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I am submitting herewith a thesis written by Donna Creel Stein entitled "Convex Cones, Polyhedral Sets and Polytopes Applied to Linear Systems and Linear Programming." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science with a major in Mathematics.

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**CONVEX CONES, POLYHEDRAL SETS
AND POLYTOPES
APPLIED TO LINEAR SYSTEMS AND LINEAR
PROGRAMMING**

A Thesis
Presented for the
Master Of Science
Degree
The University of Tennessee, Knoxville

Donna Creel Stein

May, 1998

Dedication

This thesis is dedicated to my beloved sisters

Cheryl Creel Cross and Carolyn Creel Taylor

for all the love and support that they have shown me

and posthumously

to my devoted brother Rodney Creel

who will always live in my heart.

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Lastly, I want to thank my parents and my sisters for all their prayers, love and encouragement. Above all, I want to thank the Lord Jesus Christ for answering my prayers and enabling me to complete this degree.

Abstract

This study, explores the subject of convex cones, polyhedral sets, polytopes and their relationship to linear systems and linear programming. This paper begins with the theory of convex sets. At this point necessary definitions, examples, and theorems of convexity are stated. These theorems and definitions are then used to introduce various classes of convex cones and their application to linear systems. Specifically, polyhedral cones are discussed in depth. These preliminary results are then applied to polyhedral sets and polytopes and their application to linear programming.

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Introduction

In this paper we study convex sets in n -dimensional Euclidean space and their application to linear programming. We are particularly concerned with convex cones, polyhedral sets and polytopes.

Chapter I explores the general concept of a convex set. In this chapter we present the notation and definitions involving convex sets. Several illustrations of convex and nonconvex sets are given, along with introductory theorems of convexity.

In chapter II we investigate convex cones. We begin with the definition of a cone and then present its properties. Several examples of convex cones are illustrated. We then introduce a convex polyhedral cone along with several examples. At this point we include several theorems involving convex cones. Lastly we observe that the nonnegative solutions to certain linear systems are convex cones.

Chapter III explores polyhedral sets and polytopes. We begin with definitions and examples of polyhedral sets and polytopes. We explore several theorems including Euler's hypothesis. The definition of a linear programming problem is introduced, along with the relationship between linear programming and polyhedral sets and polytopes.

Finally, chapter IV deals with the applications of convex cones and polytopes in linear programming. We end with the exploration of existence and dual theorems.

Chapter I

Convex Sets

Introduction

Chapter 1 is a summary of the introductory material of convex sets. This chapter includes necessary definitions, concepts, notation, basic theorems and examples of convex sets. The main references of chapter I are Bazaraa [2], Gass [6], Hadley [7], Lay [12], and Rockafellar [15].

Definitions and Examples Involving Convex Sets

Throughout this paper we will denote n -dimensional Euclidean space by E_n and vectors of n -tuple space will be denoted by bold-face type. The transpose of a vector, $\mathbf{x}$ will be denoted with a superscript of t . Hence we write $\mathbf{x}^t$ is the transpose of the vector $\mathbf{x}$.

Definition 1.1: We begin with the definition of Convex. A set S in E_n is said to be *convex* if the line segment joining any two points in the set also belongs to the set. In other words, if $\mathbf{x}_1$ and $\mathbf{x}_2$ are in S , then $\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$ must also belong to S for each $\lambda \in [0,1]$. An alternate definition of convex is to state that for each pair of points $\mathbf{x}_1$ and $\mathbf{x}_2$ in S that the line segment joining $\mathbf{x}_1$ and $\mathbf{x}_2$ is completely contained within the set S (i.e. $[\mathbf{x}_1 \mathbf{x}_2] \subseteq S$)

Definition 1.2: Convex Combinations. Weighted averages of the form: $\lambda \mathbf{x}_1 + (1-\lambda) \mathbf{x}_2$, where $\lambda \in [0,1]$ are referred to as *convex combinations* of $\mathbf{x}_1$ and $\mathbf{x}_2$.

Inductively, weighted averages of the form $\sum_{j=1}^k \lambda_j \mathbf{x}_j$, where $\sum_{j=1}^k \lambda_j = 1$, $\lambda_j \geq 0$, $j = 1 \dots k$ are also called *convex combinations* of $\mathbf{x}_1 \dots \mathbf{x}_k$. If λ_j is not required to be positive, the combination is an *affine combination*. If λ_j is in E_1 , the combination is a *linear combination*.

We will denote the closure of a set S by $\text{cl } S$ and the interior of a set S by $\text{int } S$.

Definition 1.3: Solid. A set is said to be *solid* if it has a nonempty interior.

Definition 1.4. Open. A set is said to be *open* if $S = \text{int } S$.

Convex sets are related to such areas of mathematics as geometry, analysis, and linear algebra. Previously, we defined a set S as *convex* if for each pair of points $\mathbf{x}_1$ and $\mathbf{x}_2$ in S the line segment joining $\mathbf{x}_1$ and $\mathbf{x}_2$ is contained in S . This definition uses the concept of betweenness; that is, all of the points between $\mathbf{x}_1$ and $\mathbf{x}_2$ are contained in S . Hence, a subset L of a real vector space is convex provided L contains every segment whose endpoints belong to L . An example of a set that is not convex can be seen in Figure 1.1 below. Clearly, the line from the point x to the point y is not totally contained within the set S . In other words, if S was convex then since x and y are in S , then $\lambda x + (1-\lambda)y$ must also belong to S for each $\lambda \in [0,1]$. But in the case of Figure 1.1, $\lambda x + (1-\lambda)y$ is clearly not all contained within S .

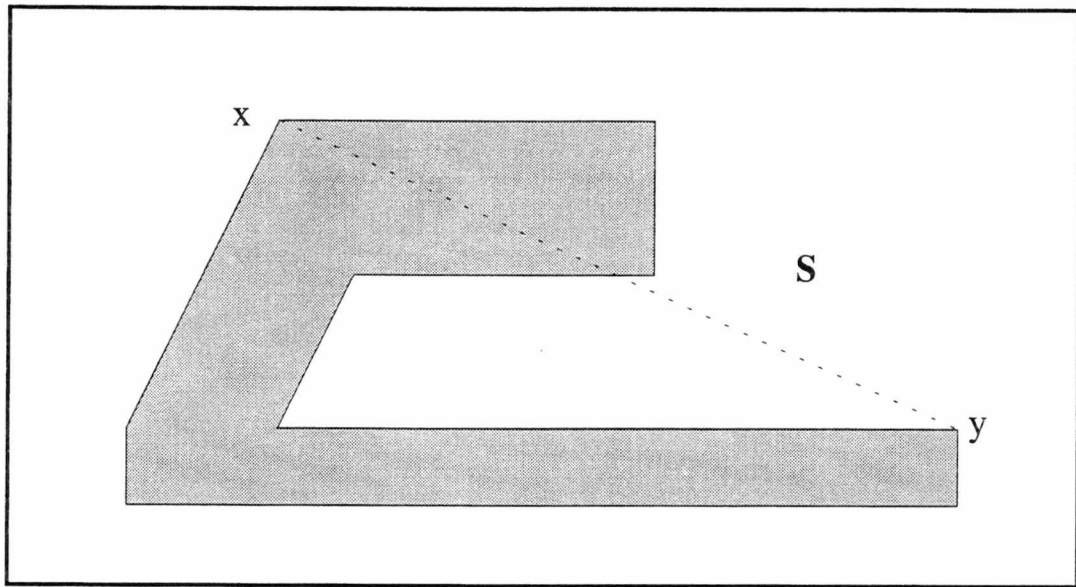


Figure 1.1 - A Nonconvex Set: S

In Chapters Three and Four we will focus on particular types of convex sets and their application to linear programming.

Definition 1.5: Boundary. x is in the *boundary* of S , denoted ∂S , if the epsilon ball about x , $N_\epsilon(x)$ contains at least one point in S and one point not in S for every $\epsilon > 0$. A set is *bounded* if it can be contained in a ball of sufficiently large radius.

Consider the rectangular box, B in Figure 1.2 on the following page. The *boundary* of the box B is the exterior edge and its faces. The boundary of B is not convex since the points x and y are both contained in the boundary (on an edge), but the points in between x and y are neither on an edge or a face of the box. That is, $\lambda x + (1-\lambda)y$ does not belong to S for each $\lambda \in [0,1]$. Yet, at the same time, the box in its entirety is convex since all the points between any two points

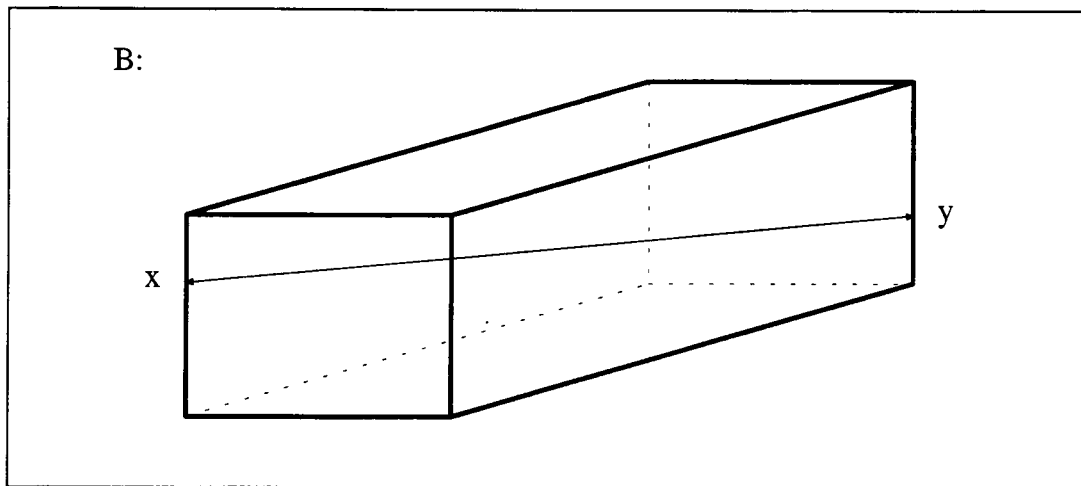


Figure 1.2 - A Set That is Both Nonconvex and Convex Set

contained in or on B are still contained within B . The box B is actually a special type of convex set, called a *Bounded Convex Polyhedron* which we shall discuss further in Chapter Three.

Definition 1.6: Half Space. An important example of a convex set is a *half space*. For any non-zero $\mathbf{b} \in E_n$ and any $\beta \in E_1$, the sets $\{\mathbf{x}: \mathbf{x}' \mathbf{b} \leq \beta\}$ and $\{\mathbf{x}: \mathbf{x}' \mathbf{b} \geq \beta\}$ are called *closed half spaces*. The sets $\{\mathbf{x}: \mathbf{x}' \mathbf{b} < \beta\}$ and $\{\mathbf{x}: \mathbf{x}' \mathbf{b} > \beta\}$ are called *open half spaces*.

Another example of a convex set is $S = \{\mathbf{x}: \mathbf{x} \text{ solves the problem below}\}$

$$\begin{array}{lll} \text{Problem P:} & \text{Minimize} & \mathbf{c}' \mathbf{x} \\ & \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & & \mathbf{x} \geq 0 \end{array}$$

Where $\mathbf{c}$ is an n vector, $\mathbf{b}$ is an m vector, $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{x}$ is an n vector. S gives all optimal solutions to the linear programming problem of minimizing the

linear function of $\mathbf{c}^t \mathbf{x}$ over the polyhedral region defined by $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$. The convex set of feasible solutions to the constraints $A\mathbf{x} = \mathbf{b}$ for general linear programming problems lie in n -dimensional Euclidean space which we refer to as the solution space. The extreme points (see Figure 1.3 below) and extreme directions of the convex set provide the basic solution. The extreme points lie in what we refer to as the intersection of the hyperplanes which we define below.

Definition 1.7: Hyperplane. A hyperplane H in E_n is a collection of points of the form $\{\mathbf{x}: \mathbf{p}^t \mathbf{x} = \alpha\}$, where $\mathbf{p}$ is a non zero vector in E_n and α is a scalar. The vector $\mathbf{p}$ is the normal vector of the hyperplane. The hyperplane H defines two closed halfspaces $H^+ = \{\mathbf{x}: \mathbf{p}^t \mathbf{x} \geq \alpha\}$ and $H^- = \{\mathbf{x}: \mathbf{p}^t \mathbf{x} \leq \alpha\}$ and the two open halfspaces $\{\mathbf{x}: \mathbf{p}^t \mathbf{x} > \alpha\}$ and $\{\mathbf{x}: \mathbf{p}^t \mathbf{x} < \alpha\}$.

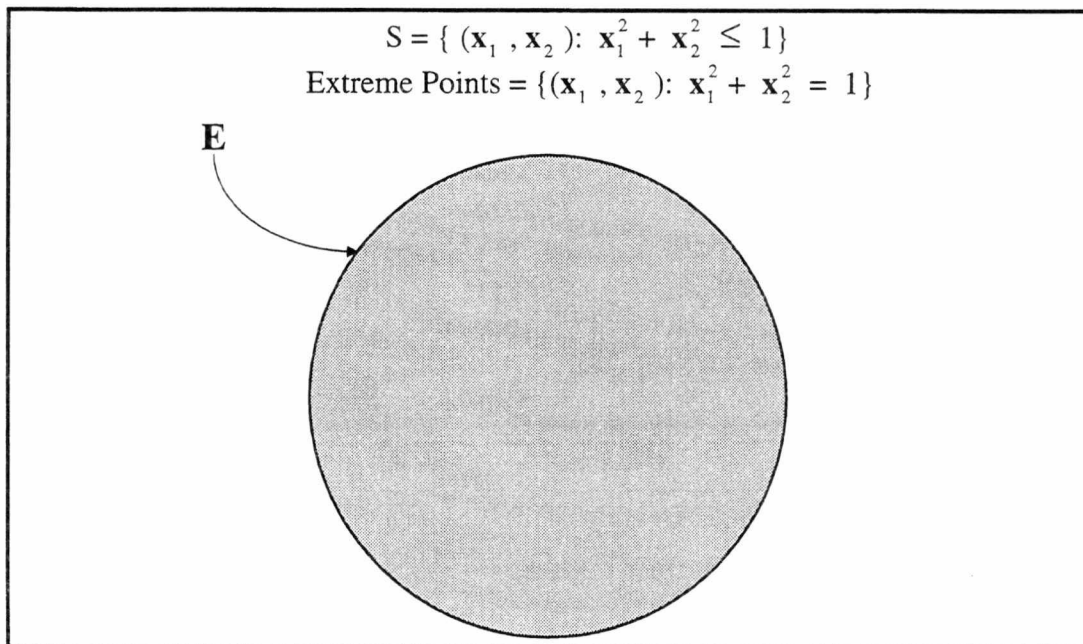


Figure 1.3 - Extreme Points of a Two Dimensional Ball

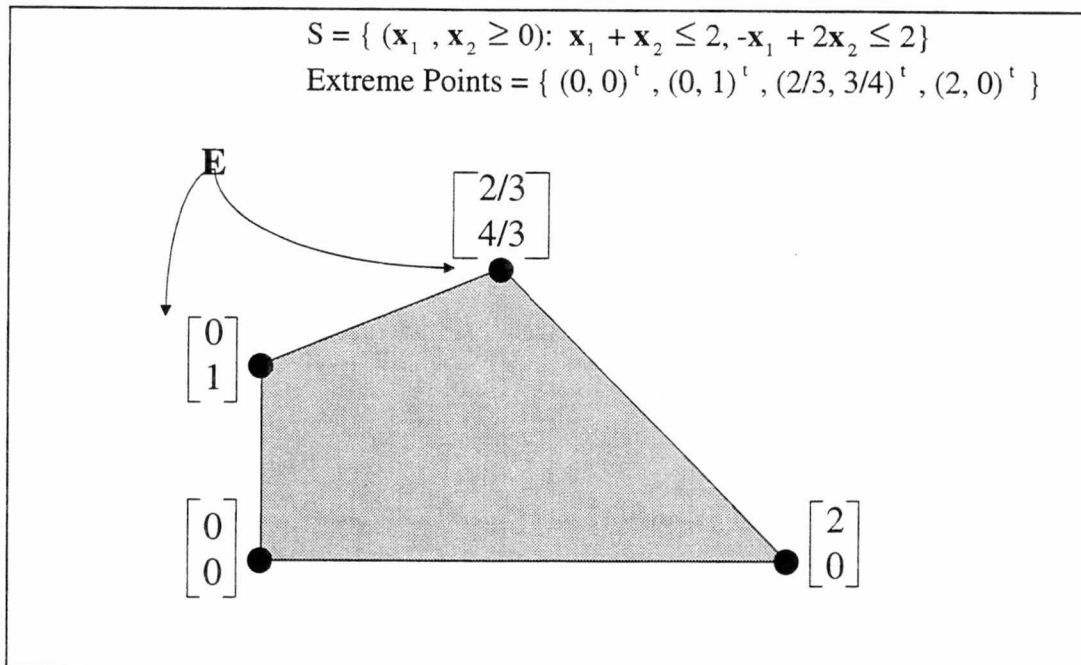


Figure 1.4 - Extreme Points of The Set S

Definition 1.8: Extreme point. Let S be a nonempty convex set in E_n . A vector $\mathbf{x} \in S$ is called an *extreme point* of S if $\mathbf{x} = \lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2$ with $\mathbf{x}_1, \mathbf{x}_2 \in S$, and $\lambda \in (0, 1)$ implies that $\mathbf{x} = \mathbf{x}_1 = \mathbf{x}_2$.

In Figure 1.3 on the previous page the extreme points of the two dimensional ball are the boundary of the circle. Another example of extreme points can be seen in Figure 1.4 above. In this case the extreme points are the vectors: $(0, 0)^t$, $(0, 1)^t$, $(2/3, 3/4)^t$, $(2, 0)^t$, and the edges or boundary.

Definition 1.9: Translate. Let $A, B \subset E_n$ and $A + B \equiv \{ \mathbf{x} + \mathbf{y} : \mathbf{x} \in A \text{ and } \mathbf{y} \in B \}$. Then the set $\mathbf{x} + A$ is the *translate* of A .

Definition 1.10: Affine. An *affine* set is a set that contains all the lines determined by pairs of its points. An affine set is a translate of a linear subspace. Consider E_3 . The affine sets in E_3 are the empty set, the one-pointed sets, the lines, the planes, and E_3 itself.

Definition 1.11: Compliment. The *compliment* of any set S in E_n is the set of all points in E_n not in S .

Definition 1.12: Separated. Two sets X and Y are *separated* by H if X lies in one of the closed half spaces and Y lies in the other, where a closed half space is the union of a hyperplane, H with its complement. Figure 1.5, below shows the hyperplane H separating the convex sets X and Y at the point x .

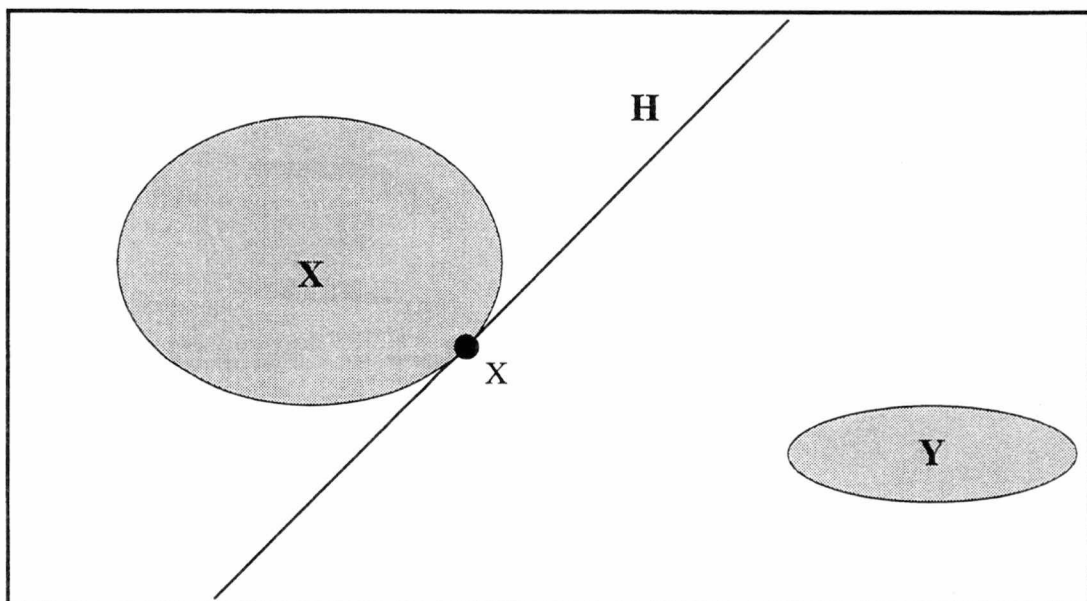


Figure 1.5 - A Hyperplane Separating Two Convex Sets

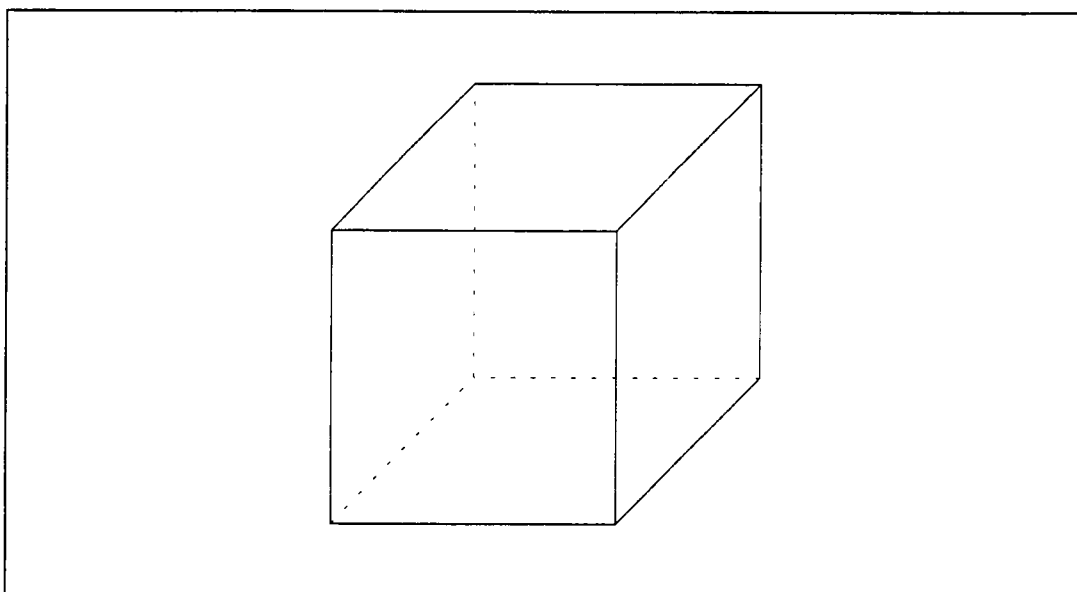


Figure 1.6 - The Faces of a Cube

Definition 1.13: Face. a *face* of the convex set A is a convex set $A' \subset A$ such that A' is not crossed by any segment in A .

For example, a cube in E_3 as shown in Figure 1.6 above has:

six two faces

twelve one faces (edges) and

eight 0-faces (extreme points).

Definition 1.14: Convex Hull. The *convex hull* of a set S is the intersection of all the convex sets which contain S . We will later go into more detail describing the convex hull in Chapter 3. See Definition 3.3.

Definition 1.15: Simplex. A *simplex* in E_n is the convex hull of $n+1$ points not

contained in any hyperplane.

Theorems And Applications Involving Convex Sets

At this point we have included several theorems and proofs related to the algebraic and geometric concepts of convex sets.

First, let us begin by listing the algebraic laws that are valid for any sets. The following four algebraic laws do not require convexity. However, we shall state them in terms of convex sets. C_i denotes a convex set in E_n and λ denotes a real number.

Algebraic Laws:

1. $C_1 + C_2 = C_2 + C_1$
2. $(C_1 + C_2) + C_3 = C_1 + (C_2 + C_3)$
3. $\lambda_1 (\lambda_2 C) = (\lambda_1 \lambda_2) C$
4. $\lambda (C_1 + C_2) = \lambda C_1 + \lambda C_2$.

Theorem 1.1. If C_1 and C_2 are convex sets in E_n then:

1. If C_1 and C_2 are convex sets in E_n then their intersection $C_1 \cap C_2$ is a convex set.
2. If C_1 and C_2 are convex sets in E_n then their sum $C_1 + C_2$, where $C_1 + C_2 = \{ \mathbf{x}_1 + \mathbf{x}_2 : \mathbf{x}_1 \in C_1, \mathbf{x}_2 \in C_2 \}$ is a convex set.
3. If C and D are convex sets in E_m and E_p , respectively then

$C \oplus D = \{ \mathbf{x} = (y, z) : y \in C, z \in D \}$ is a convex set in E_{m+p} .

Theorem 1.2. If C is a convex set and $\lambda_1 \geq 0, \lambda_2 \geq 0$, then

$$(\lambda_1 + \lambda_2) C = \lambda_1 C + \lambda_2 C$$

Proof: (i) First we show $(\lambda_1 + \lambda_2) C \subset \lambda_1 C + \lambda_2 C$.

By definition $\lambda C = \{ \lambda x : x \in C \}$. Now, let $x \in (\lambda_1 + \lambda_2) C$

$\rightarrow x \in \lambda C$ where $\lambda = \lambda_1 + \lambda_2$. $\rightarrow x \in C$ by definition.

$\rightarrow x \in \lambda_1 C$ and $x \in \lambda_2 C$. $\rightarrow x \in \lambda_1 C + \lambda_2 C$.

Hence, $(\lambda_1 + \lambda_2) C \subset \lambda_1 C + \lambda_2 C$.

ii) Showing $\lambda_1 C + \lambda_2 C \subset (\lambda_1 + \lambda_2) C$.

Let $0 < \lambda < 1$, and set $\lambda = \frac{\lambda_2}{\lambda_1 + \lambda_2}$

$$\text{Then, } (1 - \lambda) = 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{(\lambda_1 + \lambda_2) - \lambda_2}{\lambda_1 + \lambda_2} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

By convexity, $(1 - \lambda) C + \lambda C \subset C$, $0 < \lambda < 1$.

$$\rightarrow \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} \right) C + \left(\frac{\lambda_2}{\lambda_1 + \lambda_2} \right) C \subset C.$$

Multiplying by $\lambda_1 + \lambda_2$ we obtain: $\lambda_1 C + \lambda_2 C \subset (\lambda_1 + \lambda_2) C$

iii) Suppose $\lambda_1 + \lambda_2 = 0$, then clearly $\lambda_1 = 0$ and $\lambda_2 = 0$ since we define

$\lambda_1 \geq 0, \lambda_2 \geq 0$. In which case

$$\rightarrow \lambda_1 C + \lambda_2 C = 0 + 0 = 0 = (0) C = (\lambda_1 + \lambda_2) C.$$

Hence by i), ii), and iii) $\lambda_1 C + \lambda_2 C = (\lambda_1 + \lambda_2) C$. ■

Chapter II

Convex Cones

Introduction

Chapter Two develops the concept of convex cones. In this chapter we discuss various examples of convex cones, specifically polar cones and convex polyhedral cones. Several illustrations are given of convex polyhedral cones. In this chapter we develop basic theorems which show the relationship of convex cones to systems of linear equations. Later in this paper we will equate these basic concepts to linear programming. The main references in this chapter are Bazaraa [1], Gale [5], Hadley [7], Hadley [8], Kuhn-Tucker [11], and Solodvnikov [17].

Definitions and Examples Related To Convex Cones

Definition 2.1: Cone. A nonempty set C in E_n is called a *cone* with vertex zero if $x \in C$ implies that $\lambda x \in C$ for all $\lambda > 0$. If, in addition C is convex, then C is called a convex cone.

On the following page Figure 2.1 (a) is an example of a cone that is not convex since it consists of two separate parts. Clearly Figure 2.1 (b) is a typical example of a convex cone inasmuch as any line segment connecting points in the cone are contained within the cone. By this we mean that if the cone C is convex then for any arbitrary points $x, y \in C$ that $\lambda x + (1 - \lambda)y \in C$ for each $\lambda \in [0,1]$.

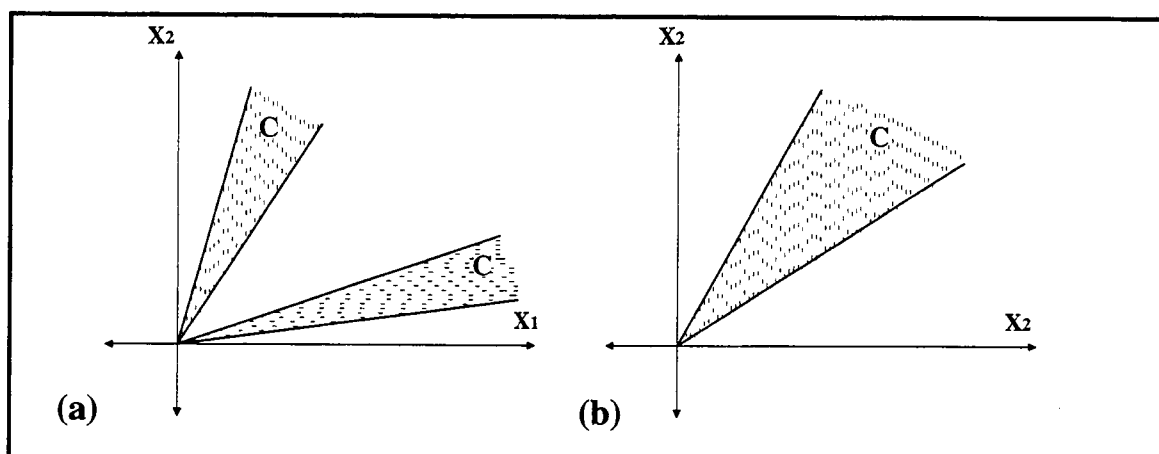


Figure 2.1 Example of a (a) - Nonconvex Cone and a (b) - Convex Cone

Other examples of convex cones are:

1. Any linear subspace L is a special case of the convex cone.
2. The set of nonnegative vectors P of E_n is a convex cone.
3. The set of all solutions of the inequality $xb \leq 0$ is a convex cone, called a half space.
4. As a generalization of 3, the set of all solutions of the homogeneous inequality $xA \leq 0$, for any matrix A forms a convex cone. We will discuss this particular example in more detail later.

We can define the following four operations of addition, dual, intersection, and negative on convex cones:

Definition 2.2: Addition. if C' and C'' are convex cones, their sum $C' + C''$ is defined by

$$C' + C'' = \{ x \mid x = x' + x'', x' \in C', x'' \in C'' \}.$$

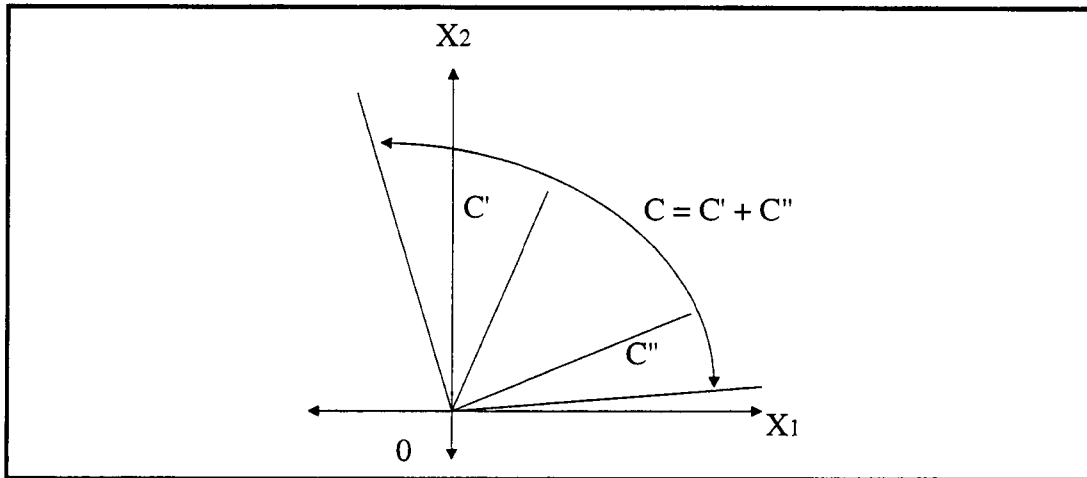


Figure 2.2 - Addition of Convex Cones

$C' + C''$ is a convex cone as seen in Figure 2.2 above.

Definition 2.3: The Dual. of a convex cone C^* is defined by

$$C^* = \{ y \mid x^t y \leq 0 \text{ for all } x \in C \}.$$

Figure 2.3 on the following page illustrates a convex cone C and it's *Dual* C^* .

Definition 2.4 Intersection. If C' and C'' are convex cones, their *intersection* $C' \cap C''$ is again a convex cone. We define the intersection by:

$$C' \cap C'' = \{ x \mid x' \in C' \text{ and } x'' \in C'' \}.$$

See Figure 2.4 on the following page.

Definition 2.5: The Negative. of a convex cone C , denoted $-C$ is the set:

$$-C = \{-x\}.$$

The negative of a convex cone can be seen in Figure 2.5 on the following page.

Both the dual and the negative of convex cones are both clearly convex cones.

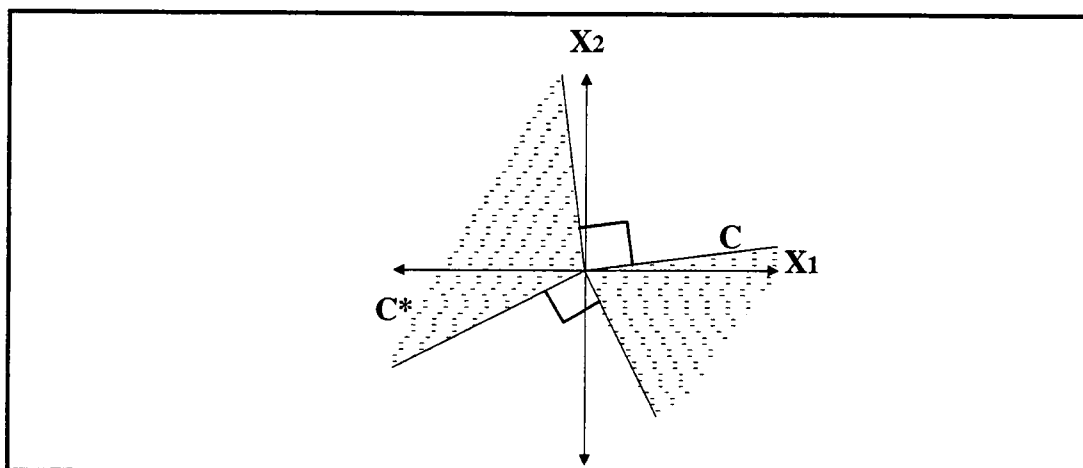


Figure 2.3 - The Dual of a Convex Cone

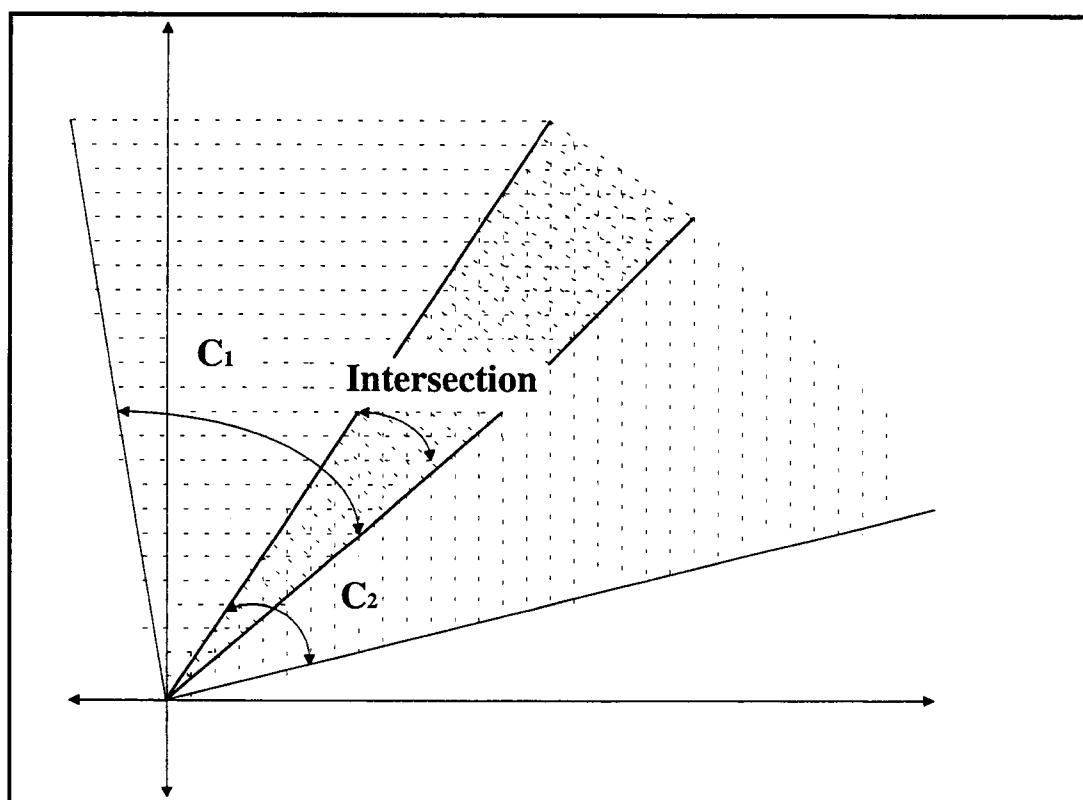


Figure 2.4 - The Intersection of Convex Cones

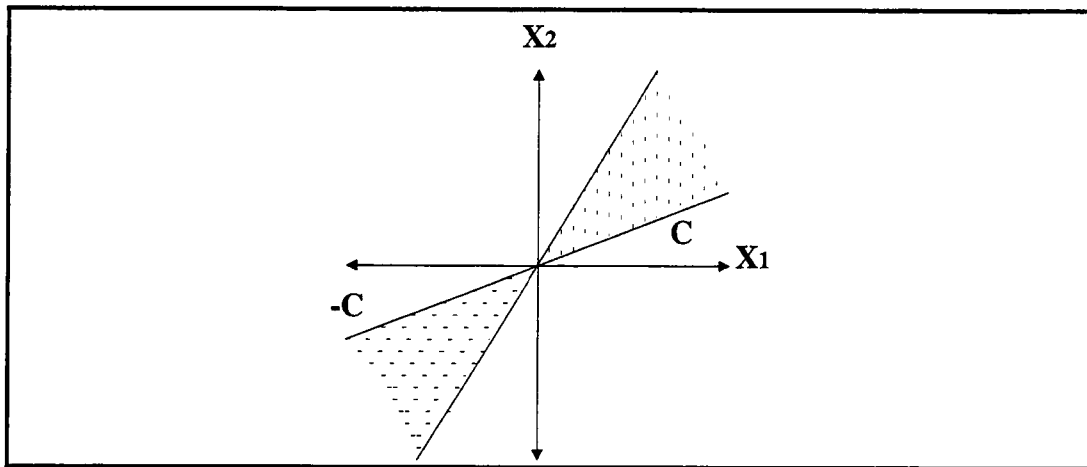


Figure 2.5 - The Negative of a Convex Cone

Definition 2.6: Span. A set S is said to *span* a convex cone C if S is a subset of C and each vector of C can be expressed as a finite linear combination of vectors of S with coefficients greater than or equal to zero.

Definition 2.7: Convex-Cone Hull. Since C is the least convex cone containing S , the cone C is referred to as the *convex cone hull* of S .

Lastly, two other kinds of cones defined below in 2.8 and 2.9 are the:

Definition 2.8: Polar Cone. Let S be a nonempty set in E_n . Then the *polar cone* of S denoted by S^+ is given by $\{p: p^T x \leq 0 \text{ for all } x \in S\}$. If S is empty, S^+ will be interpreted as E_n . To generalize this, we can say that a subset C of E_n is called a convex cone if it is closed under the operations of addition and multiplication by nonnegative scalars, that is, if $x, y \in C^+$ then $x + y \in C^+$

if $x \in C^+$ and $\alpha \geq 0$ then $\alpha x \in C^+$.

We can also define a *polar cone* by stating that if $C = \{u\}$ is a cone, then C^+ , the polar cone to C is the collection of all points $\{v\}$ such that $v^t u \geq 0$ for each v in the set and all $u \in C$. See Figure 2.6 on the following page.

Definition 2.9: Convex Polyhedral Cone. A *convex polyhedral cone* C is the sum of a finite number of half-lines, or more simply put; the cone C generated by a convex polyhedron is a convex polyhedral cone. In Euclidean geometry of three dimensions, we refer to a convex polyhedron as the intersection of finitely many closed half spaces.

Consider the region defined by the following system, where each inequality is a given half-plane:

$$a_1 x + b_1 y + c_1 z + d_1 \geq 0$$

$$a_2 x + b_2 y + c_2 z + d_2 \geq 0$$

.....

$$a_n x + b_n y + c_n z + d_n \geq 0$$

The region we have defined by the system above is the convex polyhedral cone formed by the intersection of n half-planes. If there are no points that satisfy the system then the region is empty. In Figure 2.6 on the following page, the convex polyhedron cone is defined by $n = 4$. This particular example consists of the interior and boundary points of a simple tetrahedron. It is possible for the region to extend out to infinity. In any case it is obvious that any convex polyhedron cone can be obtained by the intersection of a finite number of half-planes.

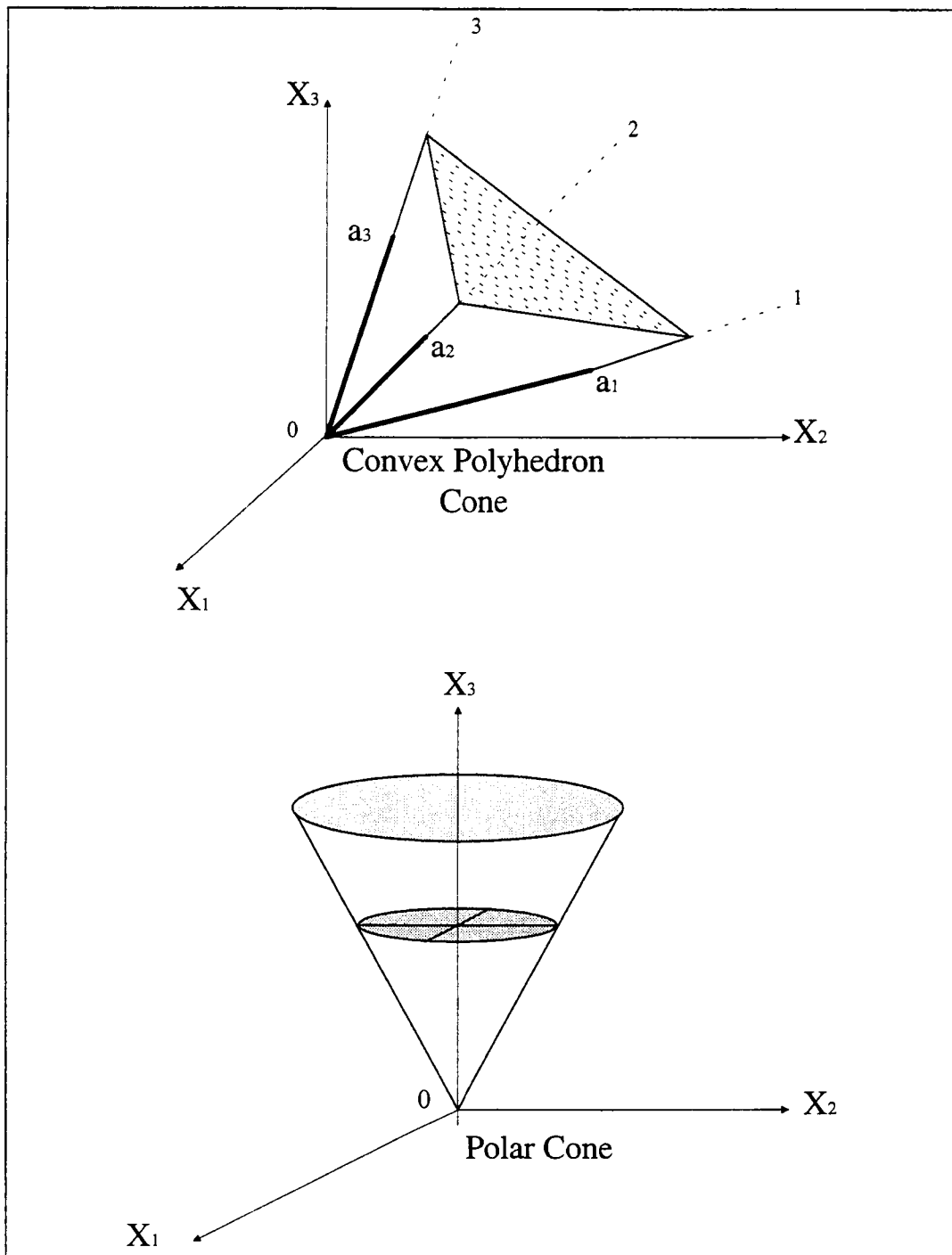


Figure 2.6 - Examples of Convex Polyhedron and Polar Cones

A convex polyhedral cone with the vertex at the origin is the feasible region of a system of homogeneous inequalities:

$$a_1 x + b_1 y + c_1 z + \geq 0$$

$$a_2 x + b_2 y + c_2 z + \geq 0$$

.....

$$a_n x + b_n y + c_n z + \geq 0.$$

Clearly, we can state that the feasible region of a homogeneous system of inequalities is a convex polyhedral cone with the vertex at the origin.

Below we list several convex polyhedral cones which are illustrated in Figure 2.7 on the following page:

1. Figure 2.7 (a) is that of a half-space. In this case the boundary plane is π and the vertex of the cone is any point P in π .
2. In Figure 2.7 (b) the convex polyhedral cone is the intersection of two half spaces whose boundary planes intersect along the straight line l . The vertex is any point P located on l .
3. The cone in Figure 2.7 (c) is the straight line l formed by the intersection of three half spaces whose boundary planes pass through the line l . The vertex is any point P on l .
4. Figure 2.7 (d) is that of a ray. The cone is a ray formed by the intersection of a straight line and a half-space. The beginning of the ray is the vertex, P .
5. The cone in Figure 2.7 (e) is the point P . The cone is obtained by the intersection of a ray and the corresponding half-space.

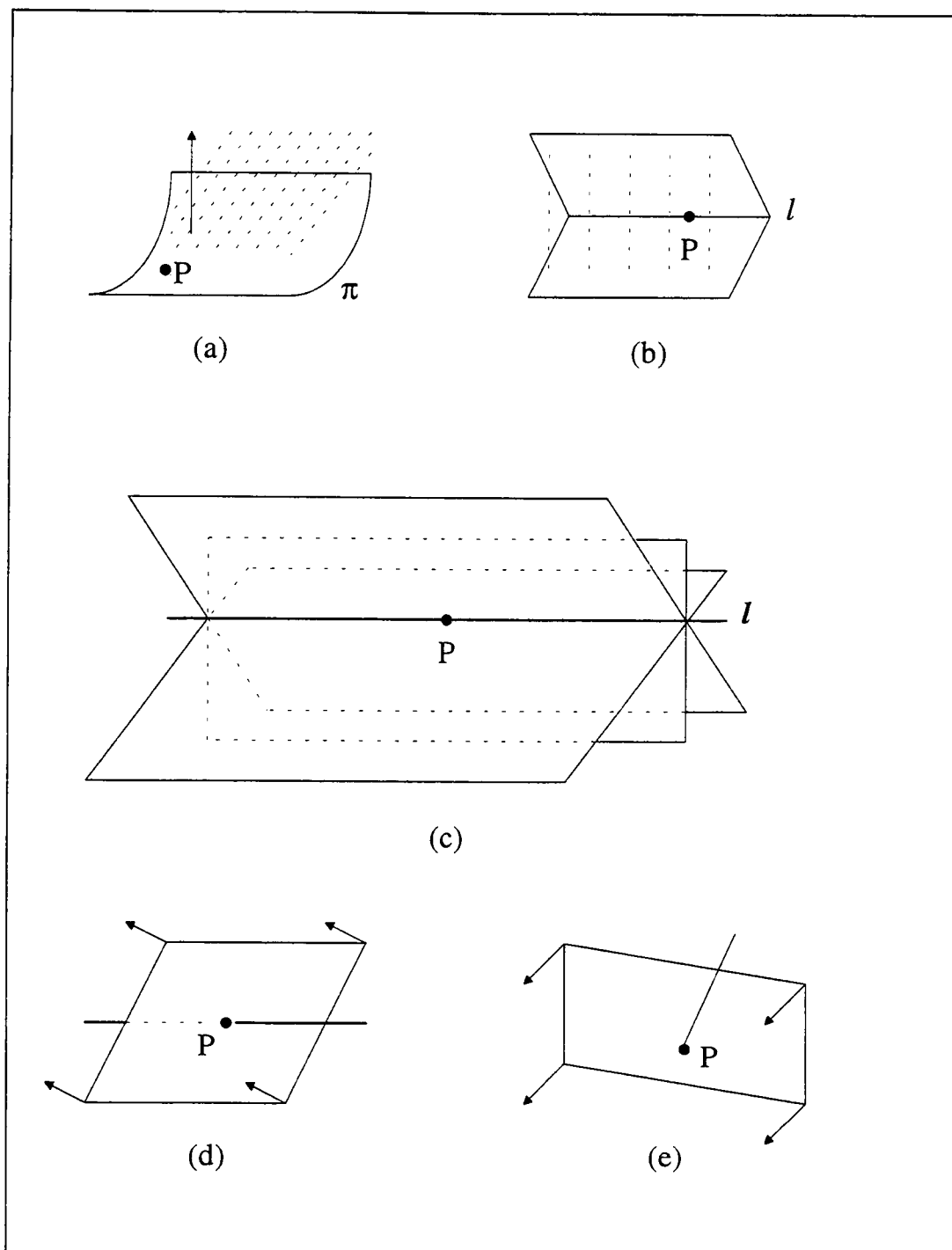


Figure 2.7 - Convex Polyhedron Cones

Theorems and Applications Involving Convex Cones

Theorem 2.1. Let C, C_1 , and C_2 be convex cones. Then

- (i) If $C_1 \subset C_2$, then $C_2^* \subset C_1^*$, (where $*$ denotes dual).
- (ii) $(C_1 + C_2)^* = C_1^* \cap C_2^*$.
- (iii) $C_1^* + C_2^* \subset (C_1 \cap C_2)^*$.
- (iv) $C \subset C^{**}$.

Proof: (i) Let $y \in C_2^*$. Then $x_2^t y \leq 0$ for all $x_2 \in C_2$, by definition of dual. Hence, $x_1^t y \leq 0$ for all $x_1 \in C_1$ since $C_1 \subset C_2$. Then $y \in C_1^*$ by definition. Thus, $C_2^* \subset C_1^*$. This is illustrated in Figure 2.8.

(ii) Let $y \in C_1^* \cap C_2^*$.

Then $x_1^t y \leq 0$ for all $x_1 \in C_1$ and $x_2^t y \leq 0$ for all $x_2 \in C_2$.

Now, if $x \in C_1 + C_2$, then $x = x_1 + x_2$ for some $x_1 \in C_1$ and $x_2 \in C_2$.

So, $x^t y = (x_1 + x_2)^t y = x_1^t y + x_2^t y \leq 0$ and $y \in (C_1 + C_2)^*$.

Therefore, $C_1^* \cap C_2^* \subset (C_1 + C_2)^*$.

Conversely, $C_1 \subset (C_1 + C_2)$ and by (i), $(C_1 + C_2)^* \subset C_1^*$.

Also, $C_2 \subset (C_1 + C_2)$ and therefore $(C_1 + C_2)^* \subset C_2^*$.

Hence, $(C_1 + C_2)^* \subset C_1^* \cap C_2^*$.

Therefore, $(C_1 + C_2)^* = C_1^* \cap C_2^*$.

(iii.) Let $y \in C_1^* + C_2^*$.

Then $y = y_1 + y_2$ for some $y_1 \in C_1^*$, $y_2 \in C_2^*$.

Since $y_1 \in C_1^*$, $x_1^t y_1 \leq 0$ for all $x_1 \in C_1$.

Similarly, $x_2^t y_2 \leq 0$ for all $x_2 \in C_2$.

if $x \in C_1 \cap C_2$, then, $x \in C_1$ and $x \in C_2$

So, $x^t y = x^t (y_1 + y_2) = x^t y_1 + x^t y_2 \leq 0$ since $x^t y_1 \leq 0$ for $x \in C_1$

and $x^t y_2 \leq 0$ for $x \in C_2$.

Hence $y \in (C_1 \cap C_2)^*$.

(iv) Let $x \in C$.

Then $x^t y \leq 0$ for all $y \in C^*$.

Therefore, $x \in C^{**}$ and $C \subset C^{**}$. ■

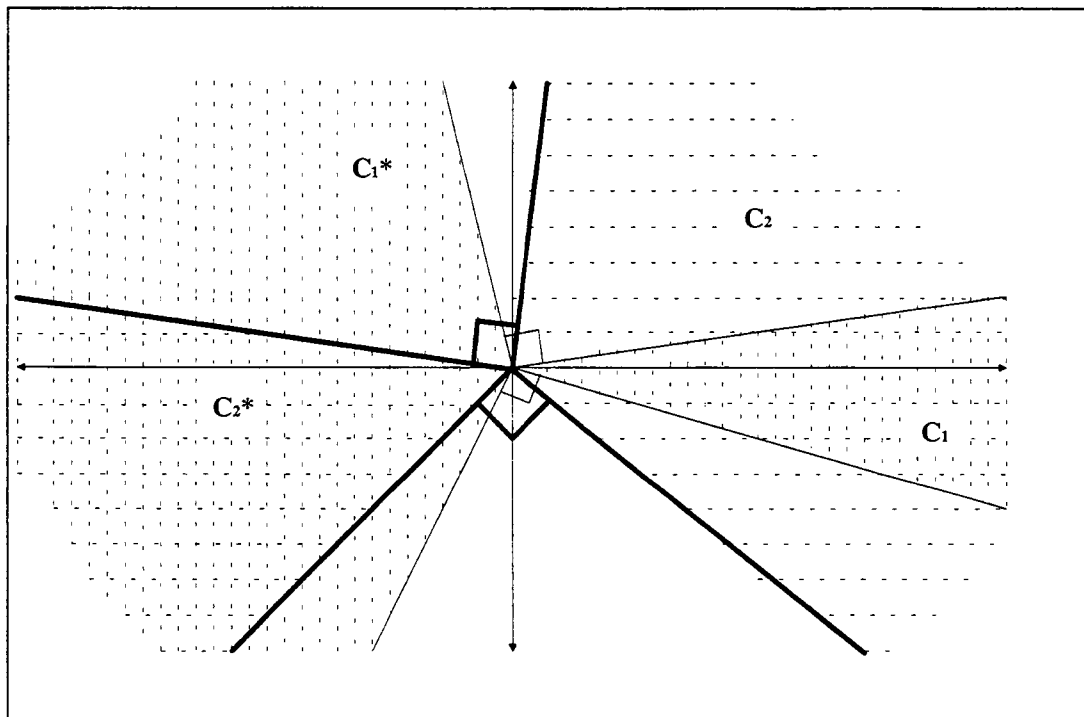


Figure 2.8 - The Dual of C_2 as a Subset of The Dual of C_1

Definition 2.10: Basic Solution. We say that a *basic solution* to $A \mathbf{x} = \mathbf{b}$, where A is an $m \times n$ matrix (where $m = \text{rows}$, $n = \text{columns}$, and $m < n$), is a solution found by setting $n-m$ variables equal to zero and solving for the remaining m variables. This is supposing that the determinant of the coefficients of the m variables is nonzero.

Theorem 2.2. the set of all nonnegative solutions of

$$(2.1) \quad A \mathbf{x} = \mathbf{0}$$

is a convex cone.

Proof: Consider the following equations:

$$(2.2) \quad \sum_{i=1}^m \lambda_i = 1 \quad \lambda_i \geq 0$$

Now any nonnegative solution of (2.1) in E_n is defined to be ≥ 0 but not equal to 0. We shall approach the proof by showing that every solution of (2.1) is a nonnegative convex combination of basic solutions, and that these solutions form a convex cone.

Suppose $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to (2.1) then $A\mathbf{x}_1 = \mathbf{0}$ and $A\mathbf{x}_2 = \mathbf{0}$. Now, by (2.2) $\lambda_i \in [0,1]$. Hence, we obtain the following :

$$\begin{aligned} 0 &= 0 + 0 \\ &= \lambda_1 A\mathbf{x}_1 + (1 - \lambda_1) A\mathbf{x}_2 \\ &= A\lambda_1 \mathbf{x}_1 + A(1 - \lambda_1) \mathbf{x}_2 \\ &= A(\lambda_1 \mathbf{x}_1 + (1 - \lambda_1) \mathbf{x}_2) \end{aligned}$$

Similarly, for any $\lambda_i \in [0, 1]$ we obtain:

$$\begin{aligned} \mathbf{0} &= \mathbf{0} + \mathbf{0} + \dots + \mathbf{0} \\ &= A(\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \dots + \lambda_n \mathbf{x}_n) \end{aligned}$$

Thus, by definition, the nonnegative solutions $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ of (2.1) form a convex set.

Now, consider any nonnegative number, s and solution $\mathbf{x}_i$ to (2.1). Then $A(s\mathbf{x}_i) = (As)\mathbf{x}_i = s(A\mathbf{x}_i) = \mathbf{0}$. Since we can multiply the solution $\mathbf{x}_i$ by any nonnegative number and still obtain $\mathbf{0}$, then by definition the nonnegative solutions $\mathbf{x}_i$ form a cone.

Hence, the set of all nonnegative solutions of (2.1) form a convex cone, completing the proof. ■

Definition 2.11: Feasible Solution. We define *feasible solution* as a vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$ which satisfies the following systems of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

and $x_j \geq 0$ where $j = 1, 2, \dots, n$

The convex set of feasible solutions to the constraints $Ax \leq b$ for the general linear programming problem lies in an n -dimensional Euclidean space, the solution

space. The extreme points of this convex set form a basic feasible solution to $Ax \leq b$, and the extreme points lie in the intersections of the hyperplanes.

We give the following example to illustrate the set of feasible solutions to the constraints $Ax \leq b$.

Example 2.1.

Consider the following system of equations:

$$5x_1 + 10x_2 \leq 50$$

$$2x_1 + x_2 \leq 10$$

$$\text{and } x_1, x_2 \geq 0$$

Solution:

Solving the equations algebraically we obtain the extreme points $(0, 5)$, $(0, 0)$, $(\frac{10}{3}, \frac{10}{3})$, and $(5, 0)$. We write the set of feasible solutions as :

$\alpha_1 (0, 5) + \alpha_2 (0, 0) + \alpha_3 (\frac{10}{3}, \frac{10}{3}) + \alpha_4 (5, 0)$ such that $\sum \alpha_i = 1$
and $\alpha_i \geq 0$. •

Geometrically we have Figure 2.9 below.

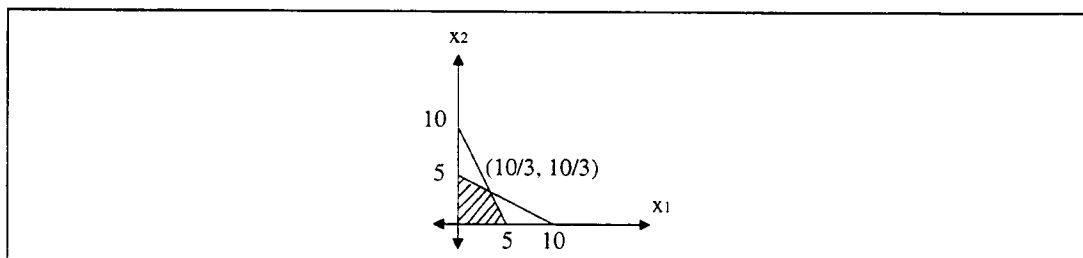


Figure 2.9 - Solution to Example 2.1

Definition 2.12: Nonnegative Orthant. We define the set of all nonnegative vectors P of E_n to be the *nonnegative orthant* of E_n .

Lemma 2.1. A linear subspace S is a convex cone.

Theorem 2.3. (Duality Theorem). If C is a convex cone, Then $C^{**} = C$.

Proof: By Theorem 2.1, $C \subset C^{**}$.

We now want to show $C^{**} \subset C$.

Suppose $\mathbf{b} \notin C$.

Since C is finite, for some matrix A , $C = \{ \mathbf{y} : \mathbf{y} = \mathbf{x}^t A \text{ for } \mathbf{x} \geq 0 \}$.

So, $\mathbf{b} = \mathbf{x}^t A$ has no nonnegative solution;

hence, there is a vector $\mathbf{y}$ such $A\mathbf{y} \leq 0$ and $\mathbf{b}\mathbf{y} > 0$.

From the first relation we have $\mathbf{x}^t A\mathbf{y} \leq 0$ for all $\mathbf{x} \geq 0$ and, $\mathbf{y} \in C^*$.

However, since $\mathbf{b}^t \mathbf{y} > 0$, $\mathbf{b} \notin C^{**}$.

Hence, we have shown that if $\mathbf{b} \notin C$, then $\mathbf{b} \notin C^{**}$.

By contrapositive, we have $C^{**} \subset C$.

Therefore, $C^{**} = C$. ■

Corollary 2.1. The intersection of a subspace S of E_n with the nonnegative orthant P of E_n is a cone.

Proof: Let $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_s$ be a basis for S^* (where S is a subspace of E_n). Let C be all nonnegative solutions of the equation $\mathbf{b}_i^t \mathbf{x} = 0$, for $i = 1, 2, \dots, s$. From

Theorem 2.2, C is a cone. If $\mathbf{x} \in S \cap P$, then $\mathbf{x} \in C$. Conversely, if $\mathbf{x} \in C$, then $\mathbf{b}_i \mathbf{x} = 0$ for all i ; hence, $\mathbf{x} \in S^{**} = S$. Therefore, $\mathbf{x} \in S \cap P$ and $C = S \cap P$. ■

Theorem 2.4 . The set of all solutions of

$$(2.6) \quad \mathbf{Ax} \leq \mathbf{0} \quad \text{is a convex cone.}$$

Proof: Let S be the subspace defined by

$$(2.7) \quad S = \{\mathbf{y}: \mathbf{y} = \mathbf{Ax}, \mathbf{x} \in \mathbb{R}_m\}$$

From Corollary 2.1 The intersection of S with the nonnegative orthant, P is a finite cone. Hence

$$(2.8) \quad S \cap P = (\mathbf{y}_1) + \dots + (\mathbf{y}_r)$$

Choose each $\mathbf{x}_i$ such that $-\mathbf{y}_i = \mathbf{Ax}_i$ and let X' be the linear subspace of solution of $\mathbf{Ax} = 0$. From Lemma 2.1 we obtain

$$(2.9) \quad X' = (\mathbf{x}'_1) + \dots + (\mathbf{x}'_r)$$

for some vectors $\mathbf{x}'_i$. Hence, the set of solutions C of (2.7) is given by $C = (\mathbf{x}_1) + \dots + (\mathbf{x}_r) + (\mathbf{x}'_1) + \dots + (\mathbf{x}'_r)$ since if $\mathbf{Ax} \leq 0$ then $\mathbf{A}(-\mathbf{x}) \in S \cap P$. Hence, by

(2.8) this implies $\mathbf{Ax} = -\sum_{i=1}^r \lambda_i \mathbf{y}_i = \sum_{i=1}^r (\mathbf{Ax}_i) \lambda_i$ where $\lambda_i \geq 0$. Thus

$$\mathbf{A}(\mathbf{x} - \sum_{i=1}^r \lambda_i \mathbf{x}_i) = 0 \text{ and from (2.9) we obtain } \mathbf{x} - \sum_{i=1}^r \lambda_i \mathbf{x}_i = \sum_{i=1}^s \mu_i \mathbf{x}'_i, \mu_i \geq 0$$

$$\text{Hence } \mathbf{x} = \sum_{i=1}^r \lambda_i \mathbf{x}_i + \sum_{i=1}^s \mu_i \mathbf{x}'_i. \quad \blacksquare$$

We use the following example to verify Theorem 2.4. We will use a geometric approach by graphing the solution space and also describe the solution in terms of vectors.

Example 2.2.

Consider the system of equations:

$$3x_1 + 2x_2 \leq 0$$

$$x_1 - 2x_2 \leq 0$$

Solution:

Geometrically, we have the system defined by the graphs of $3x_1 + 2x_2 \leq 0$ and $x_1 - 2x_2 \leq 0$ as seen in Figure 2.10 on the following page. The solution space is clearly a convex cone.

We can also think of this same example as vector multiplication, where we could rewrite each expression as the following with vectors:

$$\begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq 0 \quad \text{and} \quad \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq 0, \text{ where the product :}$$

$$\begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \text{ implies we have}$$

perpendicular vectors. For all values less than or equal to zero, we would obtain the same results as in Figure 2.10. In either case, the result is clearly a convex cone. ●

Following this we give a larger example.

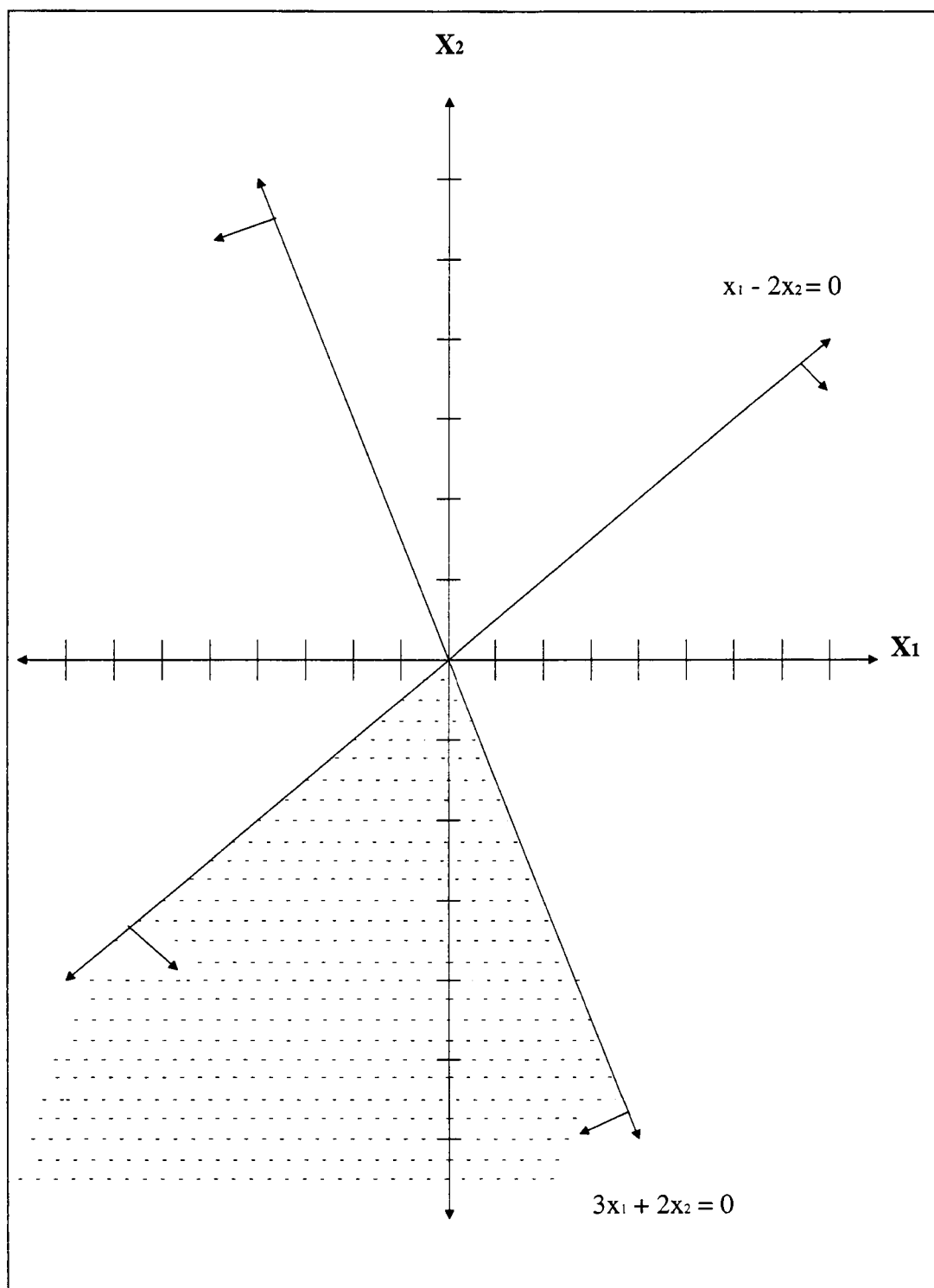


Figure 2.10 - Geometric Solution to Example 2.2

Example 2.3.

Consider the following system of equations:

$$2x_1 - 5x_2 \leq -10$$

$$x_1 - 3x_2 \leq 0$$

$$2x_1 - 4x_2 \leq -4$$

$$x_1 - 3x_2 \leq -6$$

Solution:

Geometrically we have the system defined by the graphs of the lines

$$2x_1 - 5x_2 = -10$$

$$x_1 - 3x_2 = 0$$

$$2x_1 - 4x_2 = -4$$

$$x_1 - 3x_2 = -6$$

We then find the values for which

$$x_2 \geq \frac{2}{5} x_1 + 2$$

$$x_2 \geq \frac{1}{3} x_1$$

$$x_2 \leq -\frac{1}{2} x_1 - 1$$

$$x_2 \leq -\frac{1}{3} x_1 - 2$$

In Figure 2.10 on the following page we graph the result which is clearly a convex cone. •

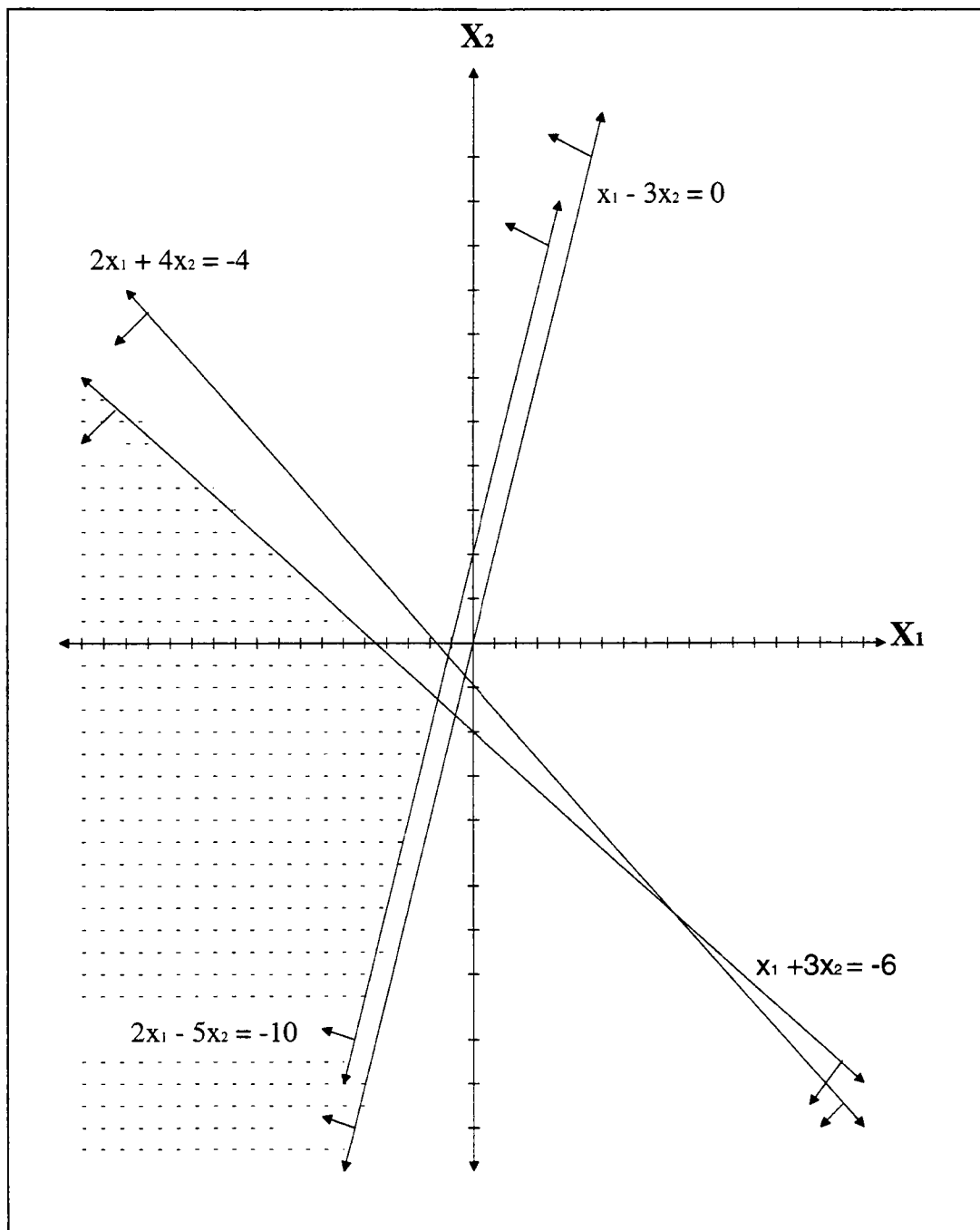


Figure 2.11 - Geometric Solution to Example 2.3

Chapter III

Polyhedral Sets and Polytopes

Introduction

Chapter 3 discusses special types of convex sets. We refer to these sets as polyhedral sets and polytopes. Polyhedral sets are the very basis of linear programming. The main references used in this chapter are Bazaraa [2], Cooper [3], Gale [5], Hadley [7], Klee [10], Kuhn-Tucker [11], Lay [12], and Mangasarian [13] .

Definitions and Examples Involving Polyhedral sets and Polytopes

Note that polyhedral sets represent an important special case of convex sets. In this case only a finite number of half-spaces is needed to represent the set.

Definition 3.1: Polyhedral set. A set S in E_n is called a *polyhedral set* if it is the intersection of a finite number of closed half-spaces: that is $S = \{x: p_i^T x \leq \alpha_i \text{ for } i = 1, \dots, m\}$, where p_i is a nonzero vector and α_i is a scalar for $i = 1, \dots, m$.

Below are some examples of polyhedral sets:

1. $S = \{x: Ax \leq b\}$
2. $S = \{x: Ax = b, x \geq 0\}$
3. $S = \{x: Ax \geq b, x \geq 0\}$

Definition 3.2: Convex Hull. The *convex hull* of S , denoted $H(S)$ or $\text{conv}(S)$ is the collection of all convex combinations of S . $H(S)$ is the minimal set that contains S .

Definition 3.3: Polytope. The convex hull of a finite or countable number of points $\mathbf{x}_1 \dots \mathbf{x}_{k+1}$ in E_n called a *polytope*.

Definition 3.4: Simplex. If the $\mathbf{x}_k$ are affinely independent (i.e.: If $0 = n_1 [\mathbf{x}_2 - \mathbf{x}_1] + n_2 [\mathbf{x}_3 - \mathbf{x}_1] + \dots + n_k [\mathbf{x}_{k+1} - \mathbf{x}_1]$, the only scalars n_i that satisfy the equation are $n_i = 0$.) then $H(\mathbf{x}_1 \dots \mathbf{x}_{k+1})$, the convex hull of $\mathbf{x}_1 \dots \mathbf{x}_{k+1}$ is called a *simplex* with vertices $\mathbf{x}_1 \dots \mathbf{x}_{k+1}$.

Note that any point x in the convex hull of a set S can be represented as a convex combination of, at most, $n+1$ points in S .

Definition 3.5: Direction. Let S be a nonempty closed convex set in E_n . A nonzero vector $\mathbf{d}$ in E_n is called a *direction*, or a *recession direction*, of S if for each $\mathbf{x} \in S$, $\mathbf{x} + \lambda \mathbf{d} \in S$ for all $\lambda \geq 0$. Two directions $\mathbf{d}'$ and $\mathbf{d}''$ are called *distinct* if $\mathbf{d}' \neq \alpha \mathbf{d}''$ for any $\alpha > 0$. A direction $\mathbf{d}$ of S is called an *extreme direction* if it cannot be written as a positive linear combination of two distinct directions.

Definition 3.6: Bounded Convex Polyhedron. We can define a *bounded convex polyhedron* S^Δ , as the convex hull of a finite nonvacuous set $S = \{P_1 \dots P_p\}$ of vectors: $S^\Delta = \{X: X = PU; U \geq 0, \sum u_i = 1\}$. By this, we mean that P is the $m \times p$ matrix with $P_1 \dots P_p$ as columns.

Theorems and Applications Involving Polyhedral Sets And Polytopes

In 1752, Euler made the first important breakthrough in the study of polyhedra with the following theorem which we refer to Euler's Theorem for Polytopes in E_3 . First, it is necessary to introduce the definition of network and a few preliminary results.

Definition 3.7: Network. A *network* on a surface in E_3 is a finite or countable set of points (nodes) and line segments such that

- (a) Each node is the endpoint of a line segment.
- (b) Each line has two nodes.
- (c) Two line segments can intersect at their endpoints only. A network divides the surface into faces, where each face is bounded by line segments.

The expression $v - e + f$ is called the Euler characteristic. For any polytope in E_3 we can build a network, beginning at the vertex which will cover the entire polytope in a finite number of steps. This process involves two kinds of steps:

- (3.1) The new edge which is adjoined adds a vertex which is not yet in the network.
- (3.2) The edge which is adjoined connects two vertices which were not yet already in the network.

Lemma 3.1. Let N be a polyhedral network. Additions to N of type (3.1) or (3.2)

do not change the Euler Characteristic.

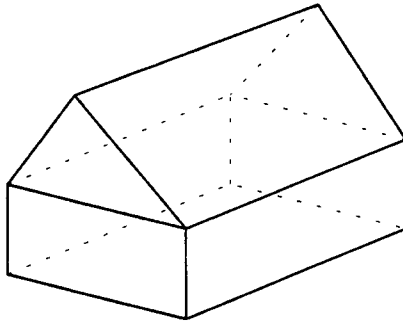
Proof: Let N' be a network obtained from N by adjoining an edge of type (3.1). Let N'' be a network obtained from N by adjoining an edge of type (3.2). Let v , e , f denote the number vertices, edges, and faces respectively. Then since type (3.1) has been added to N' the numbers for N' are $v+1$, $e+1$, f . Since additions of type (3.2) have been added to N'' , it's numbers are now v , $e+1$, $f+1$. In either case the Euler characteristic $v - e + f$ remains the same. ■

Theorem 3.1. For any 3-polytope: $v - e + f = 2$, Where v = vertices, e = edges, and f = 2-faces.

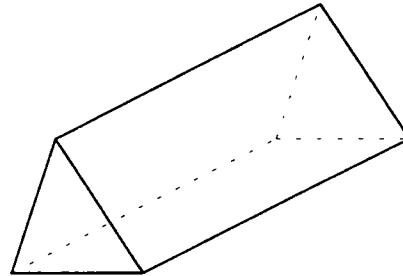
Proof: Let x be a vertex of a 3-polytope, P . The polyhedral network N that consists only of x has one vertex, no edges and one face. Hence, it has Euler characteristic 2. Consider any finite number of additions of type (3.1) or (3.2). By Lemma 3.1 N can be built into a network N' that includes all the edges and vertices of P and yet does not change the Euler characteristic. Since the faces of N' correspond to the faces of P , we have $v - e + f = 2$. ■

Consider the following diagrams in Figure 3.1 of both convex and nonconvex three dimensional solids. It is clear that in the top two figures obey Euler's Theorem for polytopes in E_3 whereas the bottom two figures do not. Since the bottom figures have an open area in the center they are clearly not convex. As a matter of interest, for nonconvex 3 dimensional solids: $v - e + f = 0$.

Convex



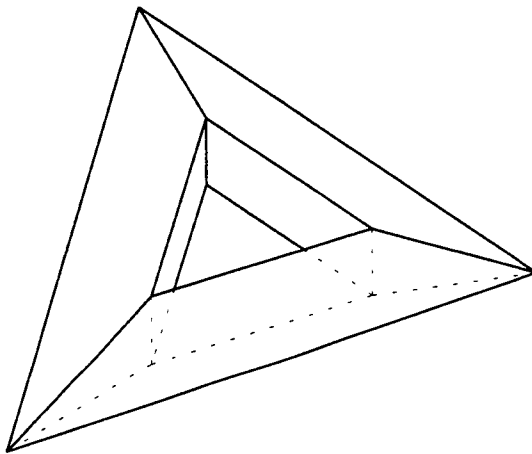
$$\begin{aligned} f &= 9, v = 10, e = 17 \\ v - e + f &= 2 \\ 10 - 17 + 9 &= 2 \end{aligned}$$



$$\begin{aligned} f &= 9, v = 6, e = 9 \\ v - e + f &= 2 \\ 6 - 5 + 9 &= 2 \end{aligned}$$

Nonconvex

$$\begin{aligned} f &= 9, v = 9, e = 18 \\ v - e + f &= 0 \\ 9 - 18 + 9 &= 0 \end{aligned}$$



$$\begin{aligned} f &= 12, v = 12, e = 24 \\ v - e + f &= 0 \\ 12 - 24 + 12 &= 0 \end{aligned}$$

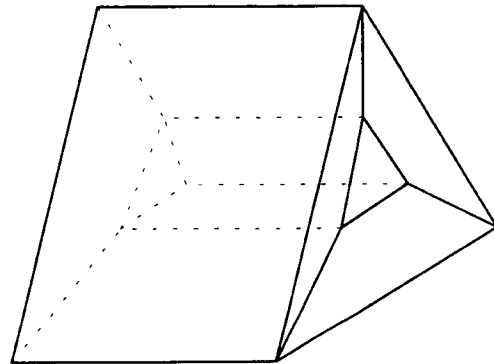


Figure 3.1 - Euler's Hypothesis

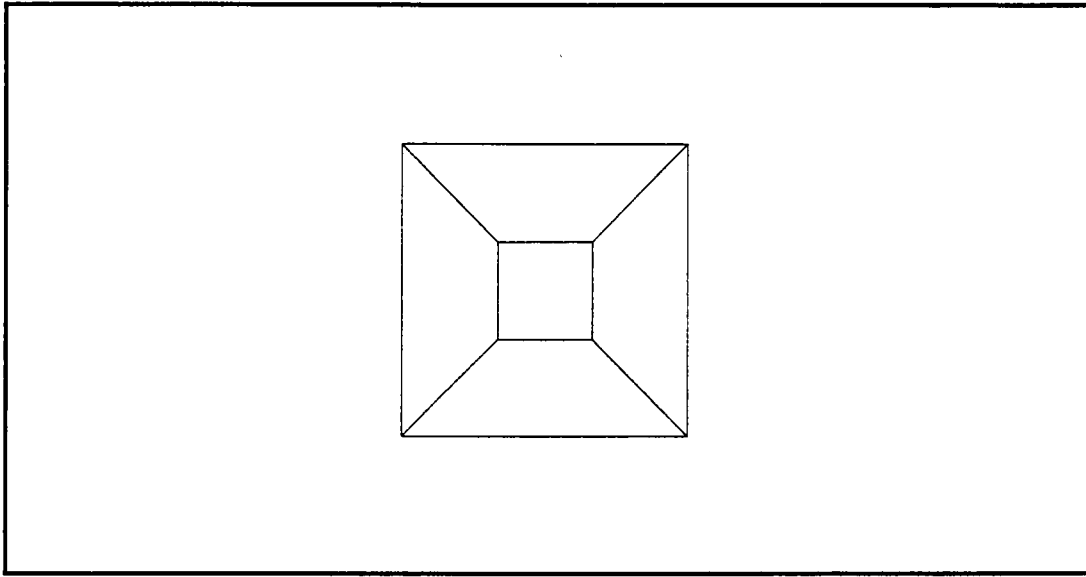


Figure 3.2 - Planar Graph of a Cube

Another landmark in the study of polyhedra is Steinitz's theorem (1934) which characterizes the graphs of 3-polytopes and their combinatorial structures formed by vertices and edges as those that are planar. The graph above in Figure 3.2 corresponds to a cube.

Other developments occurred in the 1940's. These techniques we refer to as linear programming are used to minimize linear functions on polyhedra. This process is still a topic of study in polyhedra today. An example of a discovery in this area is the fact that:

For each $k > n$ there is an n -polytope with k vertices such that each $\left\lceil \frac{n}{2} \right\rceil$ vertices determine a face. A closely related

development is the recent proof that the maximum number of vertices possessed by any n -polyhedron with $k(n-1)$ faces is:

$$\binom{k - \left\lceil \frac{(n+1)}{2} \right\rceil}{k-n} + \binom{k - \left\lceil \frac{(n+2)}{2} \right\rceil}{k-n}$$

where $\binom{a}{b} = \frac{a!}{b!(a-b)!}$.

By definition, a polyhedral set is the intersection of a finite number of half spaces- which we can refer to as the outer representation. The inner representation is then a description of the polyhedral set by means of its extreme points and extreme directions. We can then state the main result in relationship to linear programming: Let $S = \{x: Ax = b, x \geq 0\}$. Then any point in S can be represented as a convex combination of extreme points plus a nonnegative linear combination of its extreme directions. If S is bounded then it does not contain directions and any point in S can be described as a linear combination of its extreme points.

We can describe a linear programming problem as the minimization or the maximization of a linear function over a polyhedral set. The simplex method for solving linear programming problems is based on using the extreme points and directions of the polyhedral set in the problem. We will discuss this method in depth later.

Theorem 3.2. (Isoperimetric inequality) If S is the surface area and V is the volume of a convex body A in E_n and if ω is the volume of an n -dimensional ball of unit radius, then $S^n \geq n^n \omega V^{n-1}$, with equality if and only if A is a ball. Hence, for any bodies of a given volume, balls have least surface area.

Example 3.1.

Consider the case of $n=3$. In E_3 we have $S^3 \geq 3^3 \omega V^{2}$. Suppose $r = 1$ for our convex body A , where A is a ball of radius 1.

$$\text{Then } S^3 \geq n^n \omega V^{n-1},$$

$$\Rightarrow (4\pi r^2)^3 \geq 3^3 \left(\frac{4}{3} \pi r^3 \right) \left(\frac{4}{3} \pi r^3 \right)^{2} \text{ for } r = 1$$

$$\Rightarrow (4\pi \cdot 1^2)^3 \geq 3^3 \left(\frac{4}{3} \pi \cdot 1^3 \right) \left(\frac{4}{3} \pi \cdot 1^3 \right)^2$$

$$\Rightarrow (4\pi)^3 \geq 27 \left(\frac{4^3}{3^3} \right) \cdot \pi^3 = 4^3 \pi^3 \bullet$$

Example 3.2.

Another example is the case of $n = 2$. In E_2 , letting $r = 1$ we have $S^2 \geq n^n \omega V^{n-1}$,

$$\Rightarrow (2\pi r)^2 \geq 2^2 (\pi r^2) (\pi r^2)^{1} \text{ for } r = 1$$

$$\Rightarrow (2\Pi \cdot 1)^2 \geq 2^2 (\Pi \cdot 1^2)(\Pi \cdot 1^2)^{2-1}$$

$$\Rightarrow (2\Pi)^2 \geq 4(\Pi)^2 \bullet$$

We can also consider the combinatorial aspects of convex sets which are described below. According to Klee [9]:

The most extensive combinatorial developments deal with facial structure of convex polyhedra. Though terminology has not been standardized, we use the term *polyhedron* to mean a subset of E_n that is the intersection of a finite number of closed halfspaces. Of special interest are the bounded polyhedra, here called polytopes, and the polyhedral cones.

The following five conditions on a set P in E_n are equivalent:

- (i) P is a polyhedron;
- (ii) P is a closed convex set whose number of faces is finite;
- (iii) P is the convex hull of a finite system of points and rays;
- (iv) P is the vector sum $B + C = \{b + c: b \in B, c \in C\}$ of a polytope B and a polyhedral cone C ;
- (v) P is the closed convex hull of the union of a polytope B and a translate of a polyhedral cone C .

Below, in Figure 3.3 we show a 2 dimensional polyhedron, P and the associated sets B and C .

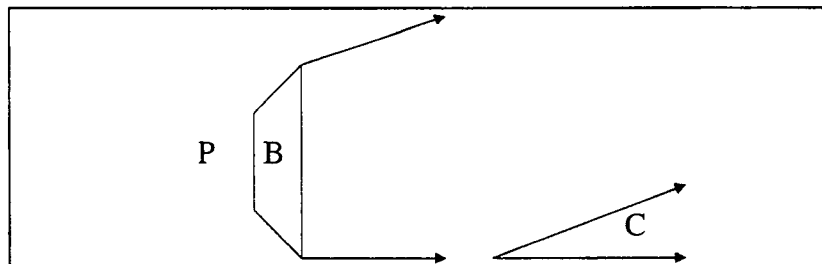


Figure 3.3 - Two Dimensional Polyhedron

Definition 3.8: Linear Programming Problems. These are Problems that deal with the efficient use of limited resources to obtain minimum use of resources or to obtain maximum profit. An example could be the minimum use of time to produce an optimal quantity of goods. *Linear Programming problems* require that the problem be stated in linear relationships. That is these relationships must be of the form $a_1 x_1 + a_2 x_2 + a_3 x_3 + \dots + a_j x_j + \dots + a_n x_n = b$, where the a_j 's and b are given and the x_j 's are unknown variables.

Definition 3.9: Line. A translate of linear subspace of E_1 is called a *line*.

Theorem 3.3. The set of all feasible solutions to a linear programming problem is a convex set. Furthermore, the set F of feasible solutions to a linear programming problem is a polyhedral set containing no lines.

Before proceeding further, it is necessary to give a geometric result which will be used in the following theorems.

Theorem 3.4. Let P be a polyhedral subset of E_n that contains no lines, and suppose the linear functional $f: E_n \rightarrow \mathbb{R}$ is bounded above on P . (i.e. $\sup_{x \in P} f(x) < \infty$.) Then there exists an extreme point x' of P such that $f(x') = \sup_{x \in P} f(x)$.

Proof: We prove the theorem by induction. Without loss of generality, assume that the dimension of P is n . If $n = 0$ the result is trivial. Assume that Theorem 3.4

holds for spaces of dimension less than n . Since f is linear then $\alpha \equiv \sup_{x \in P} f(x) = \sup_{x \in \text{bd } P} f(x)$.

Since P is a polyhedral set, we can write its boundary as $(P \cap H_1) \cup \dots \cup (P \cap H_m)$, where $H_1, \dots, H_m$ are the hyperplanes that bound the half spaces whose intersection is P . Thus, $\alpha \equiv \sup_{x \in P \cap H_i} f(x)$ for some i . Claim: $P \cap H_i$ is also a polyhedral. Note that if $n = 1$, then $P \cap H_i$ is a single point. If $n > 1$, then since P contains no lines, then $P \cap H_i \neq H_i$. In either case $P \cap H_i$ is a polyhedral containing no lines. Since the dimension of $(P \cap H_i) < n$, then by the inductive hypothesis $\alpha = f(x')$ for some extreme point in $P \cap H_i$. Any extreme point of $P \cap H_i$ is an extreme point of P . Hence, we are done. ■

Theorem 3.5. If the objective function is bounded above on F and $F \neq \emptyset$, then the linear programming problem has at least one optimal solution. Furthermore, at least one of the optimal solutions is an extreme point of F .

Proof: The result follows directly from Theorem 3.3 and Theorem 3.4. ■

An example that clearly illustrates the application of Theorem 3.5 is given below. We will solve the following problem geometrically.

Example 3.3.

$$\begin{array}{llll} \text{Maximize} & f(x_1, x_2) & \equiv & 5x_1 + 7x_2 \\ \text{subject to} & x_1 & \leq & 25 \end{array}$$

$$\begin{aligned}
 x_2 &\leq 15 \\
 x_1 + x_2 &\leq 30 \\
 \text{and } x_1 &\geq 0, \quad x_2 \geq 0
 \end{aligned}$$

Solution:

In Figure 3.4 we have graphed the constraint inequalities to obtain the shaded area as shown. The extreme points are found by solving the intersections of the pairs of linear equations. We start by graphing the lines $x_1 = 25$, $x_2 = 15$, $x_1 + x_2 = 30$.

Evaluating $f(x_1, x_2) \equiv 5x_1 + 7x_2$ at each point of intersection we obtain the values at each extreme point:

1. $(x_1, x_2) = (0, 0)$ we obtain
 $5x_1 + 7x_2 = 5 \cdot 0 + 7 \cdot 0 = 0$
2. $(x_1, x_2) = (0, 15)$ we obtain
 $5x_1 + 7x_2 = 5 \cdot 0 + 7 \cdot 15 = 105$
3. $(x_1, x_2) = (15, 15)$ we obtain
 $5x_1 + 7x_2 = 5 \cdot 15 + 7 \cdot 15 = 180$
4. $(x_1, x_2) = (25, 5)$ we obtain
 $5x_1 + 7x_2 = 5 \cdot 25 + 7 \cdot 5 = 160$
5. $(x_1, x_2) = (25, 0)$ we obtain
 $5x_1 + 7x_2 = 5 \cdot 25 + 7 \cdot 0 = 125$

Hence, we then obtain the maximum value of $5x_1 + 7x_2 = 5 \cdot 15 + 7 \cdot 15 = 180$

as shown in the following Table 3.1. This problem is referred to as the *Primal Problem*. The set of feasible solutions to the linear programming problem Example 3.3 is clearly a convex set as seen in the geometric solution of Figure 3.4. ●

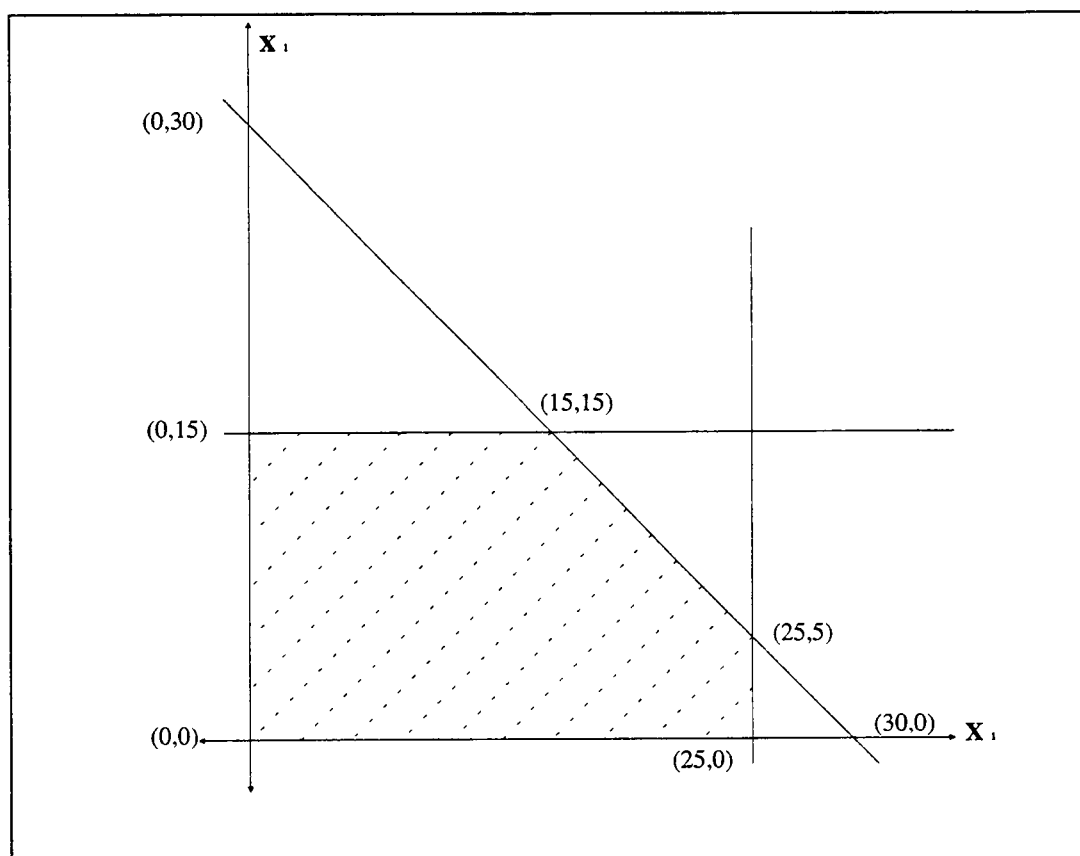


Figure 3.4 - Geometric Solution to Example 3.3

Table 3.1 - Example 3.3: Values at Each Extreme Point

(x_1, x_2)	$5x_1 + 7x_2$
$(0,0)$	0
$(0,15)$	105
$(15,15)$	180
$(25,5)$	160
$(25,0)$	125

A method for solving linear programming problems which makes use of the geometrical interpretation is the simplex method. The simplex method solves for an exact solution using an algebraic iterative procedure. Previously, we found that the set of all feasible solutions to a linear programming problem is a convex set: more precisely, a polyhedral set containing no lines. According to Theorem 3.5, at least one of the optimal solutions is an extreme point. The simplex method is computed in a finite number of steps by moving from a given extreme point along the edge of the region of feasible solutions from one extreme point to another until an optimal extreme point is reached. The simplex method determines whether a particular extreme point is optimal, and if it is not optimal, by moving along the edge, it tells us what the next extreme point will be. This process of moving along the edge of the solution space, also enables us to determine if the solution is unbounded. Below, we briefly describe the simplex algorithm.

Consider the linear programming problem:

$$\begin{array}{ll} \text{Maximize} & f(x) \equiv c^t x \\ \text{subject to} & Ax \leq b \\ \text{and} & x \geq 0 \end{array}$$

We begin by rewriting the linear program into standard form, where each inequality constraint is replaced by an equality constraint by the addition of slack variables. If the inequality is a " $\leq$ " constraint then it is converted to equality by adding a nonnegative slack variable. If the inequality is " $\geq$ " the constraint is converted to equality by the subtraction of the nonnegative slack variable. The

Table 3.2- Initial Tableau For The Simplex Procedure

				c_1	c_2	...	c_m	c_{m+1}	...	c_j	...	c_k	...	c_n
i	Basis	c	P_0	P_1	P_2	...	P_m	P_{m+1}	...	P_j	...	P_k	...	P_n
1	P_1	c_1	x_{10}	1	0	...	0	$x_{1,m+1}$	...	x_{1j}		x_{1k}	...	x_{1n}
2	P_2	c_2	x_{20}	0	1	...	0	$x_{2,m+1}$	...	x_{2j}		x_{2k}	...	x_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	P_i	c_i	x_{i0}	0	0	...	0	$x_{i,m+1}$		x_{ij}		x_{ik}	...	x_{in}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m	P_m	c_m	x_{m0}	0	0	0	1	$x_{m,m+1}$	...	x_{mj}		x_{mk}	...	x_{mn}
m+1			z_0	0	0	0	0	$z_{m+1} - c_{m+1}$	...	$z_j - c_j$		$z_k - c_k$	...	$z_n - c_n$

matrix is then arranged as in the Table 3.2 above. The unit vectors are listed as the first m vectors.

We have listed the information in the table as follows:

1. P_i are the basis vectors.
2. c_i are the coefficients of x_i in the objective function.
3. $x_{i0} = b_i$ are the constants on the right hand side of the constraints.
4. $x_{ij} = a_{ij}$ where $1 \leq i \leq m$, and $m+1 \leq j \leq n$ are the matrix coefficients of the nonbasis vectors.
5. $z_j - c_j = \sum_{i=1}^m c_i x_{ij}$ is the value of the objective function at the current extreme point.

After completing the table we begin the process of solving by the simplex method. For the initial extreme point we choose $\mathbf{X}_0 = \mathbf{P}_0 = (x_{10}, x_{20}, \dots, x_{m0})^t$ and then begin the following iteration procedure.

1. Test to see if $z_j - c_j \leq 0$ for all j . If so, then $\mathbf{X}_0$ is a minimum feasible solution. If not, continue to the next step.
2. Given that $z_j - c_j > 0$ for some j . We select the vector with the maximum value for $z_j - c_j$.
3. Select the vector that is to be removed from the basis in order to ensure feasibility. We remove the vector that minimizes $\theta = \min_{1 \leq i \leq m} (x_{i0} / x_{ik})$ for those $x_{ik} > 0$. If all $x_{ik} < 0$ the solution is unbounded. Pivot element x_{lk} where l corresponds to the vector to be removed.
4. Use Gauss-Jordan elimination to transform the tableau and obtain a new solution beginning again at step 1.

The following Example 3.4 we will solve by both the simplex method and geometrically as in the previous example. The simplex procedure will be computed by use of the Linear Interactive and Discrete Optimizer computer program, commonly referred to as LINDO. This program solves linear, integer and quadratic programming models.

Example 3.4.

$$\begin{array}{llll} \text{Maximize} & f(x_1, x_2) & = & 3x_1 + 4x_2 \\ \text{Subject to} & x_1 + 2x_2 & \leq & 25 \end{array}$$

$$\begin{aligned}
2x_1 + 3x_2 &\leq 30 \\
x_1 + x_2 &\leq 14 \\
\text{and } x_1 &\geq 0, \quad x_2 \geq 0
\end{aligned}$$

Solution:

In order to use the simplex method, we will need to rewrite the equations with slack variables in order to obtain the form $Ax = B$ for our constraints. We obtain:

$$\begin{aligned}
\text{Maximize } f(x_1, x_2) &= 3x_1 + 4x_2 \\
\text{Subject to } x_1 + 2x_2 + x_3 &= 25 \\
2x_1 + 3x_2 + x_4 &= 30 \\
x_1 + x_2 + x_5 &= 14 \\
\text{and } x_1, x_2, x_3, x_4, x_5 &\geq 0
\end{aligned}$$

Using LINDO we obtain the following output for the simplex tableaus shown Tables 3.3 through 3.5. Each tableau displays the values for each pivot, until the solution is reached. The objective value in Example 3.4 is shown to be 44 by the use of the LINDO program as seen in Table 3.5 on the following page.

In Figure 3.5 we have graphed the constraint inequalities to obtain the shaded area as shown. Solving the problem graphically we demonstrate the feasibility of the simplex method. The extreme points are (0,0), (0,10), (12,2), and (14,0) with the results in Table 3.6. Evaluating $f(x_1, x_2) = 3x_1 + 4x_2$ at each extreme point. We find the maximum value of 44 at (12,2). •

Table 3.3 -Example 3.4: Initial Simplex Tableau

B	X₁	X₂	X₃	X₄	X₅	RHS
	-3.000	-4.000	.000	.000	.000	.000
2	1.000	1.000	1.000	.000	.000	25.000
3	2.000	3.000	.000	1.00	.000	30.000
4	1.000	1.000	.000	.000	1.000	14.000

Table 3.4 - Example 3.4: X₂ enters and X₃ exits the basis

B	X₁	X₂	X₃	X₄	X₅	RHS
	-.333	.000	.000	1.333	.000	40.000
2	.333	.000	1.000	-.333	.000	15.000
X₂	.667	1.000	.000	.333	.000	10.000
4	.333	.000	.000	-.333	1.000	4.000

Table 3.5 - Example 3.4: X₁ enters and X₄ exits the basis

B	X₁	X₂	X₃	X₄	X₅	RHS
	.000	.000	.000	1.000	1.000	44.000
2	.000	.000	1.000	.000	-1.000	11.000
X₂	.000	1.000	.000	1.000	-2.000	2.000
X₁	1.000	.000	.000	-1.000	3.000	12.000

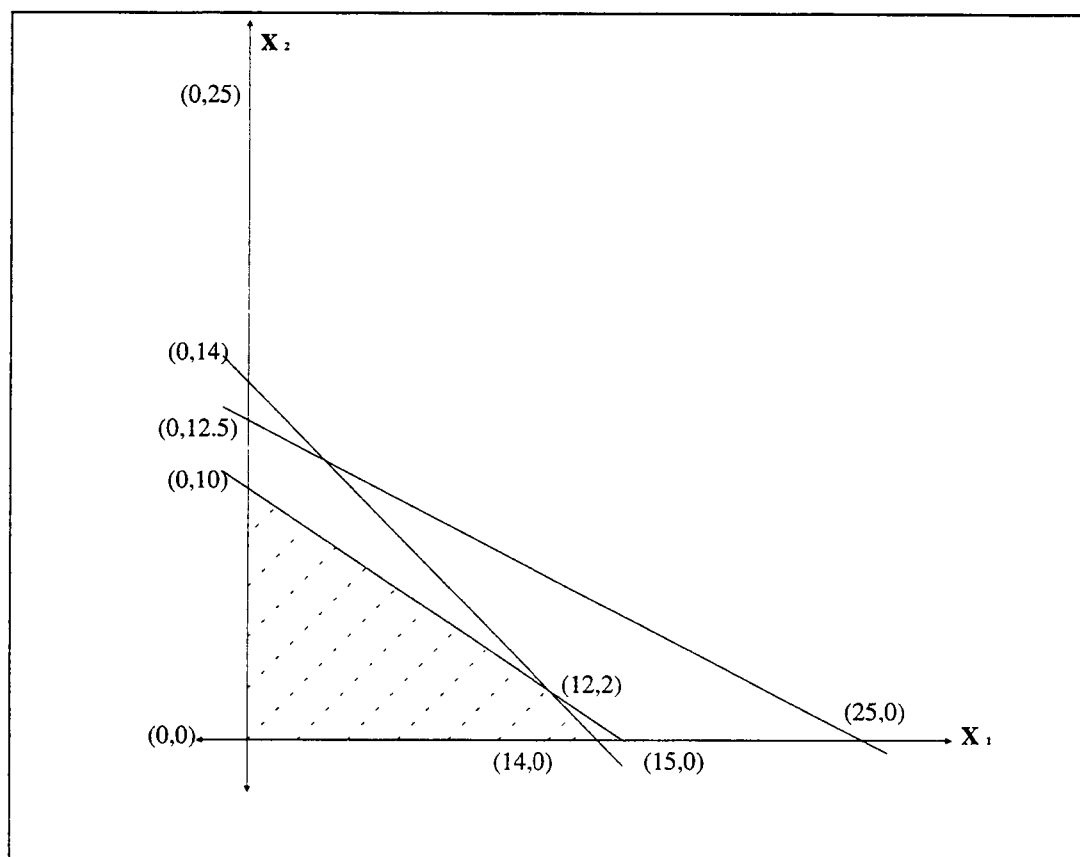


Figure 3.5 - Geometric Solution to example 3.4

Table 3.6 - Example 3.4: Values at Each Extreme Point

(x_1, x_2)	$3x_1 + 4x_2$
(0,0)	0
(0,10)	40
(12,2)	44
(14,0)	42

We will now consider another method by which the extreme points are used to find optimal solutions. This is the *Dual Linear Programming Problem*. The case we have discussed thus far is the Primal Problem. Using matrix symbolism we may write the problems as:

Primal Problem P

$$\begin{aligned}
 &\text{I.} \quad \text{Maximize } z = \mathbf{c}^t \mathbf{x}, \quad \text{Subject to} \\
 (3.3) \quad &(1) \quad \mathbf{A} \mathbf{x} \leq \mathbf{b} \\
 &(2) \quad \mathbf{x} \geq \mathbf{0}
 \end{aligned}$$

Dual Problem P*

$$\begin{aligned}
 &\text{II.} \quad \text{Minimize } w = \mathbf{b}^t \mathbf{v}, \quad \text{Subject to} \\
 (3.4) \quad &(1^*) \quad \mathbf{A}^t \mathbf{v} \geq \mathbf{c} \\
 &(2^*) \quad \mathbf{v} \geq \mathbf{0}
 \end{aligned}$$

Definition 3.8: Feasible. We say the constraints (1) and (2) are *feasible* if there exists some vector $\mathbf{x}$ satisfying them. The maximization problem is then considered feasible and $\mathbf{x}$ is the *feasible* vector for it.

Definition 3.9: Optimal. A feasible vector which produces the desired maximum for $\mathbf{c}^t \mathbf{x}$ is called an *optimal* vector for the maximization problem. The same term is defined analogously for the dual, or minimization problem.

Theorem 3.6. The dual of the dual is the primal problem.

Proof: First we consider the dual problem:

$$\begin{aligned}
 & \text{Min } w = \mathbf{b}^t \mathbf{v} && \text{subject to} \\
 (3.5) \quad & \mathbf{A}^t \mathbf{v} \geq \mathbf{c} \\
 & \mathbf{v} \geq \mathbf{0}
 \end{aligned}$$

We can write the objective function as $\text{Min } w = \text{Max } (-w) = \text{Max } w_1 = \mathbf{b}^t \mathbf{v}$. We can also multiply $\mathbf{A}^t \mathbf{v} \geq \mathbf{c}$ by -1 to obtain $-\mathbf{A}^t \mathbf{v} \leq -\mathbf{c}$. Hence, our original dual problem is equivalent to:

$$\begin{aligned}
 & \text{Max } w_1 = \mathbf{b}^t \mathbf{v} && \text{subject to} \\
 (3.6) \quad & -\mathbf{A}^t \mathbf{v} \leq -\mathbf{c} \\
 & \mathbf{v} \geq \mathbf{0}
 \end{aligned}$$

We see that (3.6) has the same form as the primal problem (3.3). Hence its dual has the form of (3.4) and is given by:

$$\begin{aligned}
 & \text{Min } z_1 = -\mathbf{c}^t \mathbf{x}, && \text{subject to} \\
 (3.7) \quad & -\mathbf{A} \mathbf{x} \geq -\mathbf{b} \\
 & \mathbf{x} \geq \mathbf{0}
 \end{aligned}$$

Note that $\text{Min } z_1 = \text{Max } (-z_1) = -\text{Max } z_1 = -\mathbf{c}^t \mathbf{x} = \text{Max } z$, and that $-\mathbf{A} \mathbf{x} \geq -\mathbf{b}$ is equivalent to $\mathbf{A} \mathbf{x} \leq \mathbf{b}$. Hence, (3.7) can be written as

$$\begin{aligned}
 & \text{Max } z = \mathbf{c}^t \mathbf{x}, && \text{subject to} \\
 (3.8) \quad & \mathbf{A} \mathbf{x} \leq \mathbf{b} \\
 & \mathbf{x} \geq \mathbf{0}
 \end{aligned}$$

which is the primal problem. ■

Theorem 3.7. (Duality Theorem) A feasible vector $\mathbf{x}$ to the primal problem is optimal if and only if there is a feasible vector $\mathbf{v}$ to the dual problem with $\mathbf{c}^t \mathbf{x} =$

$$\mathbf{b}^t \mathbf{v}.$$

Proof: First we consider an optimal solution to the primal problem and construct an optimal solution to the dual from the primal problem. Hence, this is sufficient since by Theorem 3.6, the dual of the dual is primal.

Let us first consider the primal problem (3.3). Using the simplex method, as discussed earlier, we add slack variables and convert (3.3) to the form:

$$\begin{aligned} \text{Max } z &= \mathbf{c}^t \mathbf{x} && \text{subject to} \\ (3.9) \quad \mathbf{Ax} + \mathbf{Ix}_s &= \mathbf{b} \\ \mathbf{x}, \mathbf{x}_s &\geq \mathbf{0} \end{aligned}$$

where $\mathbf{x}_s$ is the vector of slack variables. Let $\mathbf{x}_B$ be the basic optimal feasible solution to problem (3.9) and let B be the corresponding basis matrix. Since we know that $\mathbf{x}_B$ is an optimal solution then $z_j - c_j \geq 0$, for all j . If we rearrange $z_j - c_j \geq 0$ into matrix form and use the definition of z then

$$(3.10) \quad \mathbf{c}_B^t B^{-1} (\mathbf{A}, \mathbf{I}) \geq (\mathbf{c}^t, \mathbf{0}).$$

We now define $\mathbf{v}^t = \mathbf{c}_B^t B^{-1}$ and show this is an optimal feasible solution to the dual problem. Since $\mathbf{v}^t = \mathbf{c}_B^t B^{-1}$, we use (3.10) and obtain $\mathbf{v}^t \mathbf{A} \geq \mathbf{c}^t$ and $\mathbf{v}^t \geq \mathbf{0}$.

Rewriting this we obtain:

$$\begin{aligned} \mathbf{A}^t \mathbf{v} &\geq \mathbf{c} \\ (3.11) \quad \mathbf{v} &\geq \mathbf{0} \end{aligned}$$

Referring to the dual problem (3.4) and the result (3.11), clearly, $\mathbf{v}$ satisfies the both constraints.

Now, we need only show that $\mathbf{v}^t = \mathbf{c}_B^t \mathbf{B}^{-1}$ is an optimal solution to the dual problem (3.4). By assumption, $\mathbf{x}_B$ was an basic optimal feasible solution to the primal problem. Hence, $\text{Max } z = \mathbf{c}_B^t \mathbf{x}_B = \mathbf{c}_B^t \mathbf{B}^{-1} \mathbf{b} = \mathbf{v}^t \mathbf{b} = \mathbf{b}^t \mathbf{v} = \text{Min } w$. Thus we have shown by construction, that the optimal objective functions are equal. Hence, by the definition of optimal and dual solutions, the theorem is proved. ■

Next we examine the results to a linear programming problem and it's dual below.

Example 3.5

(a).The Primal problem:

$$\begin{array}{llll} \text{Maximize} & f(x_1, x_2) & = & 9x_1 + 10x_2 \\ \text{Subject to} & 5x_1 + 4x_2 & \leq & 80 \\ & 5x_1 + 8x_2 & \leq & 120 \\ \text{and} & x_1 \geq 0 & & x_2 \geq 0 \end{array}$$

(b).The dual Problem:

$$\begin{array}{llll} \text{Minimize} & f(y_1, y_2) & = & 80y_1 + 120y_2 \\ \text{Subject to} & 5y_1 + 5y_2 & \geq & 9 \\ & 4y_1 + 8y_2 & \geq & 10 \\ & y_1, y_2 & \geq & 0 \end{array}$$

Solution:

First, we solve the primal problem using LINDO, as seen in Tables 3.7 - 3.9, with the result being 172. Using LINDO the dual result is -172 (Tables 3.10-3.12).

Table 3.7 - Example 3.6 (a): Initial Simplex Tableau

B	X₁	X₂	X₃	X₄	RHS
	-9.000	-10.000	.000	.000	.000
2	5.000	4.000	1.000	.000	80.000
3	5.000	8.000	.000	1.000	120.000

Table 3.8 - Example 3.6 (a): X₁ enters and X₃ exits the basis.

B	X₁	X₂	X₃	X₄	RHS
	.000	-2.800	1.800	.000	144.000
2	1.000	.8000	.2000	.000	16.000
3	.000	4.000	-1.000	1.000	40.000

Table 3.9 - Example 3.6 (a): X₂ enters and X₄ exits the basis.

B	X₁	X₂	X₃	X₄	RHS
	.000	.000	1.100	.700	172.000
2	1.000	.000	.400	-.200	8.000
3	.000	1.000	-.250	.250	10.000

Table 3.10 - Example 3.6 (b) Dual Problem: Initial Simplex Tableau

B	X₁	X₂	X₃	X₄	RHS
	80.000	120.000	.000	.000	.000
2	-5.000	-5.000	1.000	.000	-9.000
3	-4.000	-8.000	.000	1.000	-10.000

Table 3.11 - Example 3.6 (b) Dual Problem : X₁ enters and X₄ exits the basis.

B	X₁	X₂	X₃	X₄	RHS
	.000	-40.000	.000	20.000	-200.000
2	.000	5.000	1.000	-1.250	3.500
3	1.000	2.000	.000	-.250	2.500

Table 3.12 - Example 3.6 (b) Dual Problem: X₂ enters and X₃ exits the basis.

B	X₁	X₂	X₃	X₄	RHS
	.000	.000	8.000	10.000	-172.000
2	.000	1.000	.200	-.250	.7000
3	1.000	.000	-.400	.250	1.100

Chapter IV

Applications Of Convex Cones And Polytopes

Introduction

This Chapter will include theorems dealing with applications and examples directly related to convex cones and polytopes. We will also include existence theorems and several theorems of the alternative. This chapter is based mainly on research derived from Bazaraa [3], Gale [5], Kuhn-Tucker [11], Lay [12], and Mangasarian [13], Solodovnikov [17].

Theorem 4.1. Let S be a nonempty, compact set, and let $f: S \rightarrow E_1$ be continuous on S . Then, the problem $\min \{f(x): x \in S\}$ attains its minimum, that is, there exists a minimizing solution to this problem. This result states that x' is a minimizing solution for the problem $\min \{f(x): x \in S\}$, provided that $x' \in S$ and $f(x') \leq f(x)$ for all $x \in S$.

Proof: Since S is compact it is both closed and bounded. Since f is continuous on S it is bounded below on S . Consequently, since S is nonempty, there exists a greatest lower bound $\alpha \equiv \inf \{f(x) : x \in S\}$. Let $0 < \varepsilon < 1$. Consider the set $S_k = \{x \in S: \alpha \leq f(x) + \varepsilon^k\}$ for each $k = 1, 2, \dots$. By the definition of infimum, S_k is nonempty for each k . Hence, we may construct a sequence of points $\{x_k\} \subseteq S$ by choosing any $x_k \in S$. Now, since S is bounded, there exists a convergent subsequence $\{x_{k_j}\}_{j=1}^\infty \rightarrow x'$. S is closed implies that $x' \in S$; and by the continuity of

f , since $\alpha \leq f(x_k) \leq \alpha + \varepsilon^k \quad \forall k$, then $\alpha = \lim_{k \rightarrow \infty} f(x_k) = f(x')$. Hence, there exists a solution $x' \in S$ such that $f(x') = \alpha = \{ f(x) : x \in S \}$. Thus, x is a minimizing solution. ■

Almost all optimality conditions are based on separation or support of convex sets. The following results are derived from the fact that when we are given a convex set S and a point $y \notin S$, there exists a unique point $x' \in S$ with a minimum distance from y and a hyperplane that separates y and S .

Consider the three illustrations in Figure 4.1. Each of the illustrations demonstrate an instance where a minimum does not exist. In Figure 4.1 (a) The infimum of f over (a, b) is $f(b)$, but S is not closed hence $b \notin S \rightarrow$ a minimum does not exist. Figure 4.1 (b) illustrates an example in which the infimum of f over $[a, b]$ is given by the $\lim_{x \rightarrow b^-} f(x)$. In this case the minimizing solution does not exist since the limit does not exist. Last, we consider Figure 4.1 (c) in which the minimizing solution does not exist since the function f is unbounded over the interval $[a, \infty) = \{x: x \geq a\} = S$.

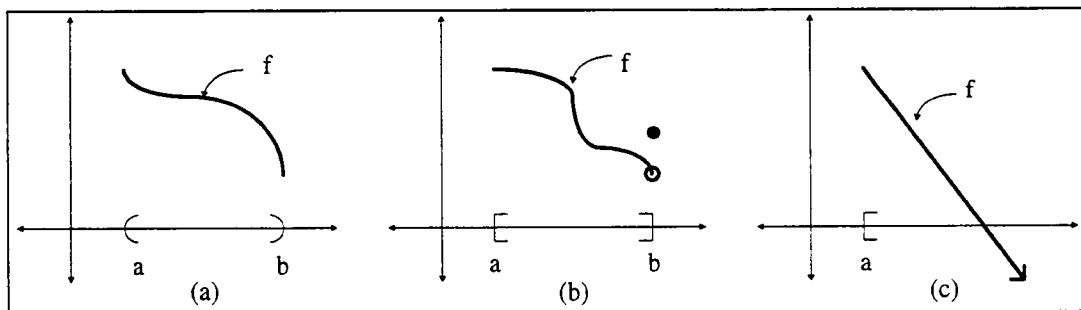


Figure 4.1. Non-existence of a minimizing solution.

Theorem 4.2. Let S be a nonempty, closed convex set in E_n and $y \notin S$. Then there exists a unique point $\underline{x} \in S$ with a minimum distance from y . Furthermore, $\underline{x}$ is the minimizing point if and only if $(y - \underline{x})^t (x - \underline{x}) \leq 0$ for all $x \in S$.

Proof: We first show the existence of a point $\underline{x} \in S$ that is a closest point. We are given that S is nonempty, hence we can assume there exists some point $x' \in S$. Consider the set S' such that $S' = \{ x : \|y - x\| \leq \|y - x'\| \}$. The infimum of $\{\|y - x\| : x \in S\}$ is equivalent to the infimum of $\{\|y - x\| : x \in S'\}$. Since the infimum of $\{\|y - x\| : x \in S'\}$ requires finding the minimum of a continuous function over S' (a nonempty compact set) then by Theorem 4.1 there exists a minimizing point $\underline{x} \in S$ closest to y .

We now show uniqueness. Now, suppose there exists some $\underline{x}' \in S$ such that $\|y - \underline{x}\| = \|y - \underline{x}'\| = \gamma$. Since S is convex then $(\underline{x} + \underline{x}') / 2 \in S$. By the triangle inequality we obtain: $\left\| y - \frac{\underline{x} + \underline{x}'}{2} \right\| \leq \frac{1}{2}\|y - \underline{x}\| + \frac{1}{2}\|y - \underline{x}'\| = \gamma \frac{1}{2}$. Suppose that strict inequality holds. This is a contradiction since $\underline{x}$ is the closest point to y . Thus equality holds and we have $y - \underline{x} = \lambda(y - \underline{x}')$ for some λ . Since $\|y - \underline{x}\| = \|y - \underline{x}'\| = \gamma$, then $|\lambda| = 1$. Now $\lambda \neq -1$ since $y = (\underline{x} + \underline{x}')/2 \in S$, contradicting our assumption that $y \notin S$. Hence $\lambda = 1$, $\underline{x} = \underline{x}'$, $\rightarrow x$ is unique.

We now show that for all $x \in S$ $(y - \underline{x})^t (x - \underline{x}) \leq 0$ is necessary and sufficient for x to be the closest point to y .

To prove sufficiency. Let $x \in S$. Then,

$$\|y - x\|^2 = \|y - \underline{x} + \underline{x} - x\|^2 = \|y - \underline{x}\|^2 + \|\underline{x} - x\|^2 + 2(\underline{x} - x)^t (y - \underline{x}) .$$

Since $\|\underline{x} - x\|^2 \geq 0$ and $(\underline{x} - x)^t (y - \underline{x}) \geq 0$ then by assumption $\|y - x\|^2 \geq \|y - \underline{x}\|^2$, and $\underline{x}$ is the minimizing point. Conversely, Suppose that $\|y - x\|^2 \geq \|y - \underline{x}\|^2$ for all $x \in S$. Since S is convex, we also know by the definition of

convex that $\underline{x} + \lambda(x - \underline{x})$ is contained in S for all $\lambda \in (0,1)$. Hence

$$(4.1) \quad \|y - \underline{x} - \lambda(x - \underline{x})\|^2 \geq \|y - \underline{x}\|^2 \text{ and}$$

$$(4.2) \quad \|y - \underline{x} - \lambda(x - \underline{x})\|^2 = \|y - \underline{x}\|^2 + \lambda^2 \|x - \underline{x}\|^2 - 2\lambda(y - \underline{x})^t (x - \underline{x})$$

Combining (4.1) and (4.2) we obtain:

$$(4.3) \quad 2\lambda(y - \underline{x})^t (x - \underline{x}) \leq \lambda^2 \|x - \underline{x}\|^2 \text{ for all } \lambda \in (0,1).$$

Then, if we divide (4.3) by λ we obtain $2(y - \underline{x})^t (x - \underline{x}) \leq \lambda \|x - \underline{x}\|^2$. Letting $\lambda \rightarrow 0^+$ then $(y - \underline{x})^t (x - \underline{x}) \leq 0$. This completes the proof. ■

Theorem 4.3. Let S be a nonempty closed convex set in E_n and $y \notin S$. Then there exists a nonzero vector p and a scalar α such that $p^t y > \alpha$ and $p^t x \leq \alpha$ for each $x \in S$.

Proof: The set S is a nonempty closed convex set in E_n and $y \notin S$. Hence, by Theorem 4.1 $\exists$ a unique minimizing point $\underline{x} \in S$ that $(x - \underline{x})^t (y - \underline{x}) \leq 0$ for each $x \in S$.

I. Letting $p = (y - \underline{x}) \neq 0$ and $\alpha = \underline{x}^t (y - \underline{x})$,

we get $(x - \underline{x})^t (y - \underline{x}) \leq 0$

$$\Leftrightarrow (y - \underline{x})^t (x - \underline{x}) \leq 0$$

$$\Leftrightarrow (y - \underline{x})^t x - (y - \underline{x})^t \underline{x} \leq 0 \text{ (by distribution)}$$

$$\Leftrightarrow p^t x - p^t \underline{x} \leq 0 \text{ (substituting } p \text{ for } (y - \underline{x}))$$

$$\Leftrightarrow \mathbf{p}^t \mathbf{x} - \alpha \leq 0$$

$$\Leftrightarrow \mathbf{p}^t \mathbf{x} \leq \alpha$$

$$\begin{aligned} \text{II. } \mathbf{p}^t \mathbf{y} - \alpha &= (\mathbf{y} - \mathbf{x})^t \mathbf{y} - \mathbf{x}^t (\mathbf{y} - \mathbf{x}) \\ &= \mathbf{y}^t (\mathbf{y} - \mathbf{x}) - \mathbf{x}^t (\mathbf{y} - \mathbf{x}) \\ &= (\mathbf{y} - \mathbf{x})^t (\mathbf{y} - \mathbf{x}) \text{ (factoring out } (\mathbf{y} - \mathbf{x})) \\ &= \|\mathbf{y} - \mathbf{x}\|^2 > 0 \end{aligned}$$

$$\Leftrightarrow \mathbf{p}^t \mathbf{y} - \alpha > 0 \Leftrightarrow \mathbf{p}^t \mathbf{y} > \alpha. \blacksquare$$

The following theorem gives the criteria for the solvability of a general set of linear equations. We will denote A as an $m \times n$ matrix, $\mathbf{x}$ as an n column vector, and $\mathbf{y}$ as an m column vector with coordinates η_j .

Theorem 4.4. Exactly one of the following alternatives hold.

$$(4.4) \quad A\mathbf{x} = \mathbf{b}$$

has a solution, or the equations

$$(4.5) \quad A^t \mathbf{y} = \mathbf{0} \qquad \mathbf{b}^t \mathbf{y} = 1$$

have a solution.

Proof: First we show that (4.4) and (4.5) cannot both be true. Assume (4.4) and (4.5) are both true. Say $\mathbf{a}^1, \dots, \mathbf{a}^n$ is a column basis for A ; $\eta_1, \dots, \eta_n$ are any real numbers and $\mathbf{b} = (b_1, \dots, b_n)$. In (4.5) we state: $A^t \mathbf{y} = \mathbf{0}$, $\mathbf{b}^t \mathbf{y} = 1$. Now, since $A\mathbf{x} = \mathbf{b}$ then $(A\mathbf{x})^t \mathbf{y} = \mathbf{b}^t \mathbf{y}$. This implies that $1 = \mathbf{b}^t \mathbf{y} = (A\mathbf{x})^t \mathbf{y} = \mathbf{x}^t (A^t \mathbf{y}) = \mathbf{x}^t \mathbf{0} = \mathbf{0}$ which is a contradiction. Hence both (4.4) and (4.5) cannot

both be true.

We now show that if (4.4) is false then (4.5) must be true. Let $\mathbf{a}_1, \dots, \mathbf{a}_r$ be a row basis for A . We can assume the vectors $\mathbf{a}_i$ and $\mathbf{b}$ are independent. If not, $\mathbf{b}$ would be a linear combination of the $\mathbf{a}_i$ which would imply (4.4) has a solution. Which is a contradiction since we have assumed (4.4) is false. Because the vectors $\mathbf{a}_1, \dots, \mathbf{a}_r$ are independent in E_n there exists a vector $\mathbf{y}$ such that $\mathbf{a}_i^t \mathbf{y} = \alpha_i$ for any numbers $\alpha_1, \dots, \alpha_r$, hence $\mathbf{a}_i^t \mathbf{y} = 0$ for $i \leq r$ where r is the number of rows and $\mathbf{b}^t \mathbf{y} = 1$. We also have for all k that $\mathbf{a}_k = \sum_{i=1}^r \lambda_i \mathbf{a}_i$ since the $\mathbf{a}_i$ are a row basis. Thus $\mathbf{y}^t \mathbf{a}_k = 0$ for all k . Hence we obtain: $A^t \mathbf{y} = \mathbf{0}$ and $\mathbf{b}^t \mathbf{y} = 1$. ■

We use the following example to verify Theorem 4.4.

Example 4.1.

Consider the following system of equations:

$$\begin{array}{rcl} 2x_1 + 3x_2 & = & 1 \\ x_1 - 3x_2 & = & 1 \\ -x_1 + x_2 & = & 0 \end{array}$$

Which we can write as:

$$\begin{bmatrix} 2 & 3 \\ 1 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Solution:

We can write the system as an augmented matrix and use Gauss-Jordan elimination to solve. Using this process we obtain the following:

$$\begin{bmatrix} 2 & 3 & | & 1 \\ 1 & -3 & | & 1 \\ -1 & 1 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{2} & | & \frac{1}{2} \\ 1 & -3 & | & 1 \\ 0 & \frac{5}{2} & | & \frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{2} & | & \frac{1}{2} \\ 0 & \frac{9}{2} & | & \frac{1}{2} \\ 0 & \frac{5}{2} & | & \frac{1}{2} \end{bmatrix} \sim$$
$$\begin{bmatrix} 1 & \frac{3}{2} & | & \frac{1}{2} \\ 0 & 9 & | & 1 \\ 0 & 5 & | & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & | & \frac{1}{3} \\ 0 & 1 & | & \frac{1}{9} \\ 0 & 0 & | & -4 \end{bmatrix}$$

Clearly, $Ax = b$ has no solution since $0 \neq -4$.

We now consider (4.5): $A^t y = 0$ and $b^t y = 1$.

We write this as:

$$\begin{bmatrix} A^t & | & 0 \\ b^t & | & 1 \end{bmatrix}$$

This is equivalent to the augmented matrix:

$$\begin{bmatrix} 2 & 1 & -1 & | & 0 \\ 3 & -3 & 1 & | & 0 \\ 1 & 1 & 0 & | & 1 \end{bmatrix}$$

Using Gauss-Jordan elimination we obtain:

$$\begin{bmatrix} 2 & 1 & -1 & | & 0 \\ 3 & -3 & 1 & | & 0 \\ 1 & 1 & 0 & | & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & | & 0 \\ 1 & -1 & -\frac{1}{3} & | & 0 \\ 1 & 1 & 0 & | & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & \frac{1}{2} & -\frac{1}{2} & | & 0 \\ 0 & -\frac{3}{2} & \frac{5}{6} & | & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & | & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & | & 0 \\ 0 & 1 & -\frac{5}{9} & | & 0 \\ 0 & 1 & 1 & | & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & | & 0 \\ 0 & 1 & -\frac{5}{9} & | & 0 \\ 0 & 0 & 1 & | & \frac{9}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & \frac{5}{7} \\ 0 & 0 & 1 & | & \frac{9}{7} \end{bmatrix}$$

Hence, we obtain: $y_1 = 1, \quad y_2 = \frac{5}{7}, \quad y_3 = \frac{9}{7}.$

Thus only one of the alternative (4.5) held for this particular example •.

Before proceeding any further it is necessary to introduce the following definitions.

Definition 4.1: Let $\mathbf{x} = (x_i)$ in E_m , then we define the following.

- (a) nonnegative. $\mathbf{x}$ is *nonnegative* if $x_i \geq 0$ for all i . This is written $\mathbf{x} \geq 0$.
- (b) positive. $\mathbf{x}$ is *positive* if $x_i > 0$ for all i . This is written $\mathbf{x} > 0$.
- (c) semipositive. $\mathbf{x}$ is *semipositive* if $\mathbf{x} \geq 0$ but $\mathbf{x} \neq 0$. We write $\mathbf{x} \geq 0$.

Theorem 4.5. (Farkus Theorem). Let A be an $m \times n$ matrix and $\mathbf{c}$ be an n vector.

Then, exactly one of the following two systems has a solution:

(4.6) $A\mathbf{x} \leq 0$ and $\mathbf{c}^t \mathbf{x} > 0$ for some $\mathbf{x} \in E_n$

(4.7) $A^t \mathbf{y} = \mathbf{c}$ and $\mathbf{y} \geq 0$ for some $\mathbf{y} \in E_m$.

Proof: Assume there exists a solution to (4.7) $\rightarrow \exists \mathbf{y} \geq \mathbf{0}$ such that $\mathbf{A}^t \mathbf{y} = \mathbf{c}$. Choose $\mathbf{x}$ such that $\mathbf{Ax} \leq \mathbf{0} \rightarrow \mathbf{c}^t \mathbf{x} = \mathbf{y}^t \mathbf{Ax} \leq 0 \rightarrow$ (4.6) has no solution. Now suppose (4.7) has no solution. Consider the set $S = \{ \mathbf{x} : \mathbf{x} = \mathbf{A}^t \mathbf{y} = \mathbf{c} \text{ and } \mathbf{y} \geq \mathbf{0} \}$. Now, S is a closed convex set by definition, and $\mathbf{c} \notin S$. By Theorem 4.3, $\exists$ a vector $\mathbf{p} \in E_n$ and a scalar α such that $\mathbf{p}^t \mathbf{c} > \alpha$ and $\mathbf{p}^t \mathbf{x} \leq \alpha \forall \mathbf{x} \in S$. $\mathbf{0} \in S \rightarrow \alpha \geq 0 \rightarrow \mathbf{p}^t \mathbf{c} > 0$, and $\alpha \geq \mathbf{p}^t \mathbf{A}^t \mathbf{y} = \mathbf{y}^t \mathbf{Ap} \forall \mathbf{y} \geq \mathbf{0} \rightarrow \mathbf{Ap} \leq \mathbf{0}$ since $\mathbf{y}$ can be made arbitrarily large. Hence, $\exists$ a vector $\mathbf{p} \in E_n$ such that $\mathbf{Ap} \leq \mathbf{0}$ and $\mathbf{c}^t \mathbf{p} > 0$. Therefore system (4.6) has a solution. ■

Theorem 4.6. (Farkus). Exactly one of the following alternatives hold.

$$(4.8) \quad \mathbf{Ax} = \mathbf{b}$$

has a nonnegative solution, or the inequalities

$$(4.9) \quad \mathbf{A}^t \mathbf{y} \geq \mathbf{0}, \quad \mathbf{b}^t \mathbf{y} < 0, \quad \text{have a solution.}$$

Proof: First we show that (4.8) and (4.9) cannot both be true. Suppose there exists a nonnegative solution $\mathbf{x}$ to (4.8) and a solution $\mathbf{y}$ to (4.9). Since $\mathbf{b} = \mathbf{Ax}$ and $\mathbf{b}^t = (\mathbf{Ax})^t$ then $0 < \mathbf{b}^t \mathbf{y} = (\mathbf{Ax})^t \mathbf{y} = (\mathbf{x}^t \mathbf{A}^t) \mathbf{y} = \mathbf{x}^t (\mathbf{A}^t \mathbf{y})$. Now, $\mathbf{x} \geq \mathbf{0}, \mathbf{A}^t \mathbf{y} \geq \mathbf{0} \rightarrow \mathbf{x}^t (\mathbf{A}^t \mathbf{y}) = \mathbf{b}^t \mathbf{y} \geq 0$. This is a contradiction to (4.9). Hence, there does not exist a nonnegative solution to both (4.8) and (4.9).

Suppose that $\mathbf{Ax} = \mathbf{b}$ has no nonnegative solution. Define the set $S \equiv \{ \mathbf{z} \in E_m : \mathbf{z} = \mathbf{Ax}, \mathbf{x} \geq \mathbf{0} \}$, and the hyperplane $H' \equiv \{ \mathbf{x} : \mathbf{x}^t \mathbf{u} = \alpha \}$ where $\mathbf{u}$ is a nonzero vector and α is a real number. Then the sets $\{\mathbf{b}\}$ and S are disjoint since H' strictly separates $\mathbf{b}$ and S in E_m . Furthermore, since $\mathbf{0} \in S$, we may also assume that $\mathbf{b}^t \mathbf{u}$

$< \alpha < 0$ and $\mathbf{z}^t \mathbf{u} > \alpha \quad \forall \mathbf{z} \in S$. Now, the plane $H \equiv \{ \mathbf{x} : \mathbf{x}^t \mathbf{u} = 0 \}$ is parallel to H' and supports S at 0. Hence $\mathbf{z}^t \mathbf{u} > 0 \quad \forall \mathbf{z} \in S$ which implies $(A\mathbf{x})^t \mathbf{u} > 0 \quad \forall \mathbf{x} \geq 0$. Thus, $\mathbf{x}^t \mathbf{u}A \geq 0 \quad \forall \mathbf{x} \geq 0$. Hence $\mathbf{u}A \geq 0$ which implies that, $\mathbf{b}^t \mathbf{u} < 0$ and $\mathbf{u}^t A \geq 0 \rightarrow \mathbf{u}^t A$ is a solution to (4.9). ■

We use the following example to verify Theorem 4.6.

Example 4.2.

Consider the following system of equations:

$$\begin{aligned}\xi_1 + 3\xi_2 - 5\xi_3 &= 2 \\ \xi_1 - 4\xi_2 - 7\xi_3 &= 3\end{aligned}$$

We will show exactly one of the alternatives (4.8) or (4.9) holds.

Solution:

First, let us consider $A^t \mathbf{y} \geq 0$ and $\mathbf{b}^t \mathbf{y} < 0$. We write this as:

$$\begin{aligned}y_1 + y_2 &\geq 0 \\ 3y_1 - 4y_2 &\geq 0 \\ -5y_1 - 7y_2 &\geq 0 \\ 2y_1 + 3y_2 &< 0\end{aligned}$$

Graphing the set of inequalities we obtain the solution in Figure 4.2 on the following page. Since (4.9) has a solution, we now show there is no solution to (4.8). If we consider the original system of equations as stated in Example 4.2. We see there are 2 rows = m and three columns = n . Since $n - m = 1$, we let $\xi_3 = 0$.

This then gives the system of equations:

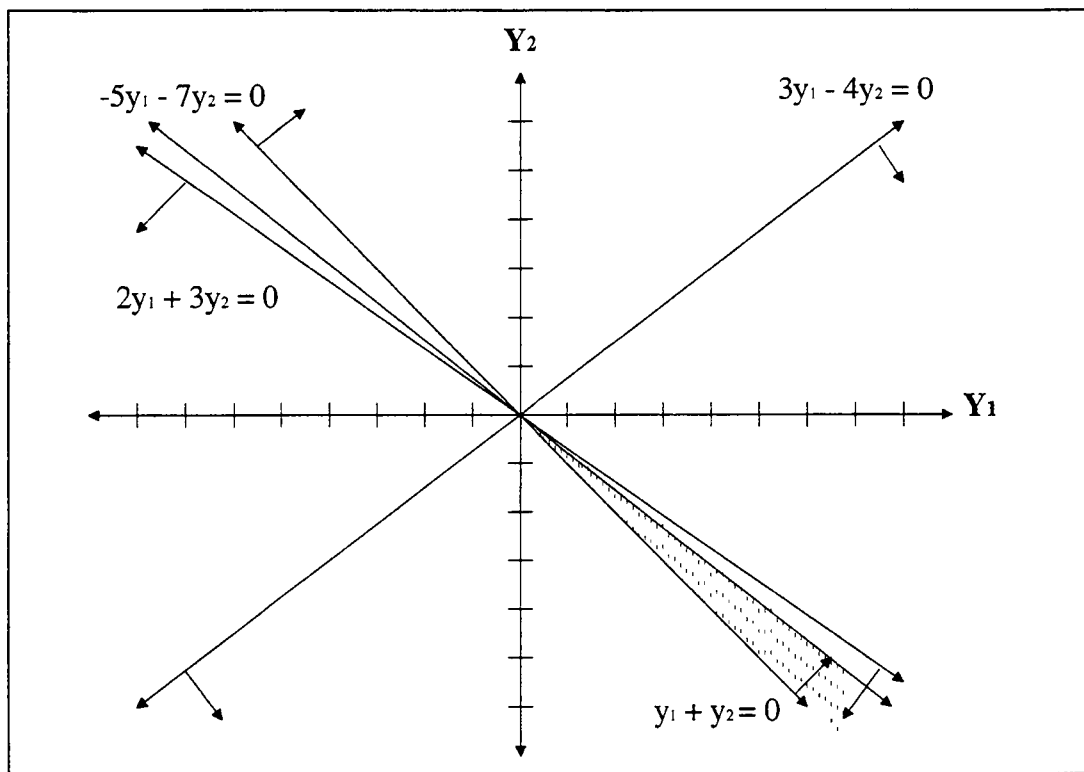


Figure 4.2 - Geometric Solution to Example 4.2

$$\xi_1 + 3\xi_2 = 2$$

$$\xi_1 - 4\xi_2 = 3$$

Solving we obtain $\xi_2 = \frac{-1}{7}$, $\xi_1 = \frac{17}{7}$.

Hence, $y = \left(\frac{17}{7}, \frac{-1}{7}, 0\right)$ which is clearly not a nonnegative solution. •

Theorem 4.7.(solutions of linear inequalities). Exactly one of the following alternatives hold.

$$(4.10) \quad A^t y \geq c$$

has a solution or the equations

$$(4.11) \quad Ax = 0, \quad c^t x = 1 \quad \text{have a nonnegative solution.}$$

in $n+1$ unknowns $x_1, x_2, \dots, x_n, t$, then there is a relationship between (4.12) and (4.13). By this we mean that if $x_1, x_2, \dots, x_n, t$ where $t > 0$ is a solution to (4.13) then :

$$(4.14) \quad x'_1 = \frac{x_1}{t}, \quad x'_2 = \frac{x_2}{t}, \quad \dots, \quad x'_n = \frac{x_n}{t}$$

is a solution of (4.12). Hence, in order to find any solution of (4.12), we must find a solution to (4.13) and transform the solution according to the formula (4.14).

We use the following Example 4.3 to verify Theorem 4.7.

Example 4.3.

Consider the following system of inequalities of the form $A^t y \geq c$.

$$\begin{array}{rcl} 4\xi_1 - 5\xi_2 & \geq & 3 \\ -2\xi_1 - 7\xi_2 & \geq & 1 \\ -2\xi_1 + \xi_2 & \geq & -2 \end{array}$$

According to Theorem 4.7 either $A^t y \geq c$ has a solution or $Ax = 0$, and $c^t x = 1$ has a solution.

Solution:

First we rewrite $Ax = 0$, $c^t x = 1$ in the matrix form:

$$\left[\begin{array}{c|c} A & 0 \\ \hline c^t & 1 \end{array} \right]$$

This is equivalent to the following augmented matrix.

$$\left[\begin{array}{ccc|c} 4 & -2 & -2 & 0 \\ -5 & -7 & 1 & 0 \\ 3 & 1 & -2 & 1 \end{array} \right]$$

Using Gauss-Jordan elimination we obtain:

$$\left[\begin{array}{ccc|c} 4 & -2 & -2 & 0 \\ -5 & -7 & 1 & 0 \\ 3 & 1 & -2 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & -\frac{7}{5} & \frac{1}{5} & 0 \\ 1 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{19}{10} & \frac{3}{10} & 0 \\ 0 & \frac{5}{6} & -\frac{1}{6} & \frac{1}{3} \end{array} \right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{19} & 0 \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{19} & 0 \\ 0 & 0 & \frac{4}{95} & \frac{2}{5} \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{19} & 0 \\ 0 & 0 & 1 & \frac{19}{2} \end{array} \right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{19}{2} \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 0 & \frac{19}{4} \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{19}{2} \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{11}{2} \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{19}{2} \end{array} \right]$$

Hence, we have the solution: $x_1 = \frac{11}{2}$, $x_2 = \frac{3}{2}$, $x_3 = \frac{19}{2}$.

Now, by use of the Solodovnikov's method described previously we will show

$A^t y \geq c$ has no solution. Note that $\xi_1 \geq 0$, $\xi_2 \geq 0$, and $t \geq 0$. First we rewrite the original inequalities to obtain:

$$L_1 : 4\xi_1 - 5\xi_2 - 3t \geq 0$$

$$L_2 : -2\xi_1 - 7\xi_2 - t \geq 0$$

$$L_3 : -2\xi_1 + \xi_2 + 2t \geq 0$$

We let $t = \xi_3$.

For convenience we set up Table 4.1 below where $L_1 = 4\xi_1 - 5\xi_2 - 3\xi_3$. We choose the nonnegative value of $L_1(\xi) = 4$ hence choose ξ_1 and now solve the following equations:

$$\xi_{12} = 5\xi_1 + 4\xi_2 = (5, 4, 0) \text{ call this } \alpha_2,$$

$$\xi_{13} = 3\xi_1 + 4\xi_3 = (3, 0, 4) \text{ call this } \alpha_3,$$

and let ξ_1 be α_1 .

From the new values above we obtain Table 4.2 on the following page where $L_2 = -2\alpha_1 - 7\alpha_2 - \alpha_3$. Since all the values are negative the process discontinues and there is no solution to the system $A^t y \geq c$.

Table 4.1- Example 4.3: Step 1 of Solodovnikov's Method.

Variable				$L_1(\xi)$
ξ_1	1	0	0	4
ξ_2	0	1	0	-5
ξ_3	0	0	1	-3

Table 4.2- Example 4.3: Step 2 of Solodovnikov's Method.

Variable				$L_2(\xi)$
ξ_1	1	0	0	-2
ξ_2	5	4	0	-38
ξ_3	3	0	41	-47

Theorem 4.8. Exactly one of the following alternatives hold.

(4.15) $Ax \leq b$

has a nonnegative solution, or the inequalities

(4.16) $A^t y \geq 0, \quad b^t y < 0,$

have a nonnegative solution.

Proof: Suppose (4.15) has no nonnegative solution. Then

(4.17) $Ax + z = b$

has no nonnegative solutions. We can rewrite (4.17) as $[A \ I] \begin{bmatrix} x \\ z \end{bmatrix} = b$, where

$[A \ I]$ is the block matrix with the left n columns = columns of A , the right m columns = columns of $m \times m$ Identity matrix and $\begin{bmatrix} x \\ z \end{bmatrix}$ = column matrix where first

n entries are coordinates of x and last m entries are coordinates of z . We know that

(4.17) has no nonnegative solutions, hence by Theorem 4.6 $\exists y$ such that $y[A \ I] \geq 0$ and $b^t u < 0$. But $y[A \ I] \geq 0$ implies that $yA \geq 0$ and $y \geq 0$. Hence, $\exists$ a nonnegative solution to (4.16). ■

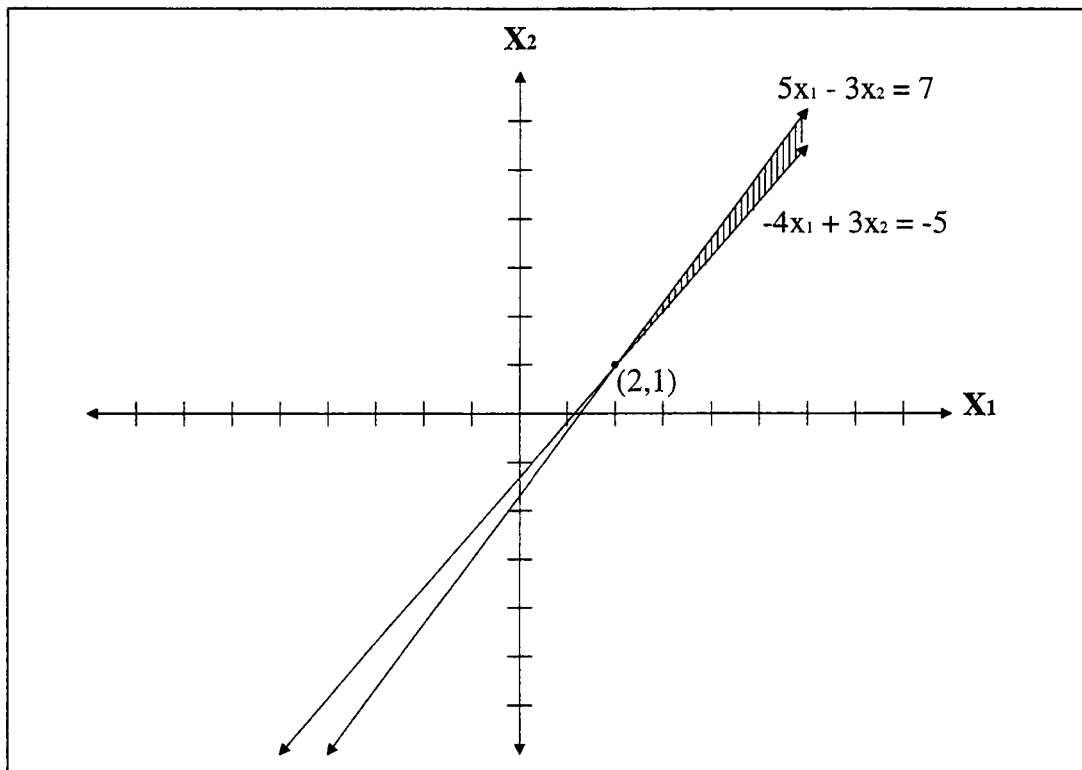


Figure 4.3 - Geometric Solution to Example 4.4

Next, we illustrate an example of Theorem 4.8.

Example 4.4.

Consider the following system of equations:

$$5x_1 - 3x_2 \leq 7$$

$$-4x_1 + 3x_2 \leq -5$$

Solution:

Writing the system in vector form we obtain:

$$\begin{bmatrix} 5 & -3 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 7 \\ -5 \end{bmatrix}$$

Solving the system graphically we obtain the result in Figure 4.3 above. The

solution for $\mathbf{x} \mathbf{A} \leq \mathbf{b}$ is clearly nonnegative. Hence, by theorem 4.8 we can conclude the inequalities $\mathbf{A}^t \mathbf{y} \geq 0, \mathbf{b}^t \mathbf{y} < 0$ have no nonnegative solution. •

Theorem 4.9. (semipositive solutions of homogeneous equations). Exactly one of the following alternatives holds.

$$(4.15) \quad \mathbf{A} \mathbf{x} = 0$$

has a semipositive solution or the inequality

$$(4.16) \quad \mathbf{y} \mathbf{A} > 0$$

has a solution.

Proof: First we show that (4.15) and (4.16) cannot both have solutions. If $\mathbf{A} \mathbf{x} = 0$, then $\mathbf{y} \mathbf{A} \mathbf{x} = 0$. But if $\mathbf{y} \mathbf{A} > 0$, then $\mathbf{y} \mathbf{A} \mathbf{x} > 0$, which is a contradiction. Hence, the two possibilities (4.15) and (4.16) are incompatible.

Now, suppose that (4.15) has no semipositive solution. Then we can conclude that $\sum a_i x_i = 0$ and $\sum x_i = 1$ have no nonnegative solution. Therefore, by Theorem 4.6 there exists a vector $\mathbf{y}$ and a number η such that $\mathbf{y} \mathbf{a}_i + \eta \geq 0$ for all i and that $0\mathbf{y} + \eta = \eta < 0$. Hence $a_i \geq -\eta$ for i . Thus 4.16 has a solution. ■

Theorem 4.10. semipositive solutions to homogeneous inequalities). Exactly one of the following alternatives holds.

$$(4.20) \quad \mathbf{A} \mathbf{x} \leq 0$$

has a semipositive solution, or the inequality

$$(4.21) \quad \mathbf{y} \mathbf{A} > 0$$

has a nonnegative solution.

Proof: As in the proof of Theorem 4.9 it is clear that the conditions (4.20) and (4.21) are incompatible.

If (4.20) has no semipositive solution then the inequalities: $\sum a_i x_i \leq 0$ and $-\sum x_i \leq -1$ have no nonnegative solutions. Hence, by Theorem 4.8 there exists $y \geq 0$ and $\eta \geq 0$ such that $ya_i - \eta \geq 0$ and $-\eta < 0$. Thus we can conclude that $ya_i \geq \eta > 0$ and $yA > 0$. ■

Theorem 4.11. A necessary and sufficient condition that one (and thus both) of the dual problems have optimal vectors is that both have feasible vectors.

Proof: (i.) First we show it is necessary for both vectors to be feasible vectors using a proof by contradiction. Suppose it is not necessary for the dual problems to have feasible vectors in order to have optimal vectors. If there are no feasible vectors, then there do not exist any vectors satisfying the constraints of the dual problem $\rightarrow$ There do not exist any optimal vectors by definition. This is a contradiction. Hence it is necessary for both vectors to be feasible vectors.

(ii.) We now show it is sufficient that both vectors be feasible. Suppose that both the maximum problem and its dual are feasible. Then $\exists$ nonnegative vectors x and y that satisfy

$$(4.22) \quad Ax \leq b \text{ and}$$

$$(4.23) \quad yA \geq c.$$

If there exist solutions to (4.22) and (4.23) then $A^{-1}Ax \leq A^{-1}b \rightarrow x \leq A^{-1}b$. And since $c \leq yA \rightarrow cx \leq yAA^{-1}b \rightarrow cx \leq yb$. If we can show x and y also satisfy (4.24) $cx - yb \geq 0$ we are done.

Suppose the system (4.22), (4.23), and (4.24) has no nonnegative solution. Then by Theorem 4.8 there exists a number θ , a nonnegative n -vector z and a nonnegative m -vector w such that

$$(4.25) \quad Az \geq \theta c$$

$$(4.26) \quad wA \geq \theta b$$

$$(4.27) \quad zb - wc \leq 0$$

Now $\theta \neq 0$, since if x and y are feasible vectors we would obtain $0 \leq xAz \leq zb$ and $0 \geq wAy \geq wc$ from equations (4.25), (4.26), (4.22), and (4.23) which contradicts (4.27). Since $\theta > 0$, w/θ and z/θ are a pair of feasible vectors so that $\frac{w}{\theta}c \leq \frac{z}{\theta}b$ or $wc \leq zb$ which contradicts (4.27). Hence, the set of inequalities (4.22), (4.23), and (4.24) has a solution which provides the desired pair of optimal vectors. ■

Theorem 4.12. If $b_1, \dots, b_k$ are positively independent, that is: $\sum_{i=1}^k \lambda_i b_i = 0$ has no semipositive solution, then the cone $C = b_1 + \dots + b_k$ is the sum of its extreme halflines.

Proof: Consider the set of vectors $\{ \mathbf{b}_1, \dots, \mathbf{b}_k \}$. Delete, if possible, some $\mathbf{b}_i$ such that $\mathbf{b}_i$ can be expressed as a nonnegative combination of the remaining vectors. Then, from the $k-1$ remaining vectors, if possible, again delete some vector which can be expressed as a nonnegative combination of the remaining vectors. Continue this process until there is no vector left in the set which can be expressed as a nonnegative combination of the remaining vectors. So we are left with a set of vectors such that $C = \sum_{i=1}^r \mathbf{b}_i$ and no vector can be expressed as a nonnegative combination of the others..

We now show that $\{ \mathbf{b}_1, \dots, \mathbf{b}_r \}$ is a set of extreme vectors. Without loss of generality suppose that $\mathbf{b}_1$ is not an extreme vector. This would imply that $\mathbf{b}_1 = \sum_{i=1}^r \lambda_i \mathbf{b}_i$, where each $\lambda_i \geq 0$. Suppose $\lambda_1 \geq 1 \rightarrow (\lambda_1 - 1) \mathbf{b}_1 + \sum_{i=2}^r \lambda_i \mathbf{b}_i = 0$.

This is a contradiction since $\{ \mathbf{b}_1, \dots, \mathbf{b}_r \}$ is a positive independent set.

Now, suppose $\lambda_1 < 1$, this implies that $\mathbf{b}_1 = \left(\frac{1}{1 - \lambda_1} \right) \sum_{i=2}^r \lambda_i \mathbf{b}_i$. But this is a

contradiction since no vector can be expressed as a nonnegative combination of the other vectors. Hence, $\mathbf{b}_1$ and all the $\mathbf{b}_i$ are extreme vectors. ■

Theorem 4.13. If A has rank of m then every solution of $A\mathbf{x} \leq 0$ is a nonnegative combination of extreme solutions.

Proof: Let $C = \mathbf{b}_1 + \dots + \mathbf{b}_k$ be the cone containing all solutions of $A\mathbf{x} \leq 0$ and assume for each $\mathbf{b}_i$ that $\sum_{i=1}^k \lambda_i \mathbf{b}_i = 0$, where $\lambda_i > 0$. If $\lambda_1 > 0$ then

$-b_1 = \frac{1}{\lambda_1} \sum_{i=2}^k \lambda_i b_i$. Hence, $A \cdot b_1 = (1/\lambda_1) \sum \lambda_i (A b_i) \leq 0$. Now, by

assumption for each b_i is an element of $C \rightarrow A b_i = 0$. This implies the rows of A are dependent which is a contradiction to our assumption that A has rank of m . Since for each b_i that $\sum_{i=1}^k \lambda_i b_i = 0$, it follows from Theorem 4.12 that the theorem is proved. ■

Before we proceed any further, we refer to three results from Mangasarian [13].

Theorem 4.14. (Tucker's First Existence Theorem.) For any given $p \times n$ matrix A , the systems $Ax \geq 0$ and $A^t y = 0, y \geq 0$ possess solution that satisfy the equation $Ax + y > 0$.

Theorem 4.15. (Tucker's Second Existence Theorem.) Let $A \neq \emptyset$ be a $p_1 \times n$ matrix and B a $p_2 \times n$ matrix. Then the systems $Ax \geq 0, Bx = 0$, and $A^t y_1 + B^t y_2 = 0, y \geq 0$ possess solutions such that $x \in R_n, y_1 \in R_{p_1}, y_2 \in R_{p_2}$ that satisfy the expression $Ax + y_1 > 0$.

Theorem 4.16. Let A, B, C , and D be $p_i \times n$ matrices where $i = 1, 2, 3, 4$ accordingly. And $A, B, C, D \neq \emptyset$. Then the system: $Ax \geq 0, Bx \geq 0, Cx \geq 0, Dx \geq 0$ and the system $A^t y_1 + B^t y_2 + C^t y_3 + D^t y_4 = 0, y_1, y_2, y_3, y_4 \geq 0$ possess solutions $x \in R_n, y_1 \in R_{p_1}, y_2 \in R_{p_2}, y_3 \in R_{p_3}, y_4 \in R_{p_4}$ that satisfy $Ax + y_1 > 0, Bx + y_2 > 0$, and $Cx + y_3 > 0$.

Theorem 4.17. (Slater's Theorem). Let A , B , C , and D be given matrices with A and B being nonvacuous. Then either

$$(4.28) \quad Ax > 0 \quad Bx \geq 0 \quad Cx \geq 0 \quad Dx = 0 \text{ has a solution or}$$

$$(4.29) \quad A^t y_1 + B^t y_2 + C^t y_3 + D^t y_4 = 0 \quad \text{with}$$

$$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0 \quad \text{or} \quad y_1 \geq 0, y_2 > 0, y_3 \geq 0$$

Proof: Assume there exists a solution to (4.28). We show this implies that if there also exists a solution to (4.29) this leads to a contradiction. Suppose (4.28) and (4.29) both have solutions. Then this implies there exists x, y_1, y_2, y_3, y_4 such that: $x A^t y_1 + x B^t y_2 + x C^t y_3 + x D^t y_4 > 0$ since $x D^t y_4 = 0$, $x C^t y_3 \geq 0$, and either $x B^t y_2 \geq 0$ and $x A^t y_1 > 0$, or $x B^t y_2 > 0$ and $x A^t y_1 \geq 0$. But this is a contradiction to (4.29) $A^t y_1 + B^t y_2 + C^t y_3 + D^t y_4 = 0$. This implies (4.28) and (4.29) cannot both hold simultaneously.

Suppose there does not exist a solution to (4.28). Then we can assume that if $Ax \geq 0, Bx \geq 0, Cx \geq 0, Dx \geq 0$ then $Ax \leq 0$ or $Bx = 0$. This implies that $A^t y_1 + B^t y_2 + C^t y_3 + D^t y_4 = 0$ with $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$ or $y_1 \geq 0, y_2 > 0, y_3 \geq 0$ by Theorem 4.16. ■

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Vita

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