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I am submitting herewith a thesis written by Ronald D. Hagan entitled "Combinatorial Interpretations and q-Analogues of the Catalan Numbers." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in mathematics.

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Combinatorial Interpretations and q -Analogues of the Catalan Numbers

A Thesis

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Abstract

The Catalan numbers form one of the more frequently encountered counting sequences in combinatorics. In this thesis, the Catalan numbers are developed in the context of their roots in two historical problems: the problem of determining the number of ways in which an n -gon can be divided into triangular regions by means of non-intersecting diagonals, and the problem of determining the number of ways in which parentheses can be inserted into a product of n factors. Several structures which are counted by the Catalan sequence are then discussed, including sub-diagonal lattice paths and trivalent trees. Finally, two q -analogues of the Catalan numbers are presented.

Table of Contents

Chapter		Page
1	Introduction.....	1
2	Parenthesizations of a Product.....	2
3	Triangulations of the n -gon.....	6
4	Sub-Diagonal Lattice Paths.....	8
5	Mountain Ranges, the Ballot Problem, Random Walks, and Binary Numbers.....	11
6	Planar Planted Trivalent Trees and Normal Planar Trees.....	13
7	The Handshake Problem.....	18
8	Sequences of $+1$'s and -1 's.....	21
9	Sequences of Positive Integers.....	24
10	q -Analogues of the Catalan Numbers.....	27
	List of References.....	38
	Vita	40

1. INTRODUCTION

In how many ways can a convex polygon of n sides be dissected into triangles by means of nonintersecting diagonals? According to Brown [2] this problem was first posed to Segner by Euler. Euler gave a solution without proof, which was later provided by Segner, along with a recurrence for the resulting sequence, in a paper in the Petersburg Transactions in 1759. The problem was next taken up in a series of four papers in Liouville's Journal for 1838-9, in which it was discussed by Lamé', Rodrigues, Binet, and Catalan.

Catalan was the first to observe that the problem of counting the triangulations of the n -gon is equivalent to the problem of counting the number of ways to insert parentheses in a product of n factors to create a unique pairwise association. It is for his work that the integer sequence which provides the solution to both problems,

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862,

takes its name as the Catalan sequence.

The sequence of Catalan numbers is one of the most frequently appearing sequences in combinatorics, having upwards of fifty distinct interpretations. We shall first derive an explicit formula for the n^{th} Catalan number, then consider some of the different counting problems whose solutions are given by the Catalan sequence, and finally discuss two q -analogues of the Catalan numbers.

2. PARENTHEZIZATIONS OF A PRODUCT

We will first consider the problem of determining the number of ways in which parentheses can be inserted to create a unique pairwise association of a product of n factors. Call this number $f(n)$. Then

n	$f(n)$	Parenthesizations of $x_1 \cdots x_n$
1	1	x_1
2	1	$x_1 x_2$
3	2	$x_1(x_2 x_3); (x_1 x_2)x_3$
4	5	$x_1(x_2(x_3 x_4)); x_1((x_2 x_3)x_4);$ $(x_1 x_2)(x_3 x_4); (x_1(x_2 x_3))x_4;$ $((x_1 x_2)x_3)x_4$

provides a partial table for the values of $f(n)$.

This problem was solved by Catalan in 1838 [4] and it is the values of $f(n)$ which form the sequence that bears his name. Thus it is in the context of this problem that we derive an expression for the n^{th} Catalan number. Consider the product

$$x_1 x_2 \cdots x_k x_{k+1} \cdots x_n.$$

By our definition of $f(n)$ the number of parenthesizations of the first k factors is given by $f(k)$ while the number of parenthesizations of the remaining $n - k$ factors is $f(n - k)$. Letting k range between 1 and $n - 1$ we then have that

$$f(n) = \sum_{k=1}^{n-1} f(k)f(n-k). \quad (2.1)$$

Set

$$F(x) = f(1)x + f(2)x^2 + \dots = \sum_{n=1}^{\infty} f(n)x^n. \quad (2.2)$$

Then

$$\begin{aligned} F^2(x) &= [f(1)f(1)]x^2 + [f(1)f(2) + f(2)f(1)]x^3 \\ &\quad + \dots + [f(1)f(n-1) + \dots + f(n-1)f(1)]x^n + \dots \\ &= f(2)x^2 + f(3)x^3 + \dots + f(n)x^n + \dots \\ &= F(x) - f(1)x = F(x) - x, \end{aligned}$$

by (2.1), so

$$F^2(x) - F(x) + x = 0.$$

Using the quadratic formula, regarding $F(x)$ as the unknown, we then have

$$F(x) = \frac{1 \pm \sqrt{1-4x}}{2}.$$

From (2.2), we see that $F(0) = 0$, so

$$F(x) = \frac{1 - \sqrt{1-4x}}{2}$$

and now by Newton's generalization of the binomial theorem,

$$\begin{aligned} F(x) &= \frac{1}{2} - \frac{1}{2}(1-4x)^{\frac{1}{2}} \\ &= \frac{1}{2} - \frac{1}{2} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4x)^n \\ &= \frac{1}{2} - \frac{1}{2} + \sum_{n=1}^{\infty} \left(-\frac{1}{2} \right) \binom{\frac{1}{2}}{n} (-4)^n x^n. \end{aligned} \quad (2.3)$$

Comparing coefficients of x^n in (2.2) and (2.3) yields

$$\begin{aligned} f(n) &= \left(-\frac{1}{2}\right) \binom{\frac{1}{2}}{n} (-4)^n \\ &= \left(-\frac{1}{2}\right) \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!} (-4)^n \\ &= \frac{1}{n} \binom{2n-2}{n-1}. \end{aligned}$$

For all $n \geq 0$ we define the n^{th} Catalan number, denoted $c(n)$, by

$$c(n) = \frac{1}{n+1} \binom{2n}{n}.$$

Then for all $n \geq 1$, $f(n) = c(n-1)$.

This result can also be obtained without the use of generating functions. Let $h(n)$ denote the number of ways to add $x_1, x_2, \dots, x_n$ in any order. Clearly $h(n) = n!f(n)$ and $h(1) = 1$. Consider the map from ways to add $x_1, x_2, \dots, x_n$ to ways to add $x_1, x_2, \dots, x_{n-1}$ that simply deletes the x_n term. A summation of the terms $x_1, x_2, \dots, x_{n-1}$ involves $n-2$ additions, a typical one of which can arise under the deletion map from any addition of the form

$$\begin{aligned} (x_n + ()) + () \\ (() + x_n) + () \\ () + (x_n + ()) \end{aligned}$$

or,

$$() + (() + x_n).$$

In addition, the pre-image of a given summation of the terms $x_1, x_2, \dots, x_{n-1}$ contains the sums obtained by adding the term x_n as the first summand or as the last summand. Therefore our map is a $4(n-2)+2 = (4n-6)$ -to-one surjection and we have the recurrence relation

$$\begin{aligned} h(1) &= 1 \\ h(n) &= (4n-6)h(n-1) \quad \text{for all } n \geq 2. \end{aligned} \quad (2.4)$$

By iterating this relation, we obtain

$$\begin{aligned} h(n) &= (4n-6)(4(n-1)-6)(4(n-2)-6)\dots \\ &= (4n-6)(4n-10)(4n-14)\dots 6 \cdot 2 \cdot 1 \\ &= \dots = \frac{(2n-2)!}{(n-1)!}. \end{aligned}$$

Thus,

$$f(n) = \frac{h(n)}{n!} = \frac{(2n-2)!}{n!(n-1)!} = \frac{1}{n} \binom{2n-2}{n-1} = c(n-1).$$

3. TRIANGULATIONS OF THE N-GON

We now turn our attention to the Euler-Segner problem of counting the number of ways in which we may triangulate an n -gon. Let $f(n)$ denote the number of ways in which a convex polygon of n sides can be dissected into triangular regions by means of non-intersecting diagonals. Then the first few values of $f(n)$ are given by figure 3.1.

Our claim is that $f(n) = c(n-2)$ for all $n \geq 3$. The proof we give is due to H. G. Forder [5]. Let us consider a triangulation of an n -sided polygon. Label the sides (excluding the base) in a clockwise fashion as $x_1, x_2, \dots, x_{n-1}$. If a diagonal spans adjacent sides x_i and x_{i+1} , label it with the product, $(x_i x_{i+1})$. Label the remaining diagonals with the product of the labels of the other two sides of the triangle above, each time enclosing the product in parentheses.

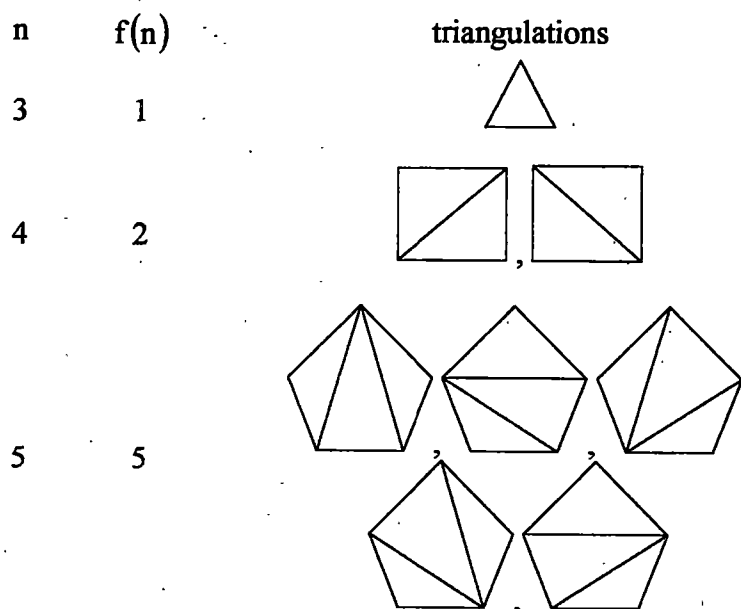


Figure 3. 1

The base is labeled last and its expression is uniquely determined by the triangulation with which we began; moreover, this expression is a unique parenthesization of the $n-1$ factors which comprise the original labels of the sides of our polygon. Notice that the process is completely reversible.

Given a parenthesization of $n-1$ factors $x_1, x_2, \dots, x_{n-1}$ we may construct an n -sided convex polygon and label the sides as before. Diagonals are then drawn so as to complete the triangles which have sides with pairwise-associated factors as labels, labeling the diagonals appropriately as we proceed. The resulting triangulation is uniquely determined by the association with which we began. Thus there is a one-to-one correspondence between the set of triangulations of a convex n -gon and the set of parenthesizations of a product of $n-1$ factors and our claim is established.

As an example of the process used to establish the correspondence, consider the triangulated octagon in figure 3.2.

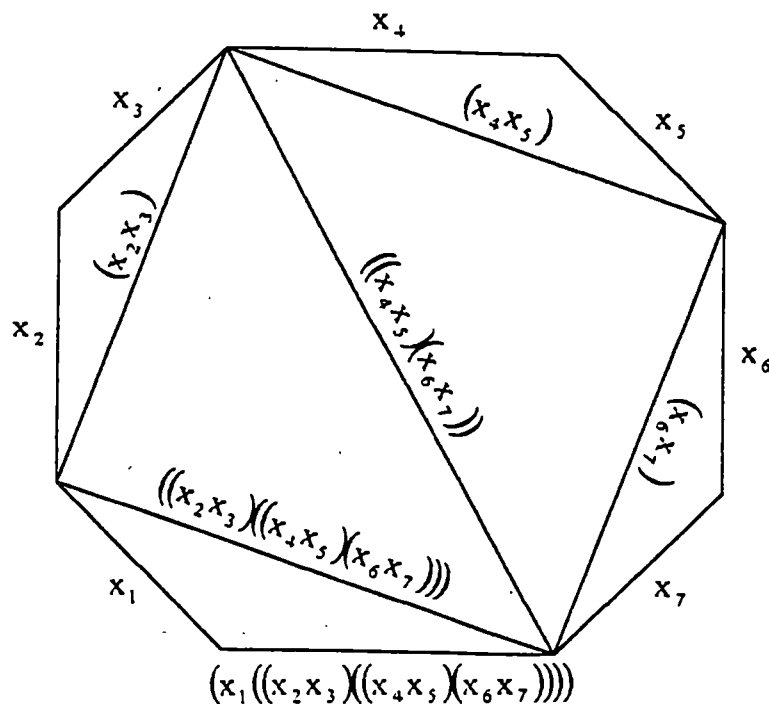


Figure 3. 2

4. SUB-DIAGONAL LATTICE PATHS

Let (m, n) be a point in $\mathcal{R}^2$ with m and n nonnegative integers. A *lattice path from $(0,0)$ to (m, n)* is a sequence of horizontal and vertical moves, each of unit length, by which one moves from $(0,0)$ to (m, n) . Obviously each such path consists of m horizontal moves (denote these by h 's) and n vertical moves (denote these by v 's) and the set of all lattice paths from $(0,0)$ to (m, n) is in one-to-one correspondence with the set of sequential arrangements of m h 's and n v 's, from which it follows that there are

$$\binom{m+n}{m} = \binom{m+n}{n} = \frac{(m+n)!}{m!n!} \quad (4.1)$$

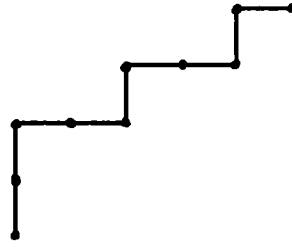
such lattice paths.

There is also a one-to-one correspondence between the set of all lattice paths from $(0,0)$ to (m, n) and the set

$$S_{m,n} := \{(a_0, a_1, \dots, a_m) : a_i \in \mathcal{N} \text{ and } 0 \leq a_0 \leq a_1 \leq \dots \leq a_m = n\}. \quad (4.2)$$

Associated with a given lattice path is the sequence $(a_0, a_1, \dots, a_m)$, where, for each $i = 0, \dots, m$, a_i is the largest y -coordinate of those points on the lattice path with x -coordinate equal to i . For example, associated with the lattice path from $(0,0)$ to $(5,4)$ given in figure 4.1 is the sequence $(2,2,3,3,4,4)$.

We now restrict our attention to the case $m \geq n$ and the problem of enumerating the lattice paths from $(0,0)$ to (m, n) that never rise above the diagonal



(2,2,3,3,4,4)

Figure 4. 1

line $y = x$, i.e., which contain no point with y-coordinate strictly greater than its x-coordinate. Thus, we are counting sequences of m h's and n v's, no initial segment of which contains strictly more v's than h's. Equivalently, we are enumerating the set

$$S_{m,n}^* := \{(a_0, a_1, \dots, a_m) : a_i \in \mathcal{N}, \text{ with } a_i \leq i, \text{ and } 0 = a_0 \leq a_1 \leq \dots \leq a_m = n\} \quad (4.3)$$

Represent a lattice path from $(0,0)$ to (m,n) as a sequential arrangement of m h's and n v's. We shall count the paths that violate the sub-diagonality condition. Such a path rises above the diagonal for the first time, reaching, say, the point $(k, k+1)$, where $0 \leq k \leq n-1$. In the h,v-sequence representing this path, change the remaining $m-k$ h's to v's and the remaining $n-k-1$ v's to h's. This yields a sequence of $n-1$ h's and $m+1$ v's, defining a lattice path from $(0,0)$ to $(n-1, m+1)$. It is easy to see that this procedure defines a one-to-one correspondence from the set of all lattice paths from $(0,0)$ to (m,n) that violate sub-diagonality to the set of all lattice paths from $(0,0)$ to $(n-1, m+1)$. Since by (4.1) there are

$$\binom{m+n}{n-1} = \binom{m+n}{m+1} = \frac{(m+n)!}{(n-1)!(m+1)!} \quad (4.4)$$

of the latter, there are

$$\binom{m+n}{m} - \binom{m+n}{n-1} = \frac{m-n+1}{m+1} \binom{m+n}{n} \quad (4.5)$$

sub-diagonal lattice paths from $(0,0)$ to (m,n) where $m \geq n$. Thus,

$$|S_{m,n}^*| = \frac{m-n+1}{m+1} \binom{m+n}{m}$$

in particular,

$$|S_{n,n}^*| = \frac{1}{n+1} \binom{2n}{n} = c(n).$$

Since, as shown in chapter two, $c(n)$ also enumerates the parenthesizations of a product $x_1, x_2, \dots, x_{n+1}$ of $n+1$ factors, it is of interest to exhibit a bijection from the set of such parenthesizations to the set of sub-diagonal lattice paths from $(0,0)$ to (n,n) , here represented by sequential arrangements of n h's and n v's, no initial segment of which contains strictly more v's than h's. A natural such bijection may be constructed as follows: Write the parenthesized product including the multiplication symbols and enclose it in an extra pair of parentheses. Now replace each “ $\bullet$ ” by an h, each “ $)$ ” by a v, and remove everything else. The result is clearly a uniquely determined sequence of n h's and n v's which satisfies our restriction that each initial segment must contain strictly more h's than v's. For example, via our construction, the parenthesization $(x_1((x_2x_3)(x_4(x_5x_6))))$ corresponds to the lattice path represented by the sequence $(h, h, v, h, h, h, v, v, v, v)$.

5. MOUNTAIN RANGES, THE BALLOT PROBLEM, RANDOM WALKS, AND BINARY NUMBERS

Numerous incarnations of the lattice path problem appear throughout the literature on the Catalan numbers. Some of these include the problems of counting the number of:

- i) mountain ranges which can be drawn using n upstrokes and n downstrokes.
- ii) ways in which the ballots can be counted in an election race between two candidates subject to the restrictions that each candidate receives exactly n votes and at no time during the tabulation does the first candidate trail the second.
- iii) ways in which a person can take $2n$ steps, each of equal length either straight forward or backward, beginning and ending with his/her back against a wall.
- iv) binary numbers of length $2n$, comprised of exactly n zeros and n ones, such that, from left to right, the number of zeros encountered is always less than, or equal to, the number of ones.

Let us first consider the problem of determining the number of mountain ranges which can be drawn using n upstrokes and n downstrokes of equal length with the restriction that we never pass below the horizon. Let $f(n)$ be the number of acceptable mountain ranges for a given n . Then for $n = 1, 2, 3$ the values of $f(n)$,

along with the corresponding ranges, are illustrated in figure 5.1.

There is an obvious bijection ϕ between the set of acceptable mountain ranges and the set of sub-diagonal lattice paths from $(0,0)$ to (n,n) which associates an upstroke with a move from (x_1, x_2) to $(x_1 + 1, x_2)$ and a downstroke with a move from (x_1, x_2) to $(x_1, x_2 + 1)$. Thus $f(n) = c(n)$ for all $n \geq 0$. The bijections from the underlying sets involved in (ii)-(iv) are all analogous. For example, to verify the fact that the solution to (ii) is also given by the n th Catalan number we modify ϕ so as to associate a vote for the first candidate with a move in the lattice from (x_1, x_2) to $(x_1 + 1, x_2)$ and a vote for the second candidate with a move from (x_1, x_2) to $(x_1, x_2 + 1)$.

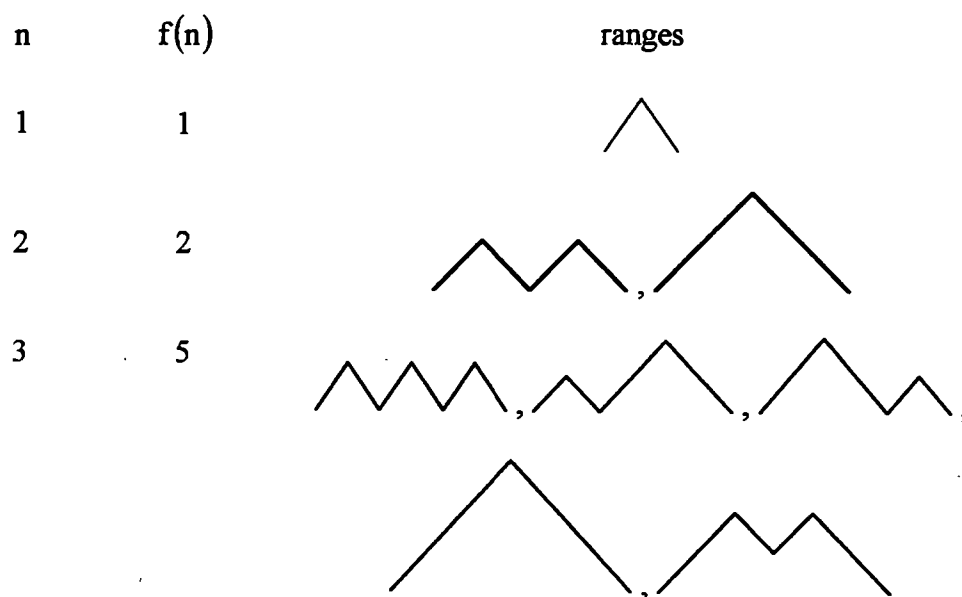


Figure 5. 1

6. PLANAR PLANTED TRIVALENT TREES AND NORMAL PLANAR TREES

A *planar tree* is a connected graph drawn in the plane which contains no circuits. By a *planted tree* we mean a tree which has a solitary vertex identified as the *root vertex* from which extends a lone edge called the “trunk”. The *valence* of a vertex is defined to be the number of edges which emanate from it. Thus, by a trivalent tree we mean a tree whose vertices are all of valence three or one. Let $f(n)$ be the number of planar planted trivalent trees with n endpoints (excluding the root vertex). We claim that $f(n) = c(n-1)$ for all n greater than one.

Suppose that we are given such a tree. We then wish to label the endpoints of the branches (excluding the root vertex) from left to right as $x_1, x_2, \dots, x_n$. Using these labels, form the product $x_1 x_2 \cdots x_n$ and associate the terms in the natural way. That is to say, if two factors label the endpoints of branches emanating from the same vertex, enclose them in parentheses. In this way each such tree corresponds to exactly one parenthesization of a product of n factors. Moreover, given a particular parenthesization of the product $x_1 x_2 \cdots x_n$ we can recover exactly one corresponding tree by reversing the process. Thus the planar trees having n endpoints which are rooted and trivalent are in one-to-one correspondence with the parenthesizations of a product of n factors and our claim is established. The trees and the corresponding parenthesizations for $n = 1, 2, 3$ are illustrated in figure 6.1.

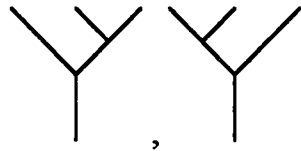
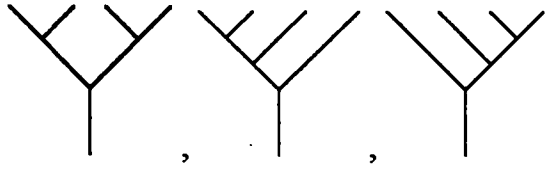
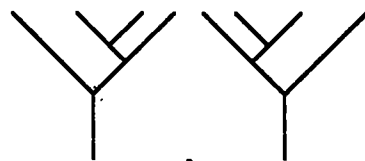
n	$f(n)$	Trivalent trees
2	1	 $(x_1 x_2)$
3	2	 $(x_1 (x_2 x_3)) \quad ((x_1 x_2) x_3)$
4	5	 $((x_1 x_2) (x_3 x_4)) \quad (((x_1 x_2) x_3) x_4) \quad (x_1 (x_2 (x_3 x_4)))$  $(x_1 ((x_2 x_3) x_4)) \quad ((x_1 (x_2 x_3)) x_4)$

Figure 6. 1

Now let us turn our attention to normal planar trees. We shall define a *normal planar tree* as a planted planar tree of n edges whose vertices (other than the root vertex) can have any valence. Let $f(n)$ denote the number of normal planar trees of n edges. Then the first few values of $f(n)$, along with the corresponding trees, are given in figure 6.2. In his article [7] Gardner attributes the following method to Frank Bernhart.

For a fixed value of n , draw the set of planted planar trivalent trees with n endpoints so that at each vertex the edges point up, down, and to the right. Each horizontal edge is then removed. If there is a trivalent vertex at the right end of the edge it is merged with the vertex at the left end, keeping all the vertical edges distinct.

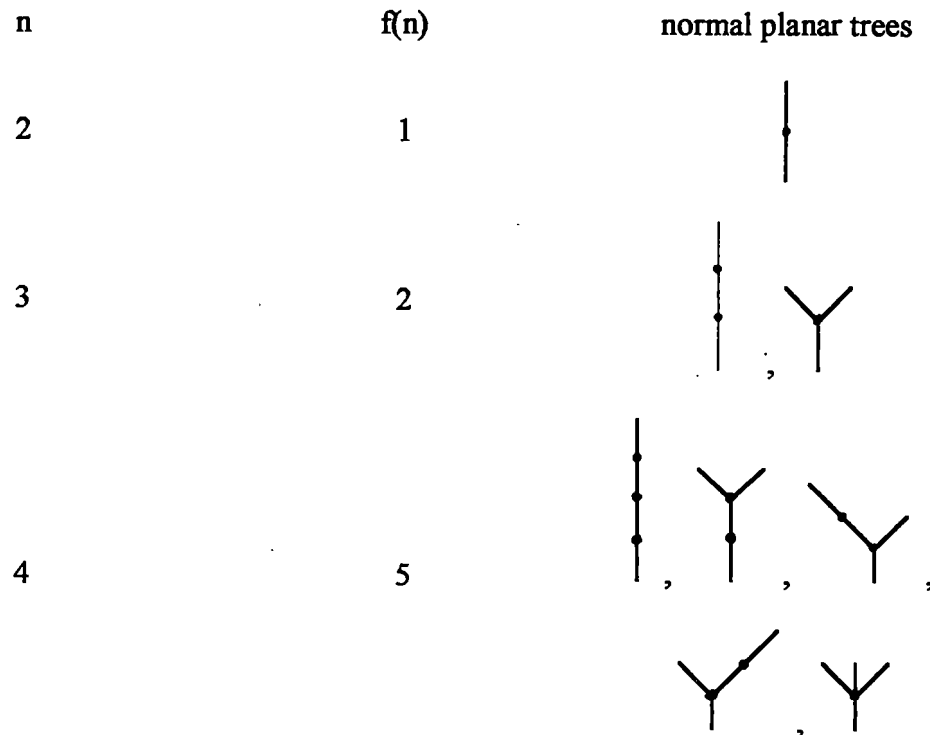


Figure 6. 2

The result of the transformation applied to a trivalent tree is a normal planar tree with n edges. This process is reversible so the set of normal planar trees is in one-to-one correspondence with the set of planted planar trivalent trees with n endpoints.

Thus $f(n) = c(n-1)$ for all $n \geq 1$. The transformation is illustrated in figure 6.3.

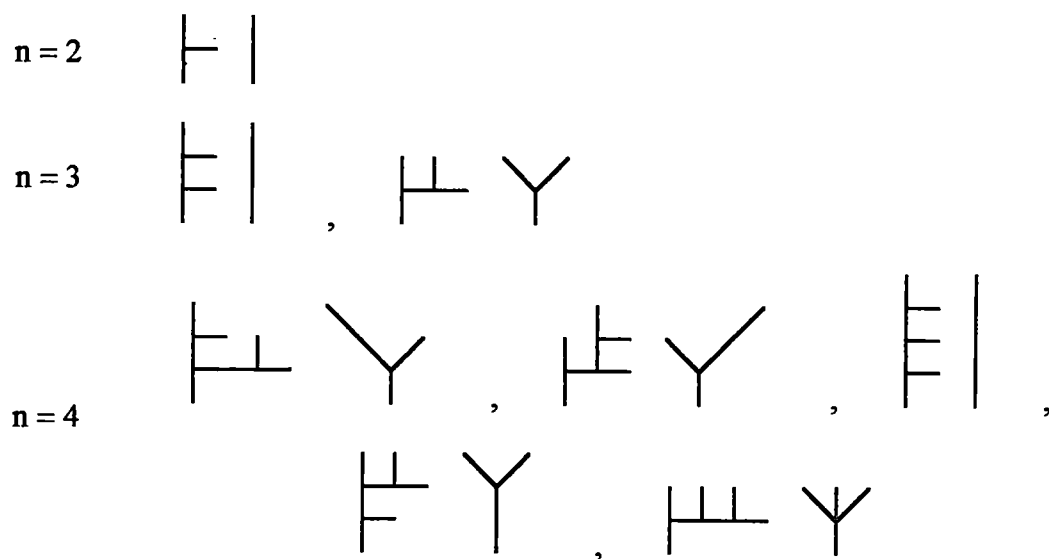


Figure 6. 3

7. The Handshake Problem

We now wish to consider another problem treated in Gardner's article. Namely, the problem of determining the number of ways in which $2n$ people seated about a circular table can shake hands without crossing arms? Equivalently, if we fix $2n$ points on a circle, we wish to count the number of ways in which we can pair the points by means of nonintersecting chords. Call this number $f(n)$. Then figure 7.1 gives the first values of $f(n)$.

Given such a pairing, form a binary number in the following manner. Beginning at the base, transverse the circle in a clockwise fashion recording a 1

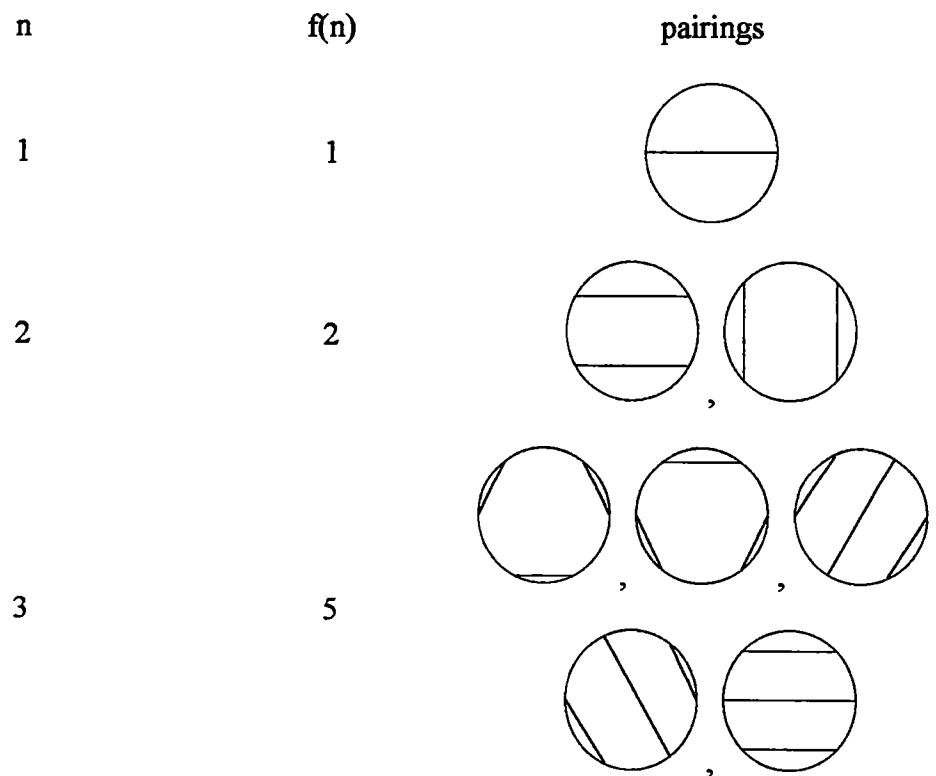


Figure 7. 1

whenever a chord is encountered for the first time, and a 0 when it encountered the second time. The result is clearly a binary number of length $2n$ which is comprised of exactly n 1's and n 0's and is such that none of its initial segments contain more 0's than 1's. Moreover, if we begin with any such binary number and $2n$ fixed points on a circle, we may determine a unique pairing of these points by connecting the points corresponding to the first 1 from the end of the binary number and the first 0 to its right with a chord, connecting the points corresponding to the second 1 from the end and the first unpaired 0 to its right with a chord, etc. Thus $f(n) = c(n)$ for all $n \geq 1$. An example of the binary numbers associated with the possible pairings for $n = 3$ is given in figure 7.2.

We may also easily show that the pairings enumerated by $f(n)$ are in one-to-one correspondence with the normal planar trees with $n + 1$ edges. Given a pairing, place a dot in each of the $n + 1$ subregions into which the circle is divided by its chords. Also, place a dot beneath the circle. Now construct a normal planar tree by using the dots as its vertices, the exterior dot serving as the root vertex. Draw the branches so that each chord is crossed by exactly one branch. The result is a unique normal planar tree of $n + 1$ edges. Since the correspondence is reversible, we have an alternative proof that $f(n) = c(n)$. The construction is illustrated in figure 7.3.

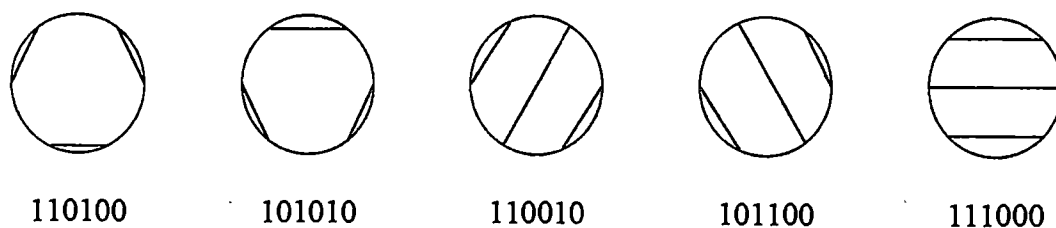


Figure 7. 2

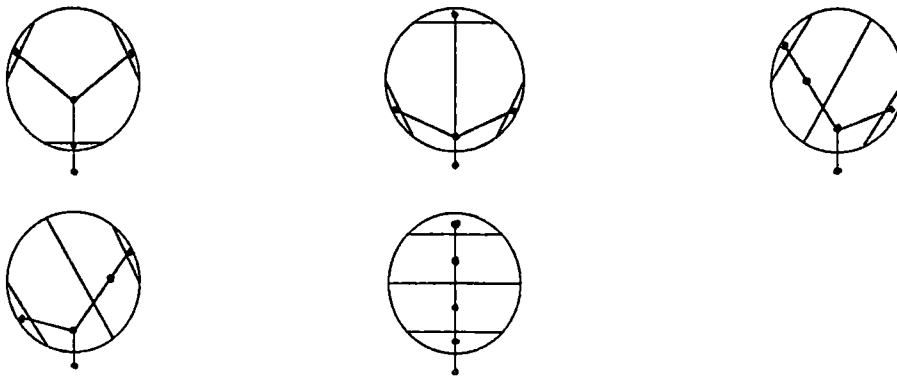


Figure 7.3

8. Sequences of +1's and -1's

Consider the set of sequences $(a_1, a_2, \dots, a_{2n})$ of +1's and -1's such that

$$a_1 + a_2 + \dots + a_{2n} = 0.$$

We wish to count the number of such sequences which have all nonnegative partial sums. Denote the number of such sequences by $f(n)$. The values of $f(n)$ for $n = 1, 2, 3$ and the acceptable sequences are given in figure 8.1. Clearly any such sequence must consist of exactly n +1's and n -1's. Moreover, any such sequence gives a sub-diagonal lattice path from $(0, 0)$ to (n, n) if we interpret a +1 as a horizontal move in the lattice and a -1 as a vertical move. Thus, $f(n) = c(n)$.

Following the method of Graham et al [8], let us now consider the problem of counting the number of sequences $(a_0, a_1, a_2, \dots, a_{2n})$ of +1's and -1's which have

n	$f(n)$	acceptable sequences
1	1	$(+1, -1)$
2	2	$(+1, +1, -1, -1), (+1, -1, +1, -1)$
3	5	$(+1, +1, +1, -1, -1, -1), (+1, +1, -1, +1, -1, -1),$ $(+1, +1, -1, -1, +1, -1), (+1, -1, +1, +1, -1, -1),$ $(+1, -1, +1, -1, +1, -1)$

Figure 8. 1

the property that

$$a_0 + a_1 + a_2 + \cdots + a_{2n} = 1$$

and which have all positive partial sums. Obviously, this number is what we have called $f(n)$, for these sequences are merely our original sequences with the additional element $a_0 = +1$. We wish to make use of a fact discovered by George Raney [9] in 1959.

Raney's lemma: If $(x_1, x_2, \dots, x_m)$ is any sequence of integers whose sum is $+1$, exactly one of the cyclic shifts

$$(x_1, x_2, \dots, x_m), (x_2, \dots, x_m, x_1), \dots, (x_m, x_1, \dots, x_{m-1})$$

has all its partial sums positive.

Let $\sigma = (x_1, x_2, \dots, x_{r+s})$ be a sequence composed of r a's and s b's, where r and s are relatively prime. Let $\psi(\sigma) = (x_2, \dots, x_{r+s}, x_1)$. We claim that $\psi^0(\sigma) = \sigma, \psi^1(\sigma), \psi^2(\sigma), \dots, \psi^{r+s-1}(\sigma)$ are all distinct. If not, there exist i and j , $0 \leq i < j \leq r+s-1$, such that $\psi^i(\sigma) = \psi^j(\sigma)$. Since ψ is clearly invertible, $\psi^k(\sigma) = \sigma$, where $k = j - i$, so that $0 < k < r+s$. This implies that σ consists of a sequence of $p = (r+s)/k$ identical blocks of length k . If each block has u a's and v b's, then $r = up$ and $s = vp$. Since $p > 1$, this contradicts the relative primality of r and s .

Consider the map ϕ from the set of all sequences of $n+1$ $+1$'s and $n-1$'s to the set of all sequences of $n+1$ $+1$'s and $n-1$'s having all positive partial sums, where ϕ

acts by mapping a sequence to its unique cyclic shift with all partial sums positive.

By Raney's lemma, ϕ is well defined. Since the pre-image of each sequence in the range of σ consists of all of the cyclic shifts of that sequence, and $(n+1, n) = 1$, ϕ is a

$2n+1$ -to-one surjection. Since there are $\binom{2n+1}{n}$ sequences with $n+1$ $+1$'s and n -1 's, we have that

$$f(n) = \binom{2n+1}{n} \frac{1}{2n+1} = \binom{2n}{n} \frac{1}{n+1} = c(n).$$

By an *m-Raney sequence* we shall mean a sequence $(a_0, a_1, \dots, a_{mn})$ of $+1$'s and $(1-m)$'s whose partial sums are all positive and whose total sum is $+1$. Let $C^{(m)}(n)$ denote the number of *m-Raney sequences* of length $mn+1$. The number of sequences with n $(1-m)$'s and $mn+1-n$ $+1$'s is given by

$$\binom{n+(mn+1-n)}{n} = \binom{mn+1}{n}.$$

Since $(n, mn+1-n) = 1$, by Raney's lemma exactly $\frac{1}{mn+1}$ of these sequences have all of their partial sums positive. Thus,

$$C^{(m)}(n) = \binom{mn+1}{n} \frac{1}{mn+1} = \binom{mn}{n} \frac{1}{(m-1)n+1}.$$

The numbers $C^{(m)}(n)$ are the so-called Fuss-Catalan numbers, treated first by N. I. Fuss in 1791. Obviously, $C^{(2)}(n) = c(n)$, the ordinary n^{th} Catalan number.

9. Sequences of Positive Integers

Let

$$P_n := \{(a_1, a_2, \dots, a_n) : a_i \in \mathcal{P}, a_i \leq i, \text{ and } 1 = a_1 \leq a_2 \leq \dots \leq a_n\}$$

$$Q_n := \{(a_1, a_2, \dots, a_n) : a_i \in \mathcal{P}, a_i \geq a_{i+1}, \text{ and } a_i \leq n+1-i\}$$

$$R_n := \{(a_1, a_2, \dots, a_{n-1}) : a_i \in \mathcal{N}, a_i \geq a_{i+1}, \text{ and } 0 \leq a_i \leq n-i\} \quad (n \geq 2)$$

$$S_n := \{(a_1, a_2, \dots, a_{n+1}) : a_i \in \mathcal{P}, a_i \leq i, \text{ and } 1 = a_1 \leq a_2 \leq \dots \leq a_{n+1} = n+1\}$$

Figure 9.1 lists the elements of these sets for $n = 1, 2, 3$. We noted in chapter 4 that the cardinality of S_n is exactly $c(n)$. As we shall see, each of the sets P_n , Q_n , and R_n also have cardinality $c(n)$.

i.) $|P_n| = c(n)$.

There is a one-to-one correspondence between the set of sub-diagonal lattice paths from $(0,0)$ to (n,n) and P_n . Given such a lattice path, we associate with each horizontal segment the integer $k+1$ where k is the y -coordinate of its endpoints. When read from left to right, we get a unique element of P_n .

ii.) $|Q_n| = c(n)$.

Consider the function $\phi: Q_n \rightarrow P_n$ given by

$$\phi((a_1, a_2, \dots, a_n)) = (a_n, a_{n-1}, \dots, a_2, a_1).$$

ϕ is obviously a well defined bijection of the sets.

n	P_n	Q_n	R_n	S_n
1	(1)	(1)		(1,2)
2	(1,1), (1,2)	(1,1), (2,1)	(1), (0)	(1,2,3), (1,1,3)
	(1,1,1), (1,1,2),	(1,1,1), (2,1,1),	(2,1), (2,0),	(1,2,3,4), (1,1,3,4),
3	(1,1,3), (1,2,2),	(3,1,1), (2,2,1),	(1,1), (1,0),	(1,1,2,4), (1,1,1,4),
	(1,2,3)	(3,2,1)	(0,0)	(1,2,2,4)

Figure 9. 1

iii.) $|R_n| = c(n)$.

There is also a one-to-one correspondence between the set of sub-diagonal lattice paths from $(0,0)$ to (n,n) and R_n for all $n \geq 2$. Given such a lattice path, each horizontal segment is assigned the integer k where k is the y -coordinate of its endpoints. When read from right to left, omitting the last 0, the resulting sequence is a uniquely determined element of R_n .

The correspondences outlined in (i) and (iii) are illustrated for $n = 3$ in figures 9.2 and 9.3, respectively.

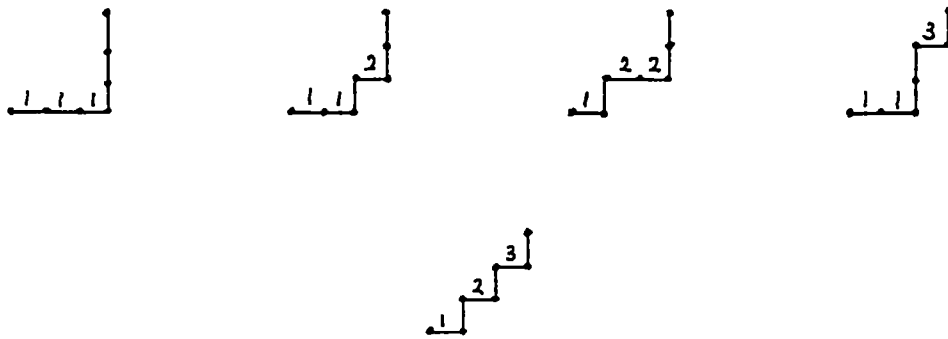


Figure 9. 2

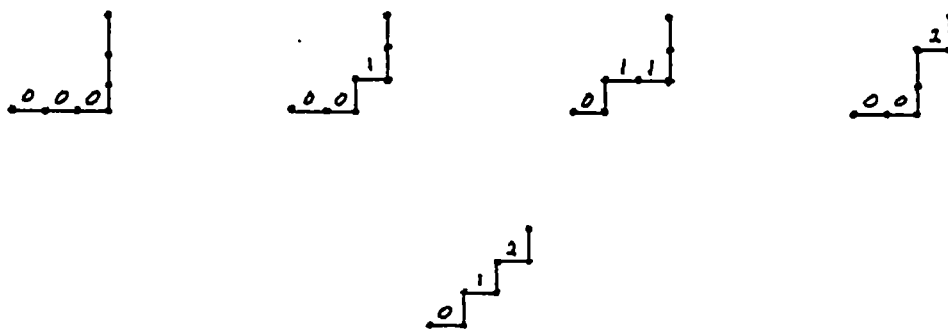


Figure 9. 3

10. q-Analogues of the Catalan Numbers

First, let us establish some preliminary results associated with the q-binomial (or Gaussian binomial) coefficient [10, pp. 25-28]. We begin by defining the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n][n-1] \cdots [n-k+1]}{[k][k-1] \cdots [1]} \quad (10.1)$$

where

$$[j] := 1 + q + \cdots + q^{j-1} \quad (10.2)$$

for all positive integers j and we take $[0] := 0$. Notice that $\begin{bmatrix} n \\ k \end{bmatrix} = \binom{n}{k}$ when $q = 1$.

It can be shown that q-analogues of many of the familiar results for the ordinary binomial coefficients are true. For example:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix} \quad (0 \leq k \leq n)$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{q^n - 1}{q^k - 1} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} = \frac{[n]}{[k]} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

Using the identity [11]

$$(1+x)(1+qx) \cdots (1+q^{n-1}x) = \sum_{k=0}^n q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} x^k \quad (10.4)$$

we may prove the q -Vandermonde theorem [11] which states that for all nonnegative integers m , n , and r

$$\begin{bmatrix} m+n \\ r \end{bmatrix} = \sum_{k \geq 0} q^{\binom{m-r}{2} + \binom{r-k}{2}} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ r-k \end{bmatrix} = \sum_{k \geq 0} q^{\binom{n-k}{2} + \binom{r-k}{2}} \begin{bmatrix} m \\ r-k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix}. \quad (10.5)$$

By (10.4)

$$(1+x)(1+qx)\cdots(1+q^{m-1}x) = \sum_{k \geq 0} q^{\binom{k}{2}} \begin{bmatrix} m \\ k \end{bmatrix} x^k \quad (10.6)$$

and

$$(1+x)(1+qx)\cdots(1+q^{n-1}x) = \sum_{j \geq 0} q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix} x^j. \quad (10.7)$$

Substitute $q^m x$ for x in (10.7), getting

$$(1+q^m x)(1+q^{m+1}x)\cdots(1+q^{m+n-1}x) = \sum_{j \geq 0} q^{\binom{j}{2} + mj} \begin{bmatrix} n \\ j \end{bmatrix} x^j. \quad (10.8)$$

Multiply (10.6) and (10.8) to obtain

$$\begin{aligned} (1+x)(1+qx)\cdots(1+q^{m+n-1}x) &= \sum_{k, j \geq 0} q^{\binom{k}{2} + \binom{j}{2} + mj} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} x^{j+k} \\ &= \sum_{r \geq 0} x^r \sum_{k \geq 0} q^{\binom{k}{2} + \binom{r-k}{2} + m(r-k)} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ r-k \end{bmatrix}. \end{aligned} \quad (10.9)$$

But (10.4) also yields

$$(1+x)(1+qx)\cdots(1+q^{m+n-1}x) = \sum_{r \geq 0} q^{\binom{r}{2}} \begin{bmatrix} m+n \\ r \end{bmatrix} x^r. \quad (10.10)$$

Comparing coefficients of x^r in (10.9) and (10.10) we then have

$$q^{\binom{r}{2}} \begin{bmatrix} m+n \\ r \end{bmatrix} = \sum_{k \geq 0} q^{\binom{k}{2} + \binom{r-k}{2} + m(r-k)} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ r-k \end{bmatrix}. \quad (10.11)$$

Divide by $q^{\binom{r}{2}}$ and the result follows.

We may further define the q -analog of the multinomial coefficient by

$$\begin{bmatrix} n \\ n_1, n_2, \dots, n_k \end{bmatrix} := \frac{n_q!}{n_{1_q}! n_{2_q}! \dots n_{k_q}!}, \text{ where } n_q! := [n]_q! = [n]_q [n-1]_q \dots [1]_q.$$

We shall say a sequence of integers $a_1 a_2 \dots a_n$ has an inversion whenever $a_i > a_j$

with $i < j$. Let $\mathfrak{S}_n$ be the symmetric group on n letters and for $\pi \in \mathfrak{S}_n$, let $i(\pi)$ denote the number of inversions in the sequence $\pi(1)\pi(2)\dots\pi(n)$. Then it can be shown [10, p. 21] that

$$\sum_{\pi \in \mathfrak{S}_n} q^{i(\pi)} = (1)(1+q)(1+q+q^2)\dots(1+q+\dots+q^{n-1}) = n_q! \quad (10.12)$$

which may be generalized as follows [10, p.27]: Let M be the multiset

$\{1^{n_1}, 2^{n_2}, \dots, r^{n_r}\}$ with $n_1 + n_2 + \dots + n_r = n$ (where n_i denotes the number of

occurrences of an element in the set) and let $\mathfrak{S}(M)$ be the set of all sequential

arrangements of the elements of M . For $\sigma \in \mathfrak{S}(M)$, let $i(\sigma)$ be the number of

inversions in the sequence σ . Consider the map $\varphi: \mathfrak{S}(M) \times \mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2} \times \dots \times \mathfrak{S}_{n_r} \rightarrow \mathfrak{S}_n$

given by $(\sigma, \pi_1, \pi_2, \dots, \pi_r) \mapsto \pi$ produced by taking the sequence $\sigma(1)\dots\sigma(n) \in \mathfrak{S}(M)$

and first replacing the n_1 1's in this sequence by $1, 2, \dots, n_1$, ordered as specified by

π_1 , then replacing the n_2 2's in the sequence by $n_1 + 1, n_1 + 2, \dots, n_1 + n_2$, ordered as

specified by π_2 , and so forth. Then clearly φ is a bijection, and since

$i(\pi) = i(\sigma) + i(\pi_1) + \dots + i(\pi_r)$, we have

$$\begin{aligned} n_q! &= \sum_{\pi \in \mathfrak{S}_n} q^{i(\pi)} = \sum_{(\sigma, \pi_1, \dots, \pi_r) \in \mathfrak{S}(M) \times \mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_r}} q^{i(\sigma) + i(\pi_1) + \dots + i(\pi_r)} \\ &= \sum_{\sigma \in \mathfrak{S}(M)} q^{i(\sigma)} \sum_{\pi_1 \in \mathfrak{S}_{n_1}} q^{i(\pi_1)} \dots \sum_{\pi_r \in \mathfrak{S}_{n_r}} q^{i(\pi_r)} \\ &= \left(\sum_{\sigma \in \mathfrak{S}(M)} q^{i(\sigma)} \right) n_{1_q}! n_{2_q}! \dots n_{r_q}! \end{aligned}$$

and so,

$$\sum_{\sigma \in \mathfrak{S}(M)} q^{i(\sigma)} = \frac{n_q!}{n_{1_q}! \dots n_{r_q}!} = \left[\begin{matrix} n \\ n_1, n_2, \dots, n_r \end{matrix} \right]. \quad (10.13)$$

Let $\mathcal{L}_{m,n}$ be the set of all lattice paths from $(0,0)$ to (m,n) and let $\alpha(\lambda)$ be the area under the path $\lambda \in \mathcal{L}_{m,n}$. Represent λ by a sequence σ with m 0's and n 1's, where a 0 represents a horizontal move in the lattice and a 1 represents a vertical move. Notice that $\alpha(\lambda) = i(\sigma)$ so by (10.13) we have the result of Pólya,

$$\sum_{\lambda \in \mathcal{L}_{m,n}} q^{\alpha(\lambda)} = \frac{(m+n)_q!}{m_q! n_q!} = \left[\begin{matrix} m+n \\ n \end{matrix} \right].$$

We wish to first consider a q -analogue of the Catalan numbers which appears in the paper of Förlinger and Hofbauer [6]. By a *Catalan word of length $2n$* we shall mean a word comprised of n 0's and n 1's, no initial segment of which contains more 1's than 0's. Such words naturally correspond to subdiagonal lattice paths from $(0,0)$ to (n,n) with (as noted above) the number of inversions in a Catalan word being equal to the area under the lattice path corresponding to that word. Let $\mathcal{C}_n$ be the set of

Catalan words of length $2n$ and let $\mathcal{L}_{n,n}^*$ be the set of all subdiagonal lattice paths from $(0,0)$ to (n,n) . Define

$$c_q(n) := \sum_{w \in \mathcal{C}_n} q^{i(w)} = \sum_{\lambda \in \mathcal{L}_{n,n}^*} q^{\alpha(\lambda)}. \quad (10.14)$$

Each word w in $\mathcal{C}_{n+1}$ may be written uniquely in the form $w = 0w_11w_2$ with $w_1 \in \mathcal{C}_k$ and $w_2 \in \mathcal{C}_{n-k}$ for some $0 \leq k \leq n$, the 1 in the representation being the first 1 in the word ending an initial segment that contains an equal number of 1's and 0's. Clearly,

$$i(w) = i(w_1) + i(w_2) + (k+1)(n-k) \quad (10.15)$$

so

$$\begin{aligned} c_q(n+1) &= \sum_{w \in \mathcal{C}_{n+1}} q^{i(w)} = \sum_{k=0}^n \sum_{\substack{w \in \mathcal{C}_{n+1}: \\ w_1 \in \mathcal{C}_k \\ w_2 \in \mathcal{C}_{n-k}}} q^{i(w)} = \sum_{k=0}^n \sum_{\substack{w_1 \in \mathcal{C}_k \\ w_2 \in \mathcal{C}_{n-k}}} q^{i(w_1) + i(w_2) + (k+1)(n-k)} \\ &= \sum_{k=0}^n q^{(k+1)(n-k)} \sum_{w_1 \in \mathcal{C}_k} q^{i(w_1)} \sum_{w_2 \in \mathcal{C}_{n-k}} q^{i(w_2)} \\ &= \sum_{k=0}^n c_q(k) c_q(n-k) q^{(k+1)(n-k)}, \quad n \geq 0 \end{aligned} \quad (10.16)$$

with $c_q(0) = 1$.

In particular, for $q = 1$ we obtain the Catalan recurrence

$$c(n+1) = \sum_{k=0}^n c(k)c(n-k).$$

The polynomials for $n=1,2,3,4$ are

n	$c_q(n)$
1	1
2	$q+1$

$$3 \quad q^3 + 2q^2 + q + 1$$

$$4 \quad q^6 + 3q^5 + 3q^4 + 3q^3 + 2q^2 + q + 1$$

Although the polynomials $c_q(n)$ satisfy the Catalan-like recurrence (10.16), they do not appear to have a simple closed form representation analogous to

$$c(n) = \frac{1}{n+1} \binom{2n}{n}. \quad (10.17)$$

Another q -analogue of $c(n)$, which we shall denote by $C_q(n)$, does have such a representation, although it does not appear to satisfy any recurrence of the type (10.16). The polynomials $C_q(n)$ are due to Carlitz, Roselle, and Scoville [3].

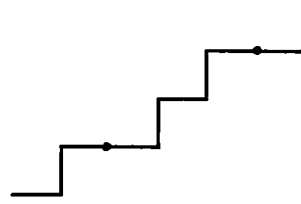
For $m, n \in \mathcal{N}$ with $m \geq n \geq 1$, let $\mathcal{L}_{m,n}^*$ be the set of all subdiagonal lattice paths from $(0,0)$ to (m,n) , and, as in chapter 4, let $\mathcal{S}_{m,n}^*$ be the set of all sequences $(a_0, \dots, a_m)$ in $\mathcal{N}$ such that $0 = a_0 \leq a_1 \leq \dots \leq a_m = n$ and $a_i \leq i$ for all $i = 0, \dots, m$. As we noted, $|\mathcal{L}_{m,n}^*| = |\mathcal{S}_{m,n}^*|$. If $\sigma = (a_0, a_1, \dots, a_m) \in \mathcal{S}_{m,n}^*$, σ is said to have a *rise* at $i \geq 1$ if $a_{i-1} < a_i$. If σ has rises at $i_1, i_2, \dots, i_r$, we define statistics ρ and δ by

$$\rho(\sigma) = r \quad (10.18)$$

and

$$\delta(\sigma) = (r+1)n + rm - \sum_{s=1}^r (a_{i_s} + i_s). \quad (10.19)$$

Notice that for the lattice path $\lambda \in \mathcal{L}_{m,n}$ corresponding to σ , $\delta(\sigma)$ is just the sum of the distances (along λ) from (m,n) to those points on λ at which the path makes a right angle turn from horizontal to vertical. For example, with $m = 6, n = 4$, and $\sigma = (0,1,1,2,3,3,4)$, $\rho(\sigma) = 4$ and $\delta(\sigma) = 20 + 24 - 2 - 5 - 7 - 10 = 20$ while in the lattice path corresponding to σ



the sum of the distances along the path from $(6,4)$ to the turning points is $9 + 6 + 4 + 1 = 20$.

Carlitz, Roselle, and Scoville now define

$$G(m, n; p, q) = \sum_{\sigma \in S_{m,n}^*} p^{\rho(\sigma)} q^{\delta(\sigma)}, \quad m \geq n \geq 1, \quad (10.20)$$

and prove that

$$G(m, m; 1, q) = \sum_{\sigma \in S_{m,m}^*} q^{\delta(\sigma)} = q^n \frac{[m-n+1]}{[m+1]} \begin{bmatrix} m+n \\ n \end{bmatrix}, \quad (10.21)$$

and hence

$$C_q(n) := G(n, n; 1, q) = \sum_{\sigma \in S_{n,n}^*} q^{\delta(\sigma)} = q^n \frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix}, \quad (10.22)$$

which reduces to (10.17) for $q = 1$. The details of their analysis follow.

If $\sigma = (a_0, a_1, \dots, a_m) \in S_{m,n}^*$, let $\sigma' := (a_0, a_1, \dots, a_{m-1})$. It is easy to check that if

$a_{m-1} = a_m$, then

$$\begin{aligned}\rho(\sigma) &= \rho(\sigma') \\ \delta(\sigma) &= \delta(\sigma') + \rho(\sigma')\end{aligned}\tag{10.23}$$

and that if $a_{m-1} = j < a_m$, then

$$\begin{aligned}\rho(\sigma) &= \rho(\sigma') + 1 \\ \delta(\sigma) &= \delta(\sigma') + (n - j + 1)\rho(\sigma') + (n - j).\end{aligned}\tag{10.24}$$

So

$$\begin{aligned}G(m, n; p, q) &= \sum_{\substack{\sigma \in S_{m,n}^* \\ a_{m-1} = a_m}} p^{\rho(\sigma)} q^{\delta(\sigma)} + \sum_{\substack{\sigma \in S_{m,n}^* \\ a_{m-1} < a_m}} p^{\rho(\sigma)} q^{\delta(\sigma)} \\ &= \sum_{\sigma' \in S_{m-1,n}^*} p^{\rho(\sigma')} q^{\delta(\sigma') + \rho(\sigma')} + \sum_{j=0}^{n-1} \sum_{\sigma' \in S_{m-1,j}^*} p^{\rho(\sigma') + 1} q^{\delta(\sigma') + (n-j+1)\rho(\sigma') + (n-j)} \\ &= G(m-1, n; pq, q) + p \sum_{j=0}^{n-1} q^{n-j} G(m-1, j; pq^{n-j+1}, q) \\ &= (1-p)G(m-1, n; pq, q) + p \sum_{j=0}^n q^{n-j} G(m-1, j; pq^{n-j+1}, q)\end{aligned}\tag{10.25}$$

In interpreting the above relations, $G(m, n; p, q)$ is assigned the value 0 whenever

$m < n$.

It follows from (10.25) that if $m \geq n \geq 2$,

$$\begin{aligned}G(m, n; p, q) &= G(m, n-1; pq, q) + G(m-1, n; pq, q) \\ &\quad - (1-pq)G(m-1, n-1; pq^2, q).\end{aligned}\tag{10.26}$$

Write

$$G(m, n; p, q) = \sum_j c(m, n, j) p^j,\tag{10.27}$$

where $c(m, n, j)$ is a polynomial in q . It is clear from (10.20) that $c(m, n, j) = 0$ if

$j < 1$ or $j > n$. By (10.26) and (10.27), if $m \geq n \geq 2$,

$$c(m, n, j) = q^j c(m, n-1, j) + q^j c(m-1, n, j) - q^{2j} c(m-1, n-1, j) + q^{2j-1} c(m-1, n-1, j-1), \quad (10.28)$$

where $c(m, n, j) = 0$ if $m < n$.

We wish to prove that for $m \geq n \geq 1$, and for every j ,

$$c(m, n, j) = q^n \frac{[m-n+1]}{[n]} q^{j(j-1)} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} m \\ j-1 \end{bmatrix} \quad (10.29)$$

Note first that the right hand side of (10.29) takes the value 0 if $j < 1$ or $j > n$, and so agrees with the value of $c(m, n, j)$ in those cases. Furthermore, it is easy to see from (10.20) that for all $m \geq 1$

$$G(m, 1; p, q) = [m]_q p, \quad (10.30)$$

and so

$$c(m, 1, 1) = [m]_q, \quad (10.31)$$

in accord with the right hand side of (10.29) for $n = j = 1$.

We may now establish (10.29) by induction on $m+n$. We have already verified that the result holds for $m+n = 2, 3$ so suppose that $m+n \geq 4$. If $n = 1$, the result follows from (10.30), so assume that $m \geq n \geq 2$. Then (10.28) and the induction hypothesis gives us

$$\begin{aligned} c(m, n, j) &= q^j q^{n-1} \frac{[m-n+2]}{[n-1]} q^{j(j-1)} \begin{bmatrix} n-1 \\ j \end{bmatrix} \begin{bmatrix} m \\ j-1 \end{bmatrix} \\ &\quad + q^j q^n \frac{[m-n]}{[n]} q^{j(j-1)} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} m-1 \\ j-1 \end{bmatrix} \\ &\quad - q^{2j} q^{n-1} \frac{[m-n+1]}{[n-1]} q^{j(j-1)} \begin{bmatrix} n-1 \\ j \end{bmatrix} \begin{bmatrix} m-1 \\ j-1 \end{bmatrix} \\ &\quad + q^{2j-1} q^{n-1} \frac{[m-n+1]}{[n-1]} q^{(j-1)(j-2)} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} \begin{bmatrix} m-1 \\ j-2 \end{bmatrix}. \end{aligned} \quad (10.32)$$

It is only a matter of straightforward but tedious algebra to verify that the right hand side of (10.32) is equal to

$$q^n \frac{[m-n+1]}{[n]} q^{j(j-1)} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} m \\ j-1 \end{bmatrix}. \quad (10.33)$$

From (10.27) and (10.29), we then have that if $m \geq n \geq 1$

$$G(m, n; p, q) = q^n \frac{[m-n+1]}{[n]} \sum_{j=1}^n p^j q^{j(j-1)} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} m \\ j-1 \end{bmatrix}. \quad (10.34)$$

Hence,

$$G(m, n; 1, q) = \sum_{\sigma \in S_{m,n}} q^{\delta(\sigma)} = q^n \frac{[m-n+1]}{[n]} \sum_{j=1}^n q^{j(j-1)} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} m \\ j-1 \end{bmatrix}. \quad (10.35)$$

But using the q -Vandermonde Theorem we obtain after two changes of variables

$$\sum_{j=1}^n q^{j(j-1)} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} m \\ j-1 \end{bmatrix} = \begin{bmatrix} m+n \\ n-1 \end{bmatrix} = \frac{[n]}{[m+1]} \begin{bmatrix} m+n \\ n \end{bmatrix} \quad (10.36)$$

so

$$G(m, n; 1, q) = q^n \frac{[m-n+1]}{[m+1]} \begin{bmatrix} m+n \\ n \end{bmatrix}$$

and

$$C_q(n) := G(n, n; 1, q) = q^n \frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix}.$$

For $n=1, 2, 3, 4$ we have

n	$C_q(n)$
1	q
2	$q^4 + q^2$

$$3 \qquad q^9 + q^7 + q^6 + q^5 + q^3$$

$$4 \qquad q^{16} + q^{14} + q^{13} + 2q^{12} + q^{11} + 2q^{10} + q^9 + 2q^8 + q^7 + q^6 + q^4$$

Interestingly, it appears that as late as 1987 there was no widely known q -analogue of the Catalan numbers which both satisfied an exact q -analogue of the Catalan recurrence and had a simple closed form expression like (10.17). The first such q -analogue was introduced by Andrews [1] using hypergeometric series.

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Vita

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