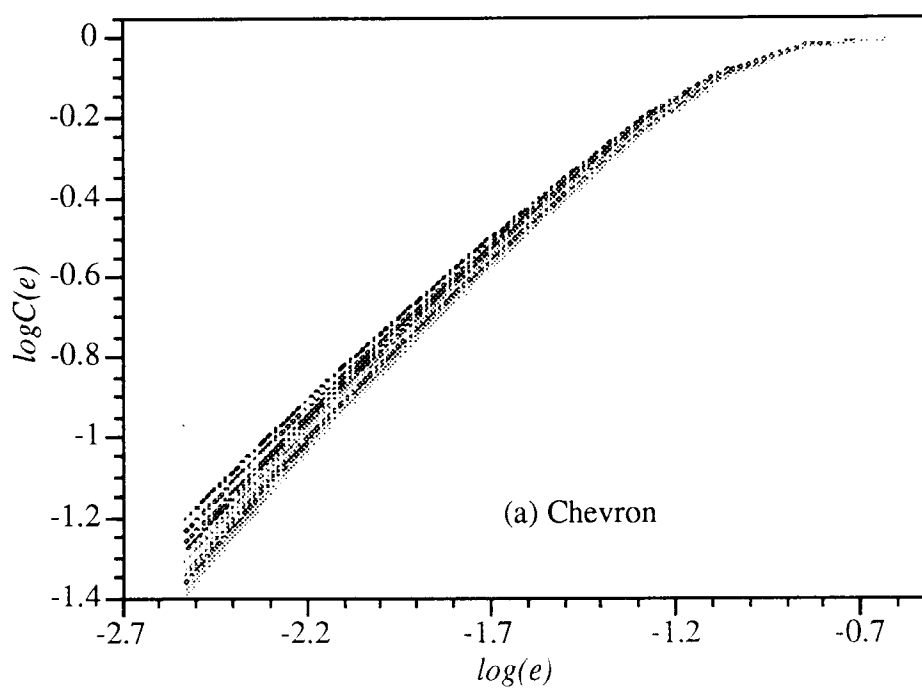


(a)

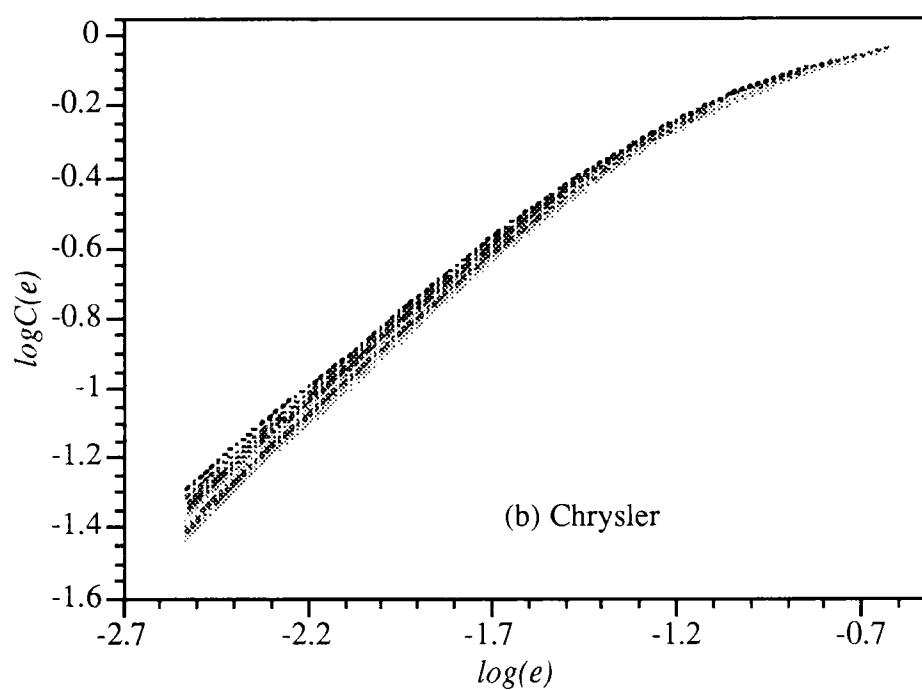


(b)

Figure C-11 GP plots of daily prices of (a) Boeing and (b) Bethlehem Steel



(a)



(b)

Figure C-12 GP plots of daily prices of (a) Chevron and (b) Chrysler

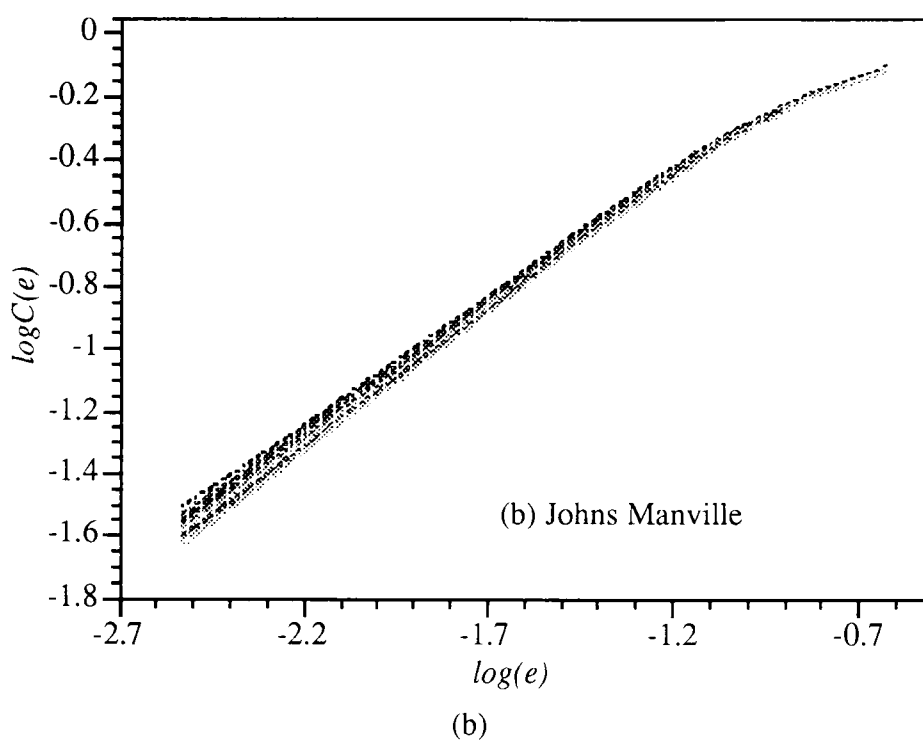
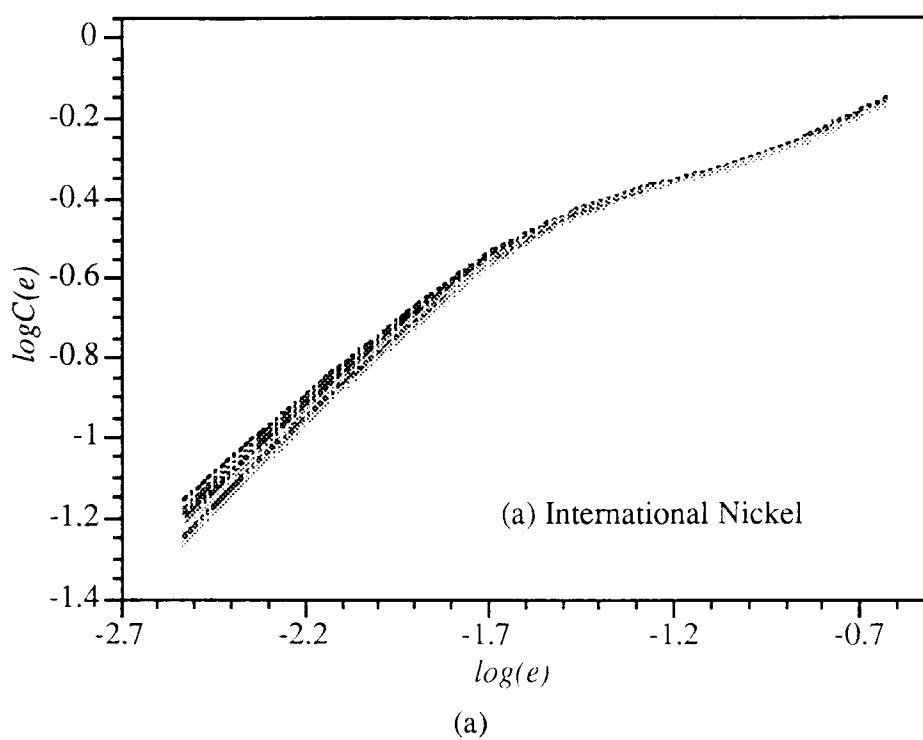


Figure C-13 GP plots of daily prices of (a) International Nickel and (b) Johns Manville

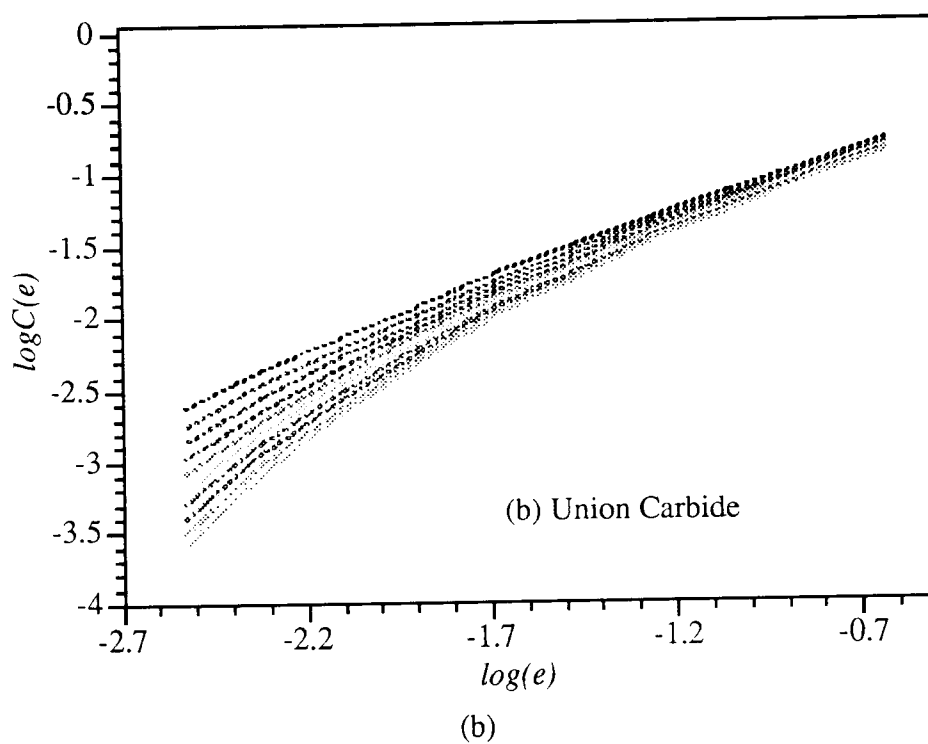
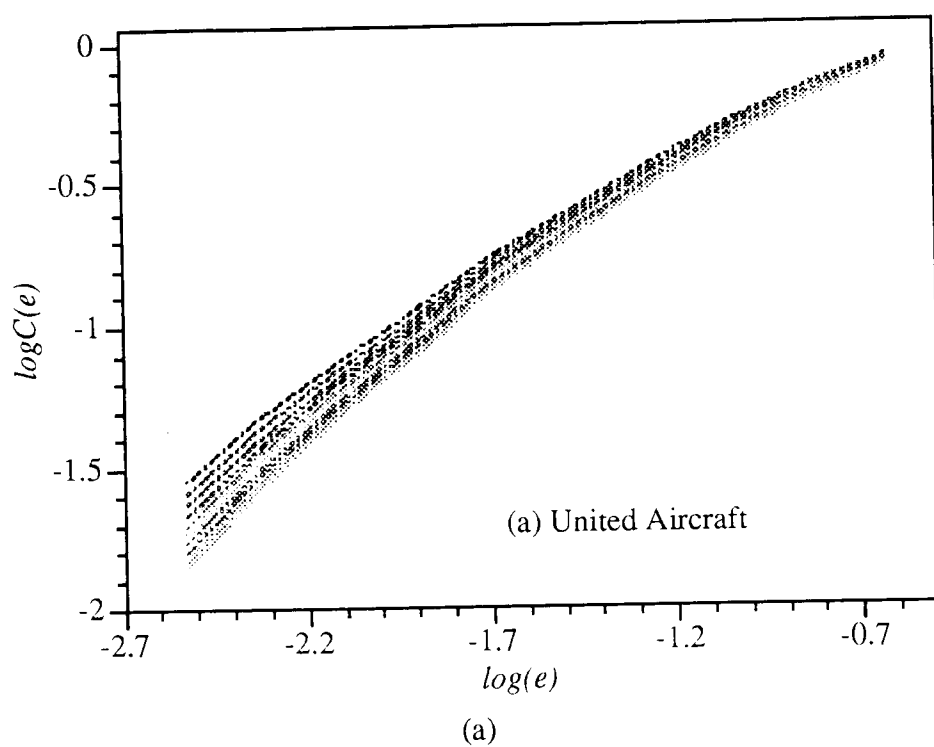
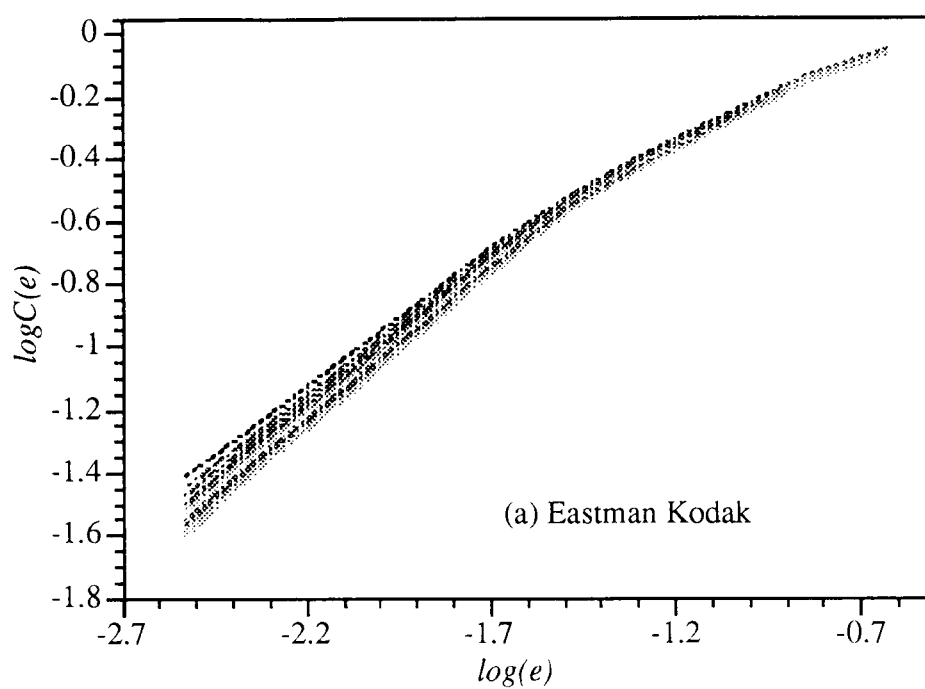
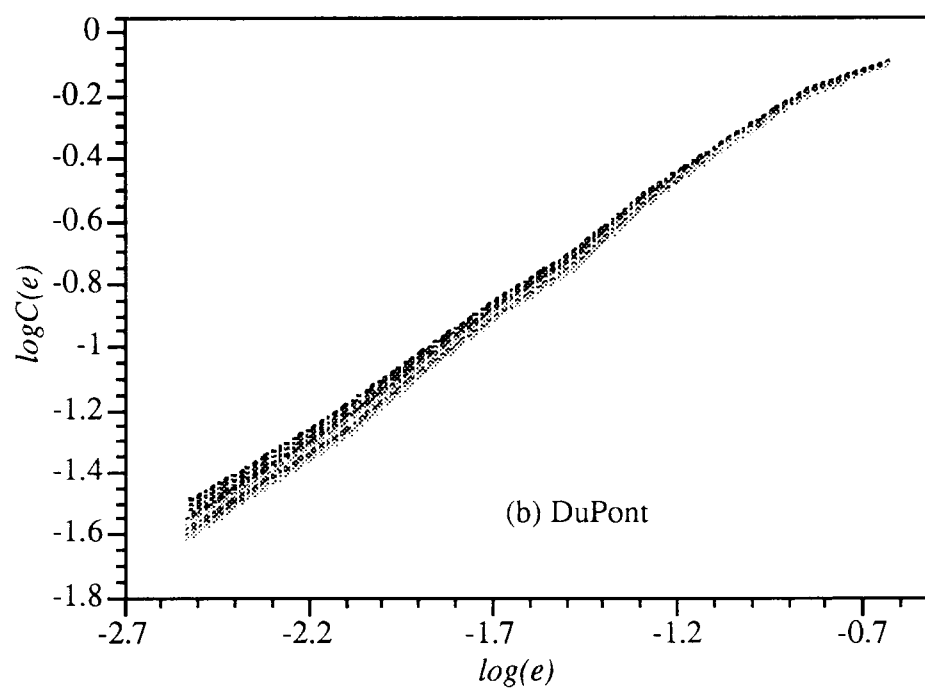


Figure C-14 GP plots of daily prices of (a) United Aircraft and (b) Union Carbide



(a)



(b)

Figure C-15 GP plots of daily prices of (a) Eastman Kodak and (b) DuPont

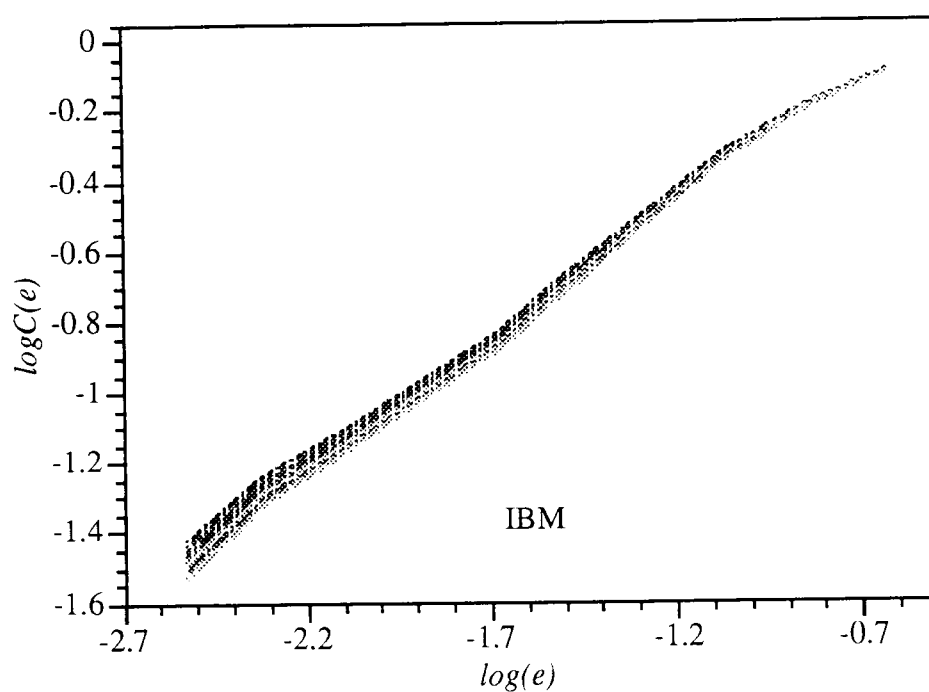


Figure C-16 GP plots of daily prices of IBM

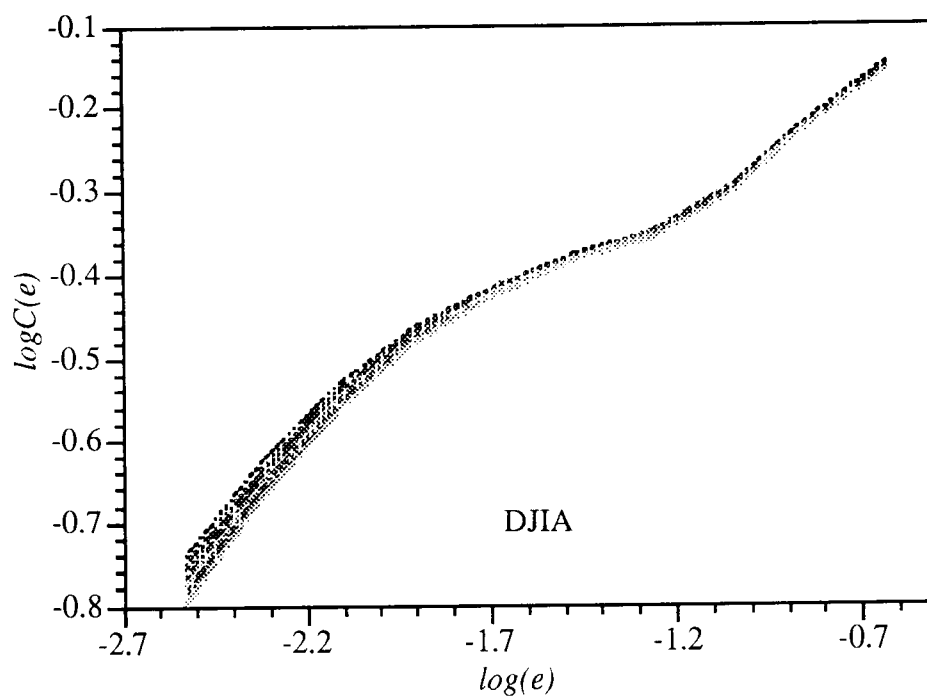


Figure C-17 GP plots of daily index levels of the DJIA

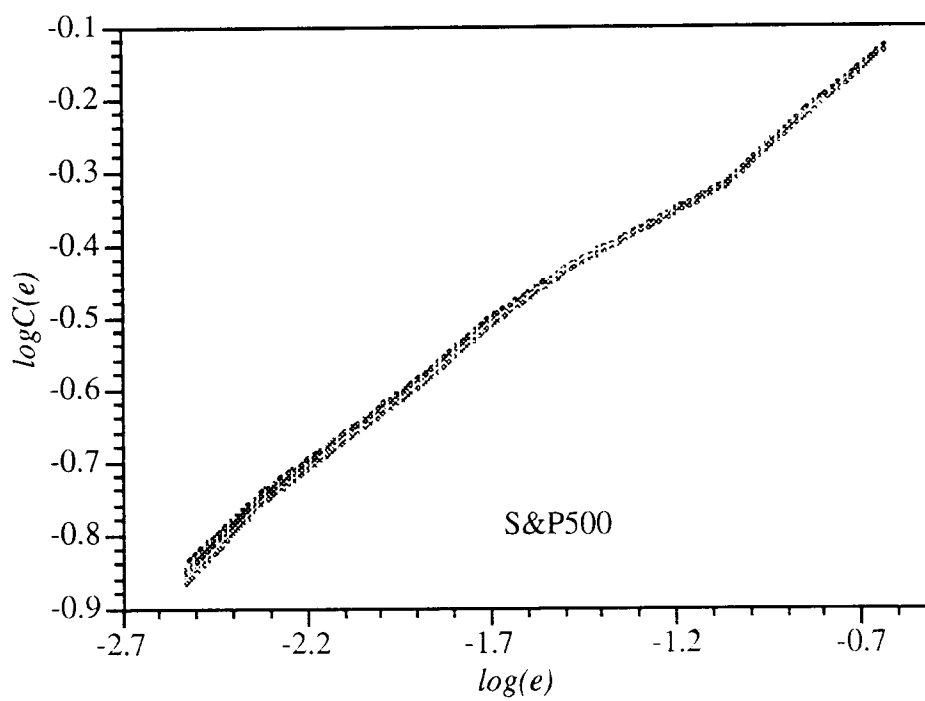


Figure C-18 GP plots of daily index levels of the S&P500

Table C-1 Correlation Integral Behavior for price series by embedding dimension

(numbers in parentheses are the standard error of the estimates)

	d = 2	d = 3	d = 4	d = 5	d = 6	d = 7	d = 8	d = 9	d = 10	d = 11
General Motors	0.68 (0.04)	0.70 (0.04)	0.71 (0.04)	0.73 (0.04)	0.74 (0.04)	0.75 (0.04)	0.76 (0.04)	0.78 (0.04)	0.79 (0.04)	0.80 (0.05)
Primamerica	0.69 (0.03)	0.71 (0.04)	0.72 (0.04)	0.74 (0.04)	0.75 (0.04)	0.76 (0.04)	0.77 (0.04)	0.78 (0.04)	0.79 (0.04)	0.80 (0.04)
Navistar Int.	0.66 (0.02)	0.67 (0.02)	0.67 (0.02)	0.68 (0.02)	0.68 (0.02)	0.69 (0.02)	0.69 (0.02)	0.70 (0.02)	0.70 (0.02)	0.70 (0.02)
Exxon	0.85 (0.02)	0.87 (0.02)	0.88 (0.02)	0.90 (0.02)	0.91 (0.02)	0.93 (0.02)	0.94 (0.02)	0.95 (0.03)	0.96 (0.03)	0.97 (0.03)
Texaco	0.54 (0.01)	0.55 (0.01)	0.56 (0.01)	0.57 (0.01)	0.58 (0.01)	0.58 (0.02)	0.59 (0.02)	0.60 (0.02)	0.60 (0.02)	0.61 (0.02)
Westinghouse	0.71 (0.02)	0.72 (0.02)	0.73 (0.02)	0.74 (0.03)	0.75 (0.03)	0.75 (0.03)	0.76 (0.03)	0.77 (0.03)	0.77 (0.03)	0.78 (0.03)
Woolworth	0.48 (0.02)	0.49 (0.02)	0.50 (0.02)	0.50 (0.02)	0.51 (0.02)	0.51 (0.02)	0.52 (0.02)	0.52 (0.02)	0.53 (0.02)	0.53 (0.02)
AT&T	0.58 (0.02)	0.59 (0.02)	0.60 (0.02)	0.61 (0.03)	0.61 (0.03)	0.62 (0.03)	0.63 (0.03)	0.63 (0.03)	0.64 (0.03)	0.64 (0.03)

Table C-1 (continued)

	d = 2	d = 3	d = 4	d = 5	d = 6	d = 7	d = 8	d = 9	d = 10	d = 11
Merck	0.70 (0.03)	0.72 (0.03)	0.74 (0.03)	0.76 (0.04)	0.77 (0.04)	0.78 (0.04)	0.80 (0.04)	0.81 (0.04)	0.82 (0.04)	0.83 (0.04)
Philip Morris	0.86 (0.02)	0.88 (0.02)	0.89 (0.02)	0.91 (0.02)	0.92 (0.03)	0.93 (0.03)	0.95 (0.03)	0.96 (0.03)	0.97 (0.03)	0.98 (0.03)
Alcoa	0.90 (0.02)	0.94 (0.02)	0.97 (0.02)	0.99 (0.02)	1.02 (0.02)	1.04 (0.02)	1.07 (0.02)	1.09 (0.02)	1.11 (0.02)	1.13 (0.02)
Coca Cola	0.78 (0.008)	0.79 (0.008)	0.81 (0.007)	0.82 (0.007)	0.83 (0.007)	0.83 (0.007)	0.84 (0.007)	0.85 (0.007)	0.86 (0.007)	0.87 (0.007)
Sears Roebuck	0.78 (0.02)	0.79 (0.03)	0.80 (0.03)	0.81 (0.03)	0.81 (0.03)	0.82 (0.03)	0.83 (0.03)	0.84 (0.03)	0.84 (0.03)	0.85 (0.03)
Procter & Gamble	0.48 (0.04)	0.50 (0.04)	0.51 (0.04)	0.52 (0.04)	0.53 (0.05)	0.54 (0.05)	0.55 (0.05)	0.56 (0.05)	0.57 (0.05)	0.58 (0.05)
International Paper	0.74 (0.02)	0.77 (0.02)	0.79 (0.02)	0.81 (0.02)	0.82 (0.02)	0.84 (0.03)	0.85 (0.03)	0.87 (0.03)	0.88 (0.03)	0.89 (0.03)
Goodyear	0.71 (0.02)	0.73 (0.02)	0.74 (0.02)	0.75 (0.02)	0.76 (0.03)	0.76 (0.03)	0.77 (0.03)	0.78 (0.03)	0.79 (0.03)	0.80 (0.03)

Table C-1 (continued)

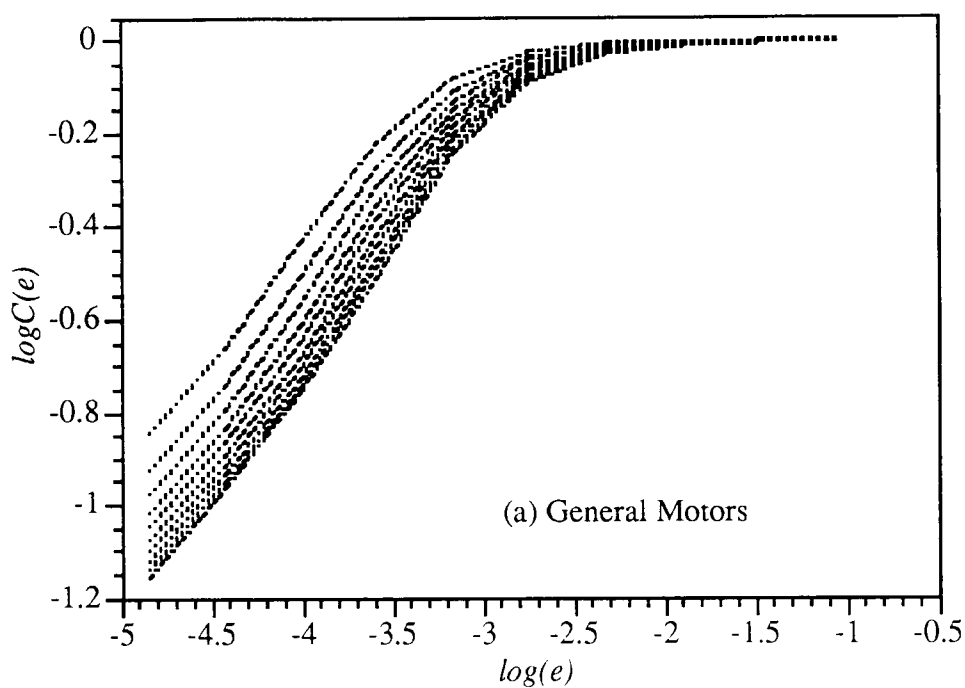
	d = 2	d = 3	d = 4	d = 5	d = 6	d = 7	d = 8	d = 9	d = 10	d = 11
Owens Illinois	0.80 (0.02)	0.81 (0.02)	0.83 (0.02)	0.84 (0.02)	0.84 (0.02)	0.86 (0.02)	0.87 (0.02)	0.88 (0.02)	0.89 (0.02)	0.90 (0.02)
American Tobacco	0.58 (0.02)	0.59 (0.02)	0.60 (0.03)	0.61 (0.03)	0.62 (0.03)	0.63 (0.03)	0.64 (0.03)	0.65 (0.03)	0.66 (0.03)	0.66 (0.03)
General Electric	0.72 (0.02)	0.74 (0.02)	0.76 (0.02)	0.77 (0.02)	0.78 (0.02)	0.80 (0.03)	0.81 (0.03)	0.82 (0.03)	0.83 (0.03)	0.84 (0.03)
Minnesota M&M	0.78 (0.01)	0.80 (0.01)	0.81 (0.01)	0.83 (0.01)	0.84 (0.01)	0.85 (0.01)	0.86 (0.01)	0.87 (0.009)	0.88 (0.009)	0.89 (0.009)
Boeing	0.45 (0.05)	0.46 (0.05)	0.47 (0.05)	0.48 (0.05)	0.48 (0.05)	0.49 (0.05)	0.50 (0.05)	0.50 (0.05)	0.51 (0.06)	0.51 (0.06)
Bethlehem Steel	0.76 (0.03)	0.79 (0.03)	0.80 (0.03)	0.82 (0.03)	0.84 (0.03)	0.85 (0.03)	0.86 (0.03)	0.87 (0.04)	0.89 (0.04)	0.90 (0.04)
Chevron	0.76 (0.03)	0.78 (0.03)	0.79 (0.03)	0.80 (0.03)	0.81 (0.03)	0.82 (0.03)	0.83 (0.03)	0.84 (0.03)	0.85 (0.03)	0.86 (0.03)
Chrysler	0.76 (0.02)	0.78 (0.02)	0.79 (0.02)	0.80 (0.02)	0.81 (0.02)	0.82 (0.03)	0.83 (0.03)	0.83 (0.03)	0.84 (0.03)	0.85 (0.03)

Table C-1 (continued)

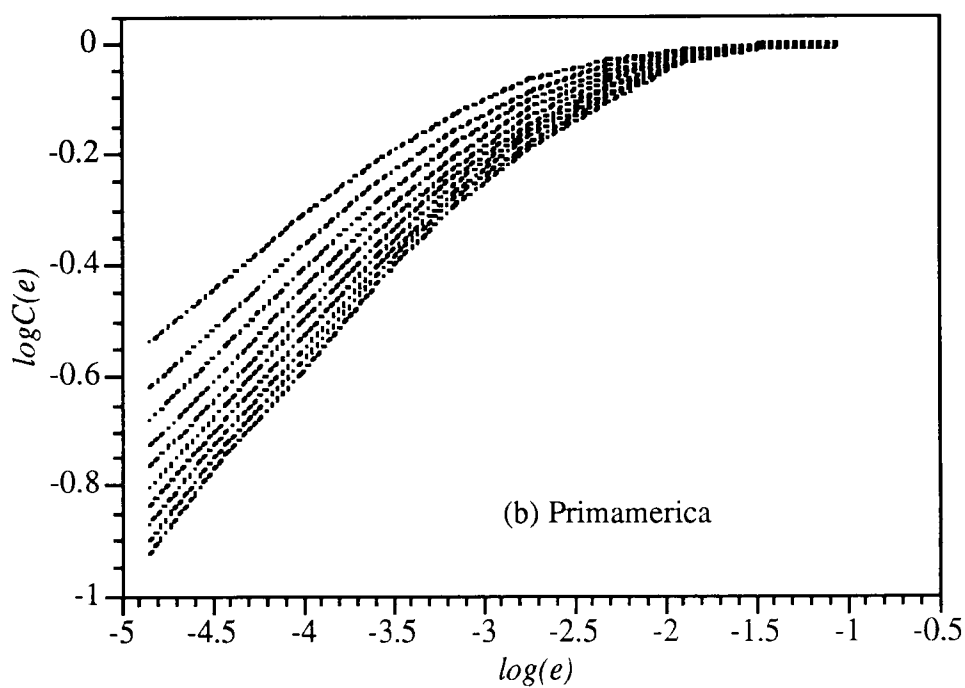
	d = 2	d = 3	d = 4	d = 5	d = 6	d = 7	d = 8	d = 9	d = 10	d = 11
International Nickel	0.70 (0.03)	0.71 (0.03)	0.72 (0.03)	0.73 (0.03)	0.74 (0.03)	0.75 (0.03)	0.76 (0.03)	0.77 (0.03)	0.78 (0.03)	0.79 (0.03)
Johns Manville	0.76 (0.01)	0.77 (0.02)	0.78 (0.02)	0.78 (0.02)	0.79 (0.02)	0.80 (0.02)	0.80 (0.02)	0.81 (0.02)	0.81 (0.02)	0.82 (0.02)
United Aircraft	0.77 (0.02)	0.80 (0.02)	0.81 (0.03)	0.83 (0.03)	0.85 (0.03)	0.86 (0.03)	0.88 (0.03)	0.89 (0.03)	0.90 (0.03)	0.92 (0.04)
Union Carbide	0.95 (0.01)	1.0 (0.02)	1.06 (0.03)	1.11 (0.03)	1.16 (0.04)	1.20 (0.05)	1.25 (0.05)	1.30 (0.06)	1.34 (0.07)	1.38 (0.07)
Eastman Kodak	0.79 (0.02)	0.81 (0.02)	0.83 (0.03)	0.84 (0.03)	0.85 (0.03)	0.86 (0.03)	0.87 (0.03)	0.88 (0.03)	0.89 (0.03)	0.90 (0.03)
DuPont	0.79 (0.01)	0.80 (0.01)	0.81 (0.01)	0.82 (0.009)	0.82 (0.009)	0.82 (0.009)	0.83 (0.009)	0.84 (0.009)	0.85 (0.009)	0.85 (0.009)
IBM	0.71 (0.01)	0.72 (0.01)	0.73 (0.01)	0.73 (0.01)	0.74 (0.01)	0.74 (0.01)	0.75 (0.01)	0.75 (0.01)	0.76 (0.01)	0.76 (0.01)
DJIA	0.18 (0.01)	0.18 (0.01)	0.18 (0.01)	0.18 (0.01)	0.18 (0.01)	0.19 (0.01)	0.19 (0.01)	0.19 (0.01)	0.19 (0.01)	0.19 (0.01)
S&P500	0.39 (0.01)	0.39 (0.01)	0.39 (0.01)	0.40 (0.01)	0.40 (0.01)	0.40 (0.01)	0.40 (0.01)			

APPENDIX D

GRASSBERGER-PROCACCIA PLOTS (GP) OF DAILY VOLUME AND CORRELATION DIMENSION ESTIMATES FOR THE INDIVIDUAL STOCKS OF THE DJIA INDEX

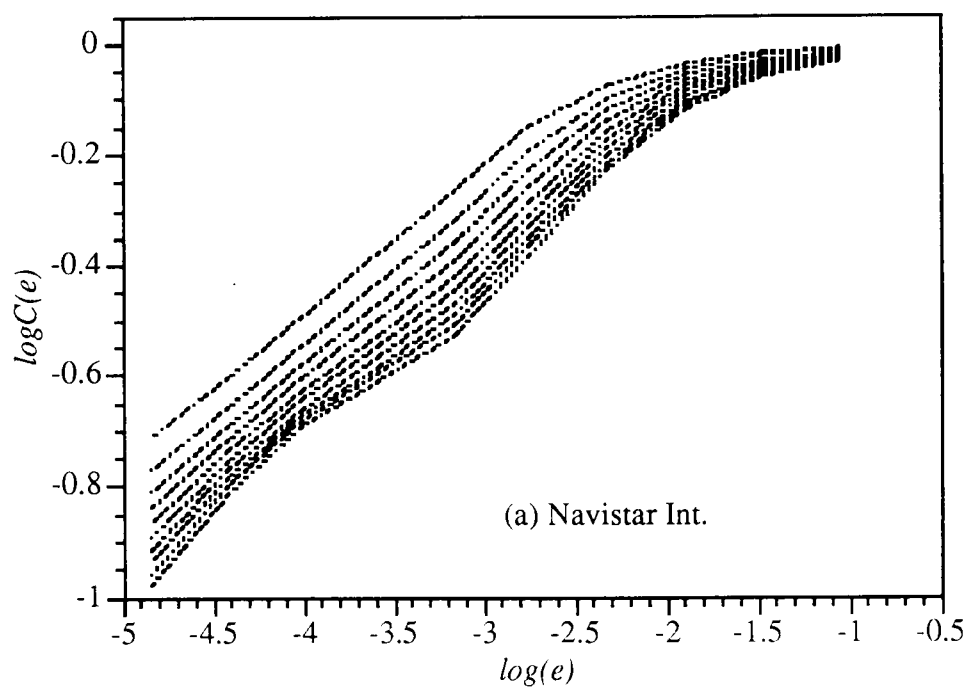


(a)

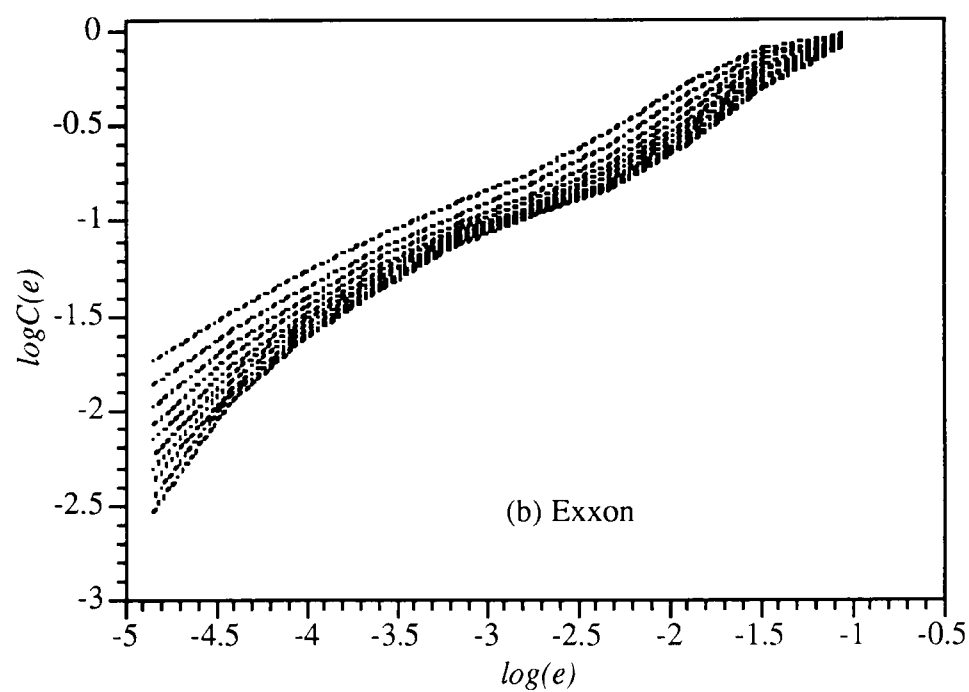


(b)

Figure D-1 GP plots of daily volume series of (a) General Motors, and (b) Primamerica

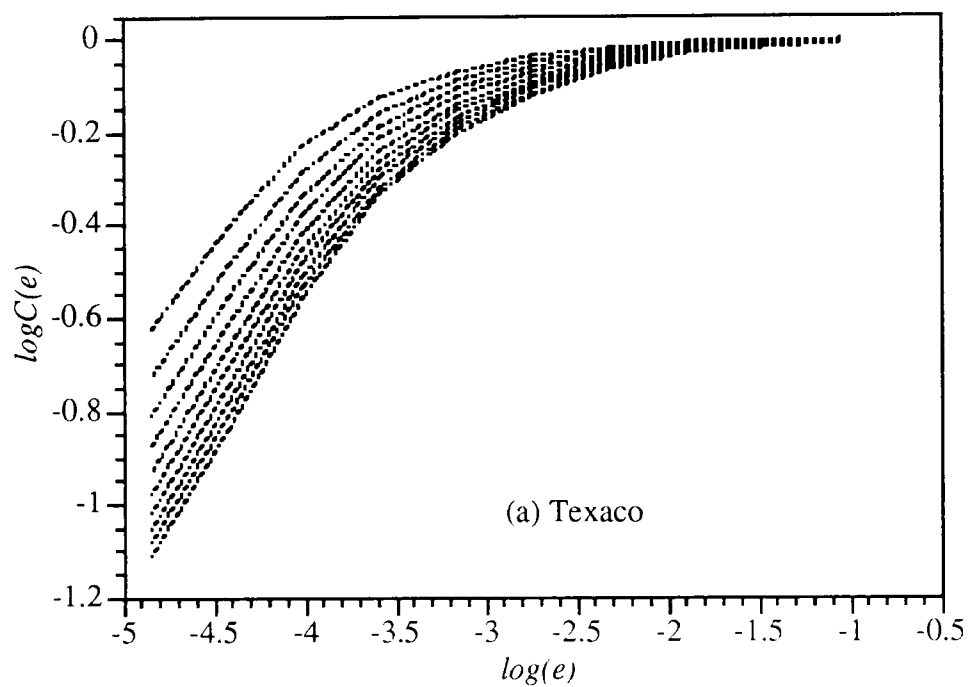


(a)

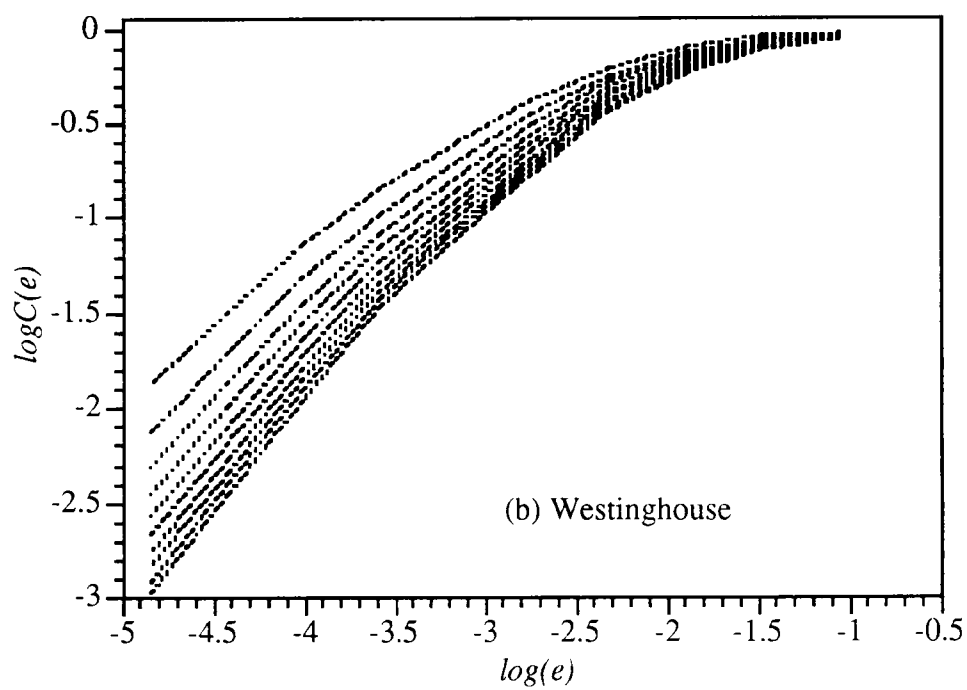


(b)

Figure D-2 GP plots of daily volume series of (a) Navistar Int. and (b) Exxon



(a)



(b)

Figure D-3 GP plots of daily volume series of (a) Texaco and (b) Westinghouse

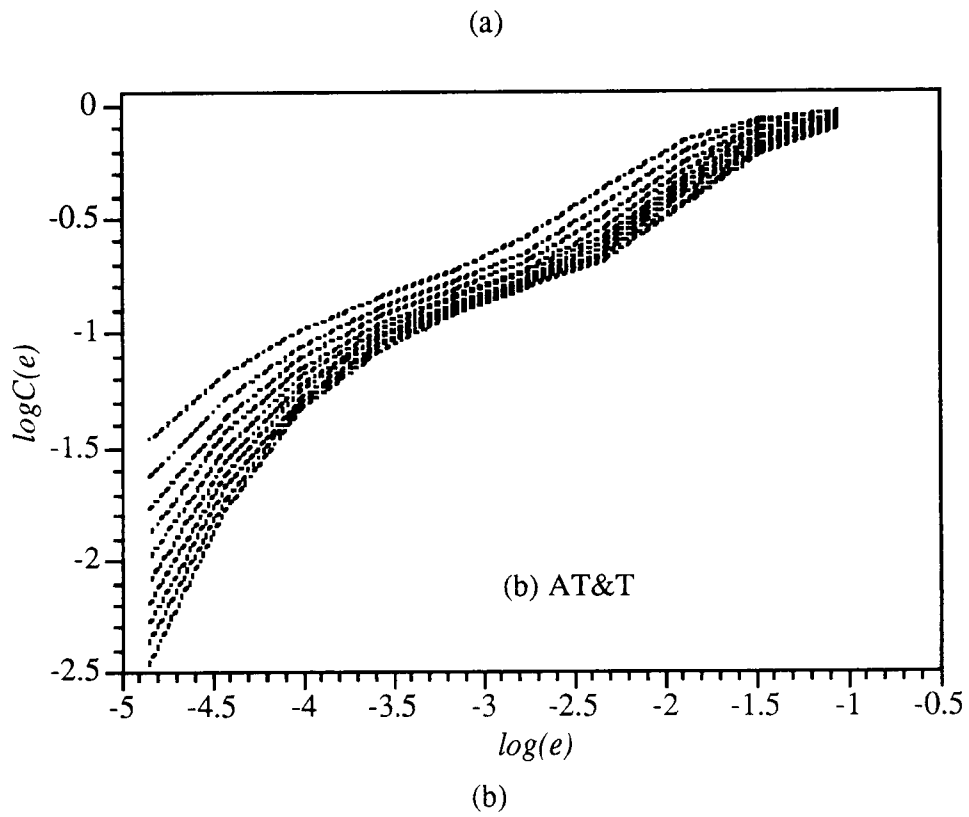
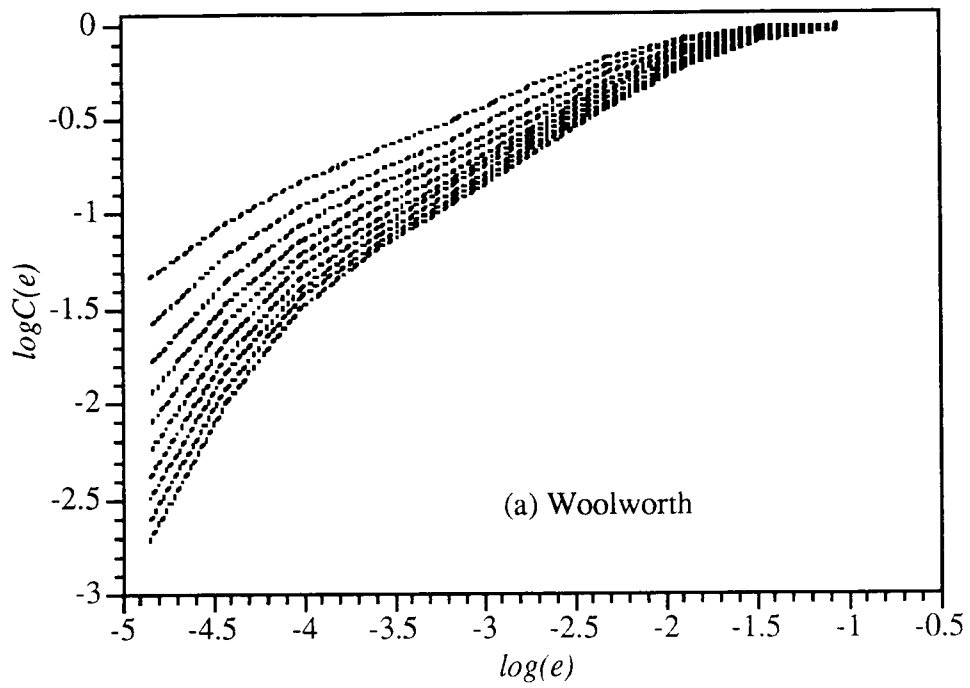
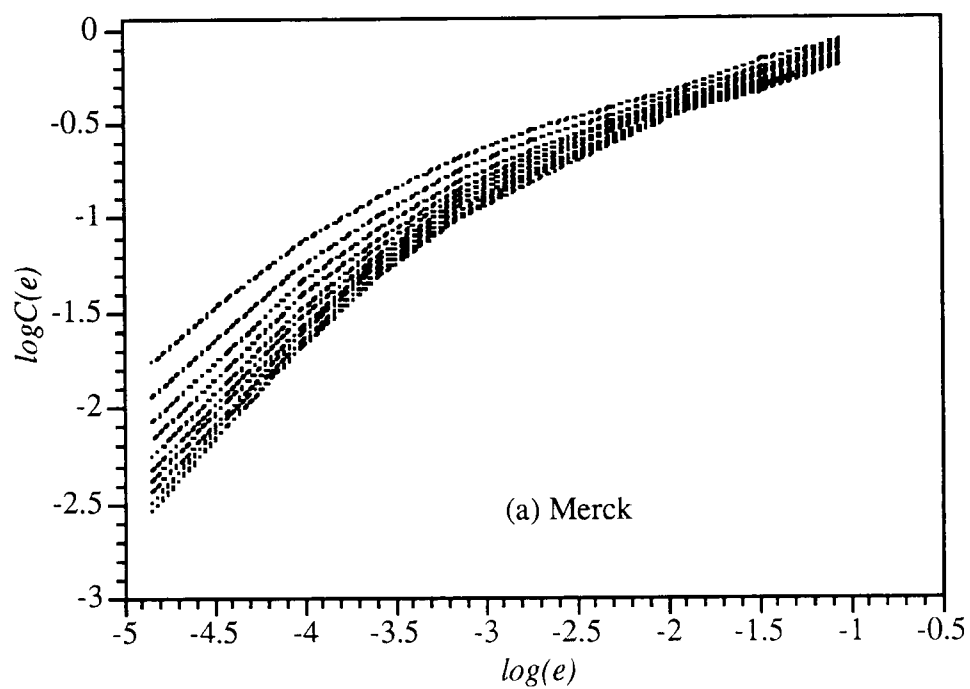
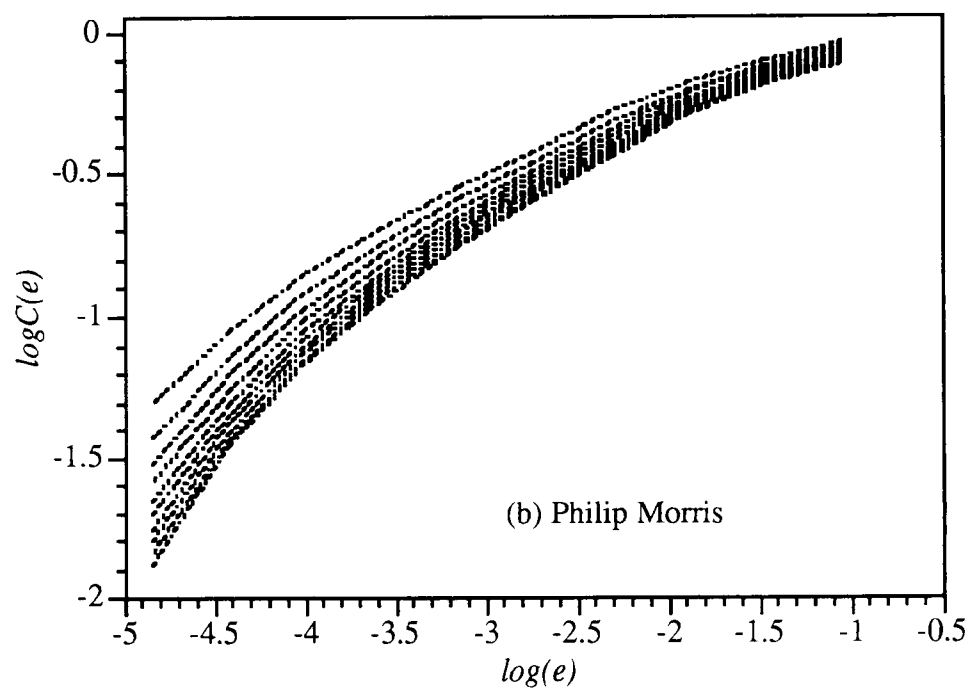


Figure D-4 GP plots of daily volume series of (a) Woolworth and (b) AT&T

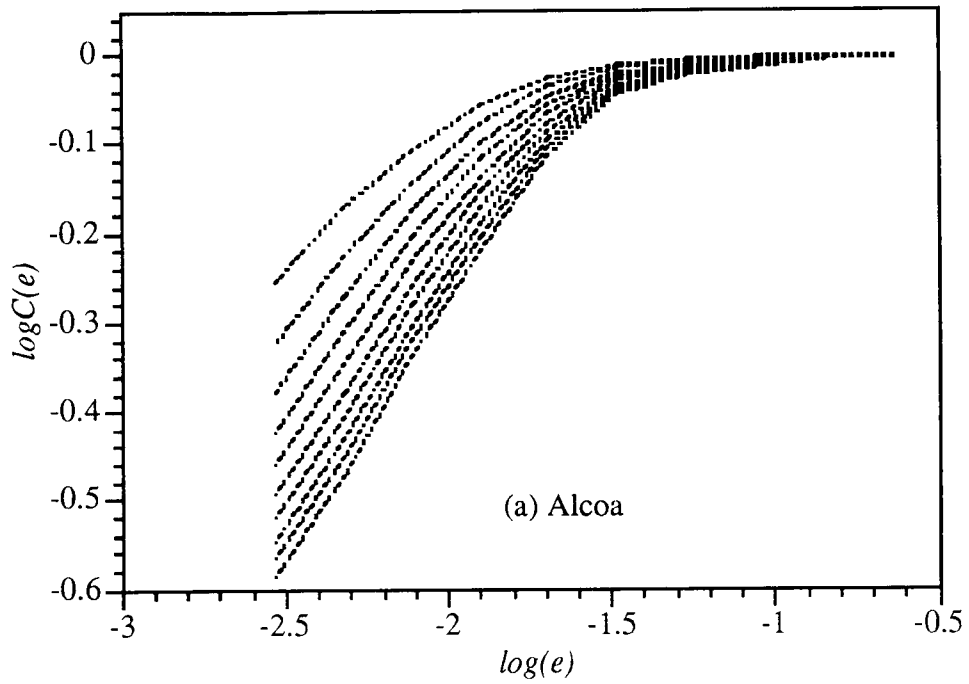


(a)

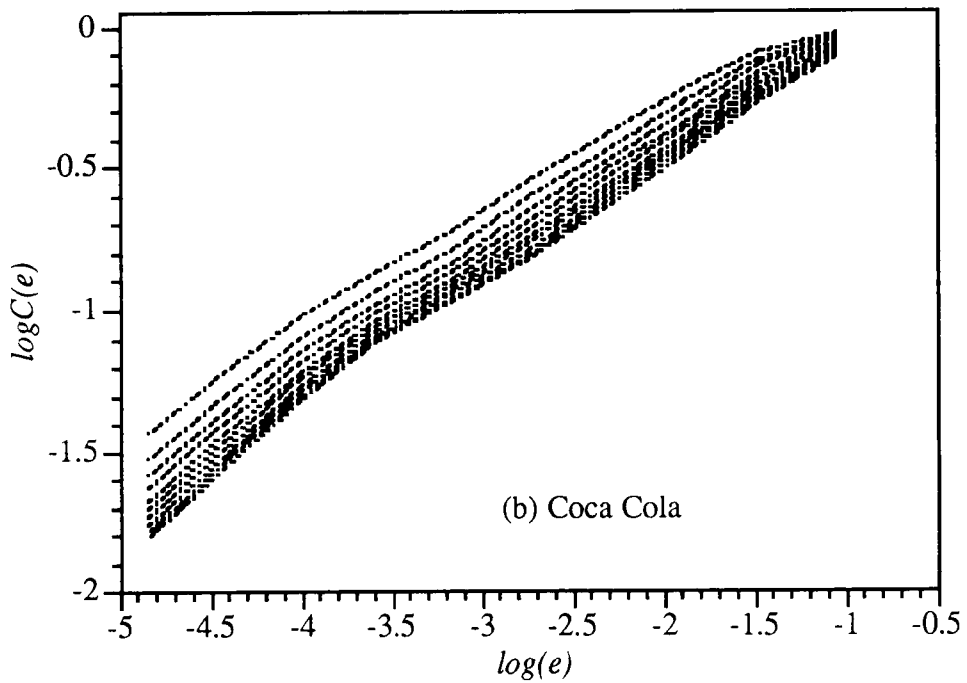


(b)

Figure D-5 GP plots of daily volume series of (a) Merck and (b) Philip Morris

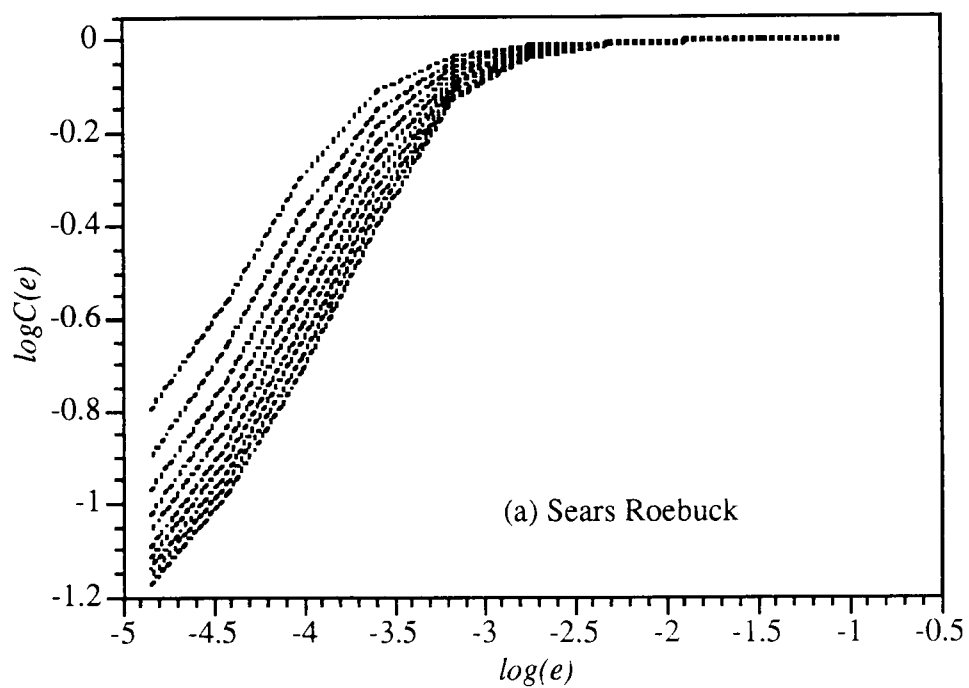


(a)

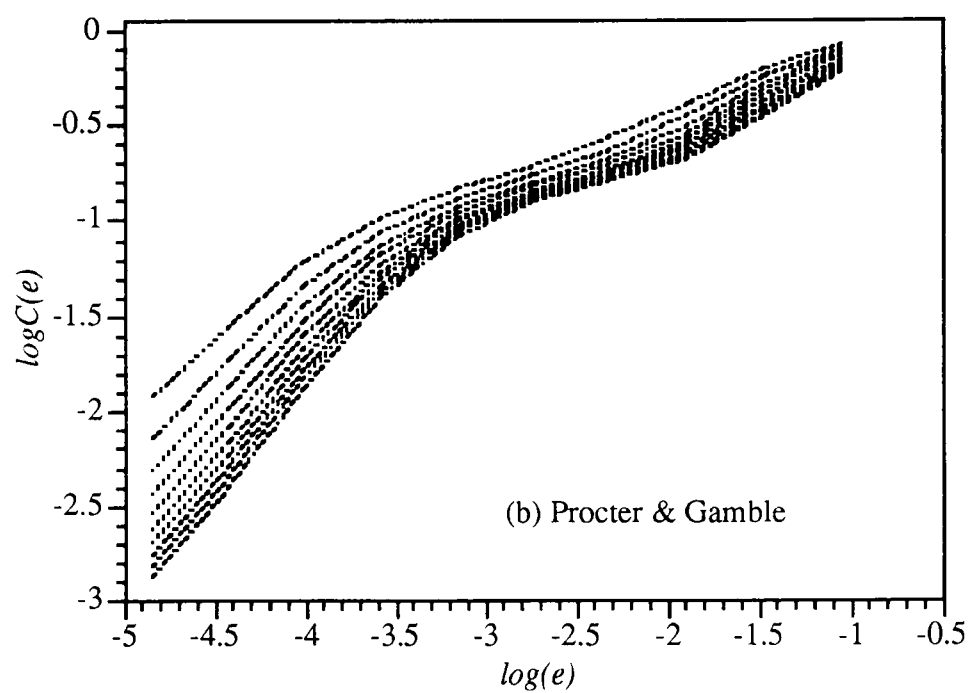


(b)

Figure D-6 GP plots of daily volume series of (a) Alcoa and (b) Coca Cola



(a)



(b)

Figure D-7 GP plots of daily volume series of (a) Sears Roebuck and (b) Procter & Gamble

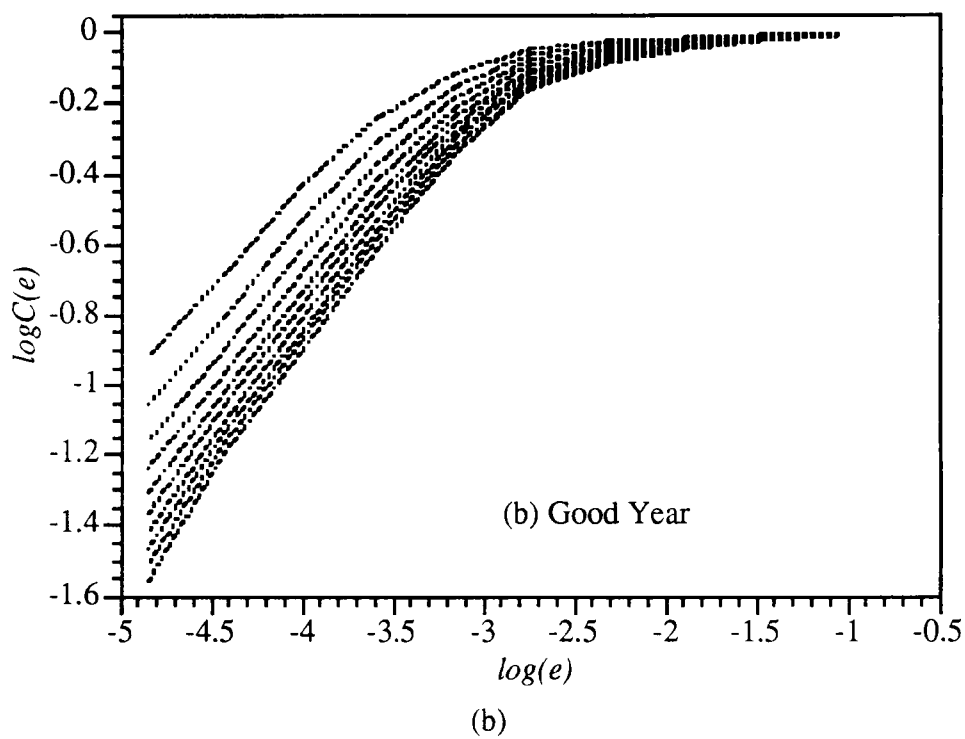
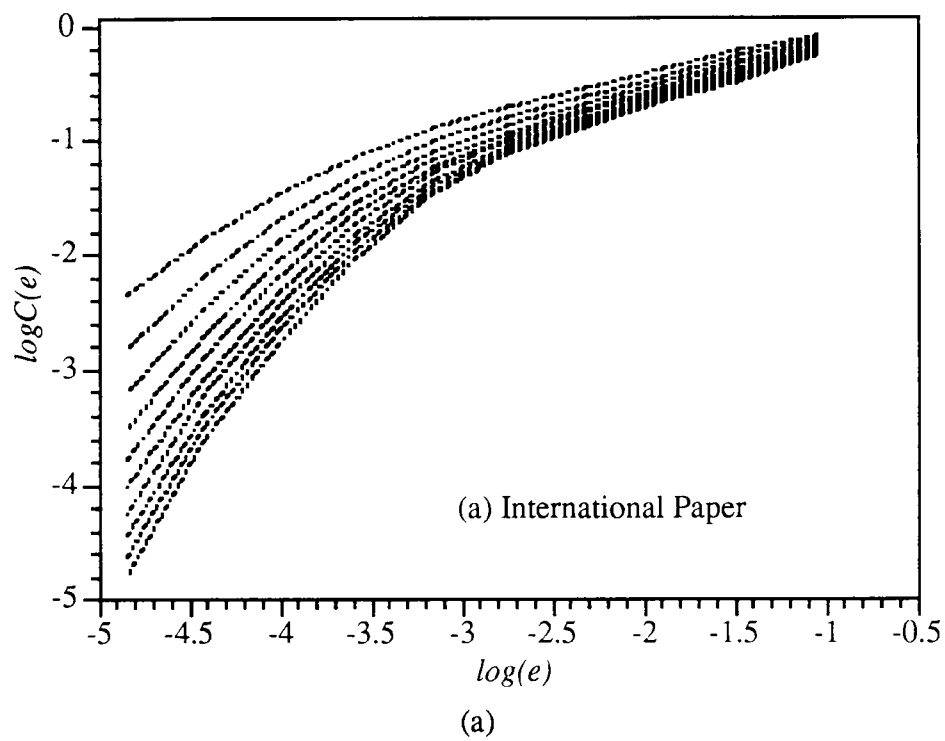
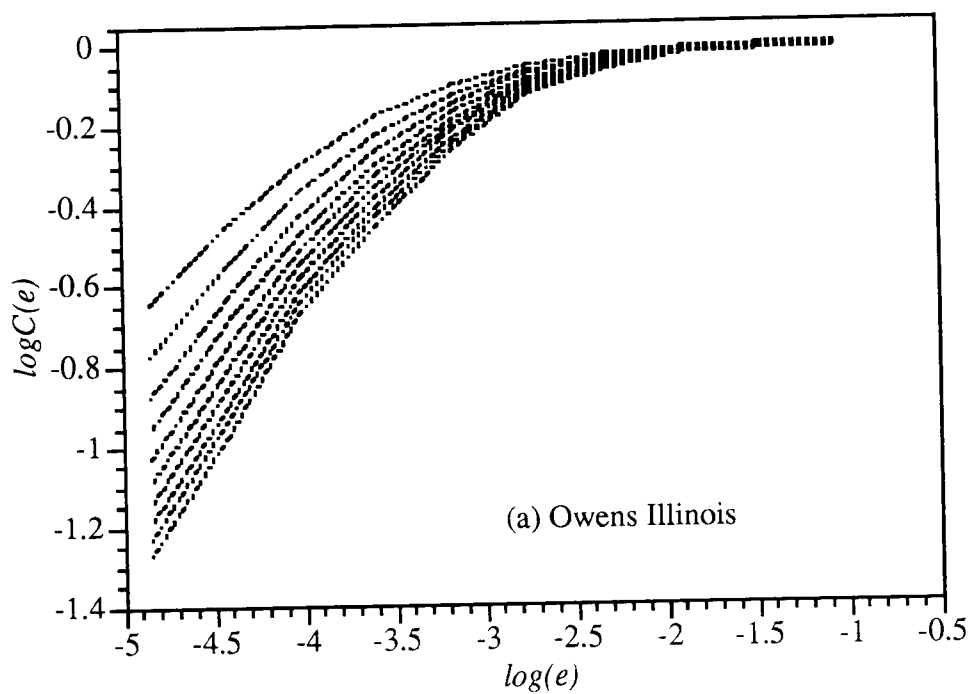
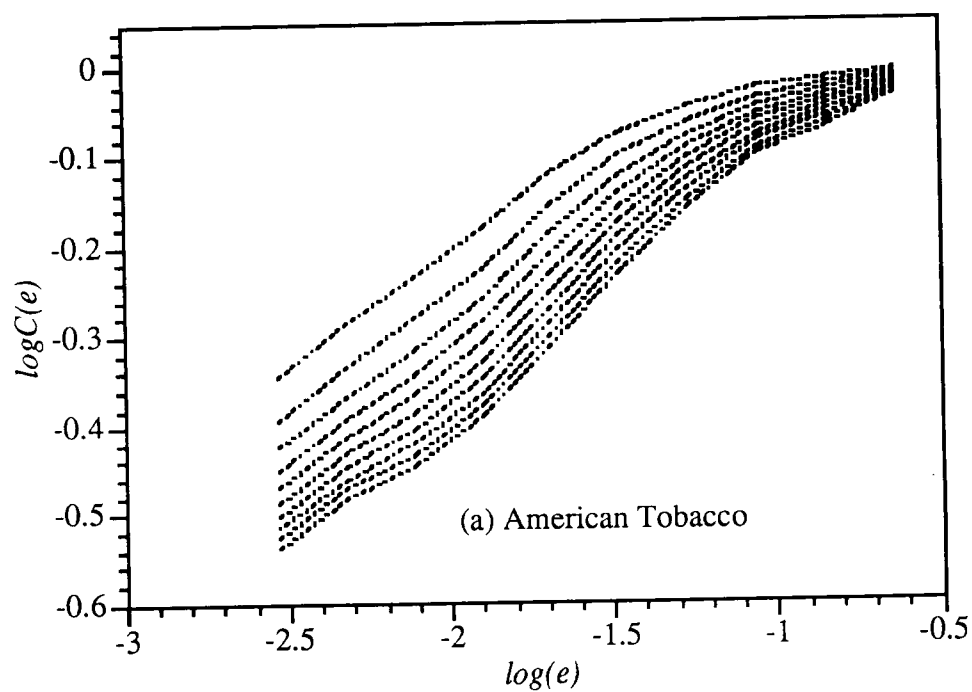


Figure D-8 GP plots of daily volume series of (a) International Paper and (b) Goodyear

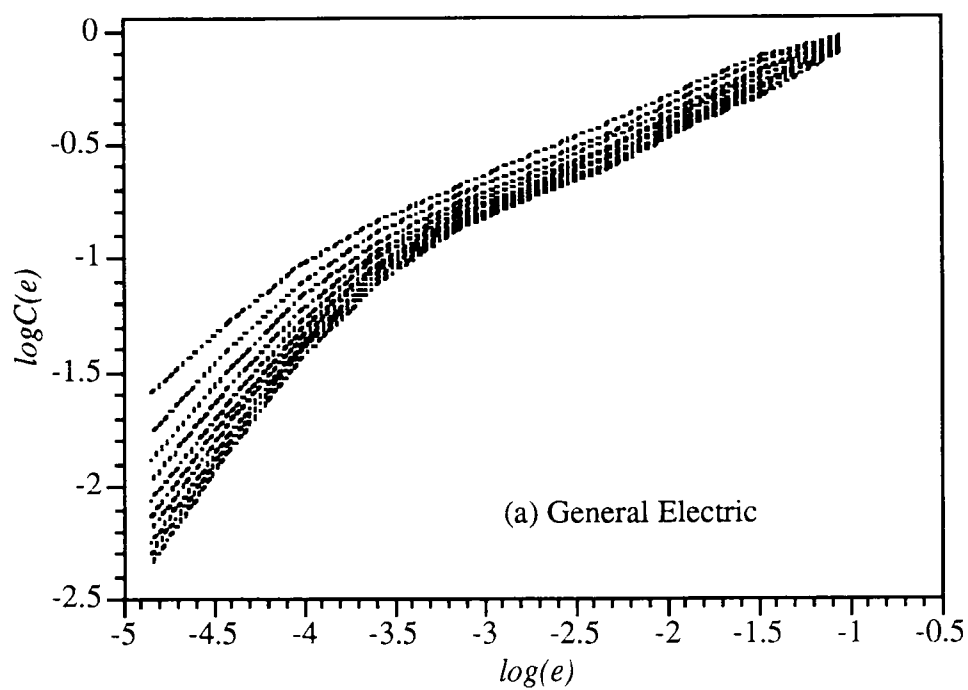


(a)

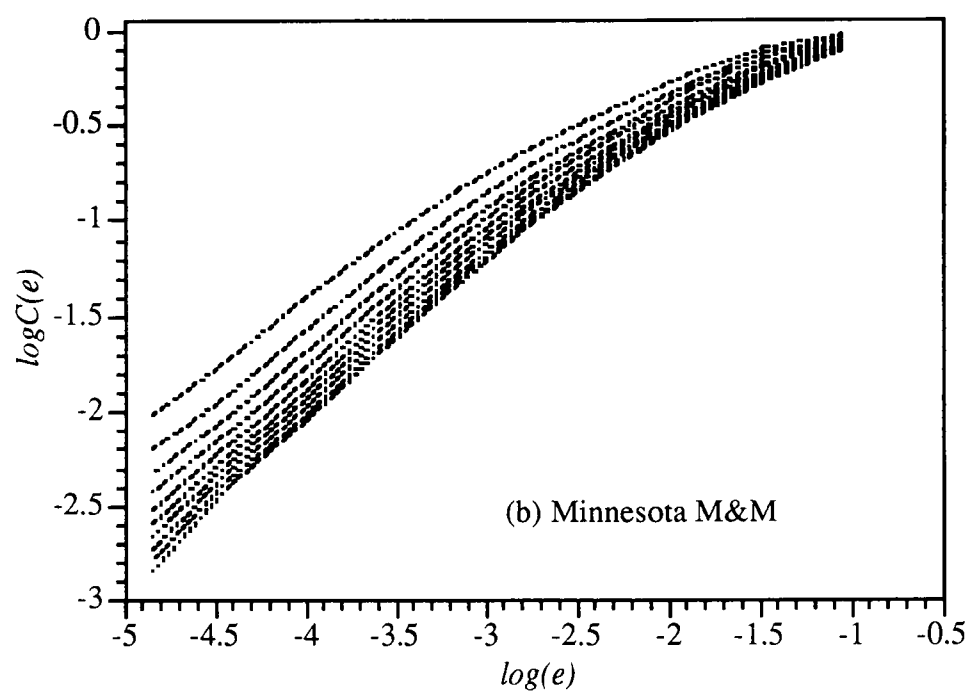


(b)

Figure D-9 GP plots of daily volume series of (a) Owens Illinois and (b) American Tobacco

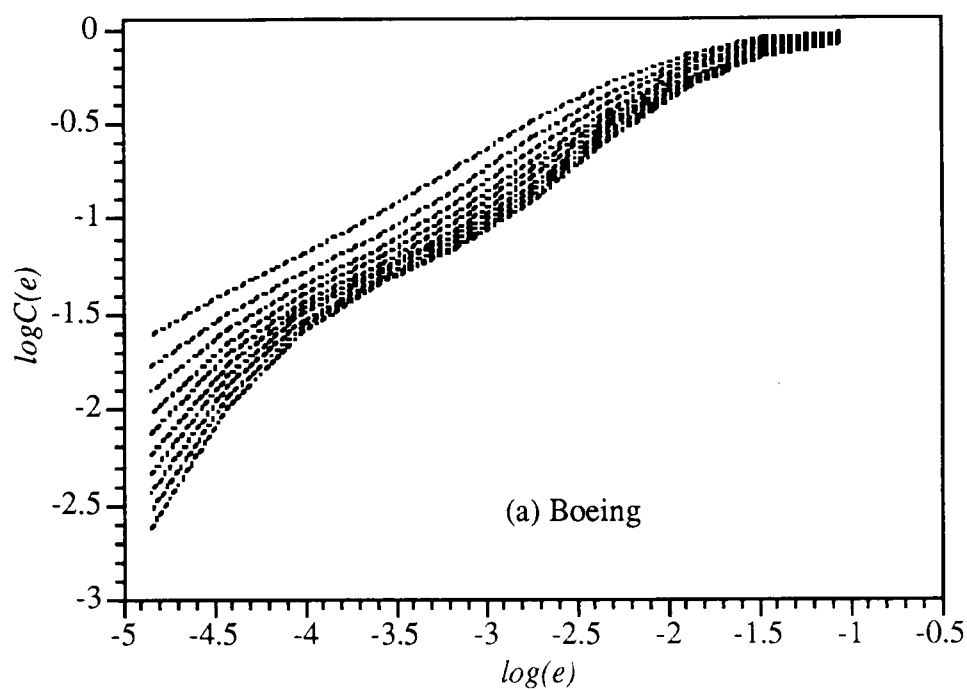


(a)

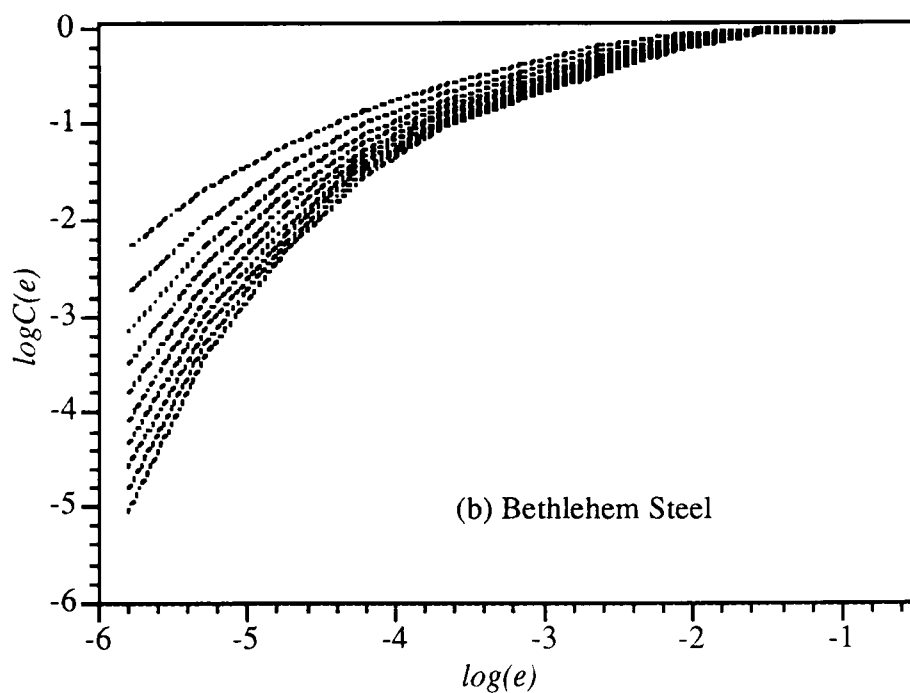


(b)

Figure D-10 GP plots of daily volume series of (a) General Electric and (b) Minnesota M&M

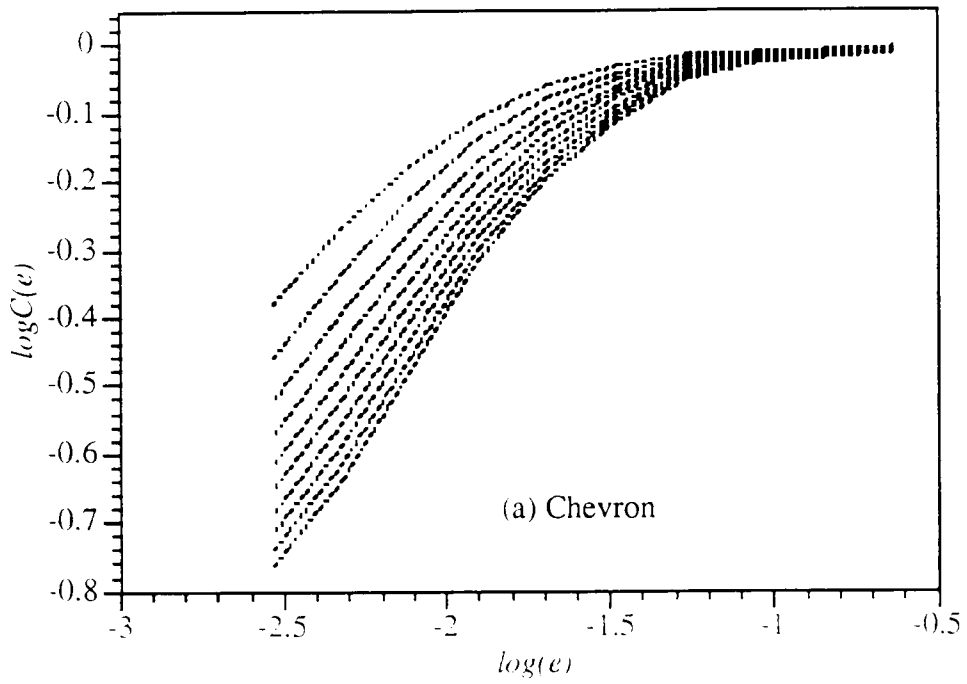


(a)

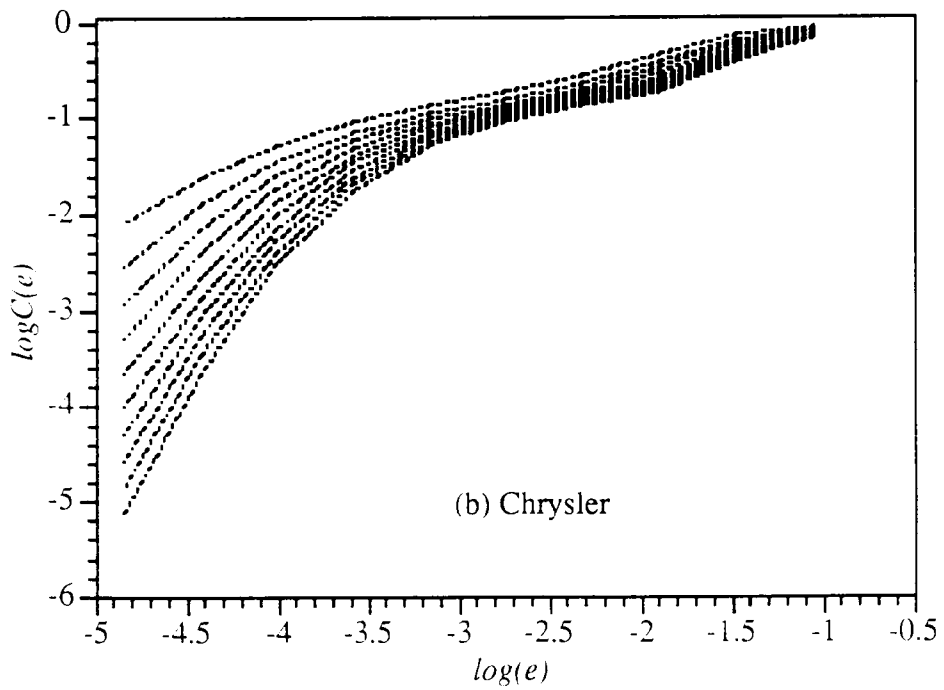


(b)

Figure D-11 GP plots of daily volume series of (a) Boeing and (b) Bethlehem Steel

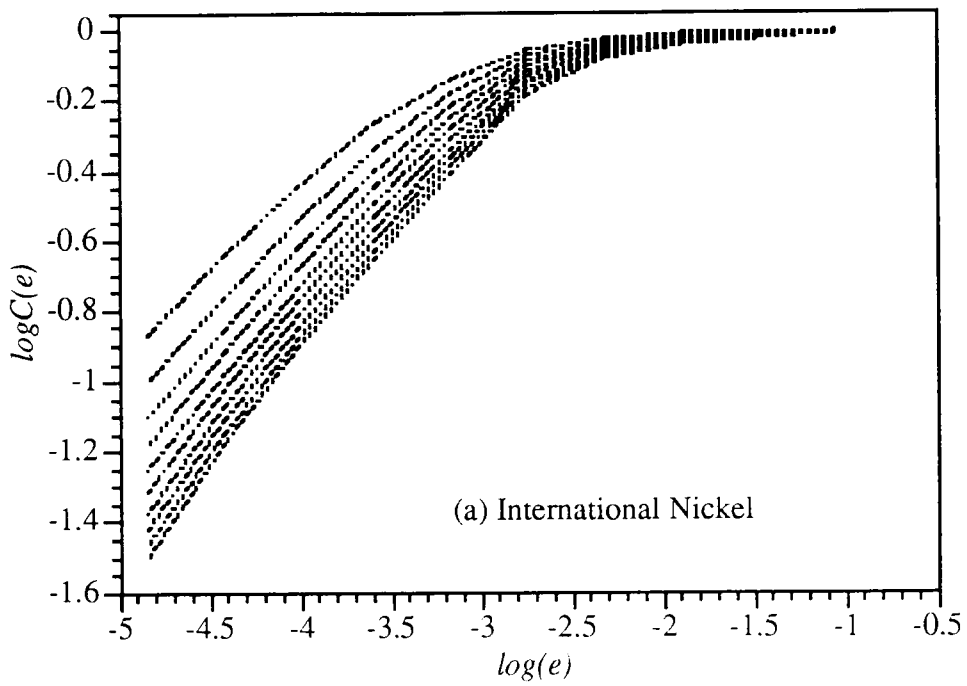


(a)

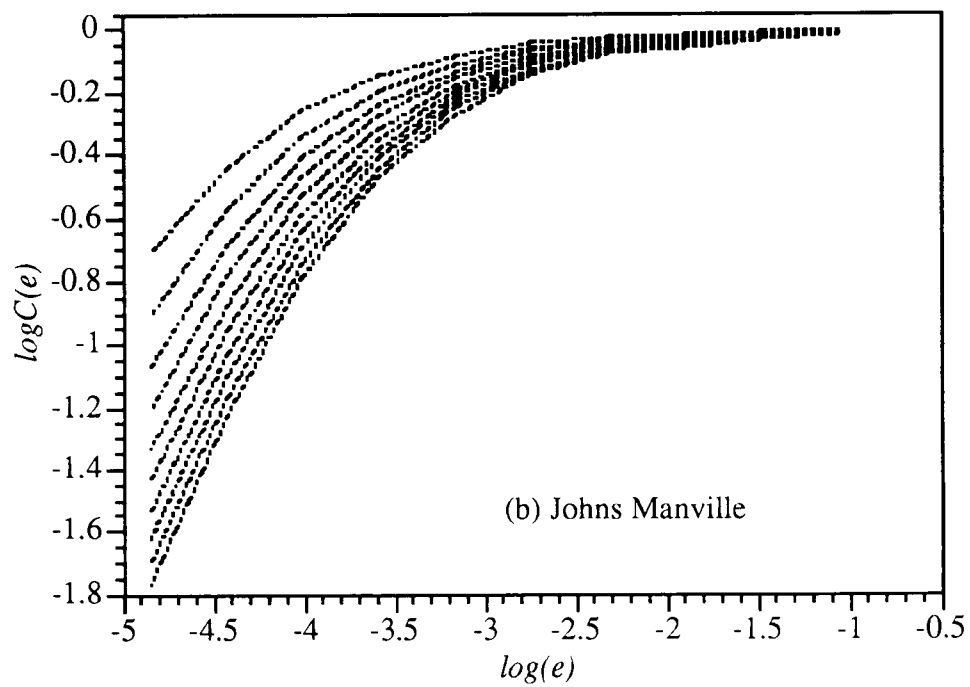


(b)

Figure D-12 GP plots of daily volume series of (a) Chevron and (b) Chrysler

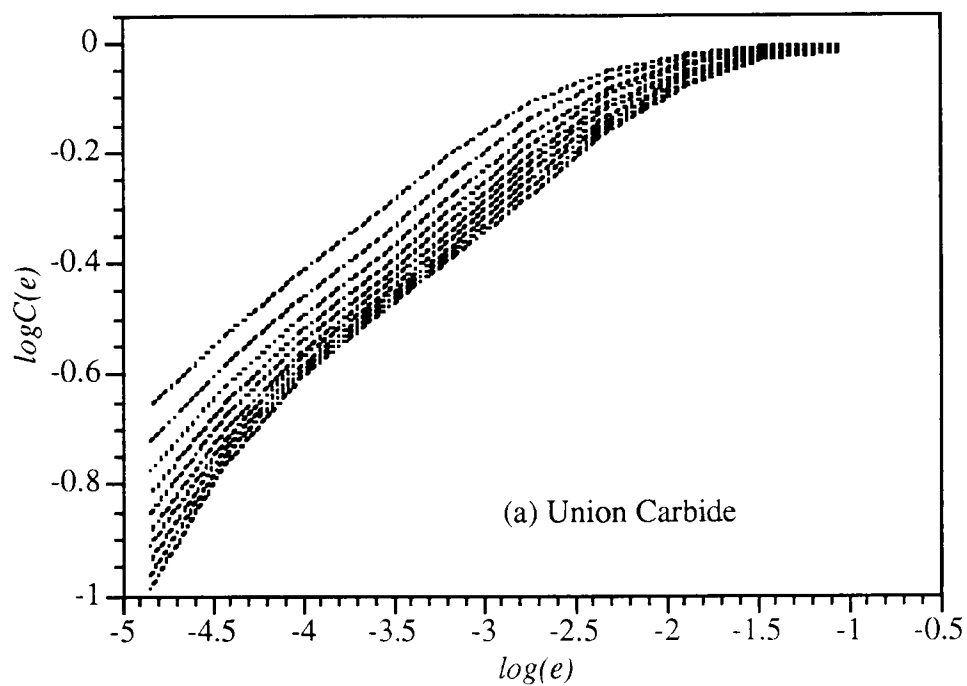


(a)

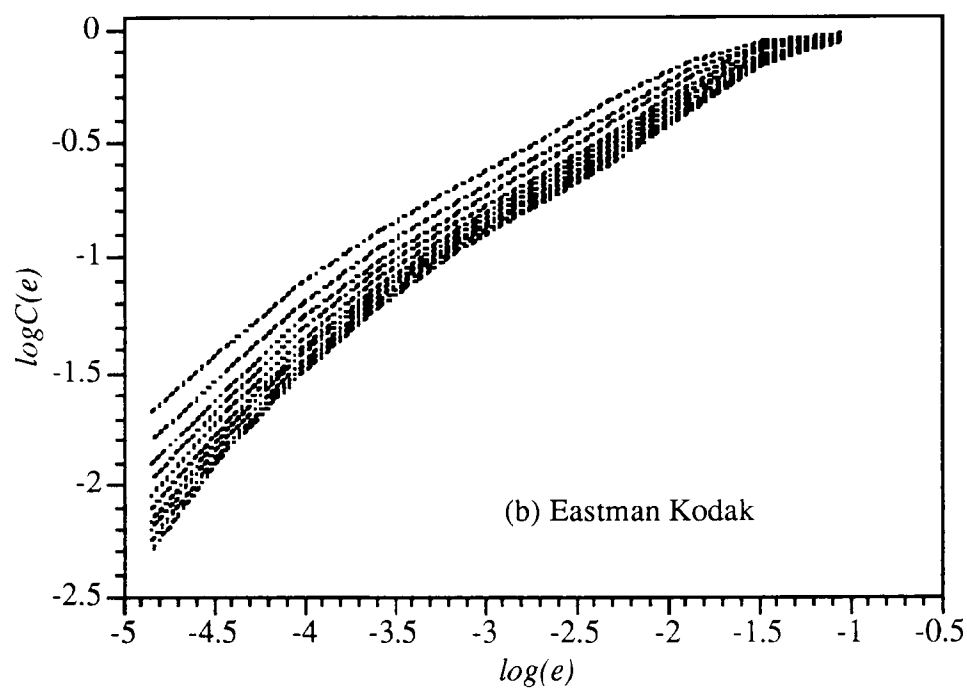


(b)

Figure D-13 GP plots of daily volume series of (a) International Nickel and (b) Johns Manville

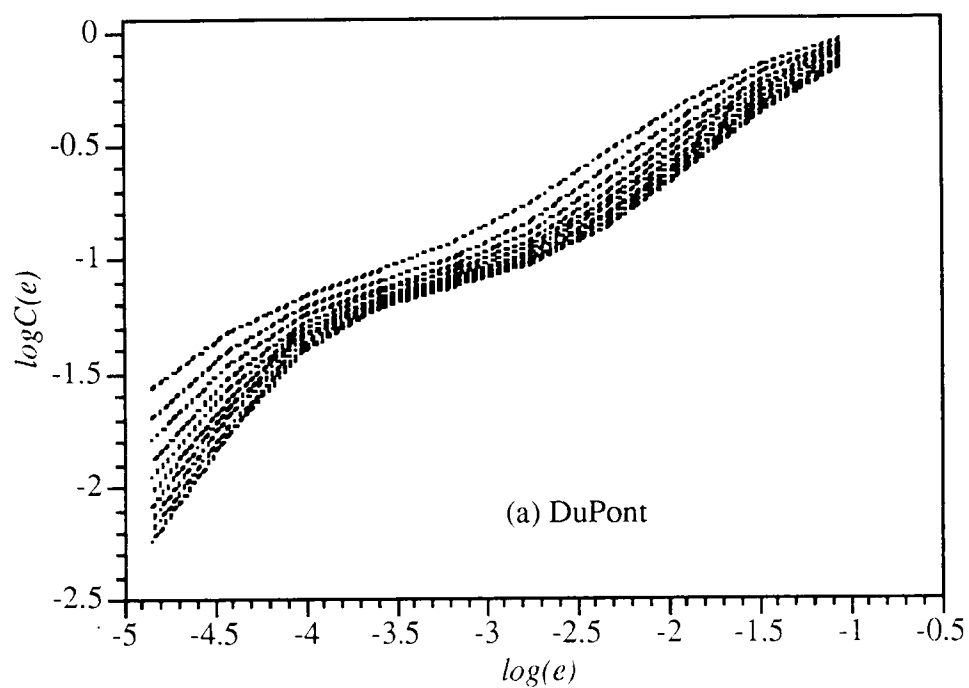


(a)

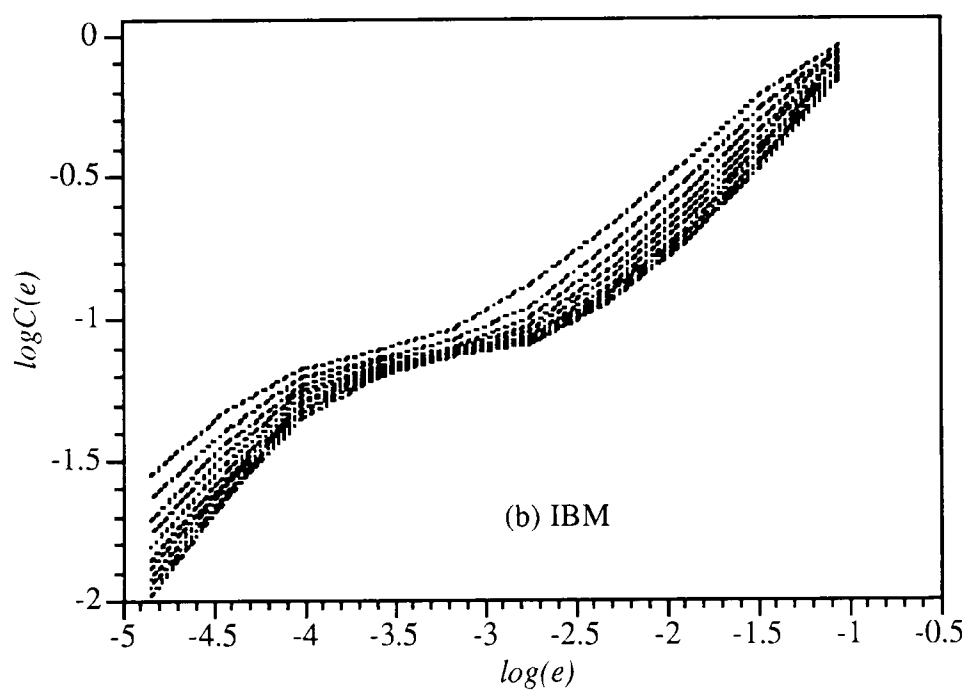


(b)

Figure D-14 GP plots of daily volume series of (a) Union Carbide and (b) Eastman Kodak



(a)



(b)

Figure D-15 GP plots of daily volume series of (a) DuPont and (b) IBM

Table D-1 Correlation Integral Behavior for volume series by embedding dimension
(numbers in parentheses are the standard error of the estimates)

	d = 2	d = 3	d = 4	d = 5	d = 6	d = 7	d = 8	d = 9	d = 10	d = 11
General Motors	0.41 (0.03)	0.44 (0.03)	0.46 (0.02)	0.48 (0.02)	0.49 (0.02)	0.50 (0.02)	0.51 (0.02)	0.51 (0.01)	0.52 (0.01)	0.52 (0.01)
Primamerica	0.20 (0.01)	0.23 (0.01)	0.25 (0.01)	0.26 (0.01)	0.28 (0.01)	0.29 (0.01)	0.30 (0.01)	0.31 (0.01)	0.31 (0.01)	0.32 (0.01)
Navistar Int.	0.24 (0.01)	0.25 (0.008)	0.26 (0.006)	0.26 (0.005)	0.27 (0.004)	0.27 (0.005)	0.27 (0.006)	0.27 (0.007)	0.28 (0.008)	0.28 (0.009)
Exxon	0.45 (0.01)	0.48 (0.01)	0.49 (0.01)	0.51 (0.01)	0.52 (0.01)	0.53 (0.02)	0.54 (0.02)	0.55 (0.02)	0.56 (0.03)	0.57 (0.03)
Texaco	0.33 (0.04)	0.38 (0.04)	0.42 (0.04)	0.45 (0.04)	0.47 (0.04)	0.49 (0.04)	0.50 (0.04)	0.52 (0.04)	0.54 (0.04)	0.55 (0.04)
Westinghouse	0.65 (0.03)	0.75 (0.04)	0.80 (0.04)	0.84 (0.04)	0.88 (0.04)	0.91 (0.04)	0.94 (0.04)	0.96 (0.04)	0.98 (0.04)	1.00 (0.04)
Woolworth	0.23 (0.03)	0.28 (0.04)	0.31 (0.04)	0.34 (0.04)	0.36 (0.04)	0.38 (0.04)	0.40 (0.04)	0.41 (0.04)	0.43 (0.04)	0.44 (0.04)
AT&T	0.38 (0.01)	0.40 (0.02)	0.43 (0.02)	0.45 (0.02)	0.46 (0.03)	0.48 (0.03)	0.50 (0.04)	0.51 (0.04)	0.52 (0.05)	0.54 (0.05)

Table D-1 (continued)

	d = 2	d = 3	d = 4	d = 5	d = 6	d = 7	d = 8	d = 9	d = 10	d = 11
Merck	0.42 (0.03)	0.47 (0.03)	0.50 (0.03)	0.52 (0.04)	0.54 (0.04)	0.55 (0.04)	0.56 (0.04)	0.58 (0.04)	0.59 (0.04)	0.60 (0.04)
Philip Morris	0.32 (0.02)	0.35 (0.02)	0.37 (0.02)	0.38 (0.02)	0.40 (0.02)	0.41 (0.03)	0.42 (0.03)	0.42 (0.03)	0.43 (0.03)	0.44 (0.03)
Alcoa	0.23 (0.02)	0.29 (0.03)	0.34 (0.03)	0.38 (0.03)	0.42 (0.03)	0.44 (0.03)	0.47 (0.03)	0.49 (0.02)	0.51 (0.02)	0.52 (0.02)
Coca Cola	0.37 (0.01)	0.39 (0.01)	0.40 (0)	0.41 (0)	0.41 (0)	0.42 (0.009)	0.42 (0.009)	0.42 (0.01)	0.43 (0.01)	0.43 (0.01)
Sears Roebuck	0.47 (0.05)	0.52 (0.04)	0.56 (0.03)	0.58 (0.03)	0.60 (0.02)	0.61 (0.02)	0.62 (0.02)	0.62 (0.02)	0.63 (0.02)	0.63 (0.02)
Procter & Gamble	0.36 (0.01)	0.37 (0.01)	0.38 (0.01)	0.38 (0.02)	0.38 (0.02)	0.38 (0.02)	0.38 (0.02)	0.38 (0.02)	0.38 (0.02)	0.38 (0.02)
International Paper	0.41 (0.01)	0.46 (0.02)	0.49 (0.02)	0.52 (0.03)	0.55 (0.03)	0.57 (0.04)	0.60 (0.04)	0.62 (0.05)	0.64 (0.05)	0.66 (0.06)
Goodyear	0.43 (0.03)	0.54 (0.02)	0.58 (0.02)	0.61 (0.02)	0.64 (0.01)	0.66 (0.01)	0.67 (0.01)	0.69 (0.01)	0.70 (0.02)	0.71 (0.02)

Table D-1 (continued)

	d = 2	d = 3	d = 4	d = 5	d = 6	d = 7	d = 8	d = 9	d = 10	d = 11
Owens Illinois	0.27 (0.03)	0.33 (0.03)	0.38 (0.03)	0.41 (0.04)	0.44 (0.04)	0.47 (0.04)	0.49 (0.04)	0.51 (0.04)	0.52 (0.04)	0.54 (0.04)
American Tobacco	0.18 (0.01)	0.21 (0.01)	0.23 (0.01)	0.24 (0.01)	0.25 (0.01)	0.26 (0.01)	0.27 (0.01)	0.27 (0.01)	0.28 (0.01)	0.28 (0.01)
General Electric	0.32 (0.009)	0.34 (0.006)	0.35 (0.005)	0.35 (0.004)	0.36 (0.004)	0.36 (0.005)	0.37 (0.006)	0.37 (0.007)	0.37 (0.009)	0.37 (0.01)
Minnesota M&M	0.53 (0.03)	0.58 (0.03)	0.62 (0.03)	0.64 (0.02)	0.66 (0.02)	0.67 (0.02)	0.69 (0.03)	0.70 (0.03)	0.71 (0.03)	0.72 (0.03)
Boeing	0.44 (0.02)	0.48 (0.02)	0.51 (0.02)	0.53 (0.02)	0.55 (0.02)	0.57 (0.03)	0.58 (0.03)	0.60 (0.04)	0.62 (0.04)	0.64 (0.04)
Bethlehem Steel	0.45 (0.05)	0.54 (0.07)	0.61 (0.08)	0.67 (0.09)	0.72 (0.10)	0.77 (0.11)	0.82 (0.12)	0.86 (0.12)	0.90 (0.13)	0.94 (0.13)
Chevron	0.33 (0.3)	0.40 (0.04)	0.45 (0.04)	0.49 (0.03)	0.53 (0.03)	0.56 (0.03)	0.58 (0.03)	0.60 (0.30)	0.62 (0.30)	0.64 (0.30)
Chrysler	0.40 (0.01)	0.43 (0.01)	0.45 (0.01)	0.47 (0.01)	0.49 (0.01)	0.51 (0.02)	0.52 (0.02)	0.54 (0.03)	0.56 (0.03)	0.58 (0.04)

Table D-1 (continued)

	d = 2	d = 3	d = 4	d = 5	d = 6	d = 7	d = 8	d = 9	d = 10	d = 11
International Nickel	0.39 (0.03)	0.45 (0.02)	0.49 (0.02)	0.51 (0.02)	0.54 (0.02)	0.56 (0.02)	0.58 (0.02)	0.59 (0.02)	0.61 (0.02)	0.62 (0.02)
Johns Manville	0.30 (0.05)	0.39 (0.06)	0.46 (0.07)	0.52 (0.08)	0.57 (0.08)	0.62 (0.08)	0.66 (0.09)	0.70 (0.09)	0.73 (0.09)	0.76 (0.09)
Union Carbide	0.24 (0.01)	0.26 (0.009)	0.27 (0.009)	0.28 (0.01)	0.28 (0.01)	0.29 (0.01)	0.29 (0.01)	0.30 (0.01)	0.30 (0.01)	0.30 (0.02)
Eastman Kodak	0.44 (0.02)	0.47 (0.02)	0.49 (0.02)	0.51 (0.02)	0.52 (0.02)	0.53 (0.02)	0.54 (0.03)	0.55 (0.03)	0.56 (0.03)	0.56 (0.03)
DuPont	0.40 (0.01)	0.41 (0.01)	0.41 (0.02)	0.41 (0.02)	0.41 (0.02)	0.41 (0.02)	0.41 (0.02)	0.41 (0.02)	0.41 (0.02)	0.41 (0.02)
IBM	0.48 (0.01)	0.49 (0.02)	0.50 (0.03)	0.49 (0.03)	0.49 (0.04)	0.48 (0.04)	0.48 (0.05)	0.47 (0.05)	0.47 (0.05)	0.47 (0.05)

APPENDIX E

CORRELATION INTEGRAL BEHAVIOR
AND
BDS STATISTIC FOR WEEKLY OBSERVATIONS
OF ELEVEN INTERNATIONAL INDICES

Table E-1 Correlation Integral values and BDS statistic for Australia

M	ϵ	Index Levels	BDS	Residuals of index levels	BDS	Returns	BDS	Residuals of Returns	BDS
2	0.5	0.32441	221.98045	0.26387	12.37628	0.11424	3.80586	0.1087	3.58537
3	0.5	0.31616	383.40286	0.16574	17.37673	0.03995	4.03621	0.03676	3.48616
4	0.5	0.3091	716.00974	0.1144	23.10215	0.01416	4.21288	0.01267	3.74322
5	0.5	0.30263	1443.98404	0.08601	31.52302	0.00533	5.36248	0.00455	4.52423
6	0.5	0.29659	3092.5926	0.06768	43.37883	0.00203	6.17336	0.00163	4.85917
7	0.5	0.29092	6932.81322	0.05532	61.32172	0.00071	5.37749	0.00054	3.91775
8	0.5	0.28574	16099.6106	0.04671	89.5645	0.00021	3.02381	0.00016	2.57096
9	0.5	0.28089	38407.6798	0.04012	133.7959	0.00006	1.25689	0.00005	2.04969
10	0.5	0.27627	93572.6845	0.0356	208.4076	0.00001	-0.3434	0.00001	0.81787
2	0.75	0.44087	201.45878	0.43998	11.30787	0.23788	4.20812	0.22539	3.64904
3	0.75	0.43443	285.58369	0.32742	14.58952	0.11956	4.43466	0.10883	3.31158
4	0.75	0.42888	424.36987	0.2581	17.85313	0.06091	4.6619	0.05348	3.54664
5	0.75	0.42375	666.59688	0.21368	21.73531	0.03204	5.39536	0.02722	4.33601
6	0.75	0.419	1097.7116	0.18382	26.57669	0.01705	5.96649	0.01392	4.77177
7	0.75	0.41455	1878.10291	0.1605	32.16405	0.00914	6.42193	0.00708	4.93045
8	0.75	0.41042	3314.12114	0.1428	39.3433	0.00491	6.76271	0.00358	4.9126
9	0.75	0.40658	5995.61697	0.12786	48.24137	0.00263	7.01103	0.0018	4.80903
10	0.75	0.4029	11065.8472	0.11602	60.04004	0.00149	7.91836	0.00097	5.50775
2	1	0.51837	196.48457	0.59271	9.99028	0.37943	4.59286	0.36167	4.0842
3	1	0.51423	252.82056	0.48573	12.13591	0.23881	4.59302	0.22125	3.86537
4	1	0.51062	335.97096	0.41248	14.26479	0.1521	4.8039	0.13692	4.0069
5	1	0.50737	466.66996	0.35981	16.32601	0.09914	5.45738	0.08709	4.72608
6	1	0.50437	673.78885	0.32218	18.68083	0.06533	5.98879	0.05575	5.17337
7	1	0.50157	1004.8771	0.29225	21.06953	0.04324	6.3687	0.03617	5.66631
8	1	0.49892	1539.47218	0.26899	23.81945	0.02877	6.69943	0.0234	5.93344
9	1	0.4964	2411.6972	0.24901	26.80112	0.01896	6.77943	0.01524	6.21876
10	1	0.49399	3848.87284	0.23288	30.36059	0.01289	7.31203	0.01027	6.93122
2	1.25	0.57424	172.61848	0.71184	9.18568	0.51755	4.86743	0.49794	4.4804
3	1.25	0.56958	208.20415	0.62084	10.40852	0.3777	4.64701	0.3561	4.19382
4	1.25	0.56545	257.06349	0.55253	11.77956	0.27764	4.69481	0.25712	4.34834
5	1.25	0.56173	329.30243	0.49954	13.02373	0.20759	5.25619	0.1891	4.9646
6	1.25	0.55831	435.91252	0.45774	14.2338	0.15644	5.70303	0.13935	5.2524
7	1.25	0.55508	593.31397	0.42326	15.36482	0.11854	6.05676	0.10343	5.53876
8	1.25	0.55204	826.86337	0.39576	16.62997	0.09043	6.4082	0.07711	5.78516
9	1.25	0.54914	1175.60355	0.3719	17.87023	0.06892	6.62221	0.05772	6.01184
10	1.25	0.54644	1700.34277	0.35222	19.26668	0.05343	7.08803	0.04417	6.55229
2	1.5	0.63357	136.5053	0.79749	9.20049	0.64204	5.41786	0.62073	4.94577
3	1.5	0.62819	155.02689	0.72571	9.67459	0.51957	5.01106	0.4941	4.60879
4	1.5	0.62356	178.86462	0.6679	10.48083	0.42263	4.9906	0.39591	4.70358
5	1.5	0.61949	212.71328	0.62142	11.37875	0.34755	5.44106	0.32103	5.20395
6	1.5	0.6158	259.92647	0.58176	12.0577	0.28691	5.72898	0.26053	5.38581
7	1.5	0.61238	325.08637	0.54774	12.66462	0.23783	5.97072	0.21196	5.4988
8	1.5	0.60918	414.76238	0.5193	13.33468	0.19791	6.19506	0.17302	5.61372
9	1.5	0.60614	538.33207	0.4937	13.91586	0.16477	6.33001	0.14163	5.72692
10	1.5	0.60322	709.16588	0.47109	14.50237	0.13849	6.63013	0.11727	6.03287

Table E-2 Correlation Integral values and BDS statistic for Belgium

M	ε	Index Levels	BDS	Residuals of index levels	BDS	Returns	BDS	Residuals of Returns	BDS
2	0.5	0.36061	134.7723	0.28609	12.8682	0.11869	3.36917	0.1162	3.03201
3	0.5	0.35548	218.1410	0.19559	18.0872	0.04343	4.07282	0.04164	3.61853
4	0.5	0.35096	377.6721	0.14797	24.4488	0.01652	4.79822	0.01507	3.74476
5	0.5	0.34686	701.6374	0.11894	33.1438	0.00631	5.06254	0.00566	4.25006
6	0.5	0.34304	1378.9667	0.10013	46.2411	0.00257	6.00099	0.00225	5.20077
7	0.5	0.3395	2830.6225	0.08687	66.3786	0.00109	7.2029	0.00093	6.33004
8	0.5	0.33624	6008.7512	0.07711	98.1732	0.00047	8.46797	0.0004	7.76859
9	0.5	0.33325	13093.1152	0.06949	148.9292	0.00019	8.58748	0.00016	8.18135
10	0.5	0.33044	29119.0485	0.06336	231.1677	0.00008	9.30815	0.00007	9.45425
2	0.75	0.46455	146.9046	0.42731	11.7609	0.23887	4.26843	0.23424	3.1294
3	0.75	0.46033	202.0793	0.3256	15.2188	0.123	4.90576	0.11864	3.92111
4	0.75	0.45654	289.8383	0.26609	18.9247	0.0651	5.50278	0.06133	4.41659
5	0.75	0.45304	437.9609	0.22582	22.9618	0.03506	5.96799	0.03252	4.98696
6	0.75	0.44976	691.9357	0.19976	28.4753	0.01941	6.62904	0.01755	5.52131
7	0.75	0.44677	1134.0922	0.18114	35.7630	0.01092	7.31576	0.0096	6.03613
8	0.75	0.44398	1914.7565	0.16654	45.3559	0.0064	8.44749	0.00549	7.0007
9	0.75	0.44141	3311.7870	0.15489	58.2960	0.00368	9.23615	0.00304	7.43823
10	0.75	0.43897	5841.2500	0.14489	75.7052	0.00215	10.2499	0.00177	8.46904
2	1	0.53113	170.9854	0.54445	11.6768	0.36689	4.84653	0.36439	3.21979
3	1	0.52838	217.1885	0.44447	13.9812	0.23197	5.45962	0.22888	4.23454
4	1	0.52582	284.4413	0.3776	16.0232	0.1499	6.02782	0.1466	4.89638
5	1	0.52339	388.7830	0.3314	18.3189	0.09821	6.45402	0.09569	5.49825
6	1	0.52106	551.7202	0.29899	21.1171	0.06619	7.19384	0.0638	6.17877
7	1	0.51899	808.2637	0.27444	24.4120	0.04508	7.86471	0.04316	6.84483
8	1	0.517	1215.5457	0.25559	28.4425	0.0313	8.70322	0.02979	7.66073
9	1	0.51505	1868.3016	0.24086	33.4457	0.02183	9.50831	0.02062	8.38865
10	1	0.51315	2924.3618	0.2279	39.4557	0.01539	10.4411	0.01445	9.23383
2	1.25	0.57853	168.8451	0.64022	12.4459	0.48923	5.20063	0.48756	3.23046
3	1.25	0.57561	203.4105	0.55112	14.0748	0.35412	5.84602	0.35099	4.28409
4	1.25	0.57289	250.8035	0.48473	15.1312	0.26062	6.38723	0.25642	4.93468
5	1.25	0.57042	320.8271	0.43621	16.4444	0.19385	6.77102	0.18988	5.47583
6	1.25	0.56811	424.0263	0.40107	18.1088	0.14689	7.35937	0.14258	6.02213
7	1.25	0.56614	576.4005	0.37358	19.9613	0.11223	7.87811	0.10804	6.50813
8	1.25	0.56421	802.0702	0.35186	22.1062	0.08733	8.60255	0.0832	7.12886
9	1.25	0.56237	1138.5051	0.33357	24.5258	0.06801	9.20531	0.06423	7.64901
10	1.25	0.56061	1643.6487	0.31787	27.3007	0.05339	9.87829	0.04982	8.15938
2	1.5	0.63958	126.3754	0.7097	12.1855	0.59879	5.80872	0.59705	3.14201
3	1.5	0.63594	143.0546	0.63238	13.6655	0.47632	6.47439	0.47224	4.30577
4	1.5	0.6325	164.4213	0.57005	14.1781	0.38437	7.08995	0.37852	5.11376
5	1.5	0.62943	194.7770	0.52081	14.7953	0.31222	7.41092	0.30639	5.68753
6	1.5	0.62658	237.0304	0.48481	15.8367	0.25663	7.85322	0.25028	6.18198
7	1.5	0.62421	295.3323	0.45565	16.9374	0.21203	8.21052	0.20559	6.56969
8	1.5	0.62194	375.2754	0.43134	18.1190	0.17742	8.73775	0.17072	7.04836
9	1.5	0.61974	484.9938	0.41002	19.3624	0.1487	9.16771	0.1426	7.50668
10	1.5	0.61766	636.1096	0.39249	20.8217	0.12491	9.56656	0.11933	7.89871

Table E-3 Correlation Integral values and BDS statistic for Canada

M	ϵ	Index Levels	BDS	Residuals of index levels	BDS	Returns	BDS	Residuals of Returns	BDS
2	0.5	0.26508	715.6151	0.16668	6.66878	0.13776	3.83404	0.13545	3.48695
3	0.5	0.25559	1425.7105	0.07938	9.03333	0.05453	4.34133	0.05326	4.23592
4	0.5	0.24747	3142.4688	0.04337	12.97701	0.02336	5.73899	0.02246	5.57991
5	0.5	0.2403	7584.9421	0.0251	17.27387	0.01016	6.57516	0.00957	6.28876
6	0.5	0.23374	19580.1077	0.01583	24.11166	0.00439	6.98643	0.00406	6.65665
7	0.5	0.22766	53085.6382	0.01079	35.55703	0.00189	7.29204	0.00172	6.87807
8	0.5	0.22191	149209.474	0.00776	54.61056	0.0008	7.29379	0.0007	6.6353
9	0.5	0.2165	431178.300	0.00577	86.51198	0.00038	8.73899	0.00027	6.05355
10	0.5	0.21131	1272866.35	0.0043	137.7804	0.00015	8.09609	0.00009	4.27379
2	0.75	0.36961	523.0852	0.29378	5.93158	0.26308	4.32287	0.25982	3.66782
3	0.75	0.36162	833.4998	0.17614	7.88822	0.14241	4.92979	0.13954	4.44461
4	0.75	0.35484	1418.5990	0.11607	10.86721	0.08048	5.88999	0.07825	5.48027
5	0.75	0.34861	2585.4587	0.079	13.34745	0.04605	6.44551	0.04435	6.03319
6	0.75	0.34276	4980.3077	0.05678	16.7128	0.02695	7.09106	0.02576	6.69278
7	0.75	0.33736	10020.6058	0.04308	21.5733	0.01605	7.80731	0.0151	7.27996
8	0.75	0.33223	20849.6570	0.03359	27.99887	0.00952	8.3551	0.00881	7.70136
9	0.75	0.32744	44551.4106	0.02749	37.88976	0.00574	9.08621	0.00522	8.28565
10	0.75	0.32287	97215.8929	0.02296	52.02439	0.00345	9.77025	0.00307	8.752
2	1	0.46339	267.1342	0.41729	5.42344	0.39567	5.2861	0.39579	4.51817
3	1	0.45524	365.6363	0.28772	6.96866	0.26104	6.0071	0.26034	5.3353
4	1	0.44841	521.7622	0.2089	8.84756	0.17645	6.67909	0.17514	5.99576
5	1	0.44239	784.0957	0.15416	10.07096	0.11979	6.92045	0.1185	6.26637
6	1	0.4368	1231.3830	0.11737	11.5749	0.08308	7.41748	0.08212	6.81576
7	1	0.43165	2005.7030	0.09243	13.55656	0.05837	7.96335	0.05745	7.32061
8	1	0.42684	3365.3171	0.07389	15.74827	0.04114	8.43521	0.04039	7.77709
9	1	0.42244	5786.6581	0.06113	18.82941	0.02922	8.97129	0.02854	8.23482
10	1	0.41835	10150.6119	0.05181	22.86631	0.02074	9.45696	0.02021	8.68591
2	1.25	0.56465	185.8130	0.53298	4.46016	0.52163	5.74554	0.52245	5.24036
3	1.25	0.55558	224.4698	0.40458	5.67831	0.39074	6.4079	0.39156	6.06073
4	1.25	0.54817	277.8109	0.31639	6.98849	0.29677	6.81803	0.2972	6.48276
5	1.25	0.54167	356.7617	0.25028	7.76615	0.22665	6.97581	0.22696	6.66651
6	1.25	0.53563	473.3419	0.20118	8.54516	0.17582	7.36006	0.17617	7.08292
7	1.25	0.53011	645.9719	0.16466	9.457	0.13796	7.81152	0.13798	7.49596
8	1.25	0.52492	902.7569	0.13624	10.38453	0.10931	8.31368	0.10899	7.93881
9	1.25	0.52013	1287.6101	0.11407	11.42417	0.08688	8.77094	0.08635	8.33679
10	1.25	0.51556	1868.2854	0.09751	12.81064	0.06906	9.16981	0.06864	8.73391
2	1.5	0.68596	137.2288	0.63306	3.64354	0.62303	5.81457	0.62294	5.41831
3	1.5	0.67764	148.3204	0.51582	4.67271	0.50633	6.59808	0.50613	6.31885
4	1.5	0.67056	162.1309	0.42673	5.53165	0.41604	7.03802	0.41548	6.76417
5	1.5	0.66439	181.8231	0.35503	5.94748	0.34332	7.18286	0.34267	6.93228
6	1.5	0.65878	208.5907	0.29813	6.35569	0.28641	7.48901	0.28569	7.25104
7	1.5	0.65355	243.9655	0.25266	6.78837	0.24046	7.78746	0.2395	7.52928
8	1.5	0.64865	290.1335	0.21643	7.29735	0.20384	8.18779	0.20269	7.90941
9	1.5	0.64411	350.1198	0.18606	7.73673	0.17317	8.52378	0.17183	8.2153
10	1.5	0.6398	427.9048	0.16163	8.28173	0.14762	8.86022	0.14647	8.56286

Table E-4 Correlation Integral values and BDS statistic for France

M	ε	Index Levels	BDS	Residuals of index levels	BDS	Returns	BDS	Residuals of Returns	BDS
2	0.5	0.35132	125.0025	0.26428	15.6075	0.10314	4.4993	0.12249	3.57013
3	0.5	0.34585	205.3180	0.17247	20.0948	0.03431	4.17155	0.04459	3.51309
4	0.5	0.34121	361.6413	0.12285	26.0531	0.01187	4.65315	0.01631	3.34355
5	0.5	0.33709	684.6405	0.09163	34.1237	0.00432	5.56813	0.00618	3.61949
6	0.5	0.33333	1372.5552	0.0707	45.9314	0.00166	6.93042	0.00253	4.59255
7	0.5	0.32985	2875.0026	0.05501	62.5675	0.00066	8.28087	0.00107	5.64052
8	0.5	0.32664	6227.9195	0.04312	86.7570	0.00025	9.16101	0.00042	5.694
9	0.5	0.32357	13842.9388	0.03402	122.5256	0.0001	10.8543	0.00019	7.23551
10	0.5	0.3206	31392.7045	0.0275	179.4877	0.00004	13.4874	0.00008	7.83266
2	0.75	0.46079	132.3467	0.41206	15.3329	0.21208	4.81668	0.2503	4.97629
3	0.75	0.45538	182.1658	0.31766	19.2489	0.10117	4.68127	0.13141	5.20413
4	0.75	0.45072	261.6756	0.26409	24.0203	0.04945	4.9905	0.06967	5.19951
5	0.75	0.44658	396.2102	0.22725	29.5608	0.02491	5.574	0.03755	5.35089
6	0.75	0.44274	627.4124	0.20044	36.7662	0.01307	6.52775	0.02102	5.95041
7	0.75	0.4392	1030.7051	0.17878	46.1223	0.00699	7.49192	0.01202	6.63268
8	0.75	0.43584	1744.1363	0.16135	58.8341	0.00379	8.44404	0.00698	7.37185
9	0.75	0.43264	3022.9969	0.14667	76.1404	0.00211	9.75644	0.00413	8.28382
10	0.75	0.42957	5342.9178	0.13495	100.6101	0.0012	11.3036	0.00246	9.26676
2	1	0.53231	149.5863	0.51829	14.2514	0.33406	4.90744	0.37667	5.47758
3	1	0.52839	189.0413	0.42565	17.2790	0.19941	4.98695	0.2413	5.87605
4	1	0.52523	246.3632	0.36968	20.5499	0.12101	5.23734	0.15638	6.01763
5	1	0.52237	334.9220	0.33221	24.1726	0.07523	5.7794	0.10306	6.3158
6	1	0.51968	472.5046	0.30437	28.3530	0.04789	6.47687	0.06973	6.92587
7	1	0.51711	687.7221	0.28271	33.4457	0.03089	7.13997	0.04753	7.44068
8	1	0.51468	1027.4024	0.26596	39.9308	0.01996	7.66447	0.03285	8.05027
9	1	0.5123	1568.4274	0.25225	48.1634	0.01296	8.17347	0.02306	8.80717
10	1	0.50995	2437.9242	0.24169	58.9450	0.00864	9.02239	0.01619	9.51121
2	1.25	0.58196	151.5834	0.59684	12.6662	0.45524	4.99207	0.50098	5.94916
3	1.25	0.57698	181.1124	0.50724	15.1261	0.31563	5.22422	0.36548	6.05904
4	1.25	0.57303	221.4746	0.4479	17.3006	0.22136	5.44429	0.26999	6.24946
5	1.25	0.56954	280.6788	0.40864	19.7762	0.15832	5.97308	0.20217	6.56197
6	1.25	0.5665	367.3392	0.37838	22.2826	0.11483	6.51418	0.15449	7.14527
7	1.25	0.5637	494.0212	0.35527	25.1802	0.08387	6.95831	0.11861	7.57725
8	1.25	0.56104	679.8461	0.33755	28.6581	0.06176	7.40653	0.09205	8.08106
9	1.25	0.5585	954.0572	0.32461	32.9809	0.04552	7.76493	0.07218	8.66148
10	1.25	0.55605	1361.3771	0.31563	38.4012	0.0341	8.32659	0.05682	9.23541
2	1.5	0.64708	119.7243	0.65929	10.9830	0.56601	4.88432	0.60605	5.72204
3	1.5	0.64176	134.1567	0.57367	13.1363	0.4353	5.30324	0.48274	5.92902
4	1.5	0.63725	152.6766	0.51329	14.7652	0.33686	5.4188	0.38844	6.17596
5	1.5	0.63309	178.8441	0.4729	16.6721	0.26452	5.87208	0.31601	6.5205
6	1.5	0.62928	215.0439	0.44046	18.3343	0.20959	6.28565	0.26137	7.11767
7	1.5	0.62577	264.4381	0.41423	20.0491	0.16695	6.61443	0.21688	7.51665
8	1.5	0.62252	331.5061	0.39231	21.9002	0.13378	6.93408	0.18106	7.90053
9	1.5	0.6194	422.4909	0.37676	24.2538	0.10707	7.12686	0.15218	8.31693
10	1.5	0.61636	546.1885	0.36598	27.1428	0.08667	7.47316	0.12815	8.68931

Table E-5 Correlation Integral values and BDS statistic for Germany

M	ε	Index Levels	BDS	Residuals of index levels	BDS	Returns	BDS	Residuals of Returns	BDS
2	0.5	0.34608	135.8093	0.20378	15.2037	0.10344	6.54479	0.10231	6.51315
3	0.5	0.33633	221.5069	0.11991	20.7368	0.03629	7.71491	0.03571	7.77545
4	0.5	0.328	386.8586	0.07682	27.3901	0.01317	8.62825	0.01286	8.65672
5	0.5	0.32048	725.5969	0.05276	37.5352	0.00502	10.0707	0.00488	10.1706
6	0.5	0.31369	1441.6557	0.03703	51.9035	0.00191	10.9958	0.0018	10.6230
7	0.5	0.30749	2994.2573	0.02669	74.1457	0.00074	12.2350	0.00065	10.8290
8	0.5	0.30187	6436.8829	0.01938	107.8070	0.00031	14.5980	0.00028	13.9469
9	0.5	0.29675	14213.2047	0.01416	159.8697	0.00011	14.7508	0.00011	14.9341
10	0.5	0.292	32049.8877	0.01052	244.1430	0.00005	17.8659	0.00005	19.4305
2	0.75	0.44268	173.6088	0.35071	16.1395	0.21325	6.55167	0.21077	6.74641
3	0.75	0.43564	244.6555	0.25435	20.7572	0.1061	7.80016	0.10436	8.02715
4	0.75	0.4295	360.6907	0.19639	25.6049	0.05441	8.73911	0.05336	9.04262
5	0.75	0.42385	561.6555	0.16096	32.6128	0.02873	9.787	0.0282	10.2462
6	0.75	0.41861	916.3282	0.13596	42.1102	0.01532	10.6561	0.01496	11.1073
7	0.75	0.41364	1552.4791	0.11722	55.3760	0.00847	12.0965	0.0082	12.5008
8	0.75	0.40896	2711.9430	0.10235	74.1047	0.00478	13.8592	0.00456	14.1043
9	0.75	0.40453	4855.4136	0.09003	100.6974	0.00276	16.0844	0.00258	16.0856
10	0.75	0.40029	8868.0595	0.08007	139.6782	0.00163	19.1394	0.00146	18.2671
2	1	0.50684	227.0280	0.472	15.8958	0.33746	6.80058	0.33413	6.85221
3	1	0.50009	294.6306	0.37714	19.6605	0.20886	8.19891	0.20618	8.35919
4	1	0.49419	395.1699	0.31634	23.1559	0.13259	9.15899	0.13034	9.30803
5	1	0.4889	554.5141	0.27676	27.6137	0.08626	10.1675	0.08484	10.4647
6	1	0.48437	810.0234	0.24788	33.0805	0.05662	10.9718	0.0556	11.3175
7	1	0.48016	1222.9821	0.22588	40.0745	0.03801	12.0865	0.03715	12.4173
8	1	0.47621	1897.6321	0.20903	49.3219	0.02593	13.4260	0.02536	13.8734
9	1	0.47248	3012.0550	0.1944	61.1751	0.01788	14.9415	0.01757	15.6156
10	1	0.46892	4871.6987	0.18218	76.8791	0.01264	17.0620	0.01248	18.0010
2	1.25	0.57072	189.1624	0.56707	15.2087	0.46172	7.2351	0.45796	7.0026
3	1.25	0.5639	228.1007	0.47755	18.3502	0.33125	8.89692	0.3273	8.75082
4	1.25	0.55784	281.5219	0.41908	21.1389	0.24315	10.0183	0.23897	9.82732
5	1.25	0.55207	360.2723	0.37967	24.2747	0.18253	11.1602	0.17861	10.9789
6	1.25	0.54663	476.3601	0.34973	27.6920	0.13842	12.0896	0.13446	11.8147
7	1.25	0.5416	647.7872	0.32655	31.7385	0.10647	13.1399	0.10261	12.7692
8	1.25	0.53685	902.0133	0.30901	36.7861	0.08297	14.3591	0.07936	13.9043
9	1.25	0.53252	1281.9210	0.2943	42.8878	0.0653	15.7361	0.06184	15.1315
10	1.25	0.52834	1853.0423	0.28181	50.3764	0.05231	17.5388	0.04912	16.8054
2	1.5	0.63208	134.9655	0.64731	14.4060	0.57533	7.54599	0.57205	7.17835
3	1.5	0.62591	153.2831	0.56357	16.8397	0.4548	9.16493	0.45121	8.9599
4	1.5	0.62043	176.7728	0.50652	18.9364	0.36562	10.1687	0.36167	9.96753
5	1.5	0.61527	210.0104	0.46643	21.0922	0.29853	11.0870	0.29511	10.9916
6	1.5	0.61038	256.3069	0.43473	23.1748	0.24591	11.8251	0.2427	11.7559
7	1.5	0.60575	320.1554	0.41017	25.5388	0.20467	12.6140	0.20135	12.5083
8	1.5	0.60133	407.9381	0.39142	28.3488	0.17204	13.4995	0.16883	13.3831
9	1.5	0.59735	529.0122	0.37582	31.5594	0.14538	14.3960	0.14223	14.2528
10	1.5	0.59367	696.4691	0.36266	35.2853	0.12427	15.5251	0.12125	15.3646

Table E-6 Correlation Integral values and BDS statistic for Italy

M	ε	Index Levels	BDS	Residuals of index levels	BDS	Returns	BDS	Residuals of Returns	BDS
2	0.5	0.38967	118.1428	0.24792	10.6870	0.12252	3.48499	0.12249	3.5701
3	0.5	0.38158	179.1619	0.15153	14.3101	0.04475	3.54738	0.04459	3.5131
4	0.5	0.37436	287.2782	0.10121	18.5261	0.0163	3.26786	0.01631	3.3436
5	0.5	0.36765	490.5813	0.07311	24.6956	0.00614	3.48085	0.00618	3.6195
6	0.5	0.36131	882.4280	0.05572	33.8001	0.00251	4.44437	0.00253	4.5926
7	0.5	0.3553	1653.7462	0.04306	46.3498	0.00105	5.36032	0.00107	5.6405
8	0.5	0.3498	3202.6313	0.03381	64.7369	0.00039	4.90919	0.00042	5.6940
9	0.5	0.34462	6362.6592	0.02716	93.2255	0.00017	6.14976	0.00019	7.2355
10	0.5	0.33977	12901.9837	0.022	136.5344	0.00007	6.54342	0.00008	7.8327
2	0.75	0.4701	153.9017	0.3983	11.3608	0.25119	5.10709	0.2503	4.9763
3	0.75	0.46457	209.4263	0.2906	14.2279	0.13216	5.31284	0.13141	5.2041
4	0.75	0.45976	296.7052	0.22403	16.9764	0.06985	5.16534	0.06967	5.1995
5	0.75	0.45532	442.2281	0.18291	20.7809	0.03772	5.34568	0.03755	5.3509
6	0.75	0.4511	688.4484	0.15697	26.1501	0.02115	5.9506	0.02102	5.9504
7	0.75	0.44706	1110.7864	0.13651	32.6849	0.0121	6.62852	0.01202	6.6327
8	0.75	0.44314	1845.0238	0.12014	41.1349	0.00701	7.34163	0.00698	7.3719
9	0.75	0.43941	3138.5537	0.10683	52.3926	0.00413	8.16601	0.00413	8.2838
10	0.75	0.43587	5444.2899	0.09579	67.5766	0.00245	9.07575	0.00246	9.2668
2	1	0.51955	211.6555	0.5193	11.6045	0.37737	5.4544	0.37667	5.4776
3	1	0.5154	272.0716	0.41436	13.7352	0.24187	5.83597	0.2413	5.8761
4	1	0.51185	361.2259	0.34171	15.3764	0.15634	5.87251	0.15638	6.0176
5	1	0.50847	501.0261	0.29088	17.3546	0.10307	6.1914	0.10306	6.3158
6	1	0.50528	722.2126	0.25826	20.3277	0.06974	6.80499	0.06973	6.9259
7	1	0.50223	1075.1456	0.2324	23.6353	0.04757	7.32924	0.04753	7.4407
8	1	0.49927	1643.8643	0.2117	27.5877	0.0329	7.94232	0.03285	8.0503
9	1	0.4964	2569.8026	0.19389	32.2482	0.02311	8.70193	0.02306	8.8072
10	1	0.49363	4092.4684	0.17955	38.1509	0.01622	9.39336	0.01619	9.5112
2	1.25	0.55823	210.6335	0.612	12.0955	0.5013	5.9273	0.50098	5.9492
3	1.25	0.55384	258.8814	0.51773	13.8540	0.36591	6.05873	0.36548	6.0590
4	1.25	0.54999	326.4808	0.44693	14.8595	0.26965	6.12587	0.26999	6.2495
5	1.25	0.54652	428.0519	0.39282	15.9244	0.202	6.47455	0.20217	6.5620
6	1.25	0.54344	581.0165	0.35609	17.7246	0.15443	7.07501	0.15449	7.1453
7	1.25	0.54064	812.0974	0.32609	19.5994	0.11862	7.52075	0.11861	7.5773
8	1.25	0.53804	1163.4267	0.30261	21.8432	0.09214	8.03869	0.09205	8.0811
9	1.25	0.53553	1701.4803	0.28201	24.3046	0.07232	8.63335	0.07218	8.6615
10	1.25	0.53312	2532.2951	0.26422	27.1483	0.05696	9.21167	0.05682	9.2354
2	1.5	0.61252	158.6913	0.68717	12.9052	0.60695	5.7892	0.60605	5.7220
3	1.5	0.60661	183.7846	0.60523	14.2980	0.48386	5.98837	0.48274	5.9290
4	1.5	0.60137	216.6015	0.54045	14.9002	0.38892	6.12017	0.38844	6.1760
5	1.5	0.59658	263.5910	0.48892	15.5476	0.31669	6.49765	0.31601	6.5205
6	1.5	0.59238	330.3036	0.45059	16.6208	0.2622	7.11656	0.26137	7.1177
7	1.5	0.58845	424.2417	0.41792	17.6678	0.21778	7.52791	0.21688	7.5167
8	1.5	0.58464	556.4121	0.39156	18.9379	0.18198	7.92148	0.18106	7.9005
9	1.5	0.58089	742.9604	0.3678	20.2477	0.15308	8.34317	0.15218	8.3169
10	1.5	0.57745	1007.9643	0.34683	21.7076	0.12902	8.71939	0.12815	8.6893

Table E-7 Correlation Integral values and BDS statistic for Japan

M	ϵ	Index Levels	BDS	Residuals of index levels	BDS	Returns	BDS	Residuals of Returns	BDS
2	0.5	0.38265	99.2853	0.33333	17.7189	0.12764	4.8366	0.12769	4.7610
3	0.5	0.37788	153.7417	0.26035	25.2251	0.0508	6.8446	0.05081	6.7772
4	0.5	0.37382	252.8343	0.21778	33.5398	0.02126	8.3810	0.0213	8.3596
5	0.5	0.37012	443.9586	0.18714	44.4345	0.00927	9.9504	0.00928	9.8973
6	0.5	0.36675	822.3960	0.16372	60.0764	0.00423	11.9897	0.00422	11.8726
7	0.5	0.36384	1589.2334	0.14439	82.8445	0.00202	14.8074	0.00201	14.6193
8	0.5	0.36115	3172.0605	0.12808	116.6000	0.00096	17.7468	0.00095	17.3835
9	0.5	0.35862	6492.5604	0.11397	167.0523	0.00046	21.2439	0.00045	20.1736
10	0.5	0.35621	13554.6349	0.10153	242.8543	0.00022	24.9227	0.0002	22.8877
2	0.75	0.47808	110.6025	0.45823	16.6995	0.25277	5.6279	0.25312	5.7390
3	0.75	0.47469	149.2255	0.38365	22.3151	0.13746	7.1552	0.13778	7.2475
4	0.75	0.47171	209.3109	0.34522	28.3451	0.07768	8.4008	0.0779	8.4731
5	0.75	0.46895	308.5853	0.31883	35.1235	0.04505	9.5012	0.04508	9.5043
6	0.75	0.46636	474.9036	0.30097	44.0538	0.0272	11.0589	0.02712	10.9802
7	0.75	0.464	757.3014	0.28755	55.9977	0.0166	12.5740	0.01651	12.4473
8	0.75	0.46175	1242.7993	0.27648	72.1209	0.01015	14.0679	0.01011	13.9514
9	0.75	0.45962	2087.8719	0.26674	94.0485	0.00625	15.7272	0.0062	15.5308
10	0.75	0.45757	3575.2454	0.25746	123.8911	0.00391	17.9030	0.00385	17.5150
2	1	0.53541	128.0932	0.54619	14.7611	0.39007	6.4553	0.39063	6.5981
3	1	0.53207	161.3297	0.46033	17.9726	0.25874	7.7544	0.25888	7.7637
4	1	0.52923	209.1868	0.41504	21.9765	0.17644	8.7716	0.17653	8.7715
5	1	0.52671	282.8276	0.38514	25.9945	0.12285	9.6858	0.12275	9.6436
6	1	0.52433	396.6731	0.36658	30.8997	0.08776	10.8329	0.08749	10.7356
7	1	0.52223	573.9913	0.35328	36.8119	0.06337	11.9393	0.06302	11.7936
8	1	0.52021	852.2415	0.34432	44.2901	0.04627	13.1512	0.04592	12.9635
9	1	0.51825	1292.8759	0.33746	53.6898	0.03395	14.4149	0.03362	14.1848
10	1	0.51638	1997.0652	0.33195	65.6023	0.025	15.7849	0.02469	15.4901
2	1.25	0.58474	132.2916	0.62796	12.5012	0.51917	7.4699	0.51993	7.7352
3	1.25	0.58091	157.8373	0.54036	14.7067	0.39304	8.7927	0.39348	8.8998
4	1.25	0.57753	192.4844	0.48606	17.2045	0.30338	9.6580	0.30372	9.7313
5	1.25	0.57441	243.2253	0.44867	19.5597	0.23797	10.4319	0.23824	10.4906
6	1.25	0.57157	317.2975	0.42189	22.0471	0.1894	11.2471	0.18953	11.2849
7	1.25	0.56904	425.3946	0.40229	24.8488	0.1521	12.0411	0.1521	12.0563
8	1.25	0.56667	583.5887	0.38978	28.3116	0.12305	12.8640	0.12297	12.8647
9	1.25	0.56444	816.4334	0.38101	32.4020	0.09977	13.6547	0.09964	13.6431
10	1.25	0.56244	1161.6005	0.37505	37.2917	0.08083	14.4010	0.08072	14.3904
2	1.5	0.63666	116.7465	0.70131	10.8822	0.63048	8.2936	0.63144	8.6476
3	1.5	0.63275	132.4871	0.62032	12.7232	0.52135	9.7610	0.52205	9.9192
4	1.5	0.62934	152.5812	0.56571	14.6265	0.43754	10.5977	0.43784	10.6622
5	1.5	0.62636	181.0706	0.52477	16.1410	0.3713	11.2357	0.37124	11.2480
6	1.5	0.6236	220.7266	0.49391	17.6316	0.3177	11.7906	0.3174	11.7735
7	1.5	0.621	275.3316	0.46928	19.1213	0.27375	12.3383	0.2732	12.2926
8	1.5	0.61855	350.3109	0.45106	20.8481	0.2373	12.9141	0.23662	12.8509
9	1.5	0.61624	453.3841	0.43676	22.7642	0.20585	13.4133	0.20498	13.3202
10	1.5	0.61406	595.5516	0.42579	24.9439	0.1789	13.9152	0.17787	13.7930

Table E-8 Correlation Integral values and BDS statistic for Switzerland

M	ε	Index Levels	BDS	Residuals of index levels	BDS	Returns	BDS	Residuals of Returns	BDS
2	0.5	0.37689	102.8531	0.19721	12.1732	0.12495	6.4649	0.12342	5.6085
3	0.5	0.36909	159.1562	0.10779	16.0690	0.04861	7.5816	0.04777	6.9411
4	0.5	0.36233	261.5042	0.06306	20.0498	0.01965	8.5359	0.01927	8.0066
5	0.5	0.3562	458.8532	0.03954	25.9278	0.00842	10.1129	0.00811	9.2977
6	0.5	0.35066	849.7740	0.02538	33.5070	0.00368	11.6886	0.00345	10.4271
7	0.5	0.34553	1641.2563	0.01622	42.8499	0.00157	12.7129	0.00144	11.0525
8	0.5	0.34094	3276.9096	0.01051	55.9121	0.00068	13.8755	0.00062	12.3827
9	0.5	0.33663	6711.5017	0.00675	72.8455	0.00027	13.9082	0.00026	13.2749
10	0.5	0.33248	14022.0609	0.0043	95.0158	0.00011	13.7108	0.00012	15.4842
2	0.75	0.44897	139.8323	0.34468	11.9078	0.24391	6.0594	0.24198	5.7029
3	0.75	0.44278	195.3562	0.2341	14.9941	0.12966	7.1971	0.12827	6.9392
4	0.75	0.43731	285.2078	0.16929	18.2292	0.07183	8.3823	0.07103	8.2155
5	0.75	0.43233	439.5069	0.12941	22.4726	0.04181	10.0193	0.04118	9.7948
6	0.75	0.42773	709.2088	0.10142	27.4958	0.02501	11.7184	0.02452	11.4260
7	0.75	0.42344	1188.1037	0.07991	33.3201	0.01498	13.1939	0.01468	12.9285
8	0.75	0.41954	2052.0855	0.06374	40.8463	0.00915	15.0581	0.00904	14.9613
9	0.75	0.41587	3631.9468	0.05106	50.4301	0.00554	16.7902	0.00549	16.7712
10	0.75	0.41235	6556.0906	0.04098	62.6889	0.00333	18.5072	0.00332	18.6955
2	1	0.50297	178.1099	0.47933	11.4853	0.37019	6.0992	0.36819	5.7316
3	1	0.49632	231.9485	0.36853	14.1885	0.23772	7.0640	0.23607	6.8249
4	1	0.49065	312.3603	0.29881	17.0162	0.15791	8.1885	0.15672	8.0116
5	1	0.4862	440.8490	0.25245	20.2109	0.10888	9.6088	0.10756	9.3315
6	1	0.48222	647.5016	0.21775	23.6212	0.07672	10.9865	0.0754	10.6173
7	1	0.47854	983.1341	0.18988	27.4037	0.05434	12.1570	0.05302	11.6551
8	1	0.4751	1534.3571	0.16731	31.9279	0.03922	13.6081	0.03817	13.0687
9	1	0.47174	2449.1506	0.14816	37.3073	0.02823	14.9476	0.02727	14.2642
10	1	0.46843	3982.7340	0.13208	43.9591	0.02031	16.3239	0.01956	15.5747
2	1.25	0.5662	165.8033	0.59258	11.2747	0.49894	6.9297	0.4949	6.4575
3	1.25	0.5592	200.7201	0.49211	13.6940	0.36745	7.7801	0.36362	7.5118
4	1.25	0.55303	248.6523	0.42518	16.0527	0.27715	8.7295	0.27394	8.5651
5	1.25	0.54771	319.8304	0.37751	18.3250	0.21399	9.7969	0.21073	9.5774
6	1.25	0.54288	425.2325	0.34126	20.6249	0.16776	10.7963	0.16486	10.5839
7	1.25	0.53838	581.4877	0.31178	23.0190	0.13195	11.5567	0.12922	11.3099
8	1.25	0.53414	814.3343	0.28698	25.6368	0.10537	12.5334	0.10308	12.3089
9	1.25	0.53004	1163.5018	0.2649	28.4820	0.08463	13.5350	0.08264	13.3090
10	1.25	0.52602	1690.7939	0.24644	31.8954	0.06833	14.6111	0.06669	14.4114
2	1.5	0.63391	126.3799	0.68356	11.0910	0.60983	7.4664	0.60697	6.9911
3	1.5	0.62642	142.9781	0.59624	13.0984	0.49111	8.1771	0.48786	7.8160
4	1.5	0.61984	164.0890	0.53635	15.1697	0.40185	8.9122	0.39887	8.6475
5	1.5	0.61404	194.0113	0.49192	16.9276	0.33391	9.7087	0.3307	9.4105
6	1.5	0.60893	235.6737	0.45817	18.6455	0.28067	10.4614	0.27767	10.1778
7	1.5	0.60431	292.9629	0.43014	20.2659	0.23701	11.0429	0.23365	10.6876
8	1.5	0.6002	371.5333	0.40652	21.9402	0.20231	11.7561	0.19901	11.3751
9	1.5	0.59638	479.2694	0.38479	23.6010	0.17384	12.5040	0.17031	12.0477
10	1.5	0.59268	627.3376	0.36603	25.4729	0.15035	13.3241	0.1468	12.8115

Table E-9 Correlation Integral values and BDS statistic for United Kingdom

M	ϵ	Index Levels	BDS	Residuals of index levels	BDS	Returns	BDS	Residuals of Returns	BDS
2	0.5	0.34953	142.2895	0.1515	8.6529	0.09127	1.8703	0.0912	1.8944
3	0.5	0.34311	234.0383	0.06783	11.3230	0.02887	3.2495	0.02881	3.2223
4	0.5	0.33745	412.4978	0.03311	14.6336	0.00925	3.7858	0.00921	3.7049
5	0.5	0.33253	781.8071	0.01691	18.1594	0.00294	3.6760	0.00293	3.6845
6	0.5	0.32804	1569.4685	0.00898	22.6495	0.001	4.6302	0.00099	4.4655
7	0.5	0.32391	3293.0382	0.00479	27.7822	0.00031	4.0000	0.0003	3.4528
8	0.5	0.32013	7148.0919	0.00261	34.6994	0.00011	5.5262	0.0001	4.2967
9	0.5	0.31658	15927.2104	0.00142	43.3378	0.00004	7.4094	0.00004	6.6005
10	0.5	0.31324	36228.0515	0.00076	53.0865	0.00001	4.9672	0.00001	5.0239
2	0.75	0.44337	177.7483	0.29628	9.3958	0.1919	1.5494	0.19223	1.6851
3	0.75	0.4378	251.4106	0.18093	11.9694	0.08713	2.8464	0.08719	2.8350
4	0.75	0.43303	372.2178	0.11818	14.9177	0.04063	3.8577	0.04063	3.8118
5	0.75	0.42876	582.3890	0.08024	17.9249	0.01948	4.8305	0.01951	4.8138
6	0.75	0.42484	954.9433	0.05611	21.4314	0.00967	5.9761	0.00972	6.0286
7	0.75	0.42128	1626.8115	0.03966	25.3656	0.00485	6.8805	0.00489	6.9748
8	0.75	0.41799	2857.8244	0.0285	30.3534	0.00251	8.0806	0.00253	8.1564
9	0.75	0.41489	5145.4215	0.02046	36.1677	0.00133	9.4991	0.00134	9.5774
10	0.75	0.41196	9451.0929	0.01478	43.4438	0.00071	10.9870	0.00072	11.0162
2	1	0.52414	196.2778	0.442	10.0061	0.31244	1.5705	0.31271	1.8149
3	1	0.52004	250.8486	0.32279	12.7386	0.17951	2.7473	0.17978	2.9226
4	1	0.51633	330.6326	0.24763	15.3706	0.10566	3.8061	0.10582	3.9326
5	1	0.51295	455.1225	0.1967	18.0454	0.06333	4.6375	0.06346	4.7554
6	1	0.50994	650.9538	0.15994	20.9269	0.03909	5.6740	0.03927	5.8307
7	1	0.50722	961.4355	0.13168	24.0575	0.02427	6.4125	0.02438	6.5538
8	1	0.50486	1458.8323	0.10969	27.7354	0.01538	7.2973	0.01543	7.4010
9	1	0.50272	2263.4064	0.09206	32.0879	0.00978	8.0456	0.0098	8.1465
10	1	0.50081	3577.8261	0.07797	37.4545	0.00625	8.7972	0.00629	8.9398
2	1.25	0.57581	186.5996	0.56643	9.7522	0.4398	1.7178	0.4405	2.0550
3	1.25	0.57198	225.1914	0.4566	12.3695	0.29784	2.8237	0.29852	3.0415
4	1.25	0.56881	278.3061	0.38051	14.4858	0.20533	3.8051	0.20596	3.9833
5	1.25	0.56605	356.8736	0.3259	16.5636	0.14338	4.5365	0.14398	4.7023
6	1.25	0.56368	472.9702	0.28405	18.5905	0.10201	5.3554	0.10258	5.5248
7	1.25	0.56154	644.5799	0.25055	20.6800	0.07299	5.9408	0.07346	6.1026
8	1.25	0.55966	899.6118	0.22337	23.0083	0.05258	6.4537	0.05295	6.6056
9	1.25	0.55797	1281.0583	0.20112	25.7116	0.03806	6.9065	0.03833	7.0423
10	1.25	0.55642	1855.6329	0.18294	28.9518	0.02761	7.2986	0.02782	7.4366
2	1.5	0.63623	133.7091	0.66538	8.8703	0.56094	1.9272	0.56118	2.2257
3	1.5	0.6315	151.6627	0.56993	11.4830	0.42658	2.9378	0.42677	3.1133
4	1.5	0.62733	174.6506	0.49801	13.0765	0.32879	3.8927	0.32903	4.0428
5	1.5	0.62348	207.2017	0.44442	14.6781	0.25572	4.5691	0.25593	4.6962
6	1.5	0.61994	252.5437	0.40141	16.0624	0.20133	5.2662	0.20171	5.4146
7	1.5	0.61663	314.9963	0.36571	17.3582	0.15939	5.7881	0.1597	5.9202
8	1.5	0.61369	400.8883	0.33597	18.7171	0.12674	6.2044	0.12698	6.3247
9	1.5	0.61112	519.1644	0.31133	20.2448	0.10104	6.5377	0.10123	6.6495
10	1.5	0.60888	682.5783	0.29121	22.0387	0.08083	6.8445	0.08111	6.9841

Table E-10 Correlation Integral values and BDS statistic for United States

M	ϵ	Index Levels	BDS	Residuals of index levels	BDS	Returns	BDS	Residuals of Returns	BDS
2	0.5	0.32621	146.5547	0.14155	11.4517	0.09275	4.4184	0.09255	4.3234
3	0.5	0.32012	252.9341	0.06526	15.9968	0.02916	4.4188	0.02902	4.2946
4	0.5	0.31517	471.5580	0.03269	20.7714	0.00911	3.9348	0.00903	3.7789
5	0.5	0.31085	949.4751	0.01766	27.5667	0.00295	4.3465	0.00291	4.1615
6	0.5	0.30691	2029.8368	0.0103	38.5680	0.00099	5.1241	0.00099	5.0967
7	0.5	0.30325	4540.3003	0.00625	55.2687	0.00036	6.6263	0.00036	6.6320
8	0.5	0.29986	10510.1263	0.00392	81.6125	0.00013	7.8990	0.00013	8.0493
9	0.5	0.29673	24982.0572	0.00251	123.8030	0.00004	7.2659	0.00004	7.4067
10	0.5	0.29378	60617.0261	0.00163	192.6130	0.00001	2.9684	0.00001	3.0479
2	0.75	0.44414	139.2251	0.27607	11.5941	0.1934	4.5197	0.19289	4.3292
3	0.75	0.43824	195.9894	0.17101	15.8092	0.08763	4.6349	0.08724	4.4828
4	0.75	0.43333	288.5192	0.11314	19.7586	0.03993	4.5503	0.03967	4.4029
5	0.75	0.429	448.6218	0.07889	24.6219	0.01841	4.6301	0.01828	4.5420
6	0.75	0.42504	730.7389	0.05739	31.1663	0.00858	4.7924	0.00851	4.7149
7	0.75	0.42149	1236.4209	0.04329	40.4102	0.00398	4.7893	0.00394	4.6890
8	0.75	0.41818	2156.6149	0.03351	53.5035	0.00186	4.8466	0.00184	4.7361
9	0.75	0.41506	3854.7195	0.02655	72.5843	0.00085	4.6186	0.00084	4.4724
10	0.75	0.41208	7027.3144	0.02136	100.4468	0.00039	4.3969	0.00038	4.2539
2	1	0.52808	147.1702	0.41156	11.2753	0.3146	4.6781	0.3138	4.4448
3	1	0.52368	186.7149	0.29868	15.0093	0.18162	4.9860	0.18076	4.7727
4	1	0.51994	244.1053	0.22838	18.2189	0.10605	5.2029	0.10526	4.9868
5	1	0.5166	333.0051	0.18138	21.6556	0.06271	5.4927	0.06207	5.2737
6	1	0.51358	471.6673	0.14829	25.7391	0.03716	5.5943	0.03667	5.3746
7	1	0.51082	689.4901	0.12456	31.0104	0.02214	5.7094	0.02179	5.4908
8	1	0.50818	1034.6057	0.10704	37.9552	0.01327	5.8442	0.01302	5.6232
9	1	0.50568	1586.7879	0.09315	46.8693	0.00812	6.2518	0.00794	5.9950
10	1	0.50329	2478.4826	0.08181	58.4266	0.00493	6.4700	0.00479	6.1486
2	1.25	0.58744	150.4926	0.53064	10.9034	0.44038	4.6600	0.43939	4.3486
3	1.25	0.5834	179.2285	0.42169	13.9784	0.29895	4.9429	0.29766	4.6597
4	1.25	0.57999	217.9729	0.34804	16.4096	0.20485	5.1574	0.20355	4.8973
5	1.25	0.5769	274.5184	0.29452	18.6347	0.14126	5.2905	0.14007	5.0465
6	1.25	0.5741	356.7665	0.25434	21.0320	0.09786	5.3791	0.09695	5.1789
7	1.25	0.57157	476.3195	0.22445	23.9844	0.06821	5.5056	0.0675	5.3273
8	1.25	0.56922	650.5895	0.2022	27.7203	0.04784	5.6719	0.04729	5.5070
9	1.25	0.56701	906.0505	0.18397	32.1144	0.03366	5.8241	0.03326	5.6789
10	1.25	0.56496	1282.9904	0.16846	37.3076	0.0238	6.0030	0.02349	5.8635
2	1.5	0.65156	123.1837	0.62978	10.3601	0.56059	5.0031	0.55951	4.6074
3	1.5	0.64558	137.6021	0.53054	12.8648	0.42769	5.3327	0.42616	4.9909
4	1.5	0.64071	155.7479	0.45826	14.6127	0.3288	5.5576	0.32701	5.2393
5	1.5	0.63653	181.3110	0.40235	16.0118	0.2541	5.6942	0.25224	5.3901
6	1.5	0.63312	216.6186	0.35766	17.3297	0.19684	5.7273	0.19514	5.4553
7	1.5	0.63016	264.5439	0.3233	18.9589	0.15338	5.8400	0.15181	5.5801
8	1.5	0.62746	329.1723	0.29691	20.9563	0.11979	5.9219	0.11838	5.6707
9	1.5	0.62514	416.4006	0.27553	23.2336	0.09357	5.9523	0.09235	5.7111
10	1.5	0.62307	534.3106	0.25706	25.7301	0.07323	5.9894	0.07219	5.7586

Table E-11 Correlation Integral values and BDS statistic for World Index

M	ϵ	Index Levels	BDS	Residuals of index levels	BDS	Returns	BDS	Residuals of Returns	BDS
2	0.5	0.40348	91.0041	0.26909	15.7934	0.10931	6.4015	0.10863	6.1373
3	0.5	0.39995	136.4408	0.18373	22.5895	0.03818	6.6435	0.03782	6.4408
4	0.5	0.39693	216.0563	0.13784	30.7173	0.0136	6.8934	0.01342	6.7059
5	0.5	0.39415	363.9468	0.10787	41.4491	0.00497	7.3996	0.0049	7.2544
6	0.5	0.3916	645.2383	0.08803	57.9250	0.00178	7.3432	0.00177	7.4128
7	0.5	0.38924	1190.9780	0.0735	83.1934	0.00062	6.8705	0.00063	7.1463
8	0.5	0.38708	2268.8708	0.06213	122.2498	0.00023	7.5114	0.00024	8.0735
9	0.5	0.38505	4430.4529	0.05297	183.5362	0.00007	5.8689	0.00008	6.9455
10	0.5	0.38309	8821.7001	0.0457	282.3703	0.00002	3.7578	0.00002	5.0619
2	0.75	0.48472	103.7751	0.42625	14.5942	0.22606	6.6613	0.22473	6.4019
3	0.75	0.48219	138.9923	0.33439	19.4712	0.11291	6.9318	0.11192	6.7536
4	0.75	0.47999	193.3657	0.28215	24.5990	0.05745	7.2298	0.05681	7.0970
5	0.75	0.478	282.5569	0.24607	30.2529	0.02975	7.6186	0.02934	7.4985
6	0.75	0.4762	430.7886	0.22043	37.5457	0.01569	8.1594	0.01545	8.0633
7	0.75	0.47455	680.1648	0.20098	47.2112	0.00849	8.9939	0.0084	9.0337
8	0.75	0.473	1104.8726	0.18511	60.0990	0.00461	9.7530	0.00458	9.8862
9	0.75	0.47151	1836.7728	0.17114	77.2325	0.00247	10.1730	0.00246	10.3887
10	0.75	0.47006	3111.8272	0.15943	100.7796	0.00131	10.4671	0.0013	10.6531
2	1	0.53759	128.0794	0.54643	13.3759	0.3598	7.4789	0.35821	7.1965
3	1	0.53532	161.2925	0.45507	16.7679	0.22543	7.7200	0.22361	7.4308
4	1	0.53344	209.1591	0.39856	19.9506	0.1439	8.1012	0.14234	7.8506
5	1	0.53175	282.7899	0.36001	23.2237	0.09341	8.5793	0.09208	8.3263
6	1	0.53026	396.7116	0.33136	26.8478	0.06186	9.2725	0.06078	9.0135
7	1	0.52886	573.9912	0.30925	31.1411	0.04151	10.0282	0.04065	9.7584
8	1	0.52759	852.2998	0.2936	36.6752	0.02793	10.6885	0.02724	10.3824
9	1	0.52643	1293.1791	0.27987	43.3093	0.01868	11.1681	0.01811	10.7998
10	1	0.52532	1997.7379	0.2682	51.5281	0.01254	11.7082	0.01209	11.2767
2	1.25	0.5969	123.1787	0.6417	12.9413	0.49193	8.1834	0.48981	7.8094
3	1.25	0.59442	145.4791	0.55668	15.2348	0.35785	8.4159	0.35516	8.0631
4	1.25	0.5924	175.4564	0.49797	17.0875	0.26373	8.6561	0.26103	8.3439
5	1.25	0.59056	219.0061	0.45534	18.8908	0.19736	9.1037	0.19479	8.8040
6	1.25	0.58885	281.9067	0.42473	20.9717	0.14995	9.7107	0.14775	9.4482
7	1.25	0.58726	372.5439	0.39944	23.1087	0.11499	10.3214	0.11304	10.0615
8	1.25	0.58582	503.4927	0.38137	25.7861	0.0889	10.9678	0.08714	10.6898
9	1.25	0.58447	693.5804	0.36601	28.7881	0.06843	11.4193	0.06689	11.1248
10	1.25	0.5832	971.1240	0.35254	32.2063	0.05309	12.0008	0.05177	11.6975
2	1.5	0.64809	106.1039	0.71529	13.0039	0.60728	8.6338	0.60554	8.2579
3	1.5	0.64606	119.4262	0.63985	14.4665	0.48666	8.7512	0.4842	8.4080
4	1.5	0.64448	136.4384	0.58301	15.4440	0.39365	8.8669	0.39088	8.5602
5	1.5	0.64313	160.4870	0.53925	16.4085	0.32255	9.2674	0.31973	8.9913
6	1.5	0.64195	193.8105	0.50779	17.7040	0.26699	9.7355	0.26444	9.5089
7	1.5	0.64085	239.3741	0.48012	18.8348	0.2225	10.1836	0.21998	9.9550
8	1.5	0.63981	301.3974	0.45825	20.1720	0.18663	10.6571	0.18423	10.4298
9	1.5	0.63885	385.8755	0.4403	21.6851	0.1564	10.9960	0.1541	10.7605
10	1.5	0.63791	501.1918	0.42467	23.3422	0.1317	11.3981	0.12955	11.1561

VITA

Efstathia Pilarinu was born in Zakynthos, Greece in 1962. She completed her High School education in 1977 and entered the Lyceum from which she graduated in 1980. The same year she ranked first in the National Examination required for entering the University. She joined the Mathematics Department at the University of Patras and graduated in June 1984 with honors. After successfully passing another National Examination she received in September 1984 a Graduate Assistantship from the Greek Government. During a three year period (1984 - 1987) she taught and did research in the area of Topology at the University of Patras. In 1987 she sought to pursue graduate studies abroad in an applied area of Mathematics. In August 1988 she was enrolled at the University of Tennessee at Knoxville having been awarded a University Scholarship at the Department of Finance of the College of Business Administration. The Research Foundation of Alexander Onassis also awarded her with a scholarship for the completion of an M.B.A. degree. In May 1992 she earned her M.B.A. degree from the College of Business Administration. Her research interests have been focused in the area of dynamic asset pricing as a result of her background in Mathematics. She has currently been offered an Analyst position in the Derivative Research/Equity Portfolio Analysis Group at Salomon Brothers Inc. in New York.