Approximation of Invariant Subspaces

A Dissertation Presented for the

Doctor of Philosophy

Degree

The University of Tennessee, Knoxville

Faruk Yilmaz

August 2017

To my family.

Acknowledgements

I would like to thank my advisor, Professor Stefan Richter, for his help, encouragement and endless support during my Ph.D degree. I am thankful that I have been able to work under his supervision. I would also like to thank my doctoral committee members Professors Carl Sundberg, Michael Frazier and Michael W. Berry.

I would like to thank the Mathematics Department at the University of Tennessee for supporting me. I would like to thank all my teachers for sharing their wisdom with me during my education. A special thanks goes to Pam Armentrout.

Last, but not least, I am grateful to my family. Thank you for your constant support and encouragement to be brave and optimistic. I could not be here without you.

Abstract

For a real number α [alpha] the Dirichlet-type spaces $\mathcal{D}_\alpha$ [script D sub alpha] are the family of Hilbert spaces consisting of all analytic functions $f(z) = \sum_{n=0}^{\infty}$ [sum over n equals zero to infinity] $\hat{f}(n)$ [f hat of n] z^n [z to the n] defined on the open unit disc $\mathbb{D}$ [unit disc] such that

$$\sum_{n=0}^{\infty}(n+1)^\alpha \mid \hat{f}(n) \mid^2$$

is finite.

For $\alpha < 0$, the spaces $\mathcal{D}_\alpha$ are known as weighted Bergman spaces. When $\alpha = 0$, then $\mathcal{D}_0 = \mathbf{H}^2$, the well known and much studied Hardy space. For $\alpha > 0$, the $\mathcal{D}_\alpha$ spaces are weighted Dirichlet spaces.

The characterization of the invariant subspaces of the multiplication operator M_z [M sub z] on the $\mathcal{D}_\alpha$ spaces depends on α, and it is partially still an open problem. The invariant subspaces of $\mathcal{D}_2$ have been characterized in 1972 by B. I. Korenblum [25].

In this dissertation we show that the invariant subspaces of $\mathcal{D}_2$ can be approximated by finite co-dimensional invariant subspaces. For the Dirichlet space $\mathbf{D} = \mathcal{D}_1$, there is no complete characterization of invariant subspaces, but we consider

$$D_E = \{f \in [\text{in}]\mathbf{D} : f^* = 0 \;\; \text{q.e. [quasi-everywhere] on E}\}$$

where $E \subseteq$ [subset]$\mathbb{T}$ [unit circle] is a Carleson thin set. In this case, we have a partial result.

In the second part of the dissertation we prove a regularity result for extremal functions in the Dirichlet space $\mathbf{D}$. If φ [phi] is an extremal function in the Dirichlet space, then we use a result of Richter and Sundberg [35] to show that for *each* point on the unit circle $\mathbb{T}$ the square of the absolute value of φ converges to its boundary value in certain tangential approach regions.

Table of Contents

1 Introduction **1**

1.1 History . . . 1

1.2 Overview . . . 2

2 The Dirichlet type $\mathcal{D}_\alpha$-Spaces **9**

2.1 Definition . . . 9

2.2 M_z-Invariant Subspaces . . . 11

2.3 Carleson thin sets . . . 15

2.4 Extremal Functions . . . 16

2.5 Some General Facts . . . 17

3 Approximation of Invariant subspaces **21**

3.1 Approximation of Invariant subspaces for the $\mathcal{D}_2$-space . . . 21

3.2 An approximation result for the Dirichlet space . . . 33

4 Tangential Limits of Extremal Functions in the Dirichlet Space **39**

4.1 Motivation . . . 39

4.2 Tangential Limits of Extremal functions . . . 42

Bibliography **50**

Vita **56**

Chapter 1

Introduction

1.1 History

Let $\mathcal{H}$ be a Banach space of analytic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Suppose that $\mathcal{H}$ is invariant with respect to the operator M_z of multiplication by the independent variable z. A (closed) subspace $\mathcal{M}$ of $\mathcal{H}$ is called invariant with respect to M_z, simply M_z-invariant, if the operator M_z maps $\mathcal{M}$ into itself, i.e. $M_z f \in \mathcal{M}$ for all $f \in \mathcal{M}$.

We say that a sequence of M_z-invariant subspaces $\mathcal{M}_n$ converges to a M_z-invariant subspace $\mathcal{M}$ if the corresponding orthogonal projections converge in the strong operator topology (SOT). If the sequence $\mathcal{M}_n$ has finite co-dimension, i.e. $\dim(\mathcal{H}/\mathcal{M}_n) < \infty$, then Shimorin [40] defines that $\mathcal{M}$ admits strong approximate spectral cosynthesis. Such an approximation process, called approximate spectral synthesis, was suggested by Nikolskii in [31]. For the weighted Bergman spaces such approximations were considered by Shimorin in [40].

1.2 Overview

For $\alpha \in \mathbb{R}$, $\mathcal{D}_\alpha$ spaces are defined

$$\mathcal{D}_\alpha = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_\alpha^2 := \sum_{n=0}^{\infty} (n+1)^\alpha \mid \hat{f}(n) \mid^2 < \infty, \quad f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \right\}.$$

For $\alpha < 0$, the spaces $\mathcal{D}_\alpha$ are known as weighted Bergman spaces. When $\alpha = 0$, then $\mathcal{D}_0 = \mathbf{H}^2$, the well known and much studied Hardy space. For $\alpha > 0$, the $\mathcal{D}_\alpha$ spaces are weighted Dirichlet spaces. This is a family of Hilbert spaces, and more information about these spaces is provided in Section 2.1. An interesting transition occurs at $\alpha = 1$. For $\alpha > 1$, $\mathcal{D}_\alpha$ spaces are contained in the disc algebra $\mathbf{A}(\mathbb{D})$ and $\mathcal{D}_\alpha$ is an algebra, while for $\alpha \leq 1$ the $\mathcal{D}_\alpha$ spaces contain unbounded functions and they are not algebras.

The multiplication operator M_z on $\mathcal{D}_\alpha$, defined by $M_z f(z) = z f(z)$ for all $f \in \mathcal{D}_\alpha$, is a bounded linear operator. A (closed) subspace $\mathcal{M}$ of $\mathcal{D}_\alpha$ is invariant with respect to multiplication operator M_z if $M_z \mathcal{M} \subseteq \mathcal{M}$. We denote the collection of M_z-invariant subspaces of $\mathcal{D}_\alpha$ by $\mathrm{Lat}(M_z, \mathcal{D}_\alpha)$. The structure of M_z-invariant subspaces of $\mathcal{D}_\alpha$ spaces depends on α and it may be very complicated. Simple invariant subspaces in all cases are zero-based invariant subspaces, denoted by $I(Z)$ where $Z \subseteq \mathbb{D}$, but not all invariant subspaces are of this type. Also, Z being discrete is certainly necessary for $I(Z) \neq (0)$, but not sufficient for any $\mathcal{D}_\alpha$ spaces. In Section 2.2, we give an overview of M_z-invariant subspaces of the $\mathcal{D}_\alpha$ spaces.

The question that we would like to bring attention to is "Can all M_z-invariant subspaces of $\mathcal{D}_\alpha$ be approximated by zero-based invariant subspaces for some finite zero sets?". It turns out that the answer to the question depends on α and it is partially still an open problem.

If $\mathcal{H}$ is a separable Hilbert space, then the unit ball in $\mathcal{B}(\mathcal{H})$, the set of all bounded linear operators on $\mathcal{H}$, is metrizable in the strong operator topology (SOT). Hence the collection of all projections onto invariant subspaces is metrizable in SOT. If

P_n is a sequence of decreasing projections, then P_n converges to P, where P is the projection onto the intersection of the ranges of P_n. Similarly, an increasing sequence of projections P_n converges to the projection onto the closed linear span of the ranges of P_n (see Section 2.5). For example, if $\{Z_n\}$ is an increasing sequence of zero sets, then the corresponding zero-based invariant subspaces $\{I(Z_n)\}$ is a decreasing sequence of subspaces and hence the corresponding sequence $\{P_n\}$ of projections is a decreasing sequence of operators. Then one can apply the above result to show that

$$I(Z_n) \to \bigcap_n I(Z_n) = I\left(\cup_n Z_n\right).$$

Also, if $\{Z_n\}$ is a decreasing sequence of zero sets, then the corresponding zero-based invariant subspaces $\{I(Z_n)\}$ is an increasing sequence of subspaces and so the corresponding sequence $\{P_n\}$ of projections is an increasing sequence of operators. Then it will imply that

$$I(Z_n) \to \operatorname{span}_n I(Z_n) \subseteq I\left(\cap_n Z_n\right).$$

Note that one may not have equality in this last inclusion.

If $\alpha \leq 1$, then an invariant subspace has finite co-dimension if and only if it is a zero-based invariant subspace for some finite subset of the unit disc $\mathbb{D}$ (see [40]). If $\alpha > 1$, then one has a similar result but one needs to allow the zeros to be in the closed unit disc. For example for $\mathcal{D}_2$ the reproducing kernels at points on the unit circle $\mathbb{T}$ exist and zero-based invariant subspaces make sense for sets with some points in the unit circle $\mathbb{T}$. It turns out that those sets do not have to be discrete, but could be Carleson thin sets (although they will only have finite co-dimension if the sets are finite). A brief description of Carleson thin sets is given in Section 2.3.

Therefore if $Z \subseteq \mathbb{D}$ is discrete, then $I(Z_j)$ converges to $I(Z)$, where $Z_j \subseteq Z_{j+1}$ contains j elements and $\cup_j Z_j = Z$. Hence, if an invariant subspace $\mathcal{M}$ can be approximated by $I(Z_n)$ for some $Z_n \subseteq \mathbb{D}$, then $\mathcal{M}$ can be approximated by finite co-dimensional invariant subspaces.

In the case of the Hardy space $\mathbf{H}^2 = \mathcal{D}_0$, $\alpha = 0$, we have a complete description of $\mathrm{Lat}(M_z, \mathbf{H}^2)$ by Beurling's theorem. Indeed, if $(0) \neq \mathcal{M} \in \mathrm{Lat}(M_z, \mathbf{H}^2)$, then $\mathcal{M} = \phi\mathbf{H}^2$ for some inner function ϕ, i.e. ϕ is in the unit ball of $\mathbf{H}^\infty$ and satisfies $|\varphi(e^{it})| = 1$ a.e. The Caratheodory-Schur Theorem states that every function in the unit ball of $\mathbf{H}^\infty$ can be approximated locally uniformly in the unit disc by a sequence of finite Blaschke products. Then one can use these two results to show easily that every invariant subspace of $\mathbf{H}^2$ can be approximated by finite co-dimensional ones (see Section 2.5).

The index of an invariant subspace plays an important role in the approximation since a sequence of index one invariant subspaces can converge only to zero or an index one invariant subspace (see Section 2.5). Non zero zero-based invariant subspaces have index one by Corollary 3.4 of [32]. If $\alpha \geq 0$, then every non-zero invariant subspace has index one. On the other hand for $\alpha < 0$ there are invariant subspaces that do not have index one. Hence not all invariant subspaces can be approximated by finite co-dimensional (zero-based) invariant subspaces. In Section 2.2 we define the index and give an overview of results concerning the index of invariant subspaces of $\mathcal{D}_\alpha$. For the weighted Bergman spaces $\mathcal{D}_\alpha$, $\alpha < 0$, Shimorin [40] showed that all index one invariant subspaces can be approximated by finite co-dimensional invariant subspaces. Therefore, in the context of the $\mathcal{D}_\alpha$ spaces, the remaining question is whether one can approximate M_z-invariant subspaces of $\mathcal{D}_\alpha$ by finite co-dimensional ones when $\alpha > 0$.

In [25], Korenblum gives a complete characterization of invariant subspaces of $\mathcal{D}_2$. Using Korenblum's description, we have the following result (see Theorem 3.1.9) which provides positive the answer that one can approximate invariant subspaces of $\mathcal{D}_2$ by finite co-dimensional invariant subspaces.

Theorem 1.2.1. *Let $\mathcal{M} \in Lat(M_z, \mathcal{D}_2)$ be a nontrivial (closed) subspace. Then there exists a sequence $\mathcal{M}_n \in Lat(M_z, \mathcal{D}_2)$, with $\dim \mathcal{M}_n^\perp < \infty$, such that $P_{\mathcal{M}_n} \to P_{\mathcal{M}}$ in* *the strong operator topology (SOT), where* $P_{\mathcal{M}_n}$ *and* $P_{\mathcal{M}}$ *are orthogonal projections onto* $\mathcal{M}_n$ *and* $\mathcal{M}$, *respectively.*

Korenblum's Theorem for $\mathcal{D}_2$ shows that non-trivial M_z-invariant subspaces of $\mathcal{D}_2$ are of the type $\mathcal{M} = U\mathcal{M}_E$, where U is an inner function, $E \subseteq \mathbb{T}$, and

$$\mathcal{M}_E := \{f \in \mathcal{D}_2 : f(z) = 0, \ \ \forall z \in E\}.$$

For details see Chapter 3. The easy case to prove Theorem 1.2.1 is when $U = B$ is a Blaschke product. Then one can set $\mathcal{M}_n = B_n\mathcal{M}_{E_n}$, where B_n is the partial Blaschke product and $\{E_n\}$ is an increasing sequence of finite sets in E such that the union of E_n is dense in E. The remaining case to finish the proof is how to approximate in the case of a general inner function U which may have a singular inner factor. This case follows from Frostman's Theorem and an argument involving extremal functions.

We should note that Korenblum's result in [26] yields a complete description of the invariant subspaces of $\mathcal{D}_{2n}$, $n \geq 1$, and an analog of Theorem 1.2.1 holds for these spaces (see Remark 3.1.10).

Our main result Theorem 1.2.1 also gives an affirmative answer to a special case of a question of J. B. Conway, and D. Hadwin which they asked in [10]. Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and T be a bounded linear operator on it. An invariant subspace $\mathcal{M} \in \mathrm{Lat}(T)$ is called stable if whenever there is a sequence of bounded operators $\{T_n\}$ on $\mathcal{H}$ such that $\|T_n - T\| \to 0$, there is a sequence $\{\mathcal{M}_n\}$ with $\mathcal{M}_n \in \mathrm{Lat}(T_n)$ for $n \geq 1$ such that $P_n \to P$ in the strong operator topology (SOT), where P_n and P are the orthogonal projections onto $\mathcal{M}_n$ and $\mathcal{M}$, respectively. If the sequence of projections can be chosen to converge in the operator norm, then we say $\mathcal{M}$ is norm-stable. The collection of stable (norm-stable) invariant subspaces of T is denoted by $\mathrm{Lat}_s(T)$ ($\mathrm{Lat}_{ns}(T)$). In [10], J. B. Conway, and D. Hadwin asked whether for any operator T, $\mathrm{Lat}_s(T)$ is the strong closure of $\mathrm{Lat}_{ns}(T)$. They proved that this is the case when T is a normal or an unweighted shift of finite multiplicity. A consequence of a general result of A. Borichev, D. Hadwin and H. Yousefi [5] is that if

T is any weighted unilateral shift operator, then an invariant subspace is norm stable if and only if it has finite co-dimension. If we denote by $\mathrm{Lat}_{fc}(M_z, \mathcal{D}_2)$ the collection of invariant subspaces with finite co-dimension, then Theorem 1.2.1 shows that the strong closure of $\mathrm{Lat}_{fc}(M_z, \mathcal{D}_2)$ is $\mathrm{Lat}(M_z, \mathcal{D}_2)$. Since $\mathrm{Lat}_{fc}(M_z, \mathcal{D}_2) \subset \mathrm{Lat}_{ns}(M_z, \mathcal{D}_2)$ (see [5]), it follows that for $\mathcal{D}_2$ the question of J. B. Conway and D. Hadwin has a positive answer.

If $\alpha > 1$, then $\mathcal{M}_E := \{f \in \mathcal{D}_2 : f(z) = 0, \quad \forall z \in E\}$ makes sense for sets E with some points in the unit circle $\mathbb{T}$. As mentioned earlier the easiest way to prove Theorem 1.2.1 for $\mathcal{M}_E$ is to use finite co-dimensional subspaces of the type $\mathcal{M}_{E_n}$ where each $E_n \subseteq E \subseteq \mathbb{T}$ is finite. If $\alpha = 1$, then there is no analogue of $\mathcal{M}_E$ for finite subset $E \subseteq \mathbb{T}$. That raises the question whether one can do the approximation for the $\mathcal{D}_2$ case without use of points in the unit circle $\mathbb{T}$ and approximate $\mathcal{M}_E$ by zero-based invariant subspaces $I(Z_n)$ for some $Z_n \subseteq \mathbb{D}$. We have following theorem which gives another proof that there exists $I_n \in \mathrm{Lat}(M_z, \mathcal{D}_2)$, with $\dim I_n^{\perp} < \infty$, such that $P_n \to P$ in SOT where P_n and P are orthogonal projections onto I_n and $\mathcal{M}_E$, respectively.

Theorem 1.2.2. *Let $\emptyset \neq E \subset \mathbb{T}$ be a Carleson thin set and $\mathcal{M}_E = \{f \in \mathcal{D}_2 : f(z) = 0, \quad \forall z \in E\}$ be a nontrivial invariant subspace of $(M_z, \mathcal{D}_2)$. Then there exists a decreasing sequence of subsets $Z_k \subseteq \mathbb{D}$ such that $\mathcal{M}_E = span_k I(Z_k)$.*

We will see that each Z_k in Theorem 1.2.2 (see Theorem 3.1.11) will contain infinitely many points, but again by decreasing property and the metrizability the space $\mathcal{M}_E$ can be approximated by finite co-dimensional invariant subspaces.

For $\alpha = 1$ we do not know all of the invariant subspaces of the Dirichlet space, but we can consider $D_E := \{f \in \mathbf{D} : f^* = 0 \text{ q.e. on } E\}$ for an arbitrary subset E of $\mathbb{T}$. Here f^* denotes the radial limit of f, and Beurling's theorem (see Theorem 4.1.1) says that each function in the Dirichlet space has non-tangential limits quasi-everywhere on the unit circle $\mathbb{T}$. A property is said to hold quasi-everywhere (q.e.) on $\mathbb{T}$ if it holds everywhere on $\mathbb{T}$ except for a subset of logarithmic capacity zero (see

[14]). It is known that D_E is a closed M_z-invariant subspace of the Dirichlet space $\mathbf{D}$ (see Section 3.2). So in particular, if $E \subset \mathbb{T}$ is a Carleson thin set, then in order to do approximation for D_E by finite co-dimensional invariant subspaces, it suffices to find a sequence of sets $Z_k \subseteq \mathbb{D}$ such that $D_E = \text{span}_k I(Z_k)$. For the Dirichlet space we have the following partial result (see Theorem 3.2.2).

Theorem 1.2.3. *Let $E \subset \mathbb{T}$ be a Carleson thin set with positive logarithmic capacity and let f_E be the corresponding outer function constructed by Korenblum that is zero on E, and smooth elsewhere. Then there exists a decreasing sequence of zero sets $Z_k \subseteq \mathbb{D}$ such that*

$$D_E \supseteq span_k I(Z_k) \supseteq [f_E].$$

It is known that $\dim \mathcal{M} \ominus z\mathcal{M} = 1$ for every nonzero M_z-invariant subspace $\mathcal{M}$ of the Dirichlet space $\mathbf{D}$ (see Theorem 2 of [34]), and every nonzero M_z-invariant subspace $\mathcal{M}$ of the Dirichlet space $\mathbf{D}$ generated by a function $\varphi \in \mathcal{M} \ominus z\mathcal{M}$, $\|\varphi\| = 1$, i.e. $\mathcal{M} = [\varphi]$ (see Theorem 7.1 of [33]). Such functions are called extremal functions and we give a brief overview of extremal functions in the Section 2.4. Therefore, the invariant subspace D_E in Theorem 1.2.3 is generated by a function, say φ, i.e. $D_E = [\varphi]$. If one could show that $[\varphi] \subseteq \text{span}_k I(Z_k)$, then the approximation for D_E by finite co-dimensional would follow. But this is the difficult case because of knowledge of the regularity of φ. The difference between f_E and φ is our lack of knowledge of the regularity of φ.

Thus we study the regularity properties of φ in Chapter 4. In [35] Richter and Sundberg proved that the radial limit of $|\varphi|$, where φ is the extremal function in the Dirichlet space, exists for every point on the unit circle $\mathbb{T}$. We extend this result and show that limit exists in approach region $\Gamma_c^\alpha(e^{i\theta})$ for every $\alpha \in [1, \infty)$ at all points $e^{i\theta} \in \mathbb{T}$, where $\Gamma_c^\alpha(e^{i\theta})$ is the tangential approach region of order α, defined as

$$\Gamma_c^\alpha(e^{i\theta}) = \{z \in \mathbb{D} : |z - e^{i\theta}|^\alpha \leq c(1 - |z|)\}, \quad c > 0, \quad \alpha \geq 1.$$

We prove the following in Theorem 4.2.3.

Theorem 1.2.4. *Let φ be an extremal function in the Dirichlet space* $\mathbf{D}$*. Then for each $e^{it} \in \mathbb{T}$, we have $|\varphi(w)|^2 \to |\varphi(e^{it})|^2$ as $w \to e^{it}$ in the tangential approach region $\Gamma_c^{\alpha}(e^{it})$.*

The main ingredient of proof of Theorem 1.2.4 is a result of Richter and Sundberg [35]. The proof of the theorem relies heavily on estimates of the reproducing kernel for the Dirichlet space and the Dominated Convergence Theorem.

Chapter 2

The Dirichlet type $\mathcal{D}_\alpha$-Spaces

2.1 Definition

Let $\mathbb{D}$ denote the open unit disc in the complex plane. For $\alpha \in \mathbb{R}$ the family of Hilbert spaces $\mathcal{D}_\alpha$ consists of all analytic functions $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ defined on $\mathbb{D}$ with the norm given by

$$\|f\|_\alpha^2 = \sum_{n=0}^{\infty} (n+1)^\alpha \mid \hat{f}(n) \mid^2 < \infty.$$

By polarization the inner product is given by

$$\langle f, g \rangle_\alpha = \sum_{n=0}^{\infty} (n+1)^\alpha \hat{f}(n) \overline{\hat{g}(n)}.$$

Given $f \in \mathcal{D}_\alpha$ and $w \in \mathbb{D}$, an application of the Cauchy-Schwarz inequality gives

$$\begin{aligned} \mid f(w) \mid = \left| \sum_{n=0}^{\infty} \hat{f}(n) w^n \right| &\leq \left(\sum_{n=0}^{\infty} (n+1)^\alpha \mid \hat{f}(n) \mid^2 \right)^{1/2} \left(\sum_{n=0}^{\infty} (n+1)^{-\alpha} |w|^{2n} \right)^{1/2} \\ &= \|f\|_\alpha^2 \left(\sum_{n=0}^{\infty} (n+1)^{-\alpha} |w|^{2n} \right)^{1/2}. \end{aligned}$$

For $\alpha > 1$, the above infinite sum is obviously finite. For $\alpha \leq 1$, one can show by an application of the ratio test that the above infinite sum is finite. This implies that the evaluation functional $f \mapsto f(w)$ is bounded. Therefore by the Riesz Representation theorem for each $w \in \mathbb{D}$ there is a vector $k_w^\alpha \in \mathcal{D}_\alpha$ such that for any $f \in \mathcal{D}_\alpha$ we have

$$\langle f, k_w^\alpha \rangle_\alpha = f(w).$$

These vectors k_w^α are called the reproducing kernels of $\mathcal{D}_\alpha$, and we say that $\mathcal{D}_\alpha$ is a reproducing kernel Hilbert space. In fact, given $w \in \mathbb{D}$, we have that

$$k_w^\alpha(z) = \sum_{n=0}^{\infty} (n+1)^{-\alpha} \overline{w}^n z^n$$

which can easily verified by using the definition of the inner product on $\mathcal{D}_\alpha$.

For $\alpha < 0$, the norm on $\mathcal{D}_\alpha$ is equivalent to

$$\int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^{-\alpha - 1} \frac{dA(z)}{\pi},$$

(see Lemma 2 in [43]).

These spaces are obviously nested in the following sense, $\mathcal{D}_\alpha \subset \mathcal{D}_\beta$ for $\alpha > \beta$. Also if $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$, then taking the derivative yields

$$f'(z) = \sum_{n=1}^{\infty} n \hat{f}(n) z^{n-1} = \sum_{n=0}^{\infty} (n+1) \hat{f}(n+1) z^n.$$

Now, it is easy to see that $f \in \mathcal{D}_\alpha$ if and only if $f' \in \mathcal{D}_{\alpha-2}$. For $\alpha > 1$, the functions in the $\mathcal{D}_\alpha$ spaces are continuous up to the boundary, hence $\mathcal{D}_\alpha \subset A(\mathbb{D}) = Hol(\mathbb{D}) \cap C(\overline{\mathbb{D}})$, the disc algebra, but this is not true for $\alpha \leq 1$. In fact more can be said, which is Hölder continuity.

Lemma 2.1.1. *If $f \in \mathcal{D}_\alpha$ for $\alpha > 1$, then f is Hölder continuous, that is, there exists an $\epsilon > 0$ and a constant $c > 0$ such that*

$$|f(z) - f(w)| \leq c|z - w|^\epsilon.$$

Proof.

$$
\begin{aligned}
|f(z)-f(w)| &= \left|\sum_{n=0}^{\infty}\hat{f}(n)z^n - \sum_{n=0}^{\infty}\hat{f}(n)w^n\right| \le \sum_{n=0}^{\infty}|\hat{f}(n)||z^n-w^n| \\
&\le \left(\sum_{n=0}^{\infty}(n+1)^{\alpha}|\hat{f}(n)|^2\right)^{1/2}\left(\sum_{n=0}^{\infty}(n+1)^{-\alpha}|z^n-w^n|^2\right)^{1/2} \\
&= \|f\|_{\alpha}\left(\sum_{n=0}^{\infty}(n+1)^{-\alpha}|z^n-w^n|^2\frac{|z-w|^{\epsilon}}{|z-w|^{\epsilon}}\right)^{1/2} \\
&\le \|f\|_{\alpha}\left(\sum_{n=0}^{\infty}(n+1)^{-\alpha}\left|\frac{z^n-w^n}{z-w}\right|^{\epsilon}|z^n-w^n|^{2-\epsilon}|z-w|^{\epsilon}\right) \\
&\le \|f\|_{\alpha}\left(\sum_{n=0}^{\infty}\frac{n^{\epsilon}}{(n+1)^{\alpha}}2^{2-\epsilon}|z-w|^{\epsilon}\right) \\
&= \left(\|f\|_{\alpha}\,2^{2-\epsilon}\sum_{n=0}^{\infty}\frac{n^{\epsilon}}{(n+1)^{\alpha}}\right)|z-w|^{\epsilon}
\end{aligned}
$$

For any $\epsilon < \alpha - 1$ the above infinite sum is finite. Hence we are done.

□

Also for $\alpha > 1$, $\mathcal{D}_\alpha$ is an algebra (see Theorem 3 in [22]), and thus $M(\mathcal{D}_\alpha) = \mathcal{D}_\alpha$, where $M(\mathcal{D}_\alpha)$ is the multiplier algebra of $\mathcal{D}_\alpha$, i.e.

$$M(\mathcal{D}_\alpha) = \{\varphi : \varphi f \in \mathcal{D}_\alpha, \forall f \in \mathcal{D}_\alpha\}.$$

A brief survey of the $\mathcal{D}_\alpha$ spaces can be found in [6].

2.2 M_z-Invariant Subspaces

The multiplication operator M_z on $\mathcal{D}_\alpha$ is given by $M_z f(z) = zf(z)$ for all $f \in \mathcal{D}_\alpha$. A simple application of the closed graph theorem shows that the multiplication operator is bounded. Also one can show that $(M_z, \mathcal{D}_\alpha)$ is unitarily equivalent to the weighted shift operator with weight sequence $\{(\frac{k+2}{k+1})^{\alpha/2}\}_{k\in\mathbb{N}_0}$ with respect to the orthonormal basis $\{(\frac{1}{k+1})^{\alpha/2}z^k\}_{k\in\mathbb{N}_0}$. A nice discussion of weighted shift operator is given by Allen Shields in [38].

We will denote this operator by M_z or $(M_z, \mathcal{D}_\alpha)$. A (closed) subspace $\mathcal{M}$ of $\mathcal{D}_\alpha$ is called invariant, if M_z maps $\mathcal{M}$ into itself. We denote the collection of invariant subspaces of $(M_z, \mathcal{D}_\alpha)$ by $\mathrm{Lat}(M_z, \mathcal{D}_\alpha)$.

If $f \in \mathcal{D}_\alpha$, we let $[f]$ be the smallest invariant subspace of the multiplication operator M_z. Thus $[f]$ is the closure of the polynomial multiples of f. A function $f \in \mathcal{D}_\alpha$ is called a cyclic vector if $[f] = \mathcal{D}_\alpha$.

If $f \in \mathcal{D}_\alpha$, then we let $Z(f) = \{\lambda \in \mathbb{D} : f(\lambda) = 0\}$, and for an invariant subspace $\mathcal{M}$ of $\mathcal{D}_\alpha$ denote by $Z(\mathcal{M})$ the set of common zeros of functions in $\mathcal{M}$, i.e.

$$Z(\mathcal{M}) = \bigcap_{f\in\mathcal{M}} Z(f).$$

It is known that for any $\lambda \in \mathbb{D}$, $M_z - \lambda$ is bounded below on $\mathcal{D}_\alpha$ (see Example 2.9 in [32]), and for an M_z-invariant subspace $\mathcal{M}$ the dimension of $\mathcal{M} \ominus (z-\lambda)\mathcal{M}$ does not depend on $\lambda \in \mathbb{D}$ (see Lemma 2.1 in [32]). We define the index of an M_z-invariant subspace $\mathcal{M}$ of $\mathcal{D}_\alpha$ to be the dimension of $\mathcal{M} \ominus z\mathcal{M}$, i.e.

$$\mathrm{ind}\mathcal{M} = \dim\mathcal{M} \ominus z\mathcal{M} = \dim\mathcal{M} \cap (z\mathcal{M})^\perp.$$

The trivial invariant subspace (0) is the invariant subspace whose index is defined to be zero.

The structure of $\mathrm{Lat}(M_z, \mathcal{D}_\alpha)$ depends on α and it may be very complicated. Simple invariant subspaces in all cases are zero-based invariant subspaces but not all invariant subspaces are of this type. Let $Z = \{z_1, z_2, z_3, ...\}$ be a finite or infinite sequence of points in the unit disc $\mathbb{D}$. We write $I(Z)$ for all functions in $\mathcal{D}_\alpha$ that are zero at each z_i, counting multiplicities. Clearly, $I(Z) \in \mathrm{Lat}(M_z, \mathcal{D}_\alpha)$. Such subspaces are called zero-based invariant subspaces. Also, Z being discrete is certainly necessary for $I(Z) \neq (0)$, but not sufficient for any $\mathcal{D}_\alpha$. It follows from Corollary 3.4 of [32] that non zero zero based invariant subspaces have index 1. **Note** : Although we will talk about invariant subspaces throughout this dissertation, we will reserve the notation $I(Z)$ for the zero-based invariant subspace.

When $\alpha = 0$, $\mathcal{D}_0 = \mathbf{H}^2$ is the Hardy space on the unit disc. By the canonical factorization theorem, every function f in $\mathbf{H}^2$ can be written as $f = BSF$, where

$$B(z) = z^s \prod_{n=1}^{\infty} \frac{\overline{z_n}}{|z_n|} \frac{z_n - z}{1 - \overline{z_n} z}$$

is the Blaschke product with zeros (z_n) and a zero of multiplicity s at 0,

$$S(z) = C \exp\bigg(-\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta) \bigg)$$

is the singular inner function for some measure μ which is a finite positive regular Borel measure on $[0, 2\pi]$ that is singular with respect to the Lebesgue measure, and C is a constant of modulus 1, and

$$F(z) = \alpha \exp\bigg(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log|f(e^{i\theta})| d\theta \bigg),$$

is an outer function with $|\alpha| = 1$.

Beurling's famous theorem gives a complete characterization of $\mathrm{Lat}(M_z, \mathbf{H}^2)$. More precisely, it says that every nontrivial M_z-invariant subspace $\mathcal{M}$ of the Hardy space $\mathbf{H}^2$ is of the form $\mathcal{M} = \phi\mathbf{H}^2 = [\phi]$, where $\phi = BS$ is an inner function with B a Blaschke product and S a singular inner function (see Chapter 17 in [37]). It also says that if $\mathcal{M}$ is a nonzero M_z-invariant subspace of the Hardy space $\mathbf{H}^2$, then $\mathrm{ind}\mathcal{M} = 1$ and $f \in \mathbf{H}^2$ is cyclic if and only if f is an outer function.

For $\alpha < 0$, $\mathcal{D}_\alpha = \mathbf{L}^2_a(\mathbb{D}, (1 - |z|^2)^{-\alpha - 1} dA)$ are the weighted Bergman spaces. Unlike the situation in the Hardy space, nonzero invariant subspaces of the weighted Bergman space can have index n where $n \in \mathbb{N} \cup \{\infty\}$. This follows from results of Apostol, Bercovici, Foiaş and Pearcy (see [3], the authors in [3] proved this fact using abstract theory). In [20] Hedenmalm constructed an invariant subspace in the Bergman space $\mathbf{L}_a^2(\mathbb{D})$ with index 2 using the span of zero set based invariant subspaces.

In the case of weighted Dirichlet space $\mathcal{D}_\alpha$, $\alpha > 0$, every non zero M_z-invariant subspace has index 1. This follows from results of Aleman [2] when $0 < \alpha < 1$, Richter and Shields [34] when $\alpha = 1$, and Richter [32, Corollary 3.8] when $\alpha > 1$.

In [25] Korenblum gives a complete description of M_z-invariant subspaces of $\mathcal{D}_2$ and in [26] he generalizes his result to M_z-invariant subspaces of $\mathcal{D}_{2n}$, $n \geq 1$. To describe Korenblum's result, consider closed subsets $K_0, ..., K_{n-1}$ of the unit circle $\mathbb{T}$ and an inner function U with the following properties:

(1) $K_0 \supset K_1 \supset ... \supset K_{n-1}$,

(2) $K_0 \setminus K_{n-1}$ consists of only isolated points,

(3) the zeros of the inner function U accumulates only in K_{n-1},

(4) the singular measure associated with U is supported on K_{n-1},

(5) $\int_{\mathbb{T}} \log(\text{dist}(z, F))|dz| > -\infty$, where the set F is the union of K_0 and the zero set of the inner function U.

Then Korenblum showed that $\mathcal{M} = UI(K_0, ..., K_{n-1})$, where

$$I(K_0, ..., K_{n-1}) = \left\{ f \in \mathcal{D}_{2n} : f^{(j)} = 0 \text{ on } K_j, j = 0, ..., n-1 \right\},$$

is a nontrivial closed M_z-invariant subspace in $\mathcal{D}_{2n}$.

Theorem 2.2.1 (Korenblum, see [26])**.** *If $\mathcal{M}$ is any nontrivial closed M_z-invariant subspace of $\mathcal{D}_{2n}$, $n \geq 1$, then there exist an inner function U and closed subsets $K_0, ..., K_{n-1}$ of the unit circle $\mathbb{T}$ as described above such that*

$$\mathcal{M} = UI(K_0, ..., K_{n-1}).$$

2.3 Carleson thin sets

Definition 2.3.1. A closed subset E of the unit circle $\mathbb{T}$ is called a Carleson thin set if

$$\int_{\mathbb{T}} \log \rho(z)|d(z)| > -\infty,$$

where $\rho(z) = \operatorname{dist}(z, E) = \inf_{\zeta \in E} |z - \zeta|$.

It is well known that the condition in Definition 2.3.1 is equivalent to $|E| = 0$ and $\sum_n |I_n| \log(\frac{1}{|I_n|}) < \infty$, where $\mathbb{T} \setminus E = \bigcup_n I_n$, I_n are open disjoint arcs (see [8]).

Carleson thin sets, also referred to as Beurling-Carleson sets, arise as boundary zero sets of analytic functions that satisfy a Lipschitz condition in $\mathbb{D}$ (see [4, 8]).

Given a Carleson thin set $E \subset \mathbb{T}$, Carleson [8], Taylor-Williams [42] constructed an outer function with some nice properties.

In [25] Korenblum proved the following lemma. For more details also see [24].

Theorem 2.3.2 (Lemma 23 in [25])**.** *Let $E \subseteq \overline{\mathbb{D}}$ be a closed set satisfying the condition*

$$\int_{\mathbb{T}} \log \rho(z)|d(z)| > -\infty,$$

where $\rho(z) = dist(z, E)$. Then there exists an outer function $\Phi(z)$ on the unit disc $\mathbb{D}$ with the following properties:

(a) $\Phi(z)$ and also $\Phi^n(z)$, $n > 0$, are infinitely differentiable functions in the closed unit disc $\overline{\mathbb{D}}$.

(b) The set of zeros of $\Phi(z)$ in $\overline{\mathbb{D}}$ coincides with the set $E \cap \mathbb{T}$.

(c) For any $n > 0$ we have $|\Phi(z)| \leq c_n \rho^n(z)$, $z \in \mathbb{T}$, where c_n are constants.

(d) For any inner function $U(z)$ whose zeros and singular measure are in E, the product $U(z)\Phi^n(z)$ is infinitely differentiable in the closed unit disc $\overline{\mathbb{D}}$ for any $n > 0$.

In particular, if $E \subseteq \mathbb{T}$ is a Carleson thin set, then Korenblum's result yields an outer function corresponding to E, denoted by $f_E(z)$, with the following properties:

(i) $f_E \in \mathbf{A}(\mathbb{D}) \cap \mathbf{C}^\infty(\mathbb{T})$ is outer with $E = Z_{\mathbb{T}}(f_E) = \{\zeta \in \mathbb{T} : f_E(\zeta) = 0\}$ and $E \subset Z_{\mathbb{T}}(f_E^{(j)}), \forall j$, i.e. f_E has zeros of infinite multiplicity at the points of E.

(ii) $|f_E| \leq 1$ on $\mathbb{D}$.

(iii) For any $N > 0$ $|f_E(z)| \leq c_N \rho(z)^N$, $z \in \mathbb{T}$.

In this dissertation, f_E will denote this function.

2.4 Extremal Functions

Let $\mathcal{H}$ be a reproducing kernel Hilbert space of analytic functions on the open unit disc $\mathbb{D}$ and let $\mathcal{M}$ be a non zero M_z-invariant subspace of $\mathcal{H}$. If $\lambda \notin Z(\mathcal{M}) \subseteq \mathbb{D}$, then the extremal problem

$$\sup\{\mathrm{Re} f(\lambda) : f \in \mathcal{M}, \|f\| \leq 1\}$$

has a unique solution $\phi \in \mathcal{M}$ with the normalization $\phi(\lambda) > 0$ (see [1, p. 136]), which will be called an extremal function for $\mathcal{M}$ at λ. It is easy to check that if a function $\phi \in \mathcal{M}$, $\phi \neq 0$, is extremal, then $\phi \in \mathcal{M} \ominus (z - \lambda)\mathcal{M}$.

We should note that the functions that some authors refer to as extremal functions are only extremal functions at $\lambda = 0$, and they have the property $\langle \varphi, z^n \varphi \rangle = 0$ for any $n \geq 1$. So we will make the following definition for the extremal function.

Definition 2.4.1. A function $\varphi \in \mathcal{H}$ is called the extremal if $\|\varphi\| = 1$ and $\langle \varphi, z^n \varphi \rangle = 0$ for all $n \geq 1$.

Extremal functions play an important role in the theory of invariant subspaces of analytic functions spaces, and they were introduced by Hedenmalm [18] to study contractive divisors on the Bergman space. Since then there have been many articles in the literature studying extremal functions (see, e.g., [11, 35, 39, 19, 12]).

In the case of the Hardy space $\mathbf{H}^2$ the extremal function at $\lambda = 0$ for the non zero invariant subspace $\mathcal{M}$ is the inner function satisfying $\mathcal{M} = \phi \mathbf{H}^2$.

Let $\mathcal{M}$ be a non zero invariant subspace of the $\mathcal{D}_2$ space, $\lambda \notin Z(\mathcal{M}) \subseteq \mathbb{D}$, and $P_{\mathcal{M}}$ be the orthogonal projection of $\mathcal{D}_2$ onto $\mathcal{M}$. Then extremal function for $\mathcal{M}$ at λ is of the form

$$\phi := \frac{P_{\mathcal{M}} k_\lambda}{\|P_{\mathcal{M}} k_\lambda\|}.$$

This is an elementary observation. In fact, one can directly check that the

$$\phi = \frac{P_{\mathcal{M}} k_\lambda}{\|P_{\mathcal{M}} k_\lambda\|}$$

is the unique solution of the extremal problem

$$\sup\{\mathrm{Re} f(\lambda) : \|f\| \leq 1, f \in \mathcal{M}\}.$$

2.5 Some General Facts

In this section, we give some general known facts that we will use throughout this dissertation. The following well-known and useful theorem will be used throughout this dissertation without extra reference.

Theorem 2.5.1 (see [6]). *Let $\mathcal{H} \subseteq Hol(\Omega)$ be a reproducing kernel Hilbert space of analytic functions on a region $\Omega \subseteq \mathbb{C}$ and let $\{f_n\} \subseteq \mathcal{H}$. Then following are equivalent:*

(a) $f_n \to f$ weakly

(b) (i) $\|f_n\| \leq M$

(ii) $f_n(z) \to f(z)$ for all $z \in \Omega$

(c) (i) $\|f_n\| \leq M$

(ii) $f_n \to f$ locally uniformly.

Proposition 2.5.2 (Proposition 1.3 in Chapter 9 of [9]). *Let $\mathcal{H}$ be a separable Hilbert space, and let $\mathcal{B}(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$. Then the strong operator topology (SOT) is metrizable on bounded subsets of $\mathcal{B}(\mathcal{H})$.*

Remark 2.5.3. Then in particular

$$\mathcal{P} := \{P : P \text{ is a projection of } \mathcal{H} \text{ onto an invariant subspace of } \mathcal{H}\}$$

is metrizable in SOT. If $\{P_n\}$ is a sequence of increasing projection, i.e. $\|P_n f\| \leq \|P_{n+1} f\|$ for all f and n, then $P_n \to P$ in SOT where P is the projection onto the span of ranP_n (see Problem 120 of [17]). Similarly, if $\{P_n\}$ is a sequence of decreasing projection, i.e. $\|P_n f\| \geq \|P_{n+1} f\|$ for all f and n, then $P_n \to P$ in SOT where P is the projection onto the intersection of ranP_n.

If $\alpha = 0$, $\mathcal{D}_0 = \mathbf{H}^2$ the Hardy space, then every M_z-invariant subspace can be approximated by finite co-dimensional M_z-invariant subspaces. Indeed, by Beurling's theorem a non trivial M_z-invariant subspace $\mathcal{M}$ of the Hardy space $\mathbf{H}^2$ is of the form $\mathcal{M} = \phi\mathbf{H}^2$ for some inner function ϕ. Then by the Caratheodory-Schur theorem (see, e.g., [36] Theorem 5.5.1), ϕ can be approximated locally uniformly in the unit disc by a sequence of finite Blaschke products B_n. Let $\{\alpha_{n_i}\}_{i=1}^{k_n}$ be the zeros of the Blaschke product B_n. Set

$$\mathcal{M}_n = B_n\mathbf{H}^2 = \{f \in \mathbf{H}^2 : f(\alpha_{n_i}) = 0, \quad i = 1, ..., k_n\}.$$

It is clear that $\mathcal{M}_n \in \mathrm{Lat}(M_z, \mathbf{H}^2)$ and $\mathcal{M}_n$ has finite co-dimension. Then by Theorem 1.1 of [16] the projection P onto $\mathcal{M}$ equals $P = M_\phi M_\phi^*$. Similarly, the projection P_n onto $\mathcal{M}_n$ equals $P_n = M_{B_n} M_{B_n}^*$. It is left to show that P_n converges to P in the SOT.

Since the set

$$\mathcal{A} = \left\{ \sum_{i=1}^{n} a_i k_{\lambda_i} : a_i \in \mathbb{C}, \quad \lambda_i \in \mathbb{D}, \quad n \in \mathbb{N} \right\},$$

where $k_{\lambda_i}(z) = \dfrac{1}{1 - \overline{\lambda_i} z}$ is the reproducing kernel for the Hardy space $\mathbf{H}^2$, is dense in $\mathbf{H}^2$, it is enough to show that $P_n k_\lambda \to P k_\lambda$ for all $\lambda \in \mathbb{D}$.

A simple computation shows that $M_\phi^* k_\lambda = \overline{\phi(\lambda)} k_\lambda$.

$$
\begin{aligned}
\|P_n k_\lambda - P k_\lambda\|^2 &= \left\| \overline{B_n(\lambda)} B_n k_\lambda - \overline{\phi(\lambda)} \phi k_\lambda \right\|^2 \\
&= \left\| \overline{B_n(\lambda)} B_n k_\lambda - \overline{\phi(\lambda)} B_n k_\lambda + \overline{\phi(\lambda)} B_n k_\lambda - \overline{\phi(\lambda)} \phi k_\lambda \right\|^2 \\
&\leq 2 \left(\left| \overline{B_n(\lambda)} - \overline{\phi(\lambda)} \right|^2 \|B_n k_\lambda\|^2 + \left| \overline{\phi(\lambda)} \right|^2 \|B_n k_\lambda - \phi k_\lambda\|^2 \right).
\end{aligned}
$$

Since $B_n \to \varphi$ weakly, we have

$$
\begin{aligned}
\|B_n k_\lambda - \phi k_\lambda\|^2 &= \langle B_n k_\lambda - \phi k_\lambda, B_n k_\lambda - \phi k_\lambda \rangle \\
&= \langle B_n k_\lambda, B_n k_\lambda \rangle - \langle B_n k_\lambda, \phi k_\lambda \rangle - \langle \phi k_\lambda, B_n k_\lambda \rangle + \langle \phi k_\lambda, \phi k_\lambda \rangle \\
&\to 0
\end{aligned}
$$

Also note that $B_n \to \phi$ locally uniformly implies that $B_n \to \phi$ pointwise and hence $\overline{B_n(\lambda)} \to \overline{\phi(\lambda)}$ for all $\lambda \in \mathbb{D}$.

Therefore, $\|P_n k_\lambda - P k_\lambda\|^2 \to 0$.

The following lemma was proved in [40]. It explains why not all invariant subspaces can be approximated by finite co-dimensional ones. Since its proof is short, we will provide it here.

Lemma 2.5.4. *The index 1 property is preserved under convergence, i.e. if $\mathcal{M}_n$ is a sequence of invariant subspaces of $\mathcal{D}_\alpha$ with index 1 and $\mathcal{M}_n \to \mathcal{M}$, where $\mathcal{M} \neq (0)$, then $\mathcal{M}$ has index 1.*

Proof. We will use Lemma 3.1 of [32]. Let $\lambda \notin Z(\mathcal{M})$ and suppose that $h \in \mathcal{D}_\alpha$ and $(z-\lambda)h \in \mathcal{M}$. We have to show that $h \in \mathcal{M}$.

Since $\mathcal{M}_n$ converges to $\mathcal{M}$, we can find a sequence $f_n \in \mathcal{M}_n$ such that $f_n \to (z-\lambda)h$ as $n \to \infty$ by Lemma 4.2 of [29]. Note that $f_n(\lambda) \to 0$.

Also since $\lambda \notin Z(\mathcal{M})$, there exists $g \in \mathcal{M}$ such that $g(\lambda) \neq 0$ and $g_n \in \mathcal{M}_n$ such that $g_n \to g$.

Define

$$h_n = f_n - \frac{f_n(\lambda)}{g_n(\lambda)} g_n.$$

We then have $h_n(\lambda) = 0$, $h_n \in \mathcal{M}_n$ and $h_n \to (z-\lambda)h$ as $n \to \infty$. Since $\mathrm{ind}\mathcal{M}_n = 1$, there exist $k_n \in \mathcal{M}_n$ such that $h_n = (z-\lambda)k_n$. We see that $(z-\lambda)k_n \to (z-\lambda)h$. $M_z - \lambda$ is bounded below, thus $k_n \to h$ and hence $h \in \mathcal{M}$.

$\square$

Theorem 2.5.5 (Frostman, see, e.g. [15])**.** *Let $\phi(z)$ be a non constant inner function on the unit disc. Then for all $\xi \in \mathbb{D}$, except possibly for a set of capacity zero, the function*

$$\phi_\xi(z) = \frac{\phi(z) - \xi}{1 - \bar{\xi}\phi(z)}$$

is a Blaschke product. In particular, there is a sequence $\xi_n \in \mathbb{D}$, $\xi_n \to 0$ such that each ϕ_{ξ_n} is a Blaschke product.

Definition 2.5.6. A Banach space $\mathcal{X}$ contained in the Hardy space $\mathbf{H}^1$ is said to have the $\mathcal{F}-$property if ${}^f/_I \in \mathcal{X}$ whenever $f \in \mathcal{X}$ and I is an inner function with ${}^f/_I \in \mathbf{H}^1$.

In [27], Korenblum and Faĭvyševskiĭ showed that for each $\alpha \geq 0$, $\mathcal{D}_\alpha$ has the $\mathcal{F}-$property. In fact, they proved that if $f \in \mathcal{D}_\alpha$ and U is an inner function that divides the inner part of f, then $U^{-1}f \in \mathcal{D}_\alpha$ and $\|U^{-1}f\|_\alpha \leq \|f\|_\alpha$ for $\alpha \geq 0$.

Definition 2.5.7. A subset Z of the unit disc $\mathbb{D}$ is called a zero set for $\mathcal{D}_\alpha$ if there exists a nonzero function $f \in \mathcal{D}_\alpha$ that vanishes on Z and nowhere else.

Chapter 3

Approximation of Invariant subspaces

3.1 Approximation of Invariant subspaces for the $\mathcal{D}_2$-space

The space $\mathcal{D}_2$ is the space of all holomorphic functions f defined on the open unit disc $\mathbb{D}$ such that $f^{'} \in \mathbf{H}^2$, and it has been studied by various authors (see, e.g., [25, 26, 24, 23, 28]). In [25] Korenblum gave a complete characterization of invariant subspaces of $(M_z, \mathcal{D}_2)$ by using the following equivalent norm on $\mathcal{D}_2$,

$$\|f\|_2^2 = \sum_{n=0}^{\infty}(1+n^2)|\hat{f}(n)|^2 = \|f\|_{H^2}^2 + \|f'\|_{H^2}^2 .$$

For the rest of this dissertation we will use this norm.

If $|\lambda| \leq 1$ and if

$$k_\lambda(z) = \sum_{n=0}^{\infty} \frac{1}{1+n^2}\overline{\lambda}^n z^n,$$

then one easily checks that $k_\lambda \in \mathcal{D}_2$ and $f(\lambda) = \langle f, k_\lambda \rangle_{\mathcal{D}_2}$ holds for all $f \in \mathcal{D}_2$.

Let E be a closed subset of the unit circle $\mathbb{T}$ and $U(z)$ be an inner function such that

(1) if $\{\alpha_k\}$ are the zeros of $U(z)$ in the unit disc, then all limit points of $\{\alpha_k\}$ belong to E,

(2) the measure defined by the singular part of $U(z)$ is supported on E.

Let $\mathcal{M}$ denotes the collection of all functions in $\mathcal{D}_2$ whose inner part is divisible by $U(z)$ and which are equal to zero on E, i.e. $\mathcal{M} = U\mathcal{M}_E$ where

$$\mathcal{M}_E = \{f \in \mathcal{D}_2 : f(z) = 0, \forall z \in E\}.$$

Clearly $\mathcal{M}$ is M_z-invariant subspace of $\mathcal{D}_2$. Non-triviality and closedness of $\mathcal{M}$ follow from the following remarks.

Remark 3.1.1 (Remark 1 in [25])**.** It follows from the result of [24] that for the non-triviality of the invariant subspace $\mathcal{M}$ it is necessary and sufficient that the condition

$$\int_{\mathbb{T}} \log(\rho(z))|dz| > -\infty$$

is satisfied, where $\rho(z) = \rho(z, F)$, $F = E \cup \{\alpha_k\}$.

Remark 3.1.2 (Remark 2 in [25])**.** The closure of the invariant subspace $\mathcal{M}$ is a simple corollary of the fact that the topology on $\mathcal{D}_2$ is stronger than the topology on the disc algebra $A(\mathbb{D})$, i.e. $\mathcal{D}_2 \subseteq A(\mathbb{D})$.

The following is the main theorem of Korenblum that gives the characterization of the $\mathrm{Lat}(M_z, \mathcal{D}_2)$.

Theorem 3.1.3 (Main Theorem in [25])**.** *Let $\mathcal{M}$ be a nontrivial closed invariant subspace of $(M_z, \mathcal{D}_2)$, let SB be greatest common divisor of inner factors of the functions $f \in \mathcal{M}$, and $E = \bigcap_{f \in \mathcal{M}} \{z \in \mathbb{T} : f(z) = 0\}$. We have $(\overline{Z(B)} \cap \mathbb{T}) \cup supp\mu \subseteq E$, where $supp\mu$ denotes the support of the measure defined by the singular inner function S. Then $\mathcal{M} = SB\mathcal{M}_E$.*

Corollary 3.1.4 (Corollary 1 in [25]). *Each closed invariant subspace of* $(M_z, \mathcal{D}_2)$ *is principal, i.e. generated by a single function.*

We start with few lemmas that we need in order to prove the main theorem.

Lemma 3.1.5. *Let* E *be a closed subset of the unit circle* $\mathbb{T}$ *and* u *be an inner function such that the zeros of* u *accumulate in* E *and the support of the measure defined by the singular part of* u *is in* E. *Then, whenever* $u_\lambda = \frac{u-\lambda}{1-\bar{\lambda}u}$ *is a Blaschke product, the zeros of* u_λ *accumulate in* E.

Proof. Let $\mathbb{T} \setminus E = \bigcup\limits_n I_n$ where I_n are disjoint open intervals. First note that u has analytic extension across each arc I_n (see, e.g., [21]). Fix $\lambda \in \mathbb{D}$. Then $1 - \overline{\lambda} u(z) \neq 0,\ \forall z \in \mathbb{D} \cup (\bigcup\limits_n I_n)$. Hence $u_\lambda(z)$ has analytic extension across each I_n, i.e. $u_\lambda(z) \in Hol(G \setminus E)$ where $\mathbb{D} \subseteq G$. If a subsequence of the zeros of u_λ converges to a point in $\mathbb{T} \setminus E$, say $\alpha_n \to e^{i\theta}$, then $u_\lambda(\alpha_n) = 0$ implies that $u(\alpha_n) = \lambda$.

So the function $g(z) = u(z) - \lambda$ has zeros at α_n that converges to a point on $G \setminus E$. Hence $g \equiv 0$. Thus $u(z) \equiv \lambda$, constant function. However, this contradicts the fact that u can not be a constant function of modulus $|\lambda| < 1$.

$\square$

Lemma 3.1.6. *Let* u *be an inner function and* $uf \in \mathcal{D}_2$. *Then* $f \in \mathcal{D}_2$, $u_\lambda f \in \mathcal{D}_2$ *and* $\|(u - u_\lambda) f\|_2 \to 0$ *as* $|\lambda| \to 0$ *where* $u_\lambda = \frac{u-\lambda}{1-\bar{\lambda}u}$, $\lambda \in \mathbb{D}$.

Proof. Let $g := uf \in \mathcal{D}_2$. Then since $\mathcal{D}_2$ has $\mathcal{F}-$property, we have $f \in \mathcal{D}_2$.

Also

$$\|u_\lambda f\|_2^2 = \|u_\lambda f\|_{H^2}^2 + \|(u_\lambda f)'\|_{H^2}^2 = \|f\|_{H^2}^2 + \|u_\lambda' f + u_\lambda f'\|_{H^2}^2$$

$$\leq \|f\|_{H^2}^2 + 2\big(\|u_\lambda' f\|_{H^2}^2 + \|u_\lambda f'\|_{H^2}^2 \big)$$

Since $f \in \mathcal{D}_2$, it is enough to show that $\|u'_\lambda f\|_{H^2} < \infty$. Note that $u'_\lambda = \frac{(1-|\lambda|^2)u'}{(1-\overline{\lambda}u)^2}$. Hence

$$\|u'_\lambda f\|^2_{H^2} = \frac{1}{2\pi}\int_0^{2\pi} |u'_\lambda(e^{i\theta})f(e^{i\theta})|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{(1-|\lambda|^2)u'(e^{i\theta})}{(1-\overline{\lambda}u(e^{i\theta}))^2} f(e^{i\theta})\right|^2 d\theta$$

$$\leq \frac{(1-|\lambda|^2)^2}{(1-|\overline{\lambda}|)^4}\frac{1}{2\pi}\int_0^{2\pi} |u'(e^{i\theta})f(e^{i\theta})|^2 d\theta < \infty$$

since $f,\ uf \in \mathcal{D}_2$ implies that $\|u'f\|^2_{H^2} < \infty$. Therefore $u_\lambda f \in \mathcal{D}_2$.

Now we will show that $\|(u-u_\lambda)f\|_2 \to 0$ as $|\lambda| \to 0$. By definition of norm we have

$$\|(u-u_\lambda)f\|^2_2 = \|(u-u_\lambda)f\|^2_{H^2} + \|((u-u_\lambda)f)'\|^2_{H^2}$$

$$\leq \|(u-u_\lambda)f\|^2_{H^2} + 2\big(\|(u-u_\lambda)'f\|^2_{H^2} + \|(u-u_\lambda)f'\|^2_{H^2}\big).$$

It is clear that $\|(u-u_\lambda)f\|^2_{H^2} \longrightarrow 0$ and $\|(u-u_\lambda)f'\|^2_{H^2} \to 0$ as $|\lambda| \to 0$ by the Dominated convergence theorem.

Furthermore, since $u_\lambda \to u$ uniformly, we have $u'_\lambda \to u'$ locally uniformly. A simple algebra shows that

$$u' - u'_\lambda = \frac{\big(|\lambda|^2 - 2\overline{\lambda}u + (\overline{\lambda}u)^2\big)u'}{(1-\overline{\lambda}u)^2}.$$

So

$$|u' - u'_\lambda| \leq \frac{2|\lambda|(|\lambda|+1)}{(1-|\lambda|)^2}|u'|$$

and hence

$$\|(u-u_\lambda)'f\|^2_{H^2} \leq \frac{4|\lambda|^2\big(|\lambda|+1\big)^2}{(1-|\lambda|)^4}\|u'f\|^2_{H^2} \longrightarrow 0, \quad \text{as} \quad |\lambda| \to 0$$

since $\|u'f\|^2_{H^2} < \infty$.

$\square$

Lemma 3.1.7. *Let $\mathcal{H} \subset Hol(\mathbb{D})$ be a reproducing kernel Hilbert space and $\mathcal{M}_n$ and $\mathcal{M}$ be closed subspaces of $\mathcal{H}$. Let P_n and P be orthogonal projections onto $\mathcal{M}_n$ and $\mathcal{M}$, respectively. If $P_n k_\lambda \to P k_\lambda$ weakly $\forall \lambda \in \mathbb{D} \setminus E$, where $E \subseteq \mathbb{D}$ is discrete, then $P_n \to P$ in SOT on $\mathcal{H}$, where $k_\lambda \in \mathcal{H}$ is the reproducing kernel at $\lambda \in \mathbb{D}$.*

Proof. First we will make the following observation: $P_n k_\lambda \to P k_\lambda$ weakly $\forall \lambda \in \mathbb{D} \setminus E$ implies that $P_n k_\lambda \to P k_\lambda$ in $\mathcal{H}$, $\forall \lambda \in \mathbb{D} \setminus E$.

$$\|P_n k_\lambda - P k_\lambda\|^2 = \langle P_n k_\lambda, P_n k_\lambda \rangle - \langle P_n k_\lambda, P k_\lambda \rangle - \langle P k_\lambda, P_n k_\lambda \rangle + \langle P k_\lambda \, P k_\lambda \rangle$$

$$\to \langle P k_\lambda, k_\lambda \rangle - \langle P k_\lambda, P k_\lambda \rangle - \langle P k_\lambda, P k_\lambda \rangle + \langle P k_\lambda \, P k_\lambda \rangle = 0.$$

Let $\epsilon > 0$ and take $f \in \mathcal{H}$. Define the set

$$\mathcal{A} = \left\{ \sum_{i=1}^{n} a_i k_{\lambda_i} : a_i \in \mathbb{C}, \lambda_i \in \mathbb{D} \setminus E \right\}.$$

Note that $\mathcal{A}$ is dense in $\mathcal{H}$. In fact, if f is perpendicular to $\mathcal{A}$, then $f(\lambda) = \langle f, k_\lambda \rangle = 0$ for all $\lambda \in \mathbb{D} \setminus E$, so that f is identically equal to the zero function.

Now choose $g \in \mathcal{A}$ such that $\|f - g\| < \frac{\epsilon}{3}$ and $\|P_n g - P g\| < \frac{\epsilon}{3}$. Then

$$\|P_n f - P f\| = \|P_n f - P_n g + P_n g - P g + P g - P f\|$$

$$\leq \|P_n (f - g)\| + \|P_n g - P g\| + \|P(g - f)\|$$

$$\leq 2 \|f - g\| + \|P_n g - P g\| < \epsilon.$$

□

Lemma 3.1.8. *Let $F = \{z_1, z_2, ..., z_n\}$ be a subset of $\mathbb{T}$, $B(z)$ be a finite Blaschke product with the zeros $\{\alpha_i\}_{i=1}^n \subseteq \mathbb{D}$, and $\mathcal{M} = B\mathcal{M}_F \in (M_z, \mathcal{D}_2)$. Then $\mathcal{M}$ has finite co-dimension.*

Proof. For simplicity we will assume that $\{\alpha_i\}_{i=1}^n$ are distinct zeros of the Blaschke product $B(z)$. A minor modification of the following argument would cover the general case. We will prove that $\mathcal{M}^\perp = \bigvee\{k_{\alpha_i}\}_{i=1}^n \cup \{k_{z_i}\}_{i=1}^n$, where k_{α_i} and k_{z_i} are the reproducing kernel at α_i and z_i,respectively.

Let

$$\mathcal{N} := \bigvee\{k_{\alpha_i}\}_{i=1}^n \cup \{k_{z_i}\}_{i=1}^n = \{\sum_{i=1}^n \lambda_i k_{\alpha_i} + \sum_{i=1}^n \beta_i k_{z_i} : \lambda_i, \beta_i \in \mathbb{C}\},$$

and let $g(z) = \sum_{i=1}^n \lambda_i k_{\alpha_i}(z) + \sum_{i=1}^n \beta_i k_{z_i}(z) \in \mathcal{N}$ and $f \in \mathcal{M}$. Then $f(z) = B(z)h(z)$, where $h(z) = 0$ on F.

Hence

$$\langle f, g\rangle = \sum_{i=1}^n \overline{\lambda_i} B_n(\alpha_i)h(\alpha_i) + \sum_{i=0}^n \overline{\beta_i} B_n(z_i)h(z_i) = 0.$$

So $g \in \mathcal{M}^\perp$, which implies that $\mathcal{N} \subset \mathcal{M}^\perp$. Taking the orthogonal complement of this yields $\mathcal{M} \subset \mathcal{N}^\perp$.

If $f \in \mathcal{N}^\perp$, then $f(\alpha_i) = 0,\ \forall i = 1, ..., n$. Set $g(z) = \frac{f(z)}{B(z)}$ where B is the finite Blaschke product with zeros $\{\alpha_i\}_{i=1}^n$. Then by the $\mathcal{F}$-property $g \in \mathcal{D}_2$. So $f = Bg$. Since $f \in \mathcal{N}^\perp$ we must also have $f(z_i) = 0$. Since $B(z_i) \neq 0$, we must have $g(z_i) = 0,\ \forall i$. Thus $f \in \mathcal{M}$. Therefore $\mathcal{N}^\perp \subset \mathcal{M}$ implies that $\mathcal{M}^\perp = \mathcal{N}$.

□

We are now ready to prove our main theorem.

Theorem 3.1.9. *Let $\mathcal{M} \in Lat(M_z, \mathcal{D}_2)$ be a nontrivial (closed) subspace. Then there exists a sequence $\mathcal{M}_n \in Lat(M_z, \mathcal{D}_2)$, with $\dim \mathcal{M}_n^\perp < \infty$, such that $P_{\mathcal{M}_n} \to P_{\mathcal{M}}$ in the strong operator topology (SOT), where $P_{\mathcal{M}_n}$ and $P_{\mathcal{M}}$ are orthogonal projections onto $\mathcal{M}_n$ and $\mathcal{M}$, respectively.*

Proof. We use the description of the invariant subspaces obtained by Korenblum. First, we will consider the invariant subspaces of the type $\mathcal{M} = B\mathcal{M}_E$, where $E \subset \mathbb{T}$ is a Carleson thin set and B is infinite Blaschke product with zeros $\{\alpha_i\}_{i=1}^\infty$ in $\mathbb{D}$ that converges to points in E, i.e.

$$B(z) = \prod_{i=1}^{\infty} \frac{\overline{\alpha_i}}{|\alpha_i|} \frac{\alpha_i - z}{1 - \overline{\alpha_i} z}.$$

Let $\{E_n\}$ be an increasing sequence of finite sets in E such that $\bigcup_n E_n$ is dense in E. Set $\mathcal{M}_n := B_n \mathcal{M}_{E_n}$, where $\mathcal{M}_{E_n} = \{f \in \mathcal{D}_2 : f(z) = 0, \forall z \in E_n\}$ and B_n is a finite Blaschke product with the zeros $\{\alpha_i\}_{i=1}^n$ in $\mathbb{D}$, i.e.

$$B_n(z) = \prod_{i=1}^{n} \frac{\overline{\alpha_i}}{|\alpha_i|} \frac{\alpha_i - z}{1 - \overline{\alpha_i} z}.$$

It follows from Remark 3.1.2 that $\mathcal{M}_n$ is closed for each n. Clearly we have $\mathcal{M}_n \in \mathrm{Lat}(M_z, \mathcal{D}_2)$. Also one can easily see that $\mathcal{M}_n \supseteq \mathcal{M}_{n+1}$, since $E_n \subseteq E_{n+1}$. It is also easy to see that $\mathcal{M}_E = \bigcap_n \mathcal{M}_{E_n}$.

We claim that $\mathcal{M} = \bigcap_n \mathcal{M}_n$. If $f \in \mathcal{M}$, then $f \in \mathcal{M}_n$, $\forall n$, so $f \in \bigcap_n \mathcal{M}_n$. Conversely, if $f \in \bigcap_n \mathcal{M}_n$, then $f(z) = 0$, $\forall z \in \bigcup_n E_n$ which is dense in E, so $f(z) = 0$ on E. Also $f \in \bigcap_n \mathcal{M}_n$ implies $f = B_n g_n$ for all n where $g_n \in \mathcal{M}_{E_n}$. By the $\mathcal{F}$-property, we also have that $\|f\|_2 \geq \|g_n\|_2$. Since $B_n \to B$ weakly, we have $g_n \to \frac{f}{B} =: g$ pointwise, hence weakly in $\mathcal{D}_2$. Fix $N \in \mathbb{N}$. Then $\forall n \geq N$ we have that $g_n \in \mathcal{M}_{E_N}$. Since $\mathcal{M}_{E_N}$ is closed, we have $g \in \mathcal{M}_{E_N}$. Hence $g \in \bigcap \mathcal{M}_{E_N} = \mathcal{M}_E$. Thus $f = Bg \in B\mathcal{M}_E = \mathcal{M}$. That proves our claim.

Let $P_{\mathcal{M}_n}$ and $P_{\mathcal{M}}$ be orthogonal projections onto $\mathcal{M}_n$ and $\mathcal{M}$, respectively. Then $P_{\mathcal{M}_n} \to P_{\mathcal{M}}$ in the SOT follows from Remark 2.5.2. By Lemma 3.1.8, we have $\dim \mathcal{M}_n{}^{\perp} < \infty$, and this finishes the proof of the first case.

Second, we will consider the general case, i.e. the invariant subspaces of the type $\mathcal{M} = U\mathcal{M}_E$ where U is inner function, E is Carleson thin set and $\mathcal{M}_E$ is as before.

Set $\mathcal{M}_\lambda = U_\lambda \mathcal{M}_E$, where $U_\lambda = \frac{U - \lambda}{1 - \overline{\lambda} U}$ is Blaschke product with simple zeros for almost all $\lambda \in \mathbb{D}$ by the Frostman's theorem. Then by Lemma 3.1.5, we know that zeros of U_λ accumulate only in E.

Let P_λ be orthogonal projection onto $\mathcal{M}_\lambda$ and P be orthogonal projection onto $\mathcal{M}$. Then using Proposition 2.5.2, it will be enough to show that $P_\lambda \to P$ in the SOT as $|\lambda| \to 0$. By Lemma 3.1.7, it suffices to show that $P_\lambda k_z \to Pk_z$ weakly $\forall z \in \mathbb{D} \setminus Z(\mathcal{M})$ as $|\lambda| \to 0$.

Fix $z \in \mathbb{D}$ such that $z \notin \mathbb{Z}(\mathcal{M})$. So $z \in \mathbb{D}$ is not a zero of the inner function U. Then extremal function for $\mathcal{M}$ at z is of the form

$$\phi := \frac{Pk_z}{\|Pk_z\|} = UF \in \mathcal{M}$$

where $F \in \mathcal{M}_E$. Then $U_\lambda F \in \mathcal{M}_\lambda$ and $\|U_\lambda F\|_2 \to \|UF\|_2$, as $|\lambda| \to 0$ by Lemma 3.1.6.

Set $S_\lambda = \frac{U_\lambda}{\|U_\lambda F\|_2}$ for $\lambda \in \mathbb{D} \setminus \{0\}$. Then $\|S_\lambda F\|_2 = 1$, $S_\lambda F \in \mathcal{M}_\lambda$ and $S_\lambda(z) \to U(z)$ pointwise as $|\lambda| \to 0$, ($S_\lambda F \to UF$ pointwise as $|\lambda| \to 0$).

Claim: For $|\lambda| < |U(z)|$ we have $\|P_\lambda k_z\|_2 \neq 0$.

Proof: If $\|P_\lambda k_z\|_2 = 0$, then $\forall f \in \mathcal{M}_\lambda$ we have

$$|f(z)| = |\langle f, k_z\rangle| = |\langle P_\lambda f, k_z\rangle| = |\langle f, P_\lambda k_z\rangle| \leq \|f\|_2 \|P_\lambda k_z\|_2 = 0.$$

This implies that $f(z) = 0$. Hence $U_\lambda(z) = 0$ since $z \in \mathbb{D}$ is fixed. Therefore $U(z) = \lambda$, which contradicts the hypothesis.

□

For $|\lambda| < |U(z)|$, let $G_\lambda \in \mathcal{M}_E$ be the function such that

$$\phi_\lambda := \frac{P_\lambda k_z}{\|P_\lambda k_z\|_2} = U_\lambda G_\lambda$$

is the extremal function for $\mathcal{M}_\lambda$. We will show that $\phi_\lambda \to \phi$ locally uniformly in $\mathbb{D}$ as $|\lambda| \to 0$.

We have, $1 = \|\phi_\lambda\|_2 = \|U_\lambda G_\lambda\|_2 \geq \|G_\lambda\|_2$ where the inequality follows from the $\mathcal{F}$-property.

So, it suffices to prove that if G is any weak limit of G_λ as $|\lambda| \to 0$, then $F = G$. Since G_λ is norm bounded, it has a weakly convergent subsequence. Thus let $G \in \mathcal{M}_E$ be such that $\exists |\lambda_n| \to 0$ with $G_{\lambda_n} \to G$ locally uniformly in $\mathbb{D}$. The extremality of $\phi_{\lambda_n} = U_{\lambda_n} G_{\lambda_n}$ in $\mathcal{M}_{\lambda_n}$ and $S_{\lambda_n} F \in \mathcal{M}_{\lambda_n}$ implies that

$$|U_{\lambda_n}(z) G_{\lambda_n}(z)| \geq |S_{\lambda_n}(z) F(z)| = \frac{|U_{\lambda_n}(z) F(z)|}{\|U_{\lambda_n} F\|_2}.$$

Since $|U_{\lambda_n}(z)| \neq 0$, we have $|G_{\lambda_n}(z)| \geq \frac{|F(z)|}{\|U_{\lambda_n} F\|_2}$ for each n. Hence $|G(z)| \geq |F(z)|$.

We also have $U_{\lambda_n}(w) G_{\lambda_n}(w) \to U(w) G(w)$ for $w \in \mathbb{D}$ and hence $U_{\lambda_n} G_{\lambda_n} \to UG$ weakly in $\mathcal{D}_2$. Thus $\|UG\|_2 \leq 1$ and $UG \in \mathcal{M}$. The extremality of $\phi = UF$ in $\mathcal{M}$ implies that $|U(z)F(z)| \geq |U(z)G(z)|$, hence $|F(z)| \geq |G(z)|$. The uniqueness of extremal function $\phi = UF$ implies that there is a $|c| = 1$ such that $UF = cUG$. From the extremal condition we have $U(z)F(z) > 0$ and $U_{\lambda_n}(z) G_{\lambda_n}(z) > 0$. Taking $n \to \infty$, we conclude that $U(z)G(z) > 0$. Therefore, $F = G$.

$\square$

Remark 3.1.10. We should note that an analog of Theorem 3.1.9 holds for $\mathcal{D}_{2n}$, $n > 1$, spaces as well. In this remark, we give a brief outline of the proof for the case of $\mathcal{D}_4$ by using Korenblum's description, Theorem 2.2.1.

The proof of an analog of Theorem 3.1.9 for $\mathcal{D}_4$ goes as follows: Let $\mathcal{M} \subseteq \mathcal{D}_4$ be a nontrivial invariant subspace. We will first assume that $\mathcal{M}$ contains an outer function. Then by the result of Korenblum's description, $\mathcal{M} = I(K_0, K_1)$, where K_0, K_1 and $I(K_0, K_1)$ are as in Theorem 2.2.1. Let $\{E_0^n\}$ and $\{E_1^n\}$ be an increasing sequence of finite sets in K_0 and K_1, respectively, such that $\bigcup\limits_n E_0^n$ is dense in E_0, $\bigcup\limits_n E_1^n$ is dense in E_1, and $E_0^n \supseteq E_1^n$ for all $n = 1, 2, \ldots$. Then set

$$\mathcal{M}_n = I(E_0^n, E_1^n) = \{f \in \mathcal{D}_4 : f = 0 \text{ on } E_0^n \text{ and } f' = 0 \text{ on } E_1^n\}.$$

It is clear that $\emptyset \neq I_n \in \mathrm{Lat}(M_z, \mathcal{D}_4)$. Since E_0^n, E_1^n are increasing, $\mathcal{M}_n$ is a decreasing sequence of invariant subspaces. Also as in the proof of Theorem 3.1.9 one shows that $\mathcal{M} = \bigcap\limits_n \mathcal{M}_n$. In order to conclude the proof, it is enough to show that $\mathcal{M}_n$ has finite co-dimension. This can be done by an argument similar to Lemma 3.1.8. The general case when $\mathcal{M} = UI(K_0, K_1)$, where U is an inner function, can be done in a way similar to what was done in the proof of Theorem 3.1.9.

As mentioned before, for $\alpha = 1$, i.e. for the Dirichlet space there is no analogue of $\mathcal{M}_E$ for finite subset $E \subset \mathbb{T}$. In the next theorem, we approximate $\mathcal{M}_E \subseteq \mathcal{D}_2$ by zero-based invariant subspace $I(Z)$ for some $Z \subseteq \mathbb{D}$ instead of approximating by using points on the unit circle.

Theorem 3.1.11. *Let $\emptyset \neq E \subset \mathbb{T}$ be a Carleson thin set and $\mathcal{M}_E = \{f \in \mathcal{D}_2 : f(z) = 0,\ \forall z \in E\}$ be the nontrivial invariant subspace of $(M_z, \mathcal{D}_2)$. Then there exists $I_n \in Lat(M_z, \mathcal{D}_2)$, with $\dim I_n^{\perp} < \infty$, such that $P_n \to P$ in SOT where P_n and P are orthogonal projections onto I_n and $\mathcal{M}_E$,respectively.*

Proof. Let $E_n = \{z_1, ..., z_n\} \subseteq E$ be such that $E_n \subseteq E_{n+1}$ and $\bigcup\limits_{n=1}^{\infty} E_n \subseteq E$ be dense. For $r < 1$ define $rE_n := \{rz_1, ..., rz_n\}$ and set

$$I(rE_n) = \{f \in \mathcal{D}_2 : f(rz_i) = 0 \ \ \forall i = 1, ..., n\}.$$

It is clear that $I(rE_n)$ is closed.

Now, choose an increasing sequence $r_n < 1$ such that $\sum_{n=1}^{\infty} n(1 - r_n) < \infty$ and consider the Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \frac{\overline{w}_n}{|w_n|} \frac{w_n - z}{1 - \overline{w}_n z}$$

where $\{w_n\}_{n=1}^{\infty} = \bigcup\limits_{n=1}^{\infty} r_n E_n = \{r_1z_1, r_2z_1, r_2z_2, r_3z_1, r_3z_2, r_3z_3, ...\}$.

Set $f(z) = B(z)f_E(z)$, where $f_E(z)$ is Korenblum's function for the Carleson thin set E (see Section 2.3). First, we claim that $f \in \mathcal{D}_2$ and $f \in \bigcap\limits_{n=1}^{\infty} I(r_nE_n)$.

By definition of the $\mathcal{D}_2$ norm we have that

$$\|f\|_2^2 = \|f\|_{H^2}^2 + \|f'\|_{H^2}^2 = \|Bf_E\|_{H^2}^2 + \|B'f_E + Bf_E'\|_{H^2}^2 .$$

Note that the first norm is obviously finite by property (ii) of the Korenblum function f_E. Also, $|f_E'B| = |f_E'|$ on $\mathbb{T}$ and the propert (i) implies that f_E' is bounded on $\overline{\mathbb{D}}$. Thus $f_E'B \in H^2$. Therefore, in order to show that $f \in \mathcal{D}_2$, it suffices to prove that $f_E B' \in H^2$. Taking the logarithmic derivative of B, we obtain

$$\frac{B'(z)}{B(z)} = \sum_{n=1}^{\infty} \frac{1-|w_n|^2}{(z-w_n)(1-\overline{w}_n z)}.$$

This implies that

$$|B'(z)| \leq \sum_{n=1}^{\infty} \frac{2(1-|w_n|)}{|z-w_n||1-\overline{w}_n|}.$$

Note that for $|z| = 1$ we have $|1-\overline{w}_n z| = |\overline{w}_n - \overline{z}|$. Hence

$$|B'(z)| \leq \sum_{n=1}^{\infty} \frac{2(1-r_n)}{|z-w_n|^2} \leq \frac{C}{d(z)^2}$$

for $z \in \mathbb{T} \setminus E$ where $d(z) = \min_{\{w_n\}} |z - w_n|$ and C is a constant.

Thus,

$$|f_E B'| \leq \frac{C c_N \rho(z)^N}{d(z)^2} \leq C_1$$

on $\mathbb{T} \setminus E$ by the property (iii) of f_E for some constant C_1. Therefore $f_E B' \in H^2$. Furthermore, by construction of f, it is clear that $f \in \bigcap_{n=1}^{\infty} I(r_n E_n)$.

Let $Z_k := \bigcup_{n=k}^{\infty} r_n E_n$ and $I(Z_k) := \{f \in \mathcal{D}_2 : f(z) = 0 \text{ for all } z \in Z_k\}$. Then one can easily show that $I(Z_k) = \bigcap_{n=k}^{\infty} I(r_n E_n)$.

Next, we want to show that

$$\bigvee_{k=1}^{\infty} I(Z_k) = \mathcal{M}_E \tag{3.1}$$

Claim1: For all k, $I(Z_k) \subset \mathcal{M}_E$.

Proof: Let $z \in \bigcup_{n=1}^{\infty} E_n$. Then there exists N such that $\forall n \geq N, \quad z \in E_n$. Hence $\forall f \in I(Z_k), \quad f(r_n z) = 0 \ \ \forall n \geq N$. Thus $f(z) = 0$ as $r_n \to 1$. Since $\bigcup_{n=1}^{\infty} E_n$ is dense in E, we get $f(z) = 0 \ \ \forall z \in E$. Therefore $f \in \mathcal{M}_E$.

□

Claim2: $[f_E] = \mathcal{M}_E$ in $\mathcal{D}_2$.

Proof: Since $f_E = 0$ on E, we have $f_E \in \mathcal{M}_E$, so $[f_E] \subset \mathcal{M}_E$. Also, by Korenbljum's theorem we have $[f_E] = G\mathcal{M}_K$ where G is the greatest common divisor of the inner parts of the functions $g \in [f_E]$ and $K = \bigcap_{g \in [f_E]} \{z \in \mathbb{T} : g(z) = 0\}$. Note that here $\mathcal{M}_K = \{g \in \mathcal{D}_2 : g(z) = 0, \forall z \in K\}$. Since f_E is outer and $f_E = 0$ on E, we must have $G \equiv 1$ and $K \subseteq E$. If there exists $z_\circ \in E \setminus K$, then $\forall g \in \mathcal{M}_K$ we have $g(z_\circ) \neq 0$ which contradicts the fact that $f_E \in \mathcal{M}_K$ and $f_E(z_\circ) = 0$. Thus $K = E$ and hence $\mathcal{M}_E = [f_E]$.

□

We can now prove (3.1). By claim 1, we have $\bigvee_{k=1}^{\infty} I(Z_k) \subset \mathcal{M}_E$. For the converse, by claim 2 it is enough to show that $f_E \in \bigvee_{k=1}^{\infty} I(Z_k)$.

Note that $I(Z_k) \subseteq I(Z_{k+1})$ since $Z_k \supseteq Z_{k+1}$. Let B_k be the Blaschke product corresponding to the set Z_k. Set $f_k = B_k f_E \in I(Z_k)$.

Since $B_k \to 1$ point-wise and $\|f_k\|_2 \leq M$ for some constant M since $f = Bf_E \in \mathcal{D}_2$, we have $f_k \to f_E$ weakly. Therefore we have (3.1).

So,

$$\mathcal{M}_E = \bigvee_{k=1}^{\infty} I(Z_k)$$

which implies that $\mathcal{M}_E^{\perp} = \bigcap_{k=1}^{\infty} I(Z_k)^{\perp}$ and $I(Z_k) \subseteq Z(I_{k+1})$ implies that $I(Z_k)^{\perp} \supseteq I(Z_{k+1})^{\perp}$.

Let P and P_k be the orthogonal projections onto $\mathcal{M}_E$ and $I(Z_k)$, respectively. Then as before it follows that $I - P_k \to I - P$ in SOT. Hence $P_k \to P$ in SOT.

We shall now establish the sequence of invariant subspaces I_n with finite co-dimension such that I_n converges to $\mathcal{M}_E$. Recall that $Z_k = \bigcup_{n=k}^{\infty} r_n E_n$.

For each k, let $B_{n,k} = \{w_1, ..., w_n\} \subseteq Z_k$ be such that $B_{n,k} \subseteq B_{n+1,k}$ and $\bigcup_{n=1}^{\infty} B_{n,k} = Z_k$. Let $I_n := I(B_{n,k})$ be the zero based invariant subspaces on $B_{n,k}$, i.e., $I(B_{n,k}) = \{f \in \mathcal{D}_2 : f(w_i) = 0, \forall i = 1, ..., n\}$. Then it is easy to see that $I(B_{n,k}) \supseteq I(B_{n+1,k})$ and $I(Z_k) = \bigcap_{n=1}^{\infty} I(B_{n,k})$. Let $P_{n,k}$ be the orthogonal projection onto $I(B_{n,k})$. Then for each k, $P_{n,k} \to P_k$ in SOT as $n \to \infty$. By Proposition 2.5.2, we can choose a subsequence $P_{n_k,k}$ such that $P_{n_k,k} \to P$ in SOT. Each $I_n = I(B_{n,k})$ has finite co-dimension follows from Lemma 3.1.8 and that concludes the proof.

$\square$

3.2 An approximation result for the Dirichlet space

The Dirichlet space $\mathbf{D}$ is the space of all anayltic functions $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ on the open unit disc $\mathbb{D}$ which have a finite Dirichlet integral

$$\mathcal{D}(f) = \iint_{\mathbb{D}} |f^{'}(z)|^2 dA(z),$$

where $dA(z) = \frac{1}{\pi} r dr dt$ denotes normalized area measure on $\mathbb{D}$.

By Parseval's formula one can compute that

$$\mathcal{D}(f) = \sum_{n=1}^{\infty} n |\hat{f}(n)|^2.$$

Then the norm on $\mathbf{D}$ becomes

$$\|f\|_{\mathbf{D}}^2 = \sum_{n=0}^{\infty} (n+1) |\hat{f}(n)|^2 = \|f\|_{\mathbf{H}^2}^2 + \mathcal{D}(f).$$

Theorem 3.2.1 (see [7]). *Let $f \in \mathbf{H}^2$ and let B be a Blaschke product with zeros $\{z_n\}$. Then*

$$\mathcal{D}(Bf) = \mathcal{D}(f) + \sum_n \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1-|z_n|^2}{|\xi - z_n|^2} |f^*(\xi)||d\xi|.$$

We do not know all of the M_z-invariant subspaces of the Dirichlet space, but we can consider

$$D_E := \{f \in \mathbf{D} : f^* = 0 \text{ q.e. on } E\}$$

for an arbitrary subset E of $\mathbb{T}$. Note that $D_E = \mathbf{D}$ if E has zero logarithmic capacity and $D_E = (0)$ if E has positive measure. Here f^* denotes the radial limit of f and it is known that radial limit exists q.e. (see Theorem 4.1.1). A property is said to hold quasi-everywhere (q.e.) on $\mathbb{T}$ if it holds everywhere on $\mathbb{T}$ except a subset of logarithmic capacity zero (see [14]).

Theorem 3.2.2 (see [6] or Theorem 9.2.3 in [14]). *For every subset $E \subset \mathbb{T}$, we have $D_E \in Lat(M_z, \mathbf{D})$.*

Now we can prove our partial approximation result for D_E. First we will prove the following lemma which provides the sequence of zero sets.

Lemma 3.2.3. *Let $E \subset \mathbb{T}$ be a compact set with Lebesgue measure $|E| = 0$. Then there exists $\mathcal{A} \subset \mathbb{D}$ a Blaschke sequence such that every point of E is a non-tangential limit point from $\mathcal{A}$.*

Proof. Let $\mathbb{T} \setminus E = \bigcup_{n=1}^{\infty} I_n$ where I_n disjoint open arcs for each n. By reordering we may assume $|I_n| \geq |I_{n+1}|$ for each n. Furthermore, we will assume $1 > |I_n|$ for each n. Of course the length of complementary subintervals I_n depends on the set E. By this further assumption perhaps we add some finitely many points to the set E but after constructing the Blaschke sequence with required property we can take out those extra points from the Blaschke sequence and E.

Let z_n be the point on the ray from origin to the midpoint of I_n such that the triangular shape made by endpoints of I_n and z_n has angular measure $120°$ at z_n, and denote this triangular shape by T_n.

Note that we have $\sum_{n=1}^{\infty} |I_n| = 2\pi$ and $\lim_{r\to 1^-} \big|\{|z| = r\} \cap T_n\big| = |I_n|$ whenever n is fixed.

For $k \in \mathbb{N}$ let $n_k \in \mathbb{N}$ be such that $\epsilon_k := \sum_{n\geq n_k} |I_n| \leq 2^{-k}$. Choose $r_k < 1$ such that

(i) $|r_k\mathbb{T} \cap T_j| \geq (1 - 2^{-k})|I_j|$ for all $1 \leq j \leq n_k$,

(ii) $(1 - r_k)n_k \leq 2^{-k}$,

(iii) $\frac{2\pi}{1-r_k} \in \mathbb{N}$.

Put $\frac{2\pi}{1-r_k}$ equally spaced points on $r_k\mathbb{T}$ for each k, and call the set of these points Z_k. For each j, $1 \leq j \leq n_k$, let P_j be the set of points that lie in $r_k\mathbb{T} \cap T_j$ and let $\mathcal{P}_k := \bigcup_{j=1}^{n_k} P_j$. Of course for each j, the exact number of points of P_j will depend on how the points are located with respect to the arc I_j, but this number will be within 1 of $\frac{2\pi}{1-r_k}\frac{|r_k\mathbb{T}\cap T_j|}{2\pi r_k}$. Now delete the points in $\mathcal{P}_k$ and let $\mathcal{A}$ be the set of the remaining points, i.e. $\mathcal{A} = \bigcup_k (Z_k \setminus \mathcal{P}_k)$.

If $n(\mathcal{P}_k)$ denotes the number of the points in $\mathcal{P}_k$, then we have

$$\sum_{j=1}^{n_k} \left\lceil \frac{2\pi}{1-r_k}\frac{|r_k\mathbb{T}\cap T_j|}{2\pi r_k} \right\rceil + n_k \geq n(P_k) \geq \sum_{j=1}^{n_k} \left\lceil \frac{2\pi}{1-r_k}\frac{|r_k\mathbb{T}\cap T_j|}{2\pi r_k} \right\rceil - n_k,$$

where $\lceil x \rceil = \min\{n \in \mathbb{Z} : n \geq x\}$.

Observe that

$$\sum_{j=1}^{n_k} \left\lceil \frac{2\pi}{1-r_k}\frac{|r_k\mathbb{T}\cap T_j|}{2\pi r_k} \right\rceil - n_k \geq \sum_{j=1}^{n_k} \frac{2\pi}{1-r_k}\frac{|r_k\mathbb{T}\cap T_j|}{2\pi} - n_k \geq \frac{1-2^{-k}}{1-r_k}\sum_{j=1}^{n_k} |I_j| - n_k.$$

Hence,

$$\sum_{k=1}^{\infty}(1-r_k)\left(\frac{2\pi}{1-r_k}-\sum_{j=1}^{n_k}\left\lceil\frac{2\pi}{1-r_k}\frac{|r_k\mathbb{T}\cap T_j|}{2\pi r_k}\right\rceil+n_k\right)$$
$$\le\sum_{k=1}^{\infty}(1-r_k)\left(\frac{2\pi}{1-r_k}-\frac{(1-2^{-k})}{1-r_k}\sum_{j=1}^{n_k}|I_j|+n_k\right)$$
$$=\sum_{k=1}^{\infty}\big(2\pi-(1-2^{-k})(2\pi-\epsilon_k)+(1-r_k)n_k\big)$$
$$=\sum_{k=1}^{\infty}\big((1-2^{-k})\epsilon_k+2^{-k}2\pi+(1-r_k)n_k\big)$$
$$\le\sum_{k=1}^{\infty}\big((1-2^{-k})2^{-k}+2^{-k}2\pi+2^{-k}\big)<\infty.$$

Therefore $\mathcal{A}$ is a Blaschke sequence.

After removing possible added points from E, notice that at each $w\in E$ we have a cone with vertex at w. This cone may not be symmetric with respect to radii but it will be contained in a non-tangential approach region. By construction we have a subsequence of $\mathcal{A}$ that is contained in the non-tangential approach region and converges to the point $w\in E$.

□

Theorem 3.2.4. *Let $E\subset\mathbb{T}$ be a Carleson thin set with positive capacity and let f_E be the corresponding outer function constructed by Korenblum that is zero on E, and smooth elsewhere. Then there exists a decreasing sequence of zero sets $Z_k\subseteq\mathbb{D}$ such that*

$$D_E\supseteq span_k I(Z_k)\supseteq[f_E].$$

Proof. It follows from Theorem 3.2.2 that D_E is a closed invariant subspace of the Dirichlet space $\mathbf{D}$, and it is nontrivial since $f_E\in D_E$. By Lemma 3.2.3 there exists $\mathcal{A}\subset\mathbb{D}$ a Blaschke sequence such that every point of E is a non-tangential limit point from $\mathcal{A}$. Let $\mathcal{A}=\{w_k\}_{k=1}^{\infty}\subseteq\mathbb{D}$, and B be the corresponding Blaschke sequence.

For each $k \geq 2$, let $Z_k = \mathcal{A} \setminus \{w_1, ..., w_{k-1}\}$ with $Z_1 = \mathcal{A}$ and consider the sequence of zero-based invariant subspaces $I(Z_k)$. Set $f(z) := B(z) f_E(z)$. Then f is in the Dirichlet space $\mathbf{D}$ and $f \in \bigcap_{k=1}^{\infty} I(Z_k)$. We can show this by using the Dirichlet integral formula given by Theorem 3.2.1. Indeed,

$$\begin{aligned}
\mathcal{D}(f) =& \mathcal{D}(B f_E) = \mathcal{D}(f_E) + \sum_{k=1}^{\infty} \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1 - |w_k|^2}{|\xi - w_k|^2} |f_E(\xi)|^2 |d\xi| \\
=& \mathcal{D}(f_E) + \sum_{k=1}^{\infty} (1 - |w_k|^2) \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1}{|\xi - w_k|^2} |f_E(\xi)|^2 |d\xi| \\
=& \mathcal{D}(f_E) + \sum_{k=1}^{\infty} (1 - |w_k|^2) \frac{1}{2\pi} \Bigg(\int_{\mathbb{T} \setminus E} \frac{1}{|\xi - w_k|^2} |f_E(\xi)|^2 |d\xi| \\
& \qquad + \int_{E} \frac{1}{|\xi - w_k|^2} |f_E(\xi)|^2 |d\xi| \Bigg) \\
=& \mathcal{D}(f_E) + \sum_{k=1}^{\infty} (1 - |w_k|^2) \frac{1}{2\pi} \sum_{n} \int_{I_n} \frac{1}{|\xi - w_k|^2} |f_E(\xi)|^2 |d\xi|
\end{aligned}$$

where $\mathbb{T} \setminus E = \bigcup_n I_n$.

Let $\xi = e^{it} \in I_n$. Then $\operatorname{dist}(e^{it}, E) \approx \operatorname{dist}(e^{it}, \partial T_n \setminus I_n) \leq |e^{it} - w_k|$. Also, from property (iii) of the Korenblum' function f_E, we have $|f_E(e^{it})|^2 \leq c_2 (\operatorname{dist}(e^{it}, E))^2$.

Putting all of this together yields

$$\mathcal{D}(f) \leq \mathcal{D}(f_E) + C \sum_{k=1}^{\infty} (1 - |w_n|^2) < \infty.$$

In order to prove that $D_E \supseteq \operatorname{span}_k I(Z_k)$, it suffices to show that for each k, $I(Z_k) \subset D_E$. Let $g \in I(Z_k)$. Since the non-tangential limits of g exist q.e. there is a subset $F \subseteq \mathbb{T}$ of logarithmic capacity zero such that $nt - \lim_{z \to w} g(z) = g(w)$ exists $\forall w \in \mathbb{T} \setminus F$. Let $w \in E \setminus F$. Then $\exists \{w_{k_n}\} \subseteq \mathcal{A}$ such that $w = nt - \lim_{n \to \infty} w_{k_n}$ by Lemma 4.1. Then $g(w_{k_n}) = 0, \quad \forall k_n \geq k$. This implies that $g(w) = 0$ as $n \to \infty$. Hence $g \in D_E$.

To finish the proof, we must show that $f_E \in \operatorname{span}_k I(Z_k)$. Let

$$f_k(z) = \frac{f(z)}{\prod\limits_{n=1}^{k} \frac{\overline{w_n}}{|w_n|} \frac{w_n - z}{1 - \overline{w_n} z}} = \frac{B(z) f_E(z)}{\prod\limits_{n=1}^{k} \frac{\overline{w_n}}{|w_n|} \frac{w_n - z}{1 - \overline{w_n} z}} = \left(\prod_{n=k+1}^{\infty} \frac{\overline{w_n}}{|w_n|} \frac{w_n - z}{1 - \overline{w_n} z} \right) f_E(z) \in I(Z_k).$$

Since $f_k \to f_E$ pointwise and f_k is norm bounded in the Dirichlet space (by $\mathcal{F}$-property), we have $f_k \to f_E$ weakly. So $f_E \in \operatorname{span}_k I(Z_k)$, which implies that $[f_E]_{\mathbf{D}} \subset \operatorname{span}_k I(Z_k)$.

$\square$

Chapter 4

Tangential Limits of Extremal Functions in the Dirichlet Space

4.1 Motivation

Given a holomorphic function f on the unit disc, $f \in \mathrm{Hol}(\mathbb{D})$, and $e^{i\theta} \in \mathbb{T}$, we define

$$f^*(e^{i\theta}) := \lim_{r \to 1^-} f(re^{i\theta}),$$

whenever this radial limit exists. If $f \in \mathbf{H}^2$, the Hardy space, then it is well known that the radial limit f^* exists a.e. on the unit circle $\mathbb{T}$, and that the limit exists in a non-tangential approach region about $e^{i\theta}$ a.e. on the unit circle $\mathbb{T}$. Since the Dirichlet space $\mathbf{D}$ is contained in the Hardy space $\mathbf{H}^2$, the same is true for functions in the Dirichlet space. In fact much more is true for the functions in the Dirichlet space.

Theorem 4.1.1 (Beurling, Theorem 3.2.1 in [14])**.** *Let $f \in \mathbf{D}$. Then there exists $E \subset \mathbb{T}$ with $c^*(E) = 0$ such that, if $e^{i\theta} \in \mathbb{T} \setminus E$, then $f^*(e^{i\theta}) := \lim_{r \to 1^-} f(re^{i\theta})$ exists, and $f(z) \to f^*(e^{i\theta})$ as $z \to e^{i\theta}$ inside each non-tangential approach region $\big\{z \in \mathbb{D} : |z - e^{i\theta}| < c(1 - |z|)\big\}$.*

Note that $c(E)$ denotes the logarithmic capacity of $E \subset \mathbb{T}$, and that $c^*(E)$ is the corresponding outer capacity. Recall that a property is said to hold quasi-everywhere (q.e.) on the unit circle $\mathbb{T}$, if it holds everywhere except on a subset of zero logarithmic capacity. Hence Beurling's theorem says that each function in the Dirichlet space has non-tangential limit quasi-everywhere on the unit circle $\mathbb{T}$.

In [14], it was proved that the Dirichlet functions have limit a.e. for certain higher order polynomially tangential and exponentially tangential approach regions.

Theorem 4.1.2 (see [30] or Theorem 3.5.1 in [14])**.** *Let $f \in \mathbf{D}$. Then, for a.e. $e^{i\theta} \in \mathbb{T}$, we have $f(z) \to f^*(e^{i\theta})$ as $z \to e^{i\theta}$ in each region*

$$|z - e^{i\theta}| < c\left(log\frac{1}{1-|z|}\right)^{-1}, \quad c > 0.$$

Before stating further results, we define polynomially tangential approach regions as follows.

Definition 4.1.3. Let $c > 0$, $\alpha \geq 1$ and $e^{i\theta} \in \mathbb{T}$. The tangential approach region of order α, denoted by $\Gamma_c^\alpha(e^{i\theta})$, at $e^{i\theta} \in \mathbb{T}$ is defined as

$$\Gamma_c^\alpha(e^{i\theta}) = \{z \in \mathbb{D} : |z - e^{i\theta}|^\alpha \leq c(1 - |z|)\}.$$

Note that if $\alpha = 1$ and $c > 1$, then the region $\Gamma_c(e^{i\theta})$ is simply a non-tangential approach region (Stolz angle)

$$\Gamma_c(e^{i\theta}) = \{z \in \mathbb{D} : |z - e^{i\theta}| \leq c(1 - |z|)\}.$$

If $c > 0$, then a simple computation shows that

$$\Gamma_c^2(e^{i\theta}) = \{z \in \mathbb{D} : |z - e^{i\theta}|^2 \leq c(1 - |z|^2)\}$$

is a closed disc internally tangent to the unit circle $\mathbb{T}$ at $e^{i\theta}$ with center $\dfrac{e^{i\theta}}{c+1}$ and radius $\dfrac{c}{c+1}$. This region is also called an oricylic approach region. If $\alpha > 1$, then $\Gamma_c^\alpha(e^{i\theta})$ is a region contained in the unit disc $\mathbb{D}$ which touches unit circle $\mathbb{T}$ at $e^{i\theta}$ tangentially. As α increases, the degree of tangency also increases.

If $w = re^{it} \in \Gamma_c^\alpha(e^{i\theta})$, then one can easily compute that

$$(1-r)^2 + 4r\Big(\sin(\frac{t-\theta}{2})\Big)^2 \leq (c(1-r))^{2/\alpha}. \tag{4.1}$$

The following Theorem of Twomey strengthens Beurling's result Theorem 4.1.1.

Theorem 4.1.4 (Theorem 5 in [44])**.** *Let $f \in \mathbf{D}$ and $\alpha > 1$. Then there exists $E \subseteq \mathbb{T}$ with $c^*(E) = 0$ such that, if $e^{i\theta} \in \mathbb{T} \setminus E$, then $f(z) \to f^*(e^{i\theta})$ as $z \to e^{i\theta}$ inside each tangential approach region $\Gamma_c^\alpha(e^{i\theta})$.*

In [35], S. Richter and C. Sundberg proved the following theorem.

Theorem 4.1.5 (Theorem 5.1 in [35])**.** *Let $\mathcal{M} \in Lat(M_z, \mathbf{D})$ and $\varphi \in \mathcal{M} \ominus z\mathcal{M}$. Then*

(a) $|\varphi(\lambda)| \leq \|\varphi\|_{\mathbf{D}}, \forall \lambda \in \mathbb{D}$

(b) $\psi(e^{it}) = \lim_{r\to 1} |\varphi(re^{it})|$ exists for every $e^{it} \in \mathbb{T}$

(c) ψ is upper semicontinuous on $\mathbb{T}$ and

$$\lim_{\lambda \to e^{it}} \sup_{\lambda \in \mathbb{D}} |\varphi(\lambda)| = \psi(e^{it}), \forall e^{it} \in \mathbb{T},$$

(d) the radial zero set of φ, $Z(\varphi)$, equals $\left\{z \in \overline{\mathbb{D}} : \limsup_{\lambda \to z} |\varphi(\lambda)| = 0\right\}$ and it is a G_δ set.

Part (b) of above theorem says that the radial limit of $|\varphi|$, where φ is the extremal function in the Dirichlet space, exists for every point $e^{it} \in \mathbb{T}$. In this chapter, we will extend this result and prove that limit exists in the tangential approach regions $\Gamma_c^\alpha(e^{i\theta})$ for every point $e^{i\theta} \in \mathbb{T}$.

4.2 Tangential Limits of Extremal functions

Let φ be the extremal function in the Dirichlet space $\mathbf{D}$, i.e. $\|\varphi\| = 1$ and $z^n\varphi \perp \varphi$ for all $n > 0$. Then by formula (5.1) in [35], we have the following equation for $w = re^{it} \in \mathbb{D}$,

$$r = \int_0^r \left(\int_{\mathbb{T}} P_{se^{it}}(z) D_z(\varphi) \frac{|dz|}{2\pi} \right) ds + r \int_{\mathbb{T}} P_w(z) |\varphi(z)|^2 \frac{|dz|}{2\pi}. \tag{4.2}$$

Note that here P_w is the Poisson kernel

$$P_w(z) = \frac{1-|w|^2}{|z-w|^2} = \frac{1}{1-w\bar{z}} + \frac{1}{1-\overline{w}z} - 1 = \sum_{k=0}^{\infty} w^k \bar{z}^k + \sum_{k=0}^{\infty} \overline{w}^k z^k - 1, \quad |z| = 1,$$

and $D_z(\varphi)$ is the local Dirichlet integral of φ at z

$$D_z(\varphi) = \int_{\mathbb{T}} \left| \frac{\varphi(\xi) - \varphi(z)}{\xi - z} \right|^2 \frac{|d\xi|}{2\pi}.$$

The first integral on the right-hand side of (4.2) is positive and increasing in r, hence its limit exists for each e^{it} and its limit is less than or equal to 1.

By an application of Hölder's inequality to the Poisson integral representation of φ or Theorem 2.12 of [13], we have

$$|\varphi(w)|^2 \le \int_{\mathbb{T}} P_w(z) |\varphi(z)|^2 \frac{|dz|}{2\pi} \le 1$$

where the second inequality follows from the equation (4.2) and hence

$$\lim_{|w|\to 1} \int_{\mathbb{T}} P_w(z) |\varphi(z)|^2 \frac{|dz|}{2\pi}$$

exists for every $e^{it} \in \mathbb{T}$.

Since the Dirichlet space is contained in VMOA (see [41]), the radial limit

$$|\varphi(e^{it})|^2 := \lim_{|w|\to 1} |\varphi(w)|^2 = \lim_{|w|\to 1} \int_{\mathbb{T}} P_w(z) |\varphi(z)|^2 \frac{|dz|}{2\pi}$$

exists for each $e^{it} \in \mathbb{T}$ and is less than or equal 1 as well. Then by equation (4.2), we have

$$|\varphi(e^{it})|^2 = 1 - \int_{\mathbb{T}} \left(\int_0^1 P_{se^{it}}(z) ds \right) D_z(\varphi) \frac{|dz|}{2\pi}. \tag{4.3}$$

Also, dividing the equation (4.2) by r, we get

$$\int_{\mathbb{T}} P_w(z) |\varphi(z)|^2 \frac{|dz|}{2\pi} = 1 - \int_{\mathbb{T}} \left(\frac{1}{r} \int_0^r P_{se^{it}}(z) ds \right) D_z(\varphi) \frac{|dz|}{2\pi}. \tag{4.4}$$

By a direct computation we obtain

$$\frac{1}{r} \int_0^r P_{se^{it}}(z) ds = 2\mathrm{Re} k_w(z) - 1, \tag{4.5}$$

where $k_w(z) = \dfrac{1}{\overline{w}z} \log \dfrac{1}{1 - \overline{w}z}$ is the reproducing kernel for the Dirichlet space $\mathbf{D}$.

Therefore, (4.4) is equal to

$$\int_{\mathbb{T}} P_w(z) |\varphi(z)|^2 \frac{|dz|}{2\pi} = 1 - \int_{\mathbb{T}} \big(2\mathrm{Re} k_w(z) - 1 \big) D_z(\varphi) \frac{|dz|}{2\pi}. \tag{4.6}$$

Proposition 4.2.1. *Let φ be an extremal function in the Dirichlet space $\mathbf{D}$. Then*

$$\sup_{0 \neq w \in \mathbb{D}} \int_{|z|=1} \frac{1}{|w|} \left| log \left(\frac{1}{1 - \overline{w}z} \right) \right| D_z(\varphi) \frac{|dz|}{2\pi} < \infty.$$

Proof. Let $w \in \mathbb{D} \setminus \{0\}$ and $z \in \mathbb{T}$ be such that $\overline{w}z = re^{i\theta}$. Since the reproducing kernel $k_w(z)$ is small for small w, we will assume that $|w| \geq \dfrac{1}{2}$. Observe that

$$\mathrm{Re} k_w(z) = \frac{\cos(\theta)}{|r|} \log \frac{1}{|1 - re^{i\theta}|} + \frac{\sin(\theta)}{|r|} \mathrm{Arg} \left(\frac{1}{1 - re^{i\theta}} \right).$$

Set

$$S = \left\{ z \in \mathbb{T} : \overline{w}z = re^{i\theta}, \left| \theta - \frac{\pi}{2} \right| < \frac{\pi}{4} \ \text{ or } \ \left| \theta - \frac{3\pi}{2} \right| < \frac{\pi}{4} \right\}.$$

Now, on the set $\mathbb{T}\backslash S$, it is easy to see $|k_w(z)|$ is comparable with the real part of the reproducing kernel $k_w(z)$, i.e. $|k_w(z)| \approx \mathrm{Re}k_w(z)$. On the set S, since $|1 - re^{i\theta}| \geq \frac{1}{2}$, it follows that $|k_w(z)| \leq C$, where C is a constant.

Notice that from equation (4.6) we have

$$\int_{\mathbb{T}} \big(2\mathrm{Re}k_w(z) - 1\big) D_z(\varphi) \frac{|dz|}{2\pi} \leq 1,$$

which implies that

$$\int_{\mathbb{T}} \mathrm{Re}k_w(z) D_z(\varphi) \frac{|dz|}{2\pi} \leq 1,$$

since φ is the extremal function in the Dirichlet space $\mathbf{D}$.

Hence

$$\int_{|z|=1} \frac{1}{|w|} \left| \log\left(\frac{1}{1 - \overline{w}z}\right) \right| D_z(\varphi) \frac{|dz|}{2\pi} = \int_{|z|=1} |k_w(z)| D_z(\varphi) \frac{|dz|}{2\pi} \leq M,$$

where M is a constant. Taking supremum over w finishes proof.

$\square$

We can now prove our main theorem.

Theorem 4.2.2. *Let φ be an extremal function in the Dirichlet space $\mathbf{D}$. Then for each $e^{it} \in \mathbb{T}$, we have $|\varphi(w)|^2 \to |\varphi(e^{it})|^2$ as $w \to e^{it}$ in the tangential approach region $\Gamma_c^\alpha(e^{it})$, for $1 \leq \alpha < \infty$.*

Proof. Without loss of generality, we assume $e^{it} = 1$ and prove that $|\varphi(w)|^2 \to |\varphi(1)|^2$ as $w = re^{i\theta} \to 1$ in the tangential approach region $\Gamma_c^\alpha(1)$. Since the Dirichlet space is contained in VMOA, it suffices to show that

$$\int_{|z|=1} P_w(z) |\varphi(z)|^2 \frac{|dz|}{2\pi} \to |\varphi(1)|^2$$

as $w = re^{i\theta} \to 1$ in the tangential approach region $\Gamma_c^\alpha(1)$.

Adding and subtracting

$$\int_{|z|=1} P_{|w|}(z)|\varphi(z)|^2 \frac{|dz|}{2\pi}$$

from

$$\left| \int_{|z|=1} P_w(z)|\varphi(z)|^2 \frac{|dz|}{2\pi} - |\varphi(1)|^2 \right|$$

and using triangle inequality, we obtain

$$\left| \int_{|z|=1} P_w(z)|\varphi(z)|^2 \frac{|dz|}{2\pi} - |\varphi(1)|^2 \right| \leq \underbrace{\left| \int_{|z|=1} P_w(z)|\varphi(z)|^2 \frac{|dz|}{2\pi} - \int_{|z|=1} P_{|w|}(z)|\varphi(z)|^2 \frac{|dz|}{2\pi} \right|}_{\mathrm{I}}$$

$$+ \underbrace{\left| \int_{|z|=1} P_{|w|}(z)|\varphi(z)|^2 \frac{|dz|}{2\pi} - |\varphi(1)|^2 \right|}_{\mathrm{II}}.$$

Notice that by Theorem (4.1.4), II $\to 0$ as $|w| \to 1$. Using the equation (4.6) for each integral in I, we will get

$$\mathrm{I} \leq \int_{|z|=1} |2\mathrm{Re}(k_r(z) - k_w(z))| \, D_z(\varphi) \frac{|dz|}{2\pi}.$$

Adding and subtracting $\dfrac{1}{rz}\log\left(\dfrac{1}{1 - zre^{-i\theta}}\right)$ to $k_r(z) - k_{re^{i\theta}}(z)$, we get

$$k_r(z) - k_{re^{i\theta}}(z) = \frac{1}{rz}\log\left(\frac{1 - zre^{-i\theta}}{1 - rz}\right) + \frac{1}{rz}(1 - e^{i\theta})\log\left(\frac{1}{1 - zre^{-i\theta}}\right).$$

Then, for $|z| = 1$

$$|\mathrm{Re}(k_r(z) - k_{re^{i\theta}}(z))| \leq \frac{1}{r}\left|\log\left(\frac{1 - zre^{-i\theta}}{1 - rz}\right)\right| + \frac{1}{r}\left|1 - e^{i\theta}\right|\left|\log\left(\frac{1}{1 - zre^{-i\theta}}\right)\right|.$$

Therefore,

$$\mathrm{I} \leq \int_{|z|=1} \frac{2}{r} \left| \log \left(\frac{1 - zre^{-i\theta}}{1 - rz} \right) \right| D_z(\varphi) \frac{|dz|}{2\pi}$$
$$+ \frac{2}{r} \left|1 - e^{i\theta}\right| \int_{|z|=1} \left| \log \left(\frac{1}{1 - zre^{-i\theta}} \right) \right| D_z(\varphi) \frac{|dz|}{2\pi} \quad (4.7)$$

By Proposition (4.2.2) the second term on the right-hand side of (4.7) will go to zero as $w = re^{i\theta} \to 1$ in the tangential approach region $\Gamma_c^\alpha(1)$. It is left to show that the first integral in (4.7) goes to zero as well. To do that, it suffices to show that

$$\int_{|z|=1} \frac{2}{r} \left| \log \left(\frac{z - re^{i\theta}}{z - r} \right) \right| D_z(\varphi) \frac{|dz|}{2\pi} \to 0$$

as $re^{i\theta} \to 1$ in the approach region $\Gamma_c^\alpha(1)$.

$$\int_{|z|=1} \frac{2}{r} \left| \log \left(\frac{z - re^{i\theta}}{z - r} \right) \right| D_z(\varphi) \frac{|dz|}{2\pi} \leq \int_{|z|=1} \frac{2}{r} \left| \log \left| \frac{z - re^{i\theta}}{z - r} \right| \right| D_z(\varphi) \frac{|dz|}{2\pi}$$
$$+ \int_{|z|=1} \frac{2}{r} \left| \mathrm{Arg} \left(\frac{z - re^{i\theta}}{z - r} \right) \right| D_z(\varphi) \frac{|dz|}{2\pi} \quad (4.8)$$

The second integral on the right-hand side of (4.8) will converge to zero as $re^{i\theta} \to 1$ by the Dominated Convergence Theorem (DCT), since $\left| \mathrm{Arg} \left(\frac{z - re^{i\theta}}{z - r} \right) \right|$ is bounded (one can see this from the picture) and will approach zero as $re^{i\theta} \to 1$ in the tangential approach region $\Gamma_c^\alpha(1)$. What we really need to show that is the first integral on the right-hand side of (4.8) converges to zero.

For $w = re^{i\theta} \in \Gamma_c^\alpha(1)$, define

$$S_w = \left\{ z \in \mathbb{T} : \left| \frac{r - w}{z - r} \right| \geq \frac{1}{2} \right\}.$$

Then we can write the first integral on the right-hand side of (4.8) as follows,

$$\int_{\mathbb{T}} \frac{2}{r}\left|\log\left|\frac{z-re^{i\theta}}{z-r}\right|\right| D_z(\varphi)\frac{|dz|}{2\pi} = \int_{S_w} \frac{2}{r}\left|\log\left|\frac{z-re^{i\theta}}{z-r}\right|\right| D_z(\varphi)\frac{|dz|}{2\pi}$$

$$+\int_{\mathbb{T}} \chi_{\mathbb{T}\setminus S_w}(z)\frac{2}{r}\left|\log\left|\frac{z-re^{i\theta}}{z-r}\right|\right| D_z(\varphi)\frac{|dz|}{2\pi}. \quad (4.9)$$

Let $f_w(z) = f_{re^{i\theta}}(z) = \chi_{\mathbb{T}\setminus S_w}(z)\frac{2}{r}\left|\log\left|\frac{z-re^{i\theta}}{z-r}\right|\right|$. If $z \in S_w$, then $f_w(z) = 0$. If $z \notin S_w$, then $\left|\frac{r-w}{z-r}\right| < \frac{1}{2}$.

Observe that on S_w,

$$\left|\frac{z-re^{i\theta}}{z-r}\right| = \left|1+\frac{r-re^{i\theta}}{z-r}\right| \leq 1+\left|\frac{r-re^{i\theta}}{z-r}\right| < \frac{3}{2},$$

and by reverse triangle inequality $\left|\frac{z-re^{i\theta}}{z-r}\right| > \frac{1}{2}$.

Then

$$\log\frac{1}{2} \leq \log\left|\frac{z-re^{i\theta}}{z-r}\right| \leq \log\frac{3}{2}.$$

Also $f_w(z) \to 0$ point-wise as $w = re^{i\theta} \to 1$ in the tangential approach region $\Gamma_c^{\alpha}(1)$. Therefore, the second integral on the right-hand side of (4.9) will approach to zero by the DCT, as $w = re^{i\theta} \to 1$ in the tangential approach region.

Multiplying and dividing the first integral on the right-hand side of (4.9) by

$$\log\frac{2}{|z-r|},$$

we obtain

$$\int_{S_w} \frac{2}{r}\left|\log\left|\frac{z-re^{i\theta}}{z-r}\right|\right| D_z(\varphi)\frac{|dz|}{2\pi} = \int_{S_w} \frac{2}{r}\frac{\left|\log\left|\frac{z-re^{i\theta}}{z-r}\right|\right|}{\log\frac{2}{|z-r|}}\log\frac{2}{|z-r|}D_z(\varphi)\frac{|dz|}{2\pi}.$$

Thus, by Proposition (4.2.2) it will be enough to show that

$$\frac{\left|\log\left|\dfrac{z-re^{i\theta}}{z-r}\right|\right|}{\log\dfrac{2}{|z-r|}}$$

is bounded on S_w. Then the result follows by the Dominated Convergence Theorem.

Since

$$\left|\log\left|\frac{z-re^{i\theta}}{z-r}\right|\right| = \left|\log\left(\frac{|z-re^{i\theta}|}{2}\frac{2}{|z-r|}\right)\right| = \left|\log\frac{|z-re^{i\theta}|}{2} + \log\frac{2}{|z-r|}\right|$$

$$\leq \log\frac{2}{|z-re^{i\theta}|} + \log\frac{2}{|z-r|},$$

then

$$\frac{\left|\log\left|\dfrac{z-re^{i\theta}}{z-r}\right|\right|}{\log\dfrac{2}{|z-r|}} \leq 1 + \frac{\log\dfrac{2}{|z-re^{i\theta}|}}{\log\dfrac{2}{|z-r|}}. \tag{4.10}$$

A direct computation shows that

$$|r-w|^2 = \left|r-re^{i\theta}\right|^2 = 4r^2\left(\sin\left(\frac{\theta}{2}\right)\right)^2$$

and (4.1) implies that $|r-w| \leq c(1-r)^{1/\alpha}$.

Also on S_w, $|z-r| \leq 2\,|r-w|$ and hence $|z-r| \leq 2c(1-r)^{1/\alpha}$.

Thus

$$\log\frac{2}{|z-r|} \geq \log\left(\frac{1}{c}\right) + \frac{1}{\alpha}\log\left(\frac{1}{1-r}\right).$$

Moreover, on the unit circle $\mathbb{T}$, $\left|z-re^{i\theta}\right| \geq (1-r)$, which implies that

$$\log\frac{2}{|z-re^{i\theta}|} \leq \log\frac{2}{(1-r)}.$$

Using above inequalities, one gets

$$(4.10) \leq 1 + \alpha \frac{\log \dfrac{2}{(1-r)}}{\log \dfrac{1}{(1-r)}},$$

which is bounded since $\dfrac{\log \dfrac{2}{(1-r)}}{\log \dfrac{1}{(1-r)}} \searrow 1$. That finishes the proof.

$\square$

Bibliography

[1] Agler, J. and McCarthy, J. E. (2002). *Pick interpolation and Hilbert function spaces*, volume 44 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI. 16

[2] Aleman, A. (1993). *The Multiplication Operator on Hilbert Spaces of Analytic Functions*. Habilitationsschrift, Hagen. 14

[3] Apostol, C., Bercovici, H., Foias, C., and Pearcy, C. (1985). Invariant subspaces, dilation theory, and the structure of the predual of a dual algebra. I. *J. Funct. Anal.*, 63(3):369–404. 14

[4] Beurling, A. (1940). Ensembles exceptionnels. *Acta Math.*, 72:1–13. 15

[5] Borichev, A., Hadwin, D., and Yousefi, H. (2013). Stable and norm-stable invariant subspaces. *J. Operator Theory*, 69(1):3–16. 5, 6

[6] Brown, L. and Shields, A. L. (1984). Cyclic vectors in the Dirichlet space. *Trans. Amer. Math. Soc.*, 285(1):269–303. 11, 17, 34

[7] Carleson, L. (1952a). On the zeros of functions with bounded dirichlet integrals. *Mathematische Zeitschrift*, 56(3):289–295. 34

[8] Carleson, L. (1952b). Sets of uniqueness for functions regular in the unit circle. *Acta Math.*, 87:325–345. 15

[9] Conway, J. B. (2013). *A course in functional analysis*, volume 96. Springer Science & Business Media. 17

[10] Conway, J. B. and Hadwin, D. (1997). Stable invariant subspaces for operators on Hilbert space. *Ann. Polon. Math.*, 66:49–61. Volume dedicated to the memory of Włodzimierz Mlak. 5

[11] Duren, P., Khavinson, D., Shapiro, H. S., and Sundberg, C. (1993). Contractive zero-divisors in Bergman spaces. *Pacific J. Math.*, 157(1):37–56. 16

[12] Duren, P. and Schuster, A. (2004). *Bergman spaces*, volume 100 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI. 16

[13] Duren, P. L. (2000). *Theory of Hp spaces*. Courier Corporation. 42

[14] El-Fallah, O., Kellay, K., Mashreghi, J., and Ransford, T. (2014). *A primer on the Dirichlet space*, volume 203 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge. 7, 34, 39, 40

[15] Garnett, J. (2007). *Bounded analytic functions*, volume 236. Springer Science & Business Media. 20

[16] Greene, D. C., Richter, S., and Sundberg, C. (2002). The structure of inner multipliers on spaces with complete nevanlinna pick kernels. *Journal of Functional Analysis*, 194(2):311–331. 18

[17] Halmos, P. R. (1982). *A Hilbert space problem book*, volume 19 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, second edition. Encyclopedia of Mathematics and its Applications, 17. 18

[18] Hedenmalm, H. (1991). A factorization theorem for square area-integrable analytic functions. *J. Reine Angew. Math.*, 422:45–68. 16

[19] Hedenmalm, H., Korenblum, B., and Zhu, K. (2000). *Theory of Bergman spaces*, volume 199 of *Graduate Texts in Mathematics*. Springer-Verlag, New York. 16

[20] Hedenmalm, P. J. H. k. (1993). An invariant subspace of the Bergman space having the codimension two property. *J. Reine Angew. Math.*, 443:1–9. 14

[21] Hoffman, K. (1988). *Banach spaces of analytic functions*. Dover Publications, Inc., New York. Reprint of the 1962 original. 23

[22] Kopp, R. P. (1969). A subcollection of algebras in a collection of Banach spaces. *Pacific J. Math.*, 30:433–435. 11

[23] Korenbljum, B. I. (1971a). A certain extremal property of outer functions. *Mat. Zametki*, 10:53–56. 21

[24] Korenbljum, B. I. (1971b). The functions that are holomorphic in the disk and smooth up to its boundary. *Dokl. Akad. Nauk SSSR*, 200:24–27. 15, 21, 22

[25] Korenbljum, B. I. (1972a). Invariant subspaces of the shift operator in a weighted Hilbert space. *Mat. Sb. (N.S.)*, 89(131):110–137, 166. v, 4, 14, 15, 21, 22, 23

[26] Korenbljum, B. I. (1972b). Invariant subspaces of the shift operator in certain weighted Hilbert spaces of sequences. *Dokl. Akad. Nauk SSSR*, 202:1258–1260. 5, 14, 21

[27] Korenbljum, B. I. and Faĭvyševskiĭ, V. M. (1972). A certain class of compression operators that are connected with the divisibility of analytic functions. *Ukrain. Mat. Ž.*, 24:692–695, 717. 20

[28] Korenbljum, B. I. and Korolevič, V. S. (1970). Analytic functions that are regular in a disc and smooth on its boundary. *Mat. Zametki*, 7:165–172. 21

[29] Luo, S. and Richter, S. (2015). Hankel operators and invariant subspaces of the Dirichlet space. *J. Lond. Math. Soc. (2)*, 91(2):423–438. 19

[30] Nagel, A., Rudin, W., and Shapiro, J. H. (1982). Tangential boundary behavior of functions in Dirichlet-type spaces. *Ann. of Math. (2)*, 116(2):331–360. 40

[31] Nikol'skii, N. K. (1984). Two problems on spectral synthesis. *Journal of Soviet Mathematics*, 26(5):2185–2186. 1

[32] Richter, S. (1987). Invariant subspaces in Banach spaces of analytic functions. *Trans. Amer. Math. Soc.*, 304(2):585–616. 4, 12, 13, 14, 19

[33] Richter, S. (1991). A representation theorem for cyclic analytic two-isometries. *Trans. Amer. Math. Soc.*, 328(1):325–349. 7

[34] Richter, S. and Shields, A. (1988). Bounded analytic functions in the Dirichlet space. *Math. Z.*, 198(2):151–159. 7, 14

[35] Richter, S. and Sundberg, C. (1994). Invariant subspaces of the Dirichlet shift and pseudocontinuations. *Trans. Amer. Math. Soc.*, 341(2):863–879. vi, 7, 8, 16, 41, 42

[36] Rudin, W. (1969). *Function theory in polydiscs.* W. A. Benjamin, Inc., New York-Amsterdam. 18

[37] Rudin, W. (1987). *Real and complex analysis.* McGraw-Hill Book Co., New York, third edition. 13

[38] Shields, A. L. (1974). Weighted shift operators and analytic function theory. pages 49–128. Math. Surveys, No. 13. 12

[39] Shimorin, S. M. (1998). Reproducing kernels and extremal functions in Dirichlet-type spaces. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 255(Issled. po Lineĭn. Oper. i Teor. Funkts. 26):198–220, 254. 16

[40] Shimorin, S. M. (2000). Approximate spectral synthesis in the Bergman space. *Duke Math. J.*, 101(1):1–39. 1, 3, 4, 19

[41] Stegenga, D. A. (1979). A geometric condition which implies BMOA. In *Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1*, Proc. Sympos. Pure Math., XXXV, Part, pages 427–430. Amer. Math. Soc., Providence, R.I. 42

[42] Taylor, B. and Williams, D. (1970). Ideals in rings of analytic functions with smooth boundary values. *Canad. J. Math*, 22:1266–1283. 15

[43] Taylor, G. D. (1966). Multipliers on D_α. *Trans. Amer. Math. Soc.*, 123:229–240. 10

[44] Twomey, J. B. (2002). Tangential boundary behaviour of harmonic and holomorphic functions. *J. London Math. Soc. (2)*, 65(1):68–84. 41

Vita

Faruk Yilmaz was born in Adiyaman, Turkey in 1982. He graduated from the Yildiz Technical University in 2006 with Bachelor's degree in Mathematics and in 2007 with Master's degree in Mathematics. Faruk started his Doctorate of Philosophy in Mathematics at the University of Tennessee, Knoxville in the Spring of 2010. Faruk is now pursuing a job in academia, where he is hoping to teach and do research.