Mathematics of Quantum Numbers

A Collection of *274* Research Papers on Quantum Numbers

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An Inequality For The Sum Of Two Quantum Numbers

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Abstract

We estimate the sum of quantum numbers in terms of the sum of their classical counterparts.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},$$

be the 2^{nd} quantization of x, so that

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim}, \tag{1}$$

$$[-x]_q^{\sim} = -[x]_q^{\sim}. \tag{2}$$

In terms of $q \neq 1 (q = 1$ is just the classical case, because

$$\lim_{q \to 1} [x]_q^{\sim} = x), \tag{3}$$

formula (1) implies that we can take

$$q > 1$$

without loss of generality. Formula (2) shows that it's enough to treat nonnegative numbers only.

Theorem 4. Let $a, b > 0$. Then

$$[a]_q^{\sim} + [b]_q^{\sim} < [a+b]_q^{\sim}, \ q > 1. \tag{5}$$

Proof. Set

$$u = q^a, \ v = q^b \Rightarrow \ u, v > 1. \tag{6}$$

Then (5) becomes, after multiplication by $q - q^{-1}$:

$$x - \frac{1}{x} + y - \frac{1}{y} \stackrel{?}{<} xy - \frac{1}{xy},$$

or

$$\frac{1}{xy} - \frac{1}{x} - \frac{1}{y} + 1 \stackrel{?}{<} xy - x - y + 1,$$

or

$$(1-\frac{1}{x})(1-\frac{1}{y}) \stackrel{?}{<} (x-1)(y-1),$$

or

$$\frac{(x-1)(y-1)}{xy} \stackrel{?}{<} (x-1)(y-1),$$

or, since $x-1>0, \quad y-1>0,$

$$\frac{1}{xy} \stackrel{?}{<} 1,$$

or

$$xy \stackrel{?}{>} 1,$$

which is true. ■

Inducting on n, we get

Corollary 7. If $a_i > 0$ for $1 \leq i \leq n$, then

$$\sum_{i=1}^{n} [a_i]_q^{\sim} < [\sum_{i=1}^{n} a_i]_q^{\sim}. \tag{8}$$

In particular, for $n=2$, we get

$$[a]_q^{\sim} + [1-a]_q^{\sim} < 1, \quad 0<a<1, \quad q \neq 1. \tag{9}$$

a known result.

Corollary 10.

$$\frac{[a]_q^{\sim}}{[2a]_q^{\sim}} < \frac{[1]}{[2]_q^{\sim}}, \ a>0, \ q \neq 1. \tag{11}$$

Proof. Take $b=a$ in (5). ■

An Inequality Between Quantum Factorials

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Abstract

Classically, $\sum_{k=1}^{n} k\, k! < (n+1)!$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x, q \in \mathbf{R}, \quad q \neq 1, q > 0$$

be the second quantization of x.

Theorem 1. For $n \in \mathbf{Z}_{\geq 1}$,

$$\sum_{k=1}^{n} ([k]^{\sim}[k]!_q^{\sim} < [n+1]!_q^{\sim}, \tag{2}$$

where

$$[k]!_q = [1]_q^{\sim} ... [k]_q^{\sim}. \tag{3}$$

Proof. We use induction on n. For $n = 1$, (2) returns:

$$1 \cdot 1! = 1 < [2]!_q^{\sim} = [2]_q^{\sim} = q + q^{-1},$$

which is true.

Now,

$$[x]_q^{\sim} = [x]_{q^{-1}}^{\sim},$$

so we can take

$$q > 1 \tag{4}$$

without loss of generality.

The inductive step in (2) amounts to:

$$[n+1]!_q^{\sim} + [n+1]_q^{\sim}[n+1]!_q^{\sim} \stackrel{?}{<} [n+2]!_q^{\sim},$$

or

$$1 + [n+1]_q^{\sim} \stackrel{?}{<} [n+2]_q^{\sim}. \tag{5}$$

Lemma 6. For $x > 0$,

$$1 + [x]_q^{\sim} < [x+1]_q^{\sim} \tag{6}$$

Proof. Multiplying by $q - q^{-1} > 0$, and calling

$$X = q^x > 1, \tag{7}$$

(6) becomes:

$$q - q^{-1} + X - X^{-1} \stackrel{?}{<} qX - q^{-1}X^{-1},$$

or

$$q^{-1}(q-1)(q+1) \stackrel{?}{=} X(q-1) - q^{-1}(1-q),$$

or

$$1 + q^{-1} \stackrel{?}{<} X + q^{-1}X^{-1},$$

or

$$q^{-1}X^{-1}(X-1) \stackrel{?}{<} X - 1,$$

or, because $X = q^x > 1$,

$$q^{-1}X^{-1} \stackrel{?}{<} 1,$$

which is true. ■
Thus, (5) is true too. ■

An Inequality For $\pi(x)$

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Abstract

We prove that $\pi(ax) < a\pi(x)$ for any $a > 1$.

Let $\pi(x)$ be the usual number-theoretic function counting the number of primes $p \leq x$.

Remark 1. For any $a > 1$,

$$\pi(ax) < a\pi(x),\ x >> 0. \tag{2}$$

Proof 1. By the P. N. T.,

$$\pi(x) - \frac{x}{logx}, \tag{3}$$

therefore

$$\pi(ax) \sim \frac{ax}{log(ax)} < \frac{ax}{logx} = a\pi(x). \quad \blacksquare$$

This, of course, is not a proof, since we are working with asymptotic expressions.

Proof 2. Set

$$p_n = nf(n), \tag{4}$$

where

$$f(n) = logn + \sum_{i=0}^{\infty} \frac{P_i}{log^i n}, \tag{5}$$

where P_i is an polynomial in $w = loglogn$ of degree $i + \delta_0^i$. I recently showed that $f(x)$ is increasing with x. Hence,

$$\pi(ax) = \frac{ax}{f(ax)} < \frac{ax}{f(x)} = a\pi(x). \quad \blacksquare$$

An Exact Asymptotic Expansion Of $\pi(x)$

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Abstract

We give an *exact* asymptotic expansion of $f\pi(x)$.

It's easy to see that we only need $\pi(p_m),\ p_m$ being the n^{th} prime.
So

$$\frac{\pi(p_m)}{p_m} = \frac{m}{p_m} = \frac{1}{f(m)}, \tag{1}$$

where

$$p_n = nf(n), \tag{2}$$

$$f(n) = logn + (loglogn - 1) + \sum_{i=1}^{\infty} \frac{P_i}{log^i n}, \tag{3}$$

P_i being a polynomial in $w = loglogn$ of degree i. Thus,

$$\pi(x) = \frac{x}{f(x)} = \frac{x}{logx + (loglogx - 1) + O(\frac{1}{logx})}, \tag{4}$$

and we see that both Legendre and Chebyghev were wrong, the former thinking that

$$\pi(x) - \frac{x}{logx - A}, A = 1.08..., \tag{5}$$

and the latter that

$$A = 1. \tag{6}$$

Both missed the important term $loglogx$.

The story doesn't end here. Expanding (4), we get:

$$\pi(x) = \frac{x}{logx}\left(1 - \frac{loglogx - 1}{logx}\right), \tag{7}$$

contradicting Dusart, [Dus 1998] in his Thesis; on p. 36 gives the formula

$$\pi(x) = \frac{x}{logx}\left[1 + \frac{1}{logx} + \frac{2}{log^2 x} + O\left(\frac{1}{log^3 x}\right)\right], \tag{8}$$

which is obviously wrong.

References

[Dus 1998] Dusart, P., *Autour de la fonction qui compte le nombre de nombres premiers*, Theses, Universite de Limoges, France (1998) http://www.unilim.fr/laco/theses/1998/T1998_01.pdf

A New Estimate For Chebyshev's $\theta(x)$

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Abstract

We establish a new estimate for Chebyshev's $\theta(x)$. Precisely, $\theta(p_n)$ is estimated in terms of n, not of p_n, as did Chebyshev and his successors. Here p_n is the prime $\#n$.

We prove in this note

Theorem 1. Let p_n denotes the n^{th} prime number. Then

$$\theta(p_n) < nlog(nlogn), \quad n > 2. \tag{2}$$

Proof. For $n = 3, 4, 5$, we have:

$$\theta(p_3) = log(2 \cdot 3 \cdot 5) = log30 = 3.40 < 3log(3log3) = 3.57, \tag{3}$$
$$\theta(p_4) = log(2 \cdot 3 \cdot 5 \cdot 7) = log210 = 5.34 < log(4log4) = 6.85, \tag{4}$$
$$\theta(p_5) = log(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11) = log(2310) = 7.74 < 5log(5log5) = 10.4. \tag{5}$$

This is the base of induction. The inductive step follows from:

$$nlog(nlogn) + logp_{n+1} \overset{?}{<} (n+1)log[(n+1)log(n+1)], \tag{6}$$

or

$$logp_{n+1} \overset{?}{<} n\{log[(n+1)log(n+1)] - log(nlogn)\} + \tag{7a}$$
$$+log[(n+1)log(n+1)]. \tag{7b}$$

But, by Rosser,

$$p_n < nlog(nlogn), \quad n \geq 6, \Rightarrow \tag{8}$$
$$logp_n < logn + loglog(nlogn) = log[nlog(nlogn)]. \tag{8'}$$

Since

$$\frac{1}{n} - \frac{1}{2n^2} < log(n+1) - logn = log(1 + \frac{1}{n}) < \frac{1}{n},$$

and

$$loglog(n+1) - loglogn \sim \frac{1}{nlogn},$$

while

$$\frac{1}{nlogn} > \frac{1}{n^2}, \tag{9}$$

we have:

$$\{\text{RHS } of(7a)\} > 1, \quad n > 1,$$

and (7) follows from:

$$logp_{n+1} \overset{?}{<} log[e(n+1)log(n+1)] \Leftrightarrow$$
$$p_{n+1} \overset{?}{<} e(n+1)log(n+1),$$

or

$$p_n < enlogn, \tag{10}$$

which is obvious, because by (8),

$$p_n < nlog(nlogn) \overset{?}{<} enlogn \Leftrightarrow$$
$$log(nlogn) - logn + loglogn \overset{?}{<} elogn \Leftrightarrow$$
$$logt \overset{?}{<} (e-1)t, \quad t = logn, \tag{11}$$

and this is true for $t > 0$, since

$$logt < t, \ t > 0. \ \blacksquare \tag{12}$$

To tie the loose end, we prove (9), accurately:

$$\{log[(n+1)log(n+1)] - log(nlogn)\} \overset{?}{>} \frac{1}{n} \ \Leftrightarrow \tag{13}$$
$$log\left[\frac{n+1}{n}\frac{log(n+1)}{logn}\right] \overset{?}{>} \frac{1}{n} \ \Leftrightarrow \tag{14}$$
$$log\left(1+\frac{1}{n}\right) + log\frac{log(n+1)}{logn} \overset{?}{>} \frac{1}{n}. \tag{15}$$

But

$$log\left(1+\frac{1}{n}\right) > \frac{1}{n} - \frac{1}{2n^2},$$
$$log(n+1) = log(n) + log\left(1+\frac{1}{n}\right) > logn + \frac{1}{n} - \frac{1}{2n^2} \ \Leftrightarrow$$
$$\frac{log(n+1)}{logn} > 1 + \frac{\frac{1}{n} - \frac{1}{2n^2}}{logn} \Rightarrow$$
$$log\frac{log(n+1)}{logn} > log\left(1 + \frac{\frac{1}{n} - \frac{1}{2n^2}}{logn}\right) > \frac{\frac{1}{n} - \frac{1}{2n^2}}{logn} - \frac{1}{2}\frac{1}{n^2}\frac{(1-\frac{1}{2n})^2}{log^2 n}, \tag{16}$$

and (15) turns into

$$\frac{\frac{1}{n}-\frac{1}{2n^2}}{logn} \overset{?}{>} \frac{1}{2n^2}+\frac{1}{2n^2}\frac{(1-\frac{1}{2n})^2}{log^2n}. \tag{17}$$

Multiplying through by $2n^2$, we get:

$$\frac{2n-1}{logn} \overset{?}{>} 1+\frac{(1-\frac{1}{2n})^2}{log^2n} \tag{18}$$

But the RHS decreases with n, while the LHS increases for n for $n \geq e$, and for $n = 2$, we have for (18):

$$\frac{2\cdot 2-1}{log2}=\frac{3}{0.6g}=4.32>1+\frac{(1-\frac{1}{4})^2}{0.69^2}=1+\frac{0.56}{0.49}=2.17, \tag{19}$$

while for $n = 3$, the LHS of (18) is $\frac{5}{log3} = 4.55$, while the LHS only decreases, so < 2.17.

An Estimate For A Finite Continuous Fraction

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Abstract

We establish two sided estimate for $\frac{1}{n}\frac{1}{n+1}+\cdots+\frac{1}{1}$.

Denote

$$S_n = \frac{1}{n+}\frac{1}{n-1+}\ldots \quad +\frac{1}{1}, \quad n \geq 1.$$

Theorem 1.

$$\frac{1}{n+1} < S_n < \frac{1}{n}. \tag{2}$$

Proof. Since

$$S_{n+1} = \frac{1}{1+n+S_n},$$

and $S_n > 0$,

so

$$S_{n+1} < \frac{1}{n+1}. \tag{3}$$

In the other direction,

$$S_{n+1} = \frac{1}{n+1+S_n} > \frac{1}{n+1+\frac{1}{n}} \overset{?}{>} \frac{1}{n+2} \quad \Leftrightarrow$$

$$n+2 \overset{?}{>} n+1+\frac{1}{n} \quad \Leftrightarrow$$

$$1 > \frac{1}{n},$$

which is obvious. ■

An Alternating Sum Of Quantum Integers Under Second Quantization

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Abstract

$\sum_{i=0}^{N}(-1)^i i$ is quantized.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0, \pm 1,$$

be the second quantization of x.

Theorem 1. We have, for $n \in \mathbf{Z}_{\geq 0}$,

$$\sum_{i=0}^{2n+1}(-1)^i[i]_q^{\sim} = -\frac{2}{([2]_{q^{1/2}}^{\sim})^2}\left[n+\frac{1}{2}\right]_q^{\sim}\left(\left[n+\frac{3}{2}\right]_q^{\sim} - \left[n+\frac{1}{2}\right]_q^{\sim}\right), \tag{2}$$

$$\sum_{i=0}^{2n}(-1)^i[i]_q^{\sim} = [n]_q^{\sim}\Big([n+1]_q^{\sim} - [n]_q^{\sim}\Big). \tag{3}$$

Proof. We use induction on n. For $n = 0$, (3) is obvious, while (2) returns:

$$-1 \stackrel{?}{=} -\frac{2}{([2]_{q^{1/2}}^{\sim})^2}\ \left[\frac{1}{2}\right]_q^{\sim}\left(\left[\frac{3}{2}\right]_q^{\sim} - \left[\frac{1}{2}\right]_q^{\sim}\right),$$

or

$$\left([2]_{q^{1/2}}^{\sim}\right)^2 = 2 + \frac{q^{1/2}-q^{-1/2}}{(q-q^{-1})^2}\left(q^{1/2}+q^{-1/2}\right)^2\left\{q^{3/2}-q^{-3/2}-q^{1/2}+q^{-1/2}\right\},$$

or, with $Q = q^{1/2}$,

$$Q^2 + Q^{-2} \stackrel{?}{=} \frac{(Q+Q^{-1})}{Q^2-Q^{-2}}\Big\{Q^3 - Q^{-3} - Q + Q^{-1}\Big\},$$

or

$$Q^2 + Q^{-2} \stackrel{?}{=} \frac{(Q^2t+Q^{-2})}{Q^2-Q^{-2}}\Big\{Q(Q^2+1) - Q^{-1}(-1+Q^{-2})\Big\},$$

or

$$(Q^2+Q^{-2})(Q-Q^{-1}) \stackrel{?}{=} \Big\{QQ(Q-Q^{-1})+Q^{-1}Q^{-1}(Q-Q^{-1})\Big\},$$

or

$$(Q^2+Q^{-2})(Q-Q^{-1}) \stackrel{?}{=} (Q^2+Q^{-2})(Q-Q^{-1}),$$

which is true.

The inductive step in (2) amounts to:

$$-[n+\frac{1}{2}]_q^{\sim}\Big([n+\frac{3}{2}]-\Big[n+\frac{1}{2}\Big]_q^{\sim}\Big)+[2n+2]_q^{\sim}-[2n+3]_q^{\sim} \stackrel{?}{=}$$
$$\stackrel{?}{=} -[n+\frac{3}{2}]_q^{\sim}\Big(\Big[n+\frac{5}{2}\Big]_q^{\sim}-\Big[n+\frac{3}{2}\Big]_q^{\sim}\Big). \tag{4}$$

Now,

$$([x+1]_q^{\sim})^2-([x]_q^{\sim})^2=[2x+1]_q^{\sim}, \forall x, \tag{5}$$

hence (4) simplifies to ($x=n+\frac{1}{2}$), with $N=n+\frac{1}{2}$:

$$[N]_q^{\sim}[N+1]_q^{\sim}+[2N+2]_q^{\sim} \stackrel{?}{=} [N+1]_q^{\sim}[N+2]_q^{\sim}, \tag{6}$$

or, with $q^N=X$:

$$(X-X^{-1})(Xq-X^{-1}q^{-1})+(q-q^{-1})(X^2q^2-X^{-2}q^{-2}) \stackrel{?}{=}$$
$$=(Xq-X^{-1}q^{-1})(Xq^2-X^{-1}q^{-2}),$$

or

$$(X^2q-q^{-1}-q+X^{-2}q^{-2})+(X^2q^3-X^{-2}q^{-1}-X^2q+X^{-2}q^{-3}) \stackrel{?}{=}$$
$$\stackrel{?}{=} (X^2q^3-q^{-2}-q+X^{-2}q^{-3}),$$

which is identically true.

Let's turn to (3). Its inductive step amounts to:

$$[n]_q^{\sim}([n+1]_q^{\sim}-[n]_q^{\sim})-[2n+1]_q^{\sim}+[2n+2]_q^{\sim} \stackrel{?}{=}$$
$$\stackrel{?}{=} [n+1]_q^{\sim}([n+2]_q^{\sim}-[n+1]_q^{\sim}),$$

or, with (5),

$$[n]_q^{\sim}[n+1]_q^{\sim}+[2n+2]_q^{\sim} \stackrel{?}{=} [n+1]_q[n+2]_q^{\sim},$$

which is (6). ■

An Alternating Sum Of Quantum Arithmetic Progression

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Abstract

For the usual arithmetic progression $a_k = a+dk$, $\sum_{k=0}^{2n}(-1)^k a_k = a+dn$, $\sum_{k=0}^{2n-1}(-1)^k a_k = -nd$. We quantize these formulae.

Let

$$[x]_q^\sim = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0, 1,$$

be the second quantization of x.

Let $a_k = [a + kd]_q^\sim$, be a quantum arithmetic progression.

Theorem 1. (i) For $n \in \mathbf{Z}_{\geq 0}$,

$$\sum_{k=0}^{2n}(-1)^k a_k = [a + dn]_q^\sim \frac{[2]_{q^{d(n+1/2)}}^\sim}{[2]_{q^{d/2}}^\sim}, \tag{2}$$

For $n \in \mathbf{Z}_{\geq 1}$,

$$\sum_{k=0}^{2n-1}(-1)^k a_k = -[dn]_q^\sim \frac{[2]_{q^{a+d(n-1/2)}}^\sim}{[2]_{q^{d/2}}^\sim} \tag{3}$$

Proof. Call $q^k - q^{-k}$ by η_k. Then the LHS of (2) is:

$$\begin{aligned}
&\frac{1}{\eta_1}\sum_{k=0}^{2n}(-1)^k(q^{a+kd} - q^{-a-kd}) = \\
&= \frac{1}{\eta_1}\left\{q^a \sum_{k=0}^{2n}(-q^d)^k - q^{-a}\sum_{k=0}^{2n}(-q^{-d})^k\right\} = \\
&= \frac{1}{\eta_1}\left\{q^a \frac{1 + q^{d(2n+1)}}{1 + q^d} - q^{-a}\frac{1 + q^{-d(2n+1)}}{1 + q^{-d}}\right\} = \\
&= \frac{[2]_{q^{d(n+1/2)}}^\sim}{[2]_{q^{d/2}}^\sim \eta_1}\left\{q^{a-\frac{d}{2}} q^{d(n+1/2)} - q^{-a+d/2} q^{-d(n+1/2)}\right\} = \\
&= \frac{[2]_{q^{d(n+1/2)}}^\sim}{[2]_{q^{d/2}}^\sim}[a + dn]_q^\sim,
\end{aligned}$$

which is (2).

For the LHS of (3), we get:

$$\begin{aligned}
&\frac{1}{\eta_1}\sum_{k=0}^{2n-1}[q^a(-q^d)^k-(-q^{-a}(-q^{-d})^k]=\\
&=\frac{1}{\eta_1}\left(q\frac{a^{1-q^{d2n}}}{1+q^d}-q^{-a}\frac{1-q^{-d2n}}{1+q^{-d}}\right)=\\
&=-\frac{\eta_{dn}}{\eta_1}\left(q^{a-\frac{d}{2}}q^{dn}+q^{-a+\frac{d}{2}}q^{-dn}\right)=\\
&=-[dn]_q^{\sim}\frac{[2]^{\sim}_{q^{d(n-1/2)+a}}}{[2]^{\sim}_{q^{d/2}}},
\end{aligned}$$

which is the RHS of (3). ■

A Hexagon Property Of Quantum Binomial Coefficients

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Abstract

Classically, $\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0, \pm 1,$$

be the 2^{nd} quantization of x.

Let

$$\begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} = \frac{[n]!_q^{\sim}}{[k]!_q^{\sim}[n-k]!_q^{\sim}},$$

where

$$[k]!_q^{\sim} = [1]_q^{\sim}...[k]_q^{\sim}, \quad k \in \mathbf{Z}_{\geq 1}; \quad [0]!_q^{\sim} = 1.$$

Theorem 1. For $n > k$,

$$\begin{bmatrix} n-1 \\ k+1 \end{bmatrix}_q^{\sim} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q^{\sim} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q^{\sim} = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q^{\sim} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q^{\sim} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q^{\sim}. \tag{2}$$

Proof. In the long-hand, (2) is:

$$\frac{[n-1]!_q^{\sim}[n]!_q^{\sim}[n+1]!_q^{\sim}}{[k-1]!_q^{\sim}[k]!_q^{\sim}[k+1]_q^{\sim}} \frac{1}{[n-k]!_q^{\sim}[n-k+1]![n+1-k]!_q^{\sim}} \stackrel{?}{=}$$

$$\stackrel{?}{=} \frac{[n-1]!_q^{\sim}[n]!_q^{\sim}[n+1]!_q^{\sim}}{[k-1]!_q^{\sim}[k]!_q^{\sim}[k+1]!_q^{\sim}} \frac{1}{[n-1-k]!_q^{\sim}[n-k+1]!_q^{\sim}[n-k]!_q^{\sim}},$$

which is true. ■

A Decomposition Of General Even Numbers, Quantized

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Abstract

We decompose general numbers of the form $2mM$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}$$

be the 2^{nd} quantization of x, so that

$$[2]_q^{\sim} = q + q^{-1}.$$

Theorem 1. For $m, M \in \mathbf{Z}_{\geq 2}$, we have:

$$[2]_q^{\sim}[m]_q^{\sim}[M]_q^{\sim} = [M-m]_q^{\sim} + [M+m]_q^{\sim} + 2\sum_{i=1}^{m-1}[M-m+2i]_q^{\sim}. \tag{2}$$

Proof. We are going to use the general formula

$$[a]_q^{\sim} + [a+?]_q^{\sim} + ... + [a+2(m-1)]_q^{\sim} = [m]_q^{\sim}[a+m-1]_q^{\sim},$$

for $a = M - m$ and $a = M - m + 2$.

Thus,

$$[M-m]_q^{\sim} + [M-m+2]_q^{\sim} + ... + [M-m+2(m-1)]_q^{\sim} = [m]_q^{\sim}[M-m+m-1]_q^{\sim} =$$
$$= [m]_q^{\sim}[M-1]_q^{\sim}, \tag{3}$$
$$[M-m+2]_q^{\sim} + ... + [M-m+2(m-1)]_q^{\sim} + [M+m]_q^{\sim} = [m]_q^{\sim}[M+1]_q^{\sim}. \tag{4}$$

The RHS of (2) is the sum of (3) and (4). The useful formula

$$[a]_q^{\sim} + [a+2]_q^{\sim} = [2]_q^{\sim}[a+1]_q^{\sim},$$

returns:

$$[m]_q^{\sim}[M-1]_q^{\sim} + [m]_q^{\sim}[M+1]_q^{\sim} = [m]_q^{\sim}[2]_q^{\sim}[M]_q^{\sim},$$

which is exactly the LHS of (2). ■

A Conjecture About Generalized Catalan Numbers

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Abstract

We define a generalized Catalan numbers and formulate a conjecture about their divisibility properties.

Let

$$c_n^k = \frac{1}{(n+k)...(n+1)}\binom{2n}{n}, \quad k \in \mathbf{Z}_{\geq 1}, \quad n \in \mathbf{Z}_{\geq 0}, \tag{1}$$

be the k^{th} generalized Catalan number, so that

$$c_n^1 = c_n = \frac{1}{n+1}\binom{2n}{n} \tag{2}$$

is the classical Catalan number, easily seen to be an integer.

Conjecture 3. There exists a sequence m_k of positive integers, $k \leq n$, such that

$$m_k c_n^k \in \mathbf{Z}, \quad \forall n \in \mathbf{Z}, \quad \forall k \in \mathbf{Z}_{\geq 1}, \quad k < n. \tag{4}$$

The Conjecture is highly counterintuitive. With $m_1 = 1$, it turns into assertion that Catalan numbers are integers. We prove the conjecture for $k = 2, \quad m_2 = 6$.

Theorem 5. We have:

$$c_n^2 = \frac{6(2n)!}{n!(n+2)!} = 3\binom{2n+2}{n} - 10\binom{2n+1}{n-1} + 8\binom{2n}{n-2}. \tag{6}$$

Proof. In the long-hand, (6) is:

$$6\frac{(2n)!}{n!(n+2)!} \stackrel{?}{=} 3\frac{(2n+2)(2n+1)(2n)!}{n!(n+2)!} - 10\frac{(2n+1)(2n)!n}{(n+2)!n!} + 8\frac{(2n)!n(n-1)}{(n+2)!n!},$$

or

$$6 \stackrel{?}{=} 3(2n+2)(2n+1) - 10(2n+1)n + 8(n-1)n,$$

or

$$6 \stackrel{?}{=} 3(4n^2+6n+2) - 10(2n^2+n) + 8(n^2-n),$$

which is true. ■

Remark 7. Unfortunately, (6) is *not* quantizable over $\mathbf{Z}[q, q^{-1}]$. So, we have no reasons to believe that our Conjecture carries into quantum realm; even though, we believe it does.

A Comparison Of Quantum Fractions

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Abstract

We compare two types of quantum fractions.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0, 1,$$

be the 2^{nd} quantization of x.

Since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Theorem 1. For $a > b > 1$,

$$\left[\frac{a}{b}\right]_q^{\sim} < \frac{[a]_q^{\sim}}{[b]_q}. \tag{2}$$

Set

$$\lambda = \frac{a}{b} > 1.$$

Then (2) becomes

$$[\lambda]_q^{\sim}[b]_q^{\sim} < [\lambda b]_q^{\sim}. \tag{3}$$

multiplying (3) by $(q - q^{-1})^2 > 0$, we get

$$(q^{\lambda} - q^{-\lambda})(q^b - q^{-b}) \stackrel{?}{>} (q^{\lambda b} - q^{-\lambda b})(q - q^{-1}),$$

or

$$q^{\lambda+b} + q^{-\lambda-b} - (q^{b-\lambda} + q^{\lambda-b}) \stackrel{?}{>} (q^{\lambda b+1} + q^{\lambda b-1}) - (q^{\lambda b-1} + q^{-\lambda b+1}),$$

or

$$(q^{\lambda b-1}+q^{-\lambda b+1})-(q^{b-\lambda}+q^{\lambda-b})\stackrel{?}{>}(q^{\lambda b+1}+q^{-\lambda b-1})-(q^{\lambda+b}+q^{-\lambda-b}),$$

or

$$q^{\lambda-b}[q^{(b-1)(\lambda+1)}-1]+q^{-1}[1-q^{(b-1)(\lambda+1)}]\stackrel{?}{>}q^{\lambda+b}[q^{(\lambda-1)(b-1)}-1]+q^{-\lambda b-1}[1-q^{(\lambda-1)(b-1)}],$$

or

$$[q^{(\lambda+1)(b-1)}-1]q^{-b+1}[q^{(\lambda-1)(b+1)}-1]\stackrel{?}{>}[q^{(\lambda-1)(b-1)}-1]q^{-\lambda b-1}[q^{(\lambda-1)(b-1)}-1],$$

or

$$[q^{(\lambda+1)(b-1)}-1][q^{(b+1)(\lambda-1)}-1]\stackrel{?}{>}q^{-2}[q^{(\lambda-1)(b-1)}-1]^2, \tag{4}$$

which is obvious, because

$$q^{(\lambda+1)(b+1)}-1>q^{(\lambda-1)(b-1)}-q\ \Leftrightarrow$$
$$Q^{\lambda+1}>Q^{\lambda-1},\ \ Q=q^{b-1}>1,$$

and like-wise with b and λ interchanged. ■

An Infinite Product of Squares In Quantum Calculus

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Abstract

Classically, $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)=\frac{1}{2}$. We quantize this.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\quad x\in\mathbf{R},\quad q\neq 0,\pm 1,$$

be the 2^{nd} quantization of x, so that

$$[x]_q^{\sim}=[x]_{q^{-1}}^{\sim},$$

and we can take

$$q>1$$

without loss of generality.

Theorem 1.

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{([n]_q^{\sim})^2}\right)=\frac{q}{[2]_q^{\sim}}. \tag{2}$$

Proof. The first term returns:

$$1-\frac{1}{([2]_q^{\sim})^2}=\frac{[3]_q^{\sim}}{[2]_q^{\sim}[2]_q^{\sim}}. \tag{3}$$

Lemma 4.

$$\prod_{k=2}^{n}\left(1-\frac{1}{([k]_q^{\sim})^2}\right)=\frac{[n+1]_q^{\sim}}{[2]_q[n]_q^{\sim}} \tag{5}$$

Proof. The base of inductive proof of (5) is (3), for $n=2$. The inductive step consists of:

$$\frac{[n+1]_q^{\sim}}{[2]_q^{\sim}[n]_q^{\sim}}\left(1-\frac{1}{([n+1]_q^{\sim})^2}\right)\stackrel{?}{=}\frac{[n+2]_q^{\sim}}{[2]_q^{\sim}[n+1]_q^{\sim}},$$

or

$$\frac{1}{[n]_q^\sim}\Big(([n+1]_q^\sim)^2-1\Big)\stackrel{?}{=}\ [n+2]_q^\sim$$

or

$$([n+1]_q^\sim)^2-1\stackrel{?}{=}[n]_q^\sim[n+2]_q^\sim,$$

or, with $X=q^n$,

$$(Xq-X^{-1}q^{-1})^2(q-q^{-1})^2\stackrel{?}{=}(X-X^{-1})(Xq^2-X^{-1}q^{-2}),$$

or

$$(X^2q^2-2+X^{-2}q^{-2})-(q^2-2+q^{-2})\stackrel{?}{=}$$
$$\stackrel{?}{=}X^2q^2-q^2+X^{-2}q^{-2},$$

and this is obvious. ■

Now, if $q>1$, it's obvious that

$$\lim_{n\to\infty}\frac{[n+1]_q^\sim}{[n]_q}=q.\quad\blacksquare$$

Another Proof Of Finitude Of Twin Primes

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Abstract

We establish a useful inequality between prime numbers from which it follows that the number of twin primes can't be infinite.

Let p_n denotes the n^{th} prime.

Theorem 1. For n large enough,

$$\frac{p_n}{n} > \frac{p_{n+1}}{n+1}. \tag{2}$$

Proof. Set

$$p_n = n(f(n)), \tag{3}$$

$$f(n) = logn + (loglogn - 1) + \sum_{i=1}^{\infty} \frac{P_i}{log^i n}, \tag{4}$$

where P_i is a polynomial in $w = loglogn$ of degree i. We also set

$$P_0 = loglogn - 1, \tag{5}$$

so

$$f(n) = logn + \sum_{i=0}^{\infty} \frac{P_i}{log^i n} \tag{6}$$

Notice that

$$\frac{p_n}{n} = f(n),$$

so our inequality says that

$$f(n) < f(n+1). \tag{7}$$

(7) is certainly true for the main term of $f(n), logn$, and for the next term, $P_0 = P_0(n)$. In the difference

$$f(n+1) - f(n) = log(n+1) - logn + [loglog(n+1) - 1] - loglogn +$$
$$+ \sum_{i \geq 1} \left(\frac{\tilde{P}_i}{log^i(n+1)} - \frac{P_i}{log^i n} \right), \tag{8}$$

with the help of relations, modulo terms of the form $1/m^2$, we have:

$$log(n+1) - logn = \frac{1}{n} \tag{9}$$

$$loglog(n+1) - loglogn = \frac{1}{nlogn}, \tag{10}$$

$$\frac{P_i(n+1)}{log^i(n+1)} - \frac{P_i(n)}{log^i n} = \frac{1}{log^i(n+1)log^i n}\Big\{log^i n\Big[P_i(n) + P_i'\frac{1}{nlogn}\Big] -$$
$$(logn + \frac{1}{n})^i P_i\Big\} = \frac{1}{log^{2i}n}\Big\{P_i log^i n + \frac{P_i'}{n}log^{i-1}n - P_i(log^i n + ilog^{i-1}n\frac{1}{n})\Big\} =$$
$$= \frac{P_i' - iP_i}{nlog^{i+1}n}, \tag{11}$$

so (8) becomes:

$$\frac{1}{n} + \frac{1}{nlogn}\Big[1 + \sum_{i=1}^{\infty}\frac{P_i' - iP_i}{log^i n}\Big], \tag{12}$$

and this is > 0 for $n >> 0$. ■

Corollary 13. The number of twin primes is finite.

Proof. If $p_n = p$ is a twin prime, $p_{n+1} = p + 2$, then

$$\frac{p+2}{n+1} - \frac{p}{n} = \frac{1}{n(n+1)}\{np + 2n - (n+1)p\} = \frac{2n-p}{n(n+1)} < 0,$$

because $2n - p < 0$, p being $\sim nlogn$. A contradiction to (2). ■

Corollary 14. For $n >> 0$,

$$p_n < nlogp_{n+1}. \tag{15}$$

Proof. Rewrite (2) as

$$n(p_{n+1} - p_n) > p_n. \tag{16}$$

I recently have proved that

$$p_{n+1} - p_n = logp_{n+1} + 0(1), \quad n >> 0. \tag{17}$$

Substituting (17) into (16), (15) follows. ■

On Weighted Sums Of Quantum Factorials

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Abstract

The classical formulae $\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$ is quantized.

We shall prove below that

$$\sum_{k=1}^{n} q^{k(k+1)/2+1}[k]_q^{\sim}[k]!_q^{\sim} = q^{n(n+1)/2}[n+1]!_q^{\sim} - 1, \tag{1}$$

where

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \tag{2}$$

$$[k]!_q^{\sim} = \prod_{i=1}^{k}[k]_q^{\sim}, \quad k \geq 1. \tag{3}$$

For $n = 1$, (1) returns:

$$q^2 \stackrel{?}{=} q[2]_q^{\sim} - 1 = q(q + q^{-1}) - 1 = q^2,$$

which is true. Here we used the fact that

$$[2]_q^{\sim} = \frac{q^2 - q^{-2}}{q - q^{-1}} = q + q^{-1}.$$

We now use induction on n for (1). The inductive step amounts to the equality

$$\begin{aligned} &q^{(n+1)(n+2)/2+1}[n+1]_q^{\sim}[n+1]!_q^{\sim} \stackrel{?}{=} [q^{(n+1)(n+2)/2}[n+2]!_q^{\sim} - 1] - \\ &-[q^{n(n+1/2)}[n+1]!_q^{\sim} - 1] = \\ &= q^{(n+1)(n+2)/2}[n+2][n+1]! - q^{n(n+1)/2}[n+1]!. \end{aligned} \tag{4}$$

Dividing this by $q^{(n+1)(n+2)/2}[n+1]!_q^{\sim}$, we arrive at

$$[n+2]_q^{\sim} \stackrel{?}{=} q^1[n+1]_q^{\sim} + q^{-(n+1)}[1]_q^{\sim},$$

which is obviously true, since, in general,

$$[a+b]_q^{\sim} = q^b[a]_q^{\sim} + q^{-a}[b]_q^{\sim}. \tag{5}$$

Remark 6. Replace (1) by q by q^{-1}, sultract, and divide by $q-q^{-1}$. We get

$$\sum_{k=1}^{n}\left[\frac{k(k+1)}{2}+1\right]_q^{\sim}[k]_q^{\sim}[k]!_q^{\sim}=\left[\frac{n(n+1)}{2}\right]_q^{\sim}[n+1]!_q^{\sim}. \tag{6}$$

This identity is new even in the classical case $q=1$:

$$\sum_{k=1}^{n}\left[\frac{k(k+1)}{2}+1\right]k\cdot k!=\frac{n}{2}(n+1)(n+1)!. \tag{7}$$

Quantum Binomials In The Second Quantization

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Abstract

The classical formula $(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$, n a positive integer, is quantized.

We have:

$$1 + x = \sum_{k=0}^{n} \begin{bmatrix} 1 \\ k \end{bmatrix}_q^{\sim} x^k, \tag{1a}$$

$$(1+qx)(1+q^{-1}x) = 1 + [2]_q^{\sim} x + x^2, \tag{1b}$$

where

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \tag{2}$$

so that

$$[2] = q + q^{-1}, \tag{3a}$$

$$[3] = q^2 + 1 + q^{-2}, ... \tag{3b}$$

We thus expect for an m-product (starting with $(1+q^m x)$) to have:

$$\prod_{i=0}^{m} (1 + q^{m-2i}x) = \sum_{k=0}^{n+1} \begin{bmatrix} m+1 \\ k \end{bmatrix}_q^{\sim} x^k, \tag{4}$$

where

$$\begin{bmatrix} m \\ k \end{bmatrix}_q^{\sim} = \frac{[m]!_q^{\sim}}{[k]!_q^{\sim}[m-k]!_q^{\sim}}, \tag{5}$$

and

$$[s]!_q^{\sim} = [1]_q^{\sim} ... [s]_q^{\sim}, \quad s \in \mathbf{Z}_{\geq 1}; \quad [0]!_q^{\sim} = 1. \tag{6}$$

We prove formula (4) by induction on m. Assuming it's true for the $(m-2)$-product:

$$\prod_{i=0}^{m-2} (1 + q^{m-2-2i}) = \sum_{k=0}^{m-1} \begin{bmatrix} m-1 \\ k \end{bmatrix}_q^{\sim} x^k, \tag{7}$$

we get:

$$\prod_{i=0}^{m}(1+q^{m-2i}x)=(1+q^m x)(1+q^{-m}x)\prod_{i=0}^{m-2}(1+q^{m-2-2i}x)=$$
$$=(1+[2]^{\sim}_{q^m}x+x^2)\sum_{k=0}^{m-1}\begin{bmatrix}m-1\\k\end{bmatrix}^{\sim}_q x^k \stackrel{?}{=}\sum_{k=0}^{m+1}\begin{bmatrix}m+1\\k\end{bmatrix}x^k,$$

which is equivalent to

$$\begin{bmatrix}m+1\\k\end{bmatrix}^{\sim}_q \stackrel{?}{=}\begin{bmatrix}m-1\\k\end{bmatrix}^{\sim}_q+[2]_{q^m}\begin{bmatrix}m-1\\k-1\end{bmatrix}^{\sim}_q+\begin{bmatrix}m-1\\k-2\end{bmatrix}^{\sim}_q, \tag{8}$$

which is immediate, based on the triple of formulae:

$$\begin{bmatrix}m+1\\k\end{bmatrix}^{\sim}_1=q^k\begin{bmatrix}m\\k\end{bmatrix}^{\sim}_q+q^{k-m-1}\begin{bmatrix}m\\k-1\end{bmatrix}^{\sim}_q, \tag{9a}$$
$$\begin{bmatrix}m\\k\end{bmatrix}=q^{-k}\begin{bmatrix}m-1\\k\end{bmatrix}^{\sim}_q+q^{m-k}\begin{bmatrix}m-1\\k-1\end{bmatrix}^{\sim}_q, \tag{9b}$$
$$\begin{bmatrix}m-1\\k-1\end{bmatrix}^{\sim}_q=q^{1-k}\begin{bmatrix}m-2\\k-1\end{bmatrix}+q^{m+2-k}\begin{bmatrix}m-2\\k-2\end{bmatrix}^{\sim}_q, \tag{9c}$$

and the formula

$$\begin{bmatrix}m+1\\k\end{bmatrix}^{\sim}_q=q^k\begin{bmatrix}m\\k\end{bmatrix}^{\sim}_q+q^{k-m-1}\begin{bmatrix}m\\k-1\end{bmatrix}^{\sim}_q. \tag{10}$$

Substituting formulae (9) into (10), we recover the desired (8). Above we used the easily verifiable formulae

$$\begin{bmatrix}m+1\\k\end{bmatrix}^{\sim}_q=q^a\begin{bmatrix}m\\k\end{bmatrix}^{\sim}_q+q^b\begin{bmatrix}m\\k-1\end{bmatrix}^{\sim}_q, \tag{11}$$

where

$$\binom{a}{b}=\binom{k}{k-m-1}\text{ or }\binom{-k}{m+1-k}, \tag{12}$$

which results from the standard identity

$$[x+y]^{\sim}_q=q^y[x]^{\sim}_q+q^{-x}[y]^{\sim}_q=q^{-y}[x]^{\sim}_q+q^x[y]^{\sim}_q, \tag{13}$$

upon dividing (11) by $[m]!^{\sim}_q/[k]!^{\sim}_q[m+1-k]!^{\sim}_q$,

On The Sum Of Quantum Factorials

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Abstract

The classical identity $1 \cdot 1! + ... + n \cdot n! = (n+1)! - 1$ is quantized.

We shall prove that

$$\sum_{i=1}^{n} q^{1+\binom{i+1}{2}} [i]_q^{\sim} [i]!_q^{\sim} = q^{\binom{n+1}{2}} [n+1]!_q^{\sim} - 1, \tag{1}$$

where

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \tag{2}$$

$$[k]!_q^{\sim} = [1]_q^{\sim} ... [k]_q^{\sim}. \tag{3}$$

For $n = 1$, (1) reduced to

$$q^2 = q(q + q^{-1}) - 1;$$

which is true. We next use induction on n. (1) follows from

$$q^{1+\binom{n+2}{2}} [n+1]_q^{\sim} [n+1]!_q^{\sim} \stackrel{?}{=} q^{\binom{n+2}{2}} [n+2]!_q^{\sim} - 1 - (q^{\binom{n+1}{2}} [n+1]!_q^{\sim} - 1).$$

Dividing both sides by $[n+1]!_q^{\sim}$, we arrive at

$$q^{1+\binom{n+2}{2}} [n+1] \stackrel{?}{=} q^{\binom{n+2}{2}} [n+2] - q^{\binom{n+1}{2}}.$$

Dividing this by $q^{\binom{n+2}{2}}$, we get

$$q[n+1]_q^{\sim} + q^{-n-1} \stackrel{?}{=} [n+2]_q^{\sim}, \tag{4}$$

which is true, since

$$[a+b]_q^{\sim} = q^b [a]_q^{\sim} + q^{-a} [b]_q^{\sim}, \quad \forall a, b \tag{5}$$

returns (4) for $a = m+1, \ell = 1$.

Triple Products in Quantum Arithmetic Progressions

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Abstract

Let $a_k = a + kd$ be an arithmetic progression. Then $\sum_{k=1}^{n} \frac{1}{a_k a_{k+1} a_{k+2}} = \frac{1}{2d}\left(\frac{1}{a_1 a_2} - \frac{1}{a_{n+1} a_{n+2}}\right)$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0, \pm 1,$$

be the 2^{nd} quantization of x.

Lemma 1. We have:

$$\frac{q^{a+kd}}{[a_k]_q^{\sim}[a_{k+1}]_q^{\sim}[a_{k+2}]_q^{\sim}] } = \frac{1}{[2d]_q^{\sim}}\left(\frac{q^{2kd}}{[a_k]_q^{\sim}[a_{k+1}]_q^{\sim}} - \frac{q^{2(k+1)d}}{[a_{k+1}]_q^{\sim}[a_{k+2}]_q^{\sim}}\right). \tag{2}$$

Proof. Proof. Simplified, (2) turns into

$$q^{-a-kd}[2d]_q^{\sim} + [a+kd]_q^{\sim} q^{2d} = [a+(k+2)d]_q^{\sim},$$

which is true because, in general,

$$[a+b]_q^{\sim} = q^b[a]_q^{\sim} + q^{-a}[b]_q^{\sim}. \quad \blacksquare$$

Summing (2) on k, we get:

$$\sum_{k=1}^{n} \frac{q^{a+kd}}{[a_k]_q^{\sim}[a_{n+1}]_q^{\sim}[a_{k+2}]_q^{\sim}} = \frac{1}{[2d]_q^{\sim}}\left(\frac{q^{2d}}{[a_q]_q^{\sim}[a_2]_q^{\sim}} - \frac{q^{-2(n+1)d}}{[a_{n+1}]_q^{\sim}[a_{n+2}]_q^{\sim}}\right). \tag{3}$$

Quantum Arithmetic Progression Whose Sums Have Divisibility Properties

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Abstract

The following classical sum formulae are quantized:

$$\sum_{i=a+1}^{a+m}(2i) = r\sum_{i=1}^{k}[2(ri+a)+r+1];$$
$$\sum_{i=a+1}^{a+m}(2i+1) = r\sum_{i=1}^{k}[2(ri+a)+a-r+2];$$
$$\sum_{i=1}^{rk}(2i+b) = r\sum_{i=1}^{1}[r(2i-1)+b+1].$$

The classical formula quoted in the Abstract, all have the property that the HS's are divisible by r, a rare occurrence. The purpose of this note is to quantize each of these 3 formulae.

Set

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \tag{1}$$
$$3_k = q^k - q^{-1} \tag{2}$$

Theorem 3. For $a \in \mathbf{Z}, r, k, \in \mathbf{Z}_{\geq 1}$, we have:

$$\sum_{i=a+1}^{a+m}[2i]_q^{\sim} = [r]_q^{\sim}\sum_{i=1}^{k}[2(ri+a)-r+1]_q^{\sim}. \tag{4}$$

Proof. We are going to use the formula [Kup 2009]: for the sum of quantum arithmetic progression

$$[a]_q^{\sim} + [a+d]_q^{\sim} + ... + [a+nd]_q^{\sim} = [n+1]_{q^{d/2}}^{\sim}[q + \frac{nd}{2}]_q^{\sim} \tag{5}$$

Thus, the LHS of (4), having rk terms, and the average value of the term

$$\{[2(a+1)] + [2(a+rk)]\}/2 = a+1+a+rk = 2a+1+rk, \tag{6}$$

sums to

$$[rk]_q^{\sim}[2a+1+rk]_q^{\sim}. \tag{7}$$

For the RHS, with k terms, and $= 2r$, the average value of the term is

$$\{[2(r+a)-r+1]+[2(rk+a)-r+1]\}\frac{1}{2} =$$
$$= -r+1+r+a+rk+a = 2a+1+rk, \tag{7}$$

so the ??? is:

$$[r]_q^{\sim}[k]_{q^r}^{\sim}[2a+1+rk]_q^{\sim}, \tag{8}$$

which is the same as (7), because, as is well known

$$[uv]_q^{\sim} = [u]_q^{\sim}[v]_{q^u}^{\sim}, \quad \forall u,v. \ \blacksquare \tag{9}$$

Theorem 10. We have

$$\sum_{i=1+a}^{rk+a}[2i+1]_q^{\sim} = [r]_q^{\sim}\sum_{i=1}^{k}[2(ri+a)-r+2]_q^{\sim}. \tag{11}$$

Proof. For the LHS, with rm terms, $d=2$, and the average term being

$$\{[2(1+a)+1]+[2(rk+a)+1]\}\frac{1}{2} = 1+(1+a)+(rk+a) = 2a+rk+2, \tag{12}$$

so the LHS is

$$[rk]_q^{\sim}[2a+rk+2]_q^{\sim}. \tag{13}$$

For the RHS, $d=2r$, we have k terms, with the average term being

$$\frac{1}{2}\{[2(r+a)-r+2]+2(rk+a)-r+2]\} =$$
$$= -r+2+(r+a)+(rk+a) = 2a+rk+2. \tag{14}$$

so, the RHS is:

$$[r]_q^{\sim}[k]_{q^r}^{\sim}[2a+rk+2]_q^{\sim},$$

which is the same as the ZHS (13). $\blacksquare$

Remark 15. Formula (4) and (11) are particular cases ($b=0,1$) of the following 4-parameter general formula

$$\sum_{i=a+1}^{a+rk}[2i+b]_q^{\sim} = [r]_q^{\sim}\sum_{i=1}^{k}[2(ri+a)-r+b+1]_q^{\sim}. \tag{16}$$

Indeed, the LHS of (16) has rk terms, $d = 2$, and the average term is:

$$\frac{1}{2}\{[2(a+1)+b]+[2(a+rk)+b]\} = b+(a+1)+(a+rk) = 2a+rk+1+b,$$

so the LHS is

$$[rk]_q^{\sim}[2a+rk+1+b]_q^{\sim}. \tag{17}$$

The RHS has a sum of k terms, $d = 2r$, and the average term is:

$$\frac{1}{2}\{[2(r+a)-r+b+1]+[2(rk+a)-r+b+1]\} =$$
$$= -r+b+1+(r+a)+(rk+a) = 2a+rk+1+b.$$

Thus, the RHS is:

$$[r]_q^{\sim}[k]_{q^r}^{\sim}[2a+rk+1+6]_q^{\sim},$$

which is the same as the LHS (17).

The third equality in the Abstract results for $a = 0, q = 1$ in formula (16).

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The Maximum Value Of The Product Of Quantum a And $1-a$

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Abstract

Classically, $a(1-a) \leq \frac{1}{4}$, with maximum achieved at $a = \frac{1}{2}$. We quantize this fact.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0, 1,$$

be the second quantization of x, so that

$$\left[\frac{1}{2}\right]_q^{\sim} = \frac{q^{1/2} - q^{-1/2}}{q - q^{-1}} = \frac{1}{q^{1/2} + q^{-1/2}} = \frac{1}{[2]_{q^{1/2}}^{\sim}} \tag{1}$$

Theorem 2. Let $0 \leq a \leq 1$. Then

$$\max_{0 \leq a \leq 1} [a]_q^{\sim}[1-a]_q^{\sim} = \left(\left[\frac{1}{2}\right]_q^{\sim}\right)^2 \tag{3}$$

Proof. We have, with $x = q^a$:

$$(q - q^{-1})^2\{LHS \text{ of } (3)\} = \left(x - \frac{1}{x}\right)\left(\frac{q}{x} - q^{-1}x^{-1}\right) =$$
$$= q - q^{-1}x^2 - \frac{q}{x^2} + q^{-1} \text{ [with } y = x^2] =$$
$$= q + q^{-1} - (q^{-1}y + \frac{q}{y}). \tag{4}$$

Thus, the derivative $\frac{\partial}{\partial y}$ of (4) is

$$-\left(q^{-1} - \frac{q}{y^2}\right). \tag{5}$$

This vanishes when

$$y^2 = g^2 \Leftrightarrow y = q \tag{6}$$

(since $y = x^2 \geq 0$), so that

$$x = \sqrt{y} = q^{1/2} \tag{7}$$

and this is *maximum*, because $[a]_q^{\sim}[1-a]_q^{\sim}$ vanishes at each end of the interval $0 \leq a \leq 1$. Thus,

$$a_{max} = \frac{1}{2} \;\Rightarrow$$
$$[a]_q^{\sim}[1-a]_q^{\sim} \leq \left(\left[\frac{1}{2}\right]_q^{\sim}\right)^2 . \;\blacksquare \tag{8}$$

The Growth Of Quantum Analog Of A Classical One

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Abstract

Classically, for $x > 1$, $\frac{x-x^{-1}}{1} < \frac{x^2-x^{-2}}{2} < \frac{x^3-x^{-3}}{3} < ....$ We generalize this

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0, \pm 1,$$

be the 2^{nd} quantization of x, so that

$$[x]_q^{\sim} = [x]_{q^{-1}}^{\sim},$$

and we can take

$$q > 1$$

without loss of generality. Calling $x = q$ in the series of inequalities in the abstract, and dividing by $q - q^{-1}$, we get:

$$\frac{[1]_q^{\sim}}{1} < \frac{[2]_q^{\sim}}{2} < \frac{[3]_q^{\sim}}{3} < ...$$

This is a particular case of

Theorem 1. For $\alpha > 1$, the function $f(\alpha) = \frac{[\alpha]_q^{\sim}}{\alpha}$ is monotonically increasing with α.

Proof. Set

$$F(\alpha) = (q - q^{-1}) f(\alpha). \tag{2}$$

Then, with $h = logq, > 0$

$$\frac{dF}{d\alpha} = \frac{d}{d\alpha}\left(\frac{q^\alpha - q^{-\alpha}}{\alpha}\right) =$$
$$= \frac{h}{\alpha}(q^\alpha + q^{-\alpha}) - \frac{q^\alpha - q^{-\alpha}}{\alpha^2} = \frac{1}{\alpha} \text{ times :}$$
$$q^\alpha\left(h - \frac{1}{\alpha}\right) + q^{-\alpha}\left(h + \frac{1}{\alpha}\right) \stackrel{?}{>} 0 \tag{3}$$

If $h \geq \frac{1}{2}$, then (3), as a sum of two nonnegative terms, one of which is positive, is positive itself. So, $\frac{df}{d\alpha} > 0$.

Now, let $h < \frac{1}{2}$, so that $0 < h < 1$. Then (3) becomes:

$$q^{-2\alpha} \stackrel{?}{>} \frac{1-h\alpha}{1+h\alpha}, \tag{4}$$

or, with $h\alpha = x$,

$$e^{-2x} \stackrel{?}{>} \frac{1-x}{1+x}, \quad 0 < x \leq 1. \tag{5}$$

But this is obvious, since

$$e^{-2x} > 1 - 2x + \frac{(2x)^2}{2} = 1 - 2x + 2x^2, \tag{6}$$

and

$$\begin{aligned} &1 - 2x + 2x^2 > \frac{1-x}{1+x} + > \\ &(1+x)(1-2x+2x^2) = 1 - 2x + 2x^2 + x(1-2x+2x^2) = \\ &= 1 - x + 2x^3 > 1 - x \ \blacksquare \end{aligned} \tag{7}$$

The Fundamental Lemma In Quantum Arithmetic

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Abstract

Classically, $a|ab, b|ab$, trivially. In quantum case, the corresponding result is highly nontrival.

Let

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0, 1,$$

be the usual quantization of x, and let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \ x \in \mathbf{R}, \ q \neq 0, \pm 1,$$

be the second quantization of x, so that

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

and we can restrict ourselves with positive numbers only.

Theorem 1. Let $a, b \in \mathbf{Z}_{\geq 1}, \ (a, b) = 1$. Then

$$\frac{[ab]_q^{\sim}}{[a]_q^{\sim}[b]_q} \ \in \mathcal{O}_2, \tag{2}$$

where

$$\mathcal{O}_2 = \left\{ \sum_{\in \mathbf{Z}} a_i q^i \ a_i \in \mathbf{Z}, \text{ finite sums}, a_{-i} = a_i \right\} \tag{3}$$

is the ring of (quantum) integers in the 2^{nd} quantization.

Proof. Because

$$[x]_q^{\sim} = q^{1-x}[x]_{q^2},$$

we can restate (2) as

$$\frac{[ab]_{q^2}}{[a]_{q^2}[b]_{q^2}} \in \mathcal{O}_1, \tag{4}$$

where

$$\mathcal{O}_1 = \left\{ \sum_{\in \mathbf{Z}} a_i q^i \mid a_i \in \mathbf{Z}, \quad \text{finite sums} \right\} \tag{5}$$

is the ring of integers in the first quantization. Changing q^2 into Q, (4) becomes:

$$\frac{Q^{ab}-1}{(Q^a-1)(Q^b-1)} \in \mathcal{O}_1(Q), \ (a,b)=1. \tag{6}$$

Now, the denominator in (6) is a monic polynomial of degree $= a+b$, with the roots

$$exp\{2\pi i \frac{n}{a}\}, \quad 0 \leq n \leq a-1, \tag{7a}$$

$$exp\{2\pi i \frac{m}{b}\}, \quad 0 \leq m \leq b-1. \tag{7b}$$

Because $(a,b) = 1$, these roots are *all* different, hence, these are *all* the roots of the denominator. The numerator, of degree ab, has these roots as its own, and because

$$ab > a+b \Leftrightarrow \tag{8a}$$

$$(a-1)(b-1) > 1, \tag{8b}$$

which happen for $a, b > 1$, (6) is proven.

The case $a = 1$, say, is obvious. ■

On The Quantum Sum Of Two Consecutive Cubes

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Abstract

The classical formula $n^3+(n+1)^3=\sum_{i=n^2+1}^{(n+1)^2} i$, is quantized.

The classical formula

$$\sum_{i=n^2+1}^{(n+1)^2} = n^3+(n+1)^3, \tag{1}$$

arriving from the arrangement [PKi 1974], prob.

$$\begin{array}{cc} 1 & 1=0^3+1^3 \\ 2\ 3\ 4 & g=1^3+2^3 \\ 5\ 6\ 7\ 8\ 9 & 35=2^3+3^3 \end{array}$$

is rather obvious:

$$\sum_{i=n^2+1}^{(n+1)^2} i$$

is the sum of an arithmetic progression with the number of terms

$$(n+1)^2-(n^2+1)+1=2n+1, \tag{2}$$

and with the average term

$$\frac{1}{2}\Big[n^2+1+(n+1)^2\Big]=\frac{1}{2}\Big[n^2+1+n^2+2n+1\Big]=n^2+n+1, \tag{3}$$

so the sum equals to

$$(2n+1)(n^2+n+1)=2n^3+2n^2+2n+n^2+n+1=2n^3+3n^2+3n+1= \\ =n^3+(n+1)^3. \tag{4}$$

A quantum version of this formula is not obvious. It is:

$$\sum_{i=n^2+1}^{(n+1)^2} [i]_q^{\sim}=[2n+1]_{q^{1/2}}^{\sim}[n^2+n+1]_q^{\sim}, \tag{5}$$

where

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}.$$

Indeed, rewriting the LHS of (5) as

$$\sum_{i=0}^{2n} [n^2 + 1 + i]_q^{\sim}, \tag{6}$$

and denoting

$$\eta_\alpha = q^\alpha - q^{-\alpha},$$

the sum (6) can be transformed as:

$$\begin{aligned} &\frac{1}{\eta_1} \sum_{i=0}^{2n} \left(q^{n^2+1+i} - q^{-n^2 1 - i} \right) = \\ &= \frac{1}{\eta_1} \{ q^{n^2 n+1} [2n+1]_q - q^{-n^2-1} [2n+1]_{q^{-1}} \}, \end{aligned} \tag{7}$$

where

$$[x]_q = \frac{1 - q^x}{1 - q}$$

is the usual (1^{st}) quantization. Then, (7) is:

$$\begin{aligned} &\frac{1}{\eta_1} \left\{ q^{n^2+1} \frac{q^{2n+1} - 1}{q - 1} - q^{-n^2-1} \quad \frac{1 - q^{2n-1}}{1 - q^{-1}} \right\} = \\ &= \frac{1}{\eta_1} \left\{ q^{n^2+1} q^{\frac{2n+1}{2}} \quad \frac{\left(q^{\frac{2n+1}{2}} - q^{\frac{-2n+1}{2}} \right)}{q^{1/2}(q^{1/2} - q^{-1/2})} - \right. \\ &\left. - q^{-n^2-1} q^{-\frac{2n+1}{2}} \quad \frac{\left(q^{\frac{2n+1}{2}} - q^{\frac{-2n+1}{2}} \right)}{q^{1/2}(q^{1/2} - q^{-1/2})} \right\} = \\ &= \frac{1}{\eta_1} \left\{ q^{n^2+n+1} - q^{-n^2-n-1} \right\} [2n+1]_{q^{1/2}}^{\sim} = \\ &= [n^2 + n + 1]_q^{\sim} [2n+1]_{q^{1/2}}^{\sim}, \end{aligned} \tag{8}$$

and this is precisely the RH (5).

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Circle Quantization

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Abstract

The (projective) circle, $x^2+y^2=z^2$, is parameterized, classically, as $(u^2-v^2)^2+(2uv)^2=(u^2+v^2)^2$, a source of all things pythagoren. We find a quantum version of this formula.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\quad x\in\mathbf{R},\quad q\neq 0,\pm 1,$$

be the 2^{nd} quantization of x.

Theorem 1. For all $u,v\in\mathbf{R}$, we have:

$$([u^2-v^2]_q^{\sim})^2+[2u^2]_q^{\sim}[2v^2]_q^{\sim}=([u^2+v^2]_q^{\sim})^2. \tag{2}$$

Proof. Setting $u^2=x, v^2=y$, we rewrite (2) as:

$$([x-y]_q^{\sim})^2+[2x]_q^{\sim}[2y]_q^{\sim}=([x+y]_q^{\sim})^2. \tag{3}$$

Setting

$$X=q^x, Y=q^y,$$

and multiplying both parts of (3) by $(q-q^{-1})^2$, we arrive at:

$$(XY^{-1}-YX^{-1})^2+(X^2-X^{-2})(Y^2-Y^{-2})\stackrel{?}{=}(XY-X^{-1}Y^{-1})^2,$$

which is obvious. ■

Setting

$$y-x=a,$$
$$y+x-b,$$

so that

$$y=\frac{a+b}{2},$$
$$x=\frac{b-a}{2},$$

we can rewrite (3) as:

$$([a]_q^{\sim})^2+[b-a]_q^{\sim}[b+a]_q^{\sim}=([b]_q^{\sim})^2. \tag{4}$$

Quantum Binomials In The Second Quantization

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Abstract

The classical formula $(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$, n a positive integer, is quantized.

We have:

$$1 + x = \sum_{k=0}^{n} \begin{bmatrix} 1 \\ k \end{bmatrix}_q^{\sim} x^k, \tag{1a}$$

$$(1+qx)(1+q^{-1}x) = 1 + [2]_q^{\sim} x + x^2, \tag{1b}$$

where

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \tag{2}$$

so that

$$[2] = q + q^{-1}, \tag{3a}$$

$$[3] = q^2 + 1 + q^{-2}, ... \tag{3b}$$

We thus expect for an m-product (starting with $(1+q^m x)$) to have:

$$\prod_{i=0}^{m} (1 + q^{m-2i}x) = \sum_{k=0}^{n+1} \begin{bmatrix} m+1 \\ k \end{bmatrix}_q^{\sim} x^k, \tag{4}$$

where

$$\begin{bmatrix} m \\ k \end{bmatrix}_q^{\sim} = \frac{[m]!_q^{\sim}}{[k]!_q^{\sim}[m-k]!_q^{\sim}}, \tag{5}$$

and

$$[s]!_q^{\sim} = [1]_q^{\sim} ... [s]_q^{\sim}, \quad s \in \mathbf{Z}_{\geq 1}; \quad [0]!_q^{\sim} = 1. \tag{6}$$

We prove formula (4) by induction on m. Assuming it's true for the $(m-2)$-product:

$$\prod_{i=0}^{m-2} (1 + q^{m-2-2i}) = \sum_{k=0}^{m-1} \begin{bmatrix} m-1 \\ k \end{bmatrix}_q^{\sim} x^k, \tag{7}$$

we get:

$$\prod_{i=0}^{m}(1+q^{m-2i}x)=(1+q^m x)(1+q^{-m}x)\prod_{i=0}^{m-2}(1+q^{m-2-2i}x)=$$
$$=(1+[2]^{\sim}_{q^m}x+x^2)\sum_{k=0}^{m-1}\begin{bmatrix}m-1\\k\end{bmatrix}^{\sim}_q x^k \stackrel{?}{=}\sum_{k=0}^{m+1}\begin{bmatrix}m+1\\k\end{bmatrix}x^k,$$

which is equivalent to

$$\begin{bmatrix}m+1\\k\end{bmatrix}^{\sim}_q \stackrel{?}{=}\begin{bmatrix}m-1\\k\end{bmatrix}^{\sim}_q+[2]_{q^m}\begin{bmatrix}m-1\\k-1\end{bmatrix}^{\sim}_q+\begin{bmatrix}m-1\\k-2\end{bmatrix}^{\sim}_q, \tag{8}$$

which is immediate, based on the triple of formulae:

$$\begin{bmatrix}m+1\\k\end{bmatrix}^{\sim}_1=q^k\begin{bmatrix}m\\k\end{bmatrix}^{\sim}_q+q^{k-m-1}\begin{bmatrix}m\\k-1\end{bmatrix}^{\sim}_q, \tag{9a}$$
$$\begin{bmatrix}m\\k\end{bmatrix}=q^{-k}\begin{bmatrix}m-1\\k\end{bmatrix}^{\sim}_q+q^{m-k}\begin{bmatrix}m-1\\k-1\end{bmatrix}^{\sim}_q, \tag{9b}$$
$$\begin{bmatrix}m-1\\k-1\end{bmatrix}^{\sim}_q=q^{1-k}\begin{bmatrix}m-2\\k-1\end{bmatrix}+q^{m+2-k}\begin{bmatrix}m-2\\k-2\end{bmatrix}^{\sim}_q, \tag{9c}$$

and the formula

$$\begin{bmatrix}m+1\\k\end{bmatrix}^{\sim}_q=q^k\begin{bmatrix}m\\k\end{bmatrix}^{\sim}_q+q^{k-m-1}\begin{bmatrix}m\\k-1\end{bmatrix}^{\sim}_q. \tag{10}$$

Substituting formulae (9) into (10), we recover the desired (8). Above we used the easily verifiable formulae

$$\begin{bmatrix}m+1\\k\end{bmatrix}^{\sim}_q=q^a\begin{bmatrix}m\\k\end{bmatrix}^{\sim}_q+q^b\begin{bmatrix}m\\k-1\end{bmatrix}^{\sim}_q, \tag{11}$$

where

$$\binom{a}{b}=\binom{k}{k-m-1}\text{ or }\binom{-k}{m+1-k}, \tag{12}$$

which results from the standard identity

$$[x+y]^{\sim}_q=q^y[x]^{\sim}_q+q^{-x}[y]^{\sim}_q=q^{-y}[x]^{\sim}_q+q^x[y]^{\sim}_q, \tag{13}$$

upon dividing (11) by $[m]!^{\sim}_q/[k]!^{\sim}_q[m+1-k]!^{\sim}_q$,

Basic Inequality Among Quantum Numbers

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Abstract

We show that inequality $0 \in \alpha < \beta$ persists when quantized.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}$$

be the 2^{nd} quantization of x, so that

$$[x]_q^{\sim} = [x]_{q^{-1}}^{\sim}. \tag{1}$$

We consider all the numbers to be real.

Theorem 2. Let, in $\mathbf{R}_{>0}$,

$$0 < \alpha < \beta. \tag{3}$$

Then

$$[\alpha]_q^{\sim} < [\beta]_q^{\sim} : \tag{4}$$

Proof. In view of (1), we can consider

$$q > 1. \tag{5}$$

Let $\triangle = \beta - \alpha > 0$, so that $\beta = \alpha + \triangle$. The inequality (4) becomes, after being multiplied by $q - q^{-1} > 0$:

$$q^{\alpha+\triangle} - q^{-\alpha-\triangle} \stackrel{?}{>} q^{\alpha} - q^{-\alpha},$$

or

$$q^{\alpha}(q^{\triangle} - 1) \stackrel{?}{>} -q^{-\alpha} + q^{-\alpha-\triangle} = q^{-\alpha-\triangle}(-q^{\triangle} + 1),$$

or which is obvious, because

$$\triangle < 0, q > 1 \Rightarrow q^{\triangle} - 10, 1 - q^{\triangle} < 0. \quad \blacksquare$$

The Fundamental Inequality Between Quantum Numbers

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Abstract

We compare the quantum sums of numbers a, b, c, d with $a + b = c + d$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbf{R}, q \neq 1,$$

be the 2^{nd} quantization of x. Let a, b, c, d be given, with

$$a + b = c + d. \tag{1}$$

Suppose

$$b > c, d,\ b > 0. \tag{2}$$

Theorem 3. Under the assumption (2),

$$[a]_q^{\sim} + [b]_q^{\sim} > [c]_q^{\sim} + [d]_q^{\sim},\quad q \neq 1. \tag{4}$$

Proof. Set

$$\triangle_1 = b - c > 0,\quad \triangle_2 = b - d = 0. \tag{5}$$

Then

$$a = c + d - b = b - \triangle_1 - \triangle_2, \tag{6}$$

and our inequality (4), after being multiplied by $q - q^{-1} > 0$, becomes:

$$q^{b-\triangle_1-\triangle_2} - q^{-b+\triangle_1+\triangle_2} + q^b - q^{-b} \stackrel{?}{>} q^{b-\triangle_1} - q^{-\triangle_1-b} + q^{b-\triangle_2} - q^{-\triangle_2-b}. \tag{7}$$

We are using the fact that

$$[x]_q^{\sim} = [x]_{q^{-1}}^{\sim},\quad \forall x, \tag{8}$$

so that we can take

$$q > 1. \tag{9}$$

Denote or

$$q^b = B > 1, q^{\triangle_1} = x > 1,\ q^{\triangle_2} = y > 1. \tag{10}$$

Then (7) becomes:

$$\frac{B}{xy} - \frac{xy}{B} + B - \frac{1}{B} \stackrel{?}{>} \frac{B}{x} - \frac{x}{B} + \frac{B}{y} - \frac{y}{B}, \tag{11}$$

or

$$B\left(\frac{1}{xy} + 1 - \frac{1}{x} - \frac{1}{y}\right) \stackrel{?}{>} \frac{1}{B}(xy + 1 - x - y),$$

or

$$B\left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{y}\right) > \frac{1}{B}(1-x)(1-y), \text{or}$$

or

$$B\frac{x-1}{x}\frac{y-1}{y} \stackrel{?}{>} \frac{1}{B}(x-1)(y-1),$$

or

$$\frac{B}{xy} \stackrel{?}{>} \frac{1}{B},$$

or

$$B^2xy \stackrel{?}{>} 1, \tag{12}$$

which is true because

$$B > 1,\ x > 1,\ y > 1. \quad \blacksquare$$

The Difference Between $O(x)$ And x

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Abstract

We show that $O(x) - x = 0(1)$.

Let

$$O(x) = \sum_{p \leq x} logp, \tag{1}$$

be the Chebyshev's O-function, where p stands for a prime.

Theorem 2. For $x > 0$,

$$O(x) = x + 0(1). \tag{3}$$

Proof. We have to show that the relations

$$O(p_m) = \sum_{i=1}^{m} logp_i, \quad O(p_{m+1}) = O(p_m) + logp_{m+1} \tag{4}$$

are consistent with (3). We have:

The Change Of The Quantum Numbers When The Underlying Classical One Changes

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Abstract

We determine how quantum numbers vary.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0, \pm 1,$$

be the 2^{nd} quantization of x, so that

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

and we can restrict ourselves to $x \geq 0$ only.

Lemma 1. For $x \geq 0$,

$$\frac{d[x]_q^{\sim}}{dx} > 0. \tag{2}$$

Proof. Since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can restrict ourselves to

$$q > 1 \tag{3}$$

without loss of generality.

Hence

$$\frac{\partial [x]_q^{\sim}}{\partial x} = \frac{\partial}{\partial x}\left(\frac{q^x - q^{-x}}{q - q^{-1}}\right) = \frac{h}{q - q^{-1}}(q^x + q^{-x}) > 0, \tag{4}$$

where

$$h = logq > 0,$$

and

$$q - q^{-1} > 0. \quad \blacksquare$$

Lemma 5. For $x \geq 1$,

$$[x]_q^{\sim} - x \tag{6}$$

grows with x.

Proof. Multiplying by $(q - q^{-1})$, we get from (6):

$$\frac{\partial}{\partial x}\Big[(q^x - q^{-x}) - x(q - q^{-1})] =$$
$$= h(q^x + q^{-x}) - (q - q^{-1}) \stackrel{?}{>} 0 \Leftrightarrow$$
$$q^x + q^{-x} \stackrel{?}{>} \frac{q - q^{-1}}{h} \Leftrightarrow$$
$$2\sum_{n\geq 0} \frac{(hx)^{2n}}{(2n)!} \stackrel{?}{>} 2\sum_{n\geq 0} \frac{h^{2n+1}}{h(2n+1)!},$$

which is obvious, because

$$x \geq 1 \Rightarrow hx \geq h,$$

and

$$\frac{1}{(2n)!} > \frac{1}{(2n+1)!}. \quad \blacksquare$$

Lemma 7. For $x \geq 1$,

$$(q - q^{-1})([x]_q^{\sim} - x) \tag{8}$$

grows with q.

Proof. We have:

$$\frac{\partial}{\partial h}\Big\{(q - q^{-1})[[x]_q^{\sim} - x]\Big\} = \frac{\partial}{\partial h}\Big((e^{hx} - e^{-hx}) - x(e^h - e^{-h})\Big) =$$
$$= x\Big\{q^x + q^{-x} - (q + q^{-1})\Big\} > 0,$$

because, as is known,

$$q + q^{-1} > Q + Q^{-1} \tag{9}$$

for $q > Q$. $\blacksquare$

Lemma 10.

$$\lim_{x\to\infty}([x]_q^{\sim} - x) = \infty, \tag{11}$$
$$\lim_{x\to\infty} \frac{[x]_q^{\sim}}{x} = \infty. \tag{12}$$

Proof. (11) is obvious, because from the Proof of Lemma 5,

$$\frac{\partial}{\partial x}([x]_q^{\sim} - x) \geq h(x-1). \tag{13}$$

(12) is also obvious:

$$\frac{[x]_q^{\sim}}{x} = \frac{q^x - q^{-x}}{(q-q^{-1})x} - \frac{1}{(q-q^{-1})}\frac{q^x}{x} \to \infty,$$

because $q > 1$. ■

1-Set Of The Parameterized Circle

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Abstract

The circle parameterization, $(t^2-1)^2+(2t)^2=(t^2+1)^2$, is quantized and then the first nontrivial order in $h=logq$ is extracted.

What is the quantum circle? The answer is still unknown. But various *parameterizations* of the circle *has been* quantized.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\quad x\in\mathbf{R},\quad q\neq 0,\pm 1,$$

be the 2^{nd} quantization of x.

As the quantum version of the parameterization

$$(t^2-1)^2+(2t)^2=(t^2+1)^2, \tag{1}$$

we take

$$([t^2+1]_q^{\sim})^2=([t^2-1]_q^{\sim})^2+([2]_q^{\sim})^2[t^2]_{q^2}^{\sim}. \tag{2}$$

Indeed, using the formula

$$([x+a]_q^{\sim})^2-([x]_q^{\sim})^2=[a]_q^{\sim}[2x+a]_q^{\sim}, \tag{3}$$

we get:

$$([t^2+1]_q^{\sim})^2-([t^2-1]_q^{\sim})^2=[2]_q^{\sim}[2t^2]_q^{\sim}=[2]_q^{\sim}[2]_q^{\sim}[t^2]_{q^2}^{\sim},$$

which is (2).

Now, with

$$h-logq,$$

$$[x]_q^{\sim}=x+\frac{h^2}{6}(x^3-x)+0(h^4). \tag{4}$$

Indeed,

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}}=\frac{e^{hx}-e^{-hx}}{e^h-e^{-h}}=\frac{2(hx+\frac{h^3x^3}{6})}{2(h+\frac{h^3}{6})}=$$

$$=(x+\frac{h^2}{6}x^3)(1-\frac{h^2}{6})=x+\frac{h^2}{6}(x^3-x).$$

We now apply the equality (4) to (2). We get

$$\{t^2+1+\frac{h^2}{6}[(t^2+1)^3-(t^2+1)]\}^2=\{t^2-1+\frac{h^2}{6}[(t^2-1)^3-(t^2-1)]\}^2+$$
$$+[2+\frac{h^2}{6}(2^3-2)]\{2t^2+\frac{h^2}{6}[(2t^2)^3-2t^2]\},$$

or

$$(t^2+1)^2+\frac{h^2}{3}[(t^2+1)^4-(t^2+1)^2]=(t^2-1)^2+\frac{h^2}{3}[(t^2-1)^4-(t^2-1)^2]+$$
$$+4t^2+\frac{h^2}{3}[(8t^6-2t^2)+6t^2].$$

Picking out $\frac{h^2}{3}$ - terms, we get:

$$[(t^2+1)^4-(t^2+1)^2]=[(t^2-1)^4-(t^2-1)^2]+8t^6+4t^2,$$

or

$$(t^2+1)^4=(t^2-1)^4+8t^2(t^4+1). \tag{5}$$

This is our candidate for the parameterization of the unknown 1-jet of the circle.

Notice that for $\tau=t^2$, (5) becomes

$$(\tau+1)^4=(\tau-1)^4+8\tau(\tau^2+1). \tag{6}$$

A Quantum Cubic Representation For 16

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Abstract

$16 = 2^2 \cdot 4$ is given a quantum cubic representation.

Theorem 1. For $n \in \mathbf{Z}_{\geq 3}$, we have

$$([2]_q^{\sim})^2[4]_{q^n}^{\sim} = [3n+3]_q^{\sim} + [3n+1]_q^{\sim} + [n+3]_q^{\sim} + [n+1]_q^{\sim} - \\ -([3n-3]_q^{\sim} + [3n-1]_q^{\sim} + [n-3]_q + [n-1]_q^{\sim}), \tag{1}$$

where

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}$$

is the symmetric 2^{nd} quantization x.

Proof. We are going to use the following useful formula

$$[m+a]_q^{\sim} - [m-a]_q^{\sim} = [2]_{q^m}^{\sim}[a]_q^{\sim}. \tag{2}$$

With it, the RHS of (1) turns into:

$$[2]_{q^{3n}}^{\sim}([3]_q^{\sim} + 1) + [2]_{q^n}^{\sim}([3]_q^{\sim} + 1),$$

and because

$$[3]_q^{\sim} + 1 = ([2]_q^{\sim})^2,$$

(1) reduces to

$$[4]_{q^n}^{\sim} \stackrel{?}{=} [2]_{q^{3n}}^{\sim} + [2]_{q^n}^{\sim},$$

or

$$[4]_q^{\sim} \stackrel{?}{=} [2]_{q^3}^{\sim} + [2]_q^{\sim},$$

which is obvious, because

$$[2]_{q^3}^{\sim} + [2]_q = (q^3 + q^{-3}) + (q + q^{-1}) = [4]_q^{\sim} \quad \blacksquare$$

Remarks On The Cramers Conjecture

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Abstract

We show that $p_{m+1} - p_m - logp_{m+1} = 0(1)$, p_m being the m^{th} prime.

The Cramers Conjecture,

$$\frac{p_{m+1} - p_m}{log^2 m} \ \rightarrow \ 1, \tag{1}$$

is too cautious. Another famous result,

$$\lim \ sup \frac{p_{m+1} - p_m}{logm} = \infty, \tag{2}$$

is shown below to be false. The means are elementary, modulo standard results in the famous Dusart thesis, the Bible of the prime-number theory.

We show below that, in fact, as $m \rightarrow \infty$,

$$p_{m+1} - p_m - logp_{m+1} = 0(1). \tag{3}$$

In particular,

$$\lim_{rn \rightarrow \infty} \frac{p_{pn+1} - p_m}{logm} = 1. \tag{4}$$

Let's prove (3), using

$$p_m = m\{logm + (loglogm - 1) + \sum_{i \geq 1} \frac{P_i}{log^i m}\}, \tag{5}$$

where P_i are polynomials in $w = loglogm$ of degree i. We also set

$$P_0 = loglogm - 1, \tag{6}$$

so that

$$degP_i = i + \delta_0^i. \tag{7}$$

Thus,

$$\begin{aligned} &p_{m+1} - p_m - logp_{m+1} = \\ &= m\{f(m+1) - f(m)\} + \{f(m+1) - logp_{m+1}\} \end{aligned} \tag{8}$$

We have, remembering that

$$log(m+1) - logm = \frac{1}{m} + ...; \tag{9}$$

$$loglog(m+1) - loglogm = \frac{1}{mlogm} + ...; \tag{10}$$

$$m\{f(m+1) - f(m)\} =$$
$$= m\{\frac{1}{m} + \frac{1}{mlogm} + ...\} = 1 + 0(1), \tag{11}$$

$$f(m) - logp_m = logm + loglogm - 1 + ... -$$
$$-\{logm + log[logm + P_0 + ...]\} =$$
$$= loglogm - 1 - log\{logm[1 + \frac{P_0}{logm} + ...]\} =$$
$$= loglogm - 1 - [loglogm + \frac{P_0}{logm} + ...] =$$
$$= -1 + 0(1). \tag{12}$$

Combining (11) with (12), we arrive at

$$p_{m+1} - p_m - logp_{m+1} = 0(1),$$

which is exactly (3).

On The Sums Of Two Consequitive Quantum Squares

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Abstract

Trivially, $n^2+(n+1)^2=2n^2+2n+1$. Nontrivially, this formula can be quantized.

We shall prove that

$$[n]_q^{\sim}[n]_{q^{2n+1}}^{\sim}+[n+1]_q^{\sim}[n+1]_{q^{2n+1}}^{\sim}=[2n^2+2n+1]_q^{\sim}, \tag{1}$$

where

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}}.$$

The LHS of (1), times $3_1 3_{2n+1}$, where $3_k-q^k-q^{-k}$, is:

$$\begin{aligned}
&\{(q^n-q^{-n})(q^{(2n+1)n}-q^{-n(2n+1)})+(q^{n+1}-q^{-n-1})(q^{(2n+1)(n+1)}-q^{-(2n+1)(n+1)})\}=\\
&=\{q^{2(n+1)n}+q^{-2(n+1)n}-q^{2n^2}-q^{-2n^2}+\\
&+q^{2(n+1)^2}+q^{-2(n+1)^2}-q^{2n(n+1)}-q^{-2(1+(n+1))}\}=\\
&=q^{2n^2}(q^{4n+2}-1)+q^{-2n^2}(q^{-4n-2}-1)=\\
&=q^{2n^2}q^{2n+1}(q^{2n+1}-q^{-2n+1})-q^{-2n^2}q^{-2n-1}(q^{2n+1}-q^{-2n-1})=\\
&=3_{2n+1}(q^{2n^2+2n+1}-q^{-2n^2-2n-1})-3_{2n+1}3_1[2n^2+2n+1]_q^{\sim},
\end{aligned}$$

which is, essentially the RHS of (1).

Quantum Powers Of 2

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Abstract

We derive a quantum decomposition of 2^n.

In the second quantization

$$x \to [x]_q^\sim = \frac{q^x - q^{-x}}{q - q^{-1}},$$
$$[2]_q^\sim = q + q^{-1}.$$

Theorem 1. For $n \in \mathbf{Z}_{\geq 1}$, we have:

$$([2]_q^\sim)^n = \sum_{k=0}^{\lfloor n/2 \rfloor} \left\{ \binom{n}{k} - \binom{n}{k-1} \right\} [n+1-2k]_q^\sim, \tag{2}$$

where

$$[k]!_q^\sim = [1]_q^\sim ... [k]_q^\sim, k \in \mathbf{Z}_{\geq 1}; \quad [0]!_q^\sim = 1.$$

Proof. We have:

$$([2]_q^\sim)^n = (q + q^{-1})^n = \sum_{k=0}^{n} \binom{n}{k} q^{n-2k}. \tag{3}$$

Denote

$$\left(\sum_i a_i q^i \right)_+ = \sum_{i \geq 0} a_i q^i.$$

It's easy to check that

$$q^{-i} + q^i = [i+1]_q^\sim - [i+1]_q^\sim, \quad i \in \mathbf{Z}^*, \tag{4}$$
$$q^0 = [1]. \tag{4a}$$

Thus, (3) returns:

$$([2]_q^\sim)^n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} ([k+1]_q^\sim - [k-1]_q^\sim), \quad n \text{ is odd.} \tag{5a}$$

For n even, $n = 2m$,

$$([2]_q^{\sim})^{2m} = \sum_{k=0}^{m-1} \binom{2m}{k} \left([k+1]_q^{\sim} - [k-1]_q^{\sim} \right) + \left\{ \binom{2m}{m} \right\} [1]_q^{\sim}, \tag{5b}$$

and

$$[1]_q^{\sim} = 1. \tag{6}$$

These are *not* formulae of the type (2). The latter comes from the following obvious observation.

Lemma 7.

Let $f(q) = \sum a_i q^i = f(q^{-1}) \leftrightarrow a_i = -a_i$. Then

$$\sum_{i=0}^{n} a_i q^{n-2i} = \sum_{i=0}^{\lfloor n/2 \rfloor} (a_i - a_{i-1})[n+1-2i]_q^{\sim}. \tag{8}$$

Proof. It's enough to remember that

$$[n] = \sum_{i=0}^{n-1} q^{n-1-2i}, n \in \mathbf{Z}_{\geq 1} \tag{9}$$

and subtract from the sum $\sum_{i=0}^{n} q^{n-2i} a_i$ one $[n+1-2i]_q^{\sim}$ term at a time. We shall also use $a_{-1} = 0$. ■

Applying The Lemma to the sum (3) we arrive at formula (6).

Notice that for a few small n, formula (2) returns:

$$\left([2]_q^{\sim} \right)^1 = [2]_q^{\sim},$$

$$\left([2]_q^{\sim} \right)^2 = [3]_q^{\sim} + 1, \tag{10a}$$

$$\left([2]_q^{\sim} \right)^3 = [4]_q^{\sim} + 2[2]_q^{\sim}, \tag{10b}$$

$$\left([2]_q^{\sim} \right)^4 = [5]_q^{\sim} + 3[3]_q^{\sim} + 2[1]_q^{\sim}. \tag{10c}$$

(0.1)

Quantum Integers In An Odd Base

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Abstract

For $n = 2$, odd base change still results in an integer. We investigate what happens for $n > 2$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0,\ 1,$$

be the 2^{nd} quantization of x, so that

$$[2]_q = q + q^{-1}. \tag{1}$$
$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim}, \tag{2}$$
$$[3]_q = q^2 + 1 + q^{-2}, ... \tag{3}$$

For n odd, $n = 2s - 1$,

$$\frac{[2]_{q^n}^{\sim}}{[2]_q^{\sim}}$$

is an integer:

$$\frac{[2]_{q^{2s-1}}^{\sim}}{[2]_q^{\sim}} = \frac{q^{2s-1} + q^{1-2s}}{q + q^{-1}} = \sum_{j=0}^{2s} q^{2s-2j}(-1)^j, \tag{4}$$

because

$$(q + q^{-1}) \sum_{js=0}^{2s} q^{2s-2j}(-1)^j = \sum_{js=0}^{2j} q^{2s+1-2j}(-1)^j + \sum_{js=0}^{2j} q^{2s-1-2j}(-1)^j =$$
$$= q^{2s+1} + \sum_{js=0}^{2s-1} q^{2s-1-2j}(-1)^{j+1} + \sum_{j=0}^{2s} q^{2s-1-2j}(-1)^j =$$
$$= q^{2s+1} + q^{2s-1-2j}(-1)^j \Big|_{j=2s} = q^{2s+1} + q^{-2s-1}. \tag{5}$$

One naturally supposes that

$$\frac{[m]_{q^{2s-1}}}{[m]_q}$$

is an integer for *all* $m \geq 1,\ s \geq 1$. It turns out to be *not* true.

We take $m=3, s=2$, so we are looking at

$$\frac{[3]^{\sim}_{q^3}}{[3]^{\sim}_{q}} = \frac{q^6+1+q^{-6}}{q^2+1+q^{-2}} = \frac{Q^3+1+Q^{-3}}{Q+1+Q^{-1}} \tag{6}$$

for $Q=q^2$. Now (6) is Q^{-2} times

$$\frac{Q^6+Q^3+1}{Q^2+Q+1},$$

and, as obvious, the polynomial Q^6+Q^3+1 is *not* divisible by the polynomial Q^3+Q+1.

On The Sum Of Inverse Quantum Factorials

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Abstract

The classical formula $\sum_{k=1}^{n} \frac{k}{[k+1]!} = 1 - \frac{1}{(n+1)!}$, is quantized.

The classical sum [PKi 1974], problem 8.2,

$$\sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!} \tag{1}$$

can be quantized thusly:

$$\sum_{k=1}^{n} \frac{q^{\binom{k}{2}-1}[k]_q^{\sim}}{[k+1]!_q^{\sim}} = 1 - \frac{q^{\binom{n+1}{2}}}{[n+1]!_q^{\sim}}, \tag{2}$$

where

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \tag{3}$$

$$[k]!_q^{\sim} = [1]_q^{\sim}...[k]_q^{\sim},\ k \in \mathbf{Z}_{\geq 1};\quad [0]!_q^{\sim} = 1. \tag{4}$$

For $n = 1$, formula (2) yields:

$$\frac{q^{-1}}{[2]_q^{\sim}} = 1 - \frac{q}{[2]_q^{\sim}}, \tag{5}$$

which is true because

$$[2]_q^{\sim} = \frac{q^2 - q^{-2}}{q - q^{-1}} = q + q^{-1}. \tag{6}$$

We use induction on n to prove (2). The inductive step amounts to the relation

$$\frac{q^{\binom{n+1}{2}-1}[n+1]_q^{\sim}}{[n+2]_q^{\sim}} \stackrel{?}{=} 1 - \frac{q^{\binom{n+2}{2}}}{[n+2]_q^{\sim}} - \left(1 - \frac{q^{\binom{n+1}{2}}}{[n+1]!_q^{\sim}}\right) =$$
$$= -\frac{q^{\binom{n+2}{2}}}{[n+2]!_q^{\sim}} + \frac{q^{\binom{n+1}{2}}}{[n+1]!_q^{\sim}},$$

or

$$q^{\binom{n+2}{2}-1}[n+1]_q^{\sim} \stackrel{?}{=} -q^{\binom{n+2}{2}} + q^{\binom{n+1}{2}}[n+2]_q^{\sim}. \tag{7}$$

Dividing by $q^{\binom{n+1}{2}}$, we arrive at

$$[n+2]_q^{\sim} \stackrel{?}{=} q^{-1}[n+1]_q^{\sim} + q^{n+1}, \tag{8}$$

which is obvious, because in general,

$$[a+b]_q^{\sim} = q^{-b}[a]_q^{\sim} + q^a[b]_q^{\sim},\quad \forall a, b. \tag{9}$$

References

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Quantum Divisibility In Exponential Families

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Abstract

In integers a, b are such that $a \equiv (mod\, p), b \equiv 1(mod\, p)$, then $a^{2n+1} + b^{2n} \equiv 0(mod\, p)$ for any nonnegative integer n. We quantize this observation.

Let $p \in \mathbf{Z}_{\geq 1}$ be given (and not necessarily a prime). If $a \equiv= 1(mod\ p), b \equiv 1(mod\ p)$ then

$$a^{2n+1} + b^{2n} \equiv -1 + 1 = 0(mod\ p).$$

Does this fact have a quantum analog?
Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0, \pm 1,$$

be the second quantization of x.

Theorem 1.

$$\left([a]_{q^2}^{\sim}\right)^{2n+1} + \left([b]_{q^2}^{\sim}\right)^{2n} \equiv 0\ (mod\ [p]_q^{\sim}). \tag{2}$$

Proof Follows from the next

Lemma 3. Suppose $k, a \in \mathbf{Z}_{\geq 0}$. Then

$$[kp + a]_{q^2}^{\sim} \equiv [a]_{q^2}^{\sim} \quad (mod\ [p]_q^{\sim}). \tag{4}$$

Proof. The LHS of (4) is:

$$q^{2p} \equiv 1 \Rightarrow q^p \equiv \pm 1.$$

since

$$[p]_q^{\sim} = \frac{q^p - q^{-p}}{q - q^{-1}},$$

$0\ (mod [p]_q)$ means

$$q^p \equiv q^{-p}, \tag{5}$$

or

$$\frac{q^{2(kp+a)}-q^{-2(kp+a)}}{q^2-q^{-2}} \equiv q^{2kp}\frac{q^{2a}-q^{-2a}}{q^2-q^{-2}} = [a]^{\sim}_{q^2}, \tag{6}$$

which is the RHS of (4). ■

Notice that for $a = \pm 1$,

$$[a]^{\sim}_{q^2} = a,$$

so our Theorem follows. ■

Remark 7. Implicit in our Proofs is the useful identity

$$[x]^{\sim}_{q^2} + [x+1]^{\sim}_{q^2} = [2x+1]^{\sim}_{q}. \tag{8}$$

Quantum Circle

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Abstract

The circle $a^2+b^2=1$ is quantized.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\quad x\in\mathbf{R},\quad q\neq 0,\pm 1,$$

be the 2^{nd} quantization of x.

We parameterize the circle

$$a^2+b^2=1 \tag{1}$$

by the rule

$$\left(\frac{2x}{x^2+1}\right)^2+\left(\frac{x^2-1}{x^2+1}\right)^2=1, \tag{2}$$

and we quantize (2) as

$$[x^2]_q^{\sim}+\left(\left[\frac{x^2-1}{2}\right]_q^{\sim}\right)^2=\left(\left[\frac{x^2+1}{2}\right]_q^{\sim}\right)^2. \tag{3}$$

Indeed, it's easy to verity that

$$\left(\left[t+1\right]_q^{\sim}\right)^2-\left(\left[t\right]_q^{\sim}\right)^2=\left[2t+1\right]_q^{\sim},\quad \forall t. \tag{4}$$

Therefore,

$$\left(\left[\frac{x^2+1}{2}\right]_q^{\sim}\right)^2-\left(\left[\frac{x^2-1}{2}\right]_q^{\sim}\right)^2=$$
$$=\left[2\cdot\frac{x^2-1}{2}+1\right]_q^{\sim}=[x^2]_q^{\sim},$$

which is (3).

Quantum Binomials In The Second Quantization

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Abstract

The classical formula $(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$, n a positive integer, is quantized.

We have:

$$1 + x = \sum_{k=0}^{n} \begin{bmatrix} 1 \\ k \end{bmatrix}_q^{\sim} x^k, \tag{1a}$$

$$(1+qx)(1+q^{-1}x) = 1 + [2]_q^{\sim} x + x^2, \tag{1b}$$

where

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \tag{2}$$

so that

$$[2] = q + q^{-1}, \tag{3a}$$

$$[3] = q^2 + 1 + q^{-2}, ... \tag{3b}$$

We thus expect for an m-product (starting with $(1+q^m x)$) to have:

$$\prod_{i=0}^{m} (1 + q^{m-2i}x) = \sum_{k=0}^{n+1} \begin{bmatrix} m+1 \\ k \end{bmatrix}_q^{\sim} x^k, \tag{4}$$

where

$$\begin{bmatrix} m \\ k \end{bmatrix}_q^{\sim} = \frac{[m]!_q^{\sim}}{[k]!_q^{\sim}[m-k]!_q^{\sim}}, \tag{5}$$

and

$$[s]!_q^{\sim} = [1]_q^{\sim} ... [s]_q^{\sim}, \quad s \in \mathbf{Z}_{\geq 1}; \quad [0]!_q^{\sim} = 1. \tag{6}$$

We prove formula (4) by induction on m. Assuming it's true for the $(m-2)$-product:

$$\prod_{i=0}^{m-2} (1 + q^{m-2-2i}) = \sum_{k=0}^{m-1} \begin{bmatrix} m-1 \\ k \end{bmatrix}_q^{\sim} x^k, \tag{7}$$

we get:

$$\prod_{i=0}^{m}(1+q^{m-2i}x)=(1+q^m x)(1+q^{-m}x)\prod_{i=0}^{m-2}(1+q^{m-2-2i}x)=$$
$$=(1+[2]^{\sim}_{q^m}x+x^2)\sum_{k=0}^{m-1}\begin{bmatrix}m-1\\k\end{bmatrix}^{\sim}_q x^k \stackrel{?}{=}\sum_{k=0}^{m+1}\begin{bmatrix}m+1\\k\end{bmatrix}x^k,$$

which is equivalent to

$$\begin{bmatrix}m+1\\k\end{bmatrix}^{\sim}_q \stackrel{?}{=}\begin{bmatrix}m-1\\k\end{bmatrix}^{\sim}_q+[2]_{q^m}\begin{bmatrix}m-1\\k-1\end{bmatrix}^{\sim}_q+\begin{bmatrix}m-1\\k-2\end{bmatrix}^{\sim}_q, \tag{8}$$

which is immediate, based on the triple of formulae:

$$\begin{bmatrix}m+1\\k\end{bmatrix}^{\sim}_1=q^k\begin{bmatrix}m\\k\end{bmatrix}^{\sim}_q+q^{k-m-1}\begin{bmatrix}m\\k-1\end{bmatrix}^{\sim}_q, \tag{9a}$$
$$\begin{bmatrix}m\\k\end{bmatrix}=q^{-k}\begin{bmatrix}m-1\\k\end{bmatrix}^{\sim}_q+q^{m-k}\begin{bmatrix}m-1\\k-1\end{bmatrix}^{\sim}_q, \tag{9b}$$
$$\begin{bmatrix}m-1\\k-1\end{bmatrix}^{\sim}_q=q^{1-k}\begin{bmatrix}m-2\\k-1\end{bmatrix}+q^{m+2-k}\begin{bmatrix}m-2\\k-2\end{bmatrix}^{\sim}_q, \tag{9c}$$

and the formula

$$\begin{bmatrix}m+1\\k\end{bmatrix}^{\sim}_q=q^k\begin{bmatrix}m\\k\end{bmatrix}^{\sim}_q+q^{k-m-1}\begin{bmatrix}m\\k-1\end{bmatrix}^{\sim}_q. \tag{10}$$

Substituting formulae (9) into (10), we recover the desired (8). Above we used the easily verifiable formulae

$$\begin{bmatrix}m+1\\k\end{bmatrix}^{\sim}_q=q^a\begin{bmatrix}m\\k\end{bmatrix}^{\sim}_q+q^b\begin{bmatrix}m\\k-1\end{bmatrix}^{\sim}_q, \tag{11}$$

where

$$\binom{a}{b}=\binom{k}{k-m-1}\text{ or }\binom{-k}{m+1-k}, \tag{12}$$

which results from the standard identity

$$[x+y]^{\sim}_q=q^y[x]^{\sim}_q+q^{-x}[y]^{\sim}_q=q^{-y}[x]^{\sim}_q+q^x[y]^{\sim}_q, \tag{13}$$

upon dividing (11) by $[m]!^{\sim}_q/[k]!^{\sim}_q[m+1-k]!^{\sim}_q$,

Quantum Analog Of The Little Fermat Theorem

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Abstract

The LFT is quantized.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, 1,$$

be the send quantization of x.

The LTF says that if p is a prime and $a \in \mathbf{Z}$ is coprime to p, then

$$a^{p-1} \equiv 1 \pmod{p}.$$

Theorem 1. Set

$$< a^n >_q^{\sim} = \prod_{k=1}^{n} [a]_{q^k}^{\sim}, \quad n \in \mathbf{Z}_{\geq 1}. \tag{2}$$

If $(a, p) = 1$, then

$$< a^{p-1} >_{q^2}^{\sim} \equiv 1 \pmod{[p]_q^{\sim}}. \tag{3}$$

Proof. $mod[n]_q^{\sim}$ means:

$$q^{2n} \equiv 1, \tag{4}$$

because

$$[n]_q^{\sim} = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

So,

$$q^{2p} \equiv 1. \tag{5}$$

Now,

$$< a^{p-1} >_{q^2}^{\sim} = \prod_{k=1}^{p-1} [a]_{q^{2k}}^{\sim} \Big/ \prod_{i=1}^{p-1} (q^{2i} - q^{-2i}) =$$

$$= \prod_{k=1}^{p-1} (q^{2ka} - q^{2ka}) \Big/ \prod_{i=1}^{p-1} (q^{2i} - q^{-2i}), \tag{6}$$

and the exponents

$$2a, 4a, ..., 2(p-1)a,$$

are, modulo $2p$, just $2, 4, ..., 2(p-1)$, because

$$2ka - 2ea = 2(k-e)a \equiv 0 \pmod{2p}$$

iff $k \equiv e \pmod{p}$. Thus, the ratio (6) is 1. This proves (3.) ■

On The Quantization Of The Sum $\sum \frac{1}{k(k+1)}$

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Abstract

The classical formula $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$ is quantized.

Theorem 1. For $n \in \mathbf{Z}_{\geq 1}$,

$$\sum_{k=1}^{n} \frac{1}{[k]_q^{\sim}[k+1]_q^{\sim}} = q - \frac{q^{n+1}}{[n+1]_q^{\sim}}, \tag{1}$$

where

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}.$$

is the second quantization of x.

Proof. The formula follows at once, if we take an obvious formula

$$[n+1]_q^{\sim} = q[n]_q^{\sim} + q^{-n}, \tag{2}$$

and divide it by $\frac{q^{-n}}{[n]_q^{\sim}[n+1]_q^{\sim}}$, resulting in the telescopic formula

$$\frac{1}{[n]_q^{\sim}[n+1]_q^{\sim}} = \frac{q^n}{[n]_q^{\sim}} - \frac{q^{n+1}}{[n+1]_q^{\sim}}. \tag{3}$$

Summing it up on n, we get (1), because

$$\frac{q^1}{[1]_q^{\sim}} = q. \quad \blacksquare$$

Quantization Of The Formula $(x+a)(y+a)-xy=a(x+y)+q^2$

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Abstract

The formula in the title is quantized.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\quad x\in\mathbf{R},\quad q\neq 1,$$

be the second quantization of x.

Theorem 1. For $x,y,a\in\mathbf{R}$,

Power Function In The Quantum Domain

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Abstract

We propose how to quantize the function $z \mapsto z^s$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0, \pm 1,$$

be the 2^{nd} quantization of x.

Quantizing the classical formula

$$\sum_{k=1}^{n} (2k-1)^3 = \binom{2n}{2}^2,$$

I found that n^3 has to be replaced by

$$[n]_{q^2}^{\sim}[n^2]_q^{\sim}.$$

This suggests that, in general,

$$x^{s+1} \mapsto [x]_{q^s}^{\sim}[x^s]_q^{\sim} \tag{1}$$

is the correct approach, and to support this contention we establish

Theorem 2. For $n \in \mathbf{Z}_{\geq 1}$,

$$[n]_{q^s}^{\sim}[n^s]_q^{\sim} = \sum_{i=0}^{n-1} [n^s + s(n-1-2i)]_q^{\sim}. \tag{3}$$

Proof. Setting

$$\eta_c = q^c - q^{-c},$$

for the RHS of (3) we get:

$$\frac{1}{\eta_1} \sum_{i=0}^{n-1} \left\{ q^{n^s + s(n-1)} q^{-2si} - q^{-[n^s + s(n-1)]} q^{2si} \right\}$$
$$= \frac{1}{\eta_1} \left\{ q^{n^s + s(n-1)} [n]_{q^{-2s}} - q^{-[n^s + s(n-1)]} [n]_{q^{2s}} \right\},$$

where

$$[x]_q = \frac{1-q^x}{1-q}$$

is the first (standard) quantization. So, the RHS of (4) is:

$$\begin{aligned}
&\frac{1}{\eta_1}\left\{q^{n^s+s(n-1)}\frac{1-q^{-2}\eta_s}{q^{-s}\eta_s} - q^{-n^s-s(n-1)}\frac{q^{2sn-1}}{q^s\eta_s}\right\} = \\
&= \frac{1}{\eta_1\eta_s}\left\{q^{n^s+sn}q^{-sn}\eta_{ns} - q^{-n^s-sn}q^{sn}\eta_{sn}\right\} \\
&= \frac{\eta_{sn}}{\eta_1\eta_s}\left\{q^{n^s} - q^{-n^s}\right\} = [n^s]_q^{\sim}[n]_{q^s}^{\sim},
\end{aligned}$$

which is the LHS of 3. ■

Ordering For Quantum Numbers

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Abstract

We order quantum numbers and compare them to the corresponding classical ones.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}$$

be the (2^{nd}) quantum version of $x, x \in \mathbf{R}$ or $\mathbf{C}$. In the latter case we fix $h = log(q)$ so that

$$q^x = e^{hx}.$$

Thereafter x is real. For $q \to 1, [x]_q^{\sim} \to x$. Let $q \neq 1$ in what follows. As real numbers, $[x]_q^{\sim}$'s acquire the natural order. Because

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

it's enough to consider positive x only.

Theorem 1.

$$[x]_q^{\sim} > x, \ q \neq 1, x > 1, \tag{2}$$
$$[x]_q^{\sim} < x, \ 0 < x < 1. \tag{3}$$

Proof. Since

$$[x]_q^{\sim} = [x]_{q^{-1}}^{\sim}, \tag{4}$$

we can take

$$q > 1 \Leftrightarrow h > 0, \tag{5}$$

without loss of generality.

Consider

$$f(x,q) = (q - q^{-1})([x]_q^{\sim} - x) = q^x - q^{-x} - x(q - q^{-1}).$$

Then

$$\frac{\partial f(x,q)}{\partial h} = x(q^x + q^{-x}) - x(q + q^{-1}) =$$
$$= x[q^x + q^{-x} - (q^{-1} + q)]. \tag{6}$$

Lemma 7.

$$\frac{\partial f}{\partial h} > 0, x > 1;\ \frac{\partial f}{\partial h} < 0, x < 1. \tag{8}$$

Proof. Since $x > 0$, we have to show that

$$q^x + q^{-x} \stackrel{?}{>} q + q^{-1}, x > 1; q^x + q^{-x} < q + q^{-1}, < x < 1, \tag{9}$$

or

$$q^x - q \stackrel{?}{>} q^{-1}q^{-x} = \frac{1}{q} - \frac{1}{q^x} = \frac{q^x - q}{qq^x}, x > 1, \tag{10}$$

$$q^x - q < \frac{q^x - 1}{qq^x},\ 0 < x < 1. \tag{11}$$

Since $q > 1$, $q^x - q > 0$ for $x > 1$, so (10) is obvious for $x < 1$, (11) returns

$$1 \stackrel{?}{>} \frac{1}{qq^x}, \tag{12}$$

which is obvious as well. ■
Since $f(x, 1) = x$, Lemma (7) implies that

$$f(x, h) > f(x, 1), \quad h > 0. \tag{13}$$

This proves (2). Similarly, for $0 < x < 1$,

$$f(x, h) < f(x, 1), \quad h > 0, \tag{14}$$

which proves (3). ■

On The Two Related Sums Of Quantum Integers

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Abstract

Classically, $\sum_{k=1}^{n} k + n^2 = \sum_{k=n+1}^{2n} k$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0, \pm 1,$$

be the 2^{nd} quantization of x, so that

$$[2]_q^{\sim} = q + q^{-1}.$$

We first notice that

$$\sum_{k=1}^{n} [k]_q^{\sim} \underset{=}{\frac{[2]_{q^k}^{\sim}}{}} \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q^{\sim} = \frac{[n+1]_q^{\sim}[n]_q^{\sim}}{[2]_q^{\sim}}, \tag{1}$$

as is easily seen by induction, together with an easily checked identity

$$[n+2]_q^{\sim} - [n]_q^{\sim} = [2]_{q^{n+1}}^{\sim}. \tag{2}$$

Next, we transform the classical identity

$$\sum_{k=1}^{n} k + n^2 = \sum_{k=n+1}^{2n} k \tag{3}$$

into

$$n^2 = \sum_{k=1}^{2n} k - 2\sum_{k=1}^{n} k = \binom{2n+1}{2} - 2\binom{n+1}{2} : \tag{4}$$

Theorem 5. For $n \in \mathbf{Z}_{\geq 0}$, we have:

$$\left([n]_q^{\sim}\right)^2 = \begin{bmatrix} 2n+1 \\ 2 \end{bmatrix}_q^{\sim} - [2]_{q^{2n}}^{\sim} \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q^{\sim}. \tag{6}$$

Proof. We rewrite (6) as

$$[n]_q^{\sim}[n]_q^{\sim} = \frac{[2n+1]_q^{\sim}[2n]_q^{\sim}}{[2]_q^{\sim}} - [2]_{q^{2n}}^{\sim} \frac{[n+1]_q^{\sim}[n]_q^{\sim}}{[2]_q^{\sim}} \tag{7}$$

Because

$$[2n]_q^{\sim} = [n]_q^{\sim}[2]_{q^n}^{\sim},$$

(7) becomes

$$[2]_q^{\sim}[n]_q^{\sim} \stackrel{?}{=} [2n+1]_q^{\sim}[2]_{q^n}^{\sim} - [2]_{q^{2n}}^{\sim}[n+1]_q^{\sim}. \tag{8}$$

Multiply (8) by $q - q^{-1}$, and call q^n X. (8) becomes:

$$(q+q^{-1})(X - X^{-1}) \stackrel{?}{=} (X^2 q - X^{-2}q^{-1})(X + X^{-1}) - (X^2 + X^{-2})(Xq - X^{-1}q^{-1}).$$

or

$$X(q+q^{-1}) - X^{-1}(q+q^{-1}) \stackrel{?}{=} (X^3 q - X^{-1}q^{-1} - Xq + X^{-3}q^{-1}) -$$
$$(X^3 q - Xq^{-1} + X^{-1}q - X^{-3}q^{-1}),$$

which is true. ■

On The Total Sum Of Quantum Binomial Coefficients

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Abstract

$\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0$ for $n>0$. We quantize this formula.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\quad x\in\mathbf{R},\quad q\neq 0,\pm 1,$$

be the 2^{nd} quantization of x.

Set

$$\begin{bmatrix} m \\ k \end{bmatrix}_q^{\sim}=\frac{[m]!_q^{\sim}}{[k]!_q^{\sim}[m-k]!_q^{\sim}},$$

where

$$[k]!_q^{\sim}=[1]_q^{\sim}...[k]_q^{\sim},\quad k\in\mathbf{Z}_{\geq 1,\pm}[0]!_q^{\sim}=1.$$

Theorem 1. For $n\in\mathbf{Z}_{\geq 1}$,

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim}q^{(n-1)^k}(-1)^k=0. \tag{2}$$

Proof. We use induction on n. For $n=1$, we have:

$$1-1=0,$$

which is true. The inductive step records, with the help of the formula

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q^{\sim}=q^k\begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim}+q^{k-n-1}\begin{bmatrix} n \\ k-1 \end{bmatrix}_q^{\sim}= \tag{3a}$$

$$=q^{-k}\begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim}+q^{n+1-k}\begin{bmatrix} n \\ k-1 \end{bmatrix}_q^{\sim}: \tag{3b}$$

$$\sum_k q^{nk}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q^{\sim}(-1)^k=\sum_k q^{(n-1)k}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^k+\sum_k q^{n+1}q^{(n-1)k}\begin{bmatrix} n \\ k-1 \end{bmatrix}](-1)^k=$$

$$=0+q^{n+1}\sum q^{(n-1)(k-1)+(n-1)}\sum_k\begin{bmatrix} n \\ k-1 \end{bmatrix}(-1)^k=$$

$$=-q^{2n}\sum_s q^{(n-1)s}\begin{bmatrix} n \\ s \end{bmatrix}(-1)^s=0.\quad\blacksquare$$

On The Sum Of Ratios Of Quantum Binomial Coefficients

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Abstract

Classically, $\sum_{k=0}^{m}\binom{n}{k}\Big/\binom{2n}{k}=2\left(1-\binom{n}{m+1}\Big/\binom{2n}{m+1}\right)$. We quantize this summation.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\quad x\in\mathbf{R},\quad q\neq 0,\pm 1,$$

be the 2^{nd} quantization of x, and

$$\begin{bmatrix}n\\k\end{bmatrix}_q^{\sim}=\frac{[n]!_q^{\sim}}{[k]!_q^{\sim}[n-k]!_q^{\sim}},$$
$$[k]!_q^{\sim}=[1]_q^{\sim}...[k]_q^{\sim},\ k\in\mathbf{Z}_{\geq 1};\quad [0]!_q^{\sim}=1.$$

Lemma 1. For $n\in\mathbf{Z}_{\geq 1}$, we have:

$$\frac{[\begin{smallmatrix}n\\k\end{smallmatrix}]_q^{\sim}}{[\begin{smallmatrix}2n\\k\end{smallmatrix}]_q^{\sim}}q^{k(n+1)}=[2]_{q^n}^{\sim}\left(\frac{[\begin{smallmatrix}n\\k\end{smallmatrix}]_q^{\sim}}{[\begin{smallmatrix}2n\\k\end{smallmatrix}]_q^{\sim}}q^{n(k+1)}-\frac{[\begin{smallmatrix}n\\k+1\end{smallmatrix}]_q^{\sim}}{[\begin{smallmatrix}2n\\k+1\end{smallmatrix}]_q^{\sim}}q^{n(k+2)}\right).\tag{2}$$

Proof. Cancelling the like terms, we arrive at

$$[2n]_q^{\sim}q^{k-n}\stackrel{?}{=}[2]_{q^n}^{\sim}\Big([2n-k]_q^{\sim}-[n-k]_q^{\sim}q^n\Big),$$

or, because

$$[2n]_q^{\sim}=[n]_q^{\sim}[2]_{q^n}^{\sim},$$

to

$$[n]_q^{\sim}q^{k-n}+[n-k]q^n\stackrel{?}{=}[2n-k]_q^{\sim},$$

which is true, because

$$[a+b]_q^{\sim}=[a]_q^{\sim}q^b+q^{-a}[b]_q^{\sim}.\quad\blacksquare$$

Corollary 3.

$$\sum_{k=0}^{m}\frac{[\begin{smallmatrix}n\\k\end{smallmatrix}]_q^{\sim}}{[\begin{smallmatrix}2n\\k\end{smallmatrix}]_q^{\sim}}q^{k(n+1)}=[2]_{q^n}^{\sim}\left(1-\frac{[\begin{smallmatrix}n\\m+1\end{smallmatrix}]_q^{\sim}}{[\begin{smallmatrix}2n\\m+1\end{smallmatrix}]_q^{\sim}}q^{n(m+2)}\right).\tag{4}$$

This is the generalization of the classical summation mentioned in the Abstract.

On The Sum Of Odd Number Of Quantum Integers

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Abstract

Classically, $\sum_{k=n}^{3n-2} k = (2n-1)^2$. We quantize this formula.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0, \pm 1,$$

be the 2^{nd} quantization of x.

We aim of quantizing the formula in the abstract, whose origin is the table

$$\begin{array}{l} 1 \\ 2\ 3\ 4 \\ 3\ 4\ 5\ 6\ 7 \\ 4\ 5\ 6\ 7\ 8\ 9\ 10, \end{array} \tag{1}$$

where the sum of elements in a row is:

$$1^2, 3^2, 5^2, 7^2,$$

Theorem 1. For $n \in \mathbf{Z}_{\geq 1}$,

$$\sum_{k=n}^{3n-2} [k]_q^{\sim} = [2n-1]_{q^{1/2}}^{\sim} [2n-1]_q^{\sim}. \tag{2}$$

Proof. Let $a_k = a + kd$ be an arithmetic progression. We use the easily established formula

$$\sum_{k=0}^{N} [a_k]_q^{\sim} = [N+1]_{q^{d/2}}^{\sim} [a + \frac{N}{2} d]_q^{\sim}. \tag{3}$$

For the LHS of (2),

$$a = n, \quad d = 1, \quad N = 2n-2, \tag{4a}$$

$$a + N\frac{d}{2} = n + (2n-2)\frac{1}{2} = 2n-1, \tag{4b}$$

and formula (2) follows. ■

On The Sum Of An Alternating Series Of Quantum Integers

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Abstract

We quantize the series $\sum_{i=1}^{n}(-1)^i i$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0, \pm 1,$$

be the second quantization of x.

Theorem 1. Let $\delta = 0, 1,\ r \in \mathbf{Z}_{\geq 0}$. Then

$$\sum_{i=0}^{2r+\delta}(-1)^i[i]_q^{\sim} = (-1)^{\delta}[r+\delta]_q^{\sim}([r+1]_q^{\sim} - [r]_q^{\sim}). \tag{2}$$

Proof. We use induction on r. For $r = 0$, (2) becomes,

$$\sum_{i=0}^{\delta}(-1)^i[i]_q^{\sim} = (-1)^{\delta}[\delta]_q^{\sim}, \tag{3}$$

which is true for $\delta = 0, 1$.

Now the inductive step $r \mapsto r+1$:

$$(-1)^{\delta}[r+\delta]_q^{\sim}([r+1]_q^{\sim} - [r]_q^{\sim}) + (-1)^{2r+\delta+1}[2r+\delta+1]_q^{\sim} + (-1)^{2r+\delta+2}[2r+\delta+2]_q^{\sim} \stackrel{?}{=}$$
$$\stackrel{?}{=} (-1)^{\delta}[r+1+\delta]_q^{\sim}([r+2]_q^{\sim} - [r+1]_q^{\sim}),$$

or

$$[r+\delta]_q^{\sim}([r+1]_q^{\sim} - [r]_q^{\sim}) + ([2r+\delta+2]_q^{\sim} - [2r+\delta+1]_q^{\sim}) \stackrel{?}{=} [r+1+\delta]_q^{\sim}([r+2]_q^{\sim} - [n-1]_q^{\sim}). \tag{4}$$

(4) is, in fact, the sum of two relations:

$$[r+\delta]_q^{\sim}[r+1]_q^{\sim} + [2r+\delta+2]_q^{\sim} \stackrel{?}{=} [r+1+\delta]_q^{\sim}[r+2]_q^{\sim}, \tag{5a}$$

$$[r+\delta]_q^{\sim}[r]_q^{\sim} + [2r+\delta+1]_q \stackrel{?}{=} [r+1+\delta]_q^{\sim}[r+1]_q^{\sim}. \tag{5b}$$

The relations $(5a, b)$ are *equivalent* to each other: we get $(5b)$ from $(5a)$ by the change: $r \to r-1, \delta \to \delta+1$. So, let's prove (5b). Call $r = x, r+\delta = y$ $(5b)$ becomes:

$$[x+1]_q^{\sim}[y+1]_q^{\sim} - [x]_q^{\sim}[y]_q^{\sim} = [x+y+1]_q^{\sim}, \tag{6}$$

which is true, well-known, and easy to verify in any case $\forall x, y \in \mathbf{R}$. ■

On The Special Quantum Arithmetic Progression

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Abstract

Let $a_0 = 1, a_k = a_0 + kd$, be an arithmetic progression. It is not difficult to show that

$$\left(\frac{1}{a_1} + \frac{2}{a_1 a_2} + ... + \frac{n}{a_1...a_n}\right) = \frac{1}{d}\left(1 - \frac{1}{a_1...a_n}\right).$$

We quantize this identity.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x, q \in \mathbf{R}, \quad q \neq 1, \ q > 0,$$

be the 2^{nd} quantization of x.

Theorem 1. For $n \in \mathbf{Z}_{\geq 1}$,

$$\sum_{k=1}^{n} \frac{[kd]_q^{\sim} q^{\binom{k}{2}d-1}}{[a_1]_q^{\sim}...[a_k]_q^{\sim}} = 1 - \frac{q^{\binom{n+1}{2}d}}{[a]_1^{\sim}...[a_n]_q^{\sim}}. \tag{2}$$

Proof. We use induction on n. For $n = 1$, (2) returns:

$$\frac{[d]_q^{\sim} q^{-1}}{[1+d]_q^{\sim}} \stackrel{?}{=} 1 - \frac{q^d}{[1+d]_q^{\sim}},$$

or

$$[1+d]_q^{\sim} = [d]_q^{\sim} q^{-1} + q^d,$$

which is true, because, in general,

$$[a+b]_q^{\sim} = q^{-b}[a]_q^{\sim} + q^a [b]_q^{\sim}, \tag{3}$$

Now, the inductive step amounts to, with

$$\sqcap_k = [a_0]_q^{\sim}...[a_k]_q^{\sim} : \tag{4}$$

$$1 - q\frac{\binom{n+1}{2}d}{\sqcap_n} + \frac{[(n+1)d]_q^{\sim} q^{\binom{n+1}{2}d} q^{-1}}{\sqcap_{n+1}} \stackrel{?}{=} 1 - \frac{q^{\binom{n+2}{2}d}}{\sqcap_{n+1}},$$

or to

$$\frac{1}{\sqcap_{n+1}}\left\{[(n+1)d]_q^{\sim}q^{-1}+q^{(n+1)d}\right\}\stackrel{?}{=}\frac{1}{\sqcap_n},$$

which is true, because, by (4),

$$[n+d]_q^{\sim}q^{-1}+1-q^{(n+1)d}=[1+(n+1)d]_q^{\sim}=[a_{n+1}]_q^{\sim}. \quad \blacksquare$$

On The Ratios of Zeroes in Quantum Calculus

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Abstract

We evaluate the ratio, at different bases, of zeroes.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0, 1,$$

be the 2^{nd} quantization of x.

Theorem 1.

$$\lim_{b \to 0} \frac{[b]_q^{\sim}}{[b]_{q^a}} = \frac{[a]_q^{\sim}}{a}. \tag{2}$$

Proof is based on the following

Lemma 3. The expression

$$\frac{[b]_q^{\sim}}{[b]_{q^a}} \tag{4}$$

is symmetric in a, b.

Proof. We have:

$$\frac{[b]_q^{\sim}}{[b]_{q^a}^{\sim}} = \frac{q^b - q^{-b}}{q - q^{-1}} \frac{q^a - q^{-a}}{q^{ab} - q^{-ab}} = \frac{[a]_q^{\sim}}{[a]_{q^b}^{\sim}} \quad \blacksquare \tag{5}$$

Letting here $b \to 0$, we get:

$$\lim_{b \to 0} \frac{[b]_q^{\sim}}{[b]_{q^a}^{\sim}} = \frac{[a]_q^{\sim}}{[a]_1^{\sim}} = \frac{[a]_q^{\sim}}{a}, \tag{6}$$

because

$$\lim_{q \to 1} [a]_q^{\sim} = \lim_{q \to 1} \frac{q^a - q^{-a}}{q - q^{-1}} = a. \tag{7}$$

Subtracting 1 from (5), we get:

$$[b]_q^{\sim} - [b]_{q^a}^{\sim} = \frac{[b]_{q^a}^{\sim}}{[a]_{q^b}^{\sim}} ([a]_q^{\sim} - [a]_{q^b}^{\sim}) \tag{8}$$

On The Ratio Of Two Quantum Numbers

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Abstract

The ratio of two quantum numbers is estimated in terms of their classical counterparts.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbf{R},$$

be the 2^{nd} quantization of x, so that

$$[-x]_q^{\sim} = -[x]_q^{\sim}, \tag{1}$$
$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim}, \tag{2}$$

We can deal with positive numbers thanks to (1), and choose

$$q > 1. \tag{3}$$

thanks to (2). We exclude $q = 1$ from consideration, because

$$\lim_{q\to 1}[x]_q^{\sim} = x. \tag{4}$$

Theorem 5. Let $a, b > 0$. Then

$$\frac{[a]_q^{\sim}}{[b]_q^{\sim}} > \frac{a}{b},\quad a > b, \tag{6}$$
$$\frac{[a]_q^{\sim}}{[b]_q^{\sim}} < \frac{a}{b},\quad a < b. \tag{7}$$

Proof. Since (7) follows from (6) by inversion, we concentrate on (6) only. Set

$$a = \alpha b,\quad \alpha > 1. \tag{8}$$

Then (6) becomes:

$$[\alpha\beta]_q^{\sim} > \alpha[b]_q^{\sim},\quad \alpha > 0, \alpha > 1, q > 1, \tag{9}$$

or, because $q - q^{-1} > 0$,

$$q^{\alpha b} - q^{-\alpha b} \stackrel{?}{>} \alpha(q^b - q^{-b}). \tag{10}$$

Set

$$f(q,b) := q^{\alpha b} - q^{-\alpha b} - \alpha(q^b - q^{-b}). \tag{11}$$

Notice that

$$f(q,0) = 0. \tag{12}$$

For $\alpha f/\partial b$, we have:

$$\partial f/\partial b = \alpha[(q^{\alpha b} + q^{-\alpha b}) - (q^b + q^{-b})]. \tag{13}$$

Lemma 14.

$$q^{\alpha b} + q^{-\alpha b} > q^b + q^{-b},\ \alpha > 1,\ b > 0. \tag{15}$$

Proof. (15) can be rewritten as:

$$q^b(q^{(\alpha - b)} - 1) \overset{?}{>} q^{-\alpha b}(-1 + q^{(\alpha-1)b}). \tag{16}$$

Since $q > 1, (\alpha - 1)b > 0$, then $q^{(\alpha-1)}b > 1$, and (16) becomes:

$$q^b \overset{?}{>} q^{-\alpha b},$$

which is obvious, because $q^b > 1, q^{-\alpha b} < 1$. ■

Thus, by the Lemma,

$$\frac{\partial f}{\partial b} > 0. \tag{17}$$

Because $f(q,0) = 0$ (12),

$$f(q,b) > 0, \quad b > 0. \tag{0.1}$$

This is exactly (10). ■

On The Quantization Of The Sum $\sum \frac{1}{k(k+1)}$

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Abstract

The classical formula $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$ is quantized.

Theorem 1. For $n \in \mathbf{Z}_{\geq 1}$,

$$\sum_{k=1}^{n} \frac{1}{[k]_q^{\sim}[k+1]_q^{\sim}} = q - \frac{q^{n+1}}{[n+1]_q^{\sim}}, \tag{1}$$

where

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}.$$

is the second quantization of x.

Proof. The formula follows at once, if we take an obvious formula

$$[n+1]_q^{\sim} = q[n]_q^{\sim} + q^{-n}, \tag{2}$$

and divide it by $\frac{q^{-n}}{[n]_q^{\sim}[n+1]_q^{\sim}}$, resulting in the telescopic formula

$$\frac{1}{[n]_q^{\sim}[n+1]_q^{\sim}} = \frac{q^n}{[n]_q^{\sim}} - \frac{q^{n+1}}{[n+1]_q^{\sim}}. \tag{3}$$

Summing it up on n, we get (1), because

$$\frac{q^1}{[1]_q^{\sim}} = q. \quad \blacksquare$$

On The Minimum Value Of The Sum Of Quantum a And $1-a$

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Abstract

Classically, $a+(1-a)=1$. We determine what happens quantumly.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\quad x,q\in\mathbf{R},\quad q\neq 1, q>0,$$

be the second quantization of x.

We want to know the extremal values of

$$[a]_q^{\sim}+[1-a]_q^{\sim},\quad 0\leq a\leq 1. \tag{1}$$

Theorem 2. For $0\leq a\leq 1$,

$$\frac{2}{[2]_{q^{1/2}}^{\sim}}\leq [a]_q^{\sim}+[1-a]_q^{\sim}\leq 1. \tag{3}$$

Proof. Denote by $f(a)$ the LHS of (1). Then, with $\eta=q-q^{-1}$,

$$f(a)=\frac{1}{\eta}(q^a-q^{-a}+q^{1-a}-q^{a-1})=\frac{1}{\eta}\Big\{q^a(1-q^{-1})-q^{-a}(1-q)\Big\}=$$
$$=\frac{1}{q^{-1}(q-1)(q+1)}\Big\{(q^{a-1}+q^{-a})(q-1)\Big\}=\frac{q}{q+1}(q^{a-1}+q^{-a}) \tag{4}$$

Therefore, with

$$h=logq, \tag{5}$$

in view of

$$[x]_q^{\sim}=[x]_{q^{-1}}^{\sim}, \tag{6}$$

and $q\neq 1$, we can take

$$q>1\ \Leftrightarrow h>0. \tag{7}$$

Hence,

$$\frac{1}{h}\frac{d}{da}(q^{a-1}+q^{-a}) = q^{a-1}-q^{-a}, \tag{8}$$

and

$$q^{a-1}-q^{-a}=0 \;\Leftrightarrow\; q^{2a-1}=1 \Leftrightarrow a=\frac{1}{2}. \tag{10}$$

At $a=\frac{1}{2}$,

$$f(1/2) = 2\left[\frac{1}{2}\right]_q^{\sim} = 2\frac{q^{1/2}-q^{1/2}}{q-q^{-1}} = \frac{2}{q^{1/2}+q^{-1/2}} < 1, \tag{11}$$

therefore we are looking at a minimum at $a=1/2$. ■

On The First Weighted Sum Of Quantum Binomial Coefficients

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Abstract

Classically, $\sum_{k=0}^{n} k\binom{n}{k}(-1)^k = 0, \quad n > 1$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0, \pm 1,$$

be the 2^{nd} quantization of x.

Set

$$\begin{bmatrix} m \\ k \end{bmatrix}_q^{\sim} = \frac{[m]!_q^{\sim}}{[k]!_q^{\sim}[m-k]!_q^{\sim}},$$
$$[k]!_q^{\sim} = [1]_q^{\sim}...[k]_q^{\sim}, \quad k \in \mathbf{Z}_{\geq 1}; \; [0]!_q^{\sim} = 1.$$

Theorem 1. For $n \in \mathbf{Z}_{\geq 1}$,

$$\sum_{k=0}^{n}(-1)^k q^{(n-2)k}[k]_q^{\sim}\begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} = 0. \tag{2}$$

Proof. We use induction on n. For $n = 2$, (2) returns:

$$-[2]_q^{\sim} + [2]_q^{\sim} \stackrel{?}{=} 0,$$

which is true.

Now, because

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q^{\sim} = q^{-k}\begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} + q^{n+1-k}\begin{bmatrix} n \\ k-1 \end{bmatrix}_q^{\sim},$$
$$\sum_{k=0}^{n+1}(-1)^k\begin{bmatrix} n+1 \\ k \end{bmatrix}[k]_q^{\sim}q^{(n-1)k} =$$
$$= \sum_k (-1)^k q^{-k}\begin{bmatrix} n \\ k \end{bmatrix}[k]_q^{\sim}q^{(n+2)k} + q^{n+1}\sum_k \begin{bmatrix} n \\ k-1 \end{bmatrix}(-1)^k[k]_q^{\sim}q^{-k+(n-1)k} =$$
$$= -q^{n+1}\sum_s \begin{bmatrix} n \\ s \end{bmatrix}(-1)^s[s+1]_q^{\sim}q^{(n-2)(s+1)} \tag{3}$$

But

$$[s+1]_q^{\sim} = q^{-1}[s]_q^{\sim} + q^s,$$

so (3) becomes

$$\sum (-1)^s \begin{bmatrix} n \\ s \end{bmatrix}_q^{\sim} q^{-s} q^{(n-1)s} \stackrel{?}{=} 0,$$

which is true because, in general,

$$\sum_{s=0}^{n} (-1)^s \begin{bmatrix} n \\ s \end{bmatrix}_q^{\sim} q^{(n-2)s} = 0, \quad n \geq 1. \quad \blacksquare \tag{4}$$

Corollary 5. For any $a \in \mathbf{R}, n \in \mathbf{Z}_{\geq 2}$,

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} [k+a]_q^{\sim} q^{(n-2)} = 0. \tag{6}$$

Proof. Since

$$[k+a]_q^{\sim} = q^k [a]_q^{\sim} + q^{-a} [k]_q^{\sim}, \tag{7}$$

(6) is reduced, by (2), to:

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} q^{(n-2)k} \stackrel{?}{=} 0,$$

and this is (4). $\blacksquare$

On The Difference Of Two Consequtive Quantum Integers

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Abstract

Classically, $(n+1)-n=1$, no matter what $n\in\mathbf{Z}$ is. Quantumly, the situation is rather different.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\quad x\in\mathbf{R},\quad q\neq 0,\pm 1, \tag{1}$$

be the second quantization of x, so that, for $n\in\mathbf{Z}_{\geq 1}$,

$$[n]_q^{\sim}=\sum_{i=0}^{n-1}q^{n-2i-1}, \tag{2}$$

and

$$[-n]_q^{\sim}=-[n]_q^{\sim}.$$

It's quite clear that $[n+1]_q^{\sim}-[n]_q^{\sim}$ is a complex object for $n\in\mathbf{Z}_{\geq 1}$. Nevertheless,

Theorem 3. For $n\in\mathbf{Z}_{\geq 1}$,

$$[n+1]_{q^2}^{\sim}-[n]_{q^2}^{\sim}=(-1)^n[2n+1]_{iq}^{\sim}, \tag{4}$$

where $i=\sqrt{-1}$ is the generator of $\mathbf{C}/\mathbf{R}$.

Proof. For the LHS of (4), we have:

$$\begin{aligned}
&\frac{1}{q^2-q^{-2}}\left\{q^{2(n+1)}-q^{-2(n+1)}-q^{2n}+q^{-2n}\right\}=\\
&=\frac{1}{q^{-2}(q^4-1)}\left\{q^{2n}(q^2-1)+q^{-2n-2}(q^2-1)\right\}\\
&=\frac{1}{(q^2+1)q^{-1}q^{-1}}\left\{q^{2n}+q^{-2n-2}\right\}=\\
&=\frac{1}{(q+q^{-1})q^{-1}}\left\{q^{2n+1}+q^{-2n-1}\right\}=\\
&=\frac{[2]_{q^{2n+1}}^{\sim}}{[2]_q^{\sim}}.
\end{aligned} \tag{5}$$

For the RHS of (4), we get:

$$(-1)^n \frac{(iq)^{2n+1} - (iq)^{-2n-1}}{iq - (iq)^{-1}} \quad [\text{because } i^2 = -1] =$$
$$= (-1)^n \frac{(-1)^n iq^{2n+1} - (-1)^n \frac{1}{i} q^{-2n-1}}{i(q + q^{-1})} =$$
$$= \frac{q^{2n+1} + q^{-2n-1}}{q + q^{-1}} = \frac{[2]^{\sim}_{q^{2n+1}}}{[2]^{\sim}_{q}},$$

and this is (5). ■

Remark 6. Formula (4) is true for *all* $n \in \mathbf{Z}$. Indeed, for $n = 0$, (4) returns:

$$1 = 1,$$

which is true, and if

$$n = -m, \quad m \in \mathbf{Z}_{\geq 1};$$

then

$$[n+1]^{\sim}_{q^2} - [n]^{\sim}_{q^2} = [-m+1]^{\sim}_{q} - [-m]^{\sim}_{q^2} =$$
$$= [m]^{\sim}_{q^2} - [m-1]^{\sim}_{q^2} \text{ [by (4)]} =$$
$$= (-1)^{m-1}[2m-1]^{\sim}_{iq} = (-1)^m [1-2m]^{\sim}_{iq} =$$
$$= (-1)^n [2n+1]^{\sim}_{q},$$

so (4) is true for negative n's also.

On Sections Of Quantum Harmonic Series

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Abstract

Classically, $\sum_{k=1}^{n} \frac{1}{n+k} > \frac{1}{2}$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x, q \in \mathbf{R}, \quad q \neq 1, q > 0,$$

be the 2^{nd} quantization of x, so that

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

and we can take

$$q > 1$$

without loss of generality.

We *can't* expect

$$\sum_{k=1}^{n} \frac{1}{[n+k]_q^{\sim}} > \frac{1}{2} \text{ or a const},$$

because

$$\sum_{n=1}^{\infty} \frac{1}{[n]}$$

converges:

$$\sum_{n=1}^{\infty} \frac{1}{[n]} = \sum \frac{q - q^{-1}}{q^n - q^{-n}},$$

and

$$\lim_{n\to\infty} \frac{q - q^{-1}}{q^n - q^{-n}} \Big/ \frac{q - q^{-1}}{q^n} = 1,$$

the series

$$\sum_{n=1}^{\infty} \frac{q - q^{-1}}{q^n}$$

being obviously convergent. Notice, that the series

$$\sum_{n=1}^{\infty} \frac{q^n}{[n]_q^\sim}$$

diverges. More generally, it's easy to see that

$$\sum_{n=1}^{\infty} \frac{q^{\alpha n}}{[n]_q^\sim}$$

diverges for $\alpha \geq 1$ and converges for $\alpha < 1$.

Theorem 1. For $n \in \mathbf{Z}_{\geq 2}$,

$$\sum_{k=1}^{n} \frac{1}{[n+k]_q^\sim} > \frac{q^{n-1/2}[2]_{q^{1/2}}^\sim}{[4]_{q^n}^\sim}. \tag{2}$$

Proof. We start with obvious observation that

$$[n+a]_q^\sim > q^a [n]_q^\sim, \ a > 0. \tag{3}$$

Indeed, multiplying by $q - q^{-1}$, and denoting

$$X = q^n, \ Q = q^a > 1,$$

we rewrite (3) as

$$XQ - X^{-1}Q^{-1} \stackrel{?}{>} Q(X - X^{-1}),$$

or

$$QX^{-1} \stackrel{?}{>} Q^{-1}X^{-1},$$

or

$$Q > Q^{-1},$$

which is true because $Q > 1$.

Next, for $a = n - k$, (3 implies:

$$[2n]_q^\sim > q^{n-k}[n+k]_q^\sim, \ \ 1 \leq k \leq n-1, \tag{4}$$

or

$$\frac{1}{[n+k]_q^\sim} > \frac{q^{n-k}}{[2n]_q^\sim}, \ \ 1 \leq k \leq n-1. \tag{5}$$

Summing on k, and adding up

$$\frac{1}{[2n]_q^\sim} = \frac{1}{[2n]_q^\sim},$$

we get:

$$\sum_{k=1}^{n} \frac{1}{[n+k]_q^{\sim}} > \frac{[n]_q}{[2n]_q^{\sim}}, \tag{6}$$

where

$$[n]_q = \frac{q^n - 1}{q - 1}$$

is the first quantization. So,

$$\frac{[n]_q}{[2n]_q^{\sim}} = \frac{q^n - 1}{(q-1)[2n]_q^{\sim}} = \frac{q^{\frac{n}{2}}(q^{n/2} - q^{-n/2})}{q^{1/2}(q^{1/2} - q^{-1/2})} \frac{1}{[2n]_q^{\sim}} =$$
$$= q^{n-1/2} \left[\frac{n}{2}\right]_q^{\sim} [2]_{q^{1/2}}^{\sim} \left[\frac{n}{2}\right]_q^{\sim} [4]_{q^{n/2}} = q^{n-1/2} \frac{[n]_{q^{1/2}}^{\sim}}{[4]_{q^{n/2}}^{\sim}}. \quad \blacksquare$$

On Second Weighted Sum of Quantum Factorials

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Abstract

The classical identity $\sum_{k=1}^{n}(k^2+1)k! = n(n+1)!$, is quantized.

We shall prove below that

$$\sum_{k=1}^{n}\{q^{\binom{k}{2}+1}([k]_q^{\sim})^2 + q^{\frac{k^2-5k+2}{2}}\}[k]!_q^{\sim} = q^{\binom{n}{2}}[n]_q^{\sim}[n+1]!_q^{\sim}, \tag{1}$$

where

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q-q^{-1}}, \tag{2}$$

$$[k]!_q^{\sim} = [1]_q^{\sim}...[k]_q^{\sim}, \quad k \in \mathbf{Z}_{\geq 1}. \tag{3}$$

Indeed, for $n=1$, (1) returns:

$$q + q^{-1} = (q+q^{-1}), \tag{4}$$

which is true. Here we used the obvious formula

$$[2]_q^{\sim} = \frac{q^{-2}-q^{-2}}{q-q^{-1}} = q+q^{-1}. \tag{5}$$

Now, the inductive step applied to (1) produces

$$\left\{q^{\binom{n+1}{2}+1}([n+1]_q^{\sim})^2 + q\frac{(n+1)(n+1-5)+2}{2}\right\}[n+1]!_q^{\sim} \stackrel{?}{=}$$
$$= \left\{q^{\binom{n+1}{2}}[n+1]_q^{\sim}[n+2]_q^{\sim} - q^{\binom{n}{2}}[n]_q^{\sim}\right\}[n+1]!_q^{\sim}.$$

Dividing this by $[n+1]!_q^{\sim}$, we arrive at

$$q^{\binom{n+1}{2}+1}([n+1]_q^{\sim})^2 + q^{\frac{(n^2-3n-2)}{2}} \stackrel{?}{=}$$
$$q^{\binom{n+1}{2}}\,[n+1]_q^{\sim}[n+2]_q^{\sim} - q^{\binom{n}{2}}[n]_q^{\sim}. \tag{6}$$

Using the easily verified formula

$$[n+2]_q^{\sim} = q[n+1]_q^{\sim} + q^{-n-1}, \tag{7}$$

the RHS of (6) becomes:

$$q^{\binom{n+1}{2}}[n+1]_q^{\sim}\{q[n+1]_q^{\sim}+q^{-n-1}\}-q^{\binom{n}{2}}[n]_q^{\sim}=q^{\binom{n+1}{2}+1}([n+1]_q^{\sim})^2+$$
$$+q^{\binom{n+1}{2}-n-1}[n+1]-q^{\binom{n}{2}}[n], \tag{8}$$

so that (7) simplifies to:

$$q^{\frac{n^2-3n-2}{2}}\stackrel{?}{=}q^{\binom{n+1}{2}-n-1}[n+1]-q^{\binom{n}{2}}[n]= \tag{9}$$
$$=q^{\binom{n}{2}}\{q^{\binom{n+1}{2}-\binom{n}{2}-n-1}[n+1]_q^{\sim}-(n)_q^{\sim}\}=$$
$$=q^{\binom{n}{2}}\{q^{-1}[n+1]_q^{\sim}-[n]_q^{\sim}\}. \tag{10}$$

But, obviously,

$$q^{-1}[n+1]_q^{\sim}-[n]_q^{\sim}=q^{-n-1},$$

so the RHS of (10) returns:

$$q^{\binom{n}{2}}q^{-n-1}=q^{\binom{n}{2}-n-1}=q^{\frac{n(n-1)-2n-2}{2}}=q^{\frac{n^2-3n-2}{2}},$$

which is the LHS of (10).

Remark 11. Exchange in (1) q by q^{-1}, subtract, and divide by $q-q^{-1}$. We get:

$$\sum_{k=1}^{n}\{[\binom{k}{2}+1]_q^{\sim}([k]_q^{\sim})^2+[\frac{k^2-5k-2}{2}]_q^{\sim}\}[k]!_q^{\sim}=$$
$$=[\binom{n}{2}]_q^{\sim}[n]_q^{\sim}[(n+1)]!_q^{\sim}. \tag{12}$$

This identity is new even in the classical case $q=1$, where it becomes:

$$\sum_{k=1}^{n}\left\{\left[\binom{k}{2}+1\right]k^2+\frac{k^2-5k^{-2}}{2}\right\}k!=\binom{n}{2}n(n+1)!. \tag{13}$$

On Rescaling A Quantum Number

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Abstract

We show how a rescaling affects the value of a quantum number.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},$$

be the 2^{nd} quantization of x, so that

$$[-x]_q^{\sim} = -[x]_q^{\sim}, \tag{1}$$
$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim}. \tag{2}$$

Formula (1) allows one to deal with nonnegative numbers only, and formula (2) shows that, apart from the classical case $q = 1$, we can take

$$q > 1$$

without loss of generality.

Now, in a previous paper, I showed that

$$\sum_{i=1}^{n} [a_i]_q^{\sim} < \left[\sum_{i=1}^{n} a_i\right]_q^{\sim}, \quad a_i > 0, \quad i \le i \le n, \quad q \ne 1. \tag{3}$$

Taking $a_i = a$ for $1 \le i \le n$, we get

$$\frac{[a]}{[na]} < \frac{1}{n}, \ n \in \mathbf{Z}_{\ge 2}. \tag{4}$$

This suggests:

Theorem 5. For $x > 1$

$$\frac{[a]_q^{\sim}}{[xa]_q^{\sim}} < \frac{1}{x}, \ x > 1, \ q \ne 1. \tag{6}$$

Proof. Denote

$$f(a, x, h) = (q - q^{-1})([xa]_q^{\sim} - x[a]_q^{\sim}) = \tag{7a}$$
$$= q^{xa} - q^{-xa} - x(q^a - q^{-a}), \quad h = log(q) > 0. \tag{7b}$$

Our theorem is equivalent to the statement:

$$f > 0 \text{ for } x > 1. \tag{8}$$

Notice that

$$f(x, a, 0) = 0. \tag{9}$$

Let's show that

$$\frac{\partial f}{\partial h} > 0, \tag{10}$$

Together with (9) it will prove (8). So,

$$\frac{\partial f}{\partial h} = xa(Q^x + Q^{-x}) - xa(Q + Q^{-1}), \quad Q = q^a > 1, \tag{11}$$

and since $x > 1, a > 0$, (10), in the form (11), becomes:

$$Q^x + Q^{-x} \overset{?}{>} Q + Q^{-1}, \; x > 1, \quad Q \neq 1, \tag{12}$$

which is obvious: it can be rewritten as

$$Q(Q^{x-1} - 1) \overset{?}{>} Q^{-x}(Q^{x-1} - 1), \tag{13}$$

or, since $Q^{x-1} > 1$ because $Q > 1$ and $x - 1 > 0$, as

$$Q \overset{?}{>} Q^{-x}, \tag{14}$$

which is obvious. ■

Corollary 15.

$$\frac{[a]_q^{\sim}}{[xa]_q^{\sim}} > \frac{1}{x}, \quad 0 < x < 1, \; q \neq 1. \tag{16}$$

Proof. Denote $y = 1/x > 1, \; A = xa$. Then (16) can be rewritten as

$$\frac{[yA]_q^{\sim}}{[A]_q^{\sim}} > y, \quad y > 1, \quad q \neq 1. \tag{17}$$

But this is (6). ■

Multiplication By 2 For Quantum Numbers

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Abstract

We derive basic inequalities in quantum numbers under multiplication by 2.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbf{R},$$

be the 2^{nd} quantization of x, so that

$$[2]_q^{\sim} = q + q^{-1}.$$

and

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$
$$[-x]_q^{\sim} = -[x]_q^{\sim}. \tag{1}$$

Theorem 2. For $m, M \in \mathbf{Z}_{>0}$, we have:

$$[2]_q^{\sim}[x]_q^{\sim} < [2x]_q^{\sim},\quad x > 1,\quad q \neq 1, \tag{3}$$
$$[2]_q^{\sim}[x]_q^{\sim} > [2x]_q^{\sim},\quad 0 < x < 1,\quad q \neq 1. \tag{4}$$

Proof. In view of (1), we can take

$$q > 1. \tag{5}$$

Then

$$q - q^{-1} > 0, \tag{6}$$

and multiplying (3), (4) by $q - q^{-1}$, we arrive at

$$(q + q^{-1})(q^x - q^{-x}) \overset{?}{<} q^{2x} - q^{-2x},\ x > 1, \tag{7}$$
$$(q + q^{-1})(q^x - q^{-x}) \overset{?}{>} q^{2x} - q^{-2x},\ 0 < x < 1. \tag{8}$$

because $x > 0, q > 1, q^x - q^{-x} > q$ so that (7), (8) can be rewritten as

$$q + q^{-1} \overset{?}{<} q^x + q^{-x},\ x > 1, \tag{9}$$
$$q + q^{-1} \overset{?}{>} q^x + q^{-x},\ 0 < x < 1, \tag{10}$$

or as

$$q^{-x}(-1 + q^{-1+x}) \overset{?}{<} q(q^{x-1} - 1),\quad x > 1. \tag{11}$$
$$q^{-x}(-1 + q^{-(1-x)}) \overset{?}{>} q(q^{x-1} - 1)\quad 0 < x < 1. \tag{12}$$

In the case $x > 1, x - 1 > 0$, and $q^{x-1} > 1$, so (11) reduces

On Quantum Triangular Numbers

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Abstract

Classically, triangular numbers $\triangle_n = \frac{n(n-1)}{2}$ satisfy the basic relation $\triangle_n + \triangle_{n+1} = n^2$. We equantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0, \pm 1,$$

be the 2^{nd} quantization of x.

It's not obvious what the quantum version of

$$\triangle_{n+1} = 1 + 2 + ... + n,$$

should be.

Theorem 1. For $n \in \mathbf{Z}_{\geq 1}$, set

$$\triangle_n = \begin{bmatrix} n \\ 2 \end{bmatrix}_q^{\sim} = \frac{[n]_q^{\sim}[n-1]_q^{\sim}}{[2]_q^{\sim}}. \tag{2}$$

Then

$$\triangle_n + \triangle_{n+1} = ([n]_q^{\sim})^2. \tag{3}$$

Proof. Our relation (3) amounts to

$$[n]_q^{\sim}[n-1]_q^{\sim} + [n+1]_q[n]_q \stackrel{?}{=} [2]_q^{\sim}[n]_q^{\sim}[n]_q^{\sim}, \tag{4}$$

or

$$[n-1]_q^{\sim} + [n+1]_q^{\sim} \stackrel{?}{=} [2]_q^{\sim}[n]_q^{\sim}, \tag{5}$$

which is obvious: in general,

$$[a]_q^{\sim} + [b]_q^{\sim} = [2]_{q^{(a-b)/2}}^{\sim} \left[\frac{a+b}{2} \right]_q^{\sim},$$

and this is (5) for

$$a = n+1, \ b = n-1. \quad \blacksquare$$

On Quantum Binomial Coefficients In The Base 2

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Abstract

We represent quantum binomial coefficients, in the base 2, a quantum analog of $\binom{n}{2}$, as a sum of simple quantum integers.

Let

$$x \to [x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}$$

be the 2^{nd} (symmetric) quantization. Numbers of the form

$$[n]_q^{\sim} = \sum_{i=0}^{n-1} q^{n-1-2i} \tag{1}$$

are called simple (quantum) integers.

The binomial coefficients at the base 2,

$$\begin{bmatrix} n \\ 2 \end{bmatrix}_q^{\sim} = \frac{[n]_q^{\sim}[n-1]_q^{\sim}}{[2]_q^{\sim}}, \tag{2}$$

are known to be sums of simple integers, but the concrete representation is unknown. It is provided below.

Theorem 3. (i) For $n \in \mathbf{Z}_{\geq 1}$, we have:

$$\begin{bmatrix} 2n \\ 2 \end{bmatrix}_q^{\sim} = \sum_{i=0}^{n-1} [1+4i]_q^{\sim}, \tag{4}$$

$$\begin{bmatrix} 2n+1 \\ 2 \end{bmatrix}_q^{\sim} = \sum_{i=0}^{n-1} [3+4i]_q^{\sim}. \tag{4}$$

Proof. We use the following useful formula for the sums of quantum arithmetic progression:

$$\sum_{k=0}^{n} [a+kd] = [n+1]_{q^{d/2}}^{\sim} [a + \frac{n}{2} d]_q^{\sim}. \tag{5}$$

Thus, the sum on the RHS of (4) becomes:

$$[n]_{q^2}^{\sim}[2n-1]_q^{\sim} = \frac{q^{2n} - q^{-2n}}{q^2 - q^{-2}} [2n-1]_q^{\sim} = \frac{[2n]_q^{\sim}[2n-1]_q^{\sim}}{[2]_q^{\sim}},$$

which is the LHS of (1).

Similarly, the sum of the RHS of (5) becomes:

$$[n]^{\sim}_{q^2}[2n+1]^{\sim}_{q} = \frac{[2n]^{\sim}_{q}}{[2]^{\sim}_{q}}[2n+1]^{\sim}_{q},$$

which is exactly the LHS of (5). ■

On Finite Sum Of Quantum Binomial Coefficients

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Abstract

Classically, $\sum_{k=0}^{\ell}\binom{k+m}{m}=\binom{\ell+1+m}{1+m}$. We quantize this formula.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\quad x\in\mathbf{R},\quad q\neq 0,1,$$

be the second quantization of x.

Set

$$\begin{bmatrix} m \\ k \end{bmatrix}=\frac{[m]!_q^{\sim}}{[k]!_q^{\sim}[m-k]!_q^{\sim}},$$

where

$$[k]!_q^{\sim}=[1]_q^{\sim}...[k]_q^{\sim},\quad k\in\mathbf{Z}_{\geq 1};\ [0]!_q^{\sim}=1.$$

Theorem 1. For $\ell m\in\mathbf{Z}_{\geq 0}$, we have:

$$\sum_{k=0}^{\ell}\begin{bmatrix} k+m \\ m \end{bmatrix}_q^{\sim}q^{k(m+2)}=\begin{bmatrix} \ell+m+q \\ m+1 \end{bmatrix}_q^{\sim}q^{\ell(m+1)}. \tag{2}$$

Proof. We use induction on ℓ. For $\ell=0$, (2) becomes

$$1=1.$$

Now, the induction step amounts to:

$$\begin{bmatrix} \ell+m+1 \\ m+1 \end{bmatrix}_q^{\sim}q^{\ell(m+1)}+\begin{bmatrix} \ell+1+m \\ m \end{bmatrix}_q q^{(\ell+1)(m+2)}\stackrel{?}{=}\begin{bmatrix} \ell+m+2 \\ m+1 \end{bmatrix}_q^{\sim}q^{(\ell+1)(m+1)},$$

or

$$\begin{bmatrix} \ell+m+1 \\ m+1 \end{bmatrix}_q^{\sim}q^{-(m+1)}+\begin{bmatrix} \ell+1+m \\ m \end{bmatrix}_q^{\sim}q^{\ell+1}\ \stackrel{?}{=}\begin{bmatrix} \ell+m+2 \\ m+1 \end{bmatrix},$$

or

$$q^{-m-1}[\ell+1]_q^{\sim}+q^{\ell+1}[m+1]_q^{\sim}\quad\stackrel{?}{=}\ [\ell+m+2]_q^{\sim},$$

which is obvious, because, in general.

$$[a+b]_q^{\sim}=q^{-b}[a]_q^{\sim}+q^a[b]_q^{\sim}.\quad\blacksquare \tag{0.1}$$

On Finite Sections Of The Quantum Harmonic Series

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Abstract

We quantize the classical identity

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = \sum_{s=1}^{n} \frac{1}{n+s}.$$

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0, \pm 1,$$

be the 2^{nd} quantization of x.

Let $\omega(n)$ be the usual arithmetic function "the number of distinct prime divisors of n":

$$n = \prod_{k=1}^{m} p_{i(k)}^{\ell(k)} \Rightarrow \omega(n) = m \text{ of all } e_k \geq 1.$$

Set

$$\epsilon(n) = \begin{cases} 0, & n \text{ even}, \\ 1, & n \text{ odd}. \end{cases}$$

Theorem 2. For $n \in \mathbf{Z}_{\geq 1}$,

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1} q^{\epsilon(k)}}{[k]_q^{\sim}} = \sum_{s=1}^{n} \frac{q^{n+s}}{[n+s]_q^{\sim}}. \tag{3}$$

Proof. We use induction on n. For $n = 1$, (3) returns:

$$\frac{q}{1} - \frac{q^{\epsilon(2)}}{[2]_q^{\sim}} = q - \frac{1}{[2]_q^{\sim}} = \frac{q^2}{[2]_q^{\sim}} \stackrel{?}{=}$$
$$\stackrel{?}{=} \frac{q^2}{[2]_q^{\sim}},$$

which is true.

Next, the induction step amounts to:

$$\frac{q^{2n+1}}{[2n+1]^{\sim}_q} - \frac{1}{[2n+2]^{\sim}_q} \stackrel{?}{=} -\frac{q^{n+1}}{[n+1]^{\sim}_q} + \frac{q^{2n+1}}{[2n+1]^{\sim}_q} + \frac{q^{2n+2}}{[2n+2]^{\sim}_q},$$

or

$$\frac{q^{n+1}}{[n+1]^{\sim}_q} \stackrel{?}{=} \frac{q^{2n+2}}{[2n+2]^{\sim}_q} + \frac{1}{[2n+2]^{\sim}_q},$$

or

$$q^{n+1} \stackrel{?}{=} \frac{(q^{2n+2}1)}{[2]^{\sim}_{q^{n+1}}}$$

which is obvious. ■

On Alternating Sums Of Squares Of Quantum Integers

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Abstract

The classical formula $\sum_{i=0}^{n}(-1)^{i-1}i^2 = (-1)^{n-1}\left(\sum_{i=0}^{n} i\right)$ is quantized.

The classical formula in the Abstract can be quantized thusly:

$$\sum_{i=0}^{n}(-1)^{i-1}([i]_q^{\sim})^2 = (-1)^{n-1}\sum_{i=0}^{n}[i]_{q^2}^{\sim}, \tag{1}$$

where

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{2}$$

To prove (1) we set $3_k = q^k - q^{-k}$, notice that (1) is true for $n = 0$, and then proceed by induction. The inductive step $n \to n+2$ amounts to:

$$(-1)^n([n+1]_q^{\sim})^2 + (-1)^{n-1}([n+2]_q^{\sim})^2 \stackrel{?}{=}$$
$$\stackrel{?}{+}\ (-1)^{n-1}\left\{[n+1]_{q^2}^{\sim} + [n+2]_{q^2}^{\sim}\right\},$$

or,

$$-([n+1]_q^{\sim})^2 + ([n+2]_q^{\sim})^2 \stackrel{?}{=} [n+1]_{q^2}^{\sim} + [n+2]_{q^2}^{\sim}. \tag{3}$$

Call

$$X = q^n.$$

Then (8) becomes

$$\frac{1}{3_1^2}\left\{-(Xq - X^{-1}q^{-1})^2 + (Xq^2 - X^{-1}q^{-2})^2\right\} = \frac{1}{3_2}\left\{X^2q^2 - X^{-2}q^{-2} + X^2q^4 - X^{-2}q^{-4}\right\} \tag{4}$$

The LHS of (4) returns:

$$\frac{1}{3_1^2}\left\{-\left(X^2q^2 - 2 + X^{-2}q^{-2}\right) + \left(X^2q^4 - 2 + X^{-2}q^{-4}\right)\right\} =$$
$$= \frac{1}{3_1^2}\left\{X^2q^2(1-q^2) - X^{-2}q^{-2}(1-q^{-2})\right\} =$$
$$= \frac{1}{3_1^2}\left\{X^2q^2q(q-q^{-1}) - X^{-2}q^{-2}q^{-1}(q-q^{-1})\right\} =$$
$$= \frac{1}{3_1}\left\{X^2q^3 - X^{-2}q^{-3}\right\} = [2x+3]_q^{\sim}. \tag{5}$$

The RHS of (4) yields:

$$\frac{1}{3_2}\Big\{X^2q^2(1+q^2)-X^{-2}q^{-2}(1+q^{-2})\Big\}=$$
$$\frac{1}{3_2}\Big\{X^2q^2q(q^{-1}+q)-X^{-2}q^{-2}q^{-1}(q+q^{-1})\Big\}=$$
$$=\frac{[2]_q^\sim}{3_2}\Big\{X^2q^3-X^{-2}q^{-3}\Big\}=\frac{1}{3_1}\Big\{X^2q^3-X^{-2}q^3\Big\}=[2x+3]_q^\sim,$$

which is the same as (5). We used in the Proof the obvious relation

$$[2]_q^\sim=\frac{3_2}{3_1}. \tag{6}$$

On A Limit In Quantum Calculus

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Abstract

$\lim_{x\to\infty}\frac{x+a}{x}=1$. Quantumly, it's no longer true.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\quad x\in\mathbf{R},\quad q\neq 0,1, \tag{1}$$

be the 2^{nd} quantization of x, so that

$$[x]_{q^{-1}}^{\sim}=[x]_q^{\sim}, \tag{2}$$

and we can take

$$q>1$$

without loss of generality.

Theorem 3. For any $x,a\in\mathbf{R}$,

$$\lim_{k\to\infty}\frac{[x+a]_q^{\sim}}{[x]_q^{\sim}}=q^a=e^{a|logq|}. \tag{4}$$

Proof. We have, with $X=q^x$:

$$\frac{[x+a]_q^{\sim}}{[x]_q^{\sim}}=\frac{Xq^a-X^{-1}q^{-a}}{X-X^{-1}}\ \Rightarrow$$

$$\lim_{X\to\infty}\frac{Xq^a-X^{-1}q^{-a}}{X-X^{-1}}\ [\text{since } q>1\Rightarrow X>1 \text{ and } X\to\infty]=q^a. \quad \blacksquare$$

Corollary 5.

$$\lim_{n\to\infty}\frac{[n+1]_q^{\sim}}{[n]_q^{\sim}}=q. \tag{6}$$

On The Sum Of Inverse Terms Of Quantum Arithmetic Progression

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Abstract

Let $u_k = \frac{1}{a_k a_{k+1}}$, where $a_k = a + kd$ form an arithmetic progression. It is well known and easy to see that $\sum_{k=1}^{n} u_k = \frac{1}{d}\left(\frac{1}{a_1} - \frac{1}{a_{n+1}}\right)$. We quantize this result.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbf{R},$$

be the second quantized x.

Let $a_k = [a + kd]_q^{\sim}$ be a quantum arithmetic progression, and let

$$u_k = \frac{q^{(1-d)k}}{a_k a_{k+1}}.$$

Theorem 1.

$$\sum_{k=0}^{n} u_k = \frac{q^a}{[a]_q^{\sim}} - \frac{q^{a+n+1}}{[a + (n+1)d]_q^{\sim}} \tag{2}$$

Proof. The formula (2) follows upon summing up the formulae

$$\frac{1}{a_k a_{k+1}} = \frac{q^{a+dk}}{a_k} - \frac{q^{a+kd+1}}{a_{k+1}}, \tag{3}$$

together with the relation

$$q^{(1-d)k} q^{a+dk}\Big|_{k\to k+1} = q^{(-d)k} q^{a+kd+1} \tag{4}$$

The later formula is immediate:

$$(1-d)k + a + kd\Big|_{k\to k+1} = a + k\Big|_{k\to k+1} = a + k + 1, \tag{5a}$$

and

$$(1-d)k + a + kd + 1 = a + k + 1. \tag{5b}$$

To prove (3), multiply both its parts by $q^{-a-kd} q_k q_{k+1}$. We get:

$$[a + (k+1)d]_q^{\sim} \stackrel{?}{=} q^{-(a+kd)} + q[a + kd]_q^{\sim}, \tag{6}$$

which is obvious in view of the formula

$$[x + d]_q^{\sim} = q^{-d}[x]_q^{\sim} + q^x [d]_q^{\sim},\ \ \forall x, d, \tag{7}$$

for $x = a + kd$.

Number 12 In Second Quantization

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Abstract

We quantize $12 = 2^2 3$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0,\ 1,$$

be the 2^{nd} quantization of (x).

Theorem 1. For $n \in \mathbf{Z}$, we have

$$([2]_q^{\sim})^2 [3]_{q^n} = 1 + [3]_q^{\sim} + [2n+1]_q^{\sim} + [2n+3]_q^{\sim} - \\ -([2n-1]_q^{\sim} + [2n-3]_q^{\sim}). \tag{2}$$

Proof. Since

$$1 + [3]_q^{\sim} = 1 + q^2 + 1 + q^{-2} = (q + q^{-1})^2 = ([2]_q^{\sim})^2,$$

our (2) becomes:

$$([2]_q^{\sim})^2([3]_{q^n}^{\sim} - 1) \stackrel{?}{=} [2n+1]_q^{\sim} + [2n+3]_q^{\sim} - \\ -([2n-1]_q + [2n-3]_q^{\sim}). \tag{3}$$

The RHS of (3) is:

$$\frac{1}{q - q^{-1}}\{q^{2n1} - q^{2n-1} + q^{2n+3} - q^{2n-3} - q^{2n-1} + q^{1-2n} - q^{2n-3} + q^{3-2n})\} = \\ = \frac{1}{q - q^{-1}}\{q^{2n+1}(1 - q^{-4}) + q^{3-2n}(1 - q^{-4}) + q^{2n+3}(1 - q^{-4}) + q^{1-2n}(1 - q^{-4})\} = \\ = \frac{q^{-2}(q^2 - q^{-2})}{q - q^{-1}}\{q^{2n-1} + q^{3n-2} + q^{2n+3} + q^{1-2n}\} = \\ = [2]_q^{\sim}\{q^{2n-1} + q^{1-2n} + q^{2n+1} + q^{-1-2n}\} = \\ = [2]_q^{\sim}\{q^{2n-1}(1 + q^2) + q^{-1-2n}(1 + q^2)\} = \\ = [2]_q^{\sim}\{q^{2n-1}q(q^{-1} + q) + q^{-1-2n}q(q^{-1} + q)\} = \\ = [2]_q^{\sim}[2]_q^{\sim}\{q^{2n} + q^{-2n}\} = ([2]_q^{\sim})^2([3]_{q^n}^{\sim} - 1),$$

which is the LHS of (3). ■

Number 8 In Second Quantization

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Abstract

We quantize $8 = 2^3$.

Let

$$[x]_q^\sim = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0, 1,$$

be the 2^{nd} quantization of x.

Theorem 1. For $n \in \mathbf{Z}$,

$$([2]_q^\sim)^2 [2]_{q^n}^\sim = [n+3]_q^\sim + [n+1]_q^\sim - ([n-3]_q^\sim + [n-1]_q^\sim). \tag{2}$$

Proof. For the RHS of (2), we have:

$$\begin{aligned}
&\frac{1}{q - q^{-1}}(q^{n+3} - q^{-n-3} + q^{n+1} - q^{-n-1} - q^{n-3} + q^{-n+3} - q^{n-1} + q^{-n+1}) = \\
&= \frac{1}{q - q^{-1}}\{q^{n-1}(q^4 - 1) + q^{-n+1}(1 - q^{-4}) + q^{n+1}(1 - q^{-4}) + q^{-n+3}(1 - q^{-4})\} = \\
&= \frac{1}{q - q^{-1}}\{q^{n-1}q^2(q^2 - q^{-2}) + q^{-n+1}q^2(q^2 - q^{-2}) + q^{n+1}q^{-2}(q^2 - q^{-2}) + \\
&+ q^{-n+3}q^{-2}(q^2 - q^{-2})\} = \\
&= [2]_q^\sim\{q^{n+1} + q^{-n-1} + q^{n-1} + q^{-n+1}\} = \\
&= [2]_q^\sim\{q^n(q + q^{-1}) + q^{-n}(q^{-1} + q^1)\} = \\
&= [2]_q^\sim [2]_q^\sim (q^n + q^{-n}) = \\
&= [2]_q^\sim [2]_q^\sim [2]_{q^n}^\sim,
\end{aligned} \tag{3}$$

because

$$[2]_q^\sim = q + q^{-1}.$$

(3) is exactly the LHS of (2). ■

Multiplication By 2 For Quantum Numbers

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Abstract

We derive basic inequalities in quantum numbers under multiplication by 2.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbf{R},$$

be the 2^{nd} quantization of x, so that

$$[2]_q^{\sim} = q + q^{-1}.$$

and

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$
$$[-x]_q^{\sim} = -[x]_q^{\sim}. \tag{1}$$

Theorem 2. For $m, M \in \mathbf{Z}_{>0}$, we have:

$$[2]_q^{\sim}[x]_q^{\sim} < [2x]_q^{\sim},\quad x > 1,\quad q \neq 1, \tag{3}$$
$$[2]_q^{\sim}[x]_q^{\sim} > [2x]_q^{\sim},\quad 0 < x < 1,\quad q \neq 1. \tag{4}$$

Proof. In view of (1), we can take

$$q > 1. \tag{5}$$

Then

$$q - q^{-1} > 0, \tag{6}$$

and multiplying (3), (4) by $q - q^{-1}$, we arrive at

$$(q + q^{-1})(q^x - q^{-x}) \overset{?}{<} q^{2x} - q^{-2x},\ x > 1, \tag{7}$$
$$(q + q^{-1})(q^x - q^{-x}) \overset{?}{>} q^{2x} - q^{-2x},\ 0 < x < 1. \tag{8}$$

because $x > 0, q > 1, q^x - q^{-x} > q$ so that (7), (8) can be rewritten as

$$q + q^{-1} \overset{?}{<} q^x + q^{-x},\ x > 1, \tag{9}$$
$$q + q^{-1} \overset{?}{>} q^x + q^{-x},\ 0 < x < 1, \tag{10}$$

or as

$$q^{-x}(-1 + q^{-1+x}) \overset{?}{<} q(q^{x-1} - 1),\quad x > 1. \tag{11}$$
$$q^{-x}(-1 + q^{-(1-x)}) \overset{?}{>} q(q^{x-1} - 1)\quad 0 < x < 1. \tag{12}$$

In the case $x > 1, x - 1 > 0$, and $q^{x-1} > 1$, so (11) reduces

Modular Arithmetic In Second Quantization

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Abstract

We determine what is a quantum analog of the classical relation $a \equiv b \ \ (mod\ n)$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0, \pm 1,$$

be the second quantization of x, so that

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim}, \tag{1}$$

$$[-x]_q^{\sim} = -[x]_q. \tag{2}$$

If

$$a \equiv b \ \ (mod\ n), \quad a, b, n \in \mathbf{Z},$$

then, in general, it is *not* true that

$$[a]_q^{\sim} \equiv [b]_q^{\sim} \ \ (mod\ [n]_q^{\sim}). \tag{3}$$

Theorem 4. The relation

$$a \equiv b \ \ (mod\ n), \quad a, b, n \in \mathbf{Z}. \tag{5}$$

implies:

$$[a]_{q^2}^{\sim} \equiv [b]_{q^2}^{\sim} \ \ (mod\ [n]_q^{\sim}). \tag{6}$$

Proof. Since

$$[n]_q = (q^n - q^{-n})/(q - q^{-1}),$$

mod $[n]_q^{\sim}$ means:

$$q^{-n} \equiv q^n,$$

or

$$q^{2n} \equiv 1. \tag{7}$$

Now, let

$$a = b + kn, \quad k \in \mathbf{Z}.$$

Then

$$[a]^{\sim}_{q^2} - [b]^{\sim}_{q^2} = [b + kn]^{\sim}_{q^2} - [b]^{\sim}_{q^2} =$$
$$= \frac{1}{q^2 - q^{-2}}(q^{2(b+kn)} - q^{2b-kn} - q^{2b} + q^{-2b}) =$$
$$= \frac{1}{q^2 - q^{-2}}\left\{q^{2b}(q^{2kn} - 1) + q^{2b}(1 - q^{-2kn})\right\} \equiv 0 \quad (mod\ [n]^{\sim}_q.) \quad \blacksquare$$

Linear Relations Among Terms Of Quantum Arithmetic Progression

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Abstract

Let $a_k = a+kd$ be an arithmetic progression. It is well known that for any $n \in \mathbf{Z}_{\geq 2}$, $\sum_{k=0}^{n}(-1^k\binom{n}{k}a_k = 0)$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 1,$$

be the 2^{nd} quantization of x.

Set

$$[k]!_q^{\sim} = [1]_q^{\sim}...[k]_q^{\sim}, \ k \in \mathbf{Z}_{\geq 1}; \quad [0]!_q^{\sim} = 1,$$
$$\begin{bmatrix} m \\ k \end{bmatrix}_q^{\sim} = \frac{[m]!_q^{\sim}}{[k]!_q^{\sim}[m-k]_q^{\sim}}.$$

Theorem 1. For any $n \in \mathbf{Z}_{\geq 2}$, we have:

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{\alpha d}}^{\sim} (-1)^k [a_k] = 0, \tag{2}$$

where

$$\alpha = \begin{cases} 1, & n \text{ even} \\ 1/2, & n \text{ odd} \end{cases}$$

Proof. We are going to prove (2) by induction on n, the cases $n = 2, 3$ - the base of induction - easily done. For $n = 2$, with

$$A = q^A, \quad Q = q^d,$$
$$[a]_q^{\sim} - [2]_{q^d}^{\sim}[a+d] + [a+2d] = \frac{1}{a - q^{-1}}\Big\{(A - A^{-1}) - (Q + Q^{-1})(AQ - Q^{-1}Q^{-1}) +$$
$$+(AQ^2 - A^1Q^{-2})\Big\} = 0,$$

and for $n=3$,

$$[a]_q^{\sim}-[3]_{q^{d/2}}^{\sim}[a+d]_q^{\sim}+[3]_{q^{d/2}}^{\sim}[a+2d]_q^{\sim}-[a+3d]_q^{\sim}=$$
$$=-[2]_{q\triangle}^{\sim}[\frac{3}{2}d]_q^{\sim}+[3]_{q^{d/2}}[2]_{q\triangle}^{\sim}[\frac{d}{2}]_q^{\sim}\sim$$
$$\sim-[\frac{d}{2}]_q[3]^{q^{d/2}}+[3]_{q^{d/2}}[\frac{d}{2}]_q^{\sim}=0,$$

where

$$\triangle=a+\frac{3}{2}d.$$

We used repeatedly above the useful formula

$$[a]_q^{\sim}+[b]_q^{\sim}=[2]_{q^{(b-a)/2}}^{\sim}\left[\frac{a+b}{2}\right]_q^{\sim}.$$

The inductive step now will come by 2: $n\to n+2$. If we rewrite (2) as

$$\frac{1}{q-q^{-1}}\sum_{k=0}^{n}\left[\begin{matrix}n\\k\end{matrix}\right]_{q^{\alpha d}}^{\sim}(-1)^k(q^{a+kd}-q^{-a-kd})=0, \tag{4}$$

and differentiate w.r.t. as:

$$\frac{d}{da}(q^a)=Lq^a,\frac{d}{da}q^{-a}=-hq^{-a},h=logq,$$

we find that (4) breaks down to *two* relations:

$$\sum_{k=0}^{n}(-1)^k\left[\begin{matrix}n\\k\end{matrix}\right]_{q^{\alpha d}}^{\sim}(q^{\epsilon d})^k(-1)^k=0,\quad \epsilon=\pm1,$$

or

$$\sum_{k=0}^{n}\left[\begin{matrix}n\\k\end{matrix}\right]_{q^{\alpha}}^{\sim}(q^{\epsilon})^k(-1)^k=0,\ \epsilon=\pm1. \tag{5}$$

It is this relation we are going to prove by induction on n.

We have:

$$\left[\begin{matrix}n+2\\k\end{matrix}\right]_q^{\sim}\equiv q^{\nu_1 k}\left(\left[\begin{matrix}n+1\\k\end{matrix}\right]_q^{\sim}+\left[\begin{matrix}n+1\\k-1\end{matrix}\right]_q^{\sim}\right),\quad \nu_1=\pm1, \tag{6}$$

where

$$a\equiv b$$

means:

$$a=q^f b$$

and f is k-independent. Repeating (6) one more time, we get:

$$q^{\nu_1 k}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q^{\sim} \equiv q^{(\nu_1+\nu_2)^k}\left(\begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q^{\sim}\right),\ \nu_2 = \pm 1, \tag{7}$$

$$q^{\nu_1 k}\begin{bmatrix} n+1 \\ k-1 \end{bmatrix}_q^{\sim} \equiv q^{(\nu_1+\nu_3)^k}\left(\begin{bmatrix} n \\ k-1 \end{bmatrix}_q^{\sim} + \begin{bmatrix} n \\ k-2 \end{bmatrix}_q^{\sim}\right),\ \nu_3 = \pm 1, \tag{8}$$

Thus, (6) becomes:

$$\begin{bmatrix} n+2 \\ k \end{bmatrix}_q^{\sim} = q^{(\nu_1+\nu_2)k}\begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} + \left(q^{(\nu_1+\nu_2)k} + q^{(\nu_1+\nu_3)k}\right)\begin{bmatrix} n \\ k-1 \end{bmatrix}_q^{\sim} + q^{(\nu_1+\nu_3)^k}\begin{bmatrix} n \\ k-2 \end{bmatrix}_q^{\sim}, \tag{9}$$

and induction step $n \to n+2$ is complete if we take

$\nu_2 = \nu_3 = -\nu_?$ ■

The Leibniz Formula For The Symmetric q-Derivatives

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Abstract

The classical Leibniz formulae for multiple derivatives, $(fg)^{(n)} = \sum_{k=0}^{n} \binom{h}{k} f^{(k)} g^{(n-k)}$, where $g^{(k)} = \left(\frac{d}{dx}\right)^k (g)$, is quantized.

Let T stand for the q multiplication of the argument:

$$(T^s f)(x) = f(q^s x),\ s \in \mathbf{Z}. \tag{1}$$

Introduce the (symmetric) q-derivative:

$$\frac{df}{d_q^{\sim} x}(x) = \frac{f(Tx) - f(T^{-1}x)}{(q - q^{-1})x}, \tag{2}$$

so that

$$\frac{dx^s}{d_q^{\sim} x} = [s]_q^{\sim} x^{s-1}, \quad s \in \mathbf{R}, \tag{3}$$

where

$$[s]_q^{\sim} = \frac{q^s - q^{-s}}{q - q^{-1}}, \quad s \in \mathbf{R} \ \ (\text{or } \mathbf{C}). \tag{4}$$

Denote

$$f^{(n)} = \left(\frac{d}{d_q^{\sim} x}\right)^{(n)} (f)., \quad n \in \mathbf{Z}_{\geq 0}. \tag{5}$$

Then the usual Leibniz formula reads

$$(fg)' = f' T(g) + T^{-1}(f) g', \tag{6}$$

with f' standing for $\frac{df}{d_q^{\sim} x}$.

Theorem 7. For $n \in \mathbf{Z}_{\geq 0}$, we have:

$$(fg)^{(n)} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} T^{-k}(f^{(n-k)}) \cdot T^{n-k}(g^{(k)}) \tag{8}$$

Proof. We use induction on n, using the obvious relation

$$(T^r f)' = q^r T^r(f'),\ r \in \mathbf{Z}. \tag{9}$$

We have:

$$\begin{aligned}
&(fg)^{(n+1)} = \sum_{s=0}^{n+1} \begin{bmatrix} n+1 \\ s \end{bmatrix}_q^{\sim} T^{-s}(f^{(n+1-s)}) \cdot T^{n+1-s}(g^{(s)}) \stackrel{?}{=} \\
&= [(fg)^{(n)}]' = \left\{ \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} T^{-k}(f^{(n-k)}) \cdot T^{n-k}(g^{(k)}) \right\}' = \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} \left\{ [T^{-k}(f^{n-k})]' [T^{n+1-k}(g^{(k)})] + [T^{-k-1}(f^{(n-k)})][T^{n-k}(g^{(k)})]' \right\} = \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} \left\{ q^{-k} [T^{-k}(f^{(n+1-k)})][T^{n+1-k}(g^{(k)}) + [T^{-k-1}(f^{(n+1)-(k+1)})][q^{n-k} T^{n+1-(k+1)}(g^{(k+1)}). \right.
\end{aligned} \tag{10}$$

Thus, (8) amounts to the equality

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q^{\sim} \stackrel{?}{=} q^{-k} \begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} + q^{n-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q^{\sim}, \tag{11}$$

which is true.

The k-Tuple Conjecture Disproved

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Abstract

We prove a general formula for prime numbers from which it follows that the k-tuple conjecture is untenable.

To V. I. Arnold, the greatest mathematician of all times.

Let p_n stand for the n^{th} prime. We prove

Theorem 1. For $n, a \in \mathbf{Z}_{\geq 1}$, a fixed,

$$p_{n+a} - p_n - a\ logp_{n+a} = 0(1), \quad n \to \infty. \tag{2}$$

Proof. We use technical tools collected under one roof in Dusart's wonderful thesis, the bible of our subject, [Dus 1998]. The first of these is the asymptotic formula of Cipolla [Cip 1902]:

$$p_n = nf(n), \tag{3}$$

$$f(n) = logn + (loglogn - 1) + \sum_{i=1}^{\infty} \frac{P_i}{log^i n} = \tag{4a}$$

$$= logn + \sum_{i=0}^{\infty} \frac{P_i}{log^i n}, \quad P_0 = loglogn - 1, \tag{4b}$$

where P_i is a polynomial in $w = loglogn$ of degree $i + \delta_0^i$.

So,

$$p_{n+a} - p_n = (n+a)f(n+a) - nf(n) = af(n+a) + \tag{5a}$$

$$+n\{f(n+a) - f(n)\}. \tag{5b}$$

Using the relations (up to higher order irrelevant $0(1/n)$ terms):

$$log(n+a) - logn = \frac{a}{n}, \tag{6a}$$

$$loglog(n+a) - loglogn = \frac{a}{nlogn}, \tag{6b}$$

we have, with $\tilde{P} := P(n+a)$:

$$\frac{\tilde{P}_i}{log^i(n+a)} - \frac{P_i}{log^i n} = \frac{1}{log^i(n+a)log^i n}\{\tilde{P}_i log^i(n) - P_i log^i(n+a)\}, \tag{7}$$

and

$$\begin{aligned}
&\tilde{P}_i log^i n - P_i log^i(n+a) = \\
&= (P_i + P_i' \frac{a}{nlogn})log^i n - P_i(logn + \frac{a}{n})^i = \\
&= P' \frac{a}{n} log^{i-1} n + P_i\{log^i n - [log^i n + \frac{ialog^{i-1}n}{n}]\} = \\
&\frac{P' - iP_i'}{n} a log^{i-1} n.
\end{aligned}$$

Hence, (7) is:

$$\begin{aligned}
&\frac{\tilde{P}_i}{log^i(n+a)} - \frac{P_i}{log^i n} = \frac{(P' - iP_i)alog^{i-1}n}{nlog^{2i}n} = \\
&= \frac{P' - iP_i}{nlog^{i+1}n} a.
\end{aligned} \tag{8}$$

Thus, (5b) is

$$n\{\frac{a}{n} + a\sum_{i\geq 0} \frac{P' - iP_i}{nlog^{i+1}(n)}\} = a + 0(1). \tag{9}$$

Next, the term $(5a)$, $a[f(n+a) - logp_{n+a}]$:

$$\begin{aligned}
&f(n) - logp_n = \\
&= logn + (loglogn - 1) + 0(1) - \{logn + log[logn + \sum_{i\geq 0} \frac{P_i}{log^i n}]\} = \\
&= loglogn - 1 - log\{logn[1 + 0(1)]\} = \\
&= loglogn - 1 - loglogn + 0(1) = -1 + 0(1).
\end{aligned} \tag{10}$$

Thus,

$$af(n+a) - alogp_{n+a} = a(-1) + 0(1). \tag{11}$$

Combining this with (9), we get (2). ■

Corollary 12. The diophantine equation

$$p_{n+a} - p_n = 2b, \tag{13}$$

for fixed $a, b \in \mathbf{Z}_{\geq 1}$, has only a *finite* number of solutions.

Proof. By (2),

$$2b - alogp_{n+a} = 0(1), \tag{14}$$

and $logp_{n+a} \to \infty$ as $n \to \infty$. ■

This disproves the k-tuple Conjecture of Hardy-Littlewood, just as the case $a = 1$ had disproved earlier the twin prime Conjecture.

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On Cubic Quantum Representation Of Integers Divisible By 8

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Abstract

The banal classical equality, $8m = 6m + 2m$, is non-banally quantized.

Let m, n be positive integers, and

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}$$

is the second quantization of x.

Theorem 1.

$$([2]_q^{\sim})^2 [2m]_{q^n}^{\sim} = \sum_{i=0}^{m-1} \{[A_i + 3]_q^{\sim} - [A_i - 3]_q^{\sim} + [A_i + 1]_q^{\sim} - [A_i - 1]_q^{\sim}\}, \tag{1}$$

where

$$A_i = (2m - 1 - 2i)n. \tag{2}$$

Proof. Using the easily verifiable

$$[A + 1]_q^{\sim} - [A - 1]_q^{\sim} = [2]_{q^A}^{\sim}, \tag{3}$$

$$[A + 3]_q^{\sim} - [A - 3]_q^{\sim} = [2]_{q^A}^{\sim} [3]_q^{\sim}, \tag{4}$$

the RHS of (1) becomes

$$([3]_q^{\sim} + 1) \sum_{i=0}^{m-1} [2]_{q^{A/i}}^{\sim}, \tag{5}$$

with $A(i) = A_i$. But

$$[3]_q^{\sim} + 1 = ([2]_q^{\sim})^2, \tag{6}$$

so (7) reduces to:

$$[2m]_{q^n}^{\sim} \stackrel{?}{=} \sum_{i=0}^{m-1} [2]_{q^{(2m-1-2i)n}}, \tag{7}$$

or

$$[2m]^{]} sim_q \stackrel{?}{=} \sum_{i=0}^{m-1} [2]^{\sim}_{q^{2m-1-2i}}. \tag{8}$$

Now, the LHS of (8) is

$$\frac{q^{2m} - q^{-2m}}{q - q^{-1}}, \tag{9}$$

while the RHS of (8), because

$$[2]^{\sim}_q = q + q^{-1}, \tag{10}$$

becomes:

$$\begin{aligned} &\sum_{i=0}^{m-1} (q^{2m-1-2i} + q^{1+2i-2m}) - a \\ &= q^{2m-1}[m]_{q^{-2}} + q^{1-2m}[m]_{q^2}, \end{aligned} \tag{11}$$

where

$$[x]_q = \frac{q^x - 1}{q - 1}$$

is the 1^{st} quantized x. Thus, (11) is:

$$\begin{aligned} &q^{2m-1}\frac{1 - q^{-2m}}{1 - q^{-2}} + q^{1-2m}\frac{q^{2m} - 1}{q^2 - 1} = \\ &= q^{-1}\frac{q^{2m-1}}{q^{-1}(q - q^{-1})} + q\frac{1 - q^{-2m}}{q(q - q^{-1})} = \\ &= \frac{1}{q - q^{-1}}(q^{2m} - q^{-2m}), \end{aligned} \tag{12}$$

and this is exactly (9). ■

Infinitesimal Content Of Quantum Formulae

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Abstract

Not every classical relation can or has been quantized. When it can, it lead to the infinite hierarchy of higher formulas, of which the original one seats at the bottom.

Consider Quantum formulae of the form

$$A = B,$$

where A and B are sums of simple quantum integers, that is, sums of

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}$$

in the 2^{nd} quantization, or

$$[x]_q = \frac{q^x - 1}{q - 1}$$

in the first one, with x taking *integer* values. A typical example is:

$$[a]_q^{\sim} + ... + [a + nd]_q^{\sim} = [n+1]_{q^{1/2}}^{\sim}[a + \frac{n+1}{2}d]_q^{\sim}, \tag{1}$$

a formula for the sum of the quantum arithmetic progression.

Of course, for $q = 1$, we recover the usual school formula

$$a + ... + (a + nd) = (n+1)(a + \frac{n+1}{2}d), \tag{2}$$

but the point is: (1) is (specific) a *deformation*, so that we get an infinite series of higher formulae by considering both parts of (1) as two ??? in $h = log\ q$ and equating the like-terms; specifically, the h-terms are of mail interest.

To proceed further, we need the formulae

$$[x]_q^{\sim} = x + h^2 \binom{x+1}{3} + O(h^4), h = logq \leftrightarrow q = e^h. \tag{3}$$

$$[x]_q = x + h\binom{x}{2} + O(h^2). \tag{4}$$

Let's derive them. For (3), we have:

$$[x]_q = \frac{e^{hx} - e^{-hx}}{e^h - e^{-h}} = \frac{2h(x + \frac{h^2}{6}x^3)}{2h(1 + \frac{h^2}{6})} + O(h^4) = (x + \frac{h^2}{6}x^3)(1 - \frac{h^2}{6}) + O(h^4) =$$

$$= x + \frac{h^2}{6}(x^3 - x) + O(h^3) = x + h^2\binom{x+1}{3} + O(h^4)$$

Similarly, for (4) we have

$$[x]_q = \frac{e^{hx-1}}{e^h-1} = \frac{h(x+\frac{h}{2}x^2)}{h(1+\frac{h}{2})} + O(h^2) = (x+\frac{h}{2}x^2)(1-\frac{h}{2}) + O(h^2) =$$
$$= x + \frac{h}{2}(x^2-x) + O(h^2) = x + h\binom{h}{2} + O(h^2)$$

formula (4) is much simpler than (3); but most of the known quantum formulae are in the second quantization form.

Let's see how this device works in the case (1) of quantum arithmetic progression. For the LHS we get:

$$\sum_{k=0}^{n}[a+kd]_q^{\sim} = \sum_{k=0}^{n}\{a+dk+h^2\binom{a+1+kd}{3}\} = (n+1)(a+\frac{n+1}{2}d)+h^2\sum_{k=0}^{n}\binom{a+1+kd}{3}+O(h^2), \tag{15}$$

while for the RHS we find:

$$[(n+1)+h^2\binom{n-2}{3}][(a+\frac{n+1}{2}d)+h^2\binom{a+1+\frac{nd}{2}}{3}]+O(h^2) = (n+1)(a+\frac{n+1}{2})d+$$
$$+h^2\{(a+\frac{n+1}{2}d)\binom{n+2}{3}+(n+1)\binom{a+1+\frac{nd}{3}}{3}\}+O(h^2).$$

Thus, we get:

$$\sum_{k=0}^{n}\binom{a+1+kd}{3} = \left(a+\frac{n+1}{2}d\right)\binom{n+2}{3}+(n+1)\binom{a+1+n\frac{d}{2}}{3} \tag{6}$$

a certainly new (and rather strange) formula.

Similar device can be applied to the multitude of other known quantum formulae.

Quantum Exponential Function

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Abstract

We quantize the usual exponential function.

The function

$$E(ax) = exp(ax) = \sum_{n=0}^{\infty} \frac{a^n x^n}{n!}. \tag{1}$$

satisfies

$$\frac{d}{dx}E(ax) = aE(ax). \tag{2}$$

Replacing $\frac{d}{dx}$ by its symmetrical quantum counterpart $\frac{d}{d_q^{\sim}x}$:

$$\frac{df}{d_q^{\sim}x} = \frac{f(qx) - f(q^{-1}x)}{q - q^{-1}},\ x \neq 0, \tag{3}$$

so that

$$\frac{d}{d_q^{\sim}x}(x^s) = [s]_q^{\sim} x^{s-1}, \quad s \in \mathbf{R} \text{ (or } \mathbf{C}), \tag{4}$$

$$[s]_q^{\sim} = \frac{q^s - q^{-s}}{q - q^{-1}}\ . \tag{5}$$

the corresponding q-version of $exp(ax)$ becomes:

$$E(ax, q) = \sum_{n=0}^{\infty} \frac{a^n x^n}{[n]!_q^{\sim}}. \tag{6}$$

Here

$$[k]!_q^{\sim} = [1]_q^{\sim}...[k]_q^{\sim}, k \in \mathbf{Z}_{\geq 1},\ [0]!_q^{\sim} = 1. \tag{7}$$

Indeed,

$$\frac{d}{d_q^{\sim}x}[E(ax)] = \sum_{n=1}^{\infty} \frac{a^n x^{n-1}}{[n-1]!} = aE(ax, q), \tag{8}$$

the desired q-analog of the classical relation (2).

Divisibility By A Product In Quantum Domain

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Abstract

If $a \equiv 0(mod\ n_i), i = 1, 2,$ and $(n_1, n_2) = 1$ then $a \equiv 0(mod\ n_1 n_2)$. We find a quantum version of this

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0, \pm 1,$$

be the second quantization of x.

We are going to use the easily verifiable fact that

$$a \equiv b(mod\ n) \ \Leftrightarrow \ [a]_{q^2}^{\sim} \equiv [b]_{q^2}^{\sim}(mod\ [n]_q^{\sim}),\ a, b, n \in \mathbf{Z}. \tag{1}$$

Theorem 2. If

$$[a]_{q^2}^{\sim} \equiv 0(mod\ [n_i]_q^{\sim}), \quad i = 1, 2, \tag{3}$$

$$(n_1, n_2) = 1, \tag{4}$$

then

$$[a]_{q^2}^{\sim} \equiv 0(mod\ [n_1 n_2]_q^{\sim}). \tag{5}$$

Proof. At $q = 1$,

$$a \equiv 0(mod\ n_1 n_2).$$

Using (1) finishes the Proof. ■

Corollary 6. If n_1 and n_2 are coprime then

$$\frac{[n_1 n_2]_{q^2}^{\sim}}{[n]_q^{\sim}[n_2]_q^{\sim}}$$

is an (quantum) integer.

Directly, this fact is rather difficult to prove.

Decomposition Of Quantum Analogs Of $m2^n$

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Abstract

$\boldsymbol{m2^n}$ is decomposed quantumly, for n, m nonnegative integers.

Let,

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},$$

so that

$$[2]_q^{\sim} = q + q^{-1}.$$

Theorem 1. For $m \in \mathbf{Z},\ n \in \mathbf{Z}_{\geq 0}$, we have:

$$[m]_q^{\sim}([2]_q^{\sim})^n = \sum_{k=0}^{n} \binom{n}{k} [n + m - 2k]_q^{\sim}. \tag{2}$$

Proof. The RHS of (2) is:

$$\begin{aligned}
&\frac{1}{q - q^{-1}} \left\{ q^{n+m} \sum_{k=0}^{n} q^{-2k} \binom{n}{k} - q^{-n-m} \sum_{k=0}^{n} q^{2k} \binom{n}{k} \right\} = \\
&= \frac{1}{q - q^{-1}} \left\{ q^{n+m}(1 + q^{-2})^n - q^{-n-m}(1 + q^2)^n \right\} = \\
&= \frac{1}{q - q^{-1}} \left\{ q^{n+m} q^{-n} (q + q^{-1})^n - q^{-n-m} q^n (q^{-1} + q)^n \right\} = \\
&= ([2]_q^{\sim})^n \frac{1}{q - q^{-1}} (q^m - q^{-m}) = ([2]_q^{\sim})^n [m]_q^{\sim},
\end{aligned}$$

which is exactly the LHS of (2). ■

Decomposition Of Numbers Divisible by Four, Quantized

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Abstract

We decompose quantum version of the numbers of the form $4m$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}$$

be the second quantized version of x, so that

$$[2]_q^{\sim} = q + q^{-1}.$$

Theorem 1. For every $m \in \mathbf{Z}$, we have

$$([2]_q^{\sim})^2 [m]_q^{\sim} = [m-2]_q^{\sim} + 2[m]_q^{\sim} + [m+2]_q^{\sim}. \tag{2}$$

Proof. We are going to use the easy formula

$$[a]_q^{\sim} + [a+2]_q^{\sim} = [2]_q^{\sim} [a+1]_q^{\sim}. \tag{3}$$

Thus,

$$[m-2]_q^{\sim} + [m]_q^{\sim} = [2]_q^{\sim} [m-1]_q^{\sim}, \tag{4}$$
$$[m]_q^{\sim} + [m+2]_q^{\sim} = [2]_q^{\sim} [m+1]_q^{\sim}. \tag{5}$$

Adding up (4) and (5), the RHS of (2) becomes:

$$[2]_q^{\sim} \Big\{ [m-1]_q^{\sim} + [m+1]_q^{\sim} \Big\} = [2]_q^{\sim} [2]_q^{\sim} [m]_q,$$

which is the LHS of (2). ■

Decomposition Of A Quantum Version Of $3 \cdot 2^n$

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Abstract

$3 \cdot 2^n$ is decomposed quantumly.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}$$

be the (2^{nd}) quantum version of x, so that for $n \in \mathbf{Z}_{\geq 1}$,

$$[n]_q^{\sim} = \sum_{i=0}^{n-1} q^{n-1-2i}. \tag{1}$$

Theorem 2. For every $n \in \mathbf{Z}_{\geq 0}$, we have

$$[3]_q^{\sim}([2]_q^{\sim})^n = \sum_{k=0}^{n} \binom{n}{k} [n+3-2k]_q^{\sim}. \tag{3}$$

Proof. The RHS of (3) is:

$$\frac{1}{q-q^{-1}} \sum_{k=0}^{n} \left\{ q^{n+3} \binom{n}{k} q^{-2k} - q^{-n-3} \binom{n}{k} q^{2k} \right\} =$$
$$= \frac{1}{q-q^{-1}} \left\{ q^{n+3}(1+q^{-2})^n - q^{-n-3}(1+q^2)^n \right\} =$$
$$= \frac{1}{q-q^{-1}} \left\{ q^{n+3} q^{-n} (q+q^{-1})^n - q^{-n-3} q^n (q^{-1}+q)^n \right\} =$$
$$= (q+q^{-1})^n \frac{1}{q-q^{-1}} \left\{ q^3 - q^{-3} \right\} = \left([2]_q^{\sim} \right)^n [3]_q^{\sim},$$

which is exactly the LHS of (3). We used the obvious fact that

$$[2]_q^{\sim} = q + q^{-1}. \quad \blacksquare$$

Cubic Two In Second Quantization

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Abstract

We decompose $([2]_q^\sim)^2[2]_{q^n}^\sim$ into a sum of 4 quantum integers.

Theorem 1. For $n \in \mathbf{Z}_{\geq 3}$, we have:

$$([2]_q^\sim)^2[2]_{q^n}^\sim = [n+3]_q^\sim + [n+1]_q^\sim - ([n-3]_q^\sim + [n-1]_q^\sim), \tag{1}$$

where

$$[x]_q^\sim = \frac{q^x - q^{-x}}{q - q^{-1}}$$

is the 2^{nd} (symmetric under exchange $q \to q^{-1}$) quantization.

Proof. Using the obvious formula

$$[a]_q^\sim + [b]_q^\sim = [2]_{q^{(b-a)/2}}^\sim \left[\frac{a+b}{2}\right]_q^\sim, \tag{2}$$

we have:

$$[n+a]_q^\sim - [n-a]_q^\sim = (n+a)_q^\sim + [a-n]_q^\sim = [2]_{q^n}^\sim [a]_q^\sim. \tag{3}$$

For $a = 3, 1$, we add up these formulae, and for the RHS of (1) get:

$$RHS = [2]_{q^n}^\sim \{[3]_q^\sim + 1\} = [2]_{q^n}[q^2 + 1 + q^{-2} + 1] = [2]_{q^n}^\sim ([2]_q^\sim)^2,$$

which is the LHS. ■

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Fibonacci Triangle In The 1^{st} Quantization

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Abstract

Consider the sums of rows of the Fibonacci triangle: $??1 = 1^3, 8 = 2^3, 27 = 3^3$, ... This is quantized.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the 1^{st} quantization of x.

$$\begin{array}{ccccc} & & [1]_q & & \\ & q[3]_q & & [5]_q & \\ q^2[7]_q & & q[q]_9 & & [11]_q \end{array}$$

Theorem 2.

$$\sum_{k=0}^{n-1} q^k [n^2 + n - 1 - 2k]_q = [n^2]_q [n]_q. \tag{3}$$

Proof. (3) is:

$$(q^n - 1) \sum_{k=0}^{n-1} q^k (q^{n^2+n-1-2k} - 1) \stackrel{?}{=} (q^{n^2} - 1)(q^n - 1),$$

or

$$(q - 1) \sum_{k=0}^{n-1} (q^{n^2+n-1-k} - q^k) \stackrel{?}{=} (q^{n^2-1})(q^n - 1),$$

or

$$(q - 1)(q^{n^2-1}) \sum_{k=0}^{n-1} q^k \stackrel{?}{=} (q^{n^2-1})(q^n - 1),$$

or

$$\sum_{k=0}^{n-1} q^k \stackrel{?}{=} \frac{q^n - 1}{q - 1} = [n]_q,$$

which is true. ■

A Semiclassical Relation In The 1^{st} Quantization

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Abstract

That relation has no classical analog.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the 1^{st} quantization of x.

Theorem 1.

$$\sum_{k=0}^{n} q^{-k}[k]_q = q^{-n}\frac{d}{d_q}([n+1]_q). \tag{2}$$

Proof. We use induction on n. For $n = 0$, (2) returns:

$$0 = 0$$

which is true.

The induction step $n \mapsto n+1$ amounts to:

$$q^{-n}\frac{d}{dq}([n+1]_q) + q^{-n-1}[n+1]_q \stackrel{?}{=} q^{-n-1}\frac{d}{dq}[n+2]_q,$$

or, with $z = (n+1)$

$$q\frac{d}{dq}[z]_q + [z]_q \stackrel{?}{=} \frac{d}{dq}([z+1]_q),$$

or, because

$$[z+1]_q = 1 + q[z]_q,$$

to

$$q\frac{d}{dq}[z]_q + [z]_q \stackrel{?}{=} \frac{d}{dq}(1 + q[z]_q) -$$
$$= [z]_q + q\frac{d}{d}([z]_q),$$

which is true. ■

An Estimate For A Linear Quantum Function

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Abstract

The classical inequality $k(1+a^{2k})-a^k \geq a^{2k-1}+...+a, a>0, k\geq 1$, is generalized and made quantum.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q-q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Dividing the inequality quoted in the abstract by a^k, and renaming a by q, we get:

$$k[2]_{q^k}^2 - 1 \geq [2k-1]_{q^{1/2}}^{\sim},$$

or

$$k[2]_{q^{2k}}^{\sim} \geq 1 + [2k-1]_q^{\sim}.$$

Theorem 1. For $x \geq 0$,

$$1 + [2x-1]_q^{\sim} \leq x[2]_{q^{2x}}^{\sim}. \tag{2}$$

Proof. Multiplying (2) by $(q-q^{-1}) > 0$, we get

$$(q-q^{-1}) + (q^{2x-1} - q^{1-2x}) \stackrel{?}{\leq} x[(q-q^{-1})(q^{2x-1}+q^{-2x})],$$

or

$$(q-q^{-1})+(q^{2x-1}-q^{1-2x}) \stackrel{?}{\leq} [(q^{2x+1}-q^{-2x-1})-(q^{2x-1}-q^{1-2x})],$$

or

$$(q-q^{-1})+(x+1)(q^{2x-1}-q^{1-2x}) \stackrel{?}{\leq} x(q^{2x+1}-q^{-2x-1}). \tag{3}$$

Set

$$h = logq > 0.$$

Then (3) is

$$2\sum_{n=0}^{\infty}\frac{h^{2n+1}}{(2n+1)!}\{1+(x+1)(2x-1)^{2n+1}\} \stackrel{?}{\leq} x2\sum_{n=0}^{\infty}\frac{h^{2n+1}}{(2n+1)!}(2x+1)^{2n+1},\ x\geq 0. \tag{4}$$

(4) follows from

$$1+(x+1)(2x-1)^{2n+1} \stackrel{?}{\leq} x(2x+1)^{2n+1}, x\geq 0, \quad n\in \mathbf{Z}_{\geq 0}. \tag{5}$$

We prove (5) by induction on n. For $n=0$, (15) returns:

$$1+(x+1)(2x-1) \stackrel{?}{\leq} x(2x+1),$$

or

$$1+(2x^2+x-1)\leq (2x^2+x),$$

which is true.

Now, the induction step. Assuming (5) is true for a given n, the next $n+1$ follows from:

$$x(2x+1)^{2n+3} \geq (2x+1)^2[1+(x+1)(2x-1)^{2n+1}] \stackrel{?}{\geq} 1+(x+1)(2x-1)^{2n+3},$$

or

$$(2x+1)^2+(x+1)(2x+1)^2(2x-1)^{2n+1} \stackrel{?}{\geq} 1+(x+1)(2x-1)^{2n+3},$$

which is obvious because $x\geq 0$:

$$(2x+1)^2\geq 1$$

and

$$(2x+1)^2(2x-1)^{2n+1}\geq (2x-1)^{2n+3} \ \Leftrightarrow\ (2x+1)^2\geq (2x-1)^2. \quad \blacksquare$$

On Quantum Linear Functions

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Abstract

If $f(x) = ax + b$, then $x_0 + x_2 = 2x, = 2f(x_0) + f(x_2)$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Let a, b be constants,

$$f(x) = [ax + b]_q^{\sim}. \tag{1}$$

Given three points, x_0, x_1, x_2, such that

$$x_1 = x_0 + d, x_2 = x_0 + 2d,$$

we get the corresponding values

$$f_0 = f(x_0), f_1 = f(x_1), f_2 = f(x_2).$$

Theorem 1.

$$f_0 + f_2 = [2]_{q^d}^{\sim} f_1. \tag{2}$$

Proof. Multiply (2) by $(q - q^{-1})$. We get:

$$(q^{x_0} - q^{-x_0}) + (q^{x_0 2d} - q^{-x_0 - 2d}) \stackrel{?}{=} (q^d + q^{-d})(q^{x_0 + d_0} - q^{-x_0 - d}) =$$
$$= q^{x_0 + 2d} q^{-x_0} + q^{x_0} - q^{-x_0 - 2d},$$

which is true. ■

On The Ratio Of Two Quantum Linear Functions

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Abstract

Classically, $\frac{\beta x+1}{x+1} < \beta$ for $\beta > 1$. We quantize this.

Let

$$[x]_q^\sim = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Since

$$[x]_{q^{-1}}^\sim = [x]_q^\sim,$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^\sim = [x]_q^\sim,$$

we can take

$$q > 1$$

without loss of generality.

Lemma 1. Let $\beta > 1$. Then

$$\frac{[\beta x + 1]_q^\sim}{[x+1]_q^\sim} < [\beta]_{q^{x+1}}^\sim. \tag{2}$$

Proof. Since $\beta > 1$,

$$[\beta x + 1]_q^\sim < [\beta x + \beta]_q^\sim, \tag{3}$$

because, in general,

$$a < b \Rightarrow [a]_q^\sim < [b]_q^\sim. \tag{4}$$

Thus,

$$\frac{[\beta x + 1]_q^\sim}{[x+1]_q^\sim} < \frac{[\beta(x+1)]_q^\sim}{[x+1]_q^\sim} = [\beta]_{q^{x+1}}^\sim,$$

and (2) is proven. ■

Corollary 5.

$$\begin{bmatrix} 2n \\ n \end{bmatrix}_q^{\sim} = \frac{[2n]!_q^{\sim}}{([n]!_q^{\sim})^2} < (< 2^n >_q^{-2})^2,\ n \in \mathbf{Z}_{\geq 2}, \tag{6}$$

where

$$[k]!_q^{\sim} = \prod_{s=1}^{k} [\gamma]_q^{\sim},\ k \in \mathbf{Z}_{\geq 1};\ [0]!_q^{\sim} = 1,$$
$$< x >_q^{\sim} = \prod_{s=1}^{n} [x]_{q^s}^{\sim},\ n \in \mathbf{Z}_{\geq 1}; < x^0 >_q^{\sim} = 1.$$

Proof. We use induction on n. For $n = 0, 1$ (6) returns equalities $1 = 1$ and $[2]_q^{\sim} = [2]_q^{\sim}$. For $n = 2$, we get:

$$\frac{[4]_q^{\sim}}{[2]_q^{\sim}[2]_q^{\sim}} = \frac{[2]_{q^2}^{\sim}}{[2]_q^{\sim}} \stackrel{?}{<} [2]_q^{\sim}[2]_{q^2}^{\sim},$$

or

$$1 < ([2]_q^{\sim})^2,$$

which is obvious. The inductive step $n \mapsto n+1$ amounts to:

$$(< 2^n >_q^{\sim})^2 \frac{[2n+2]_q^{\sim}[2n+1]_q^{\sim}}{([n+1]_q^{\sim})^2} \stackrel{?}{<} (< 2^{n+1} >_q^{\sim})^2,$$

or

$$\frac{[2m+2]_q^{\sim}[2n+1]_q^{\sim}}{[n+1]_q^{\sim}[n+1]_q^{\sim}} \stackrel{?}{<} ([2]_{q^{n+1}}^{\sim})^2,$$

or

$$\frac{[2n+1]_q^{\sim}}{[n+1]_q^{\sim}} \stackrel{?}{<} [2]_{q^{n+1}}^{\sim}. \tag{7}$$

But this is our Lemma for $\beta = 2$. ■

A Power Inequality Quantized

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Abstract

For $0 < x < 1, (1-x)^n + (1+x)^n \leq 2^n, n \in \mathbf{Z}_{\geq p}$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Theorem 1. Let $0 < x < 1$,

$$([1-x]_q^{\sim})^n + ([1+x]_q^{\sim})^n \leq ([2]_q^{\sim})^n. \tag{2}$$

Proof. We use induction on n. For $n = 1$, (2) returns:

$$(1-x)_q^{\sim} + [1+x]_q^{\sim} \stackrel{?}{\leq} [2]_q^{\sim}. \tag{3}$$

Multiplying this by $q - q^{-1} > 0$, we get:

$$(q^{1-x} - q^{x-1}) + (q^{1+x} - q^{-x-1} \stackrel{?}{\leq} q^2 - q^{-2},$$

or

$$q^x(q - q^{-1}) + q^{-x}(q - q^{-1}) \stackrel{?}{\leq} (q - q^{-1})(q + q^{-1}),$$

or

$$q^x + q^{-x} \overset{?}{\leq} q^1 + q^{-1},$$

which is true because $x < 1$ and

$$q^x + q^{-x}$$

is an *increasing* function of x.

Now, the induction step $n \mapsto n+1$, amounts to:

$$([2]_q^{\sim})^{n+1} = ([2]_q^{\sim})^n [2]_q^{\sim} \geq [2]_q^{\sim} \{([1-x]_q^{\sim})^n + [(1+x)_q^{\sim}])^n \overset{?}{\geq}\} \overset{?}{\geq} ([1-x]_q^{\sim})^{n+1} + ([1+x]_q^{\sim})^{n+1},$$

which is true because

$$[2]_q^{\sim} > [1-x]_q^{\sim}$$

and

$$[2]_q^{\sim} > [1+x]_q^{\sim},$$

since

$$2 > 1 - x$$

and

$$2 > 1 + x$$

x being < 1, and, in general,

$$a < b \Rightarrow [a]_q^{\sim} < [b]_q^{\sim}. \quad \blacksquare$$

On The Relation Between A Number And Inverse Of It In Quantum Domain

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Abstract

Classically, $x\frac{1}{x}=1$. The situation is more complex quantumly.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}}, x\in\mathbf{R},\ q\neq 0,\ \pm 1,$$

be the second quantization of x.

Since

$$[x]_{q^{-1}}^{\sim}=[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim}=[x]_q^{\sim},$$

we can take

$$q>1$$

without loss of generality.

Theorem 1. Let $x\neq 0$. Then

$$[x]_q^{\sim}\left[\frac{1}{x}\right]_q^{\sim}\geq 1. \tag{2}$$

Proof. Since

$$[-x]_q^{\sim}\left[\frac{1}{-x}\right]_q^{\sim}=[x]_q^{\sim}\left[\frac{1}{x}\right]_q^{\sim},$$

we can take $x>0$. And since

$$[x]_q^{\sim}\left[\frac{1}{x}\right]_q^{\sim}$$

is invariant with respect to the change $x \to x^{-1}$, we can take

$$x > 1,$$

the ease

$$x = 1$$

being obviously true.

Multiplying through (2) by $(q - q^{-1})^2 > 0$, we get:

$$(q^x - q^{-x})(q^{1/x} - q^{-1/x}) \stackrel{?}{\geq} (q - q^{-1})^2,$$

or

$$(q^{x+1/x} + q^{-x-1/x}) - (q^{x-1/x} - q^{1/x-x}) \stackrel{?}{>} q^x + q^{-2} - 2. \tag{3}$$

Set

$$h = log\ q > 0.$$

Then (3) becomes:

$$2\sum_{??}^{\infty} \frac{h^{2n}}{(2n)!}[(x + \frac{1}{x})^{2n} - (x - \frac{1}{x})^{2n}] \stackrel{?}{\geq} 2\sum_{n=??}^{\infty} \frac{h^{2n}}{(2n)!}2^{2n}. \tag{4}$$

Since $h > 0$, (4) follows from:

$$(x + \frac{1}{x})^{2n} - (x - \frac{1}{x})^{2n} \stackrel{?}{\geq} 2^{2n},\ n \geq 1,$$

or

$$2\sum_{k\ even} \binom{2n}{k} x^{2n-k} \left(\frac{1}{x}\right)^k \geq 2^{2n},$$

or

$$\sum_{k\ even} \binom{2n}{k} x^{2(n-k)} \stackrel{?}{\geq} q^{2n-1},$$

or

$$\sum_{k\ even} \binom{2n}{n} y^{n-k} \stackrel{?}{\geq} 2^{2n-1} m, y \geq 1,\ n \geq 1. \tag{5}$$

If n is odd, (5) is obvious. Since $y > 1$, $y^{n-k} + y^{k-n} \geq 2$, and (5) becomes:

$$2\sum_{k\ even\ <n} \binom{2n}{k} \stackrel{?}{\geq} 2^{2n-1},$$

or

$$\sum_{k\ even} \binom{2n}{k} \geq 2^{2n-1},$$

and

$$\sum_{k\ even} \binom{m}{k} = \sum_{k\ odd} \binom{m}{k} = 2^m/2. \tag{6}$$

For n even, $n = 2mg$ (5) is bounded, by the same argument, by

$$2 \sum_{k\ even<2m} \binom{4m}{k} + \binom{4m}{2m} \stackrel{?}{\geq} 2^{4m-1}$$

or

$$\sum_{k\ even} \binom{4m}{k} \stackrel{?}{\geq} 2^{4m-1},$$

which is true by (6). ■

A Simple Inequality Between Quantum Numbers

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Abstract

We prove a simple inequality between quantum numbers.

Let

$$[x]_q = \frac{q^x - 1}{q - 1},\ q \neq 0, 1,$$

be the 1^{st} quantized x.

Theorem 1. Let $q \geq 1, y > 0$. Then

$$y[2]_{q^{y+1}} \geq 2q[y]_q. \tag{2}$$

Proof. In the long-hand, (2) is:

$$y(q^{y+1} + 1) \stackrel{?}{\geq} 2q\frac{q^y - 1}{q - 1},$$

or, since $q > 1$,

$$y(q - 1)(q^{y+1} + 1) \stackrel{?}{\geq} 2q(q^y - 1),$$

or

$$y\{q^{y+2} + q - q^{y+1} - 1\} \stackrel{?}{\geq} 2(q^{y+1} - q),$$

or

$$yq^{y+2} - (y - 2)q^{y+1} + (y - 2)q \stackrel{?}{\geq} y. \tag{3}$$

At $q = 1$, (3) is an equality, and $\partial/\partial q$ of it is:

$$y(y + 2)q^{y+1} - (y - 2)(y + 1)q^y + (y - 2) \stackrel{?}{\geq} 0 \tag{4}$$

(4) follows from:

$$q^y[y(y + 2) - (y - 2)(y + 1)] + (y - 2) \stackrel{?}{\geq} 0, \tag{5}$$

or

$$q^y[(y^2+2y)-(y^2-y^{-2})]+(y-2)\stackrel{?}{\geq}0,$$

or

$$q^y(3y+2)+(y-2)\stackrel{?}{\geq}0. \tag{6}$$

Since $q>1,y$, (6) follows from

$$(3y+2)+(y-2)\stackrel{?}{\geq}0,$$

or

$$4y\stackrel{?}{\geq}0,$$

which is obvious. ■

On The Property Of Medians Under Quantization

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Abstract

The classical property of mediants is *not* preserved under quantization.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Let F_3 be a Favery series of order 3:

$$\frac{0}{1} < \frac{1}{3} < \frac{1}{2} < \frac{2}{3} < \frac{1}{1}. \tag{1}$$

Consider the pair of fractions:

$$\frac{1}{3} < \frac{1}{2}. \tag{2}$$

Their mediant is:

$$\frac{1}{3} < \frac{2}{5} < \frac{1}{2} \tag{3}$$

Quantumly,

$$\frac{1}{[3]_q^{\sim}} < \frac{1}{[2]_q^{\sim}},$$

because

$$[2]_q^{\sim} < [2]_q^{\sim},$$

since

$$2 < 3.$$

Now, quantumly, it is *not*

$$\frac{1}{[3]_q^{\sim}} < \frac{[2]_q^{\sim}}{[5]_q},$$

because it is not true that

$$[5]_q^{\sim} < [2]_q^{\sim}[3]_q^{\sim},$$

since

$$[5]_q^{\sim} \sim q^4,$$

and

$$[2]_q^{\sim}[3]_q^{\sim} \sim q^3,$$

as $q \to \infty$. ■

On A Sum Of Two Quantum Arithmetic Progressions

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Abstract

If $\{ak\}$ and $\{bk\}$ are two arithmetic progressions, then so is their sum $\{ck = ak + bk\}$. It is no longer true in quantum case, but close enough.

Let

$$[x]_q^\sim = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the 2^{nd} quantization of x.

Let $a_k = a + nd,\ b_k = b + kd'$, be two arithmetic progressions, and

$$A_k = [a_k]_q^\sim,\ B_k = [b_k]_q^\sim, \tag{1}$$

be their quantum counterparts, so

$$D_k + D_{k+2} = [2]_q^\sim D_{k+1}, \quad D = A \text{ or} B. \tag{2}$$

Then their sum,

$$X_k = A_k + B_r,$$

no longer satisfies (2), but something close does.

Theorem 3.. Set

$$C_k = q^{b_k} A_k + q^{-a_k} B_k. \tag{4}$$

Then C_k satisfies (2).

Proof. We have:

$$C_k = [c_k]_q^\sim,$$
$$c_n = a_k + b_k,$$

because, in general,

$$[a + b]_q^\sim = q^b [a]_q^\sim + q^{-a} [b]_q^\sim,$$

and this is formula (4).

Since c_k is a classical arithemtic progression, C_k is a quantum one, and therefore satisfies (2). ■

Transformations Of A 3-Term Arithmetic Progression

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Abstract

We define a quadratic transformation for any 3-term arithmetic progression.

Let α, β, γ be a 3-term arithmetic progression and let

$$P(x, y, z) = P(x, z, y) \tag{1}$$

be a homogeneous polynomial of degree > 1. Then

$$\widehat{\alpha} = P(\alpha, \beta, \gamma), \tag{2a}$$
$$\widehat{\beta} = P(\alpha, \beta, \gamma), \tag{2b}$$
$$\widehat{\gamma} = P(\gamma, \alpha, \beta) \tag{2c}$$

often have the property, if p is properly chosen, to also form an arithmetic progression.

The typical famous example is when P is cubic:

$$P(x, y, z) = x^2(y + z), \tag{3}$$

but simpler, quadratic polynomials are also possible for P.

Theorem 4. Let α, β, γ form an arithmetic progression. Then so are $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}$ given by (2) with

$$P(\alpha, \beta, \gamma) = 4\alpha^2 + 3\alpha(\beta + \gamma) - \beta\gamma. \tag{5}$$

Proof. Given that

$$\beta = \frac{\alpha + \gamma}{2},$$

we have to verify that

$$2\widehat{\beta} \stackrel{?}{=} \widehat{\alpha} + \widehat{\gamma},$$

or

$$2\{4\beta^2 + 3\beta(\alpha + \gamma) - \alpha\gamma\} \stackrel{?}{=}$$
$$\stackrel{?}{=} [4\alpha^2 + 3\alpha(\beta + \gamma) - \beta\gamma] + [4\gamma^2 + 3\gamma(\beta + \alpha) - \beta\alpha],$$

or

$$2\{(\alpha+\gamma)^2+\frac{3}{2}(\alpha+\gamma)^2-2\gamma\}\stackrel{?}{=}$$
$$\stackrel{?}{=}(\alpha^2+\gamma)^2+(\alpha\gamma+\frac{3}{2}(\alpha+\gamma)^2-\frac{1}{2}(\alpha+\gamma)^2,$$

or

$$2\{\frac{5}{3}(\alpha^2+\gamma^2)+4\alpha\gamma\}\stackrel{?}{=}$$
$$\stackrel{?}{=}5(\alpha^2+\gamma)^2+8\alpha\gamma,$$

which is obvious. ■

Three Squares In An Arithmetic Progression

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Abstract

$a^2 + (a+d)^2 + (a+2d)^2$ is never a square.

Call three terms in arithmetic progression

$$a - d, a, a + d. \tag{1}$$

The sum of their squares is:

$$(a-d)^2 + a^2 + (a+d)^2 = 3a^2 + 2d^2. \tag{2}$$

Theorem 1. If a, d are integers then

$$3a^2 + 2d^2 = u^2 \tag{3}$$

has no solutions except

$$a = d = u = 0. \tag{4}$$

Proof. We take any non-zero solution with no common factor and apply a version of Fermat's descend method. Suppose not all a, d, u are divisible by 3. The terms $3a^2, 2d^2, u^2$, give, modulo 3, remainders

$$0;\ 0 \text{ or } 2; 0 \text{ or } 1. \tag{3}$$

Thus, (3) is satisfied only when

$$d = u \equiv 0 (\text{mod } 3).$$

But then a is also divisible by 3. A contradiction. ■

Derived Terms Of A Quantum Arithmetic Progression

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Abstract

If $\{ak\}$ is an arithmetic progression and $n \in \mathbf{Z}_{\geq 1}$ is fixed, then $S_k = \sum_{i=1}^{nk} a_i/nk$ also forms an arithmetic progression. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Let $\{a_k\}$ be arithmetic progression. Fix $n \in \mathbf{Z}_{\geq 1}$, and set

$$S_k = \sum_{i=1}^{nk} a_i/nk. \tag{1}$$

Then $\{S_k\}$ is also an arithmetic progression, because

$$\frac{S_k}{kn} + \frac{S_{k+2}}{(k+2)^n} - \frac{2S_{k+1}}{(k+1)n} = [a + (k-1)\frac{dn}{2}] + [a + (3k-1)\frac{dn}{2} -$$
$$-2[a + (2_{k-1})\frac{dn}{2}] = \frac{d}{2}\{(4k-2) - 2(2k-1)\} = 0 \tag{2}$$

Interestingly enough, this fact can be quantized.

Set

$$A_k = [a_k]_q^{\sim}, \tag{3}$$
$$\overline{S}_k = \sum_{i=1}^{kn} [a_i]_q^{\sim}/[kn]_{q^{d/2}}^{\sim}, \tag{4a}$$

where

$$d = a_1 = a_0 = (A_k - A_{k-1)|_{q=1|}}. \tag{4b}$$

Then the $\overline{S}_k$'s form a quantum arithmetic progression.

Theorem 5. For any $k \in \mathbf{Z}_{\geq \mathbf{1}}$,

$$\overline{S}_k + \overline{S}_{k+2} = [2]_q^{\sim} \theta \overline{S}_{k+1}, \tag{6}$$

where

$$\mathcal{D} = \frac{nd}{2} = (\overline{S}_{k+1} - \overline{S}_k)_{q=1}. \tag{7}$$

Proof. First we establish that $\mathcal{D}$ exists. We have:

$$(S_{k+1} - \overline{S}_k|_{q=1} = \frac{\sum_{i=1}^{kn} a_i}{kn} + \frac{\sum_{j=1}^{(k+1)n} a_j}{(k+1)n} =$$

$$= -\frac{kn(a+1+\frac{k^n-1}{2}d)}{kn} + \frac{(k+1)^n\{a_1 + \frac{[(k+1)n-1]}{2}d\}}{(kn+1)n} =$$

$$\frac{d}{2}\{(kn-1) + [(kn+1)n-1\} = \frac{n}{2}d,$$

which is (7).

Next,

$$\sum_{i=1}^{N}[a_i]_q^{\sim} = [N]_{q^{d/2}}^{\sim}[\frac{a_1+a_N}{2}]_q^{\sim} =$$

$$= [N]_{q^{d/2}}^{\sim}[a_i + \frac{(N-1)^d}{2}]_q^{\sim},$$

so

$$\overline{S} = [a_1 + \frac{(nk-1)d}{2}]_q^{\sim}. \tag{8}$$

Thus, (6) is:

$$[a_1 + \frac{nk-1}{2}d]_q^{\sim} + [a_1 + \frac{n(k+2)-1}{2}d] \stackrel{?}{=} [2]_{q^{dn/2}}^{\sim}[a_1 + \frac{(k+1)n-1}{2}d]_q^{\sim},$$

or, with $a + \frac{k-1}{2}d = a$:

which is obvious because, in general,

$$[a]_q^{\sim} + [b]_q^{\sim} = [2]_{q^{(a-b)/2}[\frac{a+b}{2}]_q^{\sim}}^{\sim}. \quad \blacksquare$$

Equidistant Sums In A Quantum Arithmetic Progression

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Abstract

If $\{a_k\}$ is an arithmetic progression, then $a_k + a_p = a_{k-\ell} + a_{p+\ell}, \forall \ell \in \mathbf{Z}$. We quantize this.

Let

$$[x]_q^\sim = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Classically, if $\{a_n\}$ is an arithmetic progression then

$$a_k + a_{k+s} = a_{k-\ell} + a_{k+s+\ell}, \quad \forall \ell \in \mathbf{Z}, \quad \forall k, s \in \mathbf{Z}. \tag{1}$$

Theorem 2.

$$[a_k]_q^\sim + q^{(2\ell+s)d}[a_{k+s}]_q^\sim = q^{\ell d}[a_{k-\ell}]_q^\sim + q^{(\ell+s)d}[a_{k+s+\ell}]_q^\sim. \tag{3}$$

Proof. Since, denoting

$$A_k = [a_k]_q^\sim,$$
$$A_k = [k]_{q^d}^\sim A_1 - [k-1]_{q^d}^\sim A_0, \tag{4}$$

where

$$d = a_1 - a_0 = (A_1 - A_0)|_{q=1}, \tag{5}$$

it's enough to show that

$$[k]_q^\sim + q^{\ell+s}[k+s]_q^\sim \stackrel{?}{=} q^\ell[k-\ell]_q^\sim + q^{\ell+s}[k+s+\ell]_q^\sim, \tag{6}$$

or

$$q^k - q^{-k} + q^{2\ell+s}(q^{k+s} - q^{-k-s}) \stackrel{?}{=}$$
$$\stackrel{?}{=} q^\ell(q^{k-\ell} - q^{\ell-l}) + q^{s+\ell}(q^{k+s+\ell} - q^{-k-s-\ell}),$$

$$q^k - q^{-k} + q^{k+2\ell+2s} - q^{-k+2\ell} \stackrel{?}{=}$$
$$= q^k - q^{2\ell-k} + q^{k+2s+2\ell} - q^{-k},$$

which is obvious. ■

Sum Of The Cubes Of The Terms Of Arithmetic Progression

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Abstract

Let $a_k = a + dk$ be an arithmetic progression. We derive a simple formula for $\sum_{k=0}^{n} a_k^3$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Let

$$h = logq.$$

We apply the formula

$$[x]_q^{\sim} = x + \frac{h^2}{6}(x^3 - x) + O(h^4) \tag{1}$$

to the sum of quantum arithmetic progression:

$$\sum_{k=0}^{n} [a_k]_q^{\sim} = [n+1]_{q^{d/2}}^{\sim} [a + \frac{n}{2}d]_q^{\sim}. \tag{2}$$

We get:

$$\sum^{n} k = 0a_k + \left(\sum_{k=0}^{n} a_k^3 - \sum_{k=1}^{n}\right)\frac{h^2}{6} = \left\{(n+1) + \left(\frac{hd}{2}\right)^2 \frac{1}{6}\left[(n+1)^3 - (n+1)\right]\right\}\left\{a + \frac{n}{2}d + \frac{h^2}{6}\left(\overline{a}_n^3 - \overline{a}_n\right)\right\},$$

$$\overline{a}_n = a + \frac{n}{2}d, \tag{3}$$

or, picking $\frac{h^2}{6}$ - terms:

$$\sum_{k=1}^{n}(a_k^3 - a_k) = (n+1)(\overline{a}_n^3 - \overline{a}_n) + \frac{d^2}{4}(n+1)n(n+2)\overline{a}_n,$$

or

$$\sum_{k=0}^{n} a_k^3 = (n+1)(a + \frac{n}{2}d)^3 + \frac{3}{2}d^2\binom{n+2}{3}(a + \frac{n}{2}d). \tag{4}$$

This is our desired formula. Of course, once it is written down, it's trivial to prove it directly.

3-Term Arithmetic Progression Among Elements Of A Quantum Harmonic Series

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Abstract

The terms $\frac{1}{n+1}, \frac{1}{2k}, \frac{1}{k(k+1)}, k \in \mathbf{Z}_{\geq 2}$, form a 3-term arithmetic progression. We manage to quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the 2^{nd} quantization of x.

The elements

$$A = [a]_q^{\sim},$$
$$B = [a+d]_q^{\sim},$$
$$C = [a+2d]_q,$$

form a Quantum Arithmetic progression with the characteristic property

$$A + C = [2]_{q^d}^{\sim} B, \tag{1}$$
$$d = (B_A)|_{q=1} = (C_B)|q = 1. \tag{1a}$$

Theorem 2. Consider the elements

$$A = \frac{q^{-k}}{[k+1]_q^{\sim}} \frac{1}{[2]^s im_{q^d}[k]_q^{\sim}}, \frac{q}{[k]_q^{\sim}[k+1]_q^{\sim}}, \tag{3a}$$
$$d = \frac{-1}{2k} + \frac{1}{k+1} = \frac{-1+k}{2k(k+1)}. \tag{3b}$$

Proof. We have to check the equation (1), i.e., that

$$\frac{q}{[k+1]_q^{\sim}} + \frac{q^{-k}}{[k]_q^{\sim}[k+1]_q^{\sim}} \stackrel{?}{=} \frac{1}{[k]_q^{\sim}},$$

or

$$q[k]_q^{\sim} + q^{-k} \stackrel{?}{=} [k+1]_q^{\sim},$$

which is true because, in general,

$$[a+b]_q^{\sim} = q^{-b}[a]_q^{\sim} + q^a[b]_q^{\sim}. \quad \blacksquare$$

On The Special Quantum Arithemetic Progression

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Abstract

Let $a = 1, a_k = a + kd$, be an arithmetic progression. It is not difficult to show that

$$\left(\frac{1}{a_1} + \frac{2}{a_1 a_1} + ... + \frac{n}{a_1...a_n}\right) = \frac{1}{d}\left(1 - \frac{1}{a_1...a_n}\right).$$

We quantize this identity.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the 2^{nd} quantization of x.

Theorem 1. For $n \in \mathbf{Z}_{\geq 1}$,

$$\sum_{n=1}^{n} \frac{[nd]_q^{\sim} q^{\frac{k}{2}d-1}}{[a_1]_q^{\sim}...[a_k]_q^{\sim}} = 1 - \frac{q^{\frac{n+1}{2}d}}{[a]_q^{\sim}...[a_n]_q^{\sim}}. \tag{2}$$

Proof. We use induction on n. For $n = 1$, (2) returns:

$$\frac{[d]_q^{\sim} q^{-1}}{[1+d]_q^{\sim}} \stackrel{?}{=} 1 - \frac{q^d}{[a+d]_q^{\sim}},$$

or

$$[1+d]_q^{\sim} = [d]_q^{\sim} q^{-1} + q^d [d]_q^{\sim},$$

which is true, because, in general,

$$[a+b]_q^{\sim} = q^{-b}[a]_q^{\sim} + q^a [b]_q^{\sim}. \tag{3}$$

Now, the inductive step amounts to; with

$$\sqcap_k = [a_1]_q^{\sim}...[a_k]_q^{\sim}:$$

$$1 - \frac{q^{\frac{h+1}{2}d}}{\sqcap_n} + \frac{[(n+1)d]_q^{\sim} q^{\frac{n+1}{2}d} q^{-1}}{\sqcap_{n+1}} \stackrel{?}{=} 1 - \frac{q^{\frac{n}{2}d}}{\sqcap_{n+1}}, \tag{4}$$

or to

$$\frac{1}{\sqcap_{n+1}}\left\{[(n+1)d]_q^{\sim}q^{-1}+q^{(n+1)d}\right\}\stackrel{?}{=}\frac{1}{\sqcap_n},$$

which is true, because, by (4),

$$[n+1d]_q^{\sim}q^{-1}+1-q^{(n+1)d}=[1+(n+1)d]_q^{\sim}=[a_{n+1}]_q^{\sim}. \quad \blacksquare$$

On A Finite Sum Of Weighted Quantum Geometric Progression

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Abstract

Classically, $\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+...+\frac{2n-1}{2^n}=\frac{3-2^n-2n-3}{2^n}$. We quantize this.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}}, x\in\mathbf{R},\ q\neq 0,\ \pm 1, \tag{1}$$

be the 2^{nd} quantization of x, so that

$$[2]_q^{\sim}=q+q^{-1}. \tag{2}$$

Set

$$\begin{aligned} &t(n)=2n+1,\\ &\sqcap_n=[2]_{t(1)}^{\sim}...[2]_{t(n)}^{\sim} \end{aligned} \tag{3}$$

Theorem 4. For $n\in\mathbf{Z}_1$, we have:

$$\sum_{k=1}^{n}\frac{[2k-1]_q^{\sim}}{\sqcap_k]}q^{k^{2-k-2}}=\frac{[3][\sqcap_n]-q^{n^2}[2n+3]_q^{\sim}}{\sqcap_n}. \tag{5}$$

Proof. We use induction on n. For $n=1$, (5) returns:

$$\frac{q^-5}{\sqcap_1}\stackrel{?}{=}\frac{[3]_q^{\sim}[2]_{q^3}^{\sim}-q[5]_q^{\sim}}{\sqcap_1},$$

or

$$[3]_q^{\sim}[2]_{q^3}=q[5]_q^{\sim}+q^{-5},$$

which is true because, in general,

$$[x+1]_q^{\sim}[2]_{q^{x+1}}^{\sim}=q[2x+1]_q^{\sim}+q^{-2x-1},$$

as is easy to prove.

Now, the inductive step applied to (5) returns:

$$\{[3]_q^{\sim} \sqcap_n -q^{n^2}[2n+3]_q^{\sim}\}[2]_{q^{2n+3}} + [2n+1]_q^{\sim} q^{n^2-2n-5} \stackrel{?}{=}$$
$$\stackrel{?}{=} [3]_q^{\sim}[\sqcap_{n+1} - q^{(n+1)^2}[2n+5]_q^{\sim},$$

or

$$q^{(n+2)^2}[n+5]_q^{\sim} + [2n+1]q^{n^2=2n-5} \stackrel{?}{=} q^{n^2}[2n+3]_q^{\sim}[2]_{q^{2n+3}}^{\sim} = q^{n^2}[4n+6]_q^{\sim},$$

or

$$q^{2n+1}[2n+5]_q^{\sim} + q^{-2n-5}[2n+1]_q^{\sim} \stackrel{?}{=} [4n+6]_q^{\sim},$$

which is true because, in general,

$$[a+b]_q^{\sim} = q^b[a]_q^{\sim} + q^{-a}[b]_q^{\sim}. \quad \blacksquare$$

Sum Of An Arithmetic Progression In The 1^{st} Quantization

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Abstract

We deduce a formula for the sum.

Let

$$[x]_q^{\sim} = \frac{q^x 1}{q-1}, x, q \in \mathbf{R},\ q \neq 1,\ q > 0,$$

be the 1^{st} quantization of x.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x, q \in \mathbf{R}, q \neq 1,\ q > 0,$$

Let

$$a_k = a + kd,\ k \in \mathbf{Z},$$

be an arithmetic progression. As is known,

$$\sum_{k=0}^{n} [a_k]_q^{\sim} = [N+1]_{q^{d/2}}^{\sim} [a + N\frac{d}{2}]_q^{\sim}, \tag{1}$$

a rather simple formula in the 2^{nd} quantization.

Theorem 1.

$$\sum_{k=0}^{n} [a_k] = \frac{q^{a+\frac{Nd}{2}} [N+1]_{q^{d/2}}^{\sim} - (N+1)}{q-1}. \tag{2}$$

Proof. We have:

$$\sum_{k=0}^{N}[a_k]_q = \sum_{k=0}^{N}\frac{q^{a+kd}-1}{q-1} =$$
$$= \frac{1}{q-1}\left\{q^a\sum_{k=0}^{N}(q^a)^k - (N+1)\right\} =$$
$$= \frac{1}{q-1}\left\{q^a\frac{q^{d(N+1)}-1}{q^d-1} - (N-1)\right\} =$$
$$= \frac{1}{q-1}\left\{q^a\frac{q^{\frac{d}{2}(N+1)}}{(}q^{\frac{-d(N+1)}{2}} - q^{-d\frac{d(N+1)}{2}}q^{d/2}(q^{d/2}-q^{-d/2}) - (N+1)\right\} =$$
$$= \frac{1}{q-1}\left\{q^{a+\frac{d}{2}N}[N+1]^{\sim}_{q^{d/2}} - (N+1)\right\},$$

and this is formula (2). ■

Notice that formula (2) is *singular* in $q-1$, and, thus, has no classical analog.

Example 3. Take $N=1$. Then

$$[a]_q + [a+d]_q = \frac{q^{a+d/2}[2]^{\sim}_{q^{d/2}} - 2}{q-1} =$$
$$= \frac{1}{q-}\left\{q^{a_1 d/2}(q^{d/2}+q^{-d/2}) - 2\right\} =$$
$$= \frac{1}{q-1}\left\{q^{a+d} - 1 + q^a - 1\right\} = [a]_q + [a+d]_q.$$

More generally,

$$[a]_q + [b]_q = \frac{q^a + q^b - 2}{q-1}. \tag{4}$$

For a or $b = 0$, we recover from (4) the usual definition of a quantum number in the 1^{st} quantization. Notice that the general form of (4) is

$$\sum_{i=0}^{N}[a_i] = \frac{\sum_i q^{q_i} - (N+1)}{q-1}. \tag{5}$$

A Summation Related To A One-Parameter Family Of Quantum Arithmetic Progressions

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Abstract

For $a_k = a + kd, c \neq -2, a = \frac{2(c+1)}{c+2}, d = \frac{4}{c+2}$, we have $\sum_{k=1}^{n} \frac{k^2+ck}{a_{k-1}a_k} = \frac{1}{d}\frac{n(n+1)}{a_n}$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0, 1,$$

be the second quantization of x.

Set

$$\mathcal{D} = \frac{([2]_q^{\sim})^2}{[c+2]_q^{\sim}}, \tag{1}$$

$$A_n = \frac{[2]_q^{\sim}}{[c+2]_q^{\sim}}[c+1+2n]_q^{\sim}. \tag{2}$$

Theorem 3. For $n \in \mathbf{Z}_{\geq 1}$, we have:

$$\sum_{k=1}^{n} \frac{[k]_q^{\sim}[k+c]_q^{\sim}}{A_{k-1}A_k} = \frac{1}{\mathcal{D}}\frac{[n]_q^{\sim}[n+1]_q^{\sim}}{A_n}. \tag{4}$$

Proof. We use induction on n. For $n = 1$, (4) returns:

$$\frac{[c+1]_q^{\sim}}{A_0 A_1} \stackrel{?}{=} \frac{1}{D}\frac{[2]_q^{\sim}}{A_1},$$

or

$$D[c+1]_q^{\sim} = [2]_q^{\sim} A_0,$$

which follows from (1) and $(2)|_{n=0}$.

Next, the induction step amounts to:

$$\frac{1}{\mathcal{D}}\frac{[n]_q^{\sim}[n+1]_q^{\sim}}{A_n} + \frac{[n+1]_q^{\sim}[n+1+c]_q^{\sim}}{A_n A_{n+1}} \stackrel{?}{=} \frac{1}{\mathcal{D}}\frac{[n+1]_q^{\sim}[n+2]_q^{\sim}}{A_{n+1}},$$

or

$$\frac{1}{\mathcal{D}}\frac{[n]_q^{\sim}}{A_n}+\frac{[n+1+c]_q^{\sim}}{A_nA_{n+1}}=\frac{1}{\mathcal{D}}\frac{[n+1]_q^{\sim}}{A_{n+1}} \tag{5}$$

Now

$$A_k=f[c+1+2k]_q^{\sim} \tag{6.1}$$
$$D=f[2]_q^{\sim}, \tag{6.2}$$

where

$$f=\frac{[2]_q^{\sim}}{[c+2]_q^{\sim}}. \tag{6.3}$$

Thus, (5) becomes:

$$[n]_q^{\sim}[c+3+2n]_q^{\sim}+[2]_q^{\sim}[n+1+c]_q^{\sim}\stackrel{?}{=}[n+2]_q^{\sim}[c+1+2n]_q^{\sim}. \tag{7}$$

Multiplying this by $(q-q^{-1})^2$ and denoting

$$X=q^n,\ C=q^c,$$

we arrive at:

$$(X-X^{-1})(X^2Cq^3-X^{-2}C^{-1}q^{-3})+(q^2-q^{-2})(XCq-X^{-1}C^{-1}q^{-1})\stackrel{?}{=}$$
$$\stackrel{?}{=}(Xq^2-X^{-1}q^{-2})(X^2C_q-X^2q^{-1}C^{-1}),$$

or

$$[(X^3Cq^3+X^{-3}C^{-1}q^{-3})-(XCq^3+X^{-1}C^{-1}q^{-3}]+$$
$$+[XCq(q^2-q^{-2})-X^{-1}C^{-1}q^{-1}(q^2-q^{-2})]\stackrel{?}{=}$$
$$\stackrel{?}{=}[X^3Cq^3+X^{-3}C^{-1}q^{-3})-(XCq^{-1}+X^{-1}C^{-1}q)],$$

or

$$-Cq^3+Cq(q^2-q^{-2})\stackrel{?}{=}-Cq^{-1},$$

which is true. ■

The formulae in this paper are new even for $q=1, i-e.$, in the classical case.

A Power Inequality Quantized

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Abstract

For $0 < x < 1, (1-x)^n + (1+x)^n \leq 2^n, n \in \mathbf{Z}_{\geq p}$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Theorem 1. Let $0 < x < 1$,

$$([1-x]_q^{\sim})^n + ([1+x]_q^{\sim})^n \leq ([2]_q^{\sim})^n. \tag{2}$$

Proof. We use induction on n. For $n = 1$, (2) returns:

$$(1-x)_q^{\sim} + [1+x]_q^{\sim} \stackrel{?}{\leq} [2]_q^{\sim}. \tag{3}$$

Multiplying this by $q - q^{-1} > 0$, we get:

$$(q^{1-x} - q^{x-1}) + (q^{1+x} - q^{-x-1} \stackrel{?}{\leq} q^2 - q^{-2},$$

or

$$q^x(q - q^{-1}) + q^{-x}(q - q^{-1}) \stackrel{?}{\leq} (q - q^{-1})(q + q^{-1}),$$

or

$$q^x + q^{-x} \overset{?}{\leq} q^1 + q^{-1},$$

which is true because $x < 1$ and

$$q^x + q^{-x}$$

is an *increasing* function of x.

Now, the induction step $n \mapsto n+1$, amounts to:

$$([2]_q^{\sim})^{n+1} = ([2]_q^{\sim})^n [2]_q^{\sim} \geq [2]_q^{\sim} \{([1-x]_q^{\sim})^n + [(1+x)_q^{\sim}])^n \overset{?}{\geq}\} \overset{?}{\geq} ([1-x]_q^{\sim})^{n+1} + ([1+x]_q^{\sim})^{n+1},$$

which is true because

$$[2]_q^{\sim} > [1-x]_q^{\sim}$$

and

$$[2]_q^{\sim} > [1+x]_q^{\sim},$$

since

$$2 > 1 - x$$

and

$$2 > 1 + x$$

x being < 1, and, in general,

$$a < b \Rightarrow [a]_q^{\sim} < [b]_q^{\sim}. \quad \blacksquare$$

On The Relation Between A Number And Inverse Of It In Quantum Domain

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Abstract

Classically, $x\frac{1}{x}=1$. The situation is more complex quantumly.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}}, x\in\mathbf{R},\ q\neq 0,\ \pm 1,$$

be the second quantization of x.

Since

$$[x]_{q^{-1}}^{\sim}=[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim}=[x]_q^{\sim},$$

we can take

$$q>1$$

without loss of generality.

Theorem 1. Let $x\neq 0$. Then

$$[x]_q^{\sim}\left[\frac{1}{x}\right]_q^{\sim}\geq 1. \tag{2}$$

Proof. Since

$$[-x]_q^{\sim}\left[\frac{1}{-x}\right]_q^{\sim}=[x]_q^{\sim}\left[\frac{1}{x}\right]_q^{\sim},$$

we can take $x>0$. And since

$$[x]_q^{\sim}\left[\frac{1}{x}\right]_q^{\sim}$$

is invariant with respect to the change $x \to x^{-1}$, we can take

$$x > 1,$$

the ease

$$x = 1$$

being obviously true.

Multiplying through (2) by $(q - q^{-1})^2 > 0$, we get:

$$(q^x - q^{-x})(q^{1/x} - q^{-1/x}) \stackrel{?}{\geq} (q - q^{-1})^2,$$

or

$$(q^{x+1/x} + q^{-x-1/x}) - (q^{x-1/x} - q^{1/x-x}) \stackrel{?}{>} q^x + q^{-2} - 2. \tag{3}$$

Set

$$h = log\ q > 0.$$

Then (3) becomes:

$$2\sum_{??}^{\infty} \frac{h^{2n}}{(2n)!}[(x + \frac{1}{x})^{2n} - (x - \frac{1}{x})^{2n}] \stackrel{?}{\geq} 2\sum_{n=??}^{\infty} \frac{h^{2n}}{(2n)!}2^{2n}. \tag{4}$$

Since $h > 0$, (4) follows from:

$$(x + \frac{1}{x})^{2n} - (x - \frac{1}{x})^{2n} \stackrel{?}{\geq} 2^{2n},\ n \geq 1,$$

or

$$2\sum_{k\ even} \binom{2n}{k} x^{2n-k} \left(\frac{1}{x}\right)^k \geq 2^{2n},$$

or

$$\sum_{k\ even} \binom{2n}{k} x^{2(n-k)} \stackrel{?}{\geq} q^{2n-1},$$

or

$$\sum_{k\ even} \binom{2n}{n} y^{n-k} \stackrel{?}{\geq} 2^{2n-1} m, y \geq 1,\ n \geq 1. \tag{5}$$

If n is odd, (5) is obvious. Since $y > 1$, $y^{n-k} + y^{k-n} \geq 2$, and (5) becomes:

$$2\sum_{k\ even\ <n} \binom{2n}{k} \stackrel{?}{\geq} 2^{2n-1},$$

or

$$\sum_{k\ even} \binom{2n}{k} \geq 2^{2n-1},$$

and

$$\sum_{k\ even} \binom{m}{k} = \sum_{k\ odd} \binom{m}{k} = 2^m/2. \tag{6}$$

For n even, $n = 2mg$ (5) is bounded, by the same argument, by

$$2 \sum_{k\ even<2m} \binom{4m}{k} + \binom{4m}{2m} \stackrel{?}{\geq} 2^{4m-1}$$

or

$$\sum_{k\ even} \binom{4m}{k} \stackrel{?}{\geq} 2^{4m-1},$$

which is true by (6). ■

On Linear Relations Between Powers Of Terms Of An Arithmetic Progression

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Abstract

The relation $a_0 + a_2 = 2a_1$, is generalized for higher powers of the term $a_k = a + kd$ of an arithmetic progression.

Rewriting the relation

$$a_1 = \frac{a_) + a_2}{2}$$

as

$$a_0 - 2a_1 + a_2,$$

and noticing that

$$a_0^2 - 2a_1^2 + 3a_2^2 - a_3^2 = 0,$$

we are naturally led to the following

Theorem 1. For $m \geq 1$, the n^{th} powers of the terms a_i^m of an arithmetic progression $a_i = a + id$, satisfy:

$$\sum_{i=0}^{m+1} \binom{m+1}{i} (-1)^i a_i^m = 0. \tag{2}$$

Proof. Since

$$a_i^m = (a + id)^m = \sum_{s=0}^{m} a^{m-s} d^s i^s \binom{m}{s},$$

(2) can be rewritten as

$$\sum_{i=0}^{m+1} (-1)^i \binom{m+1}{i} \sum_s \binom{m}{s} a^{m-s} d^s i^s \stackrel{?}{=} 0,$$

or

$$\sum_{s=0}^{m} \binom{m}{s} a^{m-s} d^s \sum_{i=0}^{m+1} \binom{m+1}{i} (-1)^i i^s \stackrel{?}{=} 0,$$

which follows from

$$\sum_{i=1}^{m+1} \binom{m+1}{i} (-1)^i i^s = 0, \quad 0 \leq s \leq m, \tag{3}$$

which is:

$$\left(x\frac{d}{dx}\right)^s (x-1)^{m+1}(-1)^m|_{x=1} = 0, \tag{4}$$

which is obvious. ■

A Formula For The n^{th} Of Quantum Arithmetic Progression

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Abstract

The standard formula $a_n = na_1 - (n-1)a_0$ for the terms of a classical arithmetic progression, is quantized.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Theorem 1. For the terms of a quantum arithmetic progression,

$$a_k = [a + kd]_q^{\sim}, \quad k \in \mathbf{Z}_{\geq 0}, \tag{2}$$

we have

$$a_n = [n]_{q^d}^{\sim} a_1 - [n-1]_{q^d}^{\sim} a_0. \tag{3}$$

Proof. The formula is obvious for $d = 0$. Let now $d \neq 0$. Multiplying through (3) by $(q - q^{-1})(q^d - q^{-d})$, we get:

$$(q^d - q^{-d})(q^{a+nd} - q^{-a-nd}) \stackrel{?}{=}$$
$$\stackrel{?}{=} (q^{dn} - q^{-dn})(q^{a+d} - q^{-a-d}) - [q^{d(n-1)} - q^{-d(n-1)}](q^a - q^{-a}),$$

or

$$[q^{a+(n+1)d} + q^{-a-(n+1)d}] - [q^{a+(n-1)d} + q^{-a-(n-1)d}] \stackrel{?}{=}$$
$$\stackrel{?}{=} [q^{a+d(n+1)} + q^{-a-d(n+1)}] - [q^{a-d(n-1)} + q^{-a+d(n-1)}] =$$
$$-\{[q^{a+d(n-1)} + q^{-a-d(n-1)}] - [q^{a-d(n-1)} + q^{-a+d(n-1)}],$$

or

$$0 = 0,$$

which is obvious. ■

On Alternating Sums Of Squares Of Quantum Integers

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Abstract

The classical formula $\sum_{i=1}^{n}(-1)^{i-1}i^2 = (-1)^{n-1}(\sum_{i=1}^{n} i)$ is quantized.

The classical formula in the Abstract can be quantized thusly:

$$\sum_{i=0}^{n}(-1)^{i-1}([i]_q^{\sim})^2 = (-1)^{n-1}\sum_{i=0}^{n}[i]_{q^2}^{\sim}, \tag{1}$$

where

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{2}$$

To prove (1) we set $3_k = q^k - q^{-k}$, notice that (1) is true for $n = M$, and then proceed by induction. The inductive step $n \to n+2$ amounts to

$$(-1)^n([n+1]_q^{\sim})^2 + (-1)^{n-1}([n+2]_q^{\sim})^2 \stackrel{?}{=}$$
$$\stackrel{?}{=} (=1)^{n-1}\{[n+1]_{q^2}^{\sim} + [n+2]_{q^2}^{\sim}\},$$

or

$$-([n-1]_q^{\sim})^2([n+2]_q^{\sim})^2 \stackrel{?}{=} [n+1]_{q^2}^{\sim} + [n+2]_{q^2}^{\sim}. \tag{3}$$

Call

$$X = q^n.$$

Then (8) becomes

$$\frac{1}{3_1^2}\{(x^2q^2 - 2 + X^{-2}q^{-2}) + (X^2q^4 - 2 + X^{-2}q^{-4}])\} =$$
$$= \frac{1}{3_1^2}\{X^2q^2 - (1+q^2) - X^{-2}q^{-2}(1-q^{-2})\} =$$
$$= \frac{1}{3_q^2}\{X^2q^2q(q-q^{-1}) - X^{-2}q^{-2}q^{-1}(q-q^{-1})\} =$$
$$= \frac{1}{3_1}\{X^2q^3 - X^{-2}q^{-3}\} = [2x+3]_q^{\sim}. \tag{5}$$

the RHS of (4) yields:

$$\frac{1}{3_2}\{X^2q^2(1+q^2)-X^{-2}q^{-2}(1+q^{-2})\}=$$
$$=\frac{1}{3_2}\{X^2q^3-X^{-2}q^{-3}\}=\frac{1}{3_1}\{X^2q^3-X^{-2}q^3\}[2x+3]_q^{\sim},$$

which is the same as (5). We used in the Proof the obvious relation

$$[2]^{\sim}=\frac{3_2}{3_1}. \tag{6}$$

Sums Of Squares Of Quantum Odd Integers

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Abstract

We find a quantum analog of the classical identity

$$\sum_{i=1}^{n}(2i-1)^2 = \binom{2n+1}{3}$$

The purpose of this short note is to prove

Theorem 1. Let n be a positive integer. Then

$$\sum_{i=1}^{n}[2i-1]_q^{\sim}[2i-1]_{q^2}^{\sim} = \begin{bmatrix}2n+1\\3\end{bmatrix}_q^{\sim}, \tag{1}$$

where

$$x \to [x]_q^{\sim} = \frac{q^x - q^{-x}}{q-q^{-1}} \tag{2}$$

is the (2^{nd}) quantization of x, and

$$\begin{bmatrix}\sigma\\k\end{bmatrix}_q^{\sim} = \frac{[\sigma]_q^{\sim}...[\sigma-k+1]_q^{\sim}}{[1]_q^{\sim}...[k]_q^{\sim}}. \tag{3}$$

In particular, the RHS of (1) is:

$$\frac{[2n+1]_q^{\sim}[2n]_q^{\sim}[2n-1]_q^{\sim}}{[2]_q^{\sim}[3]_q^{\sim}}, \tag{4}$$

because $[1]_q^{\sim} = 1$.

Proof. For $n = 1$, we have

$$1 = \begin{bmatrix}3\\3\end{bmatrix},$$

which is true. We use induction on a afterward. It reduces (1) to

$$[2n+1]_q^{\sim}[2n+1]_{q^2}^{\sim} \stackrel{?}{=} \begin{bmatrix}2n+3\\3\end{bmatrix}_q^{\sim} - \begin{bmatrix}2n+1\\3\end{bmatrix}_q^{\sim} = \frac{1}{[2][3]}[2n+1]_q([2n+3]_q^{\sim}[2m+2]_q^{\sim} -$$

$$-[2n]_q^{\sim}[2n-1]_q^{\sim}),$$

or

$$[2]_q^{\sim}[3]_q^{\sim}[2n+1]_{q^2}^{\sim} \stackrel{?}{=} [2n+3]_q^{\sim}[2n+2]_q^{\sim} - [2n]_q^{\sim}[2n-1]_q^{\sim}. \tag{5}$$

Denoting $2n-1$ by x, (5) becomes:

$$[2]_q^{\sim}[3]_q^{\sim}[x+2]_{q^2}^{\sim} \stackrel{?}{=} [x+4]_q^{\sim}[x+3]_q^{\sim} - [x+1]_q^{\sim}[x]_q^{\sim}. \tag{6}$$

For the RHS of (6), with $3_k = q^k - q^{-k}$, we have:

$$\begin{aligned}
&\frac{1}{3_1^2}\{(q^{x+4}-q^{-x-4})(q^{x+3}-q^{-x-3}) - (q^{x+1}-q^{-x-1})(q^x - q^{-x})\} \text{ (with } q^x = X) = \\
&= \frac{1}{3_1^2}\{(Xq^4 - X^{-1}q^{-4})(Xq^3 - X^{-1}q^{-3}) - (Xq - X^{-1}q^{-1})(X - X^{-1})\} = \\
&= \frac{1}{3_1^2}\{X^2q^2 - q^{-1} - q + X^{-2}q^{-1}) - (X^2q - q^{-1} + X^{-1}q^{-1})\} = \\
&= \frac{1}{3_1^2}\{X^2(q^1 - q) + X^{-2}(q^{-1} - q^{-1}\} = \\
&= \frac{1}{3_1^2}\{X^2q(q^3 - q^{-3})q^3 - X^{-2}q^{-1}q^{-3}(q^3 - q^{-3})\} = \\
&= \frac{3}{3_1^2}\{X^2q^4 - X^{-2}q^{-4}\} = \frac{[3]_q^{\sim}}{3_1}[3x+2]_{q^2}^{\sim}3_2 = [2]_q^{\sim}[3]_q^{\sim}[x+2]_{q^2}^{\sim},
\end{aligned}$$

which is exactly the LHS of (6). We used the obvious relation

$$[n] = \frac{3_n}{3_1}, \quad n \in \mathbf{Z}. \quad \blacksquare \tag{7}$$

A Weighted Sum Of The Quantum Odd Numbers

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Abstract

If $a_k = 1 + 2k$, then $\sum_{k=1}^{n} \frac{k^2}{a_{k-1}a_k} = \frac{n(n+1)}{2(n+1)}$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, 1,$$

Let

$$a_k = 1 + 2k, \quad k \in \mathbf{Z}.$$

be odd numbers.

Theorem 1.

$$\sum_{k=1}^{n} \frac{([k]_q^{\sim})^2}{[a_{k+1}]_q^{\sim}[a_k]_q^{\sim}} = \frac{[n]_q^{\sim}[n+1]_q^{\sim}}{[2]_q^{\sim}[2n+1]_q^{\sim}}. \tag{2}$$

Proof. We use induction on n. For $n = 1$, (2) returns

$$\frac{1}{[3]_q^{\sim}} = \frac{[2]_q^{\sim}}{[2]_q^{\sim}[3]_q^{\sim}},$$

which is true. The inductive step amounts to:

$$\frac{[n]_q^{\sim}[n+1]_q^{\sim}}{[2]_q^{\sim}[2n+1_q^{\sim}} + \frac{([n+1]_q^{\sim})^2}{[2n+1]_q^{\sim}[2n+3]_q^{\sim}} \stackrel{?}{=} \frac{[n+1]_q^{\sim}[n+2]_q^{\sim}}{[2]_q^{\sim}[2n+3]_q^{\sim}},$$

or

$$\frac{n_q^{\sim}}{[2n+1]_q^{\sim}} + \frac{[2]_q^{\sim}[n+1]_q^{\sim}}{[2n+1]_q^{\sim}[2n+3]_q^{\sim}} \stackrel{?}{=} \frac{[n+2_q^{\sim}}{[2n+3]_q^{\sim}},$$

or

$$[n]_q^{\sim}[2n+3]_q^{\sim} + [2]_q^{\sim}[n+1]_q^{\sim} \stackrel{?}{=} [n+2]_q^{\sim}[2n+1]_q^{\sim}. \tag{3}$$

To prove (3), we multiply it by $(q - q^{-1})^s$, and denote

$$X = q^n$$

Then (3) becomes:

$$(X - X^{-1})(X^2q^3 - X^{-2}q^{-3}) + (q^2 - q^{-2})(Xq - X^{-1}q^{-1}) \stackrel{?}{=}$$
$$\stackrel{?}{=} (Xq^2 - X^{-1}q^{-2})(X^2q - X^{-2}q^{-1}),$$

or

$$[(X^3q^3 + X^{-3}q^{-3}) - (Xq^3 + X^{-1}q^{-3})] + [(q^2 - q^{-2})qX - (q^2 - q^{-2})q^{-1}X^{-1}) \stackrel{?}{=}$$
$$\stackrel{?}{=} [(X^3q^3 + X^{-3}q^{-3}) - (Xq^{-1} + X^{-1}q)],$$

or

$$-q^3 + q(q^2 - q^{-2}) \stackrel{?}{=} -q^{-1},$$

which is obvious. ■

A Sum Of A Finite Series In Quantum Domain

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Abstract

Classically, $\sum_{k=0}^{n-1}\prod_{i=0}^{k}\frac{(2n-2i)}{(2n-1-2i)}=2n,\ n\in\mathbb{Z}_{\geq 1}$. We quantize this.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-1}}{q-q^{-1}},\ x\in\mathbb{R}, q\neq 1,$$

so that

$$[x]_{q^{-1}}^{\sim}=[x]_q^{\sim}.$$

Theorem 1. For $n\geq 1$,

$$\sum_{k=0}^{n-1}q^{2n-3k-2}\prod_{i=0}^{k}\frac{[2n-2i]_q^{\sim}}{[2n-q-2i]_q^{\sim}}=[2n]_q^{\sim}. \tag{2}$$

Proof. Denote the LHS of (2) by S_n. Then,

$$S_n=\frac{[2n]_q^{\sim}}{[2n-1]_q^{\sim}}(q^{2n-2}+q^{-1}S_{n-1}), \tag{3}$$

so the assertion (2) amounts to, by induction (since (2) is obvious for $n=1$), to:

$$[2n-1]_q^{\sim}\stackrel{?}{=}q^{2n-2}+q^{-1}[2n-2], \tag{4}$$

which is true, since

$$[a+b]_q^{\sim}=q^b[a]_q^{\sim}+q^{-a}[b]_q^{\sim}, \tag{5}$$

and

$$[1]_q^{\sim}=1. \quad \blacksquare$$

On Linear Relations Between Powers Of Terms Of An Arithmetic Progression

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Abstract

The relation $a_0 + a_2 = 2a_1$, is generalized for higher powers of the term $a_k = a + kd$ of an arithmetic progression.

Rewriting the relation

$$a_1 = \frac{a_) + a_2}{2}$$

as

$$a_0 - 2a_1 + a_2,$$

and noticing that

$$a_0^2 - 2a_1^2 + 3a_2^2 - a_3^2 = 0,$$

we are naturally led to the following

Theorem 1. For $m \geq 1$, the n^{th} powers of the terms a_i^m of an arithmetic progression $a_i = a + id$, satisfy:

$$\sum_{i=0}^{m+1} \binom{m+1}{i} (-1)^i a_i^m = 0. \tag{2}$$

Proof. Since

$$a_i^m = (a + id)^m = \sum_{s=0}^{m} a^{m-s} d^s i^s \binom{m}{s},$$

(2) can be rewritten as

$$\sum_{i=0}^{m+1} (-1)^i \binom{m+1}{i} \sum_s \binom{m}{s} a^{m-s} d^s i^s \stackrel{?}{=} 0,$$

or

$$\sum_{s=0}^{m} \binom{m}{s} a^{m-s} d^s \sum_{i=0}^{m+1} \binom{m+1}{i} (-1)^i i^s \stackrel{?}{=} 0,$$

which follows from

$$\sum_{i=1}^{m+1} \binom{m+1}{i} (-1)^i i^s = 0, \quad 0 \leq s \leq m, \tag{3}$$

which is:

$$\left(x\frac{d}{dx}\right)^s (x-1)^{m+1}(-1)^m|_{x=1} = 0, \tag{4}$$

which is obvious. ■

A Formula For The n^{th} Of Quantum Arithmetic Progression

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Abstract

The standard formula $a_n = na_1 - (n-1)a_0$ for the terms of a classical arithmetic progression, is quantized.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Theorem 1. For the terms of a quantum arithmetic progression,

$$a_k = [a + kd]_q^{\sim}, \quad k \in \mathbf{Z}_{\geq 0}, \tag{2}$$

we have

$$a_n = [n]_{q^d}^{\sim} a_1 - [n-1]_{q^d}^{\sim} a_0. \tag{3}$$

Proof. The formula is obvious for $d = 0$. Let now $d \neq 0$. Multiplying through (3) by $(q - q^{-1})(q^d - q^{-d})$, we get:

$$(q^d - q^{-d})(q^{a+nd} - q^{-a-nd}) \stackrel{?}{=}$$
$$\stackrel{?}{=} (q^{dn} - q^{-dn})(q^{a+d} - q^{-a-d}) - [q^{d(n-1)} - q^{-d(n-1)}](q^a - q^{-a}),$$

or

$$[q^{a+(n+1)d} + q^{-a-(n+1)d}] - [q^{a+(n-1)d} + q^{-a-(n-1)d}] \stackrel{?}{=}$$
$$\stackrel{?}{=} [q^{a+d(n+1)} + q^{-a-d(n+1)}] - [q^{a-d(n-1)} + q^{-a+d(n-1)}] =$$
$$-\{[q^{a+d(n-1)} + q^{-a-d(n-1)}] - [q^{a-d(n-1)} + q^{-a+d(n-1)}],$$

or

$$0 = 0,$$

which is obvious. ■

Linear Relations Among Terms Of Quantum Arithmetic Progression

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Abstract

Let $a_k = a+kd$ be an arithmetic progression. It is well known that for any $n \in \mathbf{Z}_{\geq 2}$, $\sum_{k=0}^{n}(-1)^k \binom{n}{k} a_k = 0$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbf{R}, q \neq 1,$$

be the 2^{nd} quantization of x.

Set

$$[k]!_q^{\sim} = [1]_q^{\sim}...[k]_q^{\sim},\ k \in \mathbf{Z}_{\geq 1};\ [0]!_q^{\sim} = 1,$$
$$\begin{bmatrix} m \\ k \end{bmatrix}_q^{\sim} = \frac{[m]!_q^{\sim}}{[k]!_q^{\sim}[m-k]_q^{\sim}}.$$

Theorem 1. For any $n \in \mathbf{Z}_{\geq 2}$, we have:

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{\alpha d}}^{\sim} (-1)^k [a_k] = 0, \tag{2}$$

where

$$\alpha = \begin{cases} 1, & \text{n even} \\ 1/2, & \text{n odd} \end{cases}$$

Proof. We are going to prove (2) by induction on n, the cases $n = 2, 3$ - the base of induction - easily done. For $n = 2$, with

$$A = q^A,\ Q = q^d,$$
$$[a]_q^{\sim} - [2]_{q^d}^{\sim}[a+d] + [a+2d] = \frac{1}{q-q^{-1}}\{(A - A^{-1}) - (Q + Q^{-1})(AQ - A^{-1}Q^{-1}) + (AQ^2 - A^1Q^{-2})\} = 0,$$

and for $n = 3$,

$$[a]_q^{\sim} - [3]_{q^{d/2}}^{\sim}[a+d]_q^{\sim} + [3]_{q^{d/2}}^{\sim}[a+2d]_q^{\sim} - [a+3d]_q^{\sim} =$$
$$= -[2]_{q^{\triangle}}^{\sim} \left[\frac{3}{2}d\right]_q^{\sim} + [3]_{q^{d/2}}[2]_{q^{\triangle}}^{\sim} \left[\frac{d}{2}\right]_q^{\sim} \sim$$
$$\sim -\left[\frac{d}{2}\right]_q [3]_{q^{d/2}} + [3]_{q^{d/2}} \left[\frac{d}{2}\right]_q^{\sim} = 0,$$

where

$$\triangle = a + \frac{3}{2}d.$$

We used repeatedly above the useful formula

$$[a]_q^{\sim} + [b]_q^{\sim} = [2]_{q^{(b-a)/2}}^{\sim} \left[\frac{a+b}{2}\right]_q^{\sim}.$$

The inductive step now will come by $2 : n \to n+2$. If we rewrite (2) as

$$\frac{1}{q-q^{-1}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{\alpha d}}^{\sim} (-1)^k (q^{a+kd} - q^{-a-kd}) = 0, \tag{4}$$

and differentiate w.r.t.a:

$$\frac{d}{da}(q^a) = hq^a, \ \frac{d}{da}q^{-a} = -hq^{-a}, h = logq,$$

we find that (4) breaks down to *two* relations:

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_{q^{\alpha d}}^{\sim} (q^{\epsilon d})^k (-1)^k = 0, \quad \epsilon = \pm 1,$$

or

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{\alpha}}^{\sim} (q^{\epsilon})^k (-1)^k = 0, \ \epsilon = \pm 1. \tag{5}$$

It is this relation we are going to prove by induction on n.

We have:

$$\begin{bmatrix} n+2 \\ k \end{bmatrix}_q^{\sim} \equiv q^{\nu_1 k} \left(\left[\begin{matrix} n+1 \\ k \end{matrix} \right]_q^{\sim} + \begin{bmatrix} n+1 \\ k-1 \end{bmatrix}_q^{\sim} \right), \ \nu_1 = \pm 1, \tag{6}$$

where

$$a \equiv b$$

means:

$$a = q^f(b)$$

and f is k-independent. Repeating (6) one more time, we get:

$$q^{\nu_1 k} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q^{\sim} \equiv q^{(\nu_1+\nu_3)k} \left(\begin{bmatrix} n \\ k-1 \end{bmatrix}^s im_q + \begin{bmatrix} n \\ k-2 \end{bmatrix}_q^{\sim} \right), \ \nu_3 = \pm 1. \tag{8}$$

Thus, (6) becomes:

$$\begin{bmatrix} n+2 \\ k \end{bmatrix}_q^{\sim} = q^{(\nu_1+\nu_2)k} \begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} + \left(q^{(\nu+1+\nu_2)k} + q^{(\nu_1+\nu_3)k} \right) \begin{bmatrix} n \\ k-1 \end{bmatrix}_q^{\sim} + q^{(\nu_1+\nu_3)k} \begin{bmatrix} n \\ k-2 \end{bmatrix}_q^{\sim}, \tag{9}$$

and induction step $n \to n+2$ is complete if we taken

$$\nu_2 = \nu_3 = -\nu_2 \quad \blacksquare$$

A Weighted Sum Of Odd Integers In The 1^{st} Quantization Is A Quantum Square

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Abstract

I find weights $\{a_n\}$ so that the formulae $\sum_{k=1}^{N} q^{a_n}[2k-1]_q = ([N]_q)^2$ is valid, where $[x]_q = \frac{q^x-1}{q-1}, x \in \mathbf{R}, q \neq 0.1$, is the 1^{st} quantization of x.

Theorem 1. For any $N \in \mathbf{Z}_{\geq 1}$,

$$\sum_{k=1}^{N} q^{n-k}[2k-1]_q = ([N]_q)^2, \tag{2}$$

is valid.

Proof. We use induction on N. For $N = 1$, (2) returns: $1 = 1$. The inductive step $N \to N+1$, amounts, in view of the formula

$$N + 1 = qS_N + [2N+1]_q, \tag{3}$$

where S_N is the LHS of (1), to

$$q([N]_q)^2 + [2N+1]_q \stackrel{?}{=} ([N+1]_q)^2, \tag{4}$$

or to

$$[2x+1] \stackrel{?}{=} ([x+1]_q^2 - q([x]_q)^2, \tag{5}$$

or to

$$(q^{2x+1} - q^{-2x-1} \stackrel{?}{<} (q - q^{-1}) \stackrel{?}{=} (q^{x+1} - q^{-x-1})^2 - q(q^x - q^{-x})^2, \tag{6}$$

or to

$$q^{2x+2} - q^{-2x} - q^{2x} + q^{-2x-2} \stackrel{?}{=}$$
$$\stackrel{?}{=} q^{2x+2} + q^{-2x-2} - (q^{2x} - q^{-2x}), \tag{7}$$

which is obvious. ■

Sums Of Quantum Two's With Variable Bases In The Second Quantization

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Abstract

Let $[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R}$, $q \neq 0, \pm 1$, be the second quantization of x. I calculate $\sum_{i=1}^n [2]_{q^i}$ in a compact form.

Theorem 1. Let $n \in \mathbf{Z}_{\geq 1}$. Then

$$\sum_{i=1}^{n} [2]_{q^i}^{\sim} = [2n+1]_{q^{1/2}}^{\sim} - 1. \tag{2}$$

Proof. For $n = 1$, (2) returns:

$$[2]_q^{\sim} \stackrel{?}{=} [3]_{q^{1/1}}^{\sim} - 1 = (q + 1 + q^{-1}) - 1, \tag{3}$$

which is true.

The inductive step $n \to n+1$ amounts to

$$[2n+1]_{q^{1/2}}^{\sim} - 1 + [2]_{q^{n+1}}^{\sim} \stackrel{?}{=} [2n+3]_{q^{1/2}}^{\sim} - 1,$$

or to

$$q^{2x+2} - q^{-2x+1} + (q - q^{-1}) \frac{q^{2x+4} - q^{4x-2}}{q^{2x+2} - q^{-2x-2}} \stackrel{?}{=} q^{2x+3} - q^{-2x-3},$$

or to

$$q^{2x+1} - q^{-2x+1} + (q - q^{-1})(q^{2x+2} + q^{-2x-2}) \stackrel{?}{=} q^{2x+3} - q^{-2x-3},$$

or to

$$q^{2x+3} - q^{-2x+1} + q^{2x+3} + q^{-2x-1} - q^{2x+1} - q^{-2x-3} = q^{2x+3} - q^{-2x-3}$$

which is obvious. ■

On A Weighted Rational Sum Of Quantum Binomial Coefficients

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Abstract

The classical formula $\sum_{n\geq 0}(-1)^n \frac{1}{nea}\binom{r}{n} = \frac{(a-1)!r!}{(a+r)!}, a>0, r\geq 0$, is quantized.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x\in \mathbf{R},\ q\neq 0,\ \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q>1$$

without loss of generality.

For $a>0,\ n\geq 0$, set

$$\begin{aligned} &y_a^r(n) = r(n+a),\\ &x_z^r(n) = q^r a(n). \end{aligned} \tag{1}$$

Theorem 2.

$$\sum_{n\geq 0}(-1)^n \frac{x_a^r(a)}{[n+a]_q^{\sim}}\begin{bmatrix} r\\ n\end{bmatrix}_q^{\sim} = \frac{[a-1]!_q^{\sim}}{[r+1]_q^{\sim}...[a+r]_q^{\sim}}, \tag{3}$$

where

$$[k]!_q^{\sim} = [1]_q^{\sim}...[k]_q^{\sim},\ k\in \mathbf{Z}_{\geq 1},\ [0]!_q^{\sim} = 1,$$

where

$$[k]!_q^{\sim} = [1]_q^{\sim}...[k]_q^{\sim},\ k \in \mathbf{Z}_{\geq 1}; [0]!_q^{\sim} = 1,$$
$$\begin{bmatrix} m \\ k \end{bmatrix}_q^{\sim} = \frac{[m]!_q^{\sim}}{[k]!_q^{\sim}[m-k]!_q^{\sim}}.$$

Proof. Denote by $B_r(a)$ and $L(a)$ the RHS and LHS of (3), respectively.

Lemma 4.

$$S_{r+1}(a) = q^a S_r(a) - q^{r+1+a} S_r(a+1). \tag{5}$$

Proof. We have:

$$S_{r+1}(a) = \frac{[a-1]!_q^{\sim}}{[r+2]_q^{\sim}...[r+1+a]_q^{\sim}} \stackrel{?}{=} q^a S_r(a) - q^{r+1+a} S_r(a+1) =$$
$$= \frac{(a-1)!_q^{\sim}}{[r+1]_q^{\sim}...[r+a]_q^{\sim}} q^a - q^{r+1+a} \frac{[a-1]!_q^{\sim}[a]_q^{\sim}}{[r+1]_q^{\sim}...[r+a+1]_q^{\sim}},$$

or

$$[r+1]_q^{\sim} \stackrel{?}{=} q^a[r+a+1] + q^{r+1+a}[-a]_q^{\sim},$$

which is true, because, in general,

$$[x+y]_q^{\sim} = q^{-y}[x]_q^{\sim} + q^x[y]_q^{\sim}. \quad \blacksquare$$

$$[r+1]_q^{\sim} \stackrel{?}{=} q^a[r+a+1] = q^{r+1+a}[-a]_q^{\sim},$$

which is true, because, in general,

$$[x+y]_q^{\sim} = q^{-y}[x]_q^{\sim} + q^x[y]_q^{\sim}. \quad \blacksquare$$

We are going to use induction on r. For $r = 0$, (3) returns:

$$\frac{a}{[a]_q^{\sim}} \stackrel{?}{=} \frac{[a-1]!_q^{\sim}}{[a]!_q^{\sim}} = \frac{1}{[a]_q^{\sim}},$$

which is true.

We next show that the LHS $L_r(a)$ satisfies the same recursion (5) as the RHS $S_r(a)$. So,

$$L_{r+1}(a) = \sum_{s=0}(-1)^s \frac{q^{(r+1)(s+a)}}{[s+a]_q^{\sim}} \begin{bmatrix} r+1 \\ s \end{bmatrix}_q^{\sim} =$$
$$= \sum_s (-1)^s \frac{q^{(r+1)(s+a)}}{[s+a]_q^{\sim}} \left\{ q^{-s} \begin{bmatrix} r \\ s \end{bmatrix}_q^{\sim} + q^{r+1-s} \begin{bmatrix} r \\ s-1 \end{bmatrix}_q^{\sim} \right\} =$$
$$= q^a S_r(a) - q^{r+1+a} \sum_k (-1)^k \frac{q^{r(k+a+1)}}{[k+a+1]} \begin{bmatrix} r \\ k \end{bmatrix}_q^{\sim} = q^a S_r(a) - q^{a+r+1} S_r(a+1). \quad \blacksquare$$

The Classical Limit Of A Skewsymmetric Quantum Form

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Abstract

For $a, b, x + \mathbb{R}$, let $f(a, b, i) = \frac{1}{q-1}([x]_{q^a} - [x]_{a^b}$, where $[x]_q = \frac{q^x-1}{q-1}, x \in \mathbb{R}, q \neq 0, 1,$ is the first quantization of x. We calculate $\lim_{q\to 1} f(a, b, x)$.

Theorem 1. For $a, b, x \in \mathbb{R}$,

$$\lim_{q\to 1} f(a, b, x) = \frac{x^2}{2}(a - b). \tag{1}$$

Proof. Let $q = e^h, h \to 0$. Then

$$\frac{q^{ax} - 1}{q^a - 1} = \frac{1}{h^a}\left\langle hax + \frac{1}{2}(hax)^2 + ... \right\rangle =$$
$$= x + \frac{1}{2}hax^2 + ... \tag{2}$$

Therefore

$$\frac{[x]_{q^a} - [x]_{q^b}}{q - 1} = \frac{1}{h}\left\{\frac{1}{2}hx^2(a - b) + ...\right\} \to \frac{x^2}{2}(a - b) \quad \blacksquare \tag{3}$$

A Sum Of Odd Integers Is A Square In The 2^{nd} Quantization

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Abstract

Classically, $\sum_{k=1}^{N}(2k-1)=N^2$. I quantize this formula.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\ x\in\mathbb{R},\quad q\neq 0,\pm 1,$$

be the 2^{nd} quantization of x. We shall prove

Theorem 1.

$$\sum_{k=1}^{N}[2k-1]_q^{\sim}=([N]_q^{\sim})^2,\quad N\in\mathbf{Z}_{\geq 1}. \tag{1}$$

Proof. For $N=1$, (11) returns: $1=1$. We use induction on N. The inductive step amounts to

$$([N])+[2N+1]_q^{\sim}\stackrel{?}{=}([N+1]_q^{\sim})^2, \tag{2}$$

or

$$[2N+1]_q^{\sim}\stackrel{?}{=}([N+1]_q^{\sim})^2-([N]_q^{\sim})^2 \tag{3}$$

which is true because, in general,

$$([a]_q^{\sim})^2-([b]_q^{\sim})^2=[a-b]_q^{\sim}[a+b]_q^{\sim}, a,b\in\mathbb{R}> \tag{4}$$

Indeed, the LHS of (4) is

$$\left(\frac{q^a-q^{-a}}{q-q^{-1}}\right)^2-\left(\frac{q^b-q^{-b}}{q-q^{-1}}\right)^2=$$
$$\frac{1}{(q-q^{-1})^2}\left\{q^{2a}+q^{-2a}-(q^{2b}+q^{-2b})\right\}. \tag{5}$$

The RHS of (4), times $(q-q^{-1})^2$, is:

$$(q^{a-b}-q^{b-a})(q^{a+b}-q^{-a-b})=$$
$$=q^{2a}-q^{-2b}-q^{-2b}+q^{-2a}=(q^{2a}+q^{-2a})-(q^{2b}-q^{-2b}), \tag{6}$$

which is the same as (5) multiplied by $(q-q^{-1})^2$ ■.

The Sum Of The Odd Quantum Integers Is A Square In The 2^{nd} Quantization

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Abstract

Let $[x]_q = \frac{q^x-1}{q-1}, x \in \mathbb{R}, \ \ q \neq q \pm 1$, be the 2^{nd} quantization of x. $\sum_{i=1}^{N}[2K-1]_q^{\sim} = ([N]_q^{\sim})^2$.

The classical formulae

$$\sum_{K=1}^{N}(2k-1) = N \tag{1}$$

can be quantized thusly:

Theorem 1. Let $N \in \mathbb{Z}_{\geq 1}$. Then

$$\sum_{k=1}^{N}[2n-1]_q^{\sim} = ([N]_q^{\sim})^2. \tag{2}$$

Proof. We use induction on N for $N = 1$, (2) returns: $1 = 1$. The inductive step $N \to N+1$, amounts to:

$$([N]_q^{\sim})^2 + [2N+1]_q^{\sim} \stackrel{?}{=} ([N+1]_q^{\sim})^2, \tag{3}$$

or

$$[2x+1]_q^{\sim} \stackrel{?}{=} ([x+1]_q^{\sim})^2 - ([x]_q^{\sim})^2, \tag{4}$$

which follows from more general formulae:

$$([a]_q^{\sim})^2 - ([b]_q^{\sim})^2 = [a-b]_q^{\sim}[a+b]_q^{\sim}. \tag{5}$$

To prove, (5), multiply both sides of it by $(q-q^{-1})^2$. We get:

$$(q^a - q^{-a})^2 - (q^b - q^{-b})^2 \stackrel{?}{=}$$
$$\stackrel{?}{=} (q^{a-b} - q^{b-a})(q^{a+b} - q^{-a-b}). \tag{6}$$

The LHS of (6) is:

$$(q^{2a} + q^{-2a}) - (q^{2b} + q^{-2b}). \tag{7a}$$

The RHS of (6) is:

$$q^{2a} - q^{-2b} - q^{-2b} + q^{-2a}, \tag{7b}$$

which is the same as (7a). ■

Sums Of Odd Consequitive Integers In The 1^{st} Quantization

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Abstract

Let $[x]_q = \frac{q^x-1}{q-1}, x \in \mathbb{R}, \ \ q \neq 0, 1$, the first quantization of x. In that quantization, I quantize the classical relation $sum_{i=1}^{N}(2i-1) = \mathcal{N}^2$ for any integer $\mathcal{N}$.

Theorem. Let $\mathcal{N} \in \mathbb{Z}_{\geq 1}$. Then

$$\sum_{i=1}^{N} q^{N-i}[2i-1] - ([N]_q)^2. \tag{2}$$

Proof. Denote the LHS of (2) by S_N. Then

$$S_{N+1} = qS_N + [2N+1]_q, \tag{3}$$

and our conjecture (2) is:

$$S_N = ([N]_q)^2, \tag{4}$$

or, since

$$S_1 = 1 = ([1]_q)^2, \tag{5}$$

using induction N, (3) is

$$([N+1]_q^s)^2 = ([N]_q)^2 + [2N+1]_q, \tag{6}$$

or

$$[2x+1]_q = ([x+1]_q)^2 - q([x]_q)^2, \ x \in \mathbb{R}. \tag{8}$$

Rewriting (8), we get

$$[2x+1]_q(q-1)^2 = (q^{2x+1}-1)(q-1) = q^{2x+2} - q - q^{2x+1} + \stackrel{?}{=} \tag{9a}$$
$$\stackrel{?}{=} (q^{x+1}-1)^2 - q(q^x-1)^2 =$$
$$= q^{2x+2} - 2q^{x+1}q(q^{2x} - 2^x q) + 1 - q =$$
$$= q^{2x+1}(q-1) - 2q^x \cdot 0 = (q^{2x+1}-1)(q-1), \tag{9b}$$

which is (9a). ■

A Quantum Version Of A Multidimensional Analog Of Young's Inequality

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Abstract

The inequality in question reads: (1) $\sum_{i=1}^m \frac{x_i^{p_i}}{P_i}]_q >, > [c]_q ... [x_n]_q$, where $\frac{1}{P-i} + ... + \frac{1}{P_m} = 1,\ P_i, x_i > 0, [x]_q = \frac{qx-1}{q-1}$. This inequality is proved below.

Proof. By continuity, it's enough to consider the case then all the p_i are rational. Say,

$$P_i = \frac{r_i}{s_i},\ r_i s_i \in\ \mathbb{Z}_{>0},\ i = 1, ..., m. \tag{2}$$

Therefore,

$$1 = \sum_{i=1}^m \frac{1}{P_i} = \sum \frac{s_i}{r_i} = \frac{1}{\Pi_r} \sum_i \frac{s_i \Pi_r}{r_i}, \tag{3}$$

where

$$\Pi_r = r_1 ... r_m. \tag{4}$$

Hence,

$$\begin{aligned} &\left[\sum_{i=1}^m \frac{x_i^{p_i}}{P_i} \right]_q = \sum_i \left[x_i^{p_i} \frac{S_i}{r_i} \frac{1}{\Pi_r} \right]_q = \\ &= \sum_i \left[\frac{x_i^{p_i}}{\sum_j h_j} h_i \Big(\sum_k h_k \Big) \frac{1}{\Pi_r} \right]_q, \end{aligned} \tag{5}$$

where

$$n_i = S_i \frac{\Pi_r}{r_i}. \tag{6}$$

Thus,

$$\left[\sum_i \frac{x^{P_i} n_i}{(\sum_j n_j)} \Big(\sum_j n_j \Big) \frac{1}{P_r} \right]_q \geq \Pi_i \Big[x_i P_i \Big]_q = [x_1]_q ... [x_m]_q, \tag{7}$$

which is (1). Above I said:

$$\sum_j h_j = P_r, \tag{8}$$

$$P_i n_i = 1, \quad \forall_i, \tag{9}$$

and the quantum AGM inequality:

$$\left[\frac{x_1 + ...x_m}{m}\right]_q \geq [x]_q ... [x_m]_q. \tag{10}$$

A Multidimensional Analog Of The Young's Inequality

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Abstract

A multidimensional version of the classical Young's inequality is derived.

Theorem 1. Let $p_1, ..., p_n > 0$ with $\sum_{i=1}^m \frac{1}{p_i} = 1$, and let $x_1, ..., x_m > 0$. Then

$$\sum_{i=1}^m \frac{x_i^{p_i}}{p_i} \geq x_1 ... x_m. \tag{2}$$

Proof. By continuity, we can consider all the p_i's be rational; say

$$p_i = \frac{r_i}{s_i},\ r_i, s_i \in \mathbf{Z}_{>0},\quad i = 1, ..., m. \tag{3}$$

Then

$$1 = \sum_{i=1}^m \frac{1}{p_i} = \sum_i \frac{1}{r_i} = \frac{1}{\Pi_r} \sim_i s_i \frac{\Pi_r}{r_i} \tag{4}$$

where

$$\Pi_r = r_1 ... r_n \tag{5}$$

and $\frac{\Pi_r}{r_i}$ is an integer. Then

$$\sum_{i=1}^m \frac{x_1^{p_i}}{p_i} = \sum \frac{x_i^{p_i} s_i}{r_i} = \frac{1}{\Pi_r} \sum \frac{s_i \Pi_r}{r_i} x_i^{p_i} =$$
$$\frac{r_i^{\frac{s_i \Pi_r}{x} p_i}}{\sum_j \frac{s_j \P_r}{r_j}} [\text{ by } AGM] \geq$$
$$\geq \Pi_i [x_i \frac{s_i}{r_i} \Pi_r]^{1/\sum_j \frac{s_j}{r_j} \Pi_r} =$$
$$= \Pi_i [x_i^{\Pi_r}]^{1/\Pi_r} = \Pi_i x_i,$$

which is (2) ■

On The Product Of Quantum Powers

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Abstract

The classical inequality $1^1 \cdot 2^2 \cdot .n^n < n^{\frac{n(n+1)}{2}}$ is quantized.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbb{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Theorem 1. For $n \in \mathbb{Z}_{\geq 1}$,

$$([1]_q^{\sim})^1([2]_q^{\sim})^2 ... ([n]_q^{\sim})^n \leq ([n]_q^{\sim})^{n(n+1)/2}. \tag{2}$$

Proof. The inequality is as obvious as it is its classical counterpart.

Since

$$[k]_q^{\sim} \leq [n]^s im_q$$

for $k \leq n$, the LHS of (2) is $<$ than

$$([n]_q^{\sim})([n]_q^{\sim})^2 ... ([n]_q^{\sim})^n = ([n]_q^{\sim})^{n(n+1)/2}. \quad \blacksquare$$

The Proof by induction amounts to the same argument. For $n = 1$, we get

$$1 \leq 1,$$

which is true, and the induction step is:

$$([n]_q^{\sim})^{n(n+1)/2}([n+1]_q^{\sim})^{n+1} \stackrel{?}{\leq} ([n+1]_q^{\sim})^{(n+1)(n+2)/2},$$

or

$$([n]_q^{\sim})^{n(n+1)/2} \stackrel{?}{\leq} ([n+1])_q^{\sim})^{n(n+1)/2},$$

or

$$[n]_q^{\sim} \leq [n+1]_q^{\sim},$$

which is true because

$$n < n+1. \quad \blacksquare$$

The Growth Of The Sum Of Two Powers Of q

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Abstract

We show that $q^x + q^{y-x}$ decreases for $0 \le x \le y/2$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbb{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Theorem 1. Let $0 \le x \le y/2$. Then

$$f(x) = q^x + q^{y-x}$$

is decreasing with x increasing.

Proof. With

$$h = log\ q > 0,$$

we have:

$$\frac{\partial}{\partial x}(q^x + q^{y-x}) = \frac{\partial}{\partial x}(ehx + q^y e^{-h}) = h \text{ times}: \ q^x - q^{y-x},$$

and

$$q^x - q^{y-x} \stackrel{?}{\le} 0,$$

or

$$q^{2x} \le q^y,$$

which is true because $2x \le y$ and $q > 1$. ■

On A Mixture of Classical And Quantum Numbers

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Abstract

We decompose the said mixture into a sum of quantum twos.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbb{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Theorem 1. For $n\ell \mathbf{Z}_{\geq 2}$,

$$\left(\begin{bmatrix} n \\ q \end{bmatrix}_{1/2}^{\sim} \right)^2 - n = \sum_{k=0}^{n-2} [2]_{q^{1+K}}^{\sim} (n - 1 + k). \tag{2}$$

Proof. We use induction on n. For $n = 2$, (2) returns:

$$([2]_{q^{1/2}}^{\sim})^2 - 2 \stackrel{?}{=} [2]_q^{\sim},$$

which is obvious.

The induction step $n \mapsto n + 1$, with the help of the formula

$$([x+1]_q^{\sim})^2 = ([x]_q^{\sim})^2 + [2x+1]_q^{\sim},$$

amounts to

$$([2n+1]_{q^{1/2}}^{\sim}) - 1 \stackrel{?}{=} \sum_{k=0}^{n-2} [2]_{q^{1+k}}^{\sim} + [2]_{q^n}^{\sim},\ n \geq 2. \tag{3}$$

We use induction on n again. For $n = 2$, (3) returns:

$$[5]_{q^{1/2}}^{\sim} - 1 \stackrel{?}{=} [2]_q^{\sim} + [2]_{q^2}^{\sim},$$

which is obvious.

The induction step $n \mapsto n+1$ amounts to:

$$[2n+3]^{\sim}_{q^{1/2}} - [2n+1]^{\sim}_{q^{1/2}} \stackrel{?}{=} [2]^{\sim}_{q^n} + [2]^{\sim}_{q^{n+1}} - [2]^{\sim}_{q^n}, \tag{4}$$

or, because

$$[x+2]^{\sim}_q - [x]^{\sim}_q = [2]^{\sim}_{q^{n+1}}, x \in \mathbb{R},$$

(4) becomes:

$$[2]^{\sim}_{q^{n+1}} = [2]^{\sim}_{q^{n+1}},$$

which is true. ■

On The Product Of Two Successive Odd Quantum Numbers

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Abstract

The product of two successive odd numbers is a *sum* of odd numbers. This is quantized.

The table

$$\begin{array}{ccccc} & & 3 & & \\ & 3 & 5 & 7 & \\ 3 & 5 & 7 & 9 & 11, \end{array}$$

has, for the sum of it rows, $3 = 1.3, 15 = 3.5, 35 = 5.7$, ... This suggests

$$\sum_{k=0}^{2\ell}(3+2k) = (2\ell+1)(2\ell+3), \ell \in \mathbf{Z}_{\geq 0} \tag{1}$$

which is obvious.

We aim to quantize (1).

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0, \ \pm 1,$$

be the second quantization of x.

Theorem 1. For $\ell \in \mathbf{Z}_{\geq 0}$,

$$\sum_{k=0}^{2\ell}[3+2k]_q^{\sim} = [2\ell+1]_q^{\sim}[2\ell+3]_q^{\sim}. \tag{2}$$

Proof. We use the formula for the sum of quantum arithmetic progression:

$$\sum_{k=0}^{N}[a+kd]_q^{\sim} = [N+1]_{q^{d/1}}^{\sim}\left[\frac{a+(a+Nd)}{2}\right]_q^{\sim}.$$

In our case, $N = 2\ell, d = 2, \frac{a+(a+Nd)}{2} = \frac{3+(3+2\ell)}{2} = 2\ell+3$, and (2) follows. ■

A 3-Parameter Family Of Quantum Decompositions

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Abstract

The classical identity $2n^2 = \sum_{s=0}^{n-1}(2+4s)$, is generalized, quantized, and then generalized again.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbb{R},\ q \neq 0, \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take,

$$q > 1$$

without loss of generality.

Our main tool will be the formula for the sum of quantum arithmetic progression:

$$\sum_{i=0}^{N}[a+id]_q^{\sim} = [N+1]_{q^{d/2}}^{\sim}\left[\frac{a+(a+Nd}{2}\right]_q^{\sim}, \tag{1}$$

which is easy enough to verify.

We observe that

$$[1]_q^{\sim}[2]_q^{\sim} = [2]_q^{\sim},$$
$$[2]_{q^2}^{\sim}[4]_q^{\sim} = [2]_q^{\sim} + [6]_q^{\sim},$$
$$[3]_{q^2}^{\sim}[6]_q^{\sim} = [2]_q^{\sim} + [6]_q^{\sim} + [10]_q^{\sim},$$

and, in general,

$$[n]_{q^2}^{\sim}[2n]_q^{\sim} = \sum_{s=0}^{n-1}[2+4s]_q^{\sim}. \tag{2}$$

Similarly,

$$[1]_q^{\sim}[4]_q^{\sim} = [4]_q^{\sim},$$
$$[2]_{q^2}^{\sim}[6]_q^{\sim} = [4]_q^{\sim} + [8]_q^{\sim},$$
$$[3]_{q^2}^{\sim}[8]_q^{\sim} = [4]_q^{\sim}[8]_q^{\sim}[12]_q^{\sim},$$
$$...$$
$$[n]_{q^2}^{\sim}[2n+2]_q^{\sim} = \sum_{s=0}^{n-1}[4+4s]_q^{\sim}. \tag{3}$$

These observations are subsumed by the general **Theorem 4.** For $n \in \mathbf{Z}_{\geq 1}, k \in \mathbf{Z}$,

$$[n]_{q^2}^{\sim}[2n+2k]_q^{\sim} = \sum_{s=0}^{n-1}[2k+2+4s]_q^{\sim}. \tag{5}$$

Proof. Using (1) for the RHS of (5), we get:

$$[n]_{q^2}^{\sim}\left[\frac{2k+2+(2k+2+4n-4)}{2}\right]_q^{\sim} = [n]_{q^2}^{\sim}[2k+2n]_q^{\sim},$$

i.e., the LHS of (5). ■

However, formula (5) can be generated itself.

Theorem 6. For $n \in \mathbf{Z}_{\geq 1}, d, k \in \mathbf{Z}$,

$$[n]_{q^d}^{\sim}[dn+2k+2-d]_q^{\sim} = \sum_{s=0}^{n-1}[2k+2+2ds]_q^{\sim}. \tag{7}$$

Proof. Applying (1) to the RHS of (7), we get:

$$[n]_{q^d}^{\sim}\left[\frac{2k+2+(2k+2+2dn-2d)}{2}\right]_q^{\sim} =$$
$$= [n]_{q^d}^{\sim}[2k+2+dn-d]_q^{\sim},$$

i.e., the LHS of (7). ■

Quantization Of The Euler Identity

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Abstract

Euler proved that $\sum_{i=1}^n \frac{a_i-1}{a_1-a_i} = 1 - \frac{1}{a_1 \ldots a_n}$. We quantize this.

Let

$$[x]_q^\sim = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbb{R},\ q \neq 0, \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^\sim = -[x]_q^\sim,$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^\sim = [x]_q^\sim,$$

we can take,

$$q > 1$$

without loss of generality.

Theorem 1. Set

$$\prod_n = [a_1]_q^\sim \ldots [a_n]_q^\sim,$$

$$Q_n = q^{\sum_n}, \sum_n = \sum_{i=1}^n (a_i - 1);\ Q_n = 1.$$

Then

$$\sum_{i=1}^n \frac{[a_i - 1]_q^\sim}{\prod_i} Q_{i-1} = q\left(1 - \frac{Q_n}{\prod_n}\right). \tag{2}$$

Proof. We use induction on n. For $n = 1$, (2) returns, with $a = a_1$:

$$\frac{[a-1]_q^\sim}{[a]_q^\sim} \stackrel{?}{=} q\left(1 - \frac{Q_1}{[a]_q^\sim}\right),$$

or

$$q^{-1}[a-1]_q^{\sim} \stackrel{?}{=} [a]_q^{\sim} - q^{a-1},$$

or

$$q^{-1}[a-1]_q^{\sim} + q^{a-1} \stackrel{?}{=} [a]_q^{\sim},$$

which is true because, in general,

$$[x+y]_q^{\sim} = q^{-y}[x]_q^{\sim} + q^x[y]_q^{\sim}. \tag{3}$$

Now, the inductive step $h \to n+1$, amounts to:

$$q\Big(1 - \frac{Q_n}{\prod_n}\Big) + \frac{[a_{n+1}-1]_q^{\sim}}{\prod_{n+1}} Q_n \stackrel{?}{=} q(1 - \frac{Q_{n+1}}{\prod_{n+1}}\Big),$$

or

$$-[a_{n+1}]_q^{\sim} + q^{-1}[a_{n+1}-1]_q^{\sim} \stackrel{?}{=} -q^{a_{n+1}-1},$$

or

$$q^{-1}[a_{n+1}-1]_q^{\sim} + q^{a_{n_1}-1} \stackrel{?}{=} [a_{n+1}]_q^{\sim},$$

which is true by (3). ■

On The Ratio Quantum Bi-Factorials

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Abstract

$\frac{(2n-1)!!}{(2n)!!} \to 0$ as $n \to \infty$. We remark that this limit is preserved under quantization.

Let

$$[x]_q^\sim = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbb{R},\ q \neq 0, \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^\sim = -[x]_q^\sim,$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^\sim = [x]_q^\sim,$$

we can take,

$$q > 1$$

without loss of generality.

For $n \in \mathbf{Z}_{\geq 1}$, set

$$[2n-1]!!_q^\sim = \prod_{s=1}^{n} [2s-1]_q^\sim,$$
$$[2n]!!_q^\sim = \prod_{s=1}^{n} [2s]_q^\sim.$$

Theorem 1.

$$\lim_{n\to\infty} \frac{[2n-1]!!_q^\sim}{[2n]!!_q^\sim} = 0. \tag{2}$$

Proof Since

$$\frac{[2n-1]!!_q^\sim}{[2n]!!_q^\sim} = \prod_{s=1}^{n} \frac{[2s-1]_q^\sim}{[2s]_q^\sim}, \tag{3}$$

it's enough to show that

$$\frac{[2s-1]_q^{\sim}}{[2s]_q^{\sim}} < q^{-1}. \tag{4}$$

(4) follows from

Lemma 5. For $x, y > 0$,

$$\frac{[y]_q^{\sim}}{[x+y]_q^{\sim}} < q^{-x}. \tag{6}$$

Proof. (6) is:

$$[y]_q^{\sim} \stackrel{?}{<} q^{-x}[x+y]_q^{\sim}. \tag{7}$$

Multiplying this by $q - q^{-1} > 0$, we get:

$$q^y - q^{-y} \stackrel{?}{<} q^{-x}(q^{x+y} - q^{-x-y}),$$

or

$$q^{-2x-y} < q^{-y},$$

or

$$q^{-2x} < 1,$$

which is obvious because $x > 0$. ■ ■

A Quantum Inequality With No Classical Analog

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Abstract

An equality $c = c$ becomes inequality when quantized.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbb{R},\ q \neq 0, \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take,

$$q > 1$$

without loss of generality.

Theorem 1. Let $a, b > 0$. Then

$$[a]_q^{\sim}[b]_q^{\sim} \geq ([\sqrt{ab}]_q^{\sim})^2. \tag{2}$$

Proof. Multiplying through (2) by $(q - q^{-1})^2 > 0$, we get:

$$(q^a - q^{-a})(q^b - q^{-b}) \stackrel{?}{\geq} (q^{\sqrt{ab}} - q^{-\sqrt{ab}})^2,$$

or

$$(q^{a+b} + q^{-a-b}) - (q^{a-b} + q^{b-a}) \stackrel{?}{\geq} (q^{\sqrt[n]{ab}} + q^{-2\sqrt{ab}}) - 2. \tag{3}$$

Set

$$h = logq > 0.$$

Then (3) becomes

$$2\sum_{n=0}^{\infty}\frac{h^{2n}}{(2n)!}[(a+b)^{2n}-(a-b)^{2n}]\geq 2\sum n-0\frac{h^{2n}}{(2n)!}(2\sqrt{ab})^{2n}. \tag{4}$$

follows from

$$(a+b)^{2n}-(a-b)^{2n}\overset{?}{\geq}2^{2n}(ab)^n,\ n\geq 1. \tag{5}$$

Notice that for $n=1$ and for $b=a$, any n, (5) becomes an equality.
Since $a,b>0$, set

$$a=\lambda^2,\ b=B^2.$$

Then (5) becomes:

$$(A^2+B^2)^{2n}-(A^2-B^2)^{2n}\overset{?}{\geq}2^{2n}(AB)^{2n},$$

or

$$\left(\frac{A}{B}+\frac{B}{A}\right)^{2n}-\left(\frac{A}{B}-\frac{B}{A}\right)^{2n}\geq 2^{2n}, \tag{6}$$

Let $A\geq b$. Set

$$\lambda=\frac{A}{B}\geq 1.$$

Then (6) is:

$$\left(\lambda+\frac{1}{\lambda}\right)-\left(\lambda-\frac{1}{\lambda}\right)^{2n}\overset{?}{\geq}2^{2n},\ \lambda\geq 1,\quad n\geq 1, \tag{7}$$

or

$$\sum_k\binom{2n}{k}\lambda^{2n-k}\left[\left(\frac{1}{\lambda}\right)^k-\left(-\frac{1}{\lambda}\right)^k\right]\Rightarrow 2^{2n},$$

or, with

$$\Lambda^{n-k}=\lambda^2\geq 1,$$
$$\sum_{k\ odd}\binom{2n}{k}\Lambda^{n-k}\overset{?}{\geq}2^{2n-1}. \tag{8}$$

Now,

$$\Lambda^{n-k}+\Lambda^{2n-(n-k)}=\Lambda^{n-k}+\Lambda^{n+k}=\Lambda^n(\Lambda^k+\Lambda^{-k})\geq 2.$$

So, for n even (8) follows from

$$2\sum_{k\ odd\ <n}\binom{2n}{k}\overset{?}{\geq}2^{2n-1},$$

or

$$\sum_{k\ odd} \binom{2n}{k} \stackrel{?}{\geq} 2^{2n-1},$$

which is true, because

$$\sum_{k\ even} \binom{m}{k} = \sum_{k\ odd} \binom{m}{k} = 2^m/2. \tag{9}$$

If, n is odd, (8) becomes:

$$\sum_{k\ odd\ <n} \Lambda^n(\Lambda^k + \Lambda^{-k})\binom{2n}{k} + \binom{2n}{n}\Lambda^0 \stackrel{?}{\geq} 2^{2n-1},$$

or

$$\sum_{k\ odd} \binom{2n}{k} \geq 2^{2n-1},$$

which is true by (9). ■

On The Ratio Of Two Quantum Double Factorials

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Abstract

Classically, $\frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{2n+1}}$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbb{R},\ q \neq 0, \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take,

$$q > 1$$

without loss of generality.

Theorem 1. Let $a, b > 0$. Then

$$[a]_q^{\sim}[b]_q^{\sim} \geq ([\sqrt{ab}]_q^{\sim})^2. \tag{2}$$

Proof. Multiplying through (2) by $(q - q^{-1})^2 > 0$, we get:

$$(q^a - q^{-a})(q^b - q^{-b}) \stackrel{?}{\geq} (q^{\sqrt{ab}} - q^{-\sqrt{ab}})^2,$$

or

$$(q^{a+b} + q^{-a-b}) - (q^{a-b} + q^{b-a}) \stackrel{?}{\geq} (q^{\sqrt[n]{ab}} + q^{-2\sqrt{ab}}) - 2. \tag{3}$$

Set

$$h = logq > 0.$$

Then (3) becomes

$$2\sum_{n=0}^{\infty}\frac{h^{2n}}{(2n)!}[(a+b)^{2n}-(a-b)^{2n}]\geq 2\sum n-0\frac{h^{2n}}{(2n)!}(2\sqrt{ab})^{2n}. \tag{4}$$

follows from

$$(a+b)^{2n}-(a-b)^{2n}\overset{?}{\geq}2^{2n}(ab)^n,\ n\geq 1. \tag{5}$$

Notice that for $n=1$ and for $b=a$, any n, (5) becomes an equality.
Since $a,b>0$, set

$$a=\lambda^2,\ b=B^2.$$

Then (5) becomes:

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Let $A\geq b$. Set

$$\lambda=\frac{A}{B}\geq 1.$$

Then (6) is:

$$\left(\lambda+\frac{1}{\lambda}\right)-\left(\lambda-\frac{1}{\lambda}\right)^{2n}\overset{?}{\geq}2^{2n},\ \lambda\geq 1,\quad n\geq 1, \tag{7}$$

or

$$\sum_k\binom{2n}{k}\lambda^{2n-k}\left[\left(\frac{1}{\lambda}\right)^k-\left(-\frac{1}{\lambda}\right)^k\right]\Rightarrow 2^{2n},$$

or, with

$$\Lambda^{n-k}=\lambda^2\geq 1,$$
$$\sum_{k\ odd}\binom{2n}{k}\Lambda^{n-k}\overset{?}{\geq}2^{2n-1}. \tag{8}$$

Now,

$$\Lambda^{n-k}+\Lambda^{2n-(n-k)}=\Lambda^{n-k}+\Lambda^{n+k}=\Lambda^n(\Lambda^k+\Lambda^{-k})\geq 2.$$

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or

$$\sum_{k\ odd} \binom{2n}{k} \stackrel{?}{\geq} 2^{2n-1},$$

which is true, because

$$\sum_{k\ even} \binom{m}{k} = \sum_{k\ odd} \binom{m}{k} = 2^m/2. \tag{9}$$

If, n is odd, (8) becomes:

$$\sum_{k\ odd\ <n} \Lambda^n(\Lambda^k + \Lambda^{-k})\binom{2n}{k} + \binom{2n}{n}\Lambda^0 \stackrel{?}{\geq} 2^{2n-1},$$

or

$$\sum_{k\ odd} \binom{2n}{k} \geq 2^{2n-1},$$

which is true by (9). ■

On The Product Of Two Successive Odd Quantum Numbers

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Abstract

The product of two successive odd numbers is a *sum* of odd numbers. This is quantized.

The table

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has, for the sum of it rows, $3 = 1.3, 15 = 3.5, 35 = 5.7$, ... This suggests

$$\sum_{k=0}^{2\ell}(3+2k) = (2\ell+1)(2\ell+3), \ell \in \mathbf{Z}_{\geq 0} \tag{1}$$

which is obvious.

We aim to quantize (1).

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \quad q \neq 0, \ \pm 1,$$

be the second quantization of x.

Theorem 1. For $\ell \in \mathbf{Z}_{\geq 0}$,

$$\sum_{k=0}^{2\ell}[3+2k]_q^{\sim} = [2\ell+1]_q^{\sim}[2\ell+3]_q^{\sim}. \tag{2}$$

Proof. We use the formula for the sum of quantum arithmetic progression:

$$\sum_{k=0}^{N}[a+kd]_q^{\sim} = [N+1]_{q^{d/1}}^{\sim}\left[\frac{a+(a+Nd)}{2}\right]_q^{\sim}.$$

In our case, $N = 2\ell, d = 2, \frac{a+(a+Nd)}{2} = \frac{3+(3+2\ell)}{2} = 2\ell+3$, and (2) follows. ■

On A Quantum Chebyshev-Like Inequality

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Abstract

If $a_1 < ... < a_n,\ b_1, ..., b_n$ are two sequences, then it is known that $\sum_{i=1}^n (a_i b_i)^2$ is minimized when $b_1 \leq b_2 \leq ... \leq b_n$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbb{R},\ q \neq 0, \pm 1,$$

be the second quantization of x.

Let $a_1 \leq ... \leq a_n$ be given. How the sequence $b_1, ..., b_n$ must be arranged to minimize

$$S' = \sum_{i=1}^n ([a_i - b_i]_q^{\sim})^2; \tag{1}$$

Classically, at $q = 1$, it is known that the best permutation of the b_i's is such that

$$b_1 \leq b_2 \leq \leq b_n. \tag{2}$$

Theorem 3. S is minimized by the same arrangement (2).

Proof. We have to show that for $a_1 \leq a_z,\ x \leq y$,

$$([a_1 - x]_q^{\sim})^2 + ([a_2 - y]_q^{\sim})^2 \stackrel{?}{\leq} ([a_1 - y]_q^{\sim})^2 + ([a_2 - x]_q^{\sim})^2 \tag{3}$$

We use the easily verified formula

$$([x]_q^{\sim})^2 - ([y]_q^{\sim})^2 = [x - y]_q^{\sim}[x + y]_q^{\sim}. \tag{4}$$

Then (3), in the form

$$([a_1 - x]_q^{\sim})^2 - ([a_1 - y]_q^{\sim})^2 \stackrel{?}{\leq} ([a_2 - x]_q^{\sim})^2 - ([a_q - y]_q^{\sim})^2,$$

becomes

$$[y - x]_q^{\sim}[2a_1 - \sigma]_q^{\sim} \stackrel{?}{\leq} [y - x]_q^{\sim}[2a_2 - \sigma]_q^{\sim},\ \sigma = x + y,$$

or, since $y - x \geq 0$ and

$$\begin{aligned} &a > b \Rightarrow [a]_q^{\sim} > [b]_q^{\sim}, \\ &[2a_i - \sigma]_q^{\sim} \leq [2a_2 - \sigma]_q^{\sim}, \end{aligned} \tag{5}$$

which is true by (5). ■

An Estimate For Quantum Factorials

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Abstract

The classical inequality $n!^2 \geq n^2$, is quantized.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbb{R},\ q \neq 0, \pm 1,$$

be the second quantization of x.

Theorem 1. Let $0 \leq a,\ b \leq 1,\ a + b = 1$. Then

$$[a]_q^{\sim}[b]_q^{\sim} \leq \left(\left[\frac{1}{2}\right]_q^{\sim}\right)^2, \tag{2a}$$

$$[ab]_q^{\sim} \leq \left[\frac{1}{4}\right]_q^{\sim}. \tag{2b}$$

Proof. (a) Multiplying through (2a) by $(q - q^{-1})^2 > 0$, we get:

$$(q^a - q^{-a})(q^b - q^{-b})\left[(q - q^{-1})\frac{q^{1/2} - q^P - 1/2}{q - q^{-1}}\right]^2,$$

or

$$(q^{a+b} + q^{-a-b}) - (q^{a-b} + q^{b-a}) \overset{?}{\leq} (q^{1/2} - q^{-/2})^2 = q + q^{-1} - 2,$$

or

$$2 \overset{?}{\leq} q^{a-b} + q^{b-a},$$

which is obvious.

(b) Multiplying through (2b) by $q - q^{-1} > 0$, we get.

$$q^{ab} - q^{-ab} \overset{?}{\leq} q^{1/4} - q^{-1/4},$$

which follows from the Lemma below because

$ab \leq 1/4$. *blacksquare*

Lemma 3. The function

$$f(x) = q^x - q^{-x}$$

is increasing with x.

Proof. We have:

$$\frac{df}{dx} = h(q^x + q^{-x}) > 0,$$

where

$$h = logq > 0. \quad \blacksquare$$

An Inequality With Absolute Values For Quantum Numbers

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Abstract

A classical inequality $(y+2)+(y)+(y-2) \geq 4$ is quantized.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbb{R},\ q \neq 0, \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take,

$$q > 1$$

without loss of generality.

Theorem 1. For $y \in \mathbb{R}$,

$$|[y+2]_q^{\sim}| + |[y]_q^{\sim}| + |[y-2]_q^{\sim}| \geq 2[2]_q^{\sim}. \tag{2}$$

Proof. The LHS of (2) is invariant with respect to change of $y \neq 0$ into y^{-1}. (For $y = 0$, (2) turns into an equality). So, we can restruct ourselves to

$$y > 0 \tag{3}$$

only.

If $y \geq 2$, (2) is satisfied because

$$[y+2]_q^{\sim} \geq [4]_q^{\sim} 2[2]_q^{\sim},$$

as

$$\frac{[4]_q^\sim}{[2]_q^\sim} = [2]_{q^2}^\sim = q^2 + q^{-2} > 2.$$

So, let's consider the remaining range

$$0 < y < 2,$$

so that (2) becomes:

$$[y+2]_q^\sim + [y]_q^\sim + [2-y]_q^\sim \stackrel{?}{>} 2[2]_q^\sim. \tag{4}$$

Never mind $[y]_q^\sim > 0$, I claim that

$$[y+2]_q^\sim + [2-y]_q^\sim > 2[2]_q^\sim. \tag{5}$$

Indeed, multiply (5) by $q - q^{-1} > 0$. We got

$$q^{y+2} - q^{-y-2} + y^{2-y} - q^{y-2} \stackrel{?}{>} 2(q^2 - q^{-2},$$

or

$$q^y(q^2 - q^{-2}) + q^{-y}(q^2 - q^{-2}) \stackrel{?}{>} 2(q^2 - q^{-2}),$$

or

$$q^y + q^{-y} \stackrel{?}{>} 2,$$

which is obvious. ■

On The Values Of Quantum Numbers Under Addition

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Abstract

We estimate $a+b$ in terms of a and b, quantumly.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbb{R},\ q \neq 0, \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take,

$$q > 1$$

without loss of generality.

Theorem 1. Let $a, b > 0$. Then

$$[a+b]_q^{\sim} > [a]_q^{\sim} + [b]_q^{\sim}. \tag{2}$$

Proof. We use the fact that

$$q^a > 1,\ q^b > 1, \tag{3}$$

since $q > 1$.

Multiplying through (2) by $q - q^{-1} > 0$, we get:

$$q^{a+b} - q^{-a-b} \stackrel{?}{>} q^a - q^{-a} + q^b - q^{-b},$$

or

$$q^b(q^a-1)+q^{-a-b}(q^a-1) \stackrel{?}{>} q^{-a}(q^a-1)(q^a+1),$$

or

$$q^b+q^{-a-b} \stackrel{?}{>} 1+q^{-a},$$

or

$$q^{-a}(q^{a+b}-1) \stackrel{?}{>} 1-q^{-a-b}=q^{-a-b}(q^{a+b}-1),$$

or

$$q^{-a} \stackrel{?}{>} q^{-a-b},$$

or

$$1 \stackrel{?}{>} q^{-b},$$

or

$$q^b \stackrel{?}{>} 1,$$

which is true. ■

Corollary 4. If $x>y>0$ then

$$[x]_q^{\sim} > [y]_q^{\sim}. \tag{5}$$

Proof. Take $x=a+b, y=a, x>y>0$. Then

$$[a+b]_q^{\sim} = [x]_q^{\sim} > [a]_q^{\sim} = [y]_q^{\sim}. \quad ■$$

A Strengthening Of The Classical Estimate for 2^x

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Abstract

We strengthen the classical inequality $2^x > 2x+1, x \geq 3$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbb{R},\ q \neq 0, \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take,

$$q > 1$$

without loss of generality.

Recently, I conjectured that

$$([2]_q^{\sim})^x > [2]_q^{\sim}[x]_q^{\sim} + 1,\ x \geq 3, \tag{1}$$

and prove it for $x \in \mathbb{Z}_{\geq 3}$,

Theorem 2. For $n \in \mathbb{Z})\geq 3$,

$$2^n > \tag{3}$$

Proof. We use the fact that with

$$h = logq > 0,$$
$$[x]_q^{\sim} = x + \frac{h^2}{6}(x^3 - x) + 0(h^4). \tag{4}$$

Then

$$([2]_q^{\sim})^n = [2(1+\frac{h^2}{6}6)]^n = 2^n(1+\frac{h^2}{6}6n),$$
$$[2]_q^{\sim} = 2(1+\frac{h^2}{6}6),$$
$$[x]_q^{\sim} = x\{1+\frac{h^2}{6}(x^2-1)\},$$

and (1) returns:

$$2^n(1+\frac{h^2}{6}6n) > 2(1+\frac{h^2}{6}6)n\{1+\frac{h^2}{6}(n^2-1)\}+1,$$

or, picking out $h^2/6$ - terms:

$$2^n 6n > 2n(n^2-1+6),$$

or

$$2^n > 2x\frac{n^2+5}{6n},\ n \geq 3. \tag{5}$$

This is our strengthening of the classical inequality (1). It *is* strengthening for $n \geq 6$, since, for $n = 6$,

$$\frac{n^2 5}{6n}\Big|_{n=6} = 1+\frac{5}{36} > 1. \tag{6}$$

A 3-Parameter Quantum Inequality With No Classical Analog

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Abstract

Classically, $(au)\left(\frac{b}{u}\right) = ab$. We quantize this into an inequality.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbb{R},\ q \neq 0, \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take,

$$q > 1$$

without loss of generality.

Theorem 1. Let $a, b, u > 0$. Then

$$[au]_q^{\sim}\left[\frac{b}{u}\right]_q^{\sim} \geq [a]_q^{\sim}[b]_q^{\sim}. \tag{2}$$

Proof. Changing u into u^{-1} amounts to interchanging a and b. So, we can take

$$u \geq 1. \tag{3}$$

Multiplying through (2) by $(q - q^{-1})^2 > 0$, we get:

$$(q^{au} - q^{-au})(q^{b/u} - q^{-b/u}) \overset{?}{\geq} (q^a - q^{-a})(q^b - q^{-b}),$$

or

$$(q^{au+b/u}+q^{-au-b/u})-(q^{au-b/u}+q^{-au+b/u})\stackrel{?}{\geq}$$
$$\stackrel{?}{\geq}(q^{a+b}+q^{-a-b})-(q^{-a-b}+q^{b-a}). \tag{4}$$

Set

$$h=logq>0.$$

Then (4) is

$$2\sum_{n=0}^{\infty}\frac{h^{2n}}{(2n)!}\left[\left(au+\frac{b}{u}\right)^{2n}-\left(au-\frac{b}{u}\right)^{2n}\right]\stackrel{?}{\geq}2\sum_{n=0}^{\sim}\frac{h^{2n}}{(2n)!}[(a+b)]^{2n}-(a-b)^{2n}]. \tag{5}$$

(5) follows from

$$\left(au+\frac{b}{u}\right)^{2n}-\left(au-\frac{b}{u}\right)^{2n}\stackrel{?}{\geq}(a+b)^{2n}-(a-b)^{2n}, a,b>0,\ u\geq 1, \tag{6}$$

or

$$\sum_k\binom{2n}{k}(au2n-)^k\left[\left(\frac{b}{u}\right)^k-\left(-\frac{b}{u}\right)^k\right]\stackrel{?}{\geq}\sum_k\binom{2n}{k}a^{2n-k}\left[b^k-(-b)^k\right],$$

or

$$2\sum_{k\ odd}\binom{2n}{k}(au^{2n-n}\left(\frac{b}{u}\right)^k\stackrel{?}{\geq}2\sum_{k\ odd}a^{2n-k}b^k\binom{2n}{k},$$

Now, $u=1$ turns (5) into an equality, and $\partial/\partial u$ of it is

$$2n\left(au+\frac{b}{u}\right)^{2n-1}\left(au-\frac{b}{u}\right)\frac{1}{u}-\left(au-\frac{b}{u}\right)^{2n-1}\left(a+\frac{b}{u^2}\right)\stackrel{?}{\geq}0,$$

or

$$\left(au+\frac{b}{u}\right)^{2n-1}\left(au+\frac{b}{u}\right)\frac{1}{u}-\left(au+\frac{b}{u}\right)^{2n-1}\left(au+\frac{b}{u}\right)\frac{1}{4}\stackrel{?}{\geq}0,$$

or

$$\left(au+\frac{b}{u}\right)^{2n-2}\stackrel{?}{\geq}\left(au+\frac{b}{u}\right)^{2n-2},$$

which is obvious. ■

A Two-Sided Estimates For Quantum Numbers

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Abstract

We provide *two*-sided estimates for quantum numbers.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbb{R},\ q \neq 0, \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take,

$$q > 1$$

without loss of generality.

Theorem 1. For $x \geq 1$,

$$[x]_q^{\sim} \leq xq^{x-1}, \tag{2a}$$
$$[x]_q^{\sim} \leq xq^{1-x}, \tag{2b}$$

Proof. (a) Multiplying (2a) by $q - q^{-1} > 0$, we get:

$$q^x - q^{-x} \overset{?}{\leq} x(q - q^{-1})q^{x-1} = x(q^x - q^{x-2}),$$

or

$$xq^{x-2} - q^{-x} \overset{?}{\leq} (x-1)q^x,$$

or

$$q^{-1}(xq^{x-1} - q^{1-x}) \overset{?}{\leq} (x-1)q^{x-1}q, \tag{3}$$

or

$$x - q^{2(1-x)} \stackrel{?}{\leq} (x-1)q^2,$$

or

$$q^x - q^{2(1-x)} \stackrel{?}{\leq}\stackrel{?}{\leq} x(q^2-1). \tag{4}$$

Now, (4) is non equality for $x = 0$, and d/dx of it is:

$$hq^{2(1-x)} \stackrel{?}{\leq} q^2 - 1, \tag{5}$$

where

$$h = logq > 0.$$

(5) can be rewritten as:

$$\frac{h}{q-1} q^{2(1-x)} \stackrel{?}{\leq} q + 1,$$

which is obvious:

$$q - 1 = e^h - 1 > h,$$
$$q^2(1-x) < 1.$$

Thus, (5) is satisfied. (b) Multiplying (2b) by $q - q^{-1} > 0$, we get:

$$xq^{1-x}(q-q)^{-1}) = x(q^{2-x} - q^{-x}) \stackrel{?}{\leq} q^x - q^{-x},$$

or

$$xq^{2-x} - (x-1)q^{-x} \stackrel{?}{\leq} (x-1)q^{-x},$$

or

$$xq^{2-x} - (x-1)q^{-x} \stackrel{?}{\leq} q^x,$$

or

$$xq^2 - (x-1) \stackrel{?}{\leq} q^{2x}. \tag{6}$$

(5) is an equality for $q = 1$, and $\partial/\partial q^2$ of it is:

$$x \leq xq^{2(x-1)},$$

which is true because $x \geq 1$. ■

On The Ratio Of Two Quantum Double Factorials

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Abstract

Classically, $\frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{2n+1}}$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbb{R},\ q \neq 0, \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take,

$$q > 1$$

without loss of generality.

Theorem 1. Let $a, b > 0$. Then

$$[a]_q^{\sim}[b]_q^{\sim} \geq ([\sqrt{ab}]_q^{\sim})^2. \tag{2}$$

Proof. Multiplying through (2) by $(q - q^{-1})^2 > 0$, we get:

$$(q^a - q^{-a})(q^b - q^{-b}) \stackrel{?}{\geq} (q^{\sqrt{ab}} - q^{-\sqrt{ab}})^2,$$

or

$$(q^{a+b} + q^{-a-b}) - (q^{a-b} + q^{b-a}) \stackrel{?}{\geq} (q^{\sqrt[n]{ab}} + q^{-2\sqrt{ab}}) - 2. \tag{3}$$

Set

$$h = logq > 0.$$

Then (3) becomes

$$2\sum_{n=0}^{\infty}\frac{h^{2n}}{(2n)!}[(a+b)^{2n}-(a-b)^{2n}]\geq 2\sum n-0\frac{h^{2n}}{(2n)!}(2\sqrt{ab})^{2n}. \tag{4}$$

follows from

$$(a+b)^{2n}-(a-b)^{2n}\overset{?}{\geq}2^{2n}(ab)^n,\ n\geq 1. \tag{5}$$

Notice that for $n=1$ and for $b=a$, any n, (5) becomes an equality.
Since $a,b>0$, set

$$a=\lambda^2,\ b=B^2.$$

Then (5) becomes:

$$(A^2+B^2)^{2n}-(A^2-B^2)^{2n}\overset{?}{\geq}2^{2n}(AB)^{2n},$$

or

$$\left(\frac{A}{B}+\frac{B}{A}\right)^{2n}-\left(\frac{A}{B}-\frac{B}{A}\right)^{2n}\geq 2^{2n}, \tag{6}$$

Let $A\geq b$. Set

$$\lambda=\frac{A}{B}\geq 1.$$

Then (6) is:

$$\left(\lambda+\frac{1}{\lambda}\right)-\left(\lambda-\frac{1}{\lambda}\right)^{2n}\overset{?}{\geq}2^{2n},\ \lambda\geq 1,\quad n\geq 1, \tag{7}$$

or

$$\sum_k\binom{2n}{k}\lambda^{2n-k}\left[\left(\frac{1}{\lambda}\right)^k-\left(-\frac{1}{\lambda}\right)^k\right]\Rightarrow 2^{2n},$$

or, with

$$\Lambda^{n-k}=\lambda^2\geq 1,$$
$$\sum_{k\ odd}\binom{2n}{k}\Lambda^{n-k}\overset{?}{\geq}2^{2n-1}. \tag{8}$$

Now,

$$\Lambda^{n-k}+\Lambda^{2n-(n-k)}=\Lambda^{n-k}+\Lambda^{n+k}=\Lambda^n(\Lambda^k+\Lambda^{-k})\geq 2.$$

So, for n even (8) follows from

$$2\sum_{k\ odd\ <n}\binom{2n}{k}\overset{?}{\geq}2^{2n-1},$$

or

$$\sum_{k\ odd} \binom{2n}{k} \stackrel{?}{\geq} 2^{2n-1},$$

which is true, because

$$\sum_{k\ even} \binom{m}{k} = \sum_{k\ odd} \binom{m}{k} = 2^m/2. \tag{9}$$

If, n is odd, (8) becomes:

$$\sum_{k\ odd\ <n} \Lambda^n(\Lambda^k + \Lambda^{-k})\binom{2n}{k} + \binom{2n}{n}\Lambda^0 \stackrel{?}{\geq} 2^{2n-1},$$

or

$$\sum_{k\ odd} \binom{2n}{k} \geq 2^{2n-1},$$

which is true by (9). ■

On The Product Of Successive Quantum Numbers

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Abstract

Classically, $\frac{n!}{(x+1)...(x+n)} \to \infty$ as $n \to \infty$, if $x > 0$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbb{R},\ q \neq 0, \pm 1,$$

be the second quantization of x.

Since

$$[-x]^s im_q = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Theorem 1. Let $x > 0$. Then

$$\lim_{n\to\infty} \frac{[1]_q^{\sim}...[n]_q^{\sim}}{[x+1]_q^{\sim}...[x+n]_q^{\sim}} = 0. \tag{2}$$

Proof. This follows from

$$\frac{[s]_q^{\sim}}{[x+s]_q^{\sim}} < q^{-x}, \quad s, x > 0. \tag{3}$$

Indeed, (3) is:

$$(q^s - q^{-s})q^x \stackrel{?}{<} q^{x+s} - q^{-x-s},$$

or

$$q^{-x} < q^x,$$

or

$$1 < q^{2x},$$

which is true because $q > 1$ and $x > 0$.

Thus

$$\prod_{k=1}^{n} \frac{[k]_q^{\sim}}{[k+x]_q^{\sim}} < q^{-nx},$$

and

$$q^{-nx} \to 0,$$

as $n \to \infty$, because $q > 1, x > 0$. ■

3-Term Arithmetic Progression Among Elements Of A Quantum Harmonic Series

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Abstract

The terms $\frac{1}{n+1}, \frac{1}{2k}, \frac{1}{k(k+1)}, k \in \mathbf{Z}_{\geq 2}$, form a 3-term arithmetic progression. We manage to quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the 2^{nd} quantization of x.

The elements

$$A = [a]_q^{\sim},$$
$$B = [a+d]_q^{\sim},$$
$$C = [a+2d]_q,$$

form a Quantum Arithmetic progression with the characteristic property

$$A + C = [2]_{q^d}^{\sim} B, \tag{1}$$
$$d = (B_A)|_{q=1} = (C_B)|q = 1. \tag{1a}$$

Theorem 2. Consider the elements

$$A = \frac{q^{-k}}{[k+1]_q^{\sim}} \frac{1}{[2]^s im_{q^d}[k]_q^{\sim}}, \frac{q}{[k]_q^{\sim}[k+1]_q^{\sim}}, \tag{3a}$$
$$d = \frac{-1}{2k} + \frac{1}{k+1} = \frac{-1+k}{2k(k+1)}. \tag{3b}$$

Proof. We have to check the equation (1), i.e., that

$$\frac{q}{[k+1]_q^{\sim}} + \frac{q^{-k}}{[k]_q^{\sim}[k+1]_q^{\sim}} \stackrel{?}{=} \frac{1}{[k]_q^{\sim}},$$

or

$$q[k]_q^{\sim} + q^{-k} \stackrel{?}{=} [k+1]_q^{\sim},$$

which is true because, in general,

$$[a+b]_q^{\sim} = q^{-b}[a]_q^{\sim} + q^a[b]_q^{\sim}. \quad \blacksquare$$

A Basic Quantum Inequality Without A Classical Analog

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Abstract

Classically, $(a,b)-b=a$ is b-independent. It's not so ???.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\quad x\in\mathbf{R},\ q\neq 0,\pm 1,$$

is the 2^{nd} quantization of x.

Since

$$[-x]_q^{\sim}=-[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim}=[x]_q^{\sim},$$

we can take

$$q>1$$

without loss of generality.

Theorem 1. For $a>0$,

$$[a+b]_q^{\sim}-[b]_q^{\sim}\geq 2[\frac{1}{2}a]_q^{\sim}. \tag{2}$$

Proof. Multiplying through by $q-q^{-1}>0$, we have:

$$\begin{aligned}&(q-q^{-1})\{[a+b]_q^{\sim}-[b]_q^{\sim}\}=\\&=q^{a+b}-q^{-a-b}-q^b+q^{-b}=\\&=q^b(q^a-1)-q^{-a-b}(1-q^a)=\\&=(q^a-1)(q^b+q^{-a-b}).\end{aligned} \tag{3}$$

Thus,

$$\lim_{b\to\infty}([a+b]_q^{\sim}-[b]_q^{\sim})=\infty. \tag{4}$$

Now, $\partial/\partial b$ (3) is:

$$(q^a - 1)(q^b - q^{-a-b}).$$

It vanishes, ??

$$b = -\frac{a}{2}, \tag{5}$$

which is a ??? (there is no max,) as (4) shows.) Thus, the min is:

$$[a - \frac{a}{2}]_q^{\sim} - [-\frac{a}{2}]_q^{\sim} = 2[\frac{a}{2}]_q^{\sim},$$

as claimed. ■

A Bilinear In Binomial Coefficients Kernel

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Abstract

The classical identity, $\sum_{k=j}^{i}(-1)\binom{k}{j}\binom{i}{k}=0, i>j$, is quantized.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\quad x\in\mathbf{R},\ q\neq 0,\pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim}=-[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim}=[x]_q^{\sim},$$

we can take

$$q>1$$

without loss of generality.

Set

$$\begin{bmatrix} m \end{bmatrix}_q^{\sim}=\frac{[m]!_q^{\sim}}{[k]!_q^{\sim}[m-k]!_q^{\sim}},$$
$$[k]_q^{\sim}=[1]_q^{\sim}...[k]_q^{\sim}, k\in\mathbf{Z}_{\geq 1};\ [0]!_q^{\sim}=1.$$

Large**Theorem 1.** Let $j\in\mathbf{Z}_{\geq 0}, i\in\mathbf{Z}_{\geq 1}, i>j$. Then

$$\sum_{k=j}^{i}1-0^k q^{k(i-j-1)}\begin{bmatrix} k \\ j \end{bmatrix}_q^{\sim}\begin{bmatrix} i \\ k \end{bmatrix}_q^{\sim}=0. \tag{2}$$

Proof. We use induction on i. For $i=j+1$, (2) returns:

$$\begin{bmatrix} j+1 \\ j \end{bmatrix}_q^{\sim}-\begin{bmatrix} j+1 \\ i \end{bmatrix}_q^{\sim}=0,$$

which is true.

The inductive step uses the

Lemma 3.

$$\begin{bmatrix} i+1 \\ k \end{bmatrix}_q^{\sim} = q^{-k}\begin{bmatrix} i \\ k \end{bmatrix}_q^{\sim} + q^{i+1-k}\begin{bmatrix} i \\ k-1 \end{bmatrix}_q^{\sim} \tag{4}$$

Proof. (4) is:

$$\frac{[i+1]!_q^{\sim}}{[k]!_q^{\sim}[i+1]_q^{\sim}[k]!_q^{\sim}} \stackrel{?}{=} q^{-k}\frac{[i]!_q^{\sim}}{[k]!_q^{\sim}[i-k]!_q^{\sim}} + q^{i+1-k}\frac{[i]!_q^{\sim}}{[k-1]!_q^{\sim}[i+k-k]!_q^{\sim}},$$

or

$$[i+1]_q^{\sim} \stackrel{?}{=} q^{-k}[i-k+1]_q^{\sim} \quad + q^{i+1-k}[k]_q^{\sim},$$

which is true because, in general,

$$[a+b]_q^{\sim} = q^{-a}[b]_q^{\sim} + q^b[a]_q^{\sim}. \ \blacksquare$$

Thus, the inductive step $i \mapsto i+1$ of (2) becomes, denoting the LHS of (2) by $L(i,j)$:

$$\begin{aligned}
&L(i+j,j) = \sum ?????? \\
&= \sum (-1)^k q^{k(i-j-1)} q^k \begin{bmatrix} k \\ j \end{bmatrix}_q^{\sim} \left\{ q^{-k}\begin{bmatrix} k \\ k \end{bmatrix} + q^{i+q-k}\begin{bmatrix} i \\ k-1 \end{bmatrix}_q^{\sim} \right\} = \\
&= L(i,j) - q^{i+1}\sum_s q^{s(i-j-1)} q^{i-j-1}(-1)^s \begin{bmatrix} s+1 \\ j \end{bmatrix}_q^{\sim}\begin{bmatrix} i \\ s \end{bmatrix}_q^{\sim} = \\
&= L(i,j) - q^{2i-j-1}\sum_s (-1)^s q^{s(i-j-1)} \begin{bmatrix} s+1 \\ j \end{bmatrix}_q^{\sim}\begin{bmatrix} i \\ s \end{bmatrix}_q^{\sim} = \\
&= L(i,j) - q^{2i-j-1}\sum_q (-1)^s \begin{bmatrix} i \\ s \end{bmatrix} q^{s(i-j-1)} \left\{ q^{-j}\begin{bmatrix} s \\ j \end{bmatrix}_q^{\sim} + q^{s+1-j}\begin{bmatrix} s \\ j-1 \end{bmatrix}_q^{\sim} \right\} = \\
&= L(i,j) - q^{2i-j-1}\Big\{ L(i,j)q^{-j} + q^{1-j}L(i,j-1) \Big\}.
\end{aligned}$$

This concludes the induction step. $\blacksquare$

A Bound From Below For Quantum Double Factorials

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Abstract

We quantize the classical inequality, $(2n-1)!! > \frac{n^n}{n!}$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbf{R}, q \geq 0, \pm 1,$$

be the 2^{nd} quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Set

$$[k]!_q^{\sim} = [1]_q^{\sim}...[k]_q^{\sim}, k \in \mathbf{Z}_{\geq 1};\ [0]!_q^{\sim} = 1.$$
$$[2k-1]!!_q^{\sim} = [1]_q^{\sim}[3]_q^{\sim}...[2k-1]_q^{\sim}.$$

Theorem 1. For $n \in \mathbf{Z}_{\geq 2}$,

$$[2n-1]!!_q^{\sim} > \frac{([n]_q^{\sim})^n}{[n]!_q^{\sim}}. \tag{2}$$

Proof. We use induction on n. For $n = 2$, (2) returns:

$$[3]_q^{\sim} \stackrel{?}{>} \frac{([2]_q^{\sim})^2}{[2]_q^{\sim}} = [2]_q^{\sim},$$

which is true, since $3 > 2$ implies $[3]_q^{\sim} > [2]_q^{\sim}$.

The inductive step $n \mapsto h+1$ amounts to

$$[2n+1]_q^{\sim} \frac{([n]_q^{\sim})^n}{[n]!_q^{\sim}} \stackrel{?}{>} \frac{([n+1]_q^{\sim})^{n+1}}{[n+1]!_q^{\sim}},$$

or

$$[2n+1]_q^{\sim} \stackrel{?}{>} \left(\frac{[n+1]_q^{\sim}}{[n]_q^{\sim}}\right)^n, \tag{3}$$

which is true, because, in general,

$$[n+1]_q^{\sim} > \left(\frac{[n+1]_q^{\sim}}{[n]_q^{\sim}}\right)^n, \tag{4}$$

and

$$[2n+1]_q^{\sim} > [n+1]_q^{\sim} \tag{5}$$

for $n > 0$.

(4) can be seen as follows, by induction on n. For $n = 2$, (4) is:

$$[3]_q^{\sim} \stackrel{?}{>} \left(\frac{[3]_q^{\sim}}{[2]_q^{\sim}}\right)^2,$$

or

$$([2]_q^{\sim})^2 \stackrel{?}{>} [3]_q^{\sim},$$

which is obvious. And the inductive step $n \mapsto n+1$ in (4) amounts to:

$$[n+2]_q^{\sim} = \frac{[n+2]_q^{\sim}}{[n+1]_q^{\sim}}[n+1]_q^{\sim} >$$
$$> \frac{[n+2]_q^{\sim}}{[n+1]_q^{\sim}}\left(\frac{[n+1]_q^{\sim}}{[n]_q^{\sim}}\right)^n \stackrel{?}{>} \left(\frac{[n+2]_q^{\sim}}{[n+1]_q^{\sim}}\right)^{n+1} =$$

or

$$1 \stackrel{?}{>} \left(\frac{[n+2]_q^{\sim}[n]_q^{\sim}}{([n+1]_q^{\sim})^2}\right)^n,$$

or

$$([n+1]_q^{\sim})^2 \stackrel{?}{>} [n]_q^{\sim}[n+2]_q^{\sim}, \tag{6}$$

which is easy to check, by multiplying (6) by $(q-q^{-1})^2$. ■

A Bound On The Product Of Even Quantum Factorials

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Abstract

The classical bound, $\prod_{i\equiv 1}^{n}(2i)! \geq ((n+1)!)^n$, is quantized.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

is the 2^{nd} quantization of x.

Set

$$[n]!_q^{\sim} = \prod_{k=1}^{n} [k]_q^{\sim},\ n \in \mathbf{Z}_{\geq 1};\ [0]!_q^{\sim} = 1.$$

Theorem 1. For $n \in \mathbf{Z}_{\geq 1}$,

$$\prod_{i=1}^{n} [2i]!_q^{\sim} \geq ([n+1]_q^{\sim})^n. \tag{2}$$

We rewrite (2) as

$$\prod_{i=1}^{n} [2i]!_q^{\sim} [2(n+1-i)]!_q^{\sim} \stackrel{?}{>} (([n+1]!_q^{\sim})^2)^n,$$

and claim

$$[2i]!_q^{\sim} [2n+1-i]!_q^{\sim} \stackrel{?}{>} ([n+1]_q^{\sim})^2, \quad 1 \leq i \leq n. \tag{3}$$

Denote the LHS of (3) by $\prod_i, 1 \leq i \leq n$. (3) can rewritten as:

$$\prod_i \stackrel{?}{>} \prod_{(n+1)/2} . \tag{4}$$

(3) follows from:

$$\prod_i \geq \prod_{i+1}, \quad i+1 \leq \frac{n+1}{2} \ \Leftrightarrow\ i \leq \frac{n-1}{2}. \tag{5}$$

Indeed,

$$\frac{\prod_i}{\prod_{i+1}} = \frac{[2(n+1-i)]!_q^{\sim}[2(n+1-i)-1]_q^{\sim}}{[2i+1]_q^{\sim}[2i+2]_q^{\sim}} \tag{6}$$

and

$$2(n+1-i) \geq 2i+2 \Leftrightarrow 2i \leq n.$$

The observation $\prod_i = \prod_{n+1-i}$ concludes the proof. ■

A Conjectured Quantum Inequality

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Abstract

We conjecture quantum inequality for $x \in \mathbf{R}$, and prove it for $x \in \mathbf{Z}_{\geq 1}$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

is the 2^{nd} quantization of x, so that

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

and we can take

$$q > 1$$

without loss of generality.

Conjecture 1. For $x > 0$,

$$xq^{x+1/2}[2]_{q^{x+1/2}}^{\sim} > q^x[2x+1]_{q^{1/2}}^{\sim} - 1. \tag{2}$$

I can't prove this inequality. But for $x \in \mathbf{Z}_{\geq 1}$, the inequality is true and the argument is well know.

So, let $x = n \in \mathbf{Z}_{\geq 1}$. Then (2) becomes:

$$nq^{n+1/2}(q^{n+1/2} + q^{-n-1/2}) \stackrel{?}{>} q^n(q^n + q^{n-1} + ... + q^{-n}) - 1,$$

or

$$n(q^{2n+1} + 1) \stackrel{?}{>} q^{2n} + q^{2n-1} + ... + q,$$

which follows from

$$q^{2n+1} + 1 \stackrel{?}{>} q^{2n-k+1} + q^k, \quad 1 \leq k \leq n,$$

or

$$q^{2n-k-1}(q^k - 1) \stackrel{?}{>} q^{k-1},$$

or

$$q^{2n-k+1} \stackrel{?}{>} 1,$$

which is obvious, because $q > 1$ and $2n - k + 1 > 0$. ■

A Double Inequality For The Quantum Central Binomial Coefficients

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Abstract

We quantize the Schaumberger's derivation of the classical inequality, $2^{2n} > \binom{2n}{n} > 2^n$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

is the 2^{nd} quantization of x.

Set

$$\begin{bmatrix} m \\ k \end{bmatrix}_q^{\sim} = \frac{[m]!_q^{\sim}}{[k]!_q^{\sim}[m-k]!_q^{\sim}},$$
$$[k]!_q^{\sim} = [1]_q^{\sim}...[k]_q^{\sim}, k \in \mathbf{Z}_{\geq 1}; [0]!_q^{\sim} = 1.$$

Theorem 1. For $n \in \mathbf{Z}_{\geq 1}$,

$$(< 2^n >_q^{\sim})^2 > \begin{bmatrix} 2n \\ n \end{bmatrix}_q^{\sim} >< 2^n >_q^{\sim}, \tag{2}$$

where

$$< 2^n >_q^{\sim} = \prod_{i=1}^{n} [2]_{q^i}^{\sim}. \tag{3}$$

Proof. Set

$$x = [2n]!!_q^{\sim} = [2]_q^{\sim}[4]_q^{\sim}...[2n]_q^{\sim}, y = [2n-1]!!_q^{\sim} = [1]_q^{\sim}[3]_q^{\sim}...[2n-1]_q^{\sim},$$
$$2 = [n]!_q^{\sim} = [1]_q^{\sim}...[4]_q^{\sim}.$$

Then, obviously,

$$x > y > z,$$

Hence,

$$x^2 > xy > xz,$$

or

$$(<2^n>_q^{\sim})^2([n]!_q^{\sim})^2 > [2n]!_q^{\sim} > ([n]!_q^{\sim})^2, <2^n>_q^{\sim}$$

or

$$(<2^n>_q^{\sim})^2 > \begin{bmatrix} 2n \\ n \end{bmatrix}_q^{\sim} ><2^n>_q^{\sim},$$

which is (2). ■

A Double Inequality For The Quantum Factorial

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Abstract

We quantize the double inequality $\binom{n+1}{2}^n > n! > n^{n/2}, n \in \mathbf{Z}_{\geq 1}$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Set

$$[k]!_q^{\sim} = [1]_q^{\sim} ... [k]_q^{\sim},\ k \in \mathbf{Z}_{\geq 1};\ [0]!_q^{\sim} = 1.$$

Theorem 1. For $n \in \mathbf{Z}_{\geq 2}$,

$$[n]!_q^{\sim} > ([n]_q^{\sim})^{1/2}. \tag{2}$$

Proof. We use induction on n. For $n = 3$, (2) returns:

$$[3]!_q^{\sim} = [2]_q^{\sim}[3]_q^{\sim} \stackrel{?}{>} ([3]_q^{\sim})^{3/2},$$

or

$$([2]_q^{\sim})^2 \stackrel{?}{>} [3]_q^{\sim}, \tag{2'}$$

or

$$q^2+2+q^{-2} \overset{?}{>} q^2+1+q^{-2},$$

which is true.

The inductive step, $n \mapsto n+1$, amounts to

$$[n+1]_q^{\sim}([n]_q^{\sim})^{n/2} \overset{?}{>} ([n+1]_q^{\sim})^{(n+1)/2},$$

or

$$([n]_q^{\sim})^{n/2} \overset{?}{>} ([n+1]_q^{\sim})^{(n-1)/2},$$

or

$$([n]_q^{\sim})^{n} \overset{>}{>} ([n+1]^s im_q)^{n-1}, \tag{3}$$

or

$$[n+1]_q^{\sim} \overset{?}{>} \left(\frac{[n+1]_q^{\sim}}{[n]_q^{\sim}}\right)^n. \tag{4}$$

To prove (4), we use induction on n again. For $n=2$, (4) returns:

$$([2]_q^{\sim})^2 \overset{?}{>} [3]_q^{\sim},$$

and this is already verified $(2')$.

The inductive step for (4) is

$$[n+2]_q^{\sim} = \frac{[n+2]_q^{\sim}}{[n+1]_q^{\sim}}[n+1]_q^{\sim} > \frac{[n+2]_q^{\sim}}{[n+1]_q^{\sim}}\left([n+1]_q^{\sim}[n]_q^{\sim}\right)^n \overset{?}{>} \left(\frac{[n+2]_q^{\sim}}{[n+1]^{\sim}}\right)^{n+1},$$

or

$$1 \overset{?}{>} \left([h]_q^{\sim}[n+2]_q^{\sim}([n+1]_q^{\sim})^2\right)^n,$$

or

$$1 \overset{?}{>} \frac{[n]_q^{\sim}[n+2]_q^{\sim}}{([n+1]_q^{\sim})^2},$$

or

$$([n+1]_q^{\sim})^2 \overset{?}{>} [n]_q^{\sim}[n+2]_q^{\sim},$$

or

$$(q^{n+1}-q^{-n-1})^2 \overset{?}{>} (q^n-q^{-n})(q^{n+2}-q^{-n-2}),$$

or

$$q^{2(n+1)} + q^{-2(n+1)} - 2 \overset{?}{>} (q^{2n+2} + q^{-2n-2}) - (q^2 + 1^{-2}),$$

or

$$q^2 + q^{-2} \overset{?}{>} 2,$$

which is obvious. ■

Theorem 5. For $n \in \mathbf{Z}_{\geq 2}$,

$$\left(\frac{[n+1]_q^{\sim}}{[2]_q^{\sim}}\right)^n > [n]!_q^{\sim} \tag{6}$$

Proof. Induction on n doesn't work. Even for $n = 2$, (6) returns. We get:

$$([3]_q^{\sim})^2 > ([2]_q^{\sim})^3, \tag{7}$$

a rather non evident fact for $q \neq 1$.

Squaring (6), we get:

$$\left(\frac{[n+1]_q^{\sim}}{[2]_q^{\sim}}\right)^{2n} \overset{?}{>} \left(\prod_{k=1}^{n} [n+1-k]_q^{\sim}\right)^2. \tag{8}$$

(8) follows from

$$\left(\frac{[n+1]_q^{\sim}}{[2]_q^{\sim}}\right)^2 \overset{?}{>} [k]_q^{\sim}[n+1-k]_q^{\sim}, \quad 1 \leq k \leq n. \tag{9}$$

Lemma 10.

$$\max_{1 \leq k \leq n} [k]_q^{\sim}[n-1-k]_q^{\sim} \leq \left(\left[\frac{n+1}{2}\right]_q^{\sim}\right)^2. \tag{11}$$

Proof. 1 is:

$$(q^k - q^{-k})(q^{n+1-k} - q^{-n-1+k}) \overset{?}{<} \left(q^{\frac{n+1}{2}} - q^{-\frac{n+1}{2}}\right)^2,$$

or

$$(q^{n-1} + q^{-n-1}) - (q^{n+1-2k} + q^{2k-n-1}) \overset{?}{<} (q^{n+1} + q^{-n-1}) - 2,$$

or

$$2 < q^{n+1-2k} + q^{-(n+1-2k)} \tag{0.1}$$

which is obvious ■.

Thus, (9) follows from

$$\left(\frac{[n+1]_q^{\sim}}{[2]_q^{\sim}}\right)^2 \stackrel{?}{>} \left(\left[\frac{n+1}{2}\right]_q^{\sim}\right),$$

or

$$[n+1]_q^{\sim} \stackrel{?}{>} [2]_q^{\sim}\left[\frac{n+1}{2}\right]_q^{\sim},$$

or

$$[2]_{q^{(n+1)/2}}^{\sim} \stackrel{?}{>} [2]_q^{\sim},$$

which is true, because $[2]_{q^x}^{\sim} = q^x + q^{-x}$ growth with $x > 0$, and $(n+1)/2 > 1$. ■

A Finite Identity In The First Quantizatin

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Abstract

We quantize the classical identity

$$\frac{a}{a+1}+\frac{b}{(a+1)(b+1)}+\frac{c}{(a+1)(b+1)(c+1)}+...+\frac{k}{(a+1)(b+1)...(k+1)}=1-\frac{1}{(a+1)...(k+1)}.$$

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\quad x,\ q\in\mathbf{R},\ q\neq 1,\ q>0,$$

is the 2^{nd} quantization of x.

Theorem 1. Let $a,b,...,k\neq -1$. Then

$$\frac{q[a]_q}{[a+1]_q}+\frac{q[b]_q}{[a+1]_q[b+1]_q}+...+\frac{q[k]}{[a+1]_q...[k+1]_q}=1-\frac{1}{[a+1]_q...[k+1]_q}. \tag{2}$$

Proof. We have:

$$\frac{q[a]_q}{[a+1]_q}=1-\frac{1}{[a+1]_q}, \tag{3}$$

so

$$\frac{q[s]_q}{[s+1]_q}=1-\frac{1}{[s+1]_q^{\sim}}. \tag{4}$$

Multiplying (4) by $1/[a+1]_q...[b+1]_q$. We get:

$$\frac{1\quad q[s]_q}{[a+1]_q...[s+1]_q}=\frac{1}{[a+1]_q...[s+1]_q}-\frac{1}{[a+1]_q...[a+1]_q[s+1]_q}. \tag{5}$$

Adding formulas (5) for all the variables, we get a telescoping sum and, thus, recover (2). ■

A Finite Product In Quantum Domain

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Abstract

We quantize the inequality $\prod_{k=1}^{n} \frac{4k-1}{4k+1} < \sqrt{\frac{3}{4n+3}}, n \in \mathbf{Z}_{\geq 1}$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

is the 2^{nd} quantization of x.

Theorem 1. For $n \in \mathbf{Z}_{\geq 1}$,

$$\prod_{k=1}^{n} \frac{[4k-1]_q^{\sim}}{[4k+1]_q^{\sim}} \leq \sqrt{\frac{[3]_q^{\sim}}{[4n+3]_q^{\sim}}} \tag{2}$$

Proof. We use induction on n, together with simply verifyable Lemma that $a, b, c \geq 0$, a, b, c not all zero, and

$$2a \geq b + c, \tag{3}$$

implies

$$([a]_q^{\sim})^2 \geq [b]_q^{\sim}[c]_q^{\sim}. \tag{4}$$

For $n = 1$, (2) returns:

$$\frac{[3]_q^{\sim}}{[5]_q} \stackrel{?}{\leq} \sqrt{\frac{[3]_q^{\sim}}{[7]_q^{\sim}}},$$

or $([3]_q^{\sim})[7]_q^{\sim} \stackrel{?}{\leq} ([5]_q^{\sim})^2$, which is true by (3), (4).

The inductive step now amounts to:

$$\sqrt{\frac{[3]_q^{\sim}}{[4n+3]_q^{\sim}}} \cdot \frac{[4n+3]_q^{\sim}}{[4n+5]_q^{\sim}} \stackrel{?}{\leq} \sqrt{\frac{[3]_q^{\sim}}{[4n+1]_q^{\sim}}},$$

or to

$$[4n+3]_q^{\sim}[4n+7]_q^{\sim} \stackrel{?}{\leq} ([4n+5]_q^{\sim})^2,$$

which is true by (3), (4), because

$$(4n+3) + (4n+7) = 8n + 10 = 2(4n+5). \quad \blacksquare$$

A Formula For n^{th} Term Of A Quantum Arithmetic Progression

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Abstract

We derive quantum analog of a formula $a_k = a + kd,\ d = b - a, = b = a_1$.

Let $\{a_k\}$ be a quantum arithmetic progression. That means that, for any three consequitive terms a, b, c

$$a + c = [2]_{q^d} b, \tag{1}$$

where

$$d = (c - b)|_{q=1} = (b - a)|_{q=1}. \tag{2}$$

For $q = 1$, (??) becomes the usual property:

$$a, c|_{q=1} = 2b|_{q=1} \tag{3}$$

So, let

$$a, b \in \mathbf{R}[q, q^{-1}],$$
$$d = (b - a)|_{a=1}.$$

In the classical case $q = 1$, we would have:

$$a_k = a + dk = a + k(b - a) = bk - (k - 1)a. \tag{4}$$

The question is: what is the classical analog of (4)?

Let

$$a_k = x_k a + y_k b.$$

Then

$$a_{k+1} + a_{k-1} = (x_{k+1} + x_{k-1})a + (y_{k+1} + y_{k-1}) =$$
$$= [2]_{q^d} a_k = (q^d + q^{-d})(x_{ka} + ??_k),$$

whence

$$x_{k+1} = (q^d + q^{-d})x_k - x_{k-1}, \tag{5a}$$
$$y_{k+1} = (q^d + q^{-d})y_k - y_{k-1}, \tag{5b}$$

with boundary conditions

$$x_0 = 1, y_0 = 0, \tag{6a}$$
$$x_1 = 0, y_0 = 1, \tag{6b}$$

we have:

$$x_k = [k-1]^{\sim}_{q^d},$$
$$y_k = [k]^{\sim}_{q^d}.$$

Thus,

$$a_k = [k]^{\sim}_{q^d} b - [k-1]^{\sim}_{q^d} a, \tag{7}$$

a q-analog of the classical formula (4).

A Fundamental Inequality For Quantum Numbers

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Abstract

We establish a fundamental estimate from above for $\frac{q-q^{-1}}{logq}$.

Let

$$[x]_q = \frac{q^x - 1}{q-1},\ x \in \mathbf{R},\ q \neq 0, \pm 1,$$

be the 2^{nd}quantization of x.

Since

$$[x]^{\sim}_{q^{-1}} = [x]^{\sim}_q,$$

we can take

$$q > 1$$

without loss of generality.

Set

$$h = logq > 0.$$

Theorem 1.

$$\frac{q-q^{-1}}{h} < [2]^{\sim}_q. \tag{2}$$

Proof. For the LHS of (2) we have:

$$\frac{q-q^{-1}}{h} = \frac{1}{h}(e^h - e^{-h}) = \frac{2}{h}\sum_{k=0}^{\infty}\frac{h^{2h+1}}{(2h+1)!} = 2\sum_{n=0}^{\infty}\frac{h^{2n}}{(2n+1)!}, \tag{3}$$

and for the RHS of (2) we find:

$$[2]^{\sim}_q = q + q^{-1} = e^h + e^{-h} = 2\sum_{n=0}^{\infty}\frac{h^{2n}}{(2n)!}. \tag{14}$$

But

$$\frac{1}{(2n)!} > \frac{1}{(2n+1)!},\ n \in \mathbf{Z}_{\geq 0}.$$

so, (2) follows. ■

Remark 5. The condition $q > 1 \Leftrightarrow h > 0$ is immaterial for the statement (2) (but not for the Proof above). Indeed,

$$\frac{q - q^{-1}}{h} - \frac{q^{-1} - q}{-h}, \tag{6}$$

and both sides of (2) are invariant under the change $q \to q^{-1} \Leftrightarrow h \to -h$.

A Generalization Of Sun's Conjecture

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Abstract

Sun's conjecture, that every large number M of an appropriate parity, is $M = p + ax^2$, where p is a prime and $a \geq 1$, is given. I notice that x^2 can be replaced by *any* integer-valued polynomial in x, and the Conjecture still remains true.

Let $f(x)$ be a polynomial with integer coefficients; we want the number of solutions of

$$M = p + f(x). \tag{1}$$

We assume that $M - f(x)$ is *odd*. Let

$$\delta(x) = \pi(x) - \pi(x-2) = \begin{cases} 1, x \text{ odd}, & \text{a prime} \\ 0, x \text{ odd}, & \text{not a prime} \end{cases} \tag{2}$$

I recently showed that

$$\delta(x) \sim \frac{2}{logx}. \tag{3}$$

If $n(M)$ denotes the number of solutions (p, x) of (1), then

$$n(M) = \sum_x \delta(M - f(x)) = \sum_x \frac{2}{log(M-f)} \sim \frac{2}{logM} \sum \frac{1}{1 - \frac{f}{MlogM}} \sim$$
$$\sim \frac{2}{logM} \sum_x \left(1 + \frac{f(x)}{MlogM}\right). \tag{4}$$

We need to be more specific about $f(x)$. Let

$$f(x) = ax^k, \quad a \geq 1, \quad k \geq 2. \tag{5}$$

The case

$$k = 2$$

is the Sun case. Then the upper bound for x is

$$\left\lfloor \left(\frac{M}{a}\right)^{1/k} \right\rfloor \sim \left(\frac{M}{a}\right)^{1/k},$$

and (4) returns:

$$\frac{2}{logM}\sum_{x=1}^{(\frac{M}{a})^{1/k}}\left[1+\frac{ax^k}{MlogM}\right]\sim$$
$$\sim\frac{2}{logM}\left[\left(\frac{M}{a}\right)^{1/k}+\frac{a\left(\frac{M}{a}\right)^{(k+1/k)}}{(k+1)MlogM}\right]\sim$$
$$\sim\frac{2}{logM}\left[\frac{M^{1/k}}{a^{1/k}}+\frac{a^{-1/k}M^{1/k}}{(k+1)logM}\right]\sim\frac{2M^{1/k}}{logMa^{1/k}}. \tag{6}$$

Thus, the number of solutions grows to ∞ with M.

A General Quantum Inequality

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Abstract

A quantum number depends upon the number itself *and* the base. We determine how the compound object changes when the base changes.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

be the second quantization of x.

Theorem 1. Let $y > x > 0$, then

$$[a]_{q^y}^{\sim} > [a]_{q^x}^{\sim}. \tag{2}$$

If $b > 1, 1 < q' < Q'$, then

$$[b]_{Q'}^{\sim} < [b]_{q'}^{\sim}. \tag{2'}$$

Proof. Since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1 \tag{3}$$

without the loss of generality.

Let's dispose first of (2′) as it follows directly from (2). Indeed, let $b = 1/a$, then (2) yields:

$$\left[\frac{1}{b}\right]_Q^{\sim} > \left[\frac{1}{b}\right]_q^{\sim},$$

or

$$\frac{1}{[b]_{Q^{1/b}}^{\sim}} > \frac{1}{[b]_{q^{1/b}}^{\sim}},$$

or, with

$$Q' = Q^{1/b} > q^{1/b} = q' > 1.$$
$$[b]^{\sim}_{q'} > [b]^{\sim}_{Q'},$$

which is $(2')$.

The inequality (2) is:

$$(q^{ay} - q^{-ay})(q^x - q^{-x}) \stackrel{?}{>} (q^{ax} - q^{-ax})(q^y - q^{-y}),$$

or, with

$$1 < X = q^x < Y = q^y,$$

to

$$(Y^a - Y^{-a})(X - X^{-1}) \stackrel{?}{>} (X^a - X^a)(Y - Y^{-1}), Y > X > 1, a > 1. \tag{4}$$

We rewrite (4) as

$$\frac{(Y^{2a-1})}{Y^a}\frac{(X^2-1)}{X} \stackrel{?}{>} \frac{X^{2a}-1}{X^a}\frac{Y^2-1}{Y},$$

or with

$$Y^2 = u,\ X^2 = v, u > v,$$

as:

$$v\frac{a-1}{2}(u^a - 1)(v-1) \stackrel{?}{>} u\frac{a-1}{2}(v^a-1)(u-1). \tag{5}$$

Setting

$$b = \frac{a-1}{2} \ \Leftrightarrow\ a = 2b+1, b > 0, \tag{6}$$

(5) is:

$$f := v^b(v-1)(u^{2b+1}-1)u^b(u-1)(v^{2b+1}-1) \stackrel{?}{>} 0. \tag{7}$$

Since $f/u = v = 0$, it's enough to show that $\partial f/\partial u > 0$. We have:

$$\frac{\partial f}{\partial u} = u^b(v-1)(2b+1)u^{2b} - [(b+1)u^b - bu^{b-1}](v^{2b+1}-1) \stackrel{?}{>} 0,$$

or

$$(2b+1)\frac{u^{2b}}{u^{b-1}[b+1u-^b]} \stackrel{?}{>} \frac{v^{2b+1}-1}{v^b(v-1)},$$

or

$$(2b+1)\frac{u^{b+1}}{[(b+1)u-b]} \stackrel{?}{>} \frac{u^{2b+1}-1}{(v-1)v^b}. \tag{8}$$

For $v-u$, (8) returns:

$$(2b+1)\frac{u^{2n+1}}{(b+1)u-b} \overset{?}{>} \frac{u^{2b+1}-1}{u-1}$$

or

$$(2b+1)\frac{u^{2b+1}}{u^{2b+1}-u} \overset{?}{>} \frac{(b+1)u-b}{u-1},$$

or, because

$$\frac{(b+1)u-b}{u-1} < 2^{b+1} \Leftrightarrow$$
$$(b+1)u-b > (b+1)u-(b+1) \Leftrightarrow b_u > b \Leftrightarrow u > 1,$$

to

$$2b+1 \geq b+1,$$

which is true. In addition, the LHS of (8) is increasing with u:

$$\frac{\partial}{\partial u}\left[\frac{u^{b+1}}{(b+1)u-b}\right] = (b+1)\frac{u^b}{(b+1)u-b} - \frac{u^{b+1}}{[b+1u-b]^2}(b+1)] =$$
$$= (b+1)\frac{ub}{[(b+1)u-b]^2} \text{ times :}$$
$$(b+1)u-b-u = b(u-1) > 0.$$

Thus, (8) is true for $u \geq v$, in particular for $u > v$. Hence, (7), is true. ■

A Growth Property Of The Quantum Roots Of 2

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Abstract

We establish how the quantum roots of 2 grow with the base change.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x, q \in \mathbf{R},\ q \neq 1, q > 0,$$

is the 2^{nd} quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

It is known that

$$[2]_{q^x}^{\sim} = q^x + q^{-x}$$

grows with $x > 0$. We are interested in a more complex object,

$$f(x) := ([2]_{q^x}^{\sim})^{1/x}, x > 0.$$

Theorem 1. $f(x)$ decreases with x.

Proof. For $x > y > 0$, we have to show that

$$(q^x + q^{-x})^{1/x} \stackrel{?}{<} (q^y + q^{-y})^{\frac{1}{y}},$$

or

$$q(1 + q^{-2x})^{\frac{1}{x}} \stackrel{?}{<} q(1 + q^{-2y})^{1/y},$$

or

$$(1 + q^{-2x})^y \stackrel{?}{<} (1 + q^{-2y})^x,$$

which is obvious because $y < x$ and $q^{-2x} < q^{-2y}$. ■

A Homogeneous Quantization Of A Triangle

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Abstract

Triangles are quantized in such a way that classically homote??? triangles become quantumly homot????

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbf{R}, q \neq 0, \pm 1,$$

be the 2^{nd} quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Let $0 < a \leq b \leq c$ be triangle sides, subject to the condition

$$a + b > c. \tag{1}$$

Our recipe for quantization is:

$$a \to A = a - a[a]_q^{\sim}. \tag{2}$$

Theorem 3. If $a \leq b$ then $A \leq B$.

Proof. We have to show that

$$q^{-a}[a]_q^{\sim} \stackrel{?}{<} q^{-b}[b]_q^{\sim},$$

or, multiplying by $q - q^{-1} > 0$,

$$q^{-a}(q^a - q^{-a}) \stackrel{?}{<} q^{-b}(q^b - q^{-b}),$$

or

$$1 - q^{-2a} \stackrel{?}{<} 1 - q^{-2b},$$

or

$$q^{-2b} \stackrel{?}{<} q^{-2a},$$

which is true because $q > 1$ and $b > a$. ■

Notice that the ordering $a \leq b \leq c$ has not been used.

Theorem 4. If $a + b > c$ then $A + B > C$.

Proof. We have to show that

$$q^{-a} + q^{-b}[b]_q^{\sim} \stackrel{?}{>} q^{-c}[c]_q^{\sim},$$

or

$$(1 - q^{-2a}) + (1 - q^{2b}) \stackrel{?}{>} q^{-2a} + q^{-2b}. \tag{5}$$

The worst case for c is $c = a + b$. So, with

$$Q = q^2 > 1,$$

we need:

$$1 + Q^{-a-b} \stackrel{?}{>} Q^{-a} + Q^{-b},$$

or

$$1 - Q^{-a} + Q^{-b}(-Q^{-a} - 1) \stackrel{?}{>} 0,$$

or

$$1 - Q^{-b} \stackrel{?}{>} 0,$$

which is obvious. ■

Notice that iso??? triangles go, under quantization into iso?? ones.

Define two quantum triangles to be *quantumly* similar if they differ only in the base (9). Then classically similar triangles go, under quantization, into classically-quantumly similar ones. Indeed,

$$q^{-\lambda a}[\lambda a]_q^{\sim} = (q^{\lambda})^{-a}[x]_q^{\sim}[a]_{q^{\lambda}}^{\sim}. \tag{6}$$

A Hypergeometric-Like Quantum Infinite Series

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Abstract

Classically, $1+\sum_{n=0}^{\infty}\prod_{i=0}^{n}\frac{x+i}{y+i}=\frac{x}{y-x-1}$ when $y-x-1>0$ We quantize this.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\quad x\in\mathbf{R},\ q\neq 0,\pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim}=-[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim}=[x]_q^{\sim},$$

we can take

$$q>1$$

without loss of generality.

Lemma 1. Let $0<x<y, 0<\lambda=x/y<1$. Then

$$\frac{x(x+1)...(x+n)}{y(y+1)...(y+n)}\leq\lambda^{n+1}. \tag{2}$$

Proof. For $n=0$, (2) is an equality. In general, it's enough to prove that

$$\frac{x+k}{y+k}\overset{?}{<}\lambda,\quad k\in\mathbf{Z}_{\geq 1}, \tag{3}$$

which is

$$\frac{x+k}{y+k}\overset{?}{<}\frac{x}{y},$$

or

$$1+\frac{k}{x}\overset{?}{<}1+\frac{k}{y},$$

or

$$\frac{1}{y} \stackrel{?}{<} \frac{1}{x},$$

or

$$x < y$$

which is obvious. ■

Lemma 4. Let $\alpha, \triangle > 0$. Then

$$\frac{[x]_q^{\sim}}{[x+\triangle]_q^{\sim}} < q^{-\triangle}. \tag{5}$$

Proof. (5) is:

$$[x+\triangle]_q^{\sim} \stackrel{?}{>} q^{\triangle}[x]_q^{\sim},$$

or

$$q^{x+\triangle} - q^{-x-\triangle} \stackrel{?}{>} q^{\triangle}(q^x - q^{-x}),$$

or

$$q^{\triangle - x} \stackrel{?}{>} q^{-x-\triangle},$$

or

$$q^{\triangle} \stackrel{?}{>} q^{-\nabla},$$

which is true because $\triangle > 0$ and $q > 1$. ■

Corollary 6. If $y > x > 0$ then

$$\lim_{n\to\infty} \prod_{i=0}^{n} \frac{[x+i]_q^{\sim}}{[y+i]_q^{\sim}} = 0.$$

Proof. By (5), with $\triangle = y - x > 0$,

$$\prod_{i=0}^{n} \frac{[x+i]_q^{\sim}}{[y+i]_q^{\sim}} < q^{-(n+1)\triangle}, \tag{7}$$

and $\lim_{n\to\infty} q^{-n+1\triangle} = 0$. ■

The object of our attention is the series

$$1 + \frac{p}{q}\frac{p(p_r)}{q(q+r)} + \frac{p(p+r)(p+2r)}{q(q+r)(q+2r)} + ..., \quad r \geq 0, \quad q - p - r > 0, \tag{8}$$

or, passing from $\mathbf{R}p^3$ to $\mathbf{R}^2$, the series

$$s(a,b) = 1 + \frac{a}{b} + \frac{a(a+1)}{b(b+1)} + ... + \frac{a...(a+h)}{b >>>} + ..., a \geq 0,\ b > a+1. \tag{9}$$

Denote

$$x_{-1} = 1, \tag{10a}$$

$$x_k = \frac{a...(a+k)}{b(b+k)}, \tag{10b}$$

$$s_n = s_n(a,b) = \sum_{i=-1}^{n} x_i. \tag{10c}$$

Obviously,

$$s(a,b) = s_n(a,b) + x_{n+1}s(a+n+2, b+n+2). \tag{11}$$

Theorem 12.

$$??_n(a,b) = \frac{b-1}{z} - x_{n+1}\frac{b+n+1}{z}, z = b-a-1. \tag{13}$$

Proof. We have to check, first, that

$$s_{-1}(a,b) = 1 \stackrel{?}{=} \frac{b-1}{z} - \frac{a}{b}\frac{b}{z} = \frac{b-1-a}{z} = 1,$$

which is true, and next, that

$$s_n + x_{n+1} \stackrel{?}{=} s_{n+1},$$

or

$$-x_{n+1}\frac{b+n+1}{z} + x_{n+1} \stackrel{?}{=} -\frac{b+n+2}{p}x_{n+2},$$

or, since

$$x_n = \frac{a+n}{b+n}x_{n-1},$$

to

$$-(b+n+1) + (b-a-1) \stackrel{?}{=} (b+n+2)\frac{a+n+2}{b+n+2},$$

or

$$n+a+2 \stackrel{?}{=} a+n+2,$$

which is true. ■

By Lemma 1, from (13) we have:

$$\lim_{n\to\infty} x_{n+1}\frac{b+n+1}{z} = 0.$$

Hence,

$$??(a,b) = s\infty(a,b) = \frac{b-1}{b-a-1}. \tag{14}$$

We now turn to the quantum version(s). Our series is:

$$??(a,b) = 1 + q^{f_0(a,b)}\frac{[a]_q^\sim}{[b]_q^\sim} + q^{f_1(a,b)}\frac{[a]_q^\sim[a+1]_q^\sim}{[b]_q^\sim[b+1]_q^\sim} + ..., \tag{15a}$$

$$f_n(a,b) = (b+a) - n(b-a-2). \tag{15b}$$

Denote

$$X_{-1} = 1, \tag{16a}$$

$$X_k(a,b) = \prod_{k=0}^{n}\frac{[a+k]_q^\sim}{[b+k]_q^\sim}, k \in \mathbf{Z}_{\geq 0}, a \geq 0, b-a-1 > 0 \tag{16b}$$

$$S_n(a,b) = 1 + \sum_{k=0}^{n} X_k q^{f_k(a,b)},\ n \in \mathbf{Z}_{\geq 0}, \tag{16c}$$

$$S_{-1} = 1. \tag{16d}$$

Theorem 17.

$$S_n(a,b)q^a\frac{[b-1]_q^\sim}{[b-a-1]_q^\sim} - q^{\varphi_n}X_{n+1}\frac{[b+n+1]_q^\sim}{[b-a-1]_q^\sim}, \tag{18a}$$

where

$$\varphi_n = (b-1) - (n+1)(b-a-1). \tag{18b}$$

Corollary 19.

$$S(a,b) = q^a\frac{[b-1]_q^\sim}{[b-a-1]_q^\sim}. \tag{20}$$

Proof. By (7) and (186), $q^{\varphi_n}[b+n+1]_q^\sim \to 0$ as $n \to \infty$. ■

Proof of Theorem 17. For $n = -1$(18) gives

$$1 \stackrel{?}{=} q^a\frac{[b-1]_q^\sim}{[z]_q^\sim} - q^{\varphi_{-1}}\frac{[a]_q^\sim}{[b]_q^\sim}\frac{[b]_q^\sim}{[z]_q^\sim} = q^a\frac{[b-1]_q^\sim}{[z]_q^\sim} - q^{\varphi_{-1}}\frac{[a]_q^\sim}{[z]_q^\sim},$$

or

$$[z]_q^\sim = [b-a-1]_q^\sim \stackrel{?}{=} q^a[b-1]_q^\sim + q^{\varphi(-1)}[-a]_q^\sim,$$

which is true because, in general,

$$[x+y]_q^\sim = q^{-y}[x]_q^\sim + q^x[y]_q^\sim,$$

and, from (18b)

$$\varphi(-1) = b - 1.$$

Now, the inductive step $n \mapsto n+1$ in (18a), amounts to

$$S_n(a,b) + q^{f_{n+1}} X_{n+1} \stackrel{?}{=} S_{n+1}(a,b),$$

or

$$-q^{\varphi_n} X_{n+1} \frac{[b+n+1]_q^{\sim}}{[z]_q^{\sim}} + q^{f_{n+1}} X_{n+1} \stackrel{?}{=} -q^{??+1} X_{n+2} \frac{[b+n+2]}{[z]_q^{\sim}} \tag{21}$$

But

$$X_{n+2} = X_{n+1} \frac{[a+n+2]_q^{\sim}}{[b+n+2]_q^{\sim}}.$$

Hence, (21) is:

$$-q^{\varphi_n}[b+n+1]_q^{\sim} + q^{f_{n+1}}[z]_q^{\sim} \stackrel{?}{=} -q^{\varphi_{n+1}} \frac{[a+n+2]_q^{\sim}}{[b+n+2]_q^{\sim}} [b+n+2]_q^{\sim},$$

or

$$[b+n+1]_q^{\sim} \stackrel{?}{=} q^{f_{n+1}}[b-a-1]_q^{\sim} + q^{p_{n+1}-p_n}[a+n+1]_q^{\sim},$$

or

$$\begin{pmatrix} f_{n+1} - \varphi_n \\ \varphi_{n+1} - \varphi_n \end{pmatrix} = \begin{pmatrix} a+n+2 \\ -(b-a-1) \end{pmatrix}. \tag{22}$$

(22) gives:

$$\varphi_n - \varphi_{-1}(n+1)(b-a-1) = (b-1) - (n+1)(b-a-1),$$

i.e., (18b). Now (22) yields:

$$f_{n+1} = (b-1) - (n+1)(b-a-1) + a + n + 2,$$

or

$$\begin{aligned} f_n &= (b-1) - n(b-a-1) + a + n = 1 = \\ &= \{b-1+a+1\} - n\{b-a-1-1\} = \\ &= (b+a) - n(b-a-2), \end{aligned} \tag{23}$$

and this is

Remark 24. (i) Notice that the restriction $a \geq 0$ is not important, but $b-a-1 > 0$ is, (ii) For $a = 0$, the formula (20) yields 1, as expected.

A Naive Treatment Of The Hardy-Littlewood Conjecture

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Abstract

The main case of the H-L almost follows directly from the PNT (prime number theorem).

The H-L Conjecture,

$$\pi(x+y) \leq \pi(x) + \pi(y), \quad x, y \geq 3, \tag{1}$$

can be deduced directly, at least for the important case,

$$y = x^{\epsilon}, \ \epsilon > 0, \tag{2}$$

from the PNT. This is how.

Write

$$\pi(x) = \frac{x}{log(x)}(1 + 0(1)), x \to \infty, \tag{3}$$

but not in the less precise and totally worthless equivalent form

$$\pi(x) \sim \frac{x}{logx}.$$

In general, but not always, asymptotic expansions are not useful for establishing inequalities.

Now substitute (3) into (1):

$$\frac{x+y}{log(x+y)}(1+0(1)) \overset{?}{\leq} \frac{x}{logx}(1+0(1)) + \frac{y}{logy}(1+0(1)),$$

which breaks in

$$\frac{x}{log(x+y)}(1+0(1)) \overset{?}{\leq} \frac{x}{logx}(1+0(1)), \quad x, y \to \infty,$$
$$\frac{y}{log(x+y)}(1+0(1)) \overset{?}{\leq} \frac{y}{log(y)}(1+0(1)), \quad x, y \to \infty,$$

or

$$\frac{logx}{log(x+y)} \overset{?}{\leq} 1 + 0(1), \quad x, y \to \infty \tag{4a}$$
$$\frac{logy}{log(x+y))} \leq 1 + 0(1), \quad x, y \to \infty, \tag{4b}$$

which are entirely obvious one x and y tend to infinity in a *regular* manner. For example, (4) is satisfied for

$$y = x^{\epsilon}, \quad \epsilon > 0, x \to \infty, \tag{5}$$

$$y = log^{\epsilon} x, \quad \epsilon > 0, x \to \infty, \tag{6}$$

etc., depending upon how the 0(1)-terms tend to 0.

An Alternating Sum Of Binomial Coefficients With Quadratic Weight

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Abstract

Classically, $\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^2 = 0$ for $n > 2$. We quantize this formula.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbf{R},\quad q \neq 0,\ \pm 1,$$

be the 2^{nd} quantization of x.

Set

$$\begin{bmatrix} m \\ k \end{bmatrix}_q^{\sim} = \frac{[m]!_q^{\sim}}{[k]!_q^{\sim}[m-k]!_q^{\sim}},$$
$$[k]!_q^{\sim} = [1]_q^{\sim}...[k]_q^{\sim},\ k \in \mathbf{Z}_{\geq 1};\ [0]!_q^{\sim} = 1.$$

Theorem 1. For $n \in \mathbf{Z}_{\geq 1}$,

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} ([k]_q^{\sim})^2 q^{(n-3)k} = 0,\ n \geq 3. \tag{2}$$

Proof. We use induction on n. For $n = 3$, (2) returns:

$$-[3]_q^{\sim} + [3]_q^{\sim}([2]_q^{\sim})^2 - ([3]_q^{\sim})^2 \stackrel{?}{=} 0,$$

or

$$([2]_q^{\sim})^2 \stackrel{?}{=} [3]_q^{\sim} - 1,$$

which is true:

$$[2]_q^{\sim} = q + q^{-1},\quad [3]_q^{\sim} = q^2 + 1 + q^{-2}.$$

Now, because

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q^{\sim} = q^{-k}\begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} + q^{n+1}q^{-k}\begin{bmatrix} n \\ k-1 \end{bmatrix}_q^{\sim},$$

$$\sum_n (-1)^k \begin{bmatrix} n+1 \\ k \end{bmatrix}_q^{\sim} ([k]!_q^{\sim})^2 q^{(n-2)k} =$$

$$= \sum_k (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} ([k]_q^{\sim})^2 q^{(n-3)^k} + q^{n+1}\sum_k (-1)^k q^{(n-3)k}([k]_q^{\sim})^2 \begin{bmatrix} n \\ k-1 \end{bmatrix}_q^{\sim} =$$

$$= -q^{n+1}\sum_s (-1)^s ([s+1]_q^{\sim})^2 \begin{bmatrix} n \\ s \end{bmatrix}_q^{\sim} q^{(n-3)(s+1)}. \tag{3}$$

But

$$([s+1]_q^{\sim})^2 = ([s]_q^{\sim})^2 + [2s+1]_q^{\sim}.$$

so (3) is proportional to

$$\sum_s (-1)^s \begin{bmatrix} n \\ s \end{bmatrix}_q^{\sim} q^{(n-3)s}[2s+1]_q^{\sim},$$

which is known to be equal to 0. ■

An Alternating Sum Of Quantum Binomial Coefficients With Rational Weights

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Abstract

Classically, $\sum_{k=0}^{n} \frac{(-1)^k}{k+r}\binom{n}{k} = \frac{n!}{r\ldots(r+n)}, r, n \in \mathbf{Z}_{\geq 1}$. We quantize this.

Let

$$[x]_q = \frac{q^x - 1}{q-1},\ x \in \mathbf{R},\ q \neq 0, \pm 1,$$

be the 2^{nd} quantization of x.

Set

$$\begin{bmatrix} m \\ k \end{bmatrix}_q^{\sim} = \frac{[m]!_q^{\sim}}{[k]!_q^{\sim}[m-k]!_q^{\sim}},$$
$$[k]!_q^{\sim} = [1]_q^{\sim}\ldots[k]_q^{\sim},\ k \in \mathbf{Z}_{\geq 1};\ [0]!_q^{\sim} = 1.$$

Theorem 1. For $r, n \in \mathbf{Z}_{\geq 1}$,

$$\sum_{k=0}^{n} \frac{(-1)^k}{[k+r]_q^{\sim}} \begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} q^{nk} = q^{-nr} \frac{[n]!_q^{\sim}}{[r]_q^{\sim}\ldots[r+n]_q^{\sim}}. \tag{2}$$

Proof. Denote the LHS and RHS of (2) by $\mathcal{L}(n,r)$ and $\mathcal{R}(n,r)$, respectively. We prove (2) by induction on n.

For $n = 0$, (2) returns:

$$\frac{1}{[r]_q^{\sim}} = \frac{1}{[r]_q^{\sim}},$$

which is true.

Next, using the formula

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q^{\sim} = q^{-k}\left(\begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} + q^{n+1}\begin{bmatrix} n \\ k-1 \end{bmatrix}_q^{\sim}\right), \tag{3}$$

we get:

$$\mathcal{L}(n+1,r)=\sum_k \frac{(-1)^k}{[k+r]_q^\sim}q^{(n+1)k}q^{-k}\left(\begin{bmatrix}n\\k\end{bmatrix}_q^\sim+q^{n+1}\begin{bmatrix}n\\k-1\end{bmatrix}\right)=$$
$$=\mathcal{L}(n,k)-q^{2n+1}\mathcal{L}(n,r+1). \tag{4}$$

We now show that $\mathcal{R}(n,r)$ satisfies the same new version (4). We have:

$$\mathcal{R}(n+1,r)=\frac{q^{-(n+1)r[n+1]!_q^\sim}}{[r]_q^\sim...[r+n+1]}\stackrel{?}{=}\mathcal{R}(n,r)-q^{2n+1}\mathcal{R}(n,r+1)=$$
$$=q^{nr}\frac{[n]!_q^\sim}{[r]_q^\sim...[r+n]_q^\sim}-q^{-n(r+1)}q^{2n+1}\frac{[n]!_q^\sim}{[r+1]_q^\sim...[r+n+1]_q^\sim},$$

or

$$q^{-(n+1)^r}[n+1]_q^\sim\stackrel{?}{=}q^{-nr}[r+n+1]_q^\sim-q^{2n-1-r(r+1)}[r]_q^\sim,$$

or

$$q^{-r}[n+1]_q^\sim+q^{n+1}[r]_q^\sim\stackrel{?}{=}[r+n+1]_q^\sim,$$

which is true, because, in general,

$$[a+b]_q^\sim=q^{-b}=q^{-b}[a]_q^\sim+q^a[b]_q^\sim.\ \blacksquare$$

A n-Ary Goldbach Problem

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Abstract

For M and n of the same parity, the equation

$$M = p_1 + ...p_n,$$

where p_i's are prime, has $\sim \frac{2}{(n-1)!}\frac{M^{n-1}}{log^n M}$ solutions.

Let M be a positive integer, of the same parity as the positive integer $n \geq 2$.

Theorem 1. The number of solutions of the equation

$$M = p_1 + ...p_n, \tag{2}$$

is

$$\sim \frac{2}{(n-1)!}\frac{M^{n-1}}{log^n M}. \tag{3}$$

Proof. We use induction on n. For $n = 2$, (3) is my formula for the binary Goldbach problem.

Denote

$$\frac{2}{(n-1)!} = C_n, \tag{4}$$

and $\mathcal{N}_n(M)$ the number of solutions of (2). Then

$$\mathcal{N}_{n+1}(M) = \sum_p \mathcal{N}_n(M-p) = \sum_p C_n \frac{(M-p)^{n-1}}{log(M-p)^n} \sim$$

$$\sim C_n \frac{M^{n-1}}{log^n M}\sum_i (1 - \frac{p_i}{M})^{n-1} =$$

$$= C_n \frac{M^{n-1}}{log^{n-1}}\sum_{i=0}^{n-1}(-1)^k \frac{p_i^k}{M^k}\binom{n-1}{k}. \tag{5}$$

But

$$\sum_p \frac{p_i^k}{M^k} \sim \sum_i \frac{(ilogi)^k}{M^k} \sim \frac{M^{k+1}log^k M}{(K+1)M^k} \sim \frac{M}{(k+1)logM}, \tag{6}$$

hence (5) is

$$C_k \frac{M^n}{log^{n+1}} \sum_{K=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{1}{K+1}. \tag{7}$$

The latter elementary sum is known to be $\frac{1}{n}$. Thus,

$$C_{K+1} = C_n \frac{1}{n}; \tag{8}$$

together with

$$C_2 = 2,$$

one recovers (3). ■

An Inequality Between Quantum And Real Numbers

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Abstract

The classical inequality, $\frac{1+q+...+q^n}{q+...+q^{n-1}} \geq \frac{n+1}{n-1}, q > 0, n \in \mathbf{Z}_{\geq 2}$, is extended for $n \in \mathbf{R}_{>0}$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \ q \neq 0, \pm 1, \tag{1}$$

be the first quantization of x, so that, for $n \in \mathbf{Z}_2$,

$$[n]_q = 1 + q + ... + q^{n-1}. \tag{2}$$

The inequality quoted in the abstract can be rewritten as

$$(n-1)(1 + ... + q^n) \geq (n+1)(q + ...q^{n-1}),$$

or as

$$(n-1)(1 + q^n) \geq 2(q + ... + q^{n-1}),$$

or as

$$(n-1)[2]_{q^n} \geq 2q[n-1]_q. \tag{3}$$

Theorem 4. Let $q > 0, x \in \mathbf{R}_{\geq 1}$. Then

$$(x-1)[2]_{q^x} \geq 2q[x-1]_q. \tag{5}$$

Proof. In the $log - g$ hand, (5) is:

$$(x-1)(q^x + 1) \stackrel{?}{\geq} 2q\frac{q^{x-1}1}{q-1},$$

or

$$(x-1)(q^x + 1x) \stackrel{?}{\geq} \frac{2}{q-1}(q^x - q),$$

or

$$x - 1 \stackrel{?}{\geq} \frac{2}{q-1}\frac{q^x - q}{q^x + 1},$$

or

$$x-1 \stackrel{?}{\geq} \frac{2}{q-1}\left(1-\frac{[2]_q^{\sim}}{q^x+1}\right). \tag{6}$$

At $x=1$, (6) yields:

$$0=0,$$

which is true. We show that $\partial/\partial x(LHS) > \partial/\partial x(RHS)$. This will establish (6). So:

$$1 \stackrel{?}{\geq} \frac{2}{q-1}\frac{[2]_q}{(q^x+1)^2}hq^x, \quad h=logq, \tag{7}$$

or

$$1 \stackrel{?}{\geq} \frac{h}{q-1}\frac{2(1+q)q^x}{q^{2x}+2q^x+1},$$

or

$$1 \stackrel{?}{\geq} \frac{h}{q-1}\frac{2(1+q)}{q^x+2+q^{-x}}. \tag{8}$$

Now,

Lemma.

$$\frac{h}{q-1} \leq 1, \tag{9}$$

because $q=e^h$, and (9) becomes:

$$\frac{h}{e^h-1} \leq 1,$$

or

$$\frac{e^h-1}{h} \geq 1. \tag{10}$$

If $h>0$, then (11) is:

$$e^h-1 \geq h,$$

which is true for $h=9$, and

$$\frac{d}{dh}(e^h-1)=e^h \geq \frac{d}{dh}(h)=1,\ h>0.$$

If $h<0$, then (11) is:

$$e^h-1 \leq h,$$

which is true for $h = 0$, and

$$\frac{d}{dh}(e^h - 1) = e^h \leq \frac{d}{dh}(h) = 1, \ h < 0. \quad \blacksquare$$

Now, (8) follows from

$$q^x + 2 + q^{-x} \stackrel{?}{\geq} 2(1 + q),$$

or

$$q^x + q^{-x} \stackrel{?}{\geq} 2q. \tag{12}$$

This is true for $x = 1$:

$$q + q^{-1} \stackrel{?}{\geq} 2q,$$

or

$$q^{-1} \stackrel{?}{\geq} q,$$

or

$$q \leq 1.$$

For general x, (12) returns

$$q^x - 2 + q^{-x} = (q^{x/2} - q^{-x/2})^2 \stackrel{?}{\geq} 2q - 2,$$

which is true because $q \geq 1$. $\blacksquare$

Remark 13. The restriction $q < 1$ seems unnecessary. For example, it is *not* needed if $x \in \mathbf{Z}_{\geq 2}$: here (3) follows from

$$[2]_{q^n} > q^{n-k} + q^k, \quad 1 \leq k \leq n - 1. \tag{14}$$

(14) is true because it can be rewritten as:

$$q^{n-k}(q^k - 1) > q^k - 1. \tag{15}$$

If $q > 1 \Rightarrow q^k > 1$ and (15) is true; if $q < 1$, then $q^k - 1 < 0$, (15) becomes

$$q^{n-k} < 1 \tag{16}$$

and it holds because $n - k > 0$.

An Inequality For A Quantum Quadric With Zero Classical Limit

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Abstract

The classically vanishing expression, $x^2 - (\lambda x)(\frac{x}{\lambda}), \lambda \in \mathbf{R}^*$, behaves quantumly in a complex fashion.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

is the 2^{nd} quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Let $\lambda \in \mathbf{R}^*$; we can take $\lambda > 0$. We are interested in

$$f(x_1\lambda) = [x^2]_q^{\sim} - [\lambda x]_q^{\sim}[\frac{x}{\lambda}]_q^{\sim}, \tag{1}$$

which vanishes at $q = 1$ no matter what x and λ are. We have:

$$\begin{aligned} &F(x,\lambda) = (q - q^{-1})f(x,\lambda) = \\ &= (q - q^{-1})(q^{x^2} - q^{-x^2}) - \\ &= (q^{\lambda x} - q^{-\lambda x})(q^{x/\lambda} - q^{-x/\lambda}) = \\ &= (q^{x^2} + 1 + q^{-x^2-1}) - (q^{x^2-1} + q^{1-x^2}) - (q^{(+\frac{1}{\lambda})x}) + q^{-(\lambda+\frac{1}{\lambda})x}(q^{(\lambda-\frac{1}{\lambda})x} + q^{(\frac{1}{\lambda}-\lambda)x}). \end{aligned} \tag{2}$$

Theorem 1. For $x > \lambda > 1, f(x,\lambda) > 0$. For $x < \lambda < 1, f(x,\lambda) < 0$.

Proof. With

$$h = logq > 0,$$

(2) is

$$\sum_{n=0}^{\infty} \frac{h^{2n}}{(2n)!} \left\{ (x^2+1)^{2n} - (x^2-1)^{2n} - \left\{ [(\lambda + \frac{1}{x})x]^{2n} - [(\lambda - \frac{1}{\lambda})x]^{2n} \right] \right\} \right\} \stackrel{?}{>} 0,$$

which follows from

$$(x^2+1)^{2n} - (x^2-1)^{2n} \stackrel{?}{>} (\lambda + \frac{1}{x})^{2n} x^{2n} - (\lambda - \frac{1}{\lambda})^{2n} x^{2n},$$

or

$$(x + \frac{1}{x})^{2n} - (x - \frac{1}{x})^{2n} \stackrel{?}{>} (\lambda + \frac{1}{x})^{2n} x^{2n} - (\lambda - \frac{1}{\lambda})^{2n} x^{2n},$$

or

$$(x + \frac{1}{x})^{2n} - (x - \frac{1}{x})^{2n} \stackrel{?}{>} (\lambda + \frac{1}{\lambda})^{2n} - (\lambda - \frac{1}{\lambda})^{2n}. \tag{3}$$

Now, $\partial/\partial x$ of (3) is $(2n)$ times:

$$\begin{aligned}
&(1 - \frac{1}{x^2})(x + \frac{1}{x})^{2n-1} - (1 + \frac{1}{x^2})(x - \frac{1}{x})^{2n-1} = \\
&= \frac{1}{x^2} \{ (x^2-1)(x^2+1)^{2n-1} - (x^2+1)(x^2-1)^{2n-1} \} = \\
&= \frac{(x^2-1)(x^2-1)}{x^2} \{ (x^2+1)^{2n-2} - (x^2-1)^{2n-2} \} > 0.
\end{aligned}$$

for $n > 1$. (For $n = 1$, (2) is an identity $4 = 4$.) Thus, (3) increases with x and is an inequality for $x = \lambda$. This proves $f(x, \lambda) > 0$ for $x > \lambda > 1$. Similarly, $f(x, \lambda) < 0$ for $x < \lambda < 1$. ■

An Inequality For The First Quantized Numbers

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Abstract

Let $q > 0, q \neq 1$, and $0 < a < 1$. We prove that $[a]_q < ([2]_q)^{a-2}$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \ q \neq 0, \pm 1,$$

is the 2^{nd} quantization of x.

Theorem 1. Suppose $q > 0, 0 < a < 1$. Then

$$[a]_q < ([2]_q)^{a-1}. \tag{2}$$

Proof. We first check the inequality (2) is invariant with respect to the change

$$q \mapsto q^{-1}. \tag{3}$$

Indeed, suppose (2) is satisfied for a given q. Then, changing q into q^{-1}, we get:

$$[a]_{q^{-1}} = \frac{1 - \frac{1}{q^a}}{1 - \frac{1}{q}} = \frac{1}{q^{a-1}} \frac{q^{a-1}}{q-1} = q^{1-a}[a]_q <$$
$$< q^{1-a}\left(1 + \frac{1}{q}\right)^{a-1} = \left(\left[2\right]_q\right)^{a-1},$$

i.e., (2) again. Thus, without loss of generality we can taken

$$0 < q < 1. \tag{4}$$

Our inequality (2),

$$\frac{1 - q^a}{1 - q} \overset{?}{<} \frac{(1+q)^a}{1+q}, \tag{5}$$

is true for $a = 0$, where it returns:

$$\frac{0}{1-q} \overset{?}{<} \frac{1}{+q},$$

which is obvious. On the other hand, for $a = 1$, (5) returns

$$1 = 1. \tag{6}$$

It is natural to suspect then that

$$\frac{\partial}{\partial a}(RHS - LHS) \stackrel{?}{>} 0 : \tag{7}$$
$$\frac{q^n logq}{q-1} \stackrel{?}{>} \frac{log(1+q)q^a}{1+q},$$

or

$$\frac{q+1}{q-1} \stackrel{?}{<} \frac{log(1+q)}{logq},$$

or

$$1 + \frac{2}{q-1} \stackrel{?}{<} \frac{log(1+q)}{logq}$$

or

$$\frac{2}{q-1} \stackrel{?}{<} \frac{log(1+1/q)}{logq}, \quad 0 < q < 1, \tag{8}$$

or

$$\frac{2logq}{q-1} \stackrel{?}{>} log(1+\frac{1}{q}),\ 0 < q < 1, \tag{9}$$

or, because

$$log(1+x) < x, x > 0,$$

(9) follows from

$$\frac{2logq}{q-1} \stackrel{?}{>} \frac{1}{q},\ 0 < q < 1,$$

or

$$logq \stackrel{?}{<} \frac{1-q}{2q}, \quad 0 < q < 1, \tag{10}$$

which is obvious, because $logq < 0,\ \frac{1-q}{2q} > 0$.

Thus,

$$\frac{(1+q)^a}{1+q} - \frac{1-q^a}{1-q}$$

decreases from something positive at $a = 0$ to zero at $a = 1$. Therefore, it's positive for $0 < a < 1$. ■

Another Parameterization Of A Quantum Circle

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Abstract

We quantize the classical relation $(4n)^2 + (qn^2-1)^2 = (4n^2+1)^2$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

be the second quantization of x.

We use the formula

$$([a]_q^{\sim})^2 - ([b]_q^{\sim})^2 = [a-b]_q^{\sim}[a+b]_q^{\sim}. \tag{1}$$

Then

$$\begin{aligned}&([4n^2+1]_q^{\sim})^2 - ([4n^2-1]_q^{\sim})^2 = \\ &= [2]_q^{\sim}[8n^2]_q^{\sim}.\end{aligned} \tag{2}$$

Thus, the harmless classical relation,

$$([4n])^2 + (4n^2-1)^2 = (4n^2+1)^2,\ n \in \mathbf{Z},$$

becomes:

$$[2]_q^{\sim}[8n^2]_q^{\sim} + ([4n^2-1]_q^{\sim})^2 = ([4n^2+1]_q^{\sim})^2,\ n \in \mathbf{Z}. \tag{3}$$

A Proof Of Sun's Conjecture

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Abstract

Sun's conjecture, that for $a \geq 1$, every integer M of appropriate parity satisfies $M = p + ax^2$, where p is a prime, is proved.

Given a, M must be:

1) if a is even, then M is odd;

2) if a is odd, then M is even for x odd;

3) if a is odd, then M is odd for x even.

With these restrictions in mind, the Sun Conjecture is: given a, envy sufficiently large M of appropriate parity satisfies: $M = p + ax^2$, where p is a prime.

The Conjecture looks forbidding, but is, in fact, trivial.

Theorem 1. The number of solutions (p, x), given M, is $\sim$

$$\frac{2M^{V_2}}{a^{1/2} log M}. \tag{2}$$

Proof. Set

$$\delta(x) = \pi(x) - \pi(x-2), \tag{3}$$

where $\pi(x) = \sum_{p \leq x}$ is the prime-counting function. I recently proved, in setting the Goldbach conjecture, that, for x odd,

$$\delta(x) \sim \frac{2}{log x}. \tag{4}$$

Hence, if we denote by $n(M)$ the number of solutions of $M = p + ax^2$, we have:

$$n(M) = \sum_{x=1}^{u.\ell} \delta(M - ax^2),$$

where $u.\ell = [\sqrt{\frac{M}{a}}] \sim \sqrt{\frac{M}{a}}$. Thus,

$$n(M) \sim \sum_{x=1}^{\sqrt{M/a}} \frac{2}{logM - \frac{ax^2}{M}} \sim \frac{2}{logM} \sum \left(1 + \frac{ax^2}{MlogM}\right) \sim$$

$$\sim \frac{2}{logM}\left(\sqrt{\frac{M}{a}} + a\left(\sqrt{\frac{M}{a}}\right)^3_3 \frac{1}{MlogM}\right) \sim \frac{2}{logM}\left(\frac{M^{1/2}}{a^{1/2}} + \frac{M^{1/2}}{3a^{1/2}logM}\right) \sim$$

$$\sim \frac{2M^{1/2}}{a^{1/2}logM},$$

as claimed. ■

A Proper Quantization Of The Triangle Inequality

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Abstract

Triangle inequality is quantized.

Let

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbf{R},\ q \neq 0, \pm 1,$$

be the second quantization of x.

Since

$$[x]^{\sim}_{q^{-1}} = [x]^{\sim}_q,$$

we can take

$$q > 1 \tag{1}$$

without loss of generality.

Let

$$a + b > c, \tag{2}$$

for $a, b, c > 0$, be a triangle inequality. We'd like to quantize it. In a previous paper I showed that (2) sometimes leads to

$$[a]^{\sim}_q + [b]^{\sim}_q > [c]^{\sim}_q, \tag{3}$$

but at other times leads to

$$[a]^{\sim}_q + [b]_q <]c]_q. \tag{4}$$

So (3) is *not* a proper quantization route.

Theorem 5. For $a, b, c > 0$, (2) imples:

$$q^{-a}[a]^{\sim}_q + q^{-b}[b]^{\sim}_q > q^{-c}[c]^{\sim}_q. \tag{6}$$

Proof. The RHS of (6) is an increasing function of c, because

$$q^{-c}[c]^{\sim}_q = \frac{1 - q^{-2c}}{q - q^{-1}},$$

and $q > 1(1)$. Thus, (6) will satisfied if it is satisfied at max c possible, $c = a + b$:

$$q^{-a}[a]_q^{\sim} + q^{-b}[b]_q^{\sim} \stackrel{?}{>} q^{-a-b}[a+b]_q^{\sim},$$

or

$$q^{b}[a]_q^{\sim} + q^{a}[b]_q^{\sim} \stackrel{?}{>} [a+b]_q^{\sim}. \tag{7}$$

Since, in general,

$[x+y]_q^{\sim} = q^{-y}[x]_q^{\sim} + q^{x}[y]_q^{\sim}$, (7) can be rewritten as :

or

$$q^{b}[a]_q^{\sim} + q^{a}[b]_q^{\sim} \stackrel{?}{>} q^{b}[a]_q^{\sim} + q^{-a}[b]_q^{\sim},$$
$$q^{a}[b]_q^{\sim} \stackrel{?}{>} q^{-a}[b]_q^{\sim},$$

or

$$q^{a} \stackrel{?}{>} q^{-a},$$

or

$$q^{2a} \stackrel{?}{>} 1,$$

which is obvious, because $a > 0$ and $q > 1$. ■

Remark 8. Let $h = logq$ Without imposing the condition $q > 1$, we may rewrite (6) as:

$$e^{-a|logq|}[a]_q^{\sim} + e^{-b|logq|}[b]_q^{\sim} > e^{-c|logq|}[c]_q^{\sim}. \tag{9}$$

A Quadratic Quantum Inequality

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Abstract

$\left(\frac{a_1+...+a_n}{n}\right)^2 \leq \frac{a_1^2+...+a_n^2}{n}$. We quantize this.

??????

Theorem 1. Let $n \in \mathbf{Z}_{\geq 2}$. Then

$$\left(\left[\frac{a_1+...+a_n}{n}\right]_q^{\sim}\right)^2 \leq \frac{([a_1]_q^{\sim})^2+...+([a_n]_q^{\sim})^2}{n} \tag{2}$$

Proof. Multiplying through by $(q-q^{-1})^2 > 0$, and denoting

$$\sigma = a_1 + ... + a_n, \tag{3}$$

(2) becomes

$$(q^{\sigma/n} - q^{-\sigma/n})^2 \stackrel{?}{\leq} \frac{1}{n}\sum_{i=1}^{n}(q^{a_i} - q^{-a_i})^2,$$

or

$$q^{2\sigma/n} + q^{-2\sigma/n} - 2 \stackrel{?}{\leq} \frac{1}{n}\sum_{i=1}^{n}(q^{2a_i} + q^{-2a_i} - 2),$$

or

$$Q^{\sigma/n} + Q^{-\sigma/n} \stackrel{?}{\leq} \frac{1}{n}\sum_{i=1}^{n}(Q^{a_i} + Q^{-a_i}), \tag{4}$$

where $Q + q^2 > 1$.

Denote

$$A = \frac{1}{n}\sum_{i=1}^{n} Q^{a_i}, \tag{5a}$$

$$B = Q^{\sigma/n} = \sqrt[n]{Q^{\sigma}} = \sqrt[n]{\sqcap_{i=1}^{n} Q^{a_i}}. \tag{5b}$$

By the AGM inequality,

$$A \geq B > 1. \tag{6}$$

Lemma 7. If $X_1, ..., X_n > 1$ and

$$\frac{1}{n}\sum_{i=1}^{n} X_i \geq A, m \in \mathbf{Z}_{\geq 1},$$

then

$$\frac{1}{n}\sum_{i=1}^{n}(X_i + X_i^{-1}) \geq A + A^{-1}. \tag{8}$$

Proof. The function $f(x) = x + x^{-1}$ on $x > 1$ has

$$\frac{df}{dx} = 1 - \frac{1}{x^2} > 0.$$

Thus, the max of the RHS of (8), subject to the constrain

$$A \leq \frac{1}{n}\sum_{i=1}^{n} X_i,$$

is achieved for

$$A = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Hence, (8) becomes:

$$\frac{1}{n}\sum_{i=1}^{n}\left(X_i + X_i^{-1}\right) \stackrel{?}{\geq} \frac{1}{n}\sum_{i=1}^{n} X_i + \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^{-1},$$

or

$$\left(\frac{1}{n}\sum_{i=1}^{n} X_i^{-1}\right)\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \stackrel{?}{\geq} 1,$$

which follows at once for the AGM inequality. ■ ■

A Quantization Of A Simple Classical Inequality

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Abstract

Classically, $2^k > k+2$ for $k > 2$. We quantize this.

Let

$$[x]_q = \frac{q^x - 1}{q-1},\ x \in \mathbf{R},\ q \neq 0, \pm 1,$$

be the 2^{nd}quantization of x.

Since

$$[x]^{\sim}_{q^{-1}} = [x]^{\sim}_q,$$

we can take

$$q > 1$$

without loss of generality.

We want to quantize the inequality

$$2^k > k+2, \quad k \in \mathbf{Z}_{\geq 3}, \tag{1}$$

easily provable by induction on k. The natural candidate

$$([2]^{\sim}_q)^k \overset{?}{>} [k+2]^{\sim}_q \tag{2}$$

is false, since $([2]^{\sim}_q)^k = (q+q^{-1})^k = q^k ...$ while $[k+2]^{\sim}_q = q^{k+1}_{+...}$.

Theorem 3. For $k \in \mathbf{Z}_{\geq 2}$,

$$([2]^{\sim}_{q^2})^k > [k+2]^{\sim}_q. \tag{4}$$

Proof. We use induction on k. For $k = 2$, the classically forbidden case, (4) turns into:

$$([2]^{\sim}_{q^2}) \overset{?}{>} [4]^{\sim}_q = [2]^{\sim}_q [2]^{\sim}_{q^2},$$

or

$$[2]^{\sim}_{q^2} \overset{?}{>} [2]^{\sim}_q,$$

which is true, because $[2]^{\sim}_{q^a}$ grows with a.

Now the inductive step. It amounts to

$$[2]^{\sim}_{q^2}[k+2]^{\sim}_q \overset{?}{>} [k+3]^{\sim}_q,$$

or

$$(q^2+q^{-2})[k]^{\sim}_q \overset{?}{>} [k+1]^{\sim}_q,\ k \in \mathbf{Z}_{\geq 1}. \tag{5}$$

Multiplying by $q-q^{-1}>0$ (since $q>1$), (5) become:

$$(q^2+q^{-2})(q^k-q^{-k}) \overset{?}{>} q^{-k-1}(q^{2k+2}-1), \tag{6}$$

or

$$q^3(q+q^{-4}) \overset{?}{>} \frac{q^{2k+2}-1}{q^{2k}-1}. \tag{7}$$

At $k=1$, (7) is:

$$q^3(1+q^{-4}) \overset{?}{>} \frac{q^4-1}{q^2-1} = q^2+1,]$$

or

$$q^3+q^{-1} \overset{?}{>} q^2+1,$$

or

$$q^2(q^{-1}) \overset{?}{>} 1-q^{-1} = q^{-1}(q-1),$$

or

$$q^2 \overset{?}{>} q^{-1},$$

which is true because $q>1$

Now, set

$$f(x) = \frac{q^{x+2}-1}{q^x-1}.$$

Then

$$\frac{df}{dx} = h\left\{\frac{q^{x+2}}{q^x-1} - \frac{q^{x+2}-1}{(q^x-1)^2}\right\} = \frac{h}{q^x-1}\left\{q^x+2-\frac{q^{x+2}-1}{q^x-1}\right\} =$$
$$= \frac{h}{q^x-1}\left\{q^{x+2}\left[1-\frac{1}{q^x-1}\right]+\frac{1}{q^x-1}\right\} > \left[1-\frac{1}{q^x-1}\right]+\frac{1}{q^x-1} = 1.$$

Thus, $f(x)$ is increasing with x, and

$$\lim_{x\to\infty}\frac{q^{x+2}-1}{q^x-1} = q^2,$$

so (7) follows from

$$q^3(1+q^{-2}) \overset{?}{>} q^2,$$

which is obvious since $q>1$. ■

A Quantum Double-Factorial Inequality

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Abstract

We quantize the classical inequality $(2n-1)!! > \frac{n^n}{n^e}$ for $n > 1$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

be the 2^{nd}quantization of x.

Since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Set

$$[k]!_q^{\sim} = [1]_{q^{\cdot\cdot}}^{\sim}[k]_q^{\sim}, k \in \mathbf{Z}_{\geq 1}; [0]_q^{\sim} = 1;$$
$$[2k+1]!!_q^{\sim} = [1]_q^{\sim}[3]_q^{\sim}...[2k+1]_q^{\sim},\ k \in \mathbf{Z}_{\geq 0}.$$

Theorem 1. For $n \in \mathbf{Z}_{\geq 1}$,

$$[2n-1]!! > \frac{([n]_q^{\sim})^2}{[n]!_q^{\sim}}, \tag{2}$$

Proof. We use induction on n. For $n = 2$, (2) returns:

$$[3]^{\sim} \stackrel{?}{>} \frac{([2]_q^{\sim})^2}{[2]_q^{\sim}} = [2]_q^{\sim}, \tag{3}$$

which is true, because, in general,

$$a > b \Rightarrow [a]_q^{\sim} > [b]_q^{\sim}. \tag{4}$$

Now, the inductive step $n \mapsto n+1$ amounts to:

$$[2n+1]_q^{\sim} \frac{([n]_q^{\sim})^n}{[n]!_q^{\sim}} \stackrel{?}{>} \frac{([n+1]_q^{\sim})^{n+1})}{[n+1]!_q^{\sim}},$$

or

$$[2n+1]_q^{\sim} \stackrel{?}{>} \left(\frac{[n+1]_q^{\sim}}{[n]_q^{\sim}}\right)^n. \tag{5}$$

At $q=1$, the RHS of (5) is bounded by e. For $q>1$, we have to be inventive.

Since

$$[n+1]_q^{\sim} = q[n]_q^{\sim} + q^{-n}, \tag{6}$$

and

$$[2n+1]_q^{\sim} = q^{2n} + q^{2n-2} + q^{n-4} + ... \tag{7}$$

(6) follows from

$$q^{2n}(1+q^{-2}+q^{-4}) \stackrel{?}{>} \left(q + \frac{q^{-n}}{[n]_q^{\sim}}\right)^n = q^n \left(1 + \frac{q^{-n-1}}{[n]_q^{\sim}}\right)^n,$$

or

$$q^n + q^{n-2} + q^{n-4} \stackrel{?}{>} \left(1 + \frac{q^{-n-1}}{[n]_q^{\sim}}\right)^n. \tag{8}$$

Now,

$$h < [n]_q^{\sim}, \quad \forall n \geq 1,$$

so the RHS of (8) is

$$\sum_{k=0}^{n} \binom{n}{k} \frac{q^{-k(n+1)}}{[n]_q^{\sim k}}, \tag{9}$$

and

$$\binom{n}{k} \leq \frac{n^k}{k!},$$

so (9) is

$$< \sum_{k=0}^{n} \frac{q^{-k(n+1)}}{k!} = 1 + \frac{q^{-n-1}}{1!} + \frac{q^{-2(n+1)}}{2!} + \frac{q^{-n(n+1)}}{n!} < e^{-q-n-1} < e,$$

because $q>1$ and

$$q^{-n-1} < 1.$$

So, (8) is proven, because

$$q^n + q^{n-2} + q^{n-4} \geq 3 > e \text{ for } n \geq 4.$$

It remains to check the case $n = 3$:

$$[3]_q^{\sim}[5]_q^{\sim} \stackrel{?}{>} \frac{([3]_q^{\sim})^3}{[2]_q^{\sim}[3]_q^{\sim}},$$

or

$$[2]_q^{\sim}[5]_q^{\sim} \stackrel{?}{>} [3]_q^{\sim},$$

which is obvious because

$$[5]_q^{\sim} > [3]_q,$$

and

$$[2]_q^{\sim} > 2 > 1. \qquad \blacksquare$$

A Quantum Inequality For A Finite Product

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Abstract

Classically, $(m+\frac{3}{2})(m+\frac{7}{2}...(m+\frac{2k-1}{2}) > \sqrt{\frac{(m+2k)!}{m!}}$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

is the 2^{nd} quantization of x, so that

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

and we can take

$$q > 1$$

without loss of generality.

Theorem 1. Let $m, k \in \mathbf{Z}_{\geq 1}$. Then

$$\prod_{s=1}^{k} \left[m + \frac{4s-1}{2} \right]_q^{\sim} > \sqrt{\frac{[m+2k]!_q^{\sim}}{[m]!_q^{\sim}}}, \tag{2}$$

where

$$[k]!_q^{\sim} = [1]_q^{\sim}...]k]_q^{\sim}, \quad k \in \mathbf{Z}_{\geq 1}; \quad [0]!_q^{\sim} = 1.$$

Proof. We use induction on k. For $k = 1$, (2) becomes:

$$[m + \frac{3}{2}]_q^{\sim} \stackrel{?}{>} \sqrt{[m+2]_q^{\sim}[m+1]_q^{\sim}},$$

or

$$\left([m + \frac{3}{2}] \right)^2 \stackrel{?}{>} [m+1]_q^{\sim}[m+2]_q^{\sim}. \tag{3}$$

Notice that $m + 1 \geq 2$.

Lemma 4. Suppose $a > 0, b, c \geq 0$, and

$$2a \geq b + c. \tag{5}$$

Then

$$([a]_q^{\sim})^2 \geq [b]_q^{\sim}[c]_q^{\sim}. \tag{6}$$

Proof. Multiplying (γ) by $(q - q^{-1})^2 > 0$, we get:

$$(q^a - q^{-a})^2 \stackrel{?}{\geq} (q^b - q^{-b})(q^c - q^{-c}),$$

or

$$(q^a + q^{-2a} - 2) \stackrel{?}{\geq} (q^{b+c} + q^{-b-c}) - (q^{b-c} + q^{c-b}) \tag{7}$$

Since

$$q^{\triangle} + q^{-\triangle} \geq 2, \quad \forall \triangle \in \mathbf{R}, \tag{8}$$

it remains to show that

$$q^{2a} + q^{-2a} \stackrel{?}{>} q^{b+c} + q^{-(b+c)},$$

or

$$q^{b+c}(q^{2a-b-c} - 1) \stackrel{?}{\geq} q^{-2a}(q^{2a-b-c} - 1),$$

or, since $2a - b - c > 0$ by (5), $q^{2a-b-c} - 1 \geq 0$,

$$q^{b+c} \stackrel{?}{\geq} q^{-2a},$$

which is obvious. ■

Now,

$$2(m + \frac{3}{2}) = 2m + 3 = (m + 1) + (m + 2),$$

so (3) is true by the Lemma.

The inductive step consists of

$$\sqrt{\frac{[m + 2k]!_q^{\sim}}{[m]!_q^{\sim}}} \left[m + \frac{4k + 3}{2}\right]_q^{\sim} \stackrel{?}{\geq} \sqrt{\frac{[m + 2k + 2]!_q^{\sim}}{[m]!_q^{\sim}}},$$

or

$$\left[m + \frac{4k + 3}{2}\right] \stackrel{?}{\geq} [m + 2k + 2]_q^{\sim}[m + 2k + 1]_q^{\sim}, \tag{9}$$

and since

$$2(m + \frac{4k + 3}{2}) = 2m + 4k + 3 = (m + 2k + 2) + (m + 2k + 1), \text{ (9) is true by}$$

the Lemma. ■

A Quantum Inequality For The Powers Of Classical 2

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Abstract

We show that, incontrast to the classical case $q = 1, (1 + x)^n + (1 + xn^n \geq 2^n)$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \ q \neq 0, \pm 1,$$

is the 2^{nd} quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Theorem 1. Let $x > 0$. Then, for $n \in \mathbf{Z}_{\geq 1}$,

$$([1 - x]_q^{\sim})^n + ([1 + x]_q^{\sim})^n \geq 2^n. \tag{2}$$

Proof. We start with the case $n - 1$. Then

$$[1 - x]_q^{\sim} + [1 + x]_q^{\sim} = \frac{1}{q - q^{-1}} \Big\{ q^{1-x} - q^{x-1} + q^{x+1} - q^{-1-x} \} =$$

$$= \frac{1}{q - q^{-1}} \{ q^x (q - q^{-1}) + q^{-x} (q - q^{-1}) \} = q^x + q^{-x} = [2]_{q^n}^{\sim},$$

and

$$[1 - x]_q^{\sim} + [1 + x]_q^{\sim} \geq 2, \tag{3}$$

with equality iff $x = 0$. Now the inequality (2) is reduced to a purely *classical* one. Denote

$$a = [1 - x]_q^{\sim}, \quad b = [1 + x]_q^{\sim}.$$

Then (3) can be rewritten as

$$a + b \geq 2, \quad a, b \geq 0 \tag{4}$$

and we are interested in the ??? of

$$f(a, b) = a^n + b^n.$$

Obviously, this ?? is achieved at the boundary, and (2) follows. ■

A Quantum Inequality, Obvious Classically

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Abstract

Classically, $(1+a_1)...(1+a_n) > 1+a_1+...+a_n$, for $a_1>0,...,a_n>0$. We prove the quantum analog of this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q-q^{-1}}, \quad x, \in \mathbf{R},\ q \neq 0, \pm 1,$$

be the 2^{nd} quantization of x.

Theorem 1. For $a_1, ...a_n > 0, n > 1$,

$$[1+a_1]_q^{\sim}...[1+a_n]_q^{\sim} > [1+a_1+...+a_n]_q^{\sim}. \tag{2}$$

Proof. We use induction on n. For $n=2$, (2) is:

$$[1+a]_q^{\sim}[1+b]_q^{\sim} \stackrel{?}{>} [1+a+b]_q^{\sim},\ a=a_1,\ b=a_2,$$

or

$$(q^{1+a}-q^{-1-a})(q^{1+b}-q^{-1-b}) \stackrel{?}{>} (q-q^{-1})(q^{1+a+b}) - q^{-1-a-b},$$

or

$$(q^{2+a+b}+q^{-2-a-b}) - (q^{a-b}+q^{b-a}) \stackrel{?}{>}$$
$$\stackrel{?}{>} (q^{2+a+b}+q^{-2-a-b}) - (q^{a+b}+q^{-a-b}),$$

or

$$q^{a+b}+q^{-a-b} \stackrel{?}{>} q^{a-b}+q^{b-a},$$

which is true because $[2]_{q^x}^{\sim}$ is increasing with $x>0$, and $a+b \geq |a-b|$.

Suppose now (2) is true for $n=N$. Denote

$$a+1+...+a_N = \sigma,$$
$$a_1+...+a_{N+1} = \sum.$$

Then, for the LHS of (2), LHS_n, we have, with $b = a_{N+1}$:

$$LHS_{N+1} = [1+b]_q^{\sim} LHS_N >$$
$$> [1+b]_q^{\sim}[1+\sigma]_q^{\sim} \stackrel{?}{>} [1+\sum]_q^{\sim},$$

or

$$(q^{1+b} - q^{-1-b})(q^{1+\sigma} - q^{-1-\sigma}) \stackrel{?}{>} (q - q^{-1})(q^{1+\Sigma} - q^{-1-\Sigma}),$$

or

$$(q^{2+\Sigma} + q^{-2-\Sigma}) - (q^{\sigma-b} + q^{b-\sigma}) \stackrel{?}{>}$$
$$\stackrel{?}{>} (q^{2+\Sigma} + q^{-2-\Sigma}) - (q\Sigma + q^{-}\Sigma),$$

or

$$q^{\sigma+b} + q^{-\sigma-b} \stackrel{?}{>} q^{\sigma-b} + q^{b-\sigma},$$

which is true because $q^x + q^{-x}$ is an increasing function of $x \geq 0$. ■

A Inequality Between Some Special Quantum And Classical Numbers

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Abstract

Classically, $2x = (2x-1)+1$. Quantumly, the situation is more interesting.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Theorem 1. For any $x \geq 0$,

$$x[2]_{q^{2x}}^{\sim} - 1 > [2x-1]_q^{\sim}, \quad x \geq 1. \tag{2}$$

Proof. Multiplying through by $(q - q^{-1}) > 0$, we get:

$$x(q-q^{-1})(q^{2x} + a^{-2x}) - (q - q^{-1}) \stackrel{?}{>} q^{2x-1} - q^{-2x+1},$$

or

$$x\left[(q^{2x+1} - q^{-2x-1}) - (q^{2x-1} - q^{1-2x})\right] \stackrel{?}{\geq} (q^{2x-1} - q^{-2x+1}) + (q - q^{-1}),$$

or

$$x(q^{2x+1} - q^{2x-1}) \stackrel{?}{\geq} (x+1)(q^{2x-1} - q^{-2x+1}) + (q - q^{-1}), \tag{3}$$

which can also be rewritten as

$$[2x+1]_q^{\sim} \stackrel{?}{\geq} (x+1)[2x-1]_q^{\sim} + 1, x > 1, \tag{4}$$

Setting

$$h = logq > 0,$$

(3) can be rewritten as:

$$x2\sum_{n=0}^{\infty} \frac{h^{2n+1}}{(2n+1)!}(2x+1)^{2n+1} \stackrel{?}{>} (x+1)2\sum_{n=0}^{\infty} \frac{h^{2n+1}}{(2n+1)} + 2\sum_{n=0}^{\infty} \frac{h^{2n+1}}{(2n+1)}. \tag{5}$$

(5) follows from:

$$x(2x+1)^{2n+1} \stackrel{?}{\geq} (x+1)(2x-1)^{2n+1} + 1,\ n \geq 1. \tag{6}$$

(6) is an equality for $n = 0$. So let now $n \geq 1, 2n+ \geq 3$.

We first use induction on n, the base $n = 0$ has been covered. The induction step $n \mapsto n+1$ provides:

$$(x+1)(2x-1)^{2n+3} < (2x-1)^2[x(2x+1)^{2n+1} - 1] \stackrel{?}{<}$$
$$\stackrel{?}{>} (2x+1)^{2n+3} - 1,$$

or

$$x\{(2x-1)^2\}(2x+1)^{2n+1} - (2x+1)^{2n+3}] \stackrel{?}{\leq} (2x-1)^2 - 1,$$

or

$$x(2x+1)^{2n+1}\{(2x-1)^2 - (2x+1)^2\} \stackrel{?}{\leq} 4x(x-1),$$

or

$$x(2x+1)^{2n+1}\{(-2)(4x)\} \stackrel{?}{\leq} 4x(x-1), \tag{7}$$

which is obvious for $x \geq 1$ since the LHS < 0.

For $0 < x < 1$, the situation is somewhat unclear. The inequality (7) is certainly wrong for $n = 0$:

$$x(2x+1)^2 2x \stackrel{?}{\geq} x(1-x),$$

or

$$x(4x^2 + 2x + x - 1) \stackrel{?}{\geq} 0,$$

or

$$x(4x^2 + 3x - 1) \stackrel{?}{\geq} 0,$$

or

$$x(x+1)(4x-1) \stackrel{?}{\geq} 0,$$

which is false between the roots:

$$-1 \leq x \leq \frac{1}{4}. \tag{8}$$

We therefore leave the region $0 < x < 1$ in the dark, although our immediate inequality (6) is an identity for $n = 0$, irrespective of where x belongs to. Just the inductive argument breaks down on $[0, 1/4]$.

To handle, the region $[0, 1/4]$ for (6), we introduce

$$1 - 2x = t, \quad 1/2 \leq + \leq 1. \tag{9}$$

Then (6) becomes, with V standing for an unknown $<$ or $>$ sign:

$$\frac{1-t}{2}\{(2-t)^{2n+1} + t^{2n+1}\}V1 - t^{2n+1},$$

or

$$\frac{(2-t)^{2n+1} + t^{2n+1}}{2} V \frac{1 - t^{2n+1}\,^{2n+1}}{1-t} = 1 + t + ... + 4^{2n}. \tag{10}$$

Now, (10) is an equality for $n = 0$, as expected. The inductive step $n \mapsto n + 1$ gives:

$$RHS_{n+1} = RHS_n + t^{2n+1} + t^{2n+2} - V\frac{(2-t)^{2n+1}}{2} + t^{2n+1}[\frac{1}{2} + (1+t)]$$
$$V\frac{(2-t)^{2n+3}}{2} + \frac{t^{2n+1}}{2}t^2,$$

or

$$\frac{t^{2n+1}}{2}(3 + 2t - t^2)V\frac{(2-t)^{2n+1}}{2}[(t-2)^2 - 1]. \tag{11}$$

or

$$t^{2n+1}(3 + 2t - t^2)V(2-t)^{2n+1}[t^2 - tt + 3]. \tag{12}$$

But

$$t \leq 2 - t$$

on $t \in [1/2, 1]$, and

$$3 + 2t - t^2 \stackrel{?}{\geq} t^2 - t + 3,$$

or

$$3t \stackrel{?}{\geq} 2t^2,$$

or

$$3 \stackrel{?}{\geq} \frac{2}{3}t,$$

which is true. Thus, V is $<$, and (6) changes the $>$ sign into $<$ one on $x\epsilon[0, 1/4]$. ■

A Quantum Linear Function Whose Classical Limit Vanishes

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Abstract

Classically, $xy \leq 1/2$ for $x + y = 1, x, y \geq 0$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

is the 2^{nd} quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

We are interested in the linear function

$$f(x,a) = f(a,x) = [x+a]_q^{\sim} - [x]_q^{\sim} - [a]_q^{\sim},\ a, x \geq 0. \tag{0.1}$$

Since

$$[b+c]_q^{\sim} = q^{-b}[c]_q^{\sim} + q^c[b]_q^{\sim},$$

we can rewrite $f \in s$:

$$\begin{aligned} &f(x,a) = q^{-a}[x]_q^{\sim} + q^x[a]_q^{\sim} - [x]_q^{\sim} - [a]_q^{\sim} = \\ &= (q^{-a} - 1)[x]_q^{\sim} + (q^x - 1)[a]_q^{\sim} = \\ &= (q^x - 1)[a]_q^{\sim} - (1 - q^{-a})[x]_q^{\sim}. \end{aligned} \tag{2}$$

Theorem 3. $f(x,a) > 0$ for $x > a > 0$.

Proof. Notice that

$$f(a,a) = [2a]_q^{\sim} - 2[a]_q^{\sim} = [a]_q^{\sim}([2]_{q^a}^{\sim} - 2) > 0$$

because

$$[2]_{q^a}^{\sim} > 2 \text{ for } a \neq 0, \tag{4}$$

and since

$$f(x,0) = f(0,a) = 0,$$

we can safely assume that

$$ax \neq 0.$$

Now, (2) yields, for its x-derivative:

$$\frac{1}{h}\frac{\partial f(\partial c, a)}{\partial x} = q^x[a]_q^{\sim} - (1-q^{-a})\frac{[2]_{q^x}^{\sim}}{q-q^{-1}},$$

where

$$h = logq > 0.$$

Hence,

$$\begin{aligned}
&\frac{q-q^{-1}}{h}\frac{\partial f}{\partial x} = q^x(q^a - q^{-a}) - (1-q^{-a})(q^x + q^{-x}) = \\
&= q^x q^a[(1+q^{-a}) - (1+q^{-2x})] = \\
&= q^x q^a(q^{-a} - q^{-2x}) > 0,
\end{aligned}$$

because $x > 0$ and $q > 1$. This takes care of the $x > a > 0$ case, because $f(x,a) > f(a,a) > 0$. The $x < a$ case is similar, because $f(x,a) = f(a,x)$. ■

A Quantum Nestled Radicals Inequality

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Abstract

The classical inequality, $\sqrt{a}+\sqrt{a}+...+\sqrt{a}<\frac{1+\sqrt{4a+1}}{2}$, is quantized.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\quad x\in\mathbf{R},\ q\neq 0,\pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim}=-[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim}=[x]_q^{\sim},$$

we can take

$$q>1$$

without loss of generality.

Theorem 1. Let α be unspecified,

$$Q=q^{\alpha}\frac{2}{[2]_q^{\sim}}. \tag{2}$$

Then

$$\sqrt{a}+Q\sqrt{a}+...+\overline{Q\sqrt{a}}<\frac{q^{\alpha}+\sqrt{([2]_q^{\sim})^2a+q^{2\alpha}}}{[2]_q^{\sim}},\quad a\geq 0, \tag{3}$$

(n radicels on the left.)

Proof. For $n=1$, (3) is

$$\sqrt{a}\overset{?}{<}\frac{q^{\alpha}+\sqrt{([2]_q^{\sim})^2a+q^{2\alpha}}}{[2]_q^{\sim}},$$

or

$$a \overset{?}{<} \frac{q^{2\alpha}+2\sqrt{\dots}+([2]_q^{\sim})^2 a+q^{2a}}{([2]_q^{\sim})^2},$$

which is obvious.

The inductive step $n \mapsto n+1$ is:

$$\sqrt{a+Q\sqrt{\frac{q^k+\sqrt{([2]_q^{\sim})^2 a+q^{2\alpha}}}{[2]_q^{\sim}}}} \overset{?}{<} \frac{q^{\alpha}+\sqrt{([2]_q^{\sim})^2 a+q^{2\alpha}}}{[2]_q^{\sim}},$$

or

$$a+Q\sqrt{\frac{q^{\alpha}+\sqrt{([2]_q^{\sim})^2 a+q^{2\alpha}}}{[2]}} \overset{?}{<} \frac{q^{\alpha}+2q^{\alpha}\sqrt{([2]_q^{\sim})^2 a+q^{2\alpha}}+([2]_q^{\sim})^2 q+q^{2\alpha}}{([2]_q^{\sim})^2}$$

or

$$Q^2 \frac{q^{\alpha}+\sqrt{([2]_q^{\sim})^2 a+q^{2\alpha}}}{[2]_q^{\sim}} \overset{?}{<} \frac{2}{[2]_q^{\sim}} \frac{1}{[2]_q^{\sim}}\ q^{2\alpha}\ \frac{2}{[2]_q^{\sim}} \frac{1}{[2]_q^{\sim}} \sqrt{([2]_q^{\sim})^2 a+q^{2\alpha}}$$

or

$$Qz_1 \overset{?}{<} z_2, \tag{4}$$

where

$$z_1 = \frac{q^{\alpha}+\sqrt{([2]_q^{\sim})^2 a+q^{2\alpha}}}{[2]},$$
$$z_2 = \frac{q^{\alpha}+\sqrt{([2]_q^{\sim})^2 a+q^{2\alpha}} q^{-\alpha}}{[2]_q^{\sim}}. \tag{5}$$

Thus, it's enough to have

$$\alpha = 0, \tag{6}$$
$$Q \le 1, \tag{7}$$

to have (4) satisfied.

For (7), we have:

$$2q^{\alpha} \overset{?}{\le} q+q^{-1},$$

so that $\alpha = 0$ is ok. Thus,

$$\sqrt{a+\frac{2}{[2]_q^{\sim}}\sqrt{a+\dots}}+\frac{2}{[2]_q^{\sim}}\sqrt{a}\ \ <1+\sqrt{([2]_q^{\sim})^2 a+1}, \quad a \ge 0. \tag{8}$$

A Quantum Quadric Which Vanishes Classically

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Abstract

Classically, $(ax) - ax = 0$. We consider the very different quantum case.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x, q \in \mathbf{R},\ q \neq 1, q > 0,$$

is the 2^{nd} quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

We are interested in the function

$$f(x,a) = f(a,x) = [ax]_q^{\sim} - [a]_q^{\sim}[x]_q^{\sim}. \tag{1}$$

We have:

$$\begin{aligned} &F(x,a) = (q - q^{-1})^2 f(x,a) = (q - q^{-1})(q^{ax} - q^{-ax}) - (q^a - q^{-a})(q^x - q^{-x}) = \\ &= (q^{ax+1} + q^{-1-ax}) - (q^{ax-1} + q^{1-ax}) - \\ &-(q^{x+a} + q^{-x-a}) + (q^{x-a} + q^{a-x}). \end{aligned} \tag{2}$$

Set

$$h = logq > 0.$$

Then (2) is:

$$\sum_{n=0}^{\infty} \frac{h^{2n}}{(2n)!}\{(ax+1)^{2n} - (ax-1)^{2n} - (x+a)^{2n} + (x-a)^{2n}\}. \tag{3}$$

Therefore, if we want

$$[xa]_q^\sim > [x]_q^\sim [a]_q^\sim,$$

for $x, a > 1$, (3) follows from:

$$(ax+1)^{2n} - (ax-1)^{2n} - (x+a)^{2n} + (x-a)^{2n} \stackrel{?}{>} 0, \quad x, a > 1. \tag{4}$$

For $n = 1$ (4) returns:

$$4ax - 4ax = 0,$$

so $n > 1$.

Now,

$$f(1, a) = f(x, 1) = 0. \tag{5}$$

We prove (4) by induction on n, assuming only $a > 1$. Suppose (4) is true for $n < N$. Then

$$\frac{1}{2N}\frac{\partial}{\partial x}\{LHS_N\} = a(ax+1)(ax+1)^{2(N-1)} - a(ax-1)(ax-1)^{2(N-1)}$$
$$-(x+a)(x+a)^{2(N-1)} + (x-a)(x-a)^{2(N-1)} > 0,$$

because

$$a(ax+1) > a(ax-1), \tag{6a}$$
$$a(ax+1) > x+a, \tag{6b}$$
$$a(ax+1) > x-a, \tag{6c}$$

Thus, $\mathrm{LHS_N}$ of (3) grows with x for $a > 1$. Hence,

$$[ax]_q^\sim > [a]_q^\sim [x]_q^\sim, \quad a, x > 1, \tag{7a}$$
$$[ax]_q^\sim < [a]_q^\sim [x]_q^\sim, \quad a > 1, x < 1, \tag{7b}$$

and by symmetry between a and x,

$$[ax]_q^\sim < [a]_q^\sim [x]_q, \quad a < 1, x > 1. \tag{7c}$$

It remains to consider the case $0 < a, x < 1$. From (4) at $a, x \to 0$, we expect

$$[ax]_q^\sim \stackrel{?}{>} [a]_q^\sim [x]_q^\sim, \quad 0 < a, x < 1. \tag{8}$$

We prove (8) as follows. Using (4), we have:

$$2\sum_{k\ odd}^{2n} \binom{2n}{k} a^k x^k \stackrel{?}{>} 2\sum_{k\ odd}^{2n} \binom{2n}{k} x^k a^{2n-k}, \tag{9}$$

and (9) follows from the pairing

$$a^k x^k + a^{2n-k} x^{2n-k} \stackrel{?}{>} x^k a^{2n-k} + x^{2n-k} a^k,$$

or

$$a^k x^k + \frac{a^{2n} x^{2n}}{a^k x^k} \stackrel{?}{>} x^k \frac{a^{2n}}{a^k} + a^k \frac{x^{2n}}{x^k},$$

or

$$a^{2k} x^{2k} + a^{2n} x^{2n} \stackrel{?}{>} x^{2k} a^{2n} + a^{2k} x^{2n},$$

or

$$(a^{2k} - a^{2n})(x^{2k} - x^{2n}) \stackrel{?}{>} 0,$$

which is true, because $a, x < 1$, and $k \leq 2n$. ■

A Quasiclassical Rational Binomial Sum

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Abstract

We find a semiclassical version of the classical identity $\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} = \frac{2^{n+1}-1}{n+1}$.

Set

$$S_n = \sum_{k=0}^{n} q^k \binom{n}{k} = (1+q)^k = ([2]_q)^k,$$

where

$$[x]_q = \frac{q^x - 1}{q-1}, \quad x \in \mathbf{R},\ q \neq 1,$$

is the 1^{st} quantization of x.

We are interested in the rational sum

$$M_n = \sum_{k=0}^{n} \frac{q^k}{k+1}\binom{n}{k}. \tag{1}$$

We rationalize M_n with the help of the identity

$$(n+1)\frac{1}{k+1}\binom{n}{k} = \binom{n+1}{k+1}. \tag{2}$$

Thus,

$$(n+1)M_n = \sum_{k=0}^{n} q^k \binom{n+1}{k+1} = q^{-1}\Big\{ \sum_{k=0}^{n} q^{k+1}\binom{n+1}{k+1}\Big\} =$$
$$= q^{-1}\Big\{ ([2]_q^n)^{n+1} - \binom{n+1}{0}\Big\} = \Big([2]^{n+1} - 1\Big)q^{-1} \ \Rightarrow$$
$$M_n = q^{-1}\frac{([2]_q)^{n+1} - 1}{n+1}, \tag{3}$$

a $q-$generalization of the formula quoted in the abstract.

A Remark Of Smarandache 3^{rd} Conjecture On Prime Numbers

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Abstract

The 3^{rd} Smaranduche Conjecture is trivially true.

Let p_n be the n^{th} prime number, and $k > 2$ an integer.
The Smarandache 3^{rd} conjecture states:

$$P_{n+1}^{1/k} - P_n^{1/k} < \frac{2}{k}. \tag{1}$$

Let

$$1 < a_0 < a_1 < a_2 < ... \tag{2}$$

be a sequence of real positive numbers. We show that

$$a_{n+1}^{1/k} - a_n^{1/k} < \frac{2}{k} \tag{3}$$

is true if k is $>> 0$, i.e. k is large enough, if a_n does not grow too fast.
Denote

$$\alpha = 1/k.$$

Then (3) is:

$$a_{n+1}^{\alpha} - a_n^{\alpha} \stackrel{?}{<} 2\alpha,\ \alpha > 0 \tag{4}$$

But

$$a_n^{\alpha} = e^{\alpha log a_n} = 1 + \alpha log a_n + 0(\alpha^2),$$

so (4) becomes:

$$\alpha log \frac{a_{n+1}}{a_n} \stackrel{?}{<} 2\alpha,$$

which is not true if $a_{n+1} > e^2 a_n$, but it is true for prime numbers, which by Bertand postulate satisfy

$$p_{n+1}/p_n < 2.$$

A Remark On A Prime Inequality

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Abstract

We show that, p_n being the n^{th} prime, $\frac{logp_{n+1}}{logp_n} > \frac{log(n+1)}{logn}$ for $n >> 0$.

We use the asymptotic expansion

$$p_n = nf(n),$$
$$f(n) = logn + \sum_{i=0}^{\infty} \frac{P_i}{log^i n}, \quad P_0 = loglogn - 1,$$

where P_i for $i > 0$ is a polynomial in $w = loglogn$ of degree $= i$.

I proved recently that

$$f(m+1) = f(m) + \frac{1}{m}[1 + 0(1)].$$

The RHS of

$$\frac{logp_{m+1}}{logp)n} > \frac{log(m+1)}{logm}, \tag{1}$$

is, modulo $1/m^2$- terms:

$$\frac{log(m+1)}{logm} = \frac{logm + \frac{1}{m}}{logm} = 1 + \frac{1}{mlogn}. \tag{2}$$

For the LHS of (1), we get:

$$logp_m = logm + logf(m) = logm + loglogm + 0(1),$$

so the LHS of (1) is

$$\frac{log(m+1) + loglog(m+1)}{logm + loglogm} + 0(1) =$$
$$= \frac{logm + loglogm + \frac{1}{m}(1 + \frac{1}{logm})}{logm + loglogm} =$$
$$= 1 + (1 + \frac{1}{m}\frac{1}{logm + loglogm},$$

and obviously,

$$\frac{m+1}{m}\frac{1}{logm + loglogm} > \frac{1}{mlogm},$$

because

$$(m+1)logm > (logm + loglogm).$$

A Simple Inequality In The First Quantization

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Abstract

If $x > y > 0$ then 1 be $< 1/y$. We quantize this.

Let $x > y > 0, q > 1$.

Theorem 1.

$$\frac{[x-1]_1}{[x]_q} > \frac{[y-1]_q}{[y]_q}, \tag{2}$$

where

$$[x]_q = \frac{q^x - 1}{q-1}, x \in \mathbf{R},\ q \neq 0, 1,$$

is the 1^{st} quantization of x.

Proof. Since

$$[x]_q = 1 + q[x-1]_q,$$
$$[x-1]_q = q([x]_q - 1),$$

so (2) is:

$$\frac{q^{-1}[x]^q - 1}{[x]_q} \overset{?}{>} \frac{q^{-1}([y]_q - 1)}{[y]_q},$$

or

$$-\frac{1}{[x]_q} \overset{?}{>} -\frac{1}{[y]_q},$$

or

$$\frac{1}{[y]_q} \overset{?}{>} \frac{1}{x}_q$$

or

$$[y]_q \overset{?}{<} [x]_q,$$

which is obvious because $x > y > 0$. ■

A Simple Quantum Analog Of A Linear Recurrent Sequence Of Order Two

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Abstract

A linear recurrent sequence of order two is quantized.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

is the 2^{nd} quantization of x.

Let

$$x_{n+2} = ax_{n+1} + bx_n,$$
$$x_0 = u,\ x_1 = v,$$

be a second order linear recurrent sequence.

Theorem 1. Let

$$X_{n+2} = q^{-bx_n}[a]_{q^{x_{n+1}}}^{\sim} X_{n+1} + q^{a_{x_{n+1}}}[b]_{q^{x_n}} X_n, \tag{2a}$$
$$X_0 = U = [u]_q^{\sim},\ X_1 = V = [v]_q^{\sim}. \tag{2b}$$

Then

$$X_n = [x_n]_q^{\sim},\ \forall n \in \mathbf{Z}_{\geq 0}. \tag{3}$$

Proof. We use the formula

$$[c+d]_q^{\sim} = q^{-d}[c]_q^{\sim} + q^c[d]_q^{\sim}. \tag{4}$$

Therefore,

$$X_{n+2} = [X_{n+2}]_q^{\sim} = [axn+1][bx_n]_q^{\sim} =$$
$$= q^{-bx_n}[ax_{n+1}]_q^{\sim} + q^{ax_{n+1}}[bx_n]_q^{\sim} =$$
$$= q^{-bx_n}[a]_{q^x}^{\sim}[x]_{n+1}]_q^{\sim} + q^{ax_{n+1}}[b]_{q^{x_n}}^{\sim}[x_n]_q^{\sim}.$$

This is (2a). ■

Remark 5. Our quantization is by means unique, just the most simplistic.

A Space Generated By Two Distinct Primes

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Abstract

Two distinct primes $p < p'$ generate all but a finite number of elements from $\mathbf{Z}$.

Let

$$p < p'$$

be two distinct primes. We are interested in the space of linear combinations

$$ap + bp',$$

$a, b \in \mathbf{Z}_{\geq 0}$.

Theorem 1. Denote that space by S. Then S contains every integer $\geq N(p, p')$, where

$$\mathcal{N}(p, p') = (p-1)^2 p' \tag{2}$$

Proof. Let $r_0 \equiv p'(mod p), 1 \leq p' \leq p-1$. Let $\overline{k} \in \mathbf{Z}_p$ be such that $\overline{k} p_0 = 1$. Let $1 \leq k \leq p-1$ be the representative of k. Let $N \equiv r \pmod{p}$. Then $N - krp' \equiv 0 \pmod{p}$, so $N = sp + kp'$. ■

Since

$$kr \leq (p-1)^2,$$
$$v(p, p') = (p-1)^2 + p'$$

is sufficient, even though this bound is probably vastly over the true one except for $p = 2$, ■

Remark 3. The Theorem is well-known in greater generality when p and p' are any coprime numbers. The paint is that in the prime case the proof is very short.

A Strengthening Beyond Reason Of The Goldbach Conjecture

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Abstract

Let M be even, and $p_m < M < p_{m+1}$. We conjecture that at least one of the numbers $M - p_m,\ M - p_{m-1}$ is prime.

Let M be even, and

$$p_m < M < p_{m+1}.$$

The Goldbach Conjecture says that the number of solutions of the equation

$$M = p + p', \tag{1}$$

p and p' being prime, is ≥ 1. I recently showed that the Goldbach Conjecture vastly under states the matters, and that number of solutions is

$$\sim \frac{2M}{log^2 M}.$$

But what is the object whose number of solutions is 1, 2 or bounded?

Conjecture 2. Among the numbers $M - p_m$ and $M - p_{m-1}$ at least one is prime. (If $p_{m+1} = p_m + 2$, then $M > p_m + 1$).

Example/Supporting evidence. Denote $p = p_m,\ p' = p_{m-1}$. Then

$$
\begin{array}{ll}
4-(2=p)=2, & 40-(37=p)=3,\\
6-(3=p')=3, & 42-(37=p')=5,\\
8-(5=p)=3, & 44-(41=p')=3,\\
10-(7=p)=3, & 46-(43=p)=3,\\
12-(7=p')=5, & 48-(43=p')=5,\\
14-(11=p')=3, & 50-(47=p)=3,\\
16-(13=p)=3, & 52-(47=p)=5,\\
18-(13=p')=5, & 54-(47=p')=7,\\
20-(17=p')=3, & 56-(53=p)=3,\\
22-(19=p)=3, & 58-(53=p')=5,\\
24-(19=p')=5, & 60-(53=p')=7,\\
26-(23=p)=3, & 62-(59=p)=3,\\
28-(23=p)=5, & 64-(61=p)=3,\\
30-(23=p')=7, & 66-(61=p)=5,\\
32-(29=p')=3, & 68-(61=p')=7,\\
34-(29=p')-5, & 70-(67=p)=3,\\
36-(31=p)=5, & 72-(67=p')=5,\\
38-(31=p')=7, & 74-(71=p')=3.
\end{array}
$$

It's been verified numerically up to $M \leq 1.27.10^6$ by Joshua A. Kupershmidt.

A Two-Variable Quantum Inequality

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Abstract

If $xy = 1, x, y > 0$, then $x + y \geq 2$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x, q \in \mathbf{R},\ q \neq 1, q > 0,$$

is the 2^{nd} quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Theorem 1. If $x, y > 0$ and $xy = 1$ then

$$[x]_q^{\sim} + [y]_q^{\sim} \geq 2. \tag{2}$$

Proof. (2) is:

$$[x]_q^{\sim} + [\frac{1}{x}]_q^{\sim} \stackrel{?}{\geq} 2, \quad x \geq 1. \tag{3}$$

The restriction $x \geq 1$ comes from the symmetry $(x, y) \to (y, x)$ of the constraint $xy = 1$.

Now, (3) is, after multiplying it by $q - q^{-1} > 0$:

$$q^x - q^{-x} + q^{1/x} - q^{-1/x} \stackrel{?}{>} 2(q - q^{-1}). \tag{4}$$

It's easy to see that the LHS of (4) is increasing with x:

$$\frac{1}{h}\frac{\partial}{\partial x}(LHS) = q^x + q^{-x} - \frac{1}{x^2}(q^{1/x} + q^{-1/x}) \stackrel{?}{>} 0, \tag{5}$$

where

$$h = logq > 0. \tag{6}$$

(5) is obvious, because $x > 1$ and

$$q^x + q^{-x} > q^{1/x} + q^{-1/x},$$

as

$$q^t + q^{-t}$$

is an increasing function of $t > 0$. Thank you for ■.

A Very Short Proof Of The Basic Inequality Between Quantum Numbers And Their Classical Counterparts

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Abstract

We compare classical number to its quantization, with a very short proof.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

is the 2^{nd} quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Set

$$\begin{bmatrix} m \\ k \end{bmatrix}_q^{\sim} = \frac{[m]!_q^{\sim}}{[k]!_q^{\sim}[m-k]!_q^{\sim}},$$
$$[k]!_q^{\sim} = [1]_q^{\sim}...[k]_q^{\sim}, k \in \mathbf{Z}_{\geq 1};\ [0]!_q^{\sim} = 1.$$

We prove.

Theorem 1.

$$[x]_q^{\sim} > x, \quad x > 1, \tag{2a}$$
$$[x]_q^{\sim} < x, 0 < x < 1. \tag{2b}$$

Proof. (a) Multiplying through by $(q-q^{-1})>0$, we get:

$$q^x - q^{-x} \stackrel{?}{>} x(q-q^{-1}),$$

or, with

$$h = lgq > 0,$$
$$2\sum_{n=0}^{\infty} \frac{h^{2n+1}}{(2n+1)!} x^{2n+1} \stackrel{?}{>} x2\sum_{n=0}^{\infty} \frac{h^{2n+1}}{(2n+1)!},$$

which follows from:

$$x^{2n+1} \stackrel{?}{>} x,\ x>1,\ n>0, \quad \text{for } (2a),$$
$$x^{2n+1} < x, 0<x<1,\ n>0, \quad \text{for } (2b),$$

both of which are obvious. ■

A Weightless Sum Of Quantum Binomial Coefficients

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Abstract

We quantize the classical formula $\sum_{k=0}^{n}\binom{k}{r}=\binom{n+1}{p+1}$.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\quad x,\ q\in\mathbf{R},\ q\neq 1,\ q>0,$$

be the 2^{nd} quantization of x.

Set

$$\begin{bmatrix}m\\k\end{bmatrix}_q^{\sim}=\frac{[m]!_q^{\sim}}{[k]!_q^{\sim}[m-k]!_q^{\sim}},$$
$$[k]!_q^{\sim}=[1]_q^{\sim}...[k]_q^{\sim},\ k\in\mathbf{Z}_{\geq\mathbf{1}};\ [o]!_q^{\sim}=1.$$

Theorem 1. For $n\in\mathbf{Z}_{\geq 0},\ p\in\mathbf{Z}_{\geq 0}$,

$$\sum_{k=0}^{n}q^{k(p+2)-p}\begin{bmatrix}k\\p\end{bmatrix}_q^{\sim}=q^{(p+1)n}\begin{bmatrix}n+1\\p+1\end{bmatrix}_q^{\sim}. \tag{2}$$

Proof. We use induction on n. For $n=0$, (2) returns:

$$q^{-p}\delta_p^0=\delta_p^0, \tag{3}$$

which is true.

The inductive step amounts to:

$$q^{(p+1)n}\begin{bmatrix}n+1\\p+1\end{bmatrix}_q^{\sim}+q^{(n+1)(p+2)-p}\begin{bmatrix}n+1\\p\end{bmatrix}_q^{\sim}\stackrel{?}{=}q^{(p+1)(n+1)}\begin{bmatrix}n+2\\p+1\end{bmatrix}_q^{\sim},$$

or to

$$\begin{bmatrix}n+2\\p+1\end{bmatrix}_q^{\sim}\stackrel{?}{=}q^{-p-1}\begin{bmatrix}n+1\\p+1\end{bmatrix}_q^{\sim}+q^{n+1-p}\begin{bmatrix}n+1\\p\end{bmatrix}_q^{\sim}, \tag{4}$$

which is true, because, in general,

$$\begin{bmatrix}N+1\\s\end{bmatrix}_q^{\sim}=q^{-\epsilon s}\begin{bmatrix}N\\s\end{bmatrix}_q^{\sim}+q^{\epsilon(N+1-s)}\begin{bmatrix}N\\s-1\end{bmatrix}_q^{\sim},\ \epsilon=\pm 1. \tag{5}$$

■

Derived Terms Of A Quantum Arithmetic Progression

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Abstract

If $\{ak\}$ is an arithmetic progression and $n \in \mathbf{Z}_{\geq 1}$ is fixed, then $S_k = \sum_{i=1}^{nk} a_i/nk$ also forms an arithmetic progression. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R}, \ q \neq 0, \pm 1,$$

be the second quantization of x. Let $\{ak\}$ be arithmetic progression. Fix $n \in \mathbf{Z}_{\geq 1}$, and set

$$S_k = \sum_{i=1}^{nk} a_i/nk. \tag{1}$$

Then $\{S_k\}$ is also an arithmetic progression, because

$$\frac{S_k}{kn} + \frac{S_{k+2}}{(k+2)n} - \frac{2S_{k+1}}{(k+1)n} = [a + (k-1)\frac{dn}{2}] + [a + (3k-1)\frac{dn}{2}] -$$
$$-2[a + (2k-1)\frac{dn}{2}] = \frac{d}{2}\{(4k-1) - 2(2k-1)\} = 0 \tag{2}$$

Interestingly enough, this fact can be quantized. Set

$$A_k = [a_k]_q^{\sim}, \tag{3}$$
$$\overline{S}_k = \sum_{i=1}^{kn} [a_i]_q^{\sim} / [kn]_{q^{d/2}}^{\sim}, \tag{4a}$$

where

$$d = a_1 - a_0 = (A_k - A_{k-1})|_{q=1}. \tag{4b}$$

Then the $\overline{S}_k$'s form a quantum arithmetic progression.

Theorem 5. For any $k \in \mathbf{Z}_{\geq 1}$,

$$\overline{S_k} + \overline{S}_{k+2} = [2]_q^{\sim} \mathcal{D} \overline{S}_{k+1}, \tag{6}$$

where

$$\mathcal{D} = \frac{nd}{2} = (\overline{S}_{k+1} - \overline{S}_k)_{q=1}. \tag{7}$$

Proof. First we establish that $\mathcal{D}$ exists. We have:

$$(S_{k+1} - \overline{S}_k)|_{q=1} - \frac{\sum_{i=1}^{kn} a_i}{kn} + \frac{\sum_{j=1}^{(k+1)n} a_j}{(k+1)n} =$$
$$= \frac{kn(a_1 + \frac{k^n}{2}d)}{kn} + \frac{(k+1)n\{a_1 + \frac{[(k+1)n-1]}{2}d\}}{(k+1)n} = \frac{d}{2}\{(kn-1) + [(k+1)n-1]\} = \frac{n}{2}d,$$

which is (7).

Next,

$$\sum_{i=1}^{N}[a_i]_q^{\sim} = [N]_{q^{d/2}}^{\sim}\left[\frac{a_1 + a_N}{2}\right]_q^{\sim} =$$
$$= [N]_{q^{d/2}}^{\sim}[a_q + \frac{(N-1)d}{2}\Big]_q^{\sim},$$

so

$$\overline{S}_n = \left[a_1 + \frac{(nk-1)d}{2}\right]_q^{\sim}. \tag{8}$$

Thus, (6) is:

$$\left[a_1 + \frac{nk-1}{2}d\right]_q^{\sim} + \left[a_1 + \frac{n(k+2)-1}{2}d\right] \stackrel{?}{=} [2]_{q^{dn/2}}^{\sim}\left[a_1 + \frac{(k+1)n-1}{2}d\right]_q^{\sim},$$

or, with $a + \frac{???k-1}{2}d = a$:

$$[a]_q^{\sim} + [a+nd]_q^{\sim} \stackrel{?}{=} [2]_{q^{nd/2}}^{\sim}\left[a + n\frac{d}{2}\right]_q^{\sim},$$

which is obvious because, in general,

$$[a]_q^{\sim} + [b]_q^{\sim} =]2]_{q^{(a-b)/2}}^{\sim}\left[\frac{a+b}{2}\right]_q^{\sim}. \quad \blacksquare$$

Divisibility By 7 In The Quantum Domain

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Abstract

$3^{2n+1} + 2^{2n+2} \equiv (mod\ 7), n \geq \mathbf{Z}_{\geq 0}$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

be the second quantization of x.

In general, classical divisibility theorems do *not* carry into quantum domain, with rare exceptions.

Lemma 1.

$$[3]_q^{\sim} + [2]_q^{\sim}[2]_{q^5}^{\sim} = [7]_q^{\sim}. \tag{2}$$

Proof. Instead of $[3]_q^{\sim} = q^2 + 1 + q^{-2}$, let's rewrite

$$(2, 0, -2). \tag{3}$$

Then the LHS of (2) is:

$$(2,0,-2) + (1,-1)(5,-5) = (2,0,-2) + (6,-4) + (4,-6) =$$
$$= (6,4,2,0,-2,-4,-6) = [7]_q^{\sim}. \quad \blacksquare$$

Set

$$A_n = \alpha_n + \beta_n, \quad n \in \mathbf{Z}_{\geq 0}, \tag{4a}$$
$$\alpha_n = ([3]_q^{\sim}[3]_{q^2}^{\sim})^n [3]_q^{\sim}, \tag{4b}$$
$$\beta_n = [2]_q^{\sim}[2]_{q^s}^{\sim}([2]_{q^2}^{\sim})^n. \tag{4c}$$

Lemma 2.

$$[3]_q^{\sim}[3]_{q^2}^{\sim} = [7]_q^{\sim} + [2]_{q^2}^{\sim}.$$

Proof. We have:

$$(2,0.-2)(4,0,-4) = (6,2,-2) + (4,0,-4) + (2,-2,-6) =$$
$$= (6,4,2,0,-2,-4,-6) + (2,-2) = [7]_q^{\sim} + [2]_{q^2}^{\sim}. \quad \blacksquare$$

Theorem 5. For $n \in \mathbf{Z}_{\geq 0}$

$$A_n \equiv 0 (mod\ [7]_q^{\sim}). \tag{6}$$

Proof. We use induction on n. For $n = 0$, (6) returns (2). Next,

$$A_{n+1}[3]_q^{\sim}[3]_{q^2}^{\sim} + [2]_{q^2}^{\sim}\beta_n$$
$$\equiv [2]_{q^2}^{\sim}\alpha_n + [2]_{q^2}^{\sim}\beta_n = [2]_{q^2}^{\sim}(\alpha_n + \beta_n) =$$
$$= [2]_{q^2}^{\sim}A_n.$$

Thus,

$$A_{n+1} \equiv [2]_{q^2}^{\sim}A_n (mod\ [7]_q^{\sim}),$$

(6) is proven, because $A_n \equiv 0 (mod [7]_q^{\sim})$, and the inductive step is complete. $\blacksquare$

Divisibility By 17 In Quantum Domain

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Abstract

Classically, $3.5^{2n+1}+2^{3n+1}\equiv 0(mod\ 17), n\in \mathbf{Z}_{\geq 0}$. We quantize this.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\quad x\in\mathbf{R},\ q\neq 0,\pm 1,$$

be the second quantization of x.

Lemma 1.

$$[3]_{q^6}^{\sim}[5]_q^{\sim}+[2]_{q^6}^{\sim}=[17]_q^{\sim}. \tag{2}$$

Proof. This is a particular case $n=2$ of

Lemma 3. For $n\in\mathbf{Z}_{\geq 0}$,

$$[3]_{q^{2n+2}}^{\sim}[2n+1]_q^{\sim}+[2]_{q^{2n+2}}^{\sim}=[6n+5]_q^{\sim}. \tag{4}$$

Proof. Multiplying through by $q-q^{-1}$, (4) becomes:

$$(q^{4n+4}+1+q^{-4n-4})(q^{2n+1}-q^{-2n-1})+(q-q^{-1})(q^{2n+2}+q^{-2n-2})\stackrel{?}{=}q^{6n+5}-q^{6n-5},$$

or

$$(q^{6n+5}+q^{2n+1}+q^{-2n-3})-(q^{2n+3}+q^{-2n-1}+q^{-6n-5})+q^{2n+3}-q^{-2n-3}-q^{-2n+1}+q^{-2n-1}\stackrel{?}{=}$$
$$\stackrel{?}{=}q^{6n+5}-q^{-6n-5},$$

or

$$q^{2n+1}-q^{-2n-1}+q^{-(2n+3)}-q^{(2n+3)}+q^{2n+3}-q^{-2n-3}-q^{2n+1}+q^{-2n-1}\stackrel{?}{=}0,$$

or

$$\stackrel{?}{=}0,$$

which is true. ■

Theorem 5. For $n \in \mathbf{Z}_{\geq 0}$

$$[5]_q^{\sim}[3]_{q^6}^{\sim}([5]_q^{\sim}[5]_{q^3}^{\sim}) + [2]_q^{\sim}([11]_q^{\sim} - 1 - [2]_{q^6}^{\sim})^{2n} \equiv 0 (mod \ \ [17]_q^{\sim}). \tag{6}$$

Proof. We use induction on n. For $n = 0$, (6) follows from (2). For general n, we have, denoting the LHS of (6) by A_n; and

$[11]_q - 1 - [2]_{q^6}^{\sim}$ by $< 8 >$:

$$[3]_{q^6}^{\sim}[5]_q^{\sim}(([5]_q^{\sim})[5]_q^{\sim})^n + [2]_{q^6}^{\sim}(< 8 >)^n \stackrel{?}{\equiv} \ (mod\ [17]_q^{\sim}),$$

or, using (2):

$$-(([5]_q^{\sim})[5]_{q^2}^{\sim})^n + < 8 >^n \stackrel{?}{\equiv} 0 \ (mod \ \ [17]_q^{\sim}),$$

which follows from

$$([5]_q^{\sim})[5]_{q^3}^{\sim} - < 8 > \stackrel{?}{\equiv} 0 \ (mod\ [17]_q^{\sim}) \tag{7}$$

which is true, because

$$[5]_{q^3}^{\sim}([5]_q^{\sim})[5]_{q^3}^{\sim} - < 8 > = [17]_q^{\sim}. \tag{8}$$

Indeed, in short-hand notation,

$$\begin{aligned}
&[5]_q^{\sim}[5]_{q^3}^{\sim} = (1,-2,0,-2,-4)(12,6,0,-6,-12) = \\
&= (16,10,4,-2,-8) + (14,8,2,-4,-10) + (12,6,0,-6,-12) + \\
&+ (10,4,-2,18,-14) + (8,2,-4,-10,-16) = \\
&= (16,4,12,10,8,6,4,2,0,-2,-4,-6,-8,-10,-12,-14,-16) + \\
&+ (10,8,4,2,-2,-4,-8,-10) = [17]_q^{\sim} + [11]_q^{\sim} - 1 - [2]_{q^6}^{\sim}. \quad \blacksquare
\end{aligned}$$

Divisibility By 64 In The Quantum Domain

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Abstract

$3^{2n+2} - 8n - 9 \equiv 0 (mod\, 64), n \in \mathbf{Z}_{\geq 0}$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

is the 2^{nd} quantization of x.

Theorem 1. For $\in \mathbf{Z}_{\geq 0}$,

$$([3]_q^{\sim})^{2n+2} - [2]_q^{\sim}[4]_q^{\sim} n = ([3]_q^{\sim})^2 \equiv 0 (mod\ ([2]_q^{\sim}[4]_q^{\sim})^2). \tag{2}$$

Proof. We use induction on n. For $n = 0$, (2) is:

$$([3]_q^{\sim})^2 - ([3]_q^{\sim})^2 = 0.$$

Next,

$$A_{n+1} - A_n([3]_q^{\sim})^{2n+2}\{([3]_q^{\sim})^2 - 1\} - [2]_q^{\sim}[4]_q^{\sim} =$$
$$= [2]_q^{\sim}[4]_q^{\sim}\{([3]_q^{\sim})^{2n+2} - 1\}, \tag{3}$$

because

$$([3]_q^{\sim})^2 - 1 = [2]_q^{\sim}[4]_q^{\sim}. \tag{4}$$

Indeed, writing (2, 0, -2) instead of $q^2 + 1 + 2^{-2}$ for $[3]_q^{\sim}$, we have:

$$([3]_q^{\sim})^2 - 1 = (2, 0 - 2)(2, 0, -2) - (0) =$$
$$= (4, 2, 0) + (2, 0, -2) + (0, -2, -4) - (0) =$$
$$= (4, 2, 0, -2), -4) + (2, 0, -2) \stackrel{?}{=} [2]_q^{\sim}[4]_q^{\sim} =$$
$$= (1, -1)(3, 1, -1, -3) = (4, 2, 0, -2) + (2, 0, -2, -4) =$$
$$= (4, 2, 0, -2, -4) + (2, 0, -2).$$

Thus, (3) is:

$$[2]_q^{\sim}[4]_q^{\sim}\{([3]_q^{\sim})^{2n+2}\} \stackrel{?}{\equiv} 0 (mod\ ([2]_q^{\sim}[4]_q^{\sim})^2) \ \Leftrightarrow$$
$$([3]_q^{\sim})^{2n+2} - 1 \stackrel{?}{\equiv} 0 (mod\ [2]_q^{\sim}[4]_q^{\sim}),$$

which is obvious because

$$([3]_q^{\sim})^{2n+2} - 1 = (([3]_q^{\sim})^2)^{n+1} - 1 \text{ [by (4)]} \equiv$$
$$\equiv 1^{n+1} - 1 = 0 (mod\ [2]_q^{\sim}[4]_q^{\sim}). \quad \blacksquare$$

Divisibility By 9 In The Quantum Domain

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Abstract

$2^{2n}+15n-1\equiv(mod\,9)$ for $n\geq \mathbf{Z}_{\geq 0}$. We quantize this.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\quad x\in\mathbf{R},\ q\neq 0,\pm 1,$$

be the second quantization of x.

Theorem 1. For $n\in\mathbf{Z}_{\geq 0}$,

$$([2]_q^{\sim})^{2n}-[3]_q^{\sim}n-1\equiv 0(mod=([3]_q^{\sim})^2). \tag{2}$$

Proof. Denote the LHS of (2) by A_n. Using induction on n, we see at once that $A_0=A_1=0$.

Next,

$$A_{n+1}-A_n=([2]_q^{\sim})^{2n}\{([2]_q^{\sim})^2-1\}-[3]=\{([2]_q^{\sim})^{2n}-1\}[3]_q^{\sim}\stackrel{?}{\equiv}$$
$$\stackrel{?}{\equiv}0(mod\ [3]_q^{\sim}[3]_q^{\sim})\ \Leftrightarrow$$
$$([2]_q^{\sim})^{2n}-1\stackrel{?}{\equiv}0(mod\ [3]_q^{\sim}).$$

But

$$([2]_q^{\sim})^{2n}=(([2])^2)^n=([3]_q^{\sim}+1)^n\equiv 1^n\ (mod\ [3]_q^{\sim}),$$

and

$$1^n-1=1-1=0(mod\ \forall).\quad \blacksquare$$

Equidistant Sums In A Quantum Arithmetic Progression

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Abstract

If $\{a_k\}$ is an arithmetic progression, then $a_k + a_p = a_{k-\ell} + q_{p+\ell}\forall \ell \in \mathbf{Z}$. We quantize this.

Let

$$[x]_q^\sim = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

be the second quantization of x.

Classically, if $\{a_k\}$ is an arithmetic progression then

$$a_k + a_{k+s} = a_{k-\ell} + a_{k+s+\ell},\ \forall \ell \in \mathbf{Z}, \quad \forall k, s \in \mathbf{Z}. \tag{1}$$

Theorem 2.

$$[a_k]_q^\sim + q^{(2\ell+s)d}[a_{k+s}]_q^\sim = q^{\ell d}[a_{k-\ell}]_q^\sim + q^{(\ell+s)d}[a_{k+s+\ell}]_q^\sim. \tag{3}$$

Proof. Since, denoting

$$A_k = [a_k]_q^\sim, \tag{4}$$
$$A_k = [k]_{q^d}^\sim A_1 - [k-1]_{q^d}^\sim A_0, \tag{4}$$

where

$$d = a_1 - a_0 = (A_1 - A_0)|_{q=1}, \tag{5}$$

it's enough to show that

$$[k]_q^\sim + q^{2\ell+s}[k+1]_q^\sim \stackrel{?}{=} q^\ell[k-\ell]_q^\sim + q^{\ell+s}[k+s+\ell]_q^\sim, \tag{6}$$

or

$$q^k - q^{-k} + q^{2\ell+s}(q^{k+s} - q^{-k-s}) \stackrel{?}{=}$$
$$\stackrel{?}{=} q^\ell(q^{k-\ell} - q^{\ell-k}) + q^{s+\ell}(q^{k+s+\ell} - q^{-k-s-\ell}),$$

or

$$q^k - q^{-k} + q^{k+2\ell+2s} - q^{-k+2\ell} \stackrel{?}{=}$$
$$= q^k - q^{2\ell-k} + q^{k+2s+2\ell} - q^{-k},$$

which is obvious. ■

Inequality Lemmas For Quantum Functions

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Abstract

For various quantum expressions, we establish a relative order.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x, q \in \mathbf{R},\ q \neq 1, q > 0,$$

be the second quantization of x so that

$$[x]_{q-1}^{\sim} = [x]_q^{\sim},$$

and we can take

$$q > 1$$

without loss of generality.

Lemma 1. Let $q > Q > 0$. Then

$$q - q^{-1} > Q - Q^{-1} \tag{2}$$

Proof. We have:

$$q - q^{-1} - (Q - Q^{-1}) = (q - Q) - (q^{-1} - Q^{-1}) =$$
$$= (q - Q) - \frac{Q - q}{Qq} = (q - Q)(1 + \frac{1}{qQ}) > 0. \quad \blacksquare$$

Lemma 3. The Function

$$[2]_{q^m} = q^x + q^{-x} \tag{4}$$

grows with $x > 0$.

Proof. Let $h = logq > 0$. Then

$$\frac{\partial}{\partial x}(q^x + q^{-x}) = h(q^x - q^{-x}) > 0,$$

because $x > 0 \Rightarrow q^x > 1$. $\blacksquare$

Lemma 5. For $q > Q > 0$, then

$$(q + q^{-1})(Q - Q^{-1}) < (q - q^{-1})(Q + Q^{-1}). \tag{6}$$

Proof. Opening the bracket in (6), we have:

$$(qQ - q^{-1}Q^{-1}) + q^{-1}Q - qQ^{-1} \stackrel{?}{<} (qQ - q^{-1}Q^{-1}) + qQ^{-1} - q^{-1}Q,$$

or

$$q^{-1}Q \stackrel{?}{<} qQ^{-1},$$

or

$$Q^2 < q^2,$$

which is true because $Q < q$. ■

On A Family Of Selberg-Like Sums

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Abstract

We estimate the sums exactly.

Selberg in 1943 [Sel 1943] considered the sum

$$S(x,2)=\sum_{p_n\leq x}\frac{(p_{n+1}-p_n)^2}{p_n},$$

where p_n is the n^{th} prime. Under the assumption of *RC* (Rienmann's Conjecture), he deduced that

$$S(x,2)=O(log^3x).$$

Theorem 1. Let $\alpha\neq 0$. Then

$$S(x,\alpha)=\sum_{p_n\leq x}\frac{(p_{n+1}-p_n)^\alpha}{p_n}\sim\frac{log^\alpha x}{\alpha} \tag{2}$$

Proof. I recently proved that

$$p_{n+1}-p_n-logp_{n+1}+0(1)=0.$$

Hence,

$$(x,\alpha)=\sum_{p_n\leq x}\frac{(p_{n+1}-p_n)^\alpha}{p_n}\sim\sum_{p\leq\frac{x}{logx}}\frac{log^\alpha n}{nlogn}\sim\sum\frac{log^{\alpha-1}n}{n}\sim\frac{log^\alpha x}{\alpha},$$

as required. ■

The case $\alpha=0$ has drastically different asymptotics:

$$\sum_{p\leq x}\frac{1}{p}\sim loglogx.$$

Notice that

$$loglogx=\lim_{\alpha\to 0}\frac{log^a x-1}{\alpha}.$$

References

[Sel 1943] Selberg, A., *On The Normal Density of Primes In Short Intervals, and the Difference Between Consequtive Primes*, Arch. Math. Naturvid. **47** (1943) 87-105.

On A Finite Section Of An Alternating Sum Of Quantum Binomial Coefficients

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Abstract

We quantize the formula $\sum_{k=0}^{m}(-1)^k\binom{n}{k} = \binom{n-1}{m}(-1)^m, 0 \leq, < n.$

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

is the 2^{nd} quantization of x.

Set

$$\begin{bmatrix} m \\ k \end{bmatrix}_q^{\sim} = \frac{[m]!_q^{\sim}}{[k]!_q^{\sim}[m-k]!_q^{\sim}},$$
$$[k]!_q^{\sim} = [1]_q^{\sim}...[k]_q^{\sim}, k \in \mathbf{Z}_{\geq 1};\ [0]!_q^{\sim} = 1.$$

Formula in the abstract is easily proven by induction on n. For $m = n-1$, it returns

$$\sum_{k=0}^{n-1}(-1)^k\binom{n}{k} = (-1)^{n-1},$$

an hence

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = (-1)^{n-1} + (-1)^n 1 = 0,$$

so that the annoying restr???? $m < n$ can be dropped off.

Theorem 1. For $n \in \mathbf{Z}_{\geq 1},\ m \in \mathbf{Z}_{\geq 0}$,

$$\sum_{k=0}^{m}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} q^{(n-1)^k} = q^{nm} \begin{bmatrix} n-1 \\ m \end{bmatrix}_q^{\sim} (-1)^m. \tag{2}$$

Proof. We use induction on n. For $n = 1, m = 0$, (2) returns:

$$1 = 1,$$

which is true. For $n = m = 1$, (2) returns:

$$1 - 1 = 0, \tag{3}$$

which is also true. For $n = 1, m > 1$, (2) returns (3) again.

Now suppose (2) is true for $n \leq N$. Denoting the LHS and the RHS of (2) by $L(n,m)$ and $R(n,m)$, respectively, we have:

$$L(N+1,m) = \sum_{k=0}^{m} (-1)^k \begin{bmatrix} N+1 \\ k \end{bmatrix}_q^{\sim} q^{Nk}. \tag{4}$$

But

$$\begin{bmatrix} N \\ K \end{bmatrix}_q^{\sim} = \begin{bmatrix} n-1 \\ K \end{bmatrix}_q^{\sim} q^{\epsilon k} + \begin{bmatrix} N-1 \\ K-1 \end{bmatrix}_q^{\sim} q^{\epsilon(N-K)}, \ \epsilon = \pm 1. \tag{5}$$

Thus, for $\epsilon = -1$, (4) becomes:

$$\begin{aligned} L(N+1,m) &= \sum_{k=0}^{m} (-1)^k q^{Nk} \left\{ q^{-k} \begin{bmatrix} N \\ K \end{bmatrix}_q^{\sim} + q^{q^{1+N-k}} \begin{bmatrix} N \\ K-1 \end{bmatrix} \right\} = \\ &= L(N,m) + \sum_{k=0}^{m} (-1)^k q^{(N-1)k} q^{N+1} \begin{bmatrix} N \\ K-1 \end{bmatrix}_q^{\sim}, \end{aligned} \tag{6}$$

and, changing k into $s+1$, we get:

$$\begin{aligned} &\sum_{k=0}^{m} (-1)^k q^{(N-1)^k} q^{1+N} \begin{bmatrix} N-1 \\ K-1 \end{bmatrix}_q^{\sim} = \\ &= -\sum_{s=0}^{M-1} q^{(N-1)s+N+1+N-1} \begin{bmatrix} N-1 \\ s \end{bmatrix}_q^{\sim} = -q^{2N} \Big[(N-1, m-1) \Big]. \end{aligned} \tag{7}$$

Thus,

$$\begin{aligned} &L(N+1,m) = L(N,m) - q^{2N}[(N-1,m-1) \text{ (by induction)}] = \\ &= q^N \begin{bmatrix} N-1 \\ m \end{bmatrix}_q^{\sim} (-1)^m - q^{2N} (-1)^{m-1} q^{N(m-1)} \begin{bmatrix} N-1 \\ m-1 \end{bmatrix}_q^{\sim} = \\ &= (-1)^m q^{(N+1)m} \left\{ q^{-m} \begin{bmatrix} N-1 \\ m \end{bmatrix}_q^{\sim} + q^{-m+N} \begin{bmatrix} N-1 \\ m-1 \end{bmatrix} \right\} = \\ &= (-1)^m q^{(N+1)m} \begin{bmatrix} N \\ m \end{bmatrix}_q^{\sim}. \quad \blacksquare \end{aligned}$$

On A Weighted Sum Of Consecutive Quantum Integers

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Abstract

We quantize naturally the formula $\sum_{k=0}^{n} k = \binom{n+1}{2}$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x, q \in \mathbf{R},\ q \neq 1,\ q > 0,$$

be the 2^{nd} quantization of x.

Theorem 1. For $n \in \mathbf{Z}_{\geq 0}$,

$$\sum_{k=0}^{n} q^{3k-1}[k]_q^{\sim} = q^{2n}\frac{[n+1]_q^{\sim}[n]_q^{\sim}}{[2]_q^{\sim}}. \tag{2}$$

Proof. We use the induction on n. The case $n = 0$ returns:

$$0 = 0,$$

which is true.

The induction step $n \mapsto n+1$, amounts to:

$$q^{2n}\frac{[n+1]_q^{\sim}[n]_q^{\sim}}{[2]_q^{\sim}} + q^{3n+2}[n+1]_q^{\sim} \stackrel{?}{=}$$
$$= q^{2n+2}\frac{[n+2_q^{\sim}][n+1]_q^{\sim}}{[2]_q^{\sim}},$$

or

$$q^{2n}[n]_q^{\sim} + q^{3n+2}[2]_q^{\sim} \stackrel{?}{=} q^{2n+3}[n+2]_q^{\sim},$$

or

$$q^{-2}[n]_q^{\sim} + q^n[2]_q^{\sim} \stackrel{?}{=} [n+2]_q^{\sim},$$

which is true because, in general,

$$[a+b]_q^{\sim} = q^{-b}[a]_q^{\sim} + q^a[b]_q^{\sim}. \quad \blacksquare$$

On Log-Concavity Of Quantum Binomial Coefficients

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Abstract

Classically, $\binom{n}{k-1}\binom{n}{k+1} < \binom{n}{k}^2$. We quantize this.

Let

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbf{R},\ q \neq 0, \pm 1,$$

be the 2^{nd} quantization of x.

Set

$$\begin{bmatrix} m \\ k \end{bmatrix}_q^{\sim} = \frac{[m]!}{[k]!_q^{\sim}[m-k]!_q^{\sim}},\quad m \geq k \geq 0,$$
$$[k]!_q^{\sim} = [1]_q^{\sim}...[k]_q^{\sim},\ k \in \mathbf{Z}_{\geq 1};\ [0]!_q^{\sim} = 1.$$

Theorem 1. For $n \in \mathbf{Z}_1,\ \ 0 \leq k < n$,

$$\begin{bmatrix} n \\ k-1 \end{bmatrix}_q^{\sim} \begin{bmatrix} n \\ k+1 \end{bmatrix}_q^{\sim} < \left(\begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} \right)^2. \tag{2}$$

Proof. Write (2) in the long hand, we have:

$$\frac{1}{[k-1]!_q^{\sim}[n-k+1]!} \frac{1}{[k+1]![n-k-1]!} \overset{?}{<} \frac{1}{([k]!_q^{\sim})^2([n-k]!)^2}. \tag{3}$$

Multiplying (3) by

$$[k]!_q^{\sim}[k+1]!_q^{\sim}[n-k]!_q^{\sim}[n-k+1]_q^{\sim},\ \text{we arrive at :}$$
$$[k]_q^{\sim}[n-k]_q^{\sim} \quad < [k+1]_q^{\sim}[n-k+1]_q^{\sim} \tag{4}$$

which is obvious, because, for $x > 0$,

$$[x+1]_q^{\sim} > [x]_q^{\sim}. \quad \blacksquare \tag{5}$$

On Inequality For Two Products Of Quantum Linear Forms Without A Classical Analog

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Abstract

Classically, $(ax)(\beta y) = (\alpha y)(\beta x)$. It's not so quantumly.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

is the 2^{nd} quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Theorem 1. Let $\alpha > \beta > 0,\ x > y > 0$. Then

$$[\alpha x]_q^{\sim}[\beta y]_q^{\sim} > [\alpha y]_q^{\sim}[\beta x]_q^{\sim}. \tag{2}$$

Proof. Multiplying through by $(q - q^{-1})^2 > 0$, we get:

$$(q^{\alpha x} - q^{-\alpha x})(q^{\beta y} - q^{\beta y}) \overset{?}{>} (q^{\alpha y} - q^{-\alpha y})(q^{\beta x} - q^{-\beta x}),$$

or

$$\begin{aligned}&(q^{\alpha x+\beta y} + q^{-\alpha x-\beta y}) - (q^{\alpha x-\beta y} + q^{\beta y-\alpha x}) \overset{?}{>} \\ &\overset{?}{>} (q^{\alpha y+\beta x} + q^{-\alpha y-\beta x}) - (q^{\alpha y-\beta x} + q^{\beta x-\alpha y}).\end{aligned} \tag{3}$$

Lemma 4. The function

$$f(z) = q^z + q^{-z}$$

grows with z for $z > 0, q > 1$. **Proof.**

$$\frac{df}{d\tau} = h(q^2 - q^{-z}),\ h = logq > 0.$$

Since $z > 0, q^z > q^{-z}$, and $\frac{df}{dz} > 0$. ■

Now,

$$\alpha x + \beta y > \alpha y + \beta x \Leftrightarrow$$
$$\alpha(x - y) > \beta(x - y) \leq \alpha > \beta \text{ and } x > y.$$

Hence (3) follows from

$$q^{\alpha y - \beta x} + q^{\beta x - \alpha y} \overset{?}{>} q^{\alpha x - \beta y} + q^{\beta y - \alpha x},$$

which follows, by Lemma 4, from

$$\alpha y - \beta x > \beta y - \alpha x = 0$$
$$\alpha(x + y) > \beta(x + y) \Leftarrow \alpha > \beta. \quad ■$$

On Log-Concavity Of Quantum Integers

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Abstract

The usual integers are log-concave, because $(n-1)(n+1) < n^2$. We find a quantum version of this.

Let

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbf{R},\ q \neq 0, \pm 1,$$

be the 2^{nd} quantization of x.

Theorem 1. For $x \in \mathbf{Z}_{\geq 1}$,

$$[x-1]_q^{\sim}[x+1]_q^{\sim} < ([x]_q^{\sim})^2. \tag{2}$$

Proof. Multiplying (2) by $(q - q^{-1})^2 > 0$ and denoting

$$X = q^k$$

we rewrite (2) as:

$$(Xq^{-1} - X^{-1}q)(Xq - X^{-1}q^{-1}) \stackrel{?}{<} (X - X^{-1})^2,$$

or

$$(Xq^{-1} - X^{-2}) - (q^2 + q^{-2}) \stackrel{?}{<} X^2 + X^{-2} - 2,$$

or

$$2 < q^2 + q^{-2},$$

which is obvious. ■

On Lowering Powers In Quantum Domain

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Abstract

The characteristic property of lowering powers, $x^{\underline{m}} = x(x-1)..(x-m+1), m \in \mathbf{Z}_{\geq 1}$, is this: $\sum_{k=0}^{n}[k]^{\underline{m}} = \frac{(n+1)^{\underline{m+1}}}{m+1}$. We quantize the set-up.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x, q \in \mathbf{R},\ q \neq 1,\ q > 0,$$

be the 2^{nd} quantization of x.

For $km \in \mathbf{Z}_{\geq 1}$, set

$$[x]^{\underline{m}}{}_q^{\sim} = [x]_q^{\sim}[x-1]_q^{\sim}...[x-m+1]_q^{\sim}. \tag{1}$$

Theorem 2.

$$\sum_{k=0}^{n} q^{(k-1)(m+2)+2}[k]^{\underline{m}}{}_q^{\sim} = \frac{[n+1]^{\underline{m+1}}{}_q^{\sim}}{[m+1]_q^{\sim}} q^{n(m+1)}. \tag{3}$$

Proof. We use induction on n. For $n = 0$, (3) returns:

$$0 = 0,$$

because

$$1^{\underline{m+1}} = 0$$

for $m \geq 1$.

The induction step $n \mapsto n+1$ amounts to:

$$\frac{q^{n(m+1)}}{[m+1]_q^{\sim}}[n+1]_q^{\sim}...[n-m+1]_q^{\sim} + q^{n(m+2)+2}[n+1]_q^{\sim}...[n-m+2]_q^{\sim} \stackrel{?}{=}$$
$$\stackrel{?}{=} \frac{q^{(n+1)(m+1)}}{[m+1]_q^{\sim}}[n+2]_q^{\sim}...[n-m+2]_q^{\sim},$$

or

$$q^{n(m+1)}[n-m+1]_q^{\sim} + q^{n(m+2)+2}[m+1]_q^{\sim} \stackrel{?}{=} q^{(n+1)(m+1)}[n+2]_q^{\sim},$$

or

$$q^{-(m+1)}[n-m+1]_q^{\sim} + q^{n+1-m}[m+1]_q^{\sim} \stackrel{?}{=} [n+2]_q^{\sim},$$

which is true because, in general,

$$[a+b]_q^{\sim} = q^{-b}[a]_q^{\sim} + q^a[b]_q^{\sim}. \quad \blacksquare$$

On Quantum Numbers Close To One

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Abstract

We derive the first term of an asymptotic series for the quantum version of $1+\epsilon$, with $\epsilon^2=0$.

Let

$$[x]_q = \frac{q^x - q^{-x}}{q-q^{-1}},\ x\in\mathbf{R},\ q\neq 0,\pm 1,$$

be the 2^{nd} quantization of x.

Theorem 1. Modulo ϵ^2-terms,

$$[1+\epsilon]_q^{\sim} = 1+\epsilon\frac{h[2]_q^{\sim}}{q-q^{-1}}. \tag{2}$$

Proof. With

$$q=e^h,\ h=logq, \tag{3}$$

we have:

$$[1+\epsilon]_q^{\sim} = \frac{q^{1+\epsilon}-q^{-1-\epsilon}}{q-q^{-1}} = \frac{qq^{\epsilon}-q^{-1}q^{-\epsilon}}{q-q^{-1}} =$$
$$= \frac{q}{q-q^{-1}}e^{h\epsilon} - \frac{q^{-1}}{q-q^{-1}}e^{-h\epsilon} = \frac{q}{q-q^{-1}}(1+h\epsilon) - \frac{q^{-1}}{q-q^{-1}}(1-h\epsilon) =$$
$$= (1+h\epsilon)\frac{q+q^{-1}}{q-q^{-1}} = 7\frac{1+h}{q-q^{-1}}\epsilon[2]_q^{\epsilon},$$

as required.

On Ternary Goldbach Problem

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Abstract

Every odd number M, large or small, is the sum of three primes, in $\frac{M^3}{log^3 M}$ ways.

Let M be an odd number. We are interested in the number of solutions, $n_3(M)$, of the equation.

$$M = p_1 + p_2 + p_3. \tag{1}$$

Let

$$p_m \leq M < p_{m+1}, \tag{2}$$

so that

$$M \sim mlogm, \tag{3}$$

$$m \sim \frac{M}{logM}. \tag{4}$$

for large M.

Important Remark 5. We argue with "large" numbers, but this imperfection is easy remedied with the help of Dusart Thesis, the Bible of the Prime Theory Estimates. This observation applies to all my work on various prime number conjectures. I don't do it myself since I don't have time, and wishing somebody else something to do.

Now,

$$n_3(M) = \sum_{p_1} n_2(M - p_1), \tag{5}$$

and I showed, for the binary Goldbach conjecture, that

$$n_2(M) \sim \frac{2logM}{log^2 M}. \tag{6}$$

Hence, (5) returns:

$$n_3(M) \sim \sum_{p_1} \frac{2(M-p_i)}{log^2(M_{p_i})} \sim \frac{2M}{log^2 M} \sum_{p_1} (1 - \frac{p_1}{M}) \sim$$

$$\sim \frac{2M}{log^2 M}(m - \frac{m^2 log^2 m}{2M}) = \frac{2M}{log^2 M}(m - \frac{m}{2}) \sim \frac{M^2}{log^3 M},$$

as claimed. ■

On The Density of Prime Numbers

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Abstract

The density around $x >> 0$, is $1/logx$.

It is often said that the density of primes around x is $1/logx$. Still, an accurate derivation of the fact is desirable. Such a derivation is offered below.

Let $c > 0$ be an interval around $x, c = 2a, a > 0$. We look at the number of primes between $x - a$ and $x + a$.

Claim. That number is, approximately,

$$\frac{c}{logx} \tag{1}$$

Indeed, that number is

$$\pi(x + a) - \pi(x - a), \tag{2}$$

and

$$\pi(x) \sim \frac{x}{logx}$$

(by PNT, for example).

So,

$$\pi(x + a) - \pi(x - a) \sim \frac{x + a}{log(x + a)} - \frac{x - a}{log(x - a)} \sim$$
$$\sim \frac{x + a}{logx + \frac{a}{x}} - \frac{x - a}{logx - \frac{a}{x}} = \frac{1}{log^2 x - \frac{a^2}{x}} \text{ times :}$$
$$(x + a)(logx - \frac{a}{x}) - (x - a)(logx + \frac{a}{x}) = 2a(logx - 1)).$$

Thus,

$$\pi(x + a) - \pi(x - a) \sim \frac{2a(logx - 1)}{log^2 x} \sim \frac{2a}{logx} = \frac{c}{logx},$$

as required.

On The Derivative Of A Quantum Number

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Abstract

Classically, $\frac{dx}{dx} = 1$. It is not so simple in the quantum case.

Let

$$[x]_q = \frac{q^x - 1}{q - 1},\ x \in \mathbf{R},\ q \neq 0, \pm 1,$$

be the 2^{nd}quantization of x.

Since

$$[x]^{\sim}_{q^{-1}} = [x]^{\sim}_q,$$

we can take

$$q > 1$$

without loss of generality.

Theorem 1. For $x > 0$, we have:

$$\frac{d[x]^{\sim}_q}{dx} > 1, \quad x > 1, \tag{2a}$$

$$\frac{d[x]^{\sim}_q}{dx} < 1, \quad 0 < x \leq 1/\sqrt{3}. \tag{2b}$$

Proof. We have:

$$\frac{d[x]^{\sim}_q}{dx} = \frac{d}{dx}\left(\frac{q^x - q^{-x}}{q - q^{-1}}\right) = \frac{h}{q - q^{-1}}(q^x + q^{-x}), \tag{3}$$

where

$$h = logq > 0. \tag{4}$$

Now, I have proved that

$$q + q^{-1} > \frac{q - q^{-1}}{h}. \tag{5}$$

Since

$$[2]^{\sim}_{q^x}$$

increases with x, (2a) follows from (3).

It remains to establish (2b), which by (3) is

$$q^x + q^{-x} < \frac{q - q^{-1}}{h},\ 0 < x \leq 1\sqrt{3}. \tag{6}$$

For the LHS of (6), we have:

$$q^x + q^{-x} = e^{hx} + e^{-hx} = 2\sum_{h=0}^{\infty} \frac{h^{2n}x^{2n}}{(2n)!}, \tag{7a}$$

and for the RHS of (6) we find:

$$\frac{q - q^{-1}}{h} = \frac{e^h - e^{-h}}{h} = \frac{2}{h}\sum_{n=0}^{\infty} \frac{h^{2n+1}}{(2n+1)!} = 2\sum_{n=0}^{\infty} \frac{h^{2n}}{(2n+1)!} \tag{7b}$$

Thus, we need:

$$\frac{x^{2n}}{(2n)!} \overset{?}{\leq} \frac{1}{(2n+1)!}. \tag{7c}$$

For $n = 0$ it's true. For n ¿ 0, we get

$$x^{2n} \overset{?}{<} \frac{1}{2n+1},$$

or

$$x \overset{?}{<} \sqrt[2n]{\frac{1}{2n+1}} = \frac{1}{\sqrt[2n]{2n+1}}. \tag{8}$$

Now, for the function

$$f(x) = \sqrt[x]{x+1},$$

$$logf(x) = \frac{log(x+1)}{x},$$

we have:

$$\frac{d[logf(x)]}{dx} = \frac{1}{x(x+1)} - \frac{log(x+1)}{x^2} = \frac{1}{x}\left\{\frac{1}{x+1} - \frac{log(x+1)}{x}\right\} < 0.$$

Thus, $f(x)$ decreases from $\sqrt{3}$ (at $x = 2$) to 1 at $x = 0$). Thus,

$$x \leq \frac{1}{\sqrt{3}}$$

does the job. ■

Remark 9. The region

$$1/\sqrt{3} < x < 1$$

remains in the dark.

On The Difference Between Quantum And Classical Number

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Abstract

We estimate the difference mentioned in the title.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

be the 2^{nd} quantization of x.

The difference

$$[x]_q^{\sim} - x$$

Theorem 1.

$$\lim_{q \to 1} \frac{[x]_q^{\sim} - x}{(q - q^{-1})^2} = \frac{x^3 - x}{x^3 - x} 24 \tag{24}$$

Proof. We use the fact that

$$[x]_q^{\sim} = x + \frac{h^2}{6}(x^3 - x) + O(h^4), \quad h = logq. \tag{3}$$

Since

$$q - q^{-1} = e^h - e^{-h} = 2h + O(h^3), \tag{4}$$

we have:

$$(q - q^{-1}) = 4h^2 + O(h^4),$$

and

$$\frac{[x]_q^{\sim} - x}{(q - q^{-1})^2} = \frac{\frac{h^2}{6}(x^3 - x)}{4h^2} = \frac{x^3 - x}{24},$$

which is (2). ■

On The First Quantum Convoluted Sum

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Abstract

$\sum_{k=0}^{n} k(n-k) = \binom{n+1}{3}$. The quantum situation is rather different.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x,\ q \in \mathbf{R},\ q \neq 1,\ q > 0,$$

be the 2^{nd} quantization of x.

Theorem 1. For $n \in \mathbf{Z}_{\geq 0}$,

$$\sum_{k=0}^{n} [k]_q^{\sim}[n-k]_q^{\sim} = \frac{1}{(q-q^{-1})^2}\left\{(n+1)[2]_{q^n}^{\sim} - 2[n+1]_q^{\sim}\right\}. \tag{2}$$

Proof. Multiplying (2) by $(q-q^{-1})^2$, we get:

$$(q-q^{-1})^2 \sum_{k=0}^{n} [k]_q^{\sim}[n-k]_q^{\sim} = \sum_{k=0}^{n} (q^k - q^{-k})(q^{n-k} - q^{-n+k}) =$$
$$= \sum_{k=0}^{n} \left\{(q^n + q^{-n}) - (q^{n-2k} + q^{-n+2k})\right\} =$$
$$= (n+1)[2]_{q^n}^{\sim} - \tag{3a}$$
$$-\left(q^n \sum_{k=0}^{n} q^{-2k} + q^{-n} \sum_{k=0}^{n} q^{2k}\right). \tag{3b}$$

Now,

$$q^n \sum_{k=0}^{n} q^{-2k} + q^{-n} \sum_{k=0}^{n} q^{2k} =$$
$$= q^n [n+1]_{q^{-2}} + q^{-n}[n+1]_{q^2} =$$
$$= q^n \frac{1 - q^{-(2n+2)}}{1 - q^{-2}} + q^{-n} \frac{q^{2(n+1)} - 1}{q^2 - 1} =$$
$$= q^n \frac{q^{-n-1}(q^{n+1} - q^{-n-1})}{q^{-1}(q - q^{-1})} + q^{-n} \frac{q^{n+1}(q^{n+1} - q^{-n-1})}{q(q - q^{-1})} =$$
$$= [n+1]_q^{\sim} 2,$$

and (2) follows. Here

$$[x]_q = \frac{q^x - 1}{q - 1}, x, \ q \in \mathbf{R}, \quad q \neq 1, \ q > 0,$$

is the 1^{st} quantization of x. ■

On The Growth of Quantum Numbers

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Abstract

We compare two equal fraction under a base change.

Let

$$[x]_q = \frac{q^x - 1}{q - 1},\ x \in \mathbf{R},\ q \neq 1,$$

Theorem 1. Let $n \in \mathbf{Z}_{\geq 1},\ q > Q > 0$. Then

$$\frac{[n]_q}{[n+1]_q} < \frac{[n]_Q}{[n+1]_Q} \tag{2}$$

Proof. We have:

$$\frac{[m+1]_q}{[n]_q} \stackrel{?}{>} \frac{[n+1]_Q}{[n]_Q},$$

or

$$\frac{1 + ... + q^n}{1 + ... + q^{n-1}} \stackrel{?}{>} \frac{1 + Q + ... + Q^n}{1 + ... + Q^{n-1}},$$

or

$$1 + \frac{q^n}{1 + ... + q^{n-1}} \stackrel{?}{>} 1 + \frac{Q^n}{1 + ... + Q^{n-1}},$$

or

$$\frac{1 + ...q^{n-1}}{q^n} \stackrel{?}{<} \frac{1 + ... + Q^{n-1}}{Q^n},$$

or

$$\frac{1}{q^n} + ... + \frac{1}{q} < \frac{1}{Q^n} + ... + \frac{1}{Q},$$

which is true, because $q > Q \Rightarrow \frac{1}{q} < \frac{1}{Q}$. ■

On The Integer Consequtive Quantum

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Abstract

It's easy to check that $n(n+1)=\sum_{i=0}^{n-1}(2n-2i)$. We quantize this.

Let

$$[x]_q=\frac{q^x-q^{-x}}{q-q^{-1}},\ x\in\mathbf{R},\ q\neq 0,\pm 1,$$

be the 2^{nd} quantization of x.

Theorem 1. For $n\in\mathbf{Z}_{\geq 1}$,

$$[n]_q^{\sim}[n+1]_q^{\sim}=\sum_{i=0}^{n-1}[2n-2i]_q^{\sim}. \tag{2}$$

Proof. We use the formula for a sum of quantum arithmetic progression:

$$\sum_{k=0}^{m}[a+kd]_q^{\sim}=[m+1]_{q^{d/2}}^{\sim}\left[\frac{a+(a+md)}{2}\right]_q^{\sim}. \tag{3}$$

Together with

$$[x]_{q^{-1}}^{\sim}=[x]_q^{\sim}, \tag{4}$$

the RHS of (2), with the help of (3), (4), becomes:

$$\sum_{i=1}^{n-1}[2n-2i]_q^{\sim}=[n]_{q^{-1}}^{\sim}\left[\frac{2n+2}{2}\right]_q^{\sim}=[n]_q^{\sim}[n+1]_q^{\sim},$$

which is the LHS of (2). ■

On The Product Inequality in Quantum Domain

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Abstract

If ?? $-1 < a_i \leq 0$ then $(1+a_1)...(1+a_1)...(1+a_n) \geq 1+a_1+...+a_n$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

is the 2^{nd} quantization of x.

Theorem 1. Let $-1 < a_i \leq 0, i = 1, ..., n$. Then

$$[1+a_1]_q^{\sim}...[1+a_n]_q^{\sim} \geq [1+a_1+...+a_n]_q^{\sim}. \tag{2}$$

Proof. We use induction on n, the case $n = 1$ being obvious. Set

$$a = a_1 + ... + a_n, \quad a = a_{n+1}. \tag{3}$$

The inductive step amounts to:

$$[1+\sigma]_q^{\sim}[1+a]_q^{\sim} \stackrel{?}{\geq} [1+\sigma+a]_q^{\sim},\ a, \sigma \leq 0,$$

or, by multiplying by $(q - q^{-1})^2 > 0$, to

$$(q^{1+\sigma} - q^{-1-\sigma})(q^{1+a} - q^{-1-a}) \stackrel{?}{\geq} (q - q^{-1})(q^{1+\sigma+a} - q^{-1-\sigma-a}),$$

or to

$$(q^{2+\sigma+a} + q^{-2-\sigma-a}) - (q^{a-\sigma} + q^{\sigma-a}) \stackrel{?}{\geq}$$
$$\stackrel{?}{\geq} (q^{2+\sigma+a} + q^{-2-\sigma-a}) - (q^{\sigma+a} + q^{-\sigma-a}),$$

or to

$$q^{\sigma+a} + q^{\sigma-a} \stackrel{?}{\geq} q^{\sigma-a} + q^{a-\sigma},$$

or to

$$q^{\sigma}(q^a - a^{-a}) \stackrel{?}{\geq} q^{-\sigma}(q^a - a^{-a}). \tag{4}$$

Now, since

$$[x]^{\sim}_{q^{-1}} = [x]^{\sim}_{q},$$

we can take

$$q > 1$$

without loss of generality.

Since $a < 0, q^a - q^{-a} < 0$, and (4) turns into

$$q^{\sigma} \overset{?}{\leq} q^{-\sigma}$$

or to

$$q^{2\omega} \overset{?}{\leq} 1,$$

which is true because $q > 1, 2\sigma < 0$. ■

On The Product Of Two Quantum Integers As A Sum Of Odd Quantum Integers

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Abstract

It's easy to check that $n(n+2a)=\sum_{i=0}^{n-1}(2n+2a+1-2i),\ n\in\mathbf{Z}_{\geq 1}$, at $\mathbf{Z}_{\geq 0}$. We quantize this.

Let

$$[x]_q=\frac{q^x-q^{-x}}{q-q^{-1}},\ x\in\mathbf{R},\ q\neq 0,\pm 1,$$

be the 2^{nd} quantization of x.

Theorem 1. Let $a\in\mathbf{Z}_{\geq 0}, n\in\mathbf{Z}_{\geq 1}$. Then

$$[n]_q^{\sim}[n+2a]_q^{\sim}=\sum_{i=0}^{n-1}[2n+2a-1-2i]_q^{\sim}. \tag{2}$$

Proof. We use formula for the sum of quantum arithmetic progression

$$\sum_{k=0}^{n}[a+kd]_q^{\sim}=[n+1]_{q^{d/2}}^{\sim}[a+n\frac{d}{2}]_q^{\sim}. \tag{3}$$

With (3), the RHS of (2) becomes remembering that

$$[x]_{q^{-1}}^{\sim}=[x]_q^{\sim}:$$
$$[n]_q^{\sim}\Big[(2n+2a-1)+(2n+2a-1-2(n-1))2\Big]_q^{\sim}=$$
$$=[n]_q^{\sim}[n+2a]_q^{\sim},$$

and this is the LHS of (2). ■

Remark 4. Formula (2) is true for $a\in/\mathcal{R}$.

On The Quantum Harmonic Series

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Abstract

We quantize the identity: $\sum_{k=1}^{n} \frac{(???)^{k+1}}{k}\binom{n}{k} = 1 + \frac{1}{2} + ... + \frac{1}{n}$

Let

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbf{R},\ q \neq 0, \pm 1,$$

be the 2^{nd} quantization of x.

Set

$$\begin{bmatrix} m \\ k \end{bmatrix}_q^{\sim} = \frac{[m]!_q^{\sim}}{[k]!_q^{\sim}[m-k]!_q^{\sim}},$$
$$[k]!_q^{\sim} = [1]_q^{\sim}...[k]_q^{\sim},\ k \in \mathbf{Z}_{\geq 1};\ [0]!_q^{\sim} = 1.$$

Set

$$\mathcal{H}_n = \sum_{k=1}^{n} \frac{q^k}{[k]_q^{\sim}},$$

the quantum version of the harmonic series.

Theorem 1. For $n \in \mathbf{Z}_{\geq 1}$,

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{[k]_q^{\sim}} q^{nk} \begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} = \mathcal{H}_n. \tag{2}$$

Proof. We use induction on n. For $n = 1$, (2) returns:

$$q = q,$$

which is true.

Denote the LHS of (2) by $L(n)$. We have, using the formula

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q^{\sim} = q^{-k}\left(\begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} + q^{n+1}\begin{bmatrix} n \\ k-1 \end{bmatrix}_q^{\sim}\right): \tag{3}$$

$$L(n+1) = \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{[k]} q^{(n+1)k} q^{-k}\left\{\begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} + q^{n+1}\begin{bmatrix} n \\ k-1 \end{bmatrix}_q^{\sim}\right\} = L(n) +$$

$$+\sum_{k=1}^{n-1} q^{nk} q^{n+1} \frac{(-1)^{k+1}}{[k]} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q^{\sim} = L(n) -$$

$$+\sum_{s=0}^{n} q^{n+1} q^{ns+n} \frac{(-1)^s}{[s+1]_q^{\sim}} \begin{bmatrix} n \\ s \end{bmatrix}_q^{\sim} = L(n) + q^{2n+1} \frac{q^{-n}}{[n+1]_q^{\sim}}, \tag{4}$$

because

$$\sum_{k=0}^{n} \frac{(-1)^k}{[k+r]_q^{\sim}} \begin{bmatrix} n \\ k \end{bmatrix}_q^{\sim} = q^{-nr} \frac{[n]!_q^{\sim}}{[r]_q^{\sim} ... [r+n]_q^{\sim}},\ r,\ n \in \mathbf{Z}_{\geq 1},$$

as easy to check by induction on n.

Thus,

$$L(n+1) = \sum_{k=1}^{n} \frac{q^k}{[k]} + q^{n+1} \frac{1}{[n+1]} = \mathcal{H}_{n+1}. \quad \blacksquare$$

On The Ratio Of Quantum Double Factorials

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Abstract

The classical tight bounds, $\frac{1}{2\sqrt{n}} < \frac{(2n-1)!}{(2n)!!} > \frac{1}{\sqrt{2n+1}}$ are quantized.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

is the 2^{nd} quantization of x, so that

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

and we can take

$$q > 1$$

without loss of generality.

Set

$$[k]!_q^{\sim} = [1]_q^{\sim}...[k]_q^{\sim},\ k \in \mathbf{Z}_{\geq 1}; [0]!_q^{\sim} = 1,$$
$$[2k+1]!!_q^{\sim} = [1]_q^{\sim}[3]_q^{\sim}...[2k+1]_q^{\sim},\ k \in \mathbf{Z}_{\geq 0};$$
$$[2k]!!_q^{\sim} = [0]_q^{\sim} + [2]_q^{\sim}...[2k]_q^{\sim}, \quad k \in \mathbf{Z}_{\geq 0}.$$

Theorem 1. For $n \in \mathbf{Z}_{\geq 1}$,

$$\frac{1}{\sqrt{[2]_q^{\sim}[2n]_q^{\sim}}} < \frac{[2n-1]!!_q^{\sim}}{[2n]!!_q^{\sim}} < \frac{1}{\sqrt{[2n+1]_q^{\sim}}} \tag{2}$$

Proof. We are going to use the easily verifiable Lemma that if $a, b, k \geq 0$, not all a, b, c are zero, and

$$2a \geq b + c, \tag{3}$$

then

$$([a]_q^{\sim})^2 > [b]_q^{\sim}[c]_q^{\sim}. \tag{4}$$

In particular,

$$\frac{[2k-1]_q^\sim}{[2k]_q^\sim} < \frac{[2k]_q^\sim}{[2k+1]_q^\sim},\ k \in \mathbf{Z}_{\geq 1}, \tag{5}$$

because

$$[2k-1]_q^\sim [2k+1]_q^\sim < ([2k]_q^\sim)^2$$

by the Lemma with $a = 2k, b = 2k-1, c = 2k+1$.

A, ??

$$\frac{[2k-1]_q^\sim]}{[2k]_q^\sim} > \frac{[2k-2]_q^\sim}{[2k-1]_q^\sim}, \tag{6}$$

because

$$([2k-1]_q^\sim)^2 > [2k]_q^\sim [2k-2]_q^\sim,$$

by the Lemma with $a = 2k-1, b = 2k, c = 2k-2$.

Now, denote $\frac{[2n-1]!!}{[2n]!!}$ by M. We have, using (5),

$$M = \frac{[1]_q^\sim}{[2]_q^\sim}.\frac{[3]_q^\sim}{[4]_q^\sim}...\frac{[2n-1]_q^\sim}{[2n]_q^\sim} < \frac{[2]_q^\sim}{[3]_q^\sim}.\frac{[4]_q^\sim}{[5]_q^\sim}...\frac{[2n]_q^\sim}{[2n+1]_q^\sim} = \frac{1}{M[2n+1]_q^\sim},$$

so that

$$M < \frac{1}{\sqrt{[2n+1]_q^\sim}}. \tag{7}$$

Also, using (6), we get

$$M = \frac{[1]_q^\sim}{[2]_q^\sim}.\frac{[3]_q^\sim}{[4]_q^\sim}...\frac{[2n-1]_q^\sim}{[2n]_q^\sim} > \frac{[1]_q^\sim}{[2]_q^\sim}\frac{[2]_q^\sim}{[3]_q^\sim}...\frac{[2n-2]_q^\sim}{[2n-1]_q^\sim} =$$
$$= \frac{1}{[2]_q^\sim}\frac{[2n]!!_q^\sim}{[2n-1]!!_q}\frac{1}{[2n]} = \frac{1}{[2]_q^\sim [2n]_q^\sim}\frac{1}{M},$$

so that

$$M > \frac{1}{\sqrt{[2]_q^\sim [2n]_q^\sim}}. \quad \blacksquare \tag{8}$$

On The Ratio Of Two Quantum Numbers In The 1^{st} Quantization

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Abstract

We show how such a quantum ratio is connected to the classic alone.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

is the 2^{nd} quantization of x.

Theorem 1. Let $y > x > 0, q > 1$. Then

$$[y]_q^{\sim}[x]_q^{\sim} > \frac{[y]}{[x]}. \tag{2}$$

Proof. (2) is:

$$x[y]_q \overset{?}{>} y[x]_q,$$

or, with

$$h = logq > 0,$$

$$x\frac{e^{hy} - 1}{q - 1} \overset{?}{>} y\frac{e^{hx} - 1}{q - 1},$$

or, since $q > 1$,

$$x(e^{hy} - 1) \overset{?}{>} y(e^{hx} - 1),$$

or

$$x\sum_{n=1}^{\infty} \frac{h^n}{n!}y^n > y\sum_{n=1}^{\infty} \frac{h^n}{n!}x^n. \tag{3}$$

(3) follows from:

$$n = 1 :\ xy = yx,$$

$$h > 1 \quad xy^n \overset{?}{>} yx^n, \tag{4}$$

or

$$y^{n-1} \overset{?}{>} x^{n-1}, n > 1,$$

which is obvious since $y > x > 0$. ■

On The Sum Of 3 Consequitive Quantum Integers

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Abstract

Classically, $n+(n+1)+(n+2)=3(n+1)\equiv 0\ (mod\ 3)$. We quantize this.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\quad x\in\mathbf{R},\ q\neq 0,\pm 1,$$

be the second quantization of x.

Theorem 1. Let $n\in\mathbf{Z}$. Then

$$[n-1]_{q^2}^{\sim}+[n]^{\sim})q^2+[n+1]_{q^2}^{\sim}\equiv 0\ (mod\ [3]_q^{\sim}). \tag{2}$$

Proof. Denote the LHS of (2) by f. Then

$$\begin{aligned}&[q^{2(n-1)}-q^{-2(n-1)}]+[q^{2n}-q^{-2n}]+[q^{2(n+1)}]-q^{-2(n+1)}]=\\&=q^{2n}(q^{-2}+1+q^{-2})-q^{-2n}(q^2+1+q^{-1})=\\&=[3]_q^{\sim}(q^{2n}-q^{-2n}).\end{aligned} \tag{3}$$

Thus,

$$S=[3]_q^{\sim}\frac{q^{2n}-q^{-2n}}{q^2-q^{-2}}=[3]_q^{\sim}[n]_{q^2}^{\sim}.\quad\blacksquare$$

On The Sum Of Inverse Quantum Squares

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Abstract

The classical formula $\sum_{k=1}^{n} \frac{2k+1}{k^2(k+1)^2} = \frac{n(n+2)}{(n+1)^2}$, is quantized.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbf{R},\ q \neq 0, \pm 1,$$

be the second quantization of x.

Theorem 1. For $n \in \mathbf{Z}_{\geq 1}$,

$$\sum_{k=1}^{n} \frac{[2k+1]_q^{\sim}}{([k]_q^{\sim})^2([k+1]_q^{\sim})^2} = 1 - \frac{1}{([n+1]_q^{\sim})^2}. \tag{2}$$

Proof. We have:

$$\frac{1}{([k]_q^{\sim})^2} - \frac{1}{([k+1]_q^{\sim})^2} = \frac{([k+1]_q^{\sim})^2 - ([k]_q^{\sim})^2}{([k]_q^{\sim})^2([k+1]_q^{\sim})^2} = \frac{[2k+1]_q^{\sim}}{([k]_q^{\sim})^2([k+1]_q^{\sim})^2}, \tag{3}$$

because, as is easy to verify,

$$([k+1]_q^{\sim})^2 - ([k]_q^{\sim})^2 = [2k+1]_q^{\sim}. \tag{4}$$

Summing up on k formula (3), we arrive at (2). ■

Sum Of Quantum Squares

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Abstract

$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. This formula is quantized.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x, q \in \mathbf{R},\ q \neq q > 0,$$

is the second quantization of x.

Theorem 1. For $n \in \mathbf{Z}_{\geq 0}$,

$$\sum_{k=0}^{n} [k]_q^{\sim} [k]_{q^3}^{\sim} = \frac{[n]_q^{\sim} [n+1]_q^{\sim} [2n+1]_q^{\sim}}{[2]_q^{\sim} [3]_q^{\sim}}. \tag{2}$$

Proof. We use induction on n. For $n = 0$, (2) returns:

$$0 = 0,$$

which is true.

The unductive step $n \mapsto n+1$, amounts to:

$$\frac{[n]_q^{\sim} [n+1]_q^{\sim} [2n+1]_q^{\sim}}{[2]_q^{\sim} [3]_q^{\sim}} + [n+1]_q^{\sim} [n+1]_{q^3}^{\sim} \stackrel{?}{=}$$
$$\stackrel{?}{=} \frac{[n+1]_q^{\sim} [n+2]_q^{\sim} [2n+3]_q^{\sim}}{[2]_q^{\sim} [3]_q^{\sim}},$$

or

$$[n+2]_q^{\sim} [2n+3]_q^{\sim} - [n]_q^{\sim} [2n+1]_q^{\sim} \stackrel{?}{=} [2]_q^{\sim} [3]_q^{\sim} [n+1]_{q^3}^{\sim}. \tag{3}$$

Multiplying (3) by $(q - q^{-1})^2$, we get:

$$(q^{n+2} - q^{-n-2})(q^{2n+3} - q^{-2n-3}) - (q^n - q^{-n})(q^{2n+1} - q^{-2n-1}) \stackrel{?}{=}$$
$$\stackrel{?}{=} (q^2 - q^{-2})(q^3 - q^{-3}) \frac{q^{3(n+1)} - q^{-3(n+1)}}{q^3 - q^{-3}},$$

or

$$[(q^{3n+5}+q^{-3n-5})-(q^{n+1}+q^{-n-1})]-[(q^{3n+1}+q^{-3n-1})-(q^{n+1}-q^{-n-1})] \stackrel{?}{=}$$
$$\stackrel{?}{=}(q^2-q^{-2})(q^{3n+3}-q^{-3n-3}),$$

or

$$(q^{3n+5}+q^{-3n-5})-(q^{3n+1}+q^{-3n-1}) \stackrel{?}{=}$$
$$q^{3n+5}-q^{-3n-1}-q^{3n+1}+q^{(-3n-5)},$$

which is correct. ■

Representations Of Power Function In Quantum Domain

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Abstract

It's easy to check that $n^{s+1}=\sum_{i=0}^{n}[(n^s+s(n-1)-2si]$. We quantize this.

Let

$$[x]_q=\frac{q^x-q^{-x}}{q-q^{-1}},\ x\in\mathbf{R},\ q\neq 0,\pm 1,$$

be the 2^{nd} quantization of x, so that

$$[x]^{\sim}_{q^{-1}}=[x]^{\sim}_q.$$

Theorem 1. For $s\in\mathcal{R}, n\in\mathbf{Z}_{\geq 1}$,

$$[n]^{\sim}_{q^s}[n^2]^{\sim}_q=\sum_{i=0}^{n-1}[n^s+s(n-1)-2si]^{\sim}_q. \tag{2}$$

Proof. We use the formula for the sum of quantum arithmetic progression:

$$\sum_{k=0}^{m}[a+kd]^{\sim}_q=[m+1]^{\sim}_{q^{d/2}}[\frac{a+(a+nd}{2}]^{\sim}_q. \tag{3}$$

The RHS of (2), with the help of (3), returns

$$\sum_{i=0}^{n-1}[n^s+s(n-1)-2si]^{\sim}_q=[n]^{\sim}_{q^3}\left[\frac{(n^s+s(n-1)+n^s+s(n-1)-2s(n-1)}{2}\right]^{\sim}_q=$$
$$=[n]^{\sim}_{q^s}[n^s]^{\sim}_q,$$

which is the LHS of (2). ■

The Growth Of Quantum Sequence Classically Convergent To e

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Abstract

It is well know that $(1+\frac{1}{n+1}n+1)^{n+1} > (1+\frac{1}{n})^n,\ n \in \mathbf{Z}_{\geq 1}$. We find quantum versions of this sequence and its monotonicity.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

is the 2^{nd} quantization of x.

Theorem 1. For $n \in \mathbf{Z}_{\geq 1}$,

$$\left(\left[1+\frac{1}{n}\right]_q^{\sim}\right)^{n+1} > \left(\left[1+\frac{1}{n}\right]_q^{\sim}\right)^n. \tag{2}$$

Proof. We are going to use the 2^{nd} form of the quantum AGM inequality:

$$\left[\frac{a_1 + ... + a_n}{n}\right]_q^{\sim} \geq \sqrt[n]{[a_q]_q^{\sim}...[a_n]_q^{\sim}}, \tag{3}$$

for $a_1, ..., a_n \in \mathbf{R}_{\geq 0}$, with equality iff

$$a_1 = ... = a_n. \tag{4}$$

Now take:

$$a_1 = 1, a_2 = ... = a_{n+1} = 1 + \frac{1}{n}. \tag{5}$$

Then the LHS of (3) becomes:

$$\left[\frac{1 + n(1+\frac{1}{n})}{n+1}\right]_q^{\sim} = \left[1 + \frac{1}{n+1}\right]_q^{\sim}. \tag{6}$$

The RHS of (3) is:

$$\sqrt[n+1]{([1+\frac{1}{n}]_q^{\sim})^n}. \tag{7}$$

Thus,

$$\left[1+\frac{1}{n+1}\right]_q^{\sim} > \sqrt[n+1]{([1+\frac{1}{n}]_q^{\sim})^n},$$

or

$$\left(\left[1+\frac{1}{n+1}\right]_q^{\sim}\right)^{n+1} > \left(\left[1+\frac{1}{n}\right]_q^{\sim}\right). \quad \blacksquare$$

Still Another Quantum Parameterization Of The Circle

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Abstract

The formula $(4n) + (4n^2-1)^2 = (4n^2+1)^2$, is quantized.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

is the 2^{nd} quantization of x.

Theorem 1.

$$????[8x^2]_q^{\sim} + ([4x^2-1]_q^{\sim})^2 = ([4x^2+1]_q^{\sim})^2 \tag{2}$$

Proof. W use the easily verifyable formula:

$$([a]_q^{\sim})^2 - ([b]_q^{\sim})^2 = [a-b]_q^{\sim}[a+b]_q^{\sim}.$$

Then

$$([4x^2+1]_q^{\sim})^2 - ([4x^2-1]_q^{\sim})^2 = [2]_q^{\sim}[8x^2]_q^{\sim},$$

which is (2). ■

Quantum Vandermonde Summation

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Abstract

The classical Vandermonde summation $\sum_{i+j=p}\binom{n}{i}\binom{m}{j}=\binom{n+m}{p}$, is quantized.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\quad x\in\mathbf{R},\ q\neq 0,1,$$

is the 2^{nd} quantization of x.

Theorem 1.

$$\sum_{i+j=p}\begin{bmatrix}n\\i\end{bmatrix}_q^{\sim}\begin{bmatrix}m\\j\end{bmatrix}_q^{\sim}q^{-mi+hj}=\begin{bmatrix}n+m\\p\end{bmatrix}_q^{\sim},\tag{2}$$

where

$$\begin{bmatrix}n\\\end{bmatrix}_q^{\sim}=\frac{[n]!_q^{\sim}}{[k]!^{\sim}[n-k]!_q^{\sim}},$$
$$[k]!_q^{\sim}=[1]_q^{\sim}...[k]_q^{\sim}k\in\mathbf{Z}_{\geq 1};\ [0]!_q^{\sim}=1.$$

Proof. We use induction on m, the case $m=0$ being obvious. Since

$$\begin{bmatrix}n+1\\i\end{bmatrix}_q^{\sim}=\begin{bmatrix}n\\i\end{bmatrix}_q^{\sim}q^{\epsilon i}+\begin{bmatrix}n\\i-1\end{bmatrix}q^{-\epsilon(??-n-1)},\quad \epsilon=\pm 1,\tag{3}$$

because

$$[n+1]_q^{\sim}=q^{\epsilon i}[n+1-i]_q^{\sim}+[i]_q^{\sim}q^{-\epsilon(n+1-i)},\tag{4}$$

because

$$[x+y]_q^{\sim}=[x]_q^{\sim}q^y+[y]_q^{\sim}q^{-x}=[x]_q^{\sim}q^{-y}+[y]_q^{\sim}q^x.\tag{5}$$

Now, the inductive step $n \mapsto n+1$ is:

$$\sum_{i+j=p} 1 + \begin{bmatrix} n \\ i \end{bmatrix}_q^{\sim} \begin{bmatrix} m \\ j \end{bmatrix}_q^{\sim} q^{-mi+(n+1)i} =$$

$$\sum_{i+j=p} \begin{bmatrix} m \\ j \end{bmatrix}_q^{\sim} q^{mi+(n+1)j} \left\{ \begin{bmatrix} n \\ i \end{bmatrix}_q^{\sim} q^i + \begin{bmatrix} n \\ i-1 \end{bmatrix}_q^{\sim} q^{i-1-n} \right\} =$$

$$= \sum \begin{bmatrix} n \\ i \end{bmatrix}_q^{\sim} \begin{bmatrix} m \\ j \end{bmatrix}_q^{\sim} q^{-mi+nj} + q^{j+i} + \sum_{i+j=p-1} \begin{bmatrix} n \\ s \end{bmatrix}_q^{\sim} \begin{bmatrix} m \\ j \end{bmatrix}_q^{\sim} q^{-ms+nj-m+j+i-1-n} =$$

$$= q^P \begin{bmatrix} n+m \\ p \end{bmatrix}_q^{\sim} + q^{-m-n+(p-1)} \begin{bmatrix} n+m \\ p-1 \end{bmatrix}_q^{\sim} =$$

$$= \begin{bmatrix} n+m+1 \\ p \end{bmatrix}_q^{\sim},$$

as required. ■

The Number Of Primes Between Two Successive Powers

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Abstract

We show that between x^k and $(x+1)^k, k \geq 1$, there is $\sim \frac{x^{k-1}}{logx}$ primes.

Recently I proved that, on average, there is 1 prime between x and $x + logx$. The interval $(n+1)^k - n^k$ has

$$\frac{(n+1)^k - n^k}{logn^k} \sim \frac{kn^{k-1}}{klogn} = \frac{n^{k-1}}{logn}. \tag{1}$$

primes. Thus, the number of primes is being multiplied by n when k increases by 1.

Example 2. Take $n = k = 3$. Between 3^3 and 4^3 there are 8 primes: 29, 31, 37, 41, 43, 47, 53, 59. And

$$\frac{3^2}{log3} = 8.192.$$

Example 3. Take $n = 2, k = 4$. Between 16 and 81 there are 16 primes: 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79. And

$$\frac{2^3}{log2} = 15.4.$$

n doesn't have to be a prime.

Example 4. Take $n = 1.5, k = 5$ $1.5^5 = 7.5,\ 2.5^5 = 97.6$. Between 8 and 97 there are 21 primes: 11, 13, 17, ..., 79, 83, 89, 97. And

$$\frac{1.5^4}{log1.5} = \frac{5.625}{0.405} = 22.9.$$

Example 5. Take $n = 1.1, k = 6$. Then $1.1^6 = 1.77, 2.1^6 = 85.7$. There are 23 primes between 2 and 85: 2, 3, 5, 7, 11, ..., 83. And

$$\frac{1.1^5}{log1.1} = \frac{1.610}{0.0953} = 1.68.$$

We see that the agreement becomes worse as $n \to 1$, otherwise it is amazingly good.

Remark 2. Let $f(x)$ be a polynomial with $f(0) = 0$. Then we can reformulate our main result as between $f(x)$ and $f(x+1)$ there is

$$\sim \frac{f(x)}{xlogx}$$

primes.

The Quantum Analog Of $\sqrt{e} < 2$

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Abstract

$\sqrt{e} < 2$ trivially, or $\left(1 + \frac{1}{2n}\right)^n < 2$. We find quantum version of this.

Let

$$[x]_q = \frac{q^x - 1}{q - 1},\ x \in \mathbf{R},\ q \neq 0, \pm 1,$$

be the 2^{nd} quantization of x.

Since

$$[x]^{\sim}_{q^{-1}} = [x]^{\sim}_q,$$

we can take

$$q > 1$$

without loss of generality.

Take (the 2^{nd} form of) quantum AGM inequality:

$$\left[\frac{a_1 + ... + a_n}{n}\right]^{\sim}_q \geq \sqrt[n]{[a_1]^{\sim}_q ... [a_n]^{\sim}_q}. \tag{1}$$

Set

$$a_1 = 1, a_2 = ... = a_n = 1 + x, \tag{2}$$

with x specified later on. Then

$$\frac{a_1 + ... + a_n}{n} = \frac{1 + (n-1)(1+x)}{n} = 1 + \frac{n-1}{n}x, \tag{3}$$

and (1) returns:

$$\left(\left[1 + \frac{n-1}{n}x\right]^{\sim}_q\right)^n > \left(\left[1 + x\right]^{\sim}_q\right)^{n-1}. \tag{4}$$

Now take

$$x = \frac{n}{n-1}. \tag{5}$$

then (4) becomes:

$$([2]_q^{\sim})^n > \left(\left[2+\frac{1}{n-1}\right]\right)^{n-1}, \tag{6}$$

or

$$[2]_q^{\sim} > \left([2+\frac{1}{n-1}]_q^{\sim}[2]_q^{\sim}\right)^{n-1} \tag{7}$$

Now,

$$\left[2+\frac{1}{n-1}\right]_q^{\sim} = \left[2\left(1+\frac{1}{2(n-1)}\right)\right]_q^{\sim} = [2]_q^{\sim}\left[1+\frac{1}{2(n-1)}\right]_{q^2}^{\sim}, \tag{8}$$

and (7) becomes:

$$[2]_q^{\sim} > \left(\left[1+\frac{1}{2(n-1)}\right]_{q^2}^{\sim}\right)^{n-1},\ n \geq 1. \tag{9}$$

At $q=1$, (4) becomes:

$$2 > (1+\frac{1}{2n})^n,\ n>0,$$

which is obvious, because

$$2 > e^{1/2}, \tag{10}$$

and

$$\left(1+\frac{x}{n}\right)^n \tag{11}$$

increases with n for $x>0$.

The Sum Of 3 Consequitive Integers The First Of Which Is $\equiv 2 \pmod 3$, Is Divisible By 9

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Abstract

Classically, $(3n-1)+(3n)+(3n+1) = gn \equiv 0(mod 9)$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

be the second quantization of x.

Theorem 1. For $n \in \mathbf{Z}$,

$$[3n-1]_{q^2}^{\sim} + [3n]_{q^2}^{\sim} + [3n+1]_{q^2}^{\sim} \equiv 0(mod\ ([3]_q^{\sim})^2). \tag{2}$$

Proof. The LHS of (2) is:

$$\frac{1}{q^2 - q^{-2}} \left\{ [q^{2(3n-1)} - q^{-2(3n-1)}] + [q^{2-3n} - q^{-2n-3n}] + [q^{2(3n+1)} - q^{-2(3n+1)}] \right\} =$$
$$= \frac{1}{q^2 - q^{-2}} \left\{ q^{\sigma n}(q^{-2} + 1 + q^{-2}) - q^{-\sigma n}(q^2 + 1 + q^{-2}) \right\} =$$
$$= [3]_q^{\sim} \frac{(q^{\sigma n} - q^{-\sigma n})}{q^2 - q^{-2}} = [3]_q^{\sim} [3n]_{q^2}^{\sim},$$

and

$$[3n]_{q^2}^{\sim} = [3]_{q^2}^{\sim} [n]_{q^6}^{\sim},$$

so we have to show that

3-Term Arithmetic Progression Among Elements Of A Quantum Harmonic Series

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The terms $\frac{1}{k+1}, \frac{1}{2k}, \frac{1}{k(k+1)}, k \in \mathbf{Z}_{\geq 2}$, from a 3-term arithmetic progression. We manage to quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in \mathbf{R},\ q \neq 0, \pm 1,$$

be the second quantization of x.

The elements

$$A = [a]_q^{\sim},$$
$$B = [a+d]_q^{\sim}$$
$$C = [a+2d]_q,$$

form a Quantum Arithmetic progression with the characteristic property

$$A + c = [2]_{q^d}^{\sim} B, \tag{1}$$
$$d = (B-A)|_{q=1} = (C-B)|_{q=1}. \tag{1a}$$

Theorem 2. Consider the elements

$$A = \frac{q^{-k}}{[k+1]_q^{\sim}} \frac{1}{[2]_q^{\sim}[k]_q^{\sim}}, \frac{q}{[k]_q^{\sim}[k+1]_q^{\sim}}, \tag{3a}$$
$$d = \frac{-1}{2k} + \frac{1}{k+1} = \frac{-1+k}{2k(k+1)}. \tag{3b}$$

Proof. We have to check the equation (1), i.e., that

$$\frac{q}{[k+1]_q^{\sim}} + \frac{q^{-k}}{[k]_q^{\sim}[k+1]_q^{\sim}} \stackrel{?}{=} \frac{1}{[k]_q^{\sim}},$$

or

$$q[k]_q^{\sim} + q^{-k} \stackrel{?}{=} [k+1]_q^{\sim},$$

which is true because, in general,

$$[a+b]_q^{\sim} = q^{-b[a]_q^{\sim}} + q^a [b]_q^{\sim}. \quad \blacksquare$$

1-Jet Of Quantum Double-Factorials

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Abstract

We calculate 1^{st} order corrections to the quantum versions of double factorials.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Set

$$[2n+1]!!_q^{\sim} = \prod_{s=0}^{n} [2s+1]_q^{\sim},\quad n \in \mathbf{Z}_{\geq 0},$$
$$[2n]!!_q^{\sim} = \prod_{s=1}^{n} [2s]_q^{\sim},\ n \in \mathbf{Z}_{\geq 1};\quad [0]!! = 1.$$

Set

$$h = logq.$$

We are going to use the easily checked formula

$$[x]_q^{\sim} = x\{1 + \frac{h^2}{6}(x^2 - 1)\} + 0(h^4). \tag{1}$$

Theorem 2. Modulo $0(h^4)$ terms,

$$[2n+1]!!_q^{\sim} = (2n+1)\left\{1 + \frac{h^2}{6} 8\left(\frac{n+1}{3}\right)\right\}. \tag{3}$$
$$[2n]!!^{\sim} + q = (2n)!!\left\{1 + \frac{h^2}{6}\frac{3}{4n^2 + 6n - 1}\right\},\ n \in \mathbf{Z}_{\geq 1}. \tag{4}$$

Proof. (i) We have:

$$[2n+1]!!_q^{\sim} = \prod_{s=0}^{n}[2n+1]_q^{\sim} = \prod_{s=0}^{n}(2s+1)\{1+\frac{h^2}{6}[(2s+1)^2-1]\} =$$

$$= (2n+1)!!\{1+\frac{h^2}{6}\sum_{s=0}^{n}(4s^2-4s)\} =$$

$$= (2n+1)!!\Big\{1+\frac{h^2}{6}4\Big[\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}\Big]\Big\} =$$

$$= (2n+1)!!\Big\{1+\frac{h^2}{6}4\frac{n(n+1)}{6}[(2n+1)-3]\Big\} =$$

$$= (2n+1)!!\Big\{1+\frac{h^2}{6}\frac{8(n+1)n(n-1)}{6}\Big\} = (2n+1)!!\Big\{1+\frac{h^2}{6}8\binom{n+1}{3}\Big\},$$

which is (3); (ii) For (4), we have:

$$[2n]!!_q^{\sim} = \prod_{s=1}^{n}[2s]_q^{\sim} = \prod_{s=1}^{n}(2s)\Big\{1+\frac{h^2}{6}[(2s)^2-1]\} =$$

$$= (2n)!!\{1+\frac{h^2}{6}\sum_{s=1}^{n}(4s^2-1)\} = (2n)!!\Big\{1+\frac{h^2}{6}\Big[4\frac{n(n+1)(2n+1)}{6}-n\Big]\Big\}$$

$$= (2n)!!\{1+\frac{h^2}{6}\frac{n}{6}[4(n+1)(2n+1)-6]\} =$$

$$= (2n)!!\{1+\frac{h^2}{6}[4(2n^2+3n+1)-6]\} =$$

$$= (2n)!!\{1+\frac{h^2}{6}\frac{n}{3}[4n^2+6n-1]\},$$

and this is (4). ■

2 + 2 Submatrices In A General Numerical Triangle Of Arithmetic Progressions

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Abstract

Let $a_s = a + sd, s \in \mathbf{Z}$. Consider the triangle, then the determinant of its every $2x2$ submatrix is < 0.

$$\begin{array}{ccc} & a_0 & \\ a_1 & & a_2 \\ a_2 & a_2 & a_1 \end{array} \tag{0.1}$$

Denote by ℓ_n the starting entry in the n^{th} row:

$$\ell_n = a_{t_n}, \tag{1}$$

where t_n is the n^{th} triangular number:

$$t_n = \binom{n+1}{2}. \tag{2}$$

This formula is easy to derive.

Denote by $a_{n_{ji}}$ the i^{th} entry in the n^{th} row:

$$a_{n_{ji}} = \ell_n + di : \tag{3}$$

Consider an arbitrary $2x2$ sub matrix in the triangle:

$$\begin{array}{cc} (a_{n,j} & a_{n,i+k'} \\ a_{n+2k,i+k} & a_{n+2k,i+k+k'} \end{array}$$

A Bernoulli-Like Inequality For Quantum Numbers

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Abstract

The classical inequality, $(1-a)^k < \frac{1}{1+ak},\ 0<a<1, k \in \mathbf{Z}_{\geq 1}$, is quantized.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Theorem 1. Let $0<a<1,\ k \in \mathbf{Z}_{\geq 1}$, then

$$([1-a]_q^{\sim})^k < \frac{1}{[1+ka]_q^{\sim}}. \tag{2}$$

Proof. We use induction on k. For $k=1$, we get:

$$[1-a]_q^{\sim}[1+a]_q^{\sim} < 1,$$

which has been recently proved.

The inductive step $k \mapsto k+1$, yields:

$$\frac{1}{[1+ka]_q^{\sim}}[1-a]_q^{\sim} < \frac{1}{[1+(k+1)a]_q^{\sim}},$$

or

$$[1+(k+1)a]_q^{\sim}[1-a]_q^{\sim} \stackrel{?}{<} [1+ka]_q^{\sim}. \tag{3}$$

Multiplying through by $(q-q^{-1})^2>0$, (3) becomes,

$$[q^{1+(k+1)a}-q^{-1-(k+1a)}](q^{1-a}-q^{a-1}) \stackrel{?}{<} (q-q^{-1})(q^{1+ka}-q^{-1-ka}),$$

or

$$q^{2+ka}-q^{2+ka}-q^{(2+ka)}+q^{-2-ka} \stackrel{?}{<} q^{2+ka}-q^{-ka}-q^{ka}+q^{-2-ka},$$

or

$$q^{ka}+q^{-ka} \stackrel{?}{<} q^{ka}q^{(q-q^{-1})}$$

or

$$q^{-ka-1}<q^{ka+1},$$

which is obvious. ■

A Characteristic Property Of Quantum Arithmetic Progression

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Abstract

The classical arithmetical progression $a, b, c,$ is characterized by the condition $a = b = 2b$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Theorem 1. Let $a, b, c \in \mathbf{R}$. Then the conditions

$$[a]_q^{\sim} + [c]_q^{\sim} = [2]_{q^{(a-c)/2}}^{\sim} [b]_q^{\sim} \tag{2}$$

and

$$a + 2b \tag{3}$$

are equivalent.

Proof. In one direction the implication is obvious, because

$$(2)|_{q=1} = (3).$$

Now, in another direction, form (3) to (2), multiply (2) by $(q - q^{-1})$, and denote

$$b = a + d, \quad c = a + 2d. \tag{4}$$

Then (2) becomes:

$$q^a - q^{-a} + q^{ax} = (q^x + q^{-x})(q^{a+d} - q^{-a-d}) =$$
$$q^a - q^{-a} + a^{a+2d} - q^{-a-2d} \stackrel{?}{=} (q^d + q^{-d})(q^{a+d} - q^{-a-d}) =$$
$$= (q^{a+2d} - q^{-a-2d}) + (q^a - q^{-a}),$$

which is obvious. ■

A Comparison Of Quantum Quadratic Polynomials

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Abstract

Quantization of quadratic polynomials preserves partial order.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Suppose we have a pair of comparable monic quadratic polynomials:

$$p(x) = (x + a)(x + b),$$
$$r(x) = (x + c)(x + d).$$

The condition

$$p < r \tag{1}$$

means:

$$a + b = c + d =: N, \tag{2}$$
$$ab < cd, \tag{3}$$

Theorem r. For any $x \in \mathbf{R}$,

$$[x + a]_q^{\sim}[x + b]_q^{\sim} < [x + c]_q^{\sim}[x + d]_q^{\sim}. \tag{5}$$

Proof. Multiplying through by $(q-q^{-1})^2>0$, (5) becomes:

$$(q^{x+a}-q^{-x-a})(q^{x+b}-q^{-x-b}) \stackrel{?}{<} (q^{x+c}-q^{-x-c})(q^{x+d}-q^{-x-d}),$$

or

$$(q^{2x+a+b}+q^{-2x-a-b})-(q^{a-b}+q^{b-a}) \stackrel{?}{<} (q^{2x+c+d}+q^{-2x-c-d})-(q^{d-c}+q^{c-d}),$$

or, by (2),

$$q^{d-c}+q^{c-d} \stackrel{?}{<} q^{a-b}+q^{b-a}. \tag{6}$$

Since the function

$$x \to q^x+q^{-x}$$

increases with $x>0$, (6) is equivalent to:

$$|d-c| \stackrel{?}{<} |a-b|, \tag{7}$$

which follows from (2), (3) which implies that $a \le b$ say lie *outside* the interval $[c,d]$. ■

A Generalization of Sun's Conjecture

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Abstract

Sun conjectured that every large number M of an appropriate parity, is $M = p + ax^2$, where p is a prime and $a \geq 1$ is given. I notice that x^2 can be replaced by *any* polynomial in x, and the Conjecture still remains true.

Let $f(x)$ be a polynomial with integer coefficients; we want the number of solutions of

$$M = p + f(x). \tag{1}$$

We assume that $M - f(x)$ is *odd*. Let

$$\delta(x) = \pi(x) - \pi(x-2) = \begin{cases} 1, & x \text{ odd}, \quad \text{a prime} \\ 0, & x \text{ odd}, \quad \text{not a prime} \end{cases} \tag{2}$$

I recently showed that

$$\delta(x) \sim \frac{2}{logx} \tag{3}$$

If $n(M)$ denotes the number of solutions (p, x) of (1), then

$$n(M) = \sum_x \delta(M - fx) = \sum_x \frac{2}{log(M-f)} \sim \frac{2}{logM} \sum \frac{1}{1 + \frac{f}{MlogM}} \sim$$
$$\sim \frac{2}{logM} \sum_x \left(1 + \frac{f(x)}{MlogM}\right). \tag{4}$$

We need to be more specific about $f(x)$. Let

$$f(x) = ax^k, \ a \geq 1, k \geq 2. \tag{5}$$

is the Sun case. Then the upper bound for x is

$$\left\lfloor \left(\frac{M}{a}\right)^{1/k} \right\rfloor - \left(\frac{M}{a}\right)^{1/k},$$

and (4) returns:

$$\frac{2}{logM}\sum_{x=1}^{\left(\frac{M}{a}\right)^{1/k}}\left[1+\frac{aDC^k}{MlogM}\right]\sim$$
$$\sim\frac{2}{logM}\left[\left(\frac{M}{a}\right)^{1/k}+\frac{a\left(\frac{M}{a}\right)^{(K+1)/k}}{(K+1)MlogM}\right]\sim$$
$$\sim\frac{2}{logM}\left[\frac{M^{1/k}}{a^{1k}}+\frac{a^{2+R}M^{1/k}}{(K+1)logM}\right]\sim\frac{2M^{1/k}}{logMa^{1/k}}. \tag{6}$$

Thus, the number of solutions grows to ∞ with M.

A Naive Treatment Of The Hardy-Littlewood Conjecture

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Abstract

The main case of the $H-L$ almost allows directly from the PNT (prime number theorem.)

The $H-R$ Conjecture,

$$\pi(x+y) \leq \pi(x) + \pi(y),\ x, y \geq 3, \tag{1}$$

can be deduced directly, at ??? for the important case.

$$y = x^{\epsilon},\ \epsilon > 0, \tag{2}$$

from the PNT. This is how.

Write

$$\pi(x) = \frac{x}{log(x)}(1 + 0(1)), x \to \infty, \tag{3}$$

but not in the less picture and totally worthless equivalent form

$$\pi(x) \sim \frac{x}{logx}.$$

In general, but not always, asymptotic expansions are not useful for establishing inequalities.

Now substitute (3) into (1):

$$\frac{x+y}{log(x+y)}(1+0(1)) \overset{?}{\leq} \frac{x}{logx}(1+0(1)), \quad x, y \to \infty,$$

which breaks into

$$\frac{x}{log(x+y)}(1+0(1)) \overset{?}{\leq} \frac{x}{logx}(1+0(1)), \quad x, y \to \infty,$$
$$\frac{y}{log(x+y)}(1+0(1)) \overset{?}{\leq} \frac{y}{log(y)}(1+0(1)), \quad x, y \to \infty,$$

or

$$\frac{logx}{log(x+y)} \overset{?}{\leq} 1 + 0(1), \quad x, y \to \infty \tag{4a}$$
$$\frac{logy}{log(x+y)} \leq 1 + 0(1), \quad x, y \to \infty \tag{4b}$$

which are entirely obvious once x and y tend to infinity in a *regular* manner. For example, (4) is satisfied for

$$y = x^{\epsilon},\ \epsilon > 0,\ x \to \infty, \tag{5}$$

$$y = log^{\epsilon} x,\ \epsilon > 0,\ x \to \infty, \tag{6}$$

etc., depending upon how the $(x1)$ - tend to 0.

An Estimate For Quantum Odd Double-Factorials

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Abstract

We quantize the classical inequality, $(2n-1)!! < n^n, n > 1$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Theorem 1. For $n \in \mathbf{Z}_{\geq 1}$,

$$\prod_{s=0}^{n-1} [2s+1]_q^{\sim} < ([n]_q^{\sim})^n. \tag{2}$$

Proof. We are going to use the quantum AGM inequality

$$\left[\frac{a_1 + ... + a_n}{n}\right]_q^{\sim} \geq \sqrt[n]{[a_1]_q^n ... [a_n]_q^{\sim}}. \tag{3}$$

Take

$$a_i = 2_i, i = 1, ..., n.$$

then the LIS of (3) is:

$$\left[\frac{\sum_{i=1}^n 2_i}{n}\right]_q^{\sim} = [n+1]_q^{\sim}, \tag{4a}$$

while the RHS of (3) is:

$$\sqrt[n]{\prod_{s=1}^n [2s]_q^{\sim}}. \tag{4b}$$

Thus,

$$([n+1]_q^{\sim})^n > \prod_{s=1}^n [2s]_q^{\sim},$$

and this is (2). ■

An Estimate For The Ratio Of Quantum Square Roots

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Abstract

We quantize the classical inequality, $\sqrt{1+\frac{1}{x}} < 1+\frac{1}{2x}, x>0$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q-q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Theorem 1. Let $q > x > 0$. Then

$$\frac{\sqrt{[x+1]_q^{\sim}}}{\sqrt{[x]_q}} < q^{1/2} + \frac{q^{-1/2-x}}{2[x]_q^{\sim}}. \tag{2}$$

Proof.

Lemma 3.

$$[x+1]_q^{\sim} = q[x]_q^{\sim} + q^{-x}, \quad \forall x \in \mathbf{R}. \tag{4}$$

Proof. Multiplying through by $q-q^{-1}$, (4) becomes:

$$q^{x-1} - q^{x-1} \stackrel{?}{=} q(q^x - q^{-x}) + q^{-x}(q-q^{-1}),$$

which is obvious. ■

With the help of (4), (2) is

$$\sqrt{q + \frac{q^{-x}}{[x]_q^{\sim}}} \stackrel{?}{<} q^{1/2} + \frac{q^{-1/2-x}}{2[x]_q^{\sim}},$$

or

$$q + \frac{q^{-x}}{[x]_q^{\sim}} \stackrel{<}{?} (q^{1/2} + \frac{q^{-1/2-x}}{2[x]_q^{\sim}})^2,$$

or

$$q + \frac{q^{-x}}{[x]_q^{\sim}} \stackrel{?}{<} q + \frac{q^{-x}}{[x]_q^{\sim}} + \frac{q^{-1-2x}}{4([x]_q^{\sim})^2},$$

which is obvious since $q > 0$. ■

An Estimate For The Triple-Factorial In The 2^{nd} Quantization

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Abstract

$\sqrt[n]{(3n)!} < 3n(3n+1)^2$ is quantized.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x, q \in \mathbf{R},\quad q \neq 1, q > 0,$$

be the 2^{nd} quantization of x.

Since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Set

$$[k]!_q^{\sim} = \Pi_{s=1}^{k}[s]_q^{\sim},\ k \in \mathbf{Z}_{\geq 1};\quad [0]!_q^{\sim} = 1.$$

Let

$$m = 3n \geq 1.$$

Then we are to quantize

$$\left(\frac{m(m+1)^2}{4}\right)^{m/3} > m!,$$

or

$$\left(\frac{m(m+1)^2}{4}\right)^{m} > m!^3.$$

Theorem 1. For $m \geq 1$,

$$\left(\frac{[m]_q^{\sim}([m+1]_q^{\sim})^2}{([2]_q^{\sim})^2}\right)^m \geq ([m]!_q^{\sim})^3. \tag{2}$$

Proof. We use induction on m. For $m = 1$, (2) returns:

$$1 = 1,$$

which is true.

Next, the induction step $m \mapsto m + 1$ amounts to:

$$([m+1]!_q)^3 = ([m+1]_q^\sim)^3([m]!_q^\sim)^3 \leq$$
$$\leq ([m+1]_q^\sim)^3\left(\frac{[m]_q^\sim([m+1]_q^\sim)^2}{([2]_q^\sim)^2}\right)^m \stackrel{?}{\leq} \left(\frac{[m+1]_q^\sim([m+2]_q^\sim)^2}{([2]_q^\sim)^2}\right)^{m+1},$$

or

$$([2]_q^\sim)^2 \stackrel{?}{\leq} \left(\frac{[m+1]_q^\sim}{[m]_q^\sim}\right)^m \frac{([m+2]_q^\sim)^{2m+2}}{([m+1]_q^\sim)^{2m+2}}. \tag{3}$$

(3) follows from 1

$$[2]_q^\sim \stackrel{?}{\leq} \left(\frac{[m+1]_q^\sim}{[m]_q^\sim}\right)^m, \quad m \geq 2, \tag{4}$$

which is true and can be seen as follows. Rewrite (4) in a more general form as:

$$[2]_{q^x}^\sim \stackrel{?}{\leq} \left(\frac{[x+1]_q^\sim}{[x]_q^\sim}\right)^x, \quad x \geq 2, \tag{5}$$

because $[2]_{q^x}^\sim$ grows with $x > 0$.

Since

$$[x+1]_q^\sim = q[x]_q^\sim + q^{-x},$$

Bernoulli, gives:

$$\left(\frac{[x+1]_q^\sim}{[x]_q}\right)^x = \left(q + \frac{q^{-x}}{[x]_q^\sim}\right)^x = q^x\left(1 + \frac{q^{-x-1}}{[x]_q^\sim}\right)^x \geq q^x\left(1 + \frac{q^{-x-1}x}{[x]_q^\sim}\right) \stackrel{?}{\geq}$$
$$\stackrel{?}{\geq}, \; q^x + q^{-x},$$

or

$$\frac{x}{[x]_q^\sim} \stackrel{?}{\geq} q^{1-x},$$

or

$$\frac{[x]_q^\sim}{x} \stackrel{?}{\leq} q^{x-1},$$

or

$$q^x - q^{-x} \stackrel{?}{\leq} x(q - q^{-1})q^{x-1} = x(q^x - q^{x-2}),$$

or with

$$h = log\ q > 0,$$
$$2\sum \frac{h^{2s+1}}{(2s+1)!}x^{2s+1} \overset{?}{\leq} x\sum_{n=1}^{\infty}\{x^n - (x-2)^n\}. \tag{6}$$

(6) follows from:

$$2x^s \overset{?}{\leq} x^{2s+1} - (x-2)^{2s+1},$$

or, with

$$y = x - 2 \geq 0,$$
$$2(y+2)^{2s} \overset{?}{\leq} (y+2)^{2s+1} - y^{2s+1}$$

or

$$2\sum_{k=0}^{2s}\binom{2s}{k}y^k 2^{2s-k} \overset{?}{\leq} \sum_{k=0}^{2s}\binom{2s+1}{k}y^k 2^{?+1-k},$$

which follows from:

$$\binom{2s}{k} \overset{?}{\leq} \binom{2s+1}{k}, \quad 0 \leq k \leq 2s,$$

which is true because

$$\binom{n}{k} \leq \binom{n+1}{k} \Leftrightarrow$$
$$\frac{n!}{k!(n-k)} \leq \frac{(n+1)!}{k!(n+1-k)!} \Leftrightarrow$$
$$n+1-k \leq n+1. \quad \blacksquare$$

A New Inequality For The Central Binomial Coefficients

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Abstract

The classical inequality, $\binom{2n}{n} > 2^n$ for $n \geq \mathbf{Z}_{\geq 2}$, is considerably strengthened.

Theorem 1. For $n \in \mathbf{Z}_5$,

$$\binom{2n}{n} > 2.3^n \tag{2}$$

Proof. We use induction on n. For $n = s$, we have:

$$\binom{10}{5} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{2 \cdot 3 \cdot 4 \cdot 5} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{3 \cdot 4} = 3 \cdot 2 \cdot 7 \cdot 6 = 12 \cdot 42 = 504 \stackrel{?}{>} 2 \cdot 35 = \cdot 2 \cdot 243 = 486,$$

which is true

Next, the inductive step:

$$\frac{(2n+1)(2n+2)}{(n+1))n+1} \cdot 2 \cdot 3^n \stackrel{?}{>} 2 \cdot 3^{n+1},$$

or

$$2\frac{2n+1}{n+1} \stackrel{?}{>} 3,$$

or

$$4n + 2 > 3(n+1),$$

or

$$n > 1,$$

which is true. ■

A New Quantum Inequality Of AGM Type

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Abstract

Classically, $\sum_{ixa_{i+1}}^{n} \geq n$, i read mod n. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Set

$$h = logq > 0.$$

Theorem 1. Let $a_1, ..., a_n > 0$. Then

$$\left[\frac{a_1}{a_2}\right] + ... + \left[\frac{a_n}{a_1}\right] \geq n. \tag{2}$$

Equality iff $a_1 = ... = a_n$.

Proof. Call

$$\frac{a_i}{a_i + 1} = \lambda_i,\ i = 1, ..., n, \tag{3}$$

with indices read modulo n. Then (2) becomes, after multiplication by $q-q^{-1}>0$:

$$[\lambda_1]_q^{\sim}+...+[\lambda_n]_q^{\sim}\stackrel{?}{\geq} n(q-q^{-1}),$$

or

$$\sum_{i=1}^{n}(q^{\lambda}-q^{-\lambda})\geq n(q-q^{-1}),$$

or

$$\sum_{i=1}^{n}\sum_{k=0}^{\infty}\frac{h^{2k+1}}{(2k+1)!}\lambda_i^n=\sum_{k=0}^{\infty}\frac{h^{2k+1}}{(2k+1)!}\sum_{i=1}^{n}\lambda_i^k\stackrel{?}{\geq} n\sum_{k=0}^{\infty}\frac{h^{2k+1}}{(2k+1)!}. \tag{4}$$

Since $h>0$, (4) is implied by

$$\sum_{i=1}^{n}\lambda_i^k\stackrel{?}{\geq} n,$$

and this follows from the classical AGM inequality, because

$$\prod_{i=1}^{n}\lambda_i=1!\ \frac{\sum_{i=1}^{n}\lambda_i^k}{n}\geq\sqrt[n]{\prod_{k=1}^{n}\lambda_i^k=1.}\quad\blacksquare$$

An Inductive Proof Of The Formula For The Sum Of Arithmetic Progression In The 2^{nd} Quantization

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Abstract

The quantum version of the classical formula $\sum_{k=0}^{N}(a+kd) = (NH)(a+N\frac{d}{2})$ is given an inductive proof.

Theorem 1.

$$\sum_{k=0}^{N}[a+kd]_q^{\sim} = [N+1]_{q^{d/2}}^{\sim}[a+N\frac{d}{2}]_q^{\sim}. \tag{2}$$

Proof. We use induction on N. For $N=0,(2)$ returns:

$$[a]_q^{\sim} = [a]_q^{\sim},$$

which is true.

The induction step $N \mapsto N+1$ amounts to:

$$[N+1]_{q^{d/2}}^{\sim}[a+N\frac{d}{2}]_q^{\sim} + [a+(N+1)d]_q^{\sim} \stackrel{?}{=} [(N+2)]_{q^{d/2}}[a+(N+1)\frac{d}{2}]_q^{\sim},$$

or, with

$$Q = q^a, P = q^{\mathcal{D}}, \frac{d}{2} = \mathcal{D}, \tag{3}$$

to:

$$\frac{[P^{\mathcal{D}(N+1)} - P^{-\mathcal{D}(N+1)}]}{P-P^{-1}}\ \frac{(QP^N - Q^{-1}P^{-N})}{q-q^{-1}} + \frac{QD^{2(N+1)} - Q^{-1}P^{-2(N+1)}}{q-q^{-1}} \stackrel{?}{=}$$
$$\stackrel{?}{=} \frac{P^{N+2}-P^{-N-2}}{P-P^{-1}} \cdot \frac{QP^{N+1}-Q^{-1}P^{-N-1}}{q-q^{-1}},$$

or

$$(P^{N+1}-P^{-N-1})(QP^N - Q^{-1}P^{-N}) + (P-P^{-1})(QP^{2N+2} - Q^{-1}P^{-2N-2}) \stackrel{?}{=}$$
$$\stackrel{?}{=} (P^{N+2}-P^{-N-2})(QP^{N+1} - Q^{-1}P^{-N-1}). \tag{4}$$

(4) is linear in Q, Q^{-1}, and is invariant with respect to the change $Q \to Q^{-1}$, so it's enough to compare Q-coefficients:

$$(P^N + 1 - P^{-N-1}) + (P - P^{-1})P^{2N+2} \stackrel{?}{=} (P^{N+2} - P^{-N-2})P^{N+1},$$

or

$$(P^{2N+1} - P^{-1}) + (P^{2N+3} - P^{2N+1}) \stackrel{?}{=} P^{2N+3} - P^{-1},$$

which is obvious. ■

An Inequality Between Two Quantum 1st Degree Polynomials

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Abstract

Classically, $2(x+1) > 2x$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

It is obviously *not* true that

$$[2]_q^{\sim}[x+1]_q^{\sim} > [2x]_q^{\sim}.$$

Theorem 1. For $x \in \mathbf{R}$,

$$[2]_{q^{x-1}}^{\sim}[x+q]_q^{\sim} > [2x]_q^{\sim}. \tag{2}$$

Proof. Multiplying through by $q - q^{-1} > 0$, (2) becomes:

$$(q^{x-1} + q^{1-x})[q^{x+1} - q^{-1-x}] \stackrel{?}{>} q^{2x} - q^{-2x},$$

or

$$q^{2x} - q^{-2} + q^2 - q^{-2x} \stackrel{?}{>} q^{2x} - q^{-2x},$$

or

$$q^2 > q^{-2},$$

which is true because $q > 1$. ■

An Inequality For The Quantum Central Binomial Coefficients

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Abstract

Classically, $2(x+1) > 2x$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Set

$$[k]!_q^{\sim} = [1]_q^{\sim}...[k]^{\sim},\ k \in \mathbf{Z}_{\geq 1};\quad [0]! = 1,$$
$$\begin{bmatrix} m \\ k \end{bmatrix}_q^{\sim} = \frac{[m]_q^{\sim}}{[k]!_q^{\sim}[m-1]!_q^{\sim}},$$
$$< a^n >_q^{\sim} = \prod_{k=1}^{n} [a]_{q^k}^{\sim},\ n \in \mathbf{Z}_{\geq 1}.$$

Theorem 1.

$$\begin{bmatrix} 2n \\ n \end{bmatrix}_q^{\sim} > < 2^n >_q^{\sim},\quad n \in \mathbf{Z}_{\geq 2}, \tag{2}$$

Proof. We use induction on n. For $n = 2$, we get:

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q^{\sim} = \frac{[4]_q^{\sim}[3]_x^{\sim}}{[2]_q^{\sim}} = [2]_{q^2}^{\sim}[3]_q^{\sim} \overset{?}{>} < 2^2 >_q^{\sim} = [2]_q^{\sim}[2]_{q^2}^{\sim},$$

or

$$[3]_q^{\sim} > [2]_q^{\sim},$$

which is true.

The inductive step $n \mapsto n+1$ yields:

$$\begin{bmatrix} 2n+2 \\ n+1 \end{bmatrix}_q^{\sim} = \frac{[2n+2]_q^{\sim}[2n+1]_q^{\sim}}{[n+1]_q^{\sim}[n+1]_q^{\sim}} \begin{bmatrix} 2n \\ n \end{bmatrix}_q^{\sim} > \ [2]_{q^{n+1}}^{\sim} \frac{[2n+1]_q^{\sim}}{[n+1]_q^{\sim}} < 2^n >_q^2 >< 2^{n+1} >_q^{\sim},$$

or

$$\frac{[2n+1]_q^{\sim}}{[n+1]_q^{\sim}} \overset{?}{>} 1,$$

which is obvious, because

$$a > b > 0 \ \ \Rightarrow \ [a]_q^{\sim} > [b]_q^{\sim}. \qquad \blacksquare$$

A Note On A General Classical Inequality

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Abstract

We generalize the inequality equivalent to the observation that $(1+\frac{1}{n})^n$ increases with n.

It is well known that

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}, \quad n \in \mathbf{Z}_{\geq 0}. \tag{1}$$

Rewriting (1) as

$$(n+1)^{2n+1} < n^n(n+2)^{n+1},$$

one is naturally led to ask, for which a

$$(n+a)^{2n+1}n^n(n+2a)^{n+1}, \ n \geq 1. \tag{2}$$

Certainly, $a=1$ is true by the classical result, and $a=0$ turns (2) into an equality.

Theorem 3. The inequality (2) is true for any

$$a < \frac{1}{6}. \tag{4}$$

Proof. We have, taking log of (2):

$$(2n+1)log(n+a) \stackrel{?}{\leq} nlogn + (n+1)log(n+2a) \tag{5}$$

Now,

$$log(n+a) = log[n(1+\frac{a}{n})] < logn + \frac{a}{n}, \tag{6}$$

because, as is well-known,

$$log(1+x) < x.$$

Further,

$$log(n+2a) = log[n(1+\frac{2a}{n})] \pm logn + log(1+\frac{1}{\frac{n}{2a}}) >$$
$$> logn + \frac{1}{\frac{n}{2a}+1} = logn + \frac{2a}{n+2a}, \tag{7}$$

because, as is less well known,

$$log(1+\frac{1}{x}) > \frac{1}{x+1}, \quad x>0. \tag{8}$$

So, 5 follows from:

$$(2n+1)[logn+\frac{a}{n}] \stackrel{?}{<} nlogn+(n+1)[logn+\frac{2a}{n+2a}],$$

or

$$\frac{2n+1}{n}a \stackrel{?}{<} \frac{n+1}{n+2a}2a,$$

or

$$n+2a \stackrel{?}{<} \frac{n(n+1)}{2n+1}2,$$

or

$$2a \stackrel{?}{<} n\Big[\frac{2(n+1)}{2n+1}-1] = \frac{n}{2n+1}. \tag{9}$$

The RHS of (9), as $n\to\infty$, tends to $1/2$. So, $a<\frac{1}{4}$ at a minimum. But

$$\frac{n}{2n+1} = \frac{1}{2} - \frac{1}{2(2n+1)},$$

which *increases* with n. So, we have to take the minimal *possible* $n, n=1$. We get

$$2a < \frac{1}{3}, \tag{10}$$

that is, (4). ■

Another 3-Variable Quantum Inequality

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Abstract

The classical inequality $a^3+b^3+c^3+15\,abc \le 2(a+b+c)(a^2+b^2+c^2),\ a,b,c\ge 0,$ is quantized.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q-q^{-1}},\ x,q\in\mathbf{R},\ q\ne 1,\ q>0.$$

be the 2^{nd} quantization of x.

That classical inequality can be rewritten in the suggestive form

$$15abc \le (a^3+b^3+c^3) + 2\{a^2(b+c)+b^2(a+c)+c^2(a+b)\}.$$

Theorem 1. Let $[a]$ be either $[a]_q$ or $[a]_q^{\sim}$. Then, for any $\theta\ge 0$,

$$3(\theta+1)[a][b][c] \le [a]^3+[b]^3+[c]^3+\theta\Big\{[a]^2\left[\frac{\ell+c}{2}\right]+$$
$$+[b]^2\left[\frac{a+c}{2}\right]+[c]^2\left[\frac{a+b}{2}\right]\Big\}. \tag{2}$$

Proof. We use the quantum AGM inequality:

$$\left[\frac{x+y}{2}\right] \ge \sqrt{[x][y]}.$$

Thus,

$$[a]^2\left[\frac{b+c}{2}\right] \ge [a]^2\sqrt{[b][c]} = \sqrt{[a][b][c]}[a]^{3/2},$$

and the RHS of (2) is thus $\ge$ then

$$3[a][b][c]+\sqrt{[a][b][c]}\theta([a]^{3/2}+[b]^{3/2}+[c]^{3/2}) \ge$$
$$\ge 3[a][b][c]+\sqrt{[a][b][c]}\theta 3^3\sqrt{[a]^{3/2}[b]^{3/2}[c]^{3/2}} =$$
$$=(3+3\theta)[a][b][c].\ \blacksquare$$

Our original inequality was for $\theta=4$.

Another Divisibility By 64 In Quantum Domain

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Abstract

The classical criterion, $3^{2n+3}+40\,n-27\equiv 0(mod\,64), n\geq 0$, is quantized.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\ x\in\mathbf{R},\ q\neq 0,\ \pm 1,$$

be the 2^{nd} quantization of x.

Theorem 1.

$$([3]_q^{\sim})^{2n+3}+[5]_q^{\sim}[2]_q^{\sim}[4]_q^{\sim}n-([3]_q^{\sim})^3\equiv 0\ (mod\ ([2]_q^{\sim}[4]_q^{\sim})^2) \tag{2}$$

Proof. We use induction on. Denote the LHS of (2) by A_n. Then

$$A_0=0, \tag{3}$$

and

$$A_{n+1}-A_n=([3]_q^{\sim})^{2n+3}\{([3]_q^{\sim})^2-q\}+[5]_q^{\sim}[2]_q^{\sim}[4]_q^{\sim}\stackrel{?}{\equiv}0(mod\ ([2]_q^{\sim}[4]_q^{\sim})^2),$$

or, since

$$([3]_q^{\sim})^{2n+3}+[5]_q^{\sim}\stackrel{?}{\equiv}0\ (mod\ [2]_q^{\sim}[4]_q^{\sim}) \tag{4}$$

Denote by B_n the LHS of (4). Since

$$B_0=([3]_q^{\sim})^3+[5]_q^{\sim}=(q^2+1+q^{-2})^3+[5]_q^{\sim}=$$
$$[2]_q^{\sim}[4]_q^{\sim}=([2]_q^{\sim})^2[2]_{q^2}^{\sim}$$

and $[2]_q^{\sim}$ and $[2]_{q^2}^{\sim}$ are coprime in $Q[q,q^{-1}]$, we need, separately:

$$B_0\equiv 0\ (mod\ [2]_{q^2}^{\sim}), \tag{5}$$
$$B_0\equiv 0\ (mod\ ([2]_q^{\sim})^2). \tag{6}$$

Then we will be done, because

$$B_{n+1}-B_n([3]_q^{\sim})^{2n+3}[2]_q^{\sim}[4]_q^{\sim}.$$

So, since

$$[3]_q^{\sim} \equiv 1 \ (mod \ [2]_{q^2}^{\sim}),$$

we find:

$$B_0 = ([3]_q^{\sim})^3 + [5]_q^{\sim} \equiv 1 + (4, 2, 0, -2, -4) \equiv$$
$$1 + q^{-4} = q^{-2}[2]_{q^2}^{\sim} \equiv 2 \ (mod \ [2]_{q^2}^{\sim}).$$

Next,

$$[3]_q^{\sim} = ([2]_q^{\sim})^2 - 1,$$

so

$$B_0 = ([3]_q^{\sim})^3 + [5]_q^{\sim} \equiv -1 + (4, 2, 0, -2, -4) =$$
$$= (4, 2) + (-2, -4) = q^3[2]_q^{\sim} + q^{-3}[2]_q^{\sim} \stackrel{?}{\equiv} 0 \ (mod \ ([2]_q^{\sim}))^2) =$$
$$q^3 + q^{-3} \stackrel{?}{=} 0 (mod \ [2]_q^{\sim}),$$

which is obvious since

$$\frac{q^3 + q^{-3}}{[2]_q^{\sim}} = \frac{q^3 + q^{-3}}{q + q^{-1}} = q^2 - 1 + q^{-2}. \quad \blacksquare$$

Another Goldbach-Like Conjecture On Prime Numbers

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Abstract

The original Goldbach Conjecture is too timid. We formulate its strengthening.

The original Goldbach Conjecture, that every even number ≥ 4 is a sum of 2 primes, is too imprecise and easily proven. The following Conjecture is hard.

Let $P = P_{m-1}, P' = P_m, P'' = P_{m+1}$ be three successive prime numbers, where P_m is the prime A_m.

Conjecture 1. If $p'' - p' \neq 2$, (i.e., p', and p'' are not two primes) then

$$p - p + 1$$

is a prime.

It's true for $p_n \leq 1.2710^6$.

A Proof Of Sun's Conjecture

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Sun's conjecture, that for $a \geq 1$, every integer μ of appropriate parity satisfies $M = p + ax^2$, where p is a prime, is proved.

Given a, M must be:
1) if a is even, then M is odd;
2a) if a is odd, then M is even for x odd;
2b) if a is odd, then M is odd for x even.

With these restrictions in mind, the Sun Conjecture is: given a, ? sufficiently large M of appropriate parity satisfies: $M = p + ax^2$, where p is a prime.

The Conjecture looks forbidding, but is, in fact, trivial.

Theorem 1. The number of solutions (p, x), given M, is $\sim$

$$\frac{2M^{V_2}}{a^{1/2}logM}. \tag{2}$$

Proof. Set

$$\delta(x) = \pi(x) - \pi(x-2), \tag{3}$$

where $\pi(x) = \sum_{p \leq x} 1$ is the prime-counting function. I recently proved, in settling the Goldbach conjecture, that, for x odd,

$$\delta(x) \sim \frac{2}{logx}. \tag{4}$$

Hence, if we denote by $n(M)$ the number of solutions of $M = p + ax^2$, we have:

$$n(M) = \sum_{x=1}^{ub} \delta(M - ax^2),$$

where $u.b. = \left[\sqrt{\frac{M}{a}}\right] \sim \sqrt{\frac{M}{a}}$. THus,

$$n(M) \sim \sum^{\sqrt{M/a}_{x=1}} \frac{2}{logM - \frac{ax^2}{M}} \sim \frac{2}{logM} \sum \left(1 + \frac{ax^2}{MlogM}\right) \sim$$

$$\sim \frac{2}{logM}\left(\sqrt{\frac{M}{a}} + a\left(\sqrt{\frac{M}{a}}\right)^2 \frac{1}{3MlogM}\right) \sim \frac{2}{logM}\left(\frac{M^{1/2}}{a^{12}} + \frac{M^{1/2}}{3a^{12}logM}\right) \sim$$

$$\sim \frac{2M^{1/2}}{a^{1/2}logM},$$

as claimed. ■

A Quadratic Bound From Below For The Powers Of Two Under Symmetric Quantization

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Abstract

$2^n > n^2$ for $n \geq 5$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x, q \in \mathbf{R}, \quad q \neq 1, \ q > 0,$$

be the 2^{nd} (symmetric) quantization of x:

Since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Theorem 1. For $n \geq 4$,

$$([2]_{q^2}^{\sim})^n > ([n]_q^{\sim})^2. \tag{2}$$

Proof. Notice that in contrast to the classical case $n \geq s$, now $n \geq 4$.

We use induction on n; (2) is; for $n = 4$:

$$([2]_{q^2}^{\sim})^2 \stackrel{?}{>} [4]_q^{\sim} = [2]_q^{\sim}[2]_{q^2}^{\sim},$$

or

$$[2]_{q^2}^{\sim} \stackrel{?}{>} [2]_q^{\sim},$$

or

$$q^2 + q^{-2} \stackrel{?}{>} q + q^{-1},$$

or, with

$$z = q + q^{-1} > 2,$$
$$z^2 - 2 \overset{?}{>} z,$$

or

$$z^2 - z - 2 \overset{?}{>} 0,$$

or

$$(z+1)(z-2) \overset{?}{>} 0,$$

which is true for $z > 2$.

Next, the inductive step $n \mapsto n+1$ amounts to:

$$[2]^{\sim}_{q^2} \overset{?}{>} \left(\frac{[n+1]^2_q}{[h]^{\sim}_q} \right)^2. \tag{3}$$

Since

$$[n+1]^{\sim}_q = q[n]^{\sim}_q + q^{-n},$$

(3) is:

$$q^2 + q^{-2} \overset{?}{>} \left(q + \frac{q^{-n}}{[n]^{\sim}_q} \right)^2 = q^2 + \frac{2q^{-n}}{[n]^{\sim}_q} + \frac{q^{2n}}{([n]^{\sim}_q)^2},\ n \geq 4,$$

or

$$1 \overset{?}{>} \frac{2q^{2-n}}{[n]_q} + \frac{q^{2-2n}}{([n]^{\sim}_q)^2},\ n \geq 4. \tag{4}$$

Since $n \geq 4$ and $q > 1$, (4) follows from:

$$1 \overset{?}{>} \frac{2}{[4]^{\sim}_q} + \frac{1}{([4]^{\sim}_q)^2}. \tag{5}$$

Since

$$[4]^{\sim}_q > 4,$$

(5) results from

$$1 \overset{?}{>} \frac{2}{4} + \frac{1}{4^2} = \frac{1}{2} + \frac{1}{16} = \frac{9}{16},$$

which is true. ■

A Quadratic Inequality For Quantum Numbers

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Abstract

We quantize the trivial inequality $1 - a^2 < 1$, for $0 < a < 1$, in a nontrivial manner.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Theorem 1. For $0 < a < 1$,

$$[1-a]_q^{\sim}[1+a]_q^{\sim} < 1 \tag{2}$$

Proof. Multiplying through by $(q - q^{-1})^2 > 0$, we get:

$$(q^{1-a} - q^{a-1})(q^{a+1} - q^{-a-1}) \stackrel{?}{<} (q - q^{-1})^2,$$

or

$$q^2 - q^{2a} - q^{-2a} + q^{-2} \stackrel{?}{<} (q^2 + q^{-2} - 2),$$

or

$$2 < q^{2n} + q^{-2a},$$

which is true, as $q^2 \neq 1$. ■

A Quantum Bernoulli Inequality

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Abstract

The classical Bernoulli inequality is quantized.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

The classical Bernoulli inequality is:

$$(1+x)^n > 1 + nx, \quad x > -1,\ n \in \mathbf{Z}_{\geq 2}.$$

Theorem 1. Let $x > -1,\ n \in \mathbf{Z}_{\geq 2}$,

$$([1+x]_q^{\sim})^n > [1+nx]_q^{\sim}. \tag{2}$$

Proof. We use induction on n. For $n = 1$(not $n = 2$), we have:

$$([1+x]_q^{\sim})^n \ \geq [1+nx]_q^{\sim}$$

then

$$([1+x]_q^{\sim})^{n+1} > [1+(n+1)x]_q^{\sim}.$$

Indeed,

$$([1+x]_q^{\sim})^{n+1} = ([1+x]_q^{\sim})^n [1+x]_q^{\sim} \geq$$
$$\geq [1+nx]_q^{\sim}[1+x]_q^{\sim} \overset{?}{>} [1+x(n+1)]_q^{\sim},$$

or, multiplying through by $(q-q^{-1})^2>0$,

$$(q^{1+nx}-q^{-1-nx})(q^{1+x}-q^{-1-x}) \stackrel{?}{>} (q-q^{-1})[q^{1+x(n+1)}-q^{-1-x(n+1)}],$$

or

$$[q^{2+(n+1)x}+q^{-2-(n+1)x}]-[q^{(n-1)x}-q^{-(n-1)x}] \stackrel{?}{>}$$
$$\stackrel{?}{>} [q^{2+x(n+1)}+q^{-2-x(n+1)}]-[q^{x(n+1)}-q^{-x(n+1)}],$$

or

$$q^{x(n+1)}+q^{-x(n+1)} \stackrel{?}{>} q^{(n-1)x}+q^{-x(n-1)}$$

which is true because q^z+q^{-z} is an increasing function of $|z|$. ■

A Quantum Chebyshev Inequality

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Abstract

Let $\boldsymbol{a} = (a_1 \leq a_2 \leq ... \leq a_n), (\boldsymbol{b} - (b_1 \leq b_2 \leq ... \leq b_n), \sigma - a)$ permutation of the bi's. We quantize Chebyshev inequality $\boldsymbol{a}\ \boldsymbol{b}_0 \leq \boldsymbol{a}\ \boldsymbol{b}$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Set

$$h = log\ q > 0.$$

Theorem 1. Let $\boldsymbol{c} = (c_1, ...c_n) = \mathrm{b}_0$.

Then

$$[a_1 b_1]_q^{\sim} + ... + [a_n b_n]_q^{\sim} + ... + [a_n c_n]_q^{\sim} \tag{2}$$

Proof. The Theorem is predicated on the

Lemma 3.

$$[a_1 b_1]_q^{\sim} + [a_q b_2]_q^{\sim} \geq [a_1 b_2]_q^{\sim} + [a_2 b_1]_q^{\sim}. \tag{4}$$

Proof. Multiplying through by $(q-q^{-1})>0$, (4) becomes:

$$(q^{a_1b_1}-q^{-a_1b_1})+(q^{a_2b_2}-q^{-a_2b_2})\overset{?}{\geq}(q^{a_1b_2}-q^{-a_1b_2})(q^{a_2b_1}-q^{-a_2b_1}),$$

or

$$2\sum_{n=0}^{\infty}\frac{h^{2n+1}}{(2n+1)}\Big[(a_1b_1)^{2n+1}+(a_2b_2)^{2n+1}]\overset{?}{\geq}2\sum_{n=0}^{\infty}\frac{h^{2n+1}}{(2n+1)!}\Big[(a_1b_2)^{2n+1}+(a_2b_1)^{2n+1}\Big]. \quad (5)$$

Since $h>0$, (5) follows from

$$(a_1b_1)^{2n+1}+(a_2b_2)^{2n+1}\overset{?}{\geq}(a_1b_2)^{2n+1}+(a_2b_1)^{2n+1}. \quad (6)$$

Call

$$A_i=a_i^{2n+1}, B_i=b_i^{2n+1},\ i=1,2.$$

Then

$$A_1\leq A_2,\ B_1\leq B_2, \quad (7)$$

and (6) becomes:

$$A_1B_1+A_2B_2\overset{?}{\geq}A_1B_2+A_2B_1,$$

or

$$(A_1-A_2)(B_1-B_2)\overset{?}{\geq}0,$$

which is obvious in view of (7). ■■

A Ration Of Two Simple Linear Polynomials Is Estimated Form Below In The 2^{nd} Quantization

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Abstract

For $t > 0$, $\frac{1+t}{t} > 1$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x, q \in \mathbf{R}, \quad q \neq 1, \ q > 0,$$

be the 2^{nd} (symmetric) quantization of x:

Since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Theorem 1. For $t > 0$,

$$\frac{[1+t]_q^{\sim}}{[t]_q^{\sim}} > q = \lim_{t\to\infty} \frac{[1+t]_q^{\sim}}{[t]_q^{\sim}}. \tag{2}$$

Proof. (2) is:

$$[1+t]_q^{\sim} \stackrel{?}{>} q[t]_q^{\sim},$$

or

$$q^{1+t} - q^{-1-t} \stackrel{?}{>} q(q^t - q^{-t}),$$

or

$$q^{1-t} \stackrel{?}{>} q^{-1-t},$$

or

$$1 > q^{-1},$$

which is true because $q > 1$. ■

A Relation Between Quantum Quadratic Polynomials

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Abstract

The classical relation, $x^2-(x-1)(x-1)=1$, is quantized.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}}, x\in\mathbf{R},\ q\neq 0,\ \pm 1,$$

be the second quantization of x.

Classically, we trivially have:

$$x^2-(x-1)(x+1)=1.$$

This can be thought of as the series containing the Catalan equation

$$3^y-2^4=1,\quad (y,z)=2,3.$$

Theorem 1.

$$([x]_q^{\sim}-1)([x]_q^{\sim}+1)=[x-1]_q^{\sim}[x+1]_q^{\sim}. \tag{2}$$

Proof. (2) can be rewritten as

$$([x]_q^{\sim})^2-[x-1]_q^{\sim}[x+1]_q^{\sim}\stackrel{?}{=}1. \tag{3}$$

Multiplying (3) through by $(q-q^{-1})^2$, we get:

$$(q^x-q^{-x})^2-(q^{x-1}-q^{1-x})(q^{x+1}-q^{-1-x})\stackrel{?}{=}(q-q^{-1})^2,$$

or

$$q^{2x}+q^{-2x}-2-[(q^{2x}+q^{-2x})-(q^2+q^{-2})]\stackrel{?}{=}q^2+q^{-2}-2,$$

which is true. ■

A Remark On A Prime Inequality

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Abstract

We show that, p_n being the n^{th} prime, $\frac{logp_{n+1}}{logp_n} > \frac{log(n+1)}{logn}$ for $m >> 0$.

We use the asymptotic expansion

$$p_n = nf(n),$$

$$f(n) = logn + \sum_{i>0}^{n} \frac{P_i}{logi_n},\ P_0 = loglogn - 1,$$

where P_i for $i > 0$ is a polynomial in $w = loglogn$ of degree $= i$.

I proved recently that

$$f(m+1) = f(m) + \frac{1}{m}[1 + 0(1)].$$

The RHS of

$$\frac{logp_{m+1}}{logp_m} > \frac{log(m+1)}{logm}, \tag{1}$$

is, modulo $1/m^2$ - terms:

$$\frac{log(m+1)}{logm} = \frac{logm + \frac{1}{m}}{logm} = 1 + \frac{1}{mlogm}. \tag{2}$$

For the LHS of (1), we get:

$$logp_m = logm + logf(m) = logm + loglogm + 0(1),$$

so the LHS of (1) is

$$\frac{log(m+1) + loglog(m+1)}{logm + loglogm} + 0(1) =$$

$$= \frac{logm + loglogm + \frac{1}{m}(1 + \frac{1}{logm})}{logm + loglogm} =$$

$$= 1 + (1 + \frac{1}{m})\frac{1}{logm + loglogm},$$

and, obviously,

$$\frac{m+1}{m}\frac{1}{logm + loglogm} > \frac{1}{mlogm},$$

because

$$(m+1)logm > logm + loglogm.$$

A Simple Linear Inequality For The Symmetric Quantization

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Abstract

$1+t=1+t$ is quantized.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\quad x,q\in\mathbf{R},\quad q\neq 1,\ q>0,$$

be the 2^{nd} quantization of x:

Since

$$[x]_{q^{-1}}^{\sim}=[x]_q^{\sim},$$

we can take

$$q>1$$

without loss of generality.

Theorem 1. For $t>0$,

$$[1+t]_q^{\sim}>1+[t]_q^{\sim}. \tag{2}$$

Proof. (2) is:

$$q^{1+t}-q^{-1-t}\overset{?}{>}(q-q^{-1})+(q^t-q^{-t}),$$

or

$$(q-1)q^t\overset{?}{>}q^{-1}(q+1)(q-1)+q^{-1-t}(1-q),$$

or

$$q^t\overset{?}{>}1+q^{-1}-q^{-1-t},$$

or

$$q^t+q^{-1-t}\overset{?}{>}1+q^{-1},$$

or

$$q^t-1\overset{?}{>}q^{-1-t}(1-q^t),$$

or

$$1>-q^{-1-t},$$

which is obvious. ■

A Simple Quadratic Relation In The Symmetric Quantization

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Abstract

$x(x+1)-y(y+1)=(x-y)(x+y+1)$. This formula is quantized.

Let

$$[x]_q^{\sim}=\frac{q^x-x^{-x}}{q-q^{-1}},\ x\in\mathbf{R}, q\neq 1, q>0,$$

be the 2^{nd} (symmetric) quantization of x.

Theorem 1.

$$[x]_q^{\sim}[x+1]_q^{\sim}-[y]_q^{\sim}[y+1]_q=[x-y]_q^{\sim}[x+y+1]_q^{\sim}. \tag{2}$$

Proof. Multiplying through by $(q-q^{-1})^2$, we get:

$$(q^x-q^{-x})(q^{x+1}-q^{-x-1})-(q^y-q^{-y})(q^{y+1}-q^{-y-1})\stackrel{?}{=}$$
$$\stackrel{?}{=}(q^{x-y}-q^{y-x})(q^{x+y+1}-q^{-1-x-y}),$$

or

$$(q^{2x+1}+q^{2x-1})-(q+q^{-1})-\{(q^{2y+1}+q^{-2y-1})-(q+q^{-1})\}\stackrel{?}{=}$$
$$\stackrel{?}{=}(q^{2x+1}+q^{-2x+1})-(q^{2y+1}-q^{-2y-1}),$$

which is true. ■

A Space Generated By Two Distinct Primes

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Abstract

Two distinct primes $p < p'$ generate all but a finite number of elements from $\mathbf{Z}$.

Let

$$p < p'$$

be two distinct primes. We are interested in the space of linear combinations

$$ap + bp',$$
$$a, b \in \mathbf{Z}_{\geq 0}.$$

Theorem 1. Denote that space by S. Then S contains every integer $\geq N(p, p')$, where

$$\mathcal{N}(p, p') = (p-1)^2 p' \tag{2}$$

Proof. Let $r_o \equiv p'(mod\ p), 1 \leq p' \leq p-1$. Let $\bar{k} \in \mathbf{Z}_p$ be such that $\bar{k}p_0 = 1$. Let $1 \leq k \leq p-1$ be the representative of k. Let $N \equiv r(mod\ p)$. Then $N - kp' \equiv 0(mod\ p), ?_0 N = sp + krp'$

Since

$$kr \leq (p-1)^2,$$
$$\mathcal{N}(p, p') = (p-1)^2 + p'.$$

is sufficient, even though this bound is probably vastly over the true one except for $p = 2$. ■

Remark 3. The Theorem is well-known in greater generality when p and p' are any coprime numbers. The point is that in the prime case the proof is very short.

A Strengthening Beyond Reason Of The Goldbach Conjecture

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Abstract

Let M be even, and $P_m < MMP_{m+1}$. We conjecture that at least one of the numbers $M - P_m, M - P_{m-1}$, is prime.

Let M be even, and

$$P_m < M < P_{m+1}.$$

The Goldbach COnjecture says that the number of solutions of the equation

$$M = P + P', \tag{1}$$

P and P' being prime, is ≥ 1. I recently showed that the Goldbach Conjecture vastly under states the matters, and that number of solutions is

$$\sim \frac{2M}{log^2 M}.$$

But what is the object whose number of solutions is $1, 2$, or bounded?

Conjecture 2. Among the numbers $M - P_m$ and $M - P_{m-1}$ at least one is prime. (If $P_{m+1} = P_{m+2}$, then $M > P_m + 1$.)

Example/Supporting Evidence. Denote $p = p'_m, p' = p_{m-1}$, then

$$
\begin{array}{ll}
4-(2=p)=2, & 40-(37=p)=3,\\
6-(3=p')=3, & 42-(37=p')=5,\\
8-(5=p)=3, & 44-(41=p')=3,\\
10-(7-p)=3, & 46-(43=p)=3,\\
12-(7=p')=5, & 48-(43=p')=5,\\
14-(11=p')=3, & 50-(47=p)=3,\\
16-(13=p)=3, & 52-(47=p)=5,\\
18-(13=p')=5, & 54-(47=p')=7,\\
20-(17=p')=3, & 56-(53=p)=3,\\
22-(19-p')=3, & 58-(53=p)=5,\\
24-(19-p')=5, & 60-(53=p')=7,\\
26-(23=p)=3, & 62-(59=p)=3,\\
28-(23=p)=5, & 64-(61=p)=3,\\
30-(23=p')=7, & 66-(61=p)=5,\\
32-(29=p')=3, & 68-(61=p')=7,\\
34-(29=p')=5, & 70-(67=p)=3,\\
36-(31=p)=5, & 72-(67=p')=5,\\
38-(31=p')=7, & 74-(71=p')=3.
\end{array}
$$

It's been verified numerically up to $M \leq 1.27.10^6$ by Joshua A. Kupershmidt.

A Strengthening Of The Classical Bernoulli Inequality

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Abstract

Taking 1-jet of the quantum Bernoulli inequality, we strengthen the classical one.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

The classical Bernoulli inequality:

$$(1+x)^n > 1 + nx, \quad x > -1,\ n \in \mathbf{Z}_{\geq 2}. \tag{1}$$

has the following quantum version:

$$([1+x]_q^{\sim})^n > [1+nx]_q^{\sim},\ x > -1, \quad n \in \mathbf{Z}_{\geq 2},\ q > 1. \tag{2}$$

Using the formula

$$[x]_q^{\sim} = x + \frac{h^2}{6}(x^3 - x) + \mathcal{O}(h^4),$$
$$h = log\ q,$$

we transform (2), modulo $\mathcal{O}(h^4)$ terms, into:

$$< (1+x)\{1 + \frac{h^2}{6}[(1+x)^2 - 1]\} >^n < (1+hx)\{1 + \frac{h^2}{6}[(1+nx)^2 - 1]\}.$$

or

$$(1+x)^n\{1+\frac{h^2}{6}nx(x+2)\} > (1+nx)\{\frac{1+h^2}{6}nx(nx+2)\},$$

or

$$(1+x)^n(x+2) > (1+nx)(nx+2), x > -1, \quad n \in \mathbf{Z}_{\geq 2} \tag{3}$$

This is our strengthened form of the Bernoulli inequality. It *is* strengthening, because

$$x+2 < nx+2, \quad x > 0, \quad n \in \mathbf{Z}_{\geq 2}. \quad \blacksquare \tag{4}$$

A Symmetric Proof Of The Young Inequality

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Abstract

Let $p, q > 1$ with $frac1p + \frac{1}{q} = 1$. Then, for $x, y \geq 0$, $\frac{x^p}{p} + \frac{y^q}{q} \geq xy$. We provide a simple proof of that.

The Young inequality is unique in that it has no really simple analytic proof. (The geometrical one is, of course, trivial). We aim to change that.

Set

$$X = x^p,\ Y = y^q. \tag{1}$$

Then the inequality becomes:

$$\frac{1}{p}X + \frac{1}{q}Y \stackrel{?}{\geq} X^{1/p}Y^{1/q}. \tag{2}$$

For $X = Y$, (2) becomes:

$$X \geq X,$$

because

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{3}$$

Now (2) is *symmetric* with respect to the change $X \to Y, Y \to X, p \to q, q \to p$. Since $X \neq Y$, let's call X the *largest* of X, Y, and set

$$t = X/Y > 1. \tag{4}$$

Then (2), homogeneous of degree 1, becomes:

$$\frac{1}{p}t + \frac{1}{q} \stackrel{?}{\geq} t^{1/p}. \tag{5}$$

(5) is true for $t = 1$. And its t-derivative is:

$$\frac{1}{p} \stackrel{?}{\geq} \frac{1}{p}t^{1/p-1},$$

or

$$1 \geq \frac{1}{t^{1-\frac{1}{p}}},$$

which is obvious since $t > 1$ and $1 - \frac{1}{p} > 0$. Thus, (5) is true. ■

Estimate For The Product Of Quantum Factorials

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Abstract

We quantize the inequality $\prod_{k=1}^{n}[(2n)!] > [(n+1)!]^n,\ n > 1.$

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Set

$$[k]!_q^{\sim} = [1]_q^{\sim}...[k]^{\sim},\ k \in \mathbf{Z}_{\geq 1};\quad [0]! = 1,$$

Theorem 1.

$$\prod_{k=1}^{n}[2k]!_q^{\sim} > ([n+1]!_q^{\sim})^n,\quad n \in \mathbf{Z}_{\geq 2}. \tag{2}$$

Proof. We use induction on n. For $n = s$, (2) returns:

$$[2]!_q^{\sim}[4]_q^{\sim} \stackrel{?}{>} ([3]!_q^{\sim})^2,$$

or

$$[4]_q^{\sim} > [3]_q^{\sim},$$

which is true, because

$$a > b > 0 \ \Rightarrow [a]_q^{\sim} > [b]_q^{\sim}. \tag{3}$$

The induction step $n-1 \mapsto n$ is:

$$\prod_{k=1}^{n} [2k]!_q^{\sim} >$$

$$[2n]!_q^{\sim} ([n]!_q^{\sim})^{n-1} = \frac{[2n]!_q^{\sim}}{[n]!_q^{\sim}} q([n]!_q^n)^n = [2n]_q^{\sim} ... [n+1]_q^{\sim} ([n]!_q^{\sim})^n >$$

$$> ([n+1]_q^{\sim})^n ([n]_q^{\sim})^n = ([n+1]!)^n,$$

which is (2). ■

Invariance Of The Boundary Relations Of The Basic Numerical Triangle Made Up Of Two Intermingled Arithmetic Progressions

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Abstract

Let $a, b, h \in \mathbf{R}$ or $\mathbf{c}$. Let $\{x_n\} = a, b; a+h, b+h; a+2h, b+2h; ...$ consider the triangle

$$\begin{array}{ccccc} & & a & & \\ & b & & a+h & \\ b+h & & a+2h & & b+2h \\ & \cdots & & & \end{array}$$

Let ℓ_n and r_n denote the left and right boundary elements in the n^{th} row. I prove: $r_{n+1} - \ell_n$ is $a, b,$ - independent.

We need the following formulae:

$$\ell_n = h\frac{n(n+1)}{4} + \begin{cases} a, & n \equiv 0,3 \ (mod\ 4), \\ b - \frac{h}{2}, & n \equiv 1,2 \ (mod\ 4), \end{cases} \tag{1a, b}$$

$$r_n = h\frac{n(n+3)}{4} + \begin{cases} a, & n \equiv 0,1 \ (mod\ 4), \\ b - \frac{h}{2}, & n \equiv 2,3 \ (mod\ 4). \end{cases} \tag{2a, b}$$

Theorem 3.

$$r_{n+1} - \ell_n = h(n+1). \tag{4}$$

Proof. If

$$n+1 \equiv 0,1 \ (mod\ 4) \ \Rightarrow n \equiv 0,3 \ (mod\ 4).$$

Likewise, if

$$n+1 \equiv 2,3 \ (mod\ 4) \ \Rightarrow n \equiv 1,2 \ (mod\ 4).$$

Hence,

$$h^{-1}(r_{n+1} - \ell_n) = \frac{1}{4}(h+1)(n+4) - \frac{1}{4}n(n+1) =$$
$$= \frac{1}{4}\{(n^2 + 5_h + 4) - (n^2 + n)\} = \frac{1}{4}(4n+4) = n+1. \quad \blacksquare$$

Monotonicity Or x In The Symmetric Quantization

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Abstract

For $a > 0, x + a > x$. This is quantized:

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x, q \in \mathbf{R}, \quad q \neq 1, \ q > 0,$$

be the 2^{nd} quantization of x:
Since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Theorem 1. If $a > 0$ then

$$[x + a]_q^{\sim} > [x]_q^{\sim}. \tag{2}$$

Proof. (2) is:

$$q^{x+a} - q^{-x-a} \stackrel{?}{>} q^x - q^{-x},$$

or

$$q^x(q^a - 1) \stackrel{?}{>} q^{-x-a}(1 - q^a). \tag{3}$$

since $a > 0, q > 1, q^a > 1$, and (3) is:

$$q^x \stackrel{?}{>} -q^{x-a},$$

or

$$q^{2x+a} \stackrel{?}{>} -1,$$

which is obvious since the LHS > 0 and the $RHS < 0$. ■

On A Special Sum Of Consecutive Quantum Integers

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Abstract

Let $k = 2s+1$ be an odd number, $n \equiv s+1 (mod\ k)$. Then $\sum_{i=1}^{k-1}(n+i) \equiv 0 (mod\ k)$. We quantize this.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},\ x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the 2^{nd} quantization of x.

With the notation

$$k = 2s+1,\ s \in \mathbf{Z}_{\geq 1},$$
$$n \equiv s+1\ (mod\ k),$$

we have:

$$\sum_{i=0}^{k-1}[n+1]_{q^2}^{\sim} \equiv 0 (mod\ ([k]_q^{\sim})^2). \tag{1}$$

Proof. We use the formula:

$$\sum_{k=0}^{n}[a+kd]_q^{\sim} = [N+1]_{q^{d/2}}^{\sim}[a + \frac{N}{2}d]_q^{\sim}. \tag{2}$$

Then the LS of (1): is

$$\sum_{i=0}^{k-1}[k+1]_{q^2}^{\sim} = [k]_q^{\sim}\left[\frac{n+h+k-1}{2}\right]_{q^2}^{\sim}[\text{with } n = s+1+k\ell] =$$
$$= [k]_q^{\sim}\left[n + \frac{k-1}{2}\right]_{q^2}^{\sim} = [k]_q^{\sim}[s+1+(2s+1)\ell + s]_q^{\sim} =$$
$$= [k]_q^{\sim}[k+(2s+1)\ell]_{q^2}^{\sim} =$$
$$= [k]_q^{\sim}[k(1+\ell)]_{q^2}^{\sim} = [k]_q^{\sim}[k]_{q^2}^{\sim}[1+\ell]_{q^{2k}},$$

and it remains to show that

$$[k]^{\sim}_{q^2} \equiv 0 \ (mod \ [k]^{\sim}_{q}), \tag{3}$$

which is true because k is odd:

$$\frac{[k]^{\sim}_{q^2}}{[k]^{\sim}_{q}} = \frac{q^{2k}-q^{-2k}}{q^2-q^{-2}} \Big/ \frac{q^k-q^{-k}}{q-q^{-1}} = \frac{q^k+q^{-k}}{q+q^{-1}},$$

and the latter expression is in $Z[q,q^{-1}]$ because k is odd. ■

On Quantum Elementary Symmetric Functions

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Abstract

$\Pi_{i=1}^{n}(x+a_i)=\sum_{i=0}^{n}x^{n-i}\sigma_i(\mathbf{a})$. We quantize this.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}},\quad x\in\mathbf{R},\quad q\neq 0,\quad \pm 1,$$

be the second quantization of x.

Needless to say, all our results are for large M (or $m: p_m<M<P_{m+1}$).

We first sketch a rough proof, and then supply the missing details.

Let x be an integer, and

$$\delta(x)=\pi(x)-\pi(x-2). \tag{2}$$

Then

$$\delta(x)=\begin{cases} 1, x \text{ odd}, & x \quad \text{a prime}, \\ 0, x \text{ odd}, & x \quad \text{is not a prime} \end{cases} \tag{3}$$

Thus, with M even,

$$n(M)=\sum_{i=2}^{M}\delta(M-P_i),\ P_m<M<P_{m+1}. \tag{4}$$

If we could take

$$\pi(x)=\frac{x}{logx}, \tag{5}$$

the *main* asymptotic term, then

$$\delta(x)=\frac{x}{logx}-\frac{x-2}{log(x-2)}\sim\frac{1}{log^2x}\ \text{times}:$$

$$xlog(x-2)-(x-2)logx=xlog[x(1-\frac{2}{x})]-xlogx+2logx\simeq$$

$$\sim x\{logx-\frac{2}{x}\}-xlogx+2logx=2(logx-1)\sim 2logx,$$

so

$$\delta(x)\frac{2}{logx}. \tag{6}$$

On The 2^{nd} Elementary Symmetric Function Under The 1^{st} Quantization

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Abstract

We derive formula for it.

Theorem 1.

$$[x+a]_q - [x+b]_q = [x]_q[x+a+b]_q + q^x[a]_q[b]_q. \tag{2}$$

Multiplying through by $(q-1)^2$, we get

Proof. For the LHS of (2):

$$\{(a^{x+a}-1)(q^{x+b}-1)\} =$$
$$= \{q^{2x+a+b} - q^{x+a} - q^{x+b} + 1\},$$

while for the RHS of (2) we get:

$$(q^x-1)(q^{x+a+b}-1) + (q^a-1)(q^b-1)q^x =$$
$$= q^{2x+a+b} - q^x - q^{x+a+b} + 1(q^a-1)(q^b-1).$$

Thus, (2) is:

$$q^x - q^{x+a+b} - q^{x+a} - q^{x+b} \stackrel{?}{=} (q^a-1)(q^b-1)q^x,$$

or, dividing by q^x,

$$1 + q^{a+b} - q^a - q^b \stackrel{?}{=} (q^a-1)(q^b-1),$$

which is obvious. ■

On The Difference Of Quantum Square Roots

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Abstract

The classical inequality, $\frac{1}{\sqrt{n}} < \sqrt{n+1} - \sqrt{n-1}$, is strengthened and then quantized.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

The classical inequality

$$\frac{1}{\sqrt{n}} < \sqrt{n+1} - \sqrt{n-1},\ n \geq 1, \tag{1}$$

is rather inexact. A better one is:

$$\frac{\sqrt{2}}{\sqrt{n}} < \sqrt{n+1} - \sqrt{n-1}, \quad n \in \mathbf{Z}_{\geq 2}. \tag{2}$$

Theorem 3. For $N \in \mathbf{Z}_{\geq 2}$,

$$\frac{\sqrt{[2]_{q_n}^{\sim}}}{\sqrt{[n]_q^{\sim}}} < \sqrt{[n+1]_q^{\sim}} - \sqrt{[n-1]_q^{\sim}}. \tag{4}$$

Proof. Using the easily verifiable fact that

$$[n+1]_q^{\sim} - [n-1]_q^{\sim} = [2]_{q^n}^{\sim},$$

(4) is:

$$\frac{\sqrt{[2]_{q^n}^{\sim}}}{\sqrt{[n]_q^{\sim}}} \overset{?}{<} \frac{[2]_{q^n}^{\sim}}{\sqrt{[n+1]_q^{\sim}} + \sqrt{[n-1]_q^{\sim}}},$$

or

$$\sqrt{[n+1]_q^{\sim}} - \sqrt{[n-1]_q^{\sim}} \overset{?}{<} \sqrt{[2]_{q^n}^{\sim}[n]_q^{\sim}} \tag{5}$$

Now,

$$[n+1]_q^{\sim} + [n-1]_q^{\sim} = [2]_q^{\sim}[n]_q^{\sim},$$

and

$$[2]_q^{\sim} < [2]_{q^n},\ n > 1,$$

because the function

$$[2]_{q^x}^{\sim}$$

grows with x. ■

Remark 6. The proof above shows that n *doesn't* have to be an in???. The proper inequality is:

$$\frac{\sqrt{[2]_{q^x}^{\sim}}}{\sqrt{[x]_q^{\sim}}} < \sqrt{[x+1]_q^{\sim}} - \sqrt{[x-1]_q^{\sim}},\ x > 1. \tag{7}$$

On The Product Of Inverse Quantum Squares

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Abstract

We quantize the classical formula $\prod_{k=2}^{n}(1-\frac{1}{k^2})=\frac{h+1}{2n}$.

Let

$$[x]_q^{\sim}=\frac{q^x-q^{-x}}{q-q^{-1}}, x\in\mathbf{R},\ q\neq 0,\ \pm 1,$$

be the second quantization of x.

Theorem 1. For $n\in\mathbf{Z}_{\geq 2}$,

$$\prod_{k=2}^{n}\left(1-\frac{1}{([n]_q^{\sim})^2}\right)=\frac{[n+1]_q^{\sim}}{[2]_q^{\sim}[n]_q} \tag{2}$$

Proof. We use induction on n. For $n=2$, we get

$$1-\frac{1}{([2]_q^{\sim})^2}\stackrel{?}{=}\frac{[3]_q^{\sim}}{([2]_q^{\sim})^2},$$

or

$$([2]_q^{\sim})^2-1\stackrel{?}{=}[3]_q^{\sim},$$

or

$$(q+q^{-1})^2-1\stackrel{?}{=}q^2+1+q^{-2},$$

which is obvious.

The induction step $n\mapsto n+1$ is:

$$\frac{[n+1]_q^{\sim}}{[2][n]_q^{\sim}}\left(1-\frac{1}{([n+1]_q^{\sim})^2}\right)\stackrel{?}{=}\frac{[n+2]_q^{\sim}}{[2]_q^{\sim}[n+1]_q^{\sim}},$$

or

$$1-\frac{1}{([n+1]_q^{\sim})^2}\stackrel{?}{=}\frac{[n]_q^{\sim}[n+2]_q^{\sim}}{([n+1]_q^{\sim})^2}. \tag{3}$$

But it's easy to check that

$$([x]_q^{\sim})^2-[x-1]_q^{\sim}[x+1]_q^{\sim}=1,\quad\forall x\in\mathbf{R}. \tag{4}$$

Thus, (3) becomes:

$$1-\frac{1}{([n+1]_q^{\sim})^2}\stackrel{?}{=}\frac{([n+1]_q^{\sim})^2-1}{([n+1]_q^{\sim})^2},$$

which is obvious. ■

On The Sum Of Quantum Even Double-Factorials

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Abstract

We calculate inequality, $\sum_{s=1}^{n}[2s]!! < (2n+1)!!4, n \geq 2$, is quantized.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Set

$$[2n+1]!!_q^{\sim} = \prod_{s=0}^{n}[2s+1]_q^{\sim},\quad n \in \mathbf{Z}_{\geq 0},$$

$$[2n]!!_q^{\sim} = \prod_{s=1}^{n}[2s]_q^{\sim},\ n \in \mathbf{Z}_{\geq 1};\quad [0]!! = 1.$$

Theorem 1. For $n \in \mathbf{Z}_{\geq 2}$,

$$\sum_{s=1}^{n}[2s]!!_q^{\sim} < [2n+1]!!_q^{\sim} - c, \tag{2}$$

$$[2]_q^{\sim} + [2]_q^{\sim}[4]_q^{\sim} < [3]_q^{\sim}[5]_q^{\sim} - c. \tag{3}$$

Proof. (i) We use induction on n. The case $n = 2$ amounts to the proper choice of c:

$$c < [3]_q^{\sim}[5]_q^{\sim} - [2]_q^{\sim}([4]_q^{\sim} + 1) \tag{4}$$

The inductive step ($\to n+1$) turns (2) into:

$$[2n+1]!!_q^{\sim} - c + [2n+2]!! \stackrel{?}{<} [2n+3]!!_q^{\sim} - c,$$

or

$$[2n+2]!! \stackrel{?}{<} [2n+1]!!([2n+3][2n+2]_q^{\sim} - 1),$$

or

$$[2n]!!_q^{\sim}[2n+2]_q^{\sim} \stackrel{?}{<} [2n+1]([2n+3]_q^{\sim}[2n+2]_q^{\sim} - 1) \tag{5}$$

Now,

$$[2n]!!_q^{\sim} < [2n+1]!!_q^{\sim},$$

so (5) follows from

$$[2n+2]_q^{\sim} \stackrel{?}{<} [2n+3]_q^{\sim}[2n+2]_q^{\sim} - 1,$$

or

$$1 \stackrel{?}{<} [2n+2]_q^{\sim}([2n+3]_q^{\sim} - 1). \tag{6}$$

(6) follows from

$$1 \stackrel{?}{<} [2]_q^{\sim}([3]_q^{\sim} - 1) = [2]_q^{\sim}[2]_{q^2}^{\sim} = [4]_q^{\sim},$$

which is true, because $4 > 1$. ■

Principal Vertical In The Basic Numerical Triangle From A Sequence Mutilated Symmetrically Module 4

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Abstract

Let $\{x_n\} = (0, 1, 3; 4, 5, 7; ...)$ be what's left of $\mathbf{Z}_{\geq 0}$ when terms $\equiv 2 \pmod 4$ are removed. Put $\{x_n\}$ into a triangle:

$$\begin{array}{ccccc} & & & & 0 \\ & & & 1 & 3 \\ & & 4 & 5 & 7 \\ & 8 & 9 & 11 & 12 \\ 13 & 15 & 16 & 17 & 19 \end{array} \tag{0.1}$$

Let $m_n = (0, 5, 11, ...)$ denote terms standing on the middle vertical. I prove:

$$m_n = \frac{16}{3} t_n + \begin{cases} 0, & n \equiv 0, 2 \pmod 3, \\ -V_3, & n \equiv 1 \pmod 3, \end{cases}$$

where t_n is the nth triangular number.

Let $u_{n,k}$ denotes k^{th} term in the n^{th} row, so that

Quantum Elementary Symmetric Functions

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Abstract

We derive them.

We use the basic formula

$$[u+v]_q = [u]_q + q^4[v]_q, \tag{1}$$

so that

$$[x+a]_q = [x]_q + q^x[a]_q. \tag{2}$$

Denote by

$$ac = (a_1, ..., a_n), \tag{3a}$$
$$c = a_{n+1}, \tag{3b}$$
$$\Pi_n = \Pi_{i=1}^n [x+a_i]_q. \tag{3c}$$

Theorem 4. For $n \in \mathbf{Z}_{\geq 1}$,

$$\Pi_n = \sum_{i=0}^{n} ([x]_q)^{n-1} q^{ix} \sigma_i(??), \tag{5}$$

where

$$\sigma_0(a) = 1 \tag{6}$$

and, from (2),

$$\sigma_1(a) = [a]_q^{\sim}, \tag{7}$$

and, in general

$$\sigma_i(a) = 0, \quad i > 1, \tag{8a}$$
$$\sigma_i(a,c) = \sigma_i(a) + [c]_q^{\sim} \sigma_{i-1}(a). \tag{8b}$$

Proof. We use induction on n. Multiplying (5) by

$$[x+c]_q = [x]_q + q^x[c]_q, \tag{9}$$

we get:

$$\Pi_{n+1} = \Pi_n[x+c]_q = (\sum_{i=0}^{n}([x]_q)^{n-1}q^{ix}\sigma_i)([x]_q + q^x[c]_q) =$$
$$= \sum_i([x]_q)^{n+1-i}q^{ix}\sigma_i(a) + \sum[x]_q)^{n-i}q^{(i+1)x}\sigma_i(a)[c]_q \stackrel{?}{=}$$
$$= \sum([x]_q)^{n+1-j}q^{jx}\sigma_j(a,c) \quad \Leftrightarrow$$
$$\sigma_j(a,c) = \sigma_j(a) + \sigma_{j-1}(a)[c]_q,$$

which is exactly (8b). ■

Re??? $\sigma_j(a)$ by $\bar{\sigma}_j(a)$, keeping the notation $\sigma_j(a)$ for the classical ($q = 1$) case. Then we can summarize are results as

$$\bar{\sigma}_j(a) = \sigma_j([a_1]_q, ..., [a_n]_q), \tag{10}$$

an extremely satisfying result.

Some Identities Among Quantum Numbers

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Let

$$[x]_q^{\sim} = \frac{q^x - 1}{q - 1}, \quad x, q \in \mathbf{R},\ q \neq 1,\ q > 0.$$

be the 1^{st} quantization of x.

Identities among quantum integers are so far unpredictable and have to be verified by hand.

Lemma 1.

$$[n+2]_q - [2]_{q^{n+1}} = q[n]_q. \tag{2}$$

Proof. Change n into x:

$$[x+2]_q - [2]_{q^{x+1}} = q[x]_q,$$

or

$$q^{x+2} - 1 - (q-1)(q^{x+1}+1) \stackrel{?}{=} q(q^x - 1),$$

or

$$q^{x+2} - 1 - \left\{ q^{x+2} + q - q^{x+1} - 1 \right\} = q^{x+1} - q,$$

or

$$0 = 0,$$

which is true. ■

Lemma 3.

$$[7]_q^{\sim} = [3]_{q^3}^{\sim} + [2]_q^{\sim}[2]_{q^3}^{\sim}. \tag{4}$$

Proof. We have:

$$\begin{aligned}
&[3]_{q^3}^{\sim} + [2]_q^{\sim}[2]_{q^3}^{\sim} = \\
&= (q^6 + 1 + q^{-6}) + (q + q^{-1})(q^3 + q^{-3}) = \\
&= (q^6 + 1 + q^6) + (q^4 + q^2 + q^{-2} + q^{-4}) = \\
&= q^6 + q^4 + q^2 + 1 + q^{-2} + q^{-4} + q^{-6} = [7]_q^{\sim}. \ \blacksquare
\end{aligned}$$

The Growth Of $p_n/n, p_n$ Being n^{th} Prime

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Abstract

Fix $a \in \mathbf{Z}_{>1}$. We show that $\frac{l_{n+a}}{n+a} - \frac{p_n}{n} \to 0$, as $n \to \infty$.

One (or two)-sided estimate is the source of many troubles in prime number theory. Fin $a \in \mathbf{Z}_{\geq 1}$, and let

$$\triangle_n = \frac{p_{n+a}}{n+a} - \frac{p_n}{n}. \tag{1}$$

Since

$$p_n \sim nlogn,$$

one intuitively feels that

$$\triangle_n \sim log(n+a) - logn \sim \frac{a}{n} \to 0. \tag{2}$$

However, the standard estimates,

$$p_n < n(logn + loglogn), n \geq 6,$$

(due to Rosser), and

$$p_n > n(logn + loglogn - 1), n \geq 2,$$

(due to Dusart), yield:

$$\triangle_n > log(n+a) - loglog(n+a) - 1 - [logn + loglogn] > -1,$$
$$\triangle_n < log(n+a) + loglog(n+a) - [logn + loglogn - 1] < 1,$$

so

$$-1 < \triangle_n < 1,$$

a highly unsatisfactory result.

There is now a standard remedy, developed in my series of prime number papers: to use entire or truncated *full* asymptotic expansion. Thus, we write:

$$p_n = n\{logn + (loglogn - 1) + 0(\frac{1}{logn})\}. \tag{3}$$

Hence

$$\triangle_n = log(n+a) + [loglog(n+a) - 1] - [logn + (loglogn - 1)] + 0(\frac{1}{n}) = = 0(\frac{1}{n}), \tag{4}$$

and this is the desired (2).

The Number of Primes Between $n^2 + (n+1)^2$ Is $\sim \frac{n}{logn}$

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Abstract

We strengthen Ledendre's (what used to be) Conjecture, by showing that there is $\sim \frac{n}{log\ n}$ primes between n^2 and $(\sim +1)^2$.

How many primes are between n^2 and $(n+1)^2$? Legendre says, empirically, that number ≥ 1, but this vastly under states the matters.

I recently proved there is, on average, there is 1 prime between x and $x + logx$. Since, if

$$x = n^2$$
$$(n+1)^2 = (\sqrt{x}+1)^2 = x + 2\sqrt{x} + 1 \sim x + 2\sqrt{x},$$

we have to cover the interval of length $2\sqrt{x}$ by intervals of the length $log\ x$. So we get:

$$\frac{2\sqrt{x}}{logx} \sim \frac{2n}{logn^2} \sim \frac{n}{logn},$$

as claimed in the abstract.

Example 1. Take $n = 10$. Between $10^2 = 100$ and $11^2 = 121$ there are 5 primes: 101, 103, 107, 109, 113. And

$$\frac{10}{log10} = 4.34.$$

Example 2. Take $n = 998$. Between 998^2 and 999^2 there are, according to computer, 151 primes. And

$$\frac{998}{log\ 998} = 144.5$$

The Rate Of Growth Of Quantum Factorial Roots

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Abstract

We quantize the classical inequality $\sqrt[k]{k!} < \sqrt[n+1]{(k+1)!}$.

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, x \in \mathbf{R},\ q \neq 0,\ \pm 1,$$

be the second quantization of x.

Since

$$[-x]_q^{\sim} = -[x]_q^{\sim},$$

we can restrict ourselves to nonnegative numbers only, and since

$$[x]_{q^{-1}}^{\sim} = [x]_q^{\sim},$$

we can take

$$q > 1$$

without loss of generality.

Set

$$[k]!_q^{\sim} = \prod_{s=1}^{k} [s]_q^{\sim}, \quad k \in \mathbf{Z}_{\geq 1};\ [0]!_q^{\sim} = 1.$$

Theorem 1. For $k \in \mathbf{Z}_{\geq 1}$,

$$\sqrt[k]{[k]!_q^{\sim}} \ < \ \sqrt[k+1]{[k+1]!_q^{\sim}}. \tag{2}$$

Proof. Raising both sides of (2) into $k(k+1)$-th power, we get:

$$([k]!_q^{\sim})^{k+1} \stackrel{?}{<} ([k+1]!_q^{\sim})^k,$$

or

$$([k]!_q^{\sim})^k [k]!_q^{\sim} \stackrel{?}{<} ([k+1]!_q^{\sim})^k ([k]!_q^{\sim})^k,$$

or

$$[k]!_q^{\sim} \stackrel{?}{<} ([k+1]_q^{\sim})^k,$$

which is obvious because it's the product of inequalities:

$$\begin{aligned} &[1]_q^{\sim} < [k+1]_q^{\sim}, \\ &\vdots \\ &[k]_q^{\sim} < [k+1]_q^{\sim}, \end{aligned}$$

each being true because

$$a > b > 0 \Rightarrow [a]_q^{\sim} > [b]_q^{\sim}. \quad \blacksquare$$