

Numerical Methods for Stochastic Stokes and Navier-Stokes Equations

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Degree

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I dedicate this dissertation to my family and many friends. A special feeling of gratitude to my loving parents, Lam Vo and Thu Nguyen, whose words of encouragement and push for tenacity ring in my ears. My younger brother, Sang Vo, has always stood by me and is very special.

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Abstract

This dissertation consists of three main parts with each part focusing on numerical approximations of the stochastic Stokes and Navier-Stokes equations.

Part One concerns the mixed finite element methods and Chorin projection methods for solving the stochastic Stokes equations with general multiplicative noise. We propose a modified mixed finite element method for solving the Stokes equations and show that the numerical solutions converge optimally to the PDE solutions. The convergence is under energy norms (strong convergence) for the velocity and in a time-averaged norm (weak convergence) for the pressure. In addition, after establishing the error estimates in second moment, high moment estimates are also established which serve as a bridge for deriving pathwise error estimates. The second portion of Part One focuses on Chorin projection methods for solving the stochastic Stokes equation with general multiplicative noise. Two types of the Chorin projection methods, the standard Chorin and modified Chorin projection methods, are considered. We establish sub-optimal order error estimates for the velocity and pressure approximations of the standard Chorin projection method, and optimal order estimates for the modified Chorin method. Numerical experiments are also provided to verify the proved theoretical results.

Part Two studies the mixed finite element methods for solving the stochastic Navier-Stokes equations with general additive noise. We establish the strong convergence for the velocity and pressure approximations. In addition, high moment error estimates are also established which serve as a bridge for deriving pathwise error estimates for both the velocity and pressure approximations.

Part Three develops an iterative framework for parameter-dependent and random convection-diffusion problems which arise from many engineering and scientific applications.

The main idea of the framework is to reformulate the underlying PDE problem into another problem with a parameter-independent convection-diffusion coefficient and a parameter-dependent (and solution-dependent) right-hand side, a fixed-point iteration is then employed to compute the solution of the reformulated problem. The computational saving is achieved at the solver level by using the LU direct method or block Krylov subspace methods as linear solvers so many computations can be reused. Numerical experiments are presented to validate the efficiency of the iterative method.

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Chapter 1

Introduction

1.1 Background

In this chapter, we introduce some background materials related to mathematical models (i.e. partial differential equations) that are studied in this dissertation.

1.1.1 Stochastic Navier-Stokes equations

This dissertation focuses on developing and analyzing numerical methods for stochastic Stokes and Navier-Stokes equations. These equations are well-known in fluid mechanics because they describe many phenomena of scientific and engineering interests. The deterministic Navier-Stokes equations have been used to model the weather, ocean currents, water flow in a pipe and air flow around an airfoil while the stochastic counterparts are used to model turbulence and large-scale fluid flows with some uncertainty. First, we recall that the deterministic Navier-Stokes equations for incompressible and viscous fluids are given by [\[65\]](#)

$$\frac{d\mathbf{u}}{dt} = \nu \Delta \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u} - \nabla p + \mathbf{f} \quad \text{in } D_T := (0, T) \times D, \quad (1.1a)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } D_T, \quad (1.1b)$$

where D is a bounded domain in $\mathbb{R}^d$ for $d = 2, 3$. $\mathbf{u}$ and p , respectively, denote the velocity field and the pressure of fluid. $\nu > 0$ is the viscosity constant.

The system (1.1) with $\nu = 0$ was proposed by Leonhard Euler in 1757 that was used to describe the flow of incompressible and inviscid fluids. Later on, French engineer Claude-Louis Navier introduced the viscosity term for more practical and more challenging problems of viscous fluids in 1821. During the 50's of the 19th century, British physicist and mathematician Sir George Gabriel Stokes further developed the system and found exact solutions of the system in the case of two-dimensional fluids. Physically, equations (1.1) are the mathematical descriptions of the conservation of momentum and conservation of mass for fluids. The momentum equation is an application of Newton's second law of motion to the fluid, together with the constitutive relation that says the stress in the fluid is the sum of a diffusing viscous term (proportional to the gradient of velocity) and a pressure term describing incompressibility of the fluid [56].

The Navier–Stokes equations are important since they describe many physical phenomena of scientific and engineering interest. The Navier–Stokes equations also have many applications in the automobile and airspace industry as well as in medical science for modeling blood flow in human body. Moreover, (1.1) are also of great interest in pure mathematics. Even though the equations have vast applications in real world, the existence of smooth solutions is still an open problem in the case of three-dimension, which is one of the seven millennium problems in mathematics.

To obtain the stochastic counterpart of (1.1), we replace the body force $\mathbf{f}$ in (1.1a) by a stochastic perturbation noise term $\mathbf{f} + \mathbf{B}(\mathbf{u})\frac{dW}{dt}$ and then obtain the following stochastic Navier-Stokes equations

$$d\mathbf{u} = [\Delta\mathbf{u} - \mathbf{u} \cdot \nabla\mathbf{u} - \nabla p + \mathbf{f}]dt + \mathbf{B}(\mathbf{u})dW(t) \quad \text{in } D_T := (0, T) \times D, \quad (1.2a)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } D_T, \quad (1.2b)$$

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } D, \quad (1.2c)$$

In this work we mainly seek periodic-in-space solutions $(\mathbf{u}, p)$ with period L , for both equations (1.2a) and (1.2b) that is, $\mathbf{u}(t, \mathbf{x} + L\mathbf{e}_i) = \mathbf{u}(t, \mathbf{x})$ and $p(t, \mathbf{x} + L\mathbf{e}_i) = p(t, \mathbf{x})$ almost

surely and for any $(t, \mathbf{x}) \in (0, T) \times \mathbb{R}^d$ and $1 \leq i \leq d$, where $\{\mathbf{e}_i\}_{i=1}^d$ denotes the canonical basis of $\mathbb{R}^d$. The functional $\mathbf{B}$ is called as diffusion coefficient and it determines noise types of (1.2). The noise is said to be additive if $\mathbf{B}(\mathbf{u})$ is independent of $\mathbf{u}$ and multiplicative otherwise. In addition, the dependence of $\mathbf{B}$ on $\mathbf{u}$ may be linear or non-linear. The later case makes (1.2) intrinsically non-linear even after we drop the nonlinear term $\mathbf{u} \cdot \nabla \mathbf{u}$ in (1.2a) to obtain the following stochastic Stokes equations:

$$d\mathbf{u} = [\Delta \mathbf{u} - \nabla p + \mathbf{f}] dt + \mathbf{B}(\mathbf{u}) dW(t) \quad \text{in } D_T := (0, T) \times D, \quad (1.3a)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } D_T, \quad (1.3b)$$

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } D, \quad (1.3c)$$

$\{W(t); t > 0\}$ denotes an $L^2(D)$ -valued or $\mathbf{L}^2(D)$ -valued Q-Wiener process (Brownian motion) which is not differentiable for all $t > 0$ (see Section 1.2 for precise definition). Due to the lack of differentiability of $W(t)$, all the solutions $\mathbf{u}$ and p are nowhere differentiable in time.

1.1.2 Notations

Standard function and space notation will be adopted in this dissertation. For example, $\mathbf{H}^\ell(D)$ ($\ell \geq 0$) denotes the Sobolev spaces. In addition, $\mathbf{H}_{per}^\ell(D)$ is the subspace of $\mathbf{H}^\ell(D)$ consisting of $\mathbb{R}^d$ -valued periodic functions with period L in each spatial coordinate direction, and $(\cdot, \cdot) := (\cdot, \cdot)_D$ denote the standard L^2 -inner product, with induced norm $\|\cdot\|_{L^2}$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a filtered probability space with the probability measure $\mathbb{P}$, the σ -algebra $\mathcal{F}$ and the continuous filtration $\{\mathcal{F}_t\} \subset \mathcal{F}$. For a random variable v defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, let $\mathbb{E}[v]$ denote the expected value of v . For a vector space X with norm $\|\cdot\|_X$, and $1 \leq p < \infty$, we define the Bochner space $(L^p(\Omega, X); \|v\|_{L^p(\Omega, X)})$, where $\|v\|_{L^p(\Omega, X)} := (\mathbb{E}[\|v\|_X^p])^{\frac{1}{p}}$. Throughout this dissertation, unless it is stated otherwise, we shall use C to denote a generic positive constant which may depend on T , the datum functions $\mathbf{u}_0$ and $\mathbf{f}$, and the domain D but is independent of the mesh parameters.

We also define

$$\begin{aligned}\mathbb{H} &:= \{ \mathbf{v} \in \mathbf{L}_{per}^2(D); \operatorname{div} \mathbf{v} = 0 \text{ in } D \}, \\ \mathbb{V} &:= \{ \mathbf{v} \in \mathbf{H}_{per}^1(D); \operatorname{div} \mathbf{v} = 0 \text{ in } D \}.\end{aligned}$$

We recall from [37] that the (orthogonal) Helmholtz projection $\mathbf{P}_{\mathbb{H}} : \mathbf{L}_{per}^2(D) \rightarrow \mathbb{H}$ is defined by $\mathbf{P}_{\mathbb{H}} \mathbf{v} = \boldsymbol{\eta}$ for every $\mathbf{v} \in \mathbf{L}_{per}^2(D)$, where $(\boldsymbol{\eta}, \xi) \in \mathbb{H} \times H_{per}^1(D)/\mathbb{R}$ is a unique tuple such that

$$\mathbf{v} = \boldsymbol{\eta} + \nabla \xi,$$

and $\xi \in H_{per}^1(D)/\mathbb{R}$ solves the following Poisson problem (cf. [2]):

$$(\nabla \xi, \nabla q) = (\mathbf{v}, \nabla q) \quad \forall q \in H_{per}^1(D). \quad (1.4)$$

In this paper we denote by $\mathbf{A} := \mathbf{P}_{\mathbb{H}} \Delta : \mathbf{H}^2(D) \rightarrow \mathbb{H}$ the Stokes operator.

We assume that $\mathbf{B} : L^2(\Omega; \mathbf{H}_{per}^1(D)) \rightarrow L^2(\Omega; \mathbf{H}_{per}^1(D))$ is Lipschitz continuous and has linear growth, *i.e.*, there exists a constant $C > 0$ such that for all $\mathbf{v}, \mathbf{w} \in \mathbf{L}_{per}^2(D)$,

$$\|\mathbf{B}(\mathbf{v}) - \mathbf{B}(\mathbf{w})\| \leq C \|\mathbf{v} - \mathbf{w}\|, \quad (1.5a)$$

$$\|\mathbf{B}(\mathbf{v})\| \leq C(1 + \|\mathbf{v}\|), \quad (1.5b)$$

$$\|\mathcal{D}\mathbf{B}\|_* \leq C, \quad (1.5c)$$

where $\mathcal{D}\mathbf{B}$ denotes the Gateaux derivative of $\mathbf{B}$, and $\|\cdot\|_*$ is its operator norm.

1.2 Wiener process

In this section, we recall the definition of an $L^2(D)$ -valued Q-Wiener process (Brownian motion) $\{W(t); t > 0\}$.

Definition 1.2.1 ([51]). *An $L^2(D)$ -valued stochastic process $\{W(t) \mid t > 0\}$ is called a Q-Wiener process if*

- $W(0) = 0$ *a.s.*,

- $W(t)$ is a continuous function from $\mathbb{R}^+ \rightarrow L^2(D)$ for each $\omega \in \Omega$,
- $W(t)$ is $\mathcal{F}_t$ -adapted and $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for $s < t$, and
- $W(t) - W(s) \sim N(0, (t - s)Q)$ for all $0 \leq s \leq t$.

Where Q is the covariance operator of $\{W(t); t > 0\}$ which is symmetric and non-negative definite, hence, self-adjoint on $L^2(D)$.

Furthermore, a Q -Wiener process can be presented as a linear combination of the eigenvalues and eigenfunctions $\{(\lambda_j, \phi_j)\}$ of Q , which is the well-known Karhunen-Lo  ve expansion.

Theorem 1.2.2 ([51]). *Let Q be a self-adjoint covariance operator of $W(t)$. Then, $W(t)$ is a Q -Wiener process if and only if*

$$W(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \phi_j \beta_j(t), \quad a.s., \quad (1.6)$$

where $\{\beta_j(t)\}$ are identically independent distributed, $\mathcal{F}_t$ -adapted and real-valued Wiener processes. Moreover, the series converges in $L^2(\Omega, C([0, T], L^2(D)))$ for any $T > 0$.

Remark 1.2.1. *Throughout this dissertation, we focus on the case that $\mathbf{B}(\mathbf{u})$ is a vector-valued and $W(t)$ is a scalar-valued and remark that all analysis results still hold for the opposite case, namely, $B(\mathbf{u})$ is a scalar-valued and $\mathbf{W}(t)$ is a vector-valued.*

1.3 Scope of the dissertation research

This dissertation focuses on design, analysis and implementation of numerical methods for solving the stochastic Stokes and Navier-Stokes equations with multiplicative and additive noise. The Navier-Stokes equations are one of the best known systems of equations from fluid mechanics. However, it is nearly impossible to obtain their analytic solutions (velocity and pressure), seeking approximate numerical solutions is only way to solve them. The situation is even worse for their stochastic counterparts due to the contribution of the noise and the goal to obtain quantities of stochastic interests such as expected value and variance as well

as high moments. Therefore, it is imperative to develop efficient numerical methods to seek approximate numerical solutions. Below we shall outline the main issues to be addressed in this dissertation.

Firstly, this dissertation focuses on numerical analysis of the mixed finite element method (FEM) for solving the stochastic Stokes and Navier-Stokes equations with general multiplicative noise (or solution-dependent random external force) which have applications in simulating turbulence and ocean flows. Due to the noise, the PDE solutions (velocity and pressure) are not differentiable in time (i.e., no time derivative), that causes the major difficulty in analyzing rates of convergence for the numerical methods in terms of quantities of stochastic interests (instead of the numerical solution itself). Moreover, in order to compute those quantities, the stochastic Stokes and Navier-Stokes equations must be solved numerically hundreds even thousands of times, which is computationally daunting and requires efficient solvers to solve the resulting algebraic problems from the discretization. Although the PDE theory for these equations has been well-known in the 2D case, the numerical methods and analysis results were fewer and incomplete, especially in the case of non-divergence-free noise. About ten years ago, Carelli and Prohl [17] established the rates of convergence for the mixed finite element method for the stochastic Navier-Stokes equations, a sub-optimal convergence in probability for the velocity approximation was proved, but no convergence result for the pressure approximation was obtained. Thus, several natural questions have arisen: (1) How good is the mixed finite element method for those stochastic PDEs? (2) Can we achieve optimal rate of convergence with or without modifications? (3) How to measure and analyze the pressure approximation error? (4) Is the convergence in strong norms (including path-wise estimates) possible for the numerical solutions? In [34], Feng and Qiu gave the answers to the first and third questions for the mixed finite element method approximation of the stochastic Stokes equations. They proved, unlike in the deterministic case, the standard mixed FEM is not an optimal method and proposed an idea of using the stochastic inf-sup condition to obtain error estimates for the pressure approximation in a time-averaged norm. Their numerical tests show that their conclusions are sharp. Motivated by those developments, the first goal of this dissertation is to address the second and fourth questions above.

Secondly, this dissertation studies the time splitting methods for solving the stochastic Stokes equations with general multiplicative noise. About nine years ago, Carelli, Hausenblas and Prohl [16] proposed two Chorin time-splitting schemes for stochastic Stokes equations which include the standard Chorin projection method, adapted from its deterministic counterpart [65], for divergence-free multiplicative noise and a modified Chorin method for general noise. They proved the error estimates for the velocity approximations of both schemes. However, no error estimate for the pressure approximations was established in the both cases. In addition, a qualitative understanding of the standard Chorin projection method in the case of general noise has been an open question. Therefore, the next primary goals of this dissertation are to provide definite answers to these two issues and to provide numerical test results to validate the proved theoretical results.

After a stochastic PDE is discretized in time, regardless which time-stepping method is used, the resulting problem at each time step is a random PDE (i.e., its coefficients and source terms are random variables). A standard and brute force approach is to solve the random PDE using a Monte Carlo method. However, such an approach is computationally very expensive because it requires solving the underlying deterministic PDE problem (such as the Navier-Stokes) hundreds even thousands of times. Solving the resulting random PDEs is the bottle-neck of the overall numerical solution method for stochastic PDEs. Motivated by the above challenge, another goal of this dissertation is to develop and analyze a general iterative algorithm/framework for solving parameter-dependent and random convection-diffusion problems which arise from many engineering and scientific applications. It is inspired by the multi-modes method of [27, 28] and the ensemble method of [52] and extends those methods into a more general and unified framework.

1.4 Summary of main contributions

This dissertation is comprised of several research projects in the areas of numerical PDEs and organized into three integral parts.

Part One (including Chapters 2–4) of the dissertation focuses on numerical analysis of the mixed finite element method for solving the stochastic Stokes equations with general

multiplicative noise. In Chapter 2, we consider a modified mixed finite element method for solving the stochastic Stokes equations (1.3) with general multiplicative noise. As mentioned earlier, in [34] sub-optimal convergence orders $O(k^{\frac{1}{2}} + k^{-\frac{1}{2}}h)$ have been established for the standard mixed finite element approximations (velocity and pressure approximations) of the Stokes equations with multiplicative noise, where k denotes the time step size and h is the spatial mesh size. The bad factor $k^{-\frac{1}{2}}$ is caused by the non-divergence-free multiplicative noise and a consequence of the following nonuniform stability estimate for the pressure approximates $\{p^n\}_{n=1}^N$:

$$\mathbb{E} \left[k \sum_{n=1}^N \|\nabla p^n\|_{L^2}^2 \right] \leq \frac{C}{k}. \quad (1.7)$$

This blow-up estimate is the main reason only to get a sub-optimal error estimates of order $O(k^{\frac{1}{2}} + k^{-\frac{1}{2}}h)$ for the standard mixed finite element method. To circumvent this problem, in this dissertation (also see [33]), we propose a modified mixed finite element method and are able to obtain the optimal error estimates of order $O(k^{\frac{1}{2}} + h)$ for the velocity and pressure approximations. Our main idea, which is partly borrowed from [16], is first to perform the Helmholtz decomposition for the noise term at each time step and then to approximate the velocity and a pseudo pressure simultaneously via the mixed finite element method.

Chapter 3 of this dissertation derives high moment and pathwise error estimates for the standard mixed finite element method and the modified mixed finite element method that were introduced in Chapter 2. High moment and pathwise estimates are often needed to measure approximations of numerical methods for stochastic models because they are commonly used quantities of stochastic interests. Moreover, in practice, numerical simulations for the approximate solutions are done for sample paths when the Monte Carlo method is used to compute quantities of stochastic interests, which requires to use a large number of samples. However, to the best of our knowledge, no pathwise error estimate was known in the literature for the stochastic Stokes (and stochastic Navier-Stokes) equations. The pathwise error estimates would provide valuable information about the quality of each computed sample path. Our main idea for deriving the high moment error estimates for the velocity approximations is to use a bootstrap technique, adapted from [13], starting

from the second moment error estimate. The pathwise error estimate is obtained by using Kolmogorov's Theorem [19] based on the high moment error estimates. Moreover, the high moment and pathwise error estimates for the pressure approximations in the time-averaged norm are obtained with a help of the stochastic *inf-sup* condition.

Chapter 4 is concerned with the time-splitting projection method for the stochastic Stokes equations. As mentioned above, Carelli, Hausenblas and Prohl [16] considered the standard Chorin projection method in the case of a divergence-free multiplicative noise. They obtained the error estimate for the velocity approximation in the energy norm but did not address the pressure approximation. Hence, two natural questions immediately arise: Does the standard Chorin projection method still perform well in the case of general non-divergence-free noise? Is it possible to obtain error estimates for the pressure approximation? In this chapter, we provide positive answers to both questions and obtain error estimates for both the velocity and pressure approximations. We prove that the standard Chorin projection method for (1.3) with general non-divergence-free noise converges in strong norms with order $O(k^{\frac{1}{4}} + k^{-\frac{1}{2}}h)$. Moreover, our numerical tests suggest that the convergence orders are sharp, even they are not optimal. In order to obtain optimal convergence, we modify the standard Chorin scheme by using the idea of Helmholtz decomposition of the noise as used in the modified mixed finite element method in Chapter 2. We are able to show the optimal convergence of order $O(k^{\frac{1}{2}} + k^{-\frac{1}{2}}h^2)$ for both velocity and pressure approximations by the modified scheme. Like in the deterministic case, Chorin method has been popular because it separates the velocity and pressure approximations at each time step by solving two Poisson problems. Chorin method for (1.3) has the same advantage.

Part Two of this dissertation (consisting of Chapter 5) studies strong convergence of the standard mixed finite element method for the stochastic Navier-Stokes equations with additive noise (i.e. $\mathbf{B}(\mathbf{u}) \equiv \mathbf{B}$). As mentioned earlier, Carelli and Prohl [17] proved the rates of convergence in probability for the mixed finite approximations of (1.2). The main difficulty for establishing a strong convergence for any numerical method is the interplay between the nonlinearity of the Navier-Stokes equations and its stochastic noise term. To estimate the expectation of the error, another layer of integration much be evaluated, which is the main reason why the classical Gronwall inequality argument fails. To overcome this difficulty, they

defined a sequence of sub-sample spaces which converges to 1 under the probability measure. However, the error estimates on these sub-sample spaces are only partial expectations which therefore only gives a weak convergence. Recently, Bessaih and Millet [5, 6] proved the strong L^2 -convergence of the mixed finite element method by estimating the error on the complement of these sub-sample spaces, but only a very weak logarithmic convergence rate was proved. Very recently, Bessaih and Millet [7] were able to establish a strong polynomial rate convergence for the mixed finite element method in the case of divergence-free additive noise by using exponential stability estimates. Their idea is to use Gronwall inequality first to derive a pathwise ‘master’ error inequality and then to take the expectation to the inequality and to estimate the right-hand side. This idea was used earlier in [15, Section 5.3.3] to overcome the same difficulty. In this chapter, we consider a general additive noise and adapt and refine the idea of [15, 7] to derive high moment error estimates for both the velocity and pressure approximations by the mixed finite element method. We then derive their pathwise error estimates using the Kolmogorov Theorem [19].

Part Three (consisting of Chapter 6) of this dissertation studies the parameter-dependent and random variational problem:

$$A(\omega; u, v) = F(\omega; v) \quad \forall v \in V, \quad (1.8)$$

where V is a Banach space contained in a Hilbert space H . $F(\omega; \cdot) \in V^*$ and $A(\omega; \cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ represents a bilinear form which depends on a parameter $\omega \in \Omega$ and satisfies three basic conditions to ensure the well-posedness, namely, *bi-linearity*, *boundedness*, *coercivity*. Ω denotes a general parameter space which can be a deterministic or probability space, in the latter case, equation (1.8) becomes a random problem. As mentioned earlier, our goal is to develop efficient algorithms for solving (1.8).

Our main idea is to reformulate parameter-dependent problem (1.8) into another problem with parameter-independent coefficients and a parameter-dependent (and solution-dependent) right-hand side, its solution is then computed using a fixed-point iteration algorithm. Since the coefficients are parameter-independent and iterate-independent and the right-hand side is parameter-dependent and iterate-dependent, this presents an ideal set-up

for us to employ an efficient LU direct solver and/or a block Krylov subspace iterative solver at each iteration for solving a linear system with multiple right-hand vectors. It is the reuse of the LU matrix factorization and efficiently to compute matrix-matrix multiplication for all parameters that result in significant computation saving, hence, leads to an efficient overall numerical method.

The dissertation is completed with a list of future research projects in Chapter 7.

Chapter 2

Mixed finite element methods for stochastic Stokes equations

2.1 Introduction

In this chapter, we consider the fully discrete mixed finite element methods for the time-dependent stochastic Stokes equations (1.3) with general non-divergence-free multiplicative noise and the domain $D = (0, L)^2$. A periodic boundary condition with period $L > 0$ in each coordinate direction is considered in this part. $\mathbf{u}_0$ and $\mathbf{f}$ denote respectively the prescribed initial velocity and body force which are spatially periodic. For the sake of simplicity and ease of presentation, we assume $\{W(t); t \geq 0\}$ to be an $\mathbb{R}$ -valued Wiener process; see section 1.2 for further details.

The numerical analysis of the deterministic Stokes problem is well-established in the literature, see [10, 37, 44]. Well-known numerical methods include *exactly divergence-free methods*, which approximate the velocity in exactly divergence-free finite element spaces; *mixed finite element methods*, where the (discrete) *inf-sup* condition is the key criterion that distinguishes stable pairings of finite element ansatz spaces for the velocity (with more degrees of freedom) and the pressure (with less degrees of freedom); mixed methods allow a more flexible, broader application compared to *exactly divergence-free methods*, thus putting them in the center of research on numerical methods for saddle point problems in the last decades. Another class of related numerical methods are *stabilization methods* which were

initiated in [45], where the incompressibility constraint (1.3) is relaxed into

$$\operatorname{div} \mathbf{u} - \varepsilon \Delta p = 0 \quad \text{in } D_T, \quad (2.1)$$

This relaxation allows for stable pairings of equal order (nodal-based) finite element ansatz spaces for both, velocity and pressure (putting $\varepsilon = \mathcal{O}(h^2)$, where $h > 0$ is the spatial mesh size). We remark that optimal order error estimates had been obtained for all three classes of finite element methods in the deterministic setting (cf. [37, 10]), where

- *inf-sup* stable mixed finite element methods require the H^1 -regularity of the pressure in order to optimally bound the best-approximation error for the pressure, which leads to optimal order convergence; cf. [10, 37, 44],
- stabilization methods require the H^1 -regularity of the pressure for convergence; cf. [45, 58].

This chapter contributes to the numerical analysis of the stochastic Stokes problem (1.3) (i.e., $\mathbf{B} \neq \mathbf{0}$). By [50], even for smooth datum functions $\mathbf{u}_0$ and $\mathbf{f}$, the (temporal) regularity of the pressure $p \in L^1(\Omega; W^{-1,\infty}(0, T; H^1(D)/\mathbb{R}))$ is limited in general due to the driving noise. In order to motivate its impact onto the pressure, we here discuss the related question regarding *k-independent* stability estimates for the pair of random variables $(\mathbf{u}^{n+1}, p^{n+1})$ of the following time-implicit discretization of (1.3) on a uniform mesh of $[0, T]$ with the mesh size $k > 0$:

$$\mathbf{u}^{n+1} - k \Delta \mathbf{u}^{n+1} + k \nabla p^{n+1} = \mathbf{u}^n + k \mathbf{f}^{n+1} + \mathbf{B}(\mathbf{u}^n) \Delta W_{n+1} \quad \text{in } D, \quad (2.2a)$$

$$\operatorname{div} \mathbf{u}^{n+1} = 0 \quad \text{in } D, \quad (2.2b)$$

where $\Delta W_{n+1} := W(t_{n+1}) - W(t_n) \sim \mathcal{N}(0, k) = \sqrt{k} \mathcal{N}(0, 1)$ and $\mathbf{f}^{n+1} = \mathbf{f}(t_{n+1}, \cdot) \in L^2(\Omega, \mathbf{L}_{per}^2(D))$. A crucial observation for the motivation of this chapter is that the pressure gradient on the left-hand side is scaled by k , while the noise term is of order $\mathcal{O}(k^{\frac{1}{2}})$. Let us assume that the estimate (2.22) in Lemma 2.2.1 for $\{\Delta \mathbf{u}^n\}_n$ taking values in $\mathbf{L}^2(D)$ is already shown, and we now look for a uniform bound for $\{\nabla p^n\}_n$ taking values in $\mathbf{L}^2(D)$.

The strategy for deriving such a stability estimate is to fix one $\omega \in \Omega$, and to multiply (2.2) with $\nabla p^{n+1}(\omega)$: all the terms that involve the velocity vanish due to the incompressibility property and the periodic boundary condition, and we end up with

$$\frac{k}{2} \|\nabla p^{n+1}(\omega)\|_{\mathbf{L}^2}^2 \leq k \|\mathbf{f}^{n+1}\|_{\mathbf{L}^2}^2 + \left(\mathbf{B}(\mathbf{u}^n(\omega)) \Delta W_{n+1}(\omega), \nabla p^{n+1}(\omega) \right). \quad (2.3)$$

Note that the term on the right-hand side does not vanish since $\operatorname{div} \mathbf{B}(\mathbf{u}^n) \neq 0$ for a general (Lipschitz) nonlinear mapping $\mathbf{B}$. We now take expectations $\mathbb{E}[\cdot]$ on both sides, sum over all time steps, and use (2.22), the facts that $\mathbf{B}(\mathbf{u}^n)$ and ΔW_{n+1} are independent and $\mathbb{E}[|\Delta W_{n+1}|^2] \leq Ck$, and Young's inequality (with $\alpha > 0$) to obtain the estimate

$$\begin{aligned} \frac{k}{2} \sum_{n=0}^{N-1} \mathbb{E}[\|\nabla p^{n+1}\|_{\mathbf{L}^2}^2] &\leq k \sum_{n=0}^{N-1} \mathbb{E}[\|\mathbf{f}^{n+1}\|_{\mathbf{L}^2}^2] + \sum_{n=0}^{N-1} \mathbb{E}\left[\left(\mathbf{B}(\mathbf{u}^n) \Delta_{n+1} W, \nabla p^{n+1}\right)\right] \\ &\leq k \sum_{n=0}^{N-1} \mathbb{E}[\|\mathbf{f}^{n+1}\|_{\mathbf{L}^2}^2] + \alpha k \sum_{n=0}^{N-1} \mathbb{E}[\|\mathbf{B}(\mathbf{u}^n)\|_{\mathbf{L}^2}^2] + \frac{1}{4\alpha} \sum_{n=0}^{N-1} \mathbb{E}[\|\nabla p^{n+1}\|_{\mathbf{L}^2}^2]. \end{aligned}$$

Taking $\alpha = \frac{1}{k}$ allows to absorb the last term on the right-hand side to the one on the left-hand side, but the remaining term is $\sum_{n=0}^{N-1} \mathbb{E}[\|\mathbf{B}(\mathbf{u}^n)\|^2] \propto \mathcal{O}(k^{-1})$, therefore, we end up with the following k -dependent estimate:

$$\frac{k}{4} \sum_{n=0}^{N-1} \mathbb{E}[\|\nabla p^{n+1}\|_{\mathbf{L}^2}^2] \leq \frac{C}{k} + k \sum_{n=0}^{N-1} \mathbb{E}[\|\mathbf{f}^{n+1}\|_{\mathbf{L}^2}^2]. \quad (2.4)$$

The above consideration crucially affects the error analysis of a space-time discretization of (1.3):

- *Exactly divergence-free methods* require restricted settings of data, including the dimension, topology, and regularity of the spatial domain D . However, an optimal order error estimate can be proved for the velocity approximation, see [17], which uses the fact that no pressure is involved in the analysis.
- The error estimate for the velocity approximation of *inf-sup* stable mixed finite element methods in [34] was obtained based on a stability bound of type (2.4) to bound the related best-approximation error for the pressure that appears in (an auxiliary temporal

discretization of) (1.3), thus leading to a sub-optimal error estimate for the velocity of order $\mathcal{O}(k^{\frac{1}{2}} + hk^{-\frac{1}{2}})$. The computational studies in [34] suggest that this error bound is sharp.

The first goal of the chapter is to construct optimally convergent *inf-sup* stable mixed finite element methods, with “minimum” extra effort. Our main idea, which is partly borrowed from [16], is to perform the Helmholtz decomposition for the noise term at each time step first, and then to determine the new velocity and pressure iterates simultaneously via the mixed finite element method. Below we shall use the semi-discrete time-stepping scheme (2.2) to motivate our strategy. Introducing the Helmholtz decomposition of $\mathbf{B}$ as follows

$$\mathbf{B}(\mathbf{u}^n) = \nabla \xi^n + \boldsymbol{\eta}^n \quad \text{where} \quad \operatorname{div} \boldsymbol{\eta}^n = 0, \quad (2.5)$$

and setting $r^{n+1} := p^{n+1} - k^{-1} \Delta W_{n+1} \xi^n$, then (2.2) can be rewritten as

$$\mathbf{u}^{n+1} - k \Delta \mathbf{u}^{n+1} + k \nabla r^{n+1} = \mathbf{u}^n + k \mathbf{f}^{n+1} + \boldsymbol{\eta}^n \Delta W_{n+1} \quad \text{in } D, \quad (2.6a)$$

$$\operatorname{div} \mathbf{u}^{n+1} = 0 \quad \text{in } D. \quad (2.6b)$$

In contrast to estimate (2.4) for p^{n+1} , it can be shown that the pseudo pressure r^{n+1} satisfies the following improved stability estimate (see Lemma 2.2.1):

$$k \sum_{n=0}^{N-1} \mathbb{E} [\|\nabla r^{n+1}\|_{\mathbf{L}^2}^2] \leq k \sum_{n=0}^{N-1} \mathbb{E} [\|\mathbf{f}^{n+1}\|_{\mathbf{L}^2}^2], \quad (2.7)$$

which is a consequence of the divergence-free property of the modified noise term (*i.e.*, the last term on the right-hand side of (2.6a)). Conceptually, this improved stability for the pseudo pressure r^{n+1} is obtained by removing the *stochastic pressure* ξ^n from the driving noise in (2.2a). As it will be detailed in Section 2.2, any *inf-sup* stable mixed finite element discretization of (2.6) then gives optimally convergent velocity approximations (see Theorem 2.3.4), whose proof essentially relies on (2.7). We also present optimal error estimates for (temporal averages of) the pressure approximations in L^2 , which improve corresponding suboptimal estimates in [34].

We therefore conclude by saying that it is essential to identify the proper role of the semi-discrete pressures, namely, $\{p^n\}_n$ in (2.2) vs. $\{r^n\}$ in (2.6), for *inf-sup* stable mixed finite element methods for (1.3) in order to construct optimally convergent mixed methods. Moreover, this insight also suggests how to construct optimally convergent stabilization methods for (1.3) which circumvent the *inf-sup* stability criterion for mixed element methods, and hence allow a more efficient discretization such as

$$\mathbf{u}_\varepsilon^{n+1} - k\Delta\mathbf{u}_\varepsilon^{n+1} + k\nabla r_\varepsilon^{n+1} = \mathbf{u}_\varepsilon^n + k\mathbf{f}^{n+1} + \boldsymbol{\eta}_\varepsilon^n \Delta W_{n+1} \quad \text{in } D, \quad (2.8a)$$

$$\operatorname{div} \mathbf{u}_\varepsilon^{n+1} - \varepsilon \Delta r_\varepsilon^{n+1} = 0 \quad \text{in } D, \quad (2.8b)$$

for which $\varepsilon = \mathcal{O}(h^2)$ will be shown to be the optimal choice in section 2.3. The error analysis in section 2.3 verifies optimal order convergence for a standard finite element discretization of (2.8) which employs the same finite element space for approximating both, $\mathbf{u}_\varepsilon^{n+1}$ and r_ε^{n+1} ; see Theorem 2.4.3. Corresponding computational studies in section 2.5 support the conclusion that the choice of pressure in the stabilization is crucial for achieving an optimally convergent *stabilization method* for (1.3).

The remainder of this chapter is organized as follows. In section 2.1, we give exact assumptions on the data in (1.3), and recall the definition and known properties of the (strong) variational solution for problem (1.3). In section 2.2, we analyze the *Helmholtz decomposition enhanced* Euler-Maruyama time-stepping scheme (2.5)–(2.6) and its mixed finite element approximations, and establish the optimal convergence for both. section 2.3 establishes optimal convergence for the stabilized scheme (2.8) and its equal-order finite element approximations. Two-dimensional numerical experiments and computational studies are given in section 2.5 to validate the theoretical error bounds, and to computationally evidence that a proper selection of the pressure for the construction of optimally convergent mixed methods is indeed necessary.

2.1.1 Variational formulation of the stochastic Stokes equations

In this section, we shall give weak formulations of (1.2) and (1.3). We consider (1.2), (1.3) with periodic boundary condition with period $L > 0$ on the domain $D = (0, L)^2$. We first recall the solution concept for (1.3), and refer to [18, 19] for its existence and uniqueness.

Definition 2.1.1. *Given $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, let W be an $\mathbb{R}$ -valued Wiener process on it. Suppose $\mathbf{u}_0 \in L^2(\Omega, \mathbb{V})$ and $\mathbf{f} \in L^2(\Omega; L^2((0, T); \mathbf{L}_{per}^2(D)))$. An $\{\mathcal{F}_t\}$ -adapted stochastic process $\{\mathbf{u}(t); 0 \leq t \leq T\}$ is called a variational solution of (1.2) if $\mathbf{u} \in L^2(\Omega; C([0, T]; \mathbb{V})) \cap L^2(0, T; \mathbf{H}_{per}^2(D))$, and satisfies $\mathbb{P}$ -a.s. for all $t \in (0, T]$*

$$\begin{aligned} (\mathbf{u}(t), \mathbf{v}) + \nu \int_0^t (\nabla \mathbf{u}(s), \nabla \mathbf{v}) ds + \int_0^t (\mathbf{u}(s) \cdot \nabla \mathbf{u}(s), \mathbf{v}) ds \\ = (\mathbf{u}_0, \mathbf{v}) + \int_0^t (\mathbf{f}(s), \mathbf{v}) ds + \int_0^t (\mathbf{B}(\mathbf{u}(s)), \mathbf{v}) dW(s) \quad \forall \mathbf{v} \in \mathbb{V}. \end{aligned} \quad (2.9)$$

Similarly, an $\{\mathcal{F}_t\}$ -adapted stochastic process $\{\mathbf{u}(t); 0 \leq t \leq T\}$ is called a variational solution of (1.3) if $\mathbf{u} \in L^2(\Omega; C([0, T]; \mathbb{V})) \cap L^2(0, T; \mathbf{H}_{per}^2(D))$, and satisfies $\mathbb{P}$ -a.s. for all $t \in (0, T]$

$$\begin{aligned} (\mathbf{u}(t), \mathbf{v}) + \nu \int_0^t (\nabla \mathbf{u}(s), \nabla \mathbf{v}) ds = (\mathbf{u}_0, \mathbf{v}) + \int_0^t (\mathbf{f}(s), \mathbf{v}) ds \\ + \int_0^t (\mathbf{B}(\mathbf{u}(s)), \mathbf{v}) dW(s) \quad \forall \mathbf{v} \in \mathbb{V}. \end{aligned} \quad (2.10)$$

The following estimates from [16, 34] establish the Hölder continuity in time of the variational solution in various spatial norms.

Theorem 2.1.2. *Additionally suppose $\mathbf{u}_0 \in L^2(\Omega; \mathbb{V} \cap \mathbf{H}_{per}^2(D))$ and $\mathbf{f} \in L^2(\Omega, C^{\frac{1}{2}}([0, T]); \mathbf{H}_{per}^1(D))$. There exists a constant $C > 0$, such that the variational solution to problem (1.2) satisfies for $s, t \in [0, T]$*

$$\mathbb{E}[\|\mathbf{u}(t) - \mathbf{u}(s)\|_{\mathbf{L}^2}^2] + \mathbb{E}\left[\int_s^t \|\nabla(\mathbf{u}(\tau) - \mathbf{u}(s))\|_{\mathbf{L}^2}^2 d\tau\right] \leq C|t - s|, \quad (2.11a)$$

$$\mathbb{E}[\|\nabla(\mathbf{u}(t) - \mathbf{u}(s))\|_{\mathbf{L}^2}^2] + \mathbb{E}\left[\int_s^t \|\mathbf{A}(\mathbf{u}(\tau) - \mathbf{u}(s))\|_{\mathbf{L}^2}^2 d\tau\right] \leq C|t - s|. \quad (2.11b)$$

Remark 2.1.1. *To avoid the technicality of tracking the required “minimum” assumptions on $\mathbf{u}_0$ and $\mathbf{f}$ for each stability and/or error estimate, unless it is stated otherwise, we shall implicitly make the “maximum” assumption $\mathbf{u}_0 \in L^2(\Omega; \mathbb{V} \cap \mathbf{H}_{per}^2(D))$ and $\mathbf{f} \in L^2(\Omega, C^{\frac{1}{2}}([0, T]); \mathbf{H}_{per}^1(D))$ in the rest of this chapter.*

2.1.2 Definition and role of the pressure

The Definition 2.1.1 only addresses the velocity $\mathbf{u}$ in the stochastic PDEs (1.2), (1.3); a corresponding pressure which satisfies a proper formulation (see Theorem 2.1.3 below) may be constructed after the existence of a velocity field $\mathbf{u}$ has been established. We therefore consider processes

$$\mathbf{U}(t) := \int_0^t \mathbf{u}(s) ds, \quad \mathbf{F}(t) := \int_0^t \mathbf{f}(s) ds \quad \text{and} \quad \mathbf{G}(t) := \int_0^t \mathbf{B}(\mathbf{u}(s)) dW(s).$$

Evidently, $\mathbf{U} \in L^2(\Omega, L^2(0, T; \mathbf{H}_{per}^2(D)))$ and $\mathbf{G} \in L^2(\Omega, L^2(0, T; \mathbf{L}_{per}^2(D)))$, and (2.11a) therefore implies

$$(\mathbf{u}(t) - \Delta \mathbf{U}(t) - \mathbf{u}_0 - \mathbf{F}(t) - \mathbf{G}(t), \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbb{V}, t \in (0, T), \mathbb{P}\text{-a.s.} \quad (2.12)$$

By the Helmholtz decomposition [50, Theorem 4.1 and Remark 4.3], there exists a unique $P \in L^2(\Omega, L^2(0, T; H_{per}^1(D))/\mathbb{R})$ such that

$$\nabla P(t) = -[\mathbf{u}(t) - \Delta \mathbf{U}(t) - \mathbf{u}_0 - \mathbf{F}(t) - \mathbf{G}(t)] \quad \forall t \in (0, T), \mathbb{P}\text{-a.s.} \quad (2.13)$$

in the distributional sense. It is shown in [50, Section 5], that its distributional time derivative $p := \partial_t P \in L^1(\Omega; W^{-1, \infty}(0, T; H_{per}^1(D)/\mathbb{R}))$. As a consequence, we have the following result.

Theorem 2.1.3. *Let $\{\mathbf{u}(t); 0 \leq t \leq T\}$ be the variational solution of (1.3). There exists a unique adapted process $P \in L^2(\Omega, L^2(0, T; H_{per}^1(D)/\mathbb{R}))$ such that $(\mathbf{u}, P)$ satisfies $\mathbb{P}$ -a.s. for*

all $t \in (0, T]$

$$(\mathbf{u}(t), \mathbf{v}) + \int_0^t (\nabla \mathbf{u}(s), \nabla \mathbf{v}) ds - (\operatorname{div} \mathbf{v}, P(t)) \quad (2.14a)$$

$$= (\mathbf{u}_0, \mathbf{v}) + \int_0^t (\mathbf{f}(s), \mathbf{v}) ds + \int_0^t (\mathbf{B}(\mathbf{u}(s)), \mathbf{v}) dW(s) \quad \forall \mathbf{v} \in \mathbf{H}_{per}^1(D),$$

$$(\operatorname{div} \mathbf{u}, q) = 0 \quad \forall q \in L_{per}^2(D). \quad (2.14b)$$

System (2.14) can be regarded as a mixed formulation for the stochastic Stokes system (1.3), where the (time-averaged) pressure P is defined. Below, we also define another time-averaged “pressure”

$$R(t) := P(t) - \int_0^t \xi(s) dW(s), \quad (2.15)$$

where we use the Helmholtz decomposition $\mathbf{B}(\mathbf{u}(t)) = \boldsymbol{\eta}(t) + \nabla \xi(t)$, where $\xi \in H_{per}^1(D)/\mathbb{R}$ $\mathbb{P}$ -a.s. such that

$$(\nabla \xi(t), \nabla \phi) = (\mathbf{B}(\mathbf{u}(t)), \nabla \phi) \quad \forall \phi \in H_{per}^1(D). \quad (2.16)$$

Then, (2.13) can be rewritten as

$$\nabla R(t) = - \left[\mathbf{u}(t) - \Delta \mathbf{U}(t) - \mathbf{u}_0 - \mathbf{F}(t) - \int_0^t \boldsymbol{\eta}(s) dW(s) \right] \quad \forall t \in (0, T), \mathbb{P}\text{-a.s.} \quad (2.17)$$

The time averaged pseudo pressure $\{R(t); 0 \leq t \leq T\}$ will also be a target process to be approximated in our numerical methods.

2.2 Semi-discretization in time

In this section we study the stability and convergence properties of a Helmholtz decomposition enhanced Euler-Maruyama time discretization scheme that is based on (2.6), where the stochastic pressure is removed from the noise term via the Helmholtz decomposition; but its $\mathbb{V}$ -valued velocity approximation $\{\mathbf{u}^{n+1}\}_n$ still solves the original Euler-Maruyama scheme (2.2).

2.2.1 Formulation of the time-stepping scheme

In the following, let N be a positive integer, $k = \frac{T}{N}$, and $t_n = nk$ for $n = 0, 1, \dots, N$ be a uniform mesh that covers $[0, T]$.

Algorithm 1

Let $\mathbf{u}^0 = \mathbf{u}_0$. For $n = 0, 1, \dots, N - 1$ do the following steps:

Step 1: Find $\xi^n \in L^2(\Omega, H_{per}^1(D)/\mathbb{R})$ by solving

$$(\nabla \xi^n, \nabla \phi) = (\mathbf{B}(\mathbf{u}^n), \nabla \phi) \quad \forall \phi \in H_{per}^1(D). \quad (2.18)$$

Step 2: Set $\boldsymbol{\eta}^n := \mathbf{B}(\mathbf{u}^n) - \nabla \xi^n$, and find $(\mathbf{u}^{n+1}, r^{n+1}) \in L^2(\Omega, \mathbb{V} \times L_{per}^2(D)/\mathbb{R})$ by solving

$$(\mathbf{u}^{n+1}, \mathbf{v}) + k(\nabla \mathbf{u}^{n+1}, \nabla \mathbf{v}) - k(\operatorname{div} \mathbf{v}, r^{n+1}) \quad (2.19a)$$

$$= (\mathbf{u}^n, \mathbf{v}) + k(\mathbf{f}^{n+1}, \mathbf{v}) + (\boldsymbol{\eta}^n \Delta W_{n+1}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_{per}^1(D),$$

$$(\operatorname{div} \mathbf{u}^{n+1}, q) = 0 \quad \forall q \in L_{per}^2(D). \quad (2.19b)$$

Step 3: Define $p^{n+1} := r^{n+1} + k^{-1} \xi^n \Delta W_{n+1}$.

Remark 2.2.1. By the elliptic regularity theory, see [37, p. 13], the solution of (2.18) is in $\xi^n \in L^2(\Omega, H_{per}^2(D)/\mathbb{R})$, and satisfies Lebesgue-a.e.

$$-\Delta \xi^n = -\operatorname{div} \mathbf{B}(\mathbf{u}^n) \quad \text{in } D. \quad (2.20a)$$

Moreover there exists a constant $C > 0$ such that

$$\|\xi^n\|_{H^2/\mathbb{R}} \leq C \|\operatorname{div} \mathbf{B}(\mathbf{u}^n)\|_{\mathbf{L}^2}. \quad (2.21)$$

The solvability of Algorithm 1 is clear because a linear coercive elliptic PDE problem is solved at each step. *Step 1* in Algorithm 1 requires to solve a Poisson problem (2.18), which only slightly increases the computational cost if a fast solver is used to solve them. The

iterates $\{(\mathbf{u}_n, r_n)\}_n$ and $\{p_n\}_n$ defined in *Step 2* and *3* aim to approximate $\{(\mathbf{u}(t), r(t)); 0 \leq t \leq T\}$ and $\{p(t); 0 \leq t \leq T\}$, respectively. See subsection 2.2.4 for details.

2.2.2 Stability estimates

In this subsection we present some stability estimates for the time-stepping scheme given in Algorithm 1. All these estimates, in particular the estimate for $\{\nabla r^{n+1}\}_n$, will play an important role in establishing optimal order error estimates for the fully mixed finite element discretization to be given in the next section.

Lemma 2.2.1. *Let $\{(\mathbf{u}^{n+1}, r^{n+1})\}_n$ be generated by Algorithm 1. There exists a constant $C > 0$, such that*

$$\max_{1 \leq n \leq N} \mathbb{E}[\|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2] + \mathbb{E}\left[\sum_{n=1}^N \|\nabla(\mathbf{u}^n - \mathbf{u}^{n-1})\|_{\mathbf{L}^2}^2\right] + \mathbb{E}\left[k \sum_{n=1}^N \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2\right] \leq C, \quad (2.22)$$

$$\mathbb{E}\left[k \sum_{n=1}^N \|\nabla r^n\|_{\mathbf{L}^2}^2\right] \leq C. \quad (2.23)$$

Proof. See Lemma 3.1 from [17] and Lemma 2.1 from [34] for a proof of estimate (2.22). We just need to give a sketch of proof for estimate (2.23). To do that, multiply (2.6a) with ∇r^{n+1} and integration in space together with using (2.6b) and $\operatorname{div} \eta^n = 0$ then lead to

$$k \|\nabla r^{n+1}\|_{\mathbf{L}^2}^2 = k(\mathbf{f}^{n+1}, \nabla r^{n+1}) \leq \frac{k}{2} \|\mathbf{f}^{n+1}\|_{\mathbf{L}^2}^2 + \frac{k}{2} \|\nabla r^{n+1}\|_{\mathbf{L}^2}^2, \quad (2.24)$$

which gives us $k \|\nabla r^{n+1}\|_{\mathbf{L}^2}^2 \leq k \|\mathbf{f}^{n+1}\|_{\mathbf{L}^2}^2$. Then, taking the expectation and summation operator $\sum_{n=0}^{N-1}$ gives the desired estimate. $\square$

2.2.3 Error estimate for the velocity approximation

Since the velocity approximation $\{\mathbf{u}^{n+1}\}_n$ generated by Algorithm 1 also solves the original Euler-Maruyama time-stepping scheme (2.2), the following optimal order error estimate for $\{\mathbf{u}^{n+1}\}_n$ was established in [17, 34].

Theorem 2.2.2. *Let $\{(\mathbf{u}^{n+1}, r^{n+1})\}_n$ be generated by Algorithm 1. There exists a constant $C > 0$, such that*

$$\max_{1 \leq n \leq N} \left(\mathbb{E} [\|\mathbf{u}(t_n) - \mathbf{u}^n\|_{\mathbf{L}^2}^2] \right)^{\frac{1}{2}} + \left(\mathbb{E} \left[k \sum_{n=1}^N \|\nabla(\mathbf{u}(t_n) - \mathbf{u}^n)\|_{\mathbf{L}^2}^2 \right] \right)^{\frac{1}{2}} \leq C k^{\frac{1}{2}}. \quad (2.25)$$

We note that the proof of the above error estimate crucially uses the fact that $\mathbf{u}^n$ is exactly divergence-free for each $0 \leq n \leq N$.

2.2.4 Error estimates for the pressure approximation

An optimal order error estimate was obtained in [34] for $\{P(t_n)\}_n$ via the Euler-Maruyama time-stepping scheme (2.2). For the reader's convenience, we give its proof here.

Theorem 2.2.3. *Let $\{p^n; 1 \leq n \leq N\}$ be the pressure in (2.2), and $\{P(t); 0 \leq t \leq T\}$ be defined in Theorem 2.1.3. There exists a constant $C > 0$, such that*

$$\left(\mathbb{E} [\|P(t_m) - k \sum_{n=1}^m p^n\|_{L^2}^2] \right)^{\frac{1}{2}} \leq C k^{\frac{1}{2}}, \quad m = 1, 2, \dots, N. \quad (2.26)$$

Proof. Consider (2.2a), and take the sum over steps $0 \leq n \leq m-1$. We denote $\mathbf{U}^m := k \sum_{n=0}^{m-1} \mathbf{u}^{n+1}$ and $P^m := k \sum_{n=0}^{m-1} p^{n+1}$, and therefore obtain

$$\mathbf{u}^m - \mathbf{u}^0 - \Delta \mathbf{U}^m + \nabla P^m = k \sum_{n=0}^{m-1} \mathbf{f}^{n+1} + \sum_{n=0}^{m-1} \mathbf{B}(\mathbf{u}^n) \Delta W_{n+1}. \quad (2.27)$$

We subtract this equation from (2.13) at time $t = t_m$, and denote $\mathbf{E}_{\mathbf{U}}^m := \mathbf{U}(t_m) - \mathbf{U}^m \in L^2(\Omega; \mathbf{H}_{per}^1(D))$, and $E_P^m := P(t_m) - P^m \in L^2(\Omega; L_{per}^2(D))$. By the stability of the divergence

operator, there exists $\beta > 0$, such that

$$\begin{aligned}
\frac{1}{\beta} \|E_P^m\|_{L^2} &\leq \sup_{\mathbf{v} \in \mathbf{H}_{per}^1(D)} \frac{(E_P^m, \operatorname{div} \mathbf{v})}{\|\nabla \mathbf{v}\|_{\mathbf{L}^2}} \\
&\leq \|\mathbf{u}(t_m) - \mathbf{u}^m\|_{\mathbf{L}^2} + \|\nabla \mathbf{E}_U^m\|_{\mathbf{L}^2} + \left\| \sum_{n=0}^{m-1} (\mathbf{B}(\mathbf{u}(t_n)) - \mathbf{B}(\mathbf{u}^n)) \Delta W_{n+1} \right\|_{\mathbf{L}^2} \\
&\quad + \left\| \sum_{n=0}^{m-1} \int_{t_n}^{t_{n+1}} \mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}(t_n)) dW(s) \right\|_{\mathbf{L}^2} \\
&\quad + \left\| \sum_{n=0}^{m-1} \int_{t_n}^{t_{n+1}} (\mathbf{f}(s) - \mathbf{f}(t_{n+1})) ds \right\|_{\mathbf{L}^2} =: \mathbf{I} + \dots + \mathbf{V}.
\end{aligned} \tag{2.28}$$

Taking squares on both sides, and then applying expectations, Theorem 2.2.2 in combination with Hölder's inequality leads to

$$\frac{1}{\beta^2} \mathbb{E}[\|E_P^m\|_{L^2}^2] \leq Ck + \mathbb{E}[\|\mathbf{I}\|^2_{\mathbf{L}^2}] + \mathbb{E}[\|\mathbf{IV}\|_{\mathbf{L}^2}^2] + \mathbb{E}[\|\mathbf{V}\|_{\mathbf{L}^2}^2].$$

By Ito's isometry, and (1.5a), as well as (2.11a), and Theorem 2.2.2, we find the bounds

$$\mathbb{E}[\|\mathbf{I}\|^2_{\mathbf{L}^2} + \|\mathbf{IV}\|_{\mathbf{L}^2}^2] \leq C\mathbb{E}\left[k \sum_{n=0}^{m-1} \|\mathbf{u}(t_n) - \mathbf{u}^n\|_{\mathbf{L}^2}^2\right] + Ck \leq Ck,$$

and by using Cauchy-Schwarz inequality, we get

$$\mathbb{E}[\|\mathbf{V}\|_{\mathbf{L}^2}^2] \leq \mathbb{E}\left[\sum_{n=0}^{m-1} \int_{t_n}^{t_{n+1}} \|\mathbf{f}(s) - \mathbf{f}(t_{n+1})\|_{\mathbf{L}^2}^2 ds\right] \leq Ck,$$

which lead to the desired estimate (2.26). $\square$

We now consider the pressure $R^m := k \sum_{n=0}^{m-1} r^{n+1}$, where $\{r^n\}_n$ is defined by Algorithm

1. Using the new notation (2.27) can be written as

$$\mathbf{u}^m - \mathbf{u}^0 - \Delta \mathbf{U}^m + \nabla R^m = k \sum_{n=0}^{m-1} \mathbf{f}^{n+1} + \sum_{n=0}^{m-1} (\mathbf{B}(\mathbf{u}^n) - \nabla \xi^n) \Delta W_{n+1}. \tag{2.29}$$

We again subtract this equation from (2.17) at time $t = t_m$, and adapt the error notation in (2.28),

$$\frac{1}{\beta} \|E_R^m\|_{L^2} \leq \sup_{\mathbf{v} \in \mathbf{H}_{per}^1(D)} \frac{(E_R^m, \operatorname{div} \mathbf{v})}{\|\nabla \mathbf{v}\|_{\mathbf{L}^2}} \leq \text{I} + \dots + \text{IV} + \text{V} + \text{VI},$$

where V is the same as above, hence, $\mathbb{E}[\|\text{V}\|_{\mathbf{L}^2}^2] \leq Ck$, and

$$\text{VI} := \left\| \sum_{n=0}^{m-1} \nabla(\xi(t_n) - \xi^n) \Delta W_{n+1} \right\|_{\mathbf{L}^2} + \left\| \sum_{n=0}^{m-1} \int_{t_n}^{t_{n+1}} \nabla(\xi(s) - \nabla \xi(t_n)) dW(s) \right\|_{\mathbf{L}^2} =: \text{VI}_1 + \text{VI}_2.$$

By a stability result for the Poisson problems (2.16), (2.18), and property (1.5a), we easily obtain, thanks to Theorem 2.2.2,

$$\mathbb{E}[\|\text{VI}_1\|_{\mathbf{L}^2}^2] \leq C \mathbb{E} \left[k \sum_{n=0}^{m-1} \|\mathbf{u}(t_n) - \mathbf{u}^n\|_{\mathbf{L}^2}^2 \right] \leq Ck.$$

Similarly, we get $\mathbb{E}[\|\text{VI}_2\|_{\mathbf{L}^2}^2] \leq Ck$. We collect this result below.

Corollary 2.2.4. *Let $\{r^n; 1 \leq n \leq N\}$ be the discrete process from Algorithm 1. There exists a constant $C > 0$ such that*

$$\left(\mathbb{E} \left[\|R(t_m) - k \sum_{n=1}^m r^n\|_{L^2}^2 \right] \right)^{\frac{1}{2}} \leq Ck^{\frac{1}{2}}, \quad m = 1, 2, \dots, N. \quad (2.30)$$

2.3 Fully discrete, *inf-sup* stable mixed finite element method

In this section, we discretize Algorithm 1 in space via an *inf-sup* stable mixed finite element method. We choose the prototypical Taylor-Hood mixed finite element (see, *e.g.*, [37, 10]) as an example and give a detailed error analysis for the resulted fully discrete method, but we remark that the convergence analysis below also applies to general *inf-sup* stable mixed finite elements.

2.3.1 Preliminaries

Let $\mathcal{T}_h$ be a quasi-uniform triangular or rectangular mesh of $D \subset \mathbf{R}^d$ with mesh size $0 < h \ll 1$. We define the following finite element spaces:

$$\begin{aligned}\mathbb{X}_h &= \{ \mathbf{v}_h \in \mathbf{H}_{per}^1(D); \mathbf{v}_h|_K \in \mathbf{P}_2(K) \ \forall K \in \mathcal{T}_h \}, \\ W_h &= \{ q_h \in H_{per}^1(D)/\mathbb{R}; q_h|_K \in P_1(K) \ \forall K \in \mathcal{T}_h \}, \\ S_h &= \{ \phi_h \in H_{per}^1(D)/\mathbb{R}; \phi_h|_K \in P_\ell(K) \ \forall K \in \mathcal{T}_h \},\end{aligned}$$

where $\mathbf{P}_\ell(K)$ ($\ell \geq 1$) denotes the set of $\mathbb{R}^d$ -valued polynomials of degree less than or equal to ℓ over the element $K \in \mathcal{T}_h$. In general, we require that $S_h \subseteq W_h$, in particular, we choose $\ell = 1$ so that $S_h = W_h$ in this section.

We recall that the pair $(\mathbb{X}_h, W_h)$ satisfies the (discrete) *inf-sup* condition: there exists an h -independent constant $\gamma > 0$ such that

$$\sup_{\mathbf{v}_h \in \mathbb{X}_h} \frac{(\operatorname{div} \mathbf{v}_h, q_h)}{\|\nabla \mathbf{v}_h\|_{\mathbf{L}^2}} \geq \gamma \|q_h\|_{L^2} \quad \forall q_h \in W_h. \quad (2.31)$$

Next, let $\rho_h : L_{per}^2(D) \rightarrow W_h$ resp. $\mathcal{R}_h : H_{per}^1(D)/\mathbb{R} \rightarrow S_h$ denote the L^2 -resp. the Ritz-projection operators which are defined by

$$(\phi - \rho_h \phi, \chi_h) = 0 \quad \forall \phi \in L_{per}^2(D), \chi_h \in W_h, \quad (2.32)$$

$$(\nabla[\psi - \mathcal{R}_h \psi], \nabla \zeta_h) = 0 \quad \forall \psi \in H_{per}^1(D)/\mathbb{R}, \zeta_h \in S_h. \quad (2.33)$$

Then, the following approximation properties are well known (cf. [24, 37, 25]):

$$\|\phi - \rho_h \phi\|_{L^2} + h \|\nabla(\phi - \rho_h \phi)\|_{\mathbf{L}^2} \leq Ch^s \|\phi\|_{H^s} \quad \forall \phi \in H_{per}^s(D), \quad (2.34)$$

$$\|\psi - \mathcal{R}_h \psi\|_{L^2} + h \|\nabla(\psi - \mathcal{R}_h \psi)\|_{\mathbf{L}^2} \leq Ch^s \|\psi\|_{H^s} \quad \forall \psi \in H_{per}^s(D)/\mathbb{R}, \quad (2.35)$$

for $s = 1, 2$. Here, C is a positive constant independent of h .

We also consider the space $\mathbb{V}_h \subset \mathbb{X}_h$ of discretely divergence-free functions,

$$\mathbb{V}_h := \left\{ \mathbf{v}_h \in \mathbb{X}_h; (\operatorname{div} \mathbf{v}_h, q_h) = 0 \quad \forall q_h \in W_h \right\},$$

and define the $\mathbf{L}_{per}^2(D)$ -projection operator $\mathbf{P}_h : \mathbf{L}_{per}^2(D) \rightarrow \mathbb{V}_h$ by

$$(\mathbf{v} - \mathbf{P}_h \mathbf{v}, \mathbf{w}_h) = 0 \quad \forall \mathbf{v} \in L_{per}^2(D; \mathbb{R}^d), \mathbf{w}_h \in \mathbb{V}_h.$$

The following approximation properties are well-known (cf. [44]):

$$\|\mathbf{v} - \mathbf{P}_h \mathbf{v}\|_{\mathbf{L}^2} + h \|\nabla(\mathbf{v} - \mathbf{P}_h \mathbf{v})\|_{\mathbf{L}^2} \leq Ch^s \|\mathbf{v}\|_{\mathbf{H}^s} \quad \forall \mathbf{v} \in \mathbb{V} \cap \mathbf{H}_{per}^s(D) \quad (2.36)$$

for $s = 1, 2$. Here, C is again a positive constant independent of h .

2.3.2 Formulation of the fully discrete mixed finite element method

The fully discrete, *inf-sup* stable finite element below is a spatial discretization of Algorithm 1. We note that since $\mathbb{V}_h \not\subset \mathbb{V}$, in general, the mixed finite element discretization requires improved stability estimates for the semi-discrete pressure $\{r^{n+1}\}_n$ as given in Lemma 2.2.1 in order to ensure optimal convergence properties.

Algorithm 2

Let $\mathbf{u}_h^0 \in L^2(\Omega; \mathbb{X}_h)$. For $n = 0, 1, \dots, N-1$, we do the following steps:

Step 1: Determine $\xi_h^n \in L^2(\Omega; S_h)$ by solving

$$(\nabla \xi_h^n, \nabla \phi_h) = (\mathbf{B}(\mathbf{u}_h^n), \nabla \phi_h) \quad \forall \phi_h \in S_h. \quad (2.37)$$

Step 2: Set $\boldsymbol{\eta}_h^n := \mathbf{B}(\mathbf{u}_h^n) - \nabla \xi_h^n$. Find $(\mathbf{u}_h^{n+1}, r_h^{n+1}) \in L^2(\Omega, \mathbb{V}_h \times W_h)$ by solving

$$(\mathbf{u}_h^{n+1}, \mathbf{v}_h) + k(\nabla \mathbf{u}_h^{n+1}, \nabla \mathbf{v}_h) - k(\operatorname{div} \mathbf{v}_h, r_h^{n+1}) \quad (2.38a)$$

$$= (\mathbf{u}_h^n, \mathbf{v}_h) + k(\mathbf{f}^{n+1}, \mathbf{v}_h) + (\boldsymbol{\eta}_h^n \Delta W_{n+1}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbb{X}_h,$$

$$(\operatorname{div} \mathbf{u}_h^{n+1}, q_h) = 0 \quad \forall q_h \in W_h. \quad (2.38b)$$

Step 3: Define the W_h -valued random variable $p_h^{n+1} = r_h^{n+1} + k^{-1} \xi_h^n \Delta W_{n+1}$.

Remark 2.3.1. *Because of (2.37), we have $(\boldsymbol{\eta}_h^n, \nabla \phi_h) = 0$ for all $\phi_h \in S_h$, $\mathbb{P}$ -a.s. We also note that each of Step 1 and Step 2 solves a linear problem which is clearly well-posed; in particular, the well-posedness of (2.38) is ensured by the inf-sup property (2.31) of the mixed finite element spaces $\mathbb{X}_h$ and W_h .*

2.3.3 Error estimate for the velocity approximation

The main result of this section is to prove the following optimal estimate for the velocity error $\mathbf{u}^n - \mathbf{u}_h^n$.

Theorem 2.3.1. *Suppose that*

$$\mathbb{E}[\|\mathbf{u}^0 - \mathbf{u}_h^0\|_{\mathbf{L}^2}^2] \leq Ch^2.$$

Let $\{(\mathbf{u}^n, r^n); 1 \leq n \leq N\}$ and $\{(\mathbf{u}_h^n, r_h^n); 1 \leq n \leq N\}$ be respectively the solutions of Algorithm 1 and 2. Then there exists a constant $C > 0$ such that

$$\max_{1 \leq n \leq N} \left(\mathbb{E}[\|\mathbf{u}^n - \mathbf{u}_h^n\|_{\mathbf{L}^2}^2] \right)^{\frac{1}{2}} + \left(\mathbb{E} \left[k \sum_{n=1}^N \|\nabla(\mathbf{u}^n - \mathbf{u}_h^n)\|_{\mathbf{L}^2}^2 \right] \right)^{\frac{1}{2}} \leq Ch. \quad (2.39)$$

Proof. Define $\mathbf{e}_\mathbf{u}^n = \mathbf{u}^n - \mathbf{u}_h^n$ and $e_r^n = r^n - r_h^n$. It is easy to check that $\{(\mathbf{e}_\mathbf{u}^n, e_r^n)\}_n$ satisfies the following error equations $\mathbb{P}$ -a.s. for all tuple $(\mathbf{v}_h, q_h) \in \mathbb{X}_h \times W_h$,

$$(\mathbf{e}_\mathbf{u}^{n+1} - \mathbf{e}_\mathbf{u}^n, \mathbf{v}_h) + k(\nabla \mathbf{e}_\mathbf{u}^{n+1}, \nabla \mathbf{v}_h) - k(e_r^{n+1}, \operatorname{div} \mathbf{v}_h) \quad (2.40)$$

$$= ([\boldsymbol{\eta}^n - \boldsymbol{\eta}_h^n] \Delta W_{n+1}, \mathbf{v}_h),$$

$$(\operatorname{div} \mathbf{e}_\mathbf{u}^{n+1}, q_h) = 0. \quad (2.41)$$

Now for any fixed $\omega \in \Omega$, setting $\mathbf{v}_h = \mathbf{P}_h \mathbf{e}_{\mathbf{u}}^{n+1}(\omega) \in \mathbb{V}_h$ in (2.40) yields

$$\begin{aligned} (\mathbf{e}_{\mathbf{u}}^{n+1} - \mathbf{e}_{\mathbf{u}}^n, \mathbf{P}_h \mathbf{e}_{\mathbf{u}}^{n+1}) + k(\nabla \mathbf{e}_{\mathbf{u}}^{n+1}, \nabla \mathbf{P}_h \mathbf{e}_{\mathbf{u}}^{n+1}) - k(e_r^{n+1}, \operatorname{div} \mathbf{P}_h \mathbf{e}_{\mathbf{u}}^{n+1}) \\ = ([\boldsymbol{\eta}^n - \boldsymbol{\eta}_h^n] \Delta W_{n+1}, \mathbf{P}_h \mathbf{e}_{\mathbf{u}}^{n+1}). \end{aligned} \quad (2.42)$$

We now estimate each term on the left-hand side of (2.42) from below. First, by the definition of $\mathbf{P}_h$ we get

$$\begin{aligned} (\mathbf{e}_{\mathbf{u}}^{n+1} - \mathbf{e}_{\mathbf{u}}^n, \mathbf{P}_h \mathbf{e}_{\mathbf{u}}^{n+1}) &= (\mathbf{P}_h[\mathbf{e}_{\mathbf{u}}^{n+1} - \mathbf{e}_{\mathbf{u}}^n], \mathbf{P}_h \mathbf{e}_{\mathbf{u}}^{n+1}) \\ &= \frac{1}{2} [\|\mathbf{P}_h \mathbf{e}_{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{P}_h \mathbf{e}_{\mathbf{u}}^n\|_{\mathbf{L}^2}^2] + \frac{1}{2} \|\mathbf{P}_h[\mathbf{e}_{\mathbf{u}}^{n+1} - \mathbf{e}_{\mathbf{u}}^n]\|_{\mathbf{L}^2}^2. \end{aligned} \quad (2.43)$$

Next, using again the fact that $\mathbf{P}_h \mathbf{u}_h^{n+1} = \mathbf{u}_h^{n+1}$ and Schwarz inequality, we obtain

$$\begin{aligned} k(\nabla \mathbf{e}_{\mathbf{u}}^{n+1}, \nabla \mathbf{P}_h \mathbf{e}_{\mathbf{u}}^{n+1}) &= k\|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2 - k(\nabla \mathbf{e}_{\mathbf{u}}^{n+1}, \nabla[\mathbf{u}^{n+1} - \mathbf{P}_h \mathbf{u}^{n+1}]) \\ &\geq \frac{k}{2} \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2 - \frac{k}{2} \|\nabla[\mathbf{u}^{n+1} - \mathbf{P}_h \mathbf{u}^{n+1}]\|_{\mathbf{L}^2}^2 \\ &\geq \frac{k}{2} \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2 - Ckh^2 \|\mathbf{u}^{n+1}\|_{\mathbf{H}^2}^2, \end{aligned} \quad (2.44)$$

where we have used (2.36) to get the last inequality.

For the next term in (2.42), using the fact that $\mathbf{P}_h \mathbf{e}_{\mathbf{u}}^{n+1}$ takes values in $\mathbb{V}_h$, and estimates (2.34), (2.35), and (2.36), we get

$$\begin{aligned} -k(e_r^{n+1}, \operatorname{div} \mathbf{P}_h \mathbf{e}_{\mathbf{u}}^{n+1}) &= -k(r^{n+1}, \operatorname{div} \mathbf{P}_h \mathbf{e}_{\mathbf{u}}^{n+1}) \\ &= -k(r^{n+1} - \rho_h r^{n+1}, \operatorname{div} \mathbf{P}_h \mathbf{e}_{\mathbf{u}}^{n+1}) \\ &\geq -k\|r^{n+1} - \rho_h r^{n+1}\| \|\operatorname{div} \mathbf{P}_h \mathbf{e}_{\mathbf{u}}^{n+1}\| \\ &\geq -Chk \|\nabla r^{n+1}\|_{\mathbf{L}^2} \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2} \\ &\geq -\frac{k}{4} \|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2 - C^2 h^2 k \|\nabla r^{n+1}\|_{\mathbf{L}^2}^2. \end{aligned} \quad (2.45)$$

Finally, we bound the only term on the right-hand side of (2.42) from above. By the independence of the increments $\{\Delta W_{n+1}\}_n$, and its distribution, we get

$$\begin{aligned}\mathbb{E}[(\boldsymbol{\eta}^n - \boldsymbol{\eta}_h^n) \Delta W_{n+1}, \mathbf{P}_h \mathbf{e}_{\mathbf{u}}^{n+1}] &= \mathbb{E}[(\boldsymbol{\eta}^n - \boldsymbol{\eta}_h^n) \Delta W_{n+1}, \mathbf{P}_h [\mathbf{e}_{\mathbf{u}}^{n+1} - \mathbf{e}_{\mathbf{u}}^n]] \\ &\leq k \mathbb{E}[\|\boldsymbol{\eta}^n - \boldsymbol{\eta}_h^n\|_{\mathbf{L}^2}^2] + \frac{1}{4} \mathbb{E}[\|\mathbf{P}_h (\mathbf{e}_{\mathbf{u}}^{n+1} - \mathbf{e}_{\mathbf{u}}^n)\|_{\mathbf{L}^2}^2],\end{aligned}\quad (2.46)$$

and because of (1.5) and (2.36), there holds

$$\begin{aligned}\|\boldsymbol{\eta}^n - \boldsymbol{\eta}_h^n\|_{\mathbf{L}^2}^2 &\leq 2\|\mathbf{B}(\mathbf{u}^n) - \mathbf{B}(\mathbf{u}_h^n)\|_{\mathbf{L}^2}^2 + 2\|\nabla(\xi^n - \xi_h^n)\|_{\mathbf{L}^2}^2 \\ &\leq 2C\|\mathbf{e}_{\mathbf{u}}^n\|_{\mathbf{L}^2}^2 + 2\|\nabla(\xi^n - \xi_h^n)\|_{\mathbf{L}^2}^2 \\ &= 2C\|(\mathbf{u}^n - \mathbf{P}_h \mathbf{u}^n) + \mathbf{P}_h \mathbf{e}_{\mathbf{u}}^n\|_{\mathbf{L}^2}^2 + 2\|\nabla(\xi^n - \xi_h^n)\|_{\mathbf{L}^2}^2 \\ &\leq Ch^4\|\mathbf{u}^n\|_{\mathbf{H}^2}^2 + 4C\|\mathbf{P}_h \mathbf{e}_{\mathbf{u}}^n\|_{\mathbf{L}^2}^2 + 2\|\nabla(\xi^n - \xi_h^n)\|_{\mathbf{L}^2}^2.\end{aligned}\quad (2.47)$$

To control $\|\nabla(\xi^n - \xi_h^n)\|_{\mathbf{L}^2}$, we recall the definitions of ξ^n and ξ_h^n to get

$$(\nabla[\xi^n - \xi_h^n], \nabla \phi_h) = (\mathbf{B}(\mathbf{u}^n) - \mathbf{B}(\mathbf{u}_h^n), \nabla \phi_h) \quad \forall \phi_h \in S_h.$$

Setting $\phi_h = \mathcal{R}_h[\xi^n - \xi_h^n] = (\xi^n - \xi_h^n) - (\xi^n - \mathcal{R}_h \xi^n)$, properties (1.5a) and (2.35) yield

$$\begin{aligned}\|\nabla(\xi^n - \xi_h^n)\|_{\mathbf{L}^2}^2 &\leq (\nabla[\xi^n - \xi_h^n], \nabla[\xi^n - \mathcal{R}_h \xi^n]) + C\|\mathbf{e}_{\mathbf{u}}^n\| \|\nabla(\xi^n - \xi_h^n)\|_{\mathbf{L}^2} \\ &\leq \frac{1}{2} \|\nabla(\xi^n - \xi_h^n)\|_{\mathbf{L}^2}^2 + Ch^2 \|\xi^n\|_{\mathbf{H}^2/\mathbb{R}}^2 + C\|\mathbf{e}_{\mathbf{u}}^n\|_{\mathbf{L}^2}^2.\end{aligned}$$

Hence, by (2.21) in Remark 2.2.1, (2.36), and (1.5c), we get

$$\begin{aligned}\|\nabla(\xi^n - \xi_h^n)\|_{\mathbf{L}^2}^2 &\leq Ch^2 \|\xi^n\|_{\mathbf{H}^2/\mathbb{R}}^2 + C\|\mathbf{e}_{\mathbf{u}}^n\|_{\mathbf{L}^2}^2 \\ &\leq Ch^2 \|\operatorname{div} \mathbf{B}(\mathbf{u}^n)\|_{\mathbf{L}^2}^2 + C\|\mathbf{e}_{\mathbf{u}}^n\|_{\mathbf{L}^2}^2 \\ &\leq Ch^2 \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 + Ch^4 \|\mathbf{u}^n\|_{\mathbf{H}^2}^2 + C\|\mathbf{P}_h \mathbf{e}_{\mathbf{u}}^n\|_{\mathbf{L}^2}^2.\end{aligned}\quad (2.48)$$

Therefore,

$$\|\boldsymbol{\eta}^n - \boldsymbol{\eta}_h^n\|_{\mathbf{L}^2}^2 \leq C \left(h^2 \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 + h^4 \|\mathbf{u}^n\|_{\mathbf{H}^2}^2 + \|\mathbf{P}_h \mathbf{e}_\mathbf{u}^n\|_{\mathbf{L}^2}^2 \right). \quad (2.49)$$

We insert estimates (2.43)–(2.49) into (2.42), take the expectation, and apply the summation operator $\sum_{n=0}^m$ for any $0 \leq m \leq N-1$ to conclude

$$\begin{aligned} & \frac{1}{2} \mathbb{E} [\|\mathbf{P}_h \mathbf{e}_\mathbf{u}^{m+1}\|_{\mathbf{L}^2}^2] + \frac{1}{4} \sum_{n=0}^m \mathbb{E} [\|\mathbf{P}_h (\mathbf{e}_\mathbf{u}^{n+1} - \mathbf{e}_\mathbf{u}^n)\|_{\mathbf{L}^2}^2] + \frac{1}{4} \mathbb{E} \left[k \sum_{n=0}^m \|\nabla \mathbf{e}_\mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 \right] \\ & \leq Ck \sum_{n=0}^m \mathbb{E} [\|\mathbf{P}_h \mathbf{e}_\mathbf{u}^n\|_{\mathbf{L}^2}^2] + Ch^2 \mathbb{E} \left[k \sum_{n=0}^{N-1} \|\nabla r^{n+1}\|_{\mathbf{L}^2}^2 \right] \\ & \quad + Ch^2 \mathbb{E} \left[k \sum_{n=0}^{N-1} \|\nabla \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 \right] + Ch^2 \mathbb{E} \left[k \sum_{n=0}^{N-1} \|\mathbf{u}^{n+1}\|_{\mathbf{H}^2}^2 \right] \\ & \leq Ck \sum_{n=0}^m \mathbb{E} [\|\mathbf{P}_h \mathbf{e}_\mathbf{u}^n\|_{\mathbf{L}^2}^2] + Ch^2, \end{aligned} \quad (2.50)$$

where the last inequality was obtained by using estimates (2.22), (2.23) from Lemma 2.2.1.

Applying the discrete Gronwall inequality to (2.50) then leads to

$$\begin{aligned} & \frac{1}{2} \mathbb{E} [\|\mathbf{P}_h \mathbf{e}_\mathbf{u}^{m+1}\|_{\mathbf{L}^2}^2] + \frac{1}{4} \sum_{n=0}^m \mathbb{E} [\|\mathbf{P}_h (\mathbf{e}_\mathbf{u}^{n+1} - \mathbf{e}_\mathbf{u}^n)\|_{\mathbf{L}^2}^2] + \frac{1}{4} \mathbb{E} \left[k \sum_{n=0}^m \|\nabla \mathbf{e}_\mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 \right] \\ & \leq \exp(CT) Ch^2 \quad (1 \leq m \leq N). \end{aligned} \quad (2.51)$$

Finally, the desired estimate (2.39) follows from an application of the triangle inequality on $\mathbf{e}_\mathbf{u}^{m+1} = (\mathbf{u}^{m+1} - \mathbf{P}_h \mathbf{u}^{m+1}) + \mathbf{P}_h \mathbf{e}_\mathbf{u}^{m+1}$ and using (2.51) and (2.36). The proof is complete. $\square$

2.3.4 Error estimates for the pressure approximation

In this subsection, we derive some error estimates for both, $r^n - r_h^n$ and $p^n - p_h^n$. The argumentation parallels the one in section 2.2.4, and uses the *inf-sup* condition (2.31), in particular.

Theorem 2.3.2. *Suppose that*

$$\mathbb{E}[\|\mathbf{u}^0 - \mathbf{u}_h^0\|_{\mathbf{L}^2}^2] \leq Ch^2.$$

Let $\{(\mathbf{u}^n, r^n); 1 \leq n \leq N\}$ and $\{(\mathbf{u}_h^n, r_h^n); 1 \leq n \leq N\}$ be respectively the solutions of Algorithm 1 and 2. There exists a constant $C > 0$, such that

$$\left(\mathbb{E}\left[\left\|k \sum_{n=1}^N (r^n - r_h^n)\right\|_{L^2}^2\right]\right)^{\frac{1}{2}} \leq Ch.$$

Proof. Summing (2.40) (after lowering the index by one) over $1 \leq n \leq m \leq N$ leads to

$$\begin{aligned} (\mathbf{e}_{\mathbf{u}}^m, \mathbf{v}_h) + k \sum_{n=1}^m (\nabla \mathbf{e}_{\mathbf{u}}^n, \nabla \mathbf{v}_h) - k \sum_{n=1}^m (\operatorname{div} \mathbf{v}_h, e_r^n) \\ = (\mathbf{e}_{\mathbf{u}}^0, \mathbf{v}_h) + \sum_{n=1}^m ([\boldsymbol{\eta}^{n-1} - \boldsymbol{\eta}_h^{n-1}] \Delta W_n, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbb{X}_h. \end{aligned}$$

By (2.31), we conclude (compare with (2.28))

$$\frac{1}{3\gamma} \|k \sum_{n=1}^m e_r^n\|_{L^2} \leq \|\mathbf{e}_{\mathbf{u}}^m\|_{\mathbf{L}^2} + \|\mathbf{e}_{\mathbf{u}}^0\|_{\mathbf{L}^2} + \|k \sum_{n=1}^m \nabla \mathbf{e}_{\mathbf{u}}^m\|_{\mathbf{L}^2} + \left\| \sum_{n=1}^m (\boldsymbol{\eta}^{n-1} - \boldsymbol{\eta}_h^{n-1}) \Delta W_n \right\|_{\mathbf{L}^2}.$$

Taking expectations after squaring both sides, using estimates (2.47), (2.48), (2.49), and Theorem 2.3.1 yield the desired result. $\square$

The following result now is a simply corollary of Theorem 2.3.2.

Corollary 2.3.3. *Let $\{p_h^n\}_n$ be the solution in Algorithm 2. Then there exists a constant $C > 0$, such that*

$$\left(\mathbb{E}\left[\left\|k \sum_{n=1}^N (p^n - p_h^n)\right\|_{L^2}^2\right]\right)^{\frac{1}{2}} \leq Ch.$$

2.3.5 Space-time error estimates for Algorithm 2

Theorems 2.2.2, 2.2.3, 2.3.1, 2.3.2, and Corollaries 2.2.4, 2.3.3 now provide the following global error estimates.

Theorem 2.3.4. *Let $(\mathbf{u}, P)$ solve (1.3), and $\{(\mathbf{u}_h^n, r_h^n, p_h^n); 1 \leq n \leq N\}$ solves Algorithm 2. There exists a constant $C > 0$,*

$$\begin{aligned} \text{(i)} \quad & \max_{1 \leq n \leq N} \left(\mathbb{E} [\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|_{\mathbf{L}^2}^2] \right)^{\frac{1}{2}} + \left(\mathbb{E} \left[k \sum_{n=1}^N \|\nabla(\mathbf{u}(t_n) - \mathbf{u}_h^n)\|_{\mathbf{L}^2}^2 \right] \right)^{\frac{1}{2}} \leq C(k^{\frac{1}{2}} + h), \\ \text{(ii)} \quad & \left(\mathbb{E} \left[\|R(t_m) - k \sum_{n=1}^m r_h^n\|_{L^2}^2 \right] \right)^{\frac{1}{2}} + \left(\mathbb{E} \left[\|P(t_m) - k \sum_{n=1}^m p_h^n\|_{L^2}^2 \right] \right)^{\frac{1}{2}} \leq C(k^{\frac{1}{2}} + h), \end{aligned}$$

for all $1 \leq m \leq N$.

2.4 Stabilization methods for the stochastic Stokes equations

The scheme in section 2.3 requires *inf-sup* stable pairings $(\mathbb{X}_h, W_h)$, for which the Taylor-Hood mixed finite element is one example. By recalling its definition in subsection 2.3.1, we observe that the dimension of $\mathbb{X}_h$ exceeds that of W_h . The motivation for the stabilization methods in [45] is to relax the *inf-sup* stability criterion for pairings of ansatz spaces in order to allow for equal-order ansatz spaces for both velocity and pressure approximates; see [45, 10, 37, 24] for further details.

Below we replace $\mathbb{X}_h$ defined in subsection 2.3.1 by

$$\mathbb{Y}_h = \{ \mathbf{v}_h \in \mathbf{H}_{per}^1(D); \mathbf{v}_h|_K \in \mathbf{P}_1(K) \ \forall K \in \mathcal{T}_h \},$$

to which we associate the L_{per}^2 -projection operator $\mathbf{Q}_h : \mathbf{L}_{per}^2(D) \rightarrow \mathbb{Y}_h$ by

$$(\mathbf{v} - \mathbf{Q}_h \mathbf{v}, \mathbf{w}_h) = 0 \quad \forall \mathbf{v} \in \mathbf{L}_{per}^2(D), \ \mathbf{w}_h \in \mathbb{Y}_h,$$

which satisfies the following approximation property (cf. [24]):

$$\|\mathbf{v} - \mathbf{Q}_h \mathbf{v}\|_{\mathbf{L}^2} + h \|\nabla(\mathbf{v} - \mathbf{Q}_h \mathbf{v})\|_{\mathbf{L}^2} \leq Ch^s \|\mathbf{v}\|_{\mathbf{H}^s} \quad \forall \mathbf{v} \in \mathbf{H}_{per}^s(D) \quad (2.52)$$

for $s = 1, 2$. Here, $C > 0$ is a constant independent of h . Moreover, let W_h be the same as in section 2.3, and $\tilde{\mathcal{R}}_h$ denote the Ritz projection from $H_{per}^1(D)/\mathbb{R}$ to W_h . Again, we take $S_h = W_h$ in this section.

In this section, we consider the equal-order pairing $(\mathbb{Y}_h, W_h)$ to discretize (1.3) based on (2.8a)–(2.8b), which violates the *inf-sup* condition; in fact, the following estimate is known to hold (cf. [45]): there exists $\delta > 0$ independent of $h > 0$, such that

$$\frac{1}{\delta^2} \|q_h\|^2 \leq \sup_{\mathbf{v}_h \in \mathbb{Y}_h} \frac{|(q_h, \operatorname{div} \mathbf{v}_h)|^2}{\|\nabla \mathbf{v}_h\|_{\mathbf{L}^2}^2} + h^2 \|\nabla q_h\|_{\mathbf{L}^2}^2 \quad \forall q_h \in W_h. \quad (2.53)$$

Equation (2.53) can be regarded as the reason why this pairing still performs optimally when applied to the Stokes problem, where $\varepsilon = \mathcal{O}(h^2)$. Below we show that such a strategy can be again successful for the stochastic Stokes problem (1.3), if the *proper* pressure is chosen for the perturbation, and that using the Helmholtz projection of the noise provides such an approach.

To prepare for the analysis, we start with a modification of Algorithm 1 that perturbs the incompressibility constraint.

Algorithm 3

Let $0 < \varepsilon \ll 1$ and $\mathbf{u}_\varepsilon^0 = \mathbf{u}_0$. For $n = 0, 1, \dots, N-1$, do the following steps:

Step 1: Find $\xi_\varepsilon^n \in L^2(\Omega, H_{per}^1(D)/\mathbb{R})$ by solving

$$(\nabla \xi_\varepsilon^n, \nabla \phi) = (\mathbf{B}(\mathbf{u}_\varepsilon^n), \nabla \phi) \quad \forall \phi \in H_{per}^1(D). \quad (2.54)$$

Step 2: Set $\boldsymbol{\eta}_\varepsilon^n := \mathbf{B}(\mathbf{u}_\varepsilon^n) - \nabla \xi_\varepsilon^n$, and find $(\mathbf{u}_\varepsilon^{n+1}, r_\varepsilon^{n+1}) \in L^2(\Omega, \mathbf{H}_{per}^1(D) \times H_{per}^1(D)/\mathbb{R})$ by solving

$$(\mathbf{u}_\varepsilon^{n+1}, \mathbf{v}) + k(\nabla \mathbf{u}_\varepsilon^{n+1}, \nabla \mathbf{v}) - k(\operatorname{div} \mathbf{v}, r_\varepsilon^{n+1}) \quad (2.55a)$$

$$= (\mathbf{u}_\varepsilon^n, \mathbf{v}) + k(\mathbf{f}^{n+1}, \mathbf{v}) + (\boldsymbol{\eta}_\varepsilon^n \Delta W_{n+1}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_{per}^1(D),$$

$$(\operatorname{div} \mathbf{u}_\varepsilon^{n+1}, q) + \varepsilon(\nabla r_\varepsilon^{n+1}, \nabla q) = 0 \quad \forall q \in H_{per}^1(D). \quad (2.55b)$$

Step 3: Define $p_\varepsilon^{n+1} := r_\varepsilon^{n+1} + k^{-1} \xi_\varepsilon^n \Delta W_{n+1}$.

Because each step involves a coercive linear problem, Algorithm 3 has a unique solution. The first energy estimate can be obtained as follow:

Lemma 2.4.1. *Let $\{(u_\varepsilon^n, r_\varepsilon^n)\}_n$ be the solution of Algorithm 3. Then there exists a constant $C > 0$, such that*

$$\begin{aligned} \max_{0 \leq n \leq N-1} \frac{1}{2} \mathbb{E}[\|\mathbf{u}_\varepsilon^{n+1}\|^2] + k \sum_{n=0}^{N-1} \mathbb{E}[\|\nabla \mathbf{u}_\varepsilon^{n+1}\|^2 + \varepsilon \|\nabla r_\varepsilon^{n+1}\|^2] \\ \leq C \left(\mathbb{E}[\|\mathbf{u}_0\|^2] + k \sum_{n=0}^{N-1} \mathbb{E}[\|\mathbf{f}^{n+1}\|^2] \right). \end{aligned} \quad (2.56)$$

Proof. By fixing one $\omega \in \Omega$ and choosing $(\mathbf{v}, q) = (\mathbf{u}_\varepsilon^{n+1}(\omega), r_\varepsilon^{n+1}(\omega))$ in (2.55a), we then obtain the identity

$$\begin{aligned} \frac{1}{2} \left(\|\mathbf{u}_\varepsilon^{n+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{u}_\varepsilon^n\|_{\mathbf{L}^2}^2 + \|\mathbf{u}_\varepsilon^{n+1} - \mathbf{u}_\varepsilon^n\|_{\mathbf{L}^2}^2 \right) \\ + \frac{k}{2} \|\nabla \mathbf{u}_\varepsilon^{n+1}\|_{\mathbf{L}^2}^2 + k\varepsilon \|\nabla r_\varepsilon^{n+1}\|_{\mathbf{L}^2}^2 = \frac{Ck}{2} \|\mathbf{f}^{n+1}\|_{\mathbf{L}^2}^2 + (\boldsymbol{\eta}_\varepsilon^n \Delta W_{n+1}, \mathbf{u}_\varepsilon^{n+1}). \end{aligned} \quad (2.57)$$

Taking expectations, applying the summation operator $\sum_{n=0}^m$ for any $0 \leq m \leq N-1$, and using the independence of the increments $\{\Delta W_{n+1}\}_n$ yield

$$\begin{aligned} \mathbb{E}[(\boldsymbol{\eta}_\varepsilon^n \Delta W_{n+1}, \mathbf{u}_\varepsilon^{n+1})] &= \mathbb{E}[(\boldsymbol{\eta}_\varepsilon^n \Delta W_{n+1}, [\mathbf{u}_\varepsilon^{n+1} - \mathbf{u}_\varepsilon^n])] \\ &\leq k \mathbb{E}[\|\boldsymbol{\eta}_\varepsilon^n\|_{\mathbf{L}^2}^2] + \frac{1}{4} \mathbb{E}[\|\mathbf{u}_\varepsilon^{n+1} - \mathbf{u}_\varepsilon^n\|_{\mathbf{L}^2}^2]. \end{aligned} \quad (2.58)$$

Because of (2.54) and (1.5b), we have

$$\mathbb{E}[\|\nabla \xi_\varepsilon^n\|_{\mathbf{L}^2}^2] \leq \mathbb{E}[\|\mathbf{B}(\mathbf{u}_\varepsilon^n)\|_{\mathbf{L}^2}^2] \leq C(1 + \mathbb{E}[\|\mathbf{u}_\varepsilon^n\|_{\mathbf{L}^2}^2]),$$

and therefore $\mathbb{E}[\|\boldsymbol{\eta}_\varepsilon^n\|_{\mathbf{L}^2}^2] \leq C(1 + \mathbb{E}[\|\mathbf{u}_\varepsilon^n\|_{\mathbf{L}^2}^2])$ in (2.58). We insert these auxiliary estimates into (2.57), take expectations, sum over all time steps, and use the discrete Gronwall inequality to get the desired estimate. $\square$

Note that the estimate for $\{\nabla r_\varepsilon^{n+1}\}$ is scaled by $\varepsilon > 0$, which is too weak in the following to verify optimal error estimates for a spatial discretization of Algorithm 3. The following

stability result therefore sharpens the estimate (2.56); its proof crucially exploits again the fact that each $\boldsymbol{\eta}_\varepsilon^n$ is a $\mathbb{H}$ -valued random variable.

Lemma 2.4.2. *Let $\{(u_\varepsilon^n, r_\varepsilon^n)\}_n$ be the solution of Algorithm 3. Then there exists a constant $C > 0$, such that*

$$\max_{1 \leq n \leq N} \mathbb{E}[\|\nabla \mathbf{u}_\varepsilon^n\|_{\mathbf{L}^2}^2] + k \sum_{n=1}^N \mathbb{E}[\|\Delta \mathbf{u}_\varepsilon^n\|_{\mathbf{L}^2}^2] + k \sum_{n=1}^N \mathbb{E}[\|\nabla r_\varepsilon^n\|_{\mathbf{L}^2}^2] \leq C \quad (2.59)$$

Proof. Step 1: We adapt the argumentation from [16, Thm. 3.1], and interpret problem (2.8) — with $\boldsymbol{\eta}^n$ being replaced by $\boldsymbol{\eta}_\varepsilon^n$ — as perturbation of problem (2.6). Subtracting the corresponding equations of both systems and denoting $\mathbf{e}_\mathbf{u}^{n+1} := \mathbf{u}^{n+1} - \mathbf{u}_\varepsilon^{n+1}$ resp. $e_r^{n+1} := r^{n+1} - r_\varepsilon^{n+1}$ yield

$$\mathbf{e}_\mathbf{u}^{n+1} - k \Delta \mathbf{e}_\mathbf{u}^{n+1} + k \nabla e_r^{n+1} = \mathbf{e}_\mathbf{u}^n + [\boldsymbol{\eta}^n - \boldsymbol{\eta}_\varepsilon^n] \Delta W_{n+1} \quad \text{in } D, \quad (2.60a)$$

$$\operatorname{div} \mathbf{e}_\mathbf{u}^{n+1} - \varepsilon \Delta e_r^{n+1} = -\varepsilon \Delta r^{n+1} \quad \text{in } D. \quad (2.60b)$$

Now fix one $\omega \in \Omega$, test (2.60a) with $\mathbf{v} = \mathbf{e}_\mathbf{u}^{n+1}(\omega)$, and (2.60b) with $q = e_r^{n+1}(\omega)$, and afterwards sum both equations, we then conclude

$$\begin{aligned} & \frac{1}{2} \left(\|\mathbf{e}_\mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{e}_\mathbf{u}^n\|_{\mathbf{L}^2}^2 + \|\mathbf{e}_\mathbf{u}^{n+1} - \mathbf{e}_\mathbf{u}^n\|_{\mathbf{L}^2}^2 \right) + k \|\nabla \mathbf{e}_\mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 + k \varepsilon \|\nabla e_r^{n+1}\|_{\mathbf{L}^2}^2 \\ & = ([\boldsymbol{\eta}^n - \boldsymbol{\eta}_\varepsilon^n] \Delta W_{n+1}, \mathbf{e}_\mathbf{u}^{n+1}) + k \varepsilon (\nabla r^{n+1}, \nabla e_r^{n+1}). \end{aligned} \quad (2.61)$$

Using Young's inequality, hiding one part of the last term into the corresponding term on the left-hand side, using the independence of ΔW_{n+1} and $\mathbf{e}_\mathbf{u}^n$, ΔW_{n+1} as well as of $[\boldsymbol{\eta}^n - \boldsymbol{\eta}_\varepsilon^n]$, and utilizing (1.5), we obtain

$$\begin{aligned} \mathbb{E}[([\boldsymbol{\eta}^n - \boldsymbol{\eta}_\varepsilon^n] \Delta W_{n+1}, \mathbf{e}_\mathbf{u}^{n+1})] &= \mathbb{E}[([\boldsymbol{\eta}^n - \boldsymbol{\eta}_\varepsilon^n] \Delta W_{n+1}, \mathbf{e}_\mathbf{u}^{n+1} - \mathbf{e}_\mathbf{u}^n)] \\ &\leq k \mathbb{E}[\|\boldsymbol{\eta}^n - \boldsymbol{\eta}_\varepsilon^n\|_{\mathbf{L}^2}^2] + \frac{1}{2} \mathbb{E}[\|\mathbf{e}_\mathbf{u}^{n+1} - \mathbf{e}_\mathbf{u}^n\|_{\mathbf{L}^2}^2] \\ &\leq Ck \left(\mathbb{E}[\|\mathbf{e}_\mathbf{u}^n\|_{\mathbf{L}^2}^2] + \mathbb{E}[\|\nabla(\xi^n - \xi_\varepsilon^n)\|_{\mathbf{L}^2}^2] \right) + \frac{1}{2} \mathbb{E}[\|\mathbf{e}_\mathbf{u}^{n+1} - \mathbf{e}_\mathbf{u}^n\|_{\mathbf{L}^2}^2]. \end{aligned}$$

Subtracting (2.54) from (2.18) and using (1.5), we get

$$\|\nabla(\xi^n - \xi_\varepsilon^n)\|_{\mathbf{L}^2} \leq \|\mathbf{B}(\mathbf{u}^n) - \mathbf{B}(\mathbf{u}^\varepsilon)\|_{\mathbf{L}^2} \leq C\|\mathbf{e}_\mathbf{u}^n\|_{\mathbf{L}^2}.$$

Hence, we then conclude from (2.61) with the help of the discrete Gronwall inequality, and (2.23) that

$$\begin{aligned} & \frac{1}{2}\mathbb{E}[\|\mathbf{e}_\mathbf{u}^{m+1}\|_{\mathbf{L}^2}^2] + \sum_{n=0}^m \mathbb{E}[\|\mathbf{e}_\mathbf{u}^{n+1} - \mathbf{e}_\mathbf{u}^n\|_{\mathbf{L}^2}^2] + k \sum_{n=0}^m \mathbb{E}[\|\nabla \mathbf{e}_\mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 + \varepsilon \|\nabla e_r^{n+1}\|_{\mathbf{L}^2}^2] \\ & \leq C\varepsilon k \sum_{n=0}^m \mathbb{E}[\|\nabla r^{n+1}\|_{\mathbf{L}^2}^2] \leq C\varepsilon \quad (0 \leq m \leq N). \end{aligned}$$

Consequently, by (2.23) we conclude that

$$k \sum_{n=0}^N \mathbb{E}[\|\nabla e_r^{n+1}\|_{\mathbf{L}^2}^2] \leq C \quad \text{implies} \quad k \sum_{n=0}^N \mathbb{E}[\|\nabla r_\varepsilon^{n+1}\|_{\mathbf{L}^2}^2] \leq C.$$

Step 2: Fix one $\omega \in \Omega$ in (2.8a), multiply the equation with $-\Delta \mathbf{u}_\varepsilon^{n+1}(\omega)$, integrate, perform an integration by parts on the last term, and use the periodicity of $\boldsymbol{\eta}_\varepsilon^n$ and $\mathbf{u}_\varepsilon^{n+1}$, we get

$$\begin{aligned} & \frac{1}{2}\mathbb{E}\left[\|\nabla \mathbf{u}_\varepsilon^{n+1}\|_{\mathbf{L}^2}^2 - \|\nabla \mathbf{u}_\varepsilon^n\|_{\mathbf{L}^2}^2 + \|\nabla(\mathbf{u}_\varepsilon^{n+1} - \mathbf{u}_\varepsilon^n)\|_{\mathbf{L}^2}^2 + k\|\Delta \mathbf{u}_\varepsilon^{n+1}\|_{\mathbf{L}^2}^2\right] \\ & \leq k\mathbb{E}[\|\nabla r_\varepsilon^{n+1}\|_{\mathbf{L}^2}^2] + \mathbb{E}\left[\int_D \Delta W_{n+1} \nabla \boldsymbol{\eta}_\varepsilon^n \cdot \nabla(\mathbf{u}_\varepsilon^{n+1} - \mathbf{u}_\varepsilon^n) d\mathbf{x}\right]. \end{aligned} \quad (2.62)$$

To bound the last term above, we use Schwarz inequality, the fact that $\boldsymbol{\eta}_\varepsilon^n = \mathbf{B}(\mathbf{u}_\varepsilon^n) - \nabla \xi_\varepsilon^n$, (1.5), and (2.21) to get

$$\begin{aligned} \mathbb{E}\left[\int_D \Delta W_{n+1} \nabla \boldsymbol{\eta}_\varepsilon^n \cdot \nabla(\mathbf{u}_\varepsilon^{n+1} - \mathbf{u}_\varepsilon^n) d\mathbf{x}\right] & \leq \frac{1}{4}\mathbb{E}[\|\nabla(\mathbf{u}_\varepsilon^{n+1} - \mathbf{u}_\varepsilon^n)\|_{\mathbf{L}^2}^2] \\ & \quad + Ck\mathbb{E}[\|\nabla \mathbf{u}_\varepsilon^n\|_{\mathbf{L}^2}^2]. \end{aligned} \quad (2.63)$$

Substituting (2.63) into (2.62), summing over all time steps, and using (2.56) and the result of *Step 1*, we get the desired estimate (2.59). The proof is complete. $\square$

From *Step 1* of the above proof, we also obtain the following result.

Theorem 2.4.3. *Let $\{(\mathbf{u}^{n+1}, r^{n+1})\}_n$ and $\{(\mathbf{u}_\varepsilon^n, r_\varepsilon^n)\}_n$ be the solutions of Algorithm 1 and 3, respectively. Then there exists a constant $C > 0$, such that*

$$\begin{aligned} \max_{1 \leq n \leq N} \mathbb{E}[\|\mathbf{u}^n - \mathbf{u}_\varepsilon^n\|_{\mathbf{L}^2}^2] + k \sum_{n=1}^N \mathbb{E}[\|\nabla(\mathbf{u}^n - \mathbf{u}_\varepsilon^n)\|_{\mathbf{L}^2}^2] \\ + \varepsilon \|\nabla(r^n - r_\varepsilon^n)\|_{\mathbf{L}^2}^2] \leq C\varepsilon. \end{aligned} \quad (2.64)$$

We are ready to bound the error between the pressures $\{r^n\}_n$ and $\{r_\varepsilon^n\}_n$; the proof of it uses (2.60a) after summation in time, and follows along the lines of the proof of Theorem 2.2.3, using the stability of the divergence operator (cf. estimate (2.28)), and Theorem 2.4.3.

Theorem 2.4.4. *Let $\{r^n; 1 \leq n \leq N\}$ be generated by Algorithm 1 and $\{r_\varepsilon^n; 1 \leq n \leq N\}$ by Algorithm 3. There exists a constant $C > 0$, such that for $m = 1, 2, \dots, N$,*

$$\left(\mathbb{E} \left[\left\| k \sum_{n=1}^m r^n - k \sum_{n=1}^m r_\varepsilon^n \right\|_{L^2}^2 + \left\| k \sum_{n=1}^m p^n - k \sum_{n=1}^m p_\varepsilon^n \right\|_{L^2}^2 \right] \right)^{\frac{1}{2}} \leq C\varepsilon.$$

Next, we present the following modification of Algorithm 2.

Algorithm 4

Let $0 < \varepsilon \ll 1$ and $\mathbf{u}_{\varepsilon,h}^0 \in L^2(\Omega; \mathbb{Y}_h)$. For $n = 0, 1, \dots, N-1$, do the following steps:

Step 1: Determine $\xi_{\varepsilon,h}^n \in L^2(\Omega; S_h)$ from

$$(\nabla \xi_{\varepsilon,h}^n, \nabla \phi_h) = (\mathbf{B}(\mathbf{u}_{\varepsilon,h}^n), \nabla \phi_h) \quad \forall \phi_h \in S_h. \quad (2.65)$$

Step 2: Set $\boldsymbol{\eta}_{\varepsilon,h}^n := \mathbf{B}(\mathbf{u}_{\varepsilon,h}^n) - \nabla \xi_{\varepsilon,h}^n$. Find $(\mathbf{u}_{\varepsilon,h}^{n+1}, r_{\varepsilon,h}^{n+1}) \in L^2(\Omega, \mathbb{Y}_h \times W_h)$ by solving

$$(\mathbf{u}_{\varepsilon,h}^{n+1}, \mathbf{v}_h) + k(\nabla \mathbf{u}_{\varepsilon,h}^{n+1}, \nabla \mathbf{v}_h) - k(\operatorname{div} \mathbf{v}_h, r_{\varepsilon,h}^{n+1}) \quad (2.66a)$$

$$= (\mathbf{u}_{\varepsilon,h}^n, \mathbf{v}_h) + k(\mathbf{f}^{n+1}, \mathbf{v}_h) + (\boldsymbol{\eta}_{\varepsilon,h}^n \Delta W_{n+1}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbb{Y}_h,$$

$$(\operatorname{div} \mathbf{u}_{\varepsilon,h}^{n+1}, q_h) + \varepsilon(\nabla \mathbf{u}_{\varepsilon,h}^{n+1}, \nabla q_h) = 0 \quad \forall q_h \in W_h. \quad (2.66b)$$

Step 3: Define the W_h -valued random variable $p_{\varepsilon,h}^{n+1} = r_{\varepsilon,h}^{n+1} + k^{-1} \xi_{\varepsilon,h}^n \Delta W_{n+1}$.

The main result of this section is the following estimate for the velocity error $\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon,h}^n$.

Theorem 2.4.5. *Suppose*

$$\mathbb{E}[\|\mathbf{u}^0 - \mathbf{u}_{\varepsilon,h}^0\|_{\mathbf{L}^2}^2] \leq Ch^2.$$

Let $\{(\mathbf{u}_\varepsilon^n, r_\varepsilon^n); 1 \leq n \leq N\}$ and $\{(\mathbf{u}_{\varepsilon,h}^n, r_{\varepsilon,h}^n); 1 \leq n \leq N\}$ be the solutions of Algorithm 3 and 4, respectively. Then there exists a constant $C > 0$, such that

$$\begin{aligned} \max_{1 \leq n \leq N} \left(\mathbb{E}[\|\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon,h}^n\|_{\mathbf{L}^2}^2] \right)^{\frac{1}{2}} + \left(\mathbb{E} \left[k \sum_{n=1}^N \|\nabla(\mathbf{u}_\varepsilon^n - \mathbf{u}_{\varepsilon,h}^n)\|_{\mathbf{L}^2}^2 \right] \right)^{\frac{1}{2}} \\ \leq C \left(h + \frac{h^2}{\sqrt{\varepsilon}} \right). \end{aligned} \quad (2.67)$$

This estimate suggests that $\varepsilon = \mathcal{O}(h^2)$ is the optimal choice of ε .

Proof. Let $\mathbf{e}_\mathbf{u}^{n+1} := \mathbf{u}_\varepsilon^{n+1} - \mathbf{u}_{\varepsilon,h}^{n+1}$ and $e_r^{n+1} := r_\varepsilon^{n+1} - r_{\varepsilon,h}^{n+1}$. Then $\{(\mathbf{e}_\mathbf{u}^n, e_r^n)\}_n$ satisfies the following error equations $\mathbb{P}$ -a.s. for all tuple $(\mathbf{v}_h, q_h) \in \mathbb{Y}_h \times W_h$,

$$(\mathbf{e}_\mathbf{u}^{n+1} - \mathbf{e}_\mathbf{u}^n, \mathbf{v}_h) + k(\nabla \mathbf{e}_\mathbf{u}^{n+1}, \nabla \mathbf{v}_h) + k(\nabla e_r^{n+1}, \mathbf{v}_h) = ([\boldsymbol{\eta}_\varepsilon^n - \boldsymbol{\eta}_{\varepsilon,h}^n] \Delta W_{n+1}, \mathbf{v}_h), \quad (2.68)$$

$$(\operatorname{div} \mathbf{e}_\mathbf{u}^{n+1}, q_h) + \varepsilon(\nabla e_r^{n+1}, \nabla q_h) = 0. \quad (2.69)$$

Now consider (2.68)–(2.69) for $\omega \in \Omega$ fixed, and choose

$$(\mathbf{v}_h, q_h) = (\mathbf{Q}_h \mathbf{e}_\mathbf{u}^{n+1}(\omega), \tilde{\mathcal{R}}_h e_r^{n+1}(\omega)) \in \mathbb{Y}_h \times W_h;$$

we then deduce

$$\begin{aligned} (\mathbf{e}_\mathbf{u}^{n+1} - \mathbf{e}_\mathbf{u}^n, \mathbf{Q}_h \mathbf{e}_\mathbf{u}^{n+1}) + k(\nabla \mathbf{e}_\mathbf{u}^{n+1}, \nabla \mathbf{Q}_h \mathbf{e}_\mathbf{u}^{n+1}) - k(e_r^{n+1}, \operatorname{div} \mathbf{Q}_h \mathbf{e}_\mathbf{u}^{n+1}) \\ = ([\boldsymbol{\eta}_\varepsilon^n - \boldsymbol{\eta}_{\varepsilon,h}^n] \Delta W_{n+1}, \mathbf{Q}_h \mathbf{e}_\mathbf{u}^{n+1}). \end{aligned} \quad (2.70)$$

We can adopt the corresponding arguments in (2.43) and (2.44), and use Lemma 2.4.2 to treat the first two terms in (2.70), and also the argument around (2.46) may easily be adopted to the present setting. But a different treatment is required to deal with the last term on the left-hand side of (2.70) because it involves the error in the pressure. We rewrite this term as

follows:

$$\begin{aligned}
\text{II} &:= \left(e_r^{n+1}, \operatorname{div} [\mathbf{e}_{\mathbf{u}}^{n+1} + (\mathbf{Q}_h \mathbf{u}_{\varepsilon}^{n+1} - \mathbf{u}_{\varepsilon}^{n+1})] \right) \\
&= (\tilde{\mathcal{R}}_h e_r^{n+1}, \operatorname{div} \mathbf{e}_{\mathbf{u}}^{n+1}) + (r_{\varepsilon}^{n+1} - \tilde{\mathcal{R}}_h r_{\varepsilon}^{n+1}, \operatorname{div} \mathbf{e}_{\mathbf{u}}^{n+1}) + (e_r^{n+1}, \operatorname{div} [\mathbf{Q}_h \mathbf{u}_{\varepsilon}^{n+1} - \mathbf{u}_{\varepsilon}^{n+1}]) \\
&=: \text{II}_1 + \text{II}_2 + \text{II}_3.
\end{aligned}$$

We estimate II_2 with the help of (2.52) and using Lemma 2.4.2,

$$\mathbb{E}[|\text{II}_2|] \leq Ch^2 \mathbb{E}[\|\nabla r_{\varepsilon}^{n+1}\|_{\mathbf{L}^2}^2] + \frac{1}{4} \mathbb{E}[\|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2].$$

Integrating by parts in II_3 , using (2.52) and again Lemma 2.4.2 yield

$$\begin{aligned}
|\text{II}_3| &= |(\tilde{\mathcal{R}}_h e_r^{n+1}, \operatorname{div} [\mathbf{Q}_h \mathbf{u}_{\varepsilon}^{n+1} - \mathbf{u}_{\varepsilon}^{n+1}])| + |(r_{\varepsilon}^{n+1} - \tilde{\mathcal{R}}_h r_{\varepsilon}^{n+1}, \operatorname{div} [\mathbf{Q}_h \mathbf{u}_{\varepsilon}^{n+1} - \mathbf{u}_{\varepsilon}^{n+1}])| \\
&\leq \frac{\varepsilon}{4} \|\nabla \tilde{\mathcal{R}}_h e_r^{n+1}\|_{\mathbf{L}^2}^2 + \frac{Ch^4}{\varepsilon} \|\Delta \mathbf{u}_{\varepsilon}^{n+1}\|_{\mathbf{L}^2}^2 + Ch^2 \|\nabla r_{\varepsilon}^{n+1}\|_{\mathbf{L}^2} \|\Delta \mathbf{u}_{\varepsilon}^{n+1}\|_{\mathbf{L}^2}.
\end{aligned}$$

Because of (2.69), we have

$$\text{II}_1 = -\varepsilon (\nabla e_r^{n+1}, \nabla [\tilde{\mathcal{R}}_h e_r^{n+1}]) = -\varepsilon \|\nabla \tilde{\mathcal{R}}_h e_r^{n+1}\|_{\mathbf{L}^2}^2.$$

Putting the above auxiliary estimates together, we obtain that there exists some h - and ε -independent constant $C > 0$ such that

$$\begin{aligned}
&\frac{1}{2} \mathbb{E}[\|\mathbf{Q}_h \mathbf{e}_{\mathbf{u}}^{m+1}\|_{\mathbf{L}^2}^2] + \frac{1}{4} \sum_{n=0}^m \mathbb{E}[\|\mathbf{Q}_h (\mathbf{e}_{\mathbf{u}}^{n+1} - \mathbf{e}_{\mathbf{u}}^n)\|_{\mathbf{L}^2}^2] \\
&\quad + \frac{1}{4} k \sum_{n=0}^m \mathbb{E}[\|\nabla \mathbf{e}_{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2 + \varepsilon \|\nabla \tilde{\mathcal{R}}_h e_r^{n+1}\|_{\mathbf{L}^2}^2] \leq C \left(h^2 + \frac{h^4}{\varepsilon} \right)
\end{aligned} \tag{2.71}$$

for every $0 \leq m \leq N$. The desired estimates (2.67) then follows from an application of the discrete Gronwall inequality. The proof is complete. $\square$

The last result gives an estimate for the pressure approximation error. Using (2.53), equation (2.68) after taking a summation in n , and Lemma 2.4.2, we obtain

Theorem 2.4.6. *Let $\{r_\varepsilon^n; 1 \leq n \leq N\}$ be the pressure in Algorithm 3, and $\{r_{\varepsilon,h}^n; 1 \leq n \leq N\}$ be the pressure in Algorithm 4. There exists a constant $C > 0$, such that for all $1 \leq m \leq N$*

$$\left(\mathbb{E} \left[\left\| k \sum_{n=1}^m r_\varepsilon^n - k \sum_{n=1}^m r_{\varepsilon,h}^n \right\|_{L^2}^2 + \left\| k \sum_{n=1}^m p_\varepsilon^n - k \sum_{n=1}^m p_{\varepsilon,h}^n \right\|_{L^2}^2 \right] \right)^{\frac{1}{2}} \leq C \left(h + \frac{h^2}{\sqrt{\varepsilon}} \right).$$

To sum up the results in this section, we have shown the following error estimates for Algorithm 4.

Theorem 2.4.7. *Let $(\mathbf{u}, P)$ be the solution of (1.3) and $\{(\mathbf{u}_{\varepsilon,h}^n, r_{\varepsilon,h}^n, p_{\varepsilon,h}^n); 1 \leq n \leq N\}$ be the solution of Algorithm 4. There exists a constant $C > 0$, such that*

$$\begin{aligned} \text{(i)} \quad & \max_{1 \leq n \leq N} \left(\mathbb{E} \left[\left\| \mathbf{u}(t_n) - \mathbf{u}_{\varepsilon,h}^n \right\|_{\mathbf{L}^2}^2 \right] \right)^{\frac{1}{2}} + \left(\mathbb{E} \left[k \sum_{n=1}^N \left\| \nabla(\mathbf{u}(t_n) - \mathbf{u}_{\varepsilon,h}^n) \right\|_{\mathbf{L}^2}^2 \right] \right)^{\frac{1}{2}} \\ & \leq C \left(k^{\frac{1}{2}} + h + \frac{h^2}{\sqrt{\varepsilon}} \right), \\ \text{(ii)} \quad & \left(\mathbb{E} \left[\left\| R(t_m) - k \sum_{n=1}^m r_{\varepsilon,h}^n \right\|_{L^2}^2 \right] \right)^{\frac{1}{2}} + \left(\mathbb{E} \left[\left\| P(t_m) - k \sum_{n=1}^m p_{\varepsilon,h}^n \right\|_{L^2}^2 \right] \right)^{\frac{1}{2}} \\ & \leq C \left(k^{\frac{1}{2}} + h + \frac{h^2}{\sqrt{\varepsilon}} \right). \end{aligned}$$

2.5 Computational experiments

We present computational results to validate the theoretical error estimates in Theorem 2.3.4, and evidence how crucial the numerical treatment of the pressure part in the noise is to obtain an optimally convergent mixed method for (1.3). Our computations are done using the software packages FreeFem++ [42] and Matlab, and the physical domain of all experiments is taken to be $D = (0, 1)^2$, i.e., $L = 1$.

Specifically, we use Algorithm 2 to compute the solution of the following initial-(Dirichlet) boundary value problem:

$$d\mathbf{u} = [\Delta\mathbf{u} - \nabla p + \mathbf{f}]dt + B(\mathbf{u})d\mathbf{W}(t) \quad \text{in } D_T := (0, T) \times D, \quad (2.72a)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } D_T, \quad (2.72b)$$

$$\mathbf{u} = 0 \quad \text{on } \partial D_T := (0, T) \times \partial D, \quad (2.72c)$$

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } D, \quad (2.72d)$$

Test 1. Let $\mathbf{u}_0 = (0, 0)$, and $B(\mathbf{u}) = 5(u_1^2 + u_2^2 + 1)^{\frac{1}{2}}$, which is non-solenoidal. We choose $\mathbf{W}$ in (1.3) to be a $\mathbb{R}^J$ -valued Wiener process, with increment

$$\mathbf{W}^J(t_{n+1}, \mathbf{x}) - \mathbf{W}^J(t_n, \mathbf{x}) = \sqrt{k_0} \sum_{j_1=1}^J \sum_{j_2=1}^J \sqrt{\lambda_{j_1, j_2}} \mathbf{e}_{j_1, j_2}(\mathbf{x}) \xi_{j_1, j_2}^n, \quad (2.73)$$

where $\mathbf{x} = (x, y) \in D$, $\xi_{j_1, j_2}^n \sim N(0, 1)$, $\lambda_{j_1, j_2} = \frac{1}{(j_1 + j_2)^2}$, and

$$\mathbf{e}_{j_1, j_2}(\mathbf{x}) = (\sin(j_1 \pi x) \sin(j_2 \pi y), \sin(j_1 \pi x) \sin(j_2 \pi y)). \quad (2.74)$$

We use the following parameters: $J = 4$ and $T = 1$, and take $N_p = 400$ to be the number of realizations in this test. In addition, the body force is $\mathbf{f} = (f_1, f_2)$ with

$$f_1(x, y) = \pi \cos(t) \sin(2\pi y) \sin(\pi x) \sin(\pi x) - 2\pi^3 \sin(t) \sin(2\pi y) (2 \cos(2\pi x) - 1) \quad (2.75)$$

$$- \pi \sin(t) \sin(\pi x) \sin(\pi y),$$

$$f_2(x, y) = -\pi \cos(t) \sin(2\pi x) \sin(\pi y) \sin(\pi y) - 2\pi^3 \sin(t) \sin(2\pi x) (1 - 2 \cos(2\pi y))$$

$$+ \pi \sin(t) \cos(\pi x) \cos(\pi y).$$

Let k_0 and k denote the fine and regular time step sizes which are used to generate the numerical true solution and a computed solution, clearly $k_0 < k$. Moreover, $(\mathbf{u}_h^n(\tau), p_h^n(\tau))$ denote the numerical solution at the time step t_n using the time step size τ ; below, $\tau = k_0$

or k . For any $1 \leq n \leq N$, we use the following numerical integration formulas:

$$\begin{aligned}\mathcal{E}_{\mathbf{u},0}^n &:= \left(\mathbb{E} \left[\|\mathbf{u}(t_n) - \mathbf{u}_h^n(k)\|_{\mathbf{L}^2}^2 \right] \right)^{\frac{1}{2}} \approx \left(\frac{1}{N_p} \sum_{\ell=1}^{N_p} \|\mathbf{u}_h^n(k_0, \omega_\ell) - \mathbf{u}_h^n(k, \omega_\ell)\|_{\mathbf{L}^2}^2 \right)^{\frac{1}{2}}, \\ \mathcal{E}_{\mathbf{u},1}^n &:= \left(\mathbb{E} \left[\|\nabla(\mathbf{u}(t_n) - \mathbf{u}_h^n(k))\|_{\mathbf{L}^2}^2 \right] \right)^{\frac{1}{2}} \approx \left(\frac{1}{N_p} \sum_{\ell=1}^{N_p} \|\nabla(\mathbf{u}_h^n(k_0, \omega_\ell) - \mathbf{u}_h^n(k, \omega_\ell))\|_{\mathbf{L}^2}^2 \right)^{\frac{1}{2}}, \\ \mathcal{E}_{p,av}^N &:= \left(\mathbb{E} \left[\left\| P(T) - k \sum_{n=1}^{\frac{T}{k}} p_h^n(k) \right\|_{L^2}^2 \right] \right)^{\frac{1}{2}} \\ &\approx \left(\frac{1}{N_p} \sum_{\ell=1}^{N_p} \left\| k_0 \sum_{n=1}^{\frac{T}{k_0}} p_h^n(k_0, \omega_\ell) - k \sum_{n=1}^{\frac{T}{k_i}} p_h^n(k, \omega_\ell) \right\|_{L^2}^2 \right)^{\frac{1}{2}},\end{aligned}$$

and

$$\mathcal{E}_{p,0}^n := \left(\mathbb{E} \left[\|p(t_n) - p_h^n(k)\|^2 \right] \right)^{\frac{1}{2}} \approx \left(\frac{1}{N_p} \sum_{\ell=1}^{N_p} \|p_h^n(k_0, \omega_\ell) - p_h^n(k, \omega_\ell)\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

The definitions of $\mathcal{E}_{r,av}^N$ and $\mathcal{E}_{r,0}^N$ are similar.

We then implement Algorithm 2 and verify the convergence orders of the time and spatial discretizations proved in Theorem 2.3.4.

To generate a numerical exact solution for computing the orders of convergence, we use $k_0 = \frac{1}{1024}$ and $h = \frac{1}{40}$ as fine mesh sizes to compute such a solution. Then, to compute the convergence order of the time discretization for the velocity, we fix $h = \frac{1}{40}$ and then compute the numerical solution with following time mesh sizes: $k = \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}$. The errors in the L^2 -norm ($\mathcal{E}_{\mathbf{u},0}^n$) and H^1 -norm ($\mathcal{E}_{\mathbf{u},1}^n$) are shown in Table 2.1. The numerical results verify the convergence order $O(k^{\frac{1}{2}})$ which is stated in Theorem 2.3.4.

Tables 2.2 and 2.3 display respectively the L^2 -norm errors ($\mathcal{E}_{\alpha,av}^N$) and ($\mathcal{E}_{\alpha,0}^N$) ($\alpha = r$ and p) of the time-averaged pressure approximations using different time mesh sizes. The numerical results indicate the convergence rate $O(k^{\frac{1}{2}})$ which is predicted in Theorem 2.3.4. We also present the standard L^2 -norm errors $\mathcal{E}_{r,0}^N$ and $\mathcal{E}_{p,0}^N$ in Table 2.2 and 2.3 respectively for comparison purposes, for which we observe a significantly slower rate. It should be noted that our convergence theory does not conclude such a convergence behavior.

To verify the spatial convergence order for the velocity approximation, we fix $k = \frac{1}{512}$ and use different spatial mesh sizes $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$ to compute the errors $\mathcal{E}_{\mathbf{u},0}^n$ and $\mathcal{E}_{\mathbf{u},1}^n$. Table 2.4 contains the computational results which verify first order convergence rate for $\mathcal{E}_{\mathbf{u},1}^n$ as stated in Theorem 2.3.4 and second order convergence for $\mathcal{E}_{\mathbf{u},0}^n$.

To verify the convergence rate for the pressure approximation, we fix $k = \frac{1}{512}$ and use different spatial mesh sizes: $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$. Tables 2.5 displays the error $\mathcal{E}_{p,av}^N$ of the pressure approximation. It suggests that $\mathcal{E}_{r,av}^N, \mathcal{E}_{p,av}^N$ converge with an order slightly better than $O(h)$ as stated in Theorem 2.3.4.

Test 2. In this test, we use Algorithm 2 to solve the driven cavity problem with stochastic forcing, which is described by system (1.3) with the following non-homogeneous boundary condition:

$$\mathbf{u}(x_1, x_2) = \begin{cases} (1, 0), & x_2 = 1, 0 < x_1 < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathbf{W}$ be the Wiener process as in (2.73), and $B \equiv 1$, *i.e.*, the noise is additive. We use the following parameters in the test: $T = 1$, $h = \frac{1}{20}$, $k = 0.01$, and the number of the realizations is $N_p = 1001$.

Figure 2.1 displays the expected values of the computed stochastic velocity $\mathbf{u}_h^N$ and pressure p_h^N ; expectedly, they behave similarly as their deterministic counterparts do. On the other hand, individual realizations of the computed stochastic velocity $\mathbf{u}_h^N$ and pressure p_h^N given in Figures 2.2–2.4 show quite different behaviors from their deterministic counterparts.

Table 2.1: Algorithm 2: Time discretization errors for the velocity $\{\mathbf{u}_h^n\}_n$.

k	$\mathcal{E}_{\mathbf{u},0}^n$	order	$\mathcal{E}_{\mathbf{u},1}^n$	order
1/32	0.0298875		0.479939	
1/64	0.0238283	0.3269	0.406732	0.2388
1/128	0.016284	0.5490	0.0300667	0.4359
1/256	0.00969576	0.7482	0.197304	0.6078

Table 2.2: Algorithm 2: Time discretization errors for the pressure $\{r_h^n\}_n$.

k	$\mathcal{E}_{r,av}^N$	order	$\mathcal{E}_{r,0}^N$	order
1/32	0.0258463		1.27249	
1/64	0.0167279	0.6277	1.1632	0.1296
1/128	0.0117858	0.5052	0.974548	0.2553
1/256	0.00972497	0.2773	0.755733	0.3669

Table 2.3: Algorithm 2: Time discretization errors for the pressure approximation $\{p_h^n\}_n$.

k	$\mathcal{E}_{p,av}^N$	order	$\mathcal{E}_{p,0}^N$	order
1/32	0.139535		18.2813	
1/64	0.0818481	0.7696	17.9973	0.0226
1/128	0.056455	0.5358	17.0739	0.0760
1/256	0.036139	0.6435	15.527	0.1370

Table 2.4: Algorithm 2: Spatial discretization errors for the velocity approximation $\{\mathbf{u}_h^n\}_n$.

h	$\mathcal{E}_{\mathbf{u},0}^n$	order	$\mathcal{E}_{\mathbf{u},1}^n$	order
1/4	0.484335		5.64812	
1/8	0.141512	1.7751	2.98187	0.9216
1/16	0.0352466	2.0054	1.48554	1.0052
1/32	0.00786841	2.1633	0.804592	0.8847

Table 2.5: Algorithm 2: Spatial discretization errors for the pressure approximation.

h	$\mathcal{E}_{r,av}^N$	order	$\mathcal{E}_{p,av}^N$	order
1/4	7.55676		7.61385	
1/8	2.44124	1.7751	2.45072	1.6354
1/16	0.764419	1.6779	0.765958	1.6779
1/32	0.251246	1.6053	0.251472	1.6069

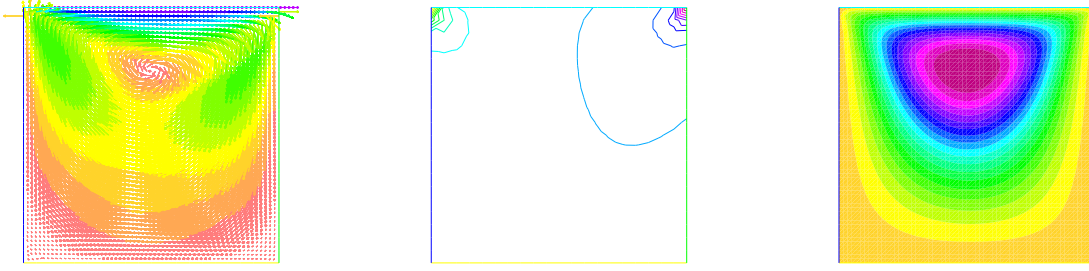


Figure 2.1: Test 2: (a) The expected value of $\{\mathbf{u}_h^N\}_n$. (b) Level-lines of the expected value of $\{p_h^N\}_n$. (c) The streamlines of the expected value of $\{\mathbf{u}_h^N\}_n$.

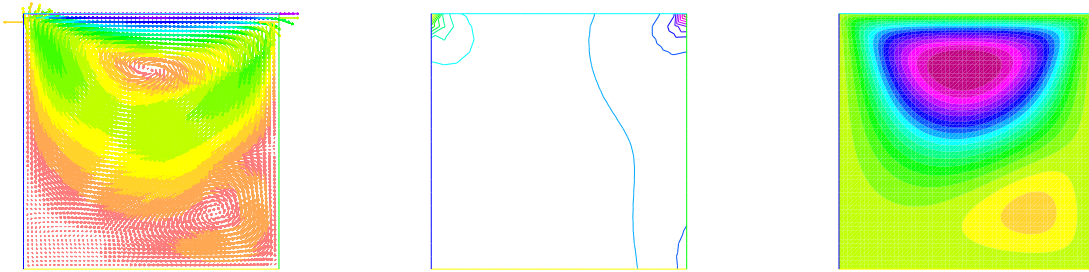


Figure 2.2: First realization of (a) the velocity $\{\mathbf{u}_h^N\}_n$; (b) the pressure $\{p_h^N\}_n$; (c) the streamline of $\{\mathbf{u}_h^N\}_n$.

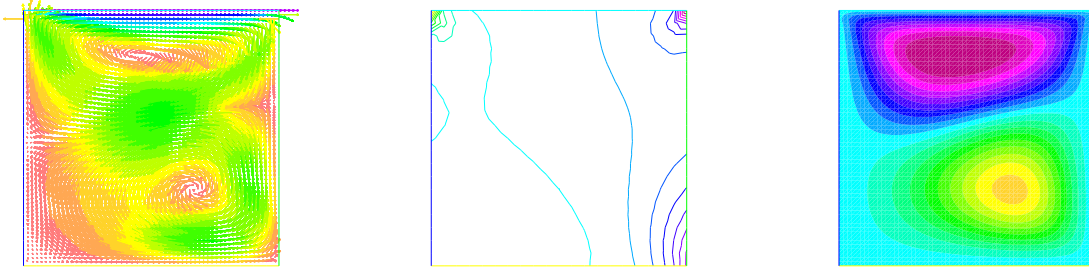


Figure 2.3: Second realization of (a) the velocity $\{\mathbf{u}_h^N\}_n$; (b) the pressure $\{p_h^N\}_n$; (c) the streamline of $\{\mathbf{u}_h^N\}_n$.

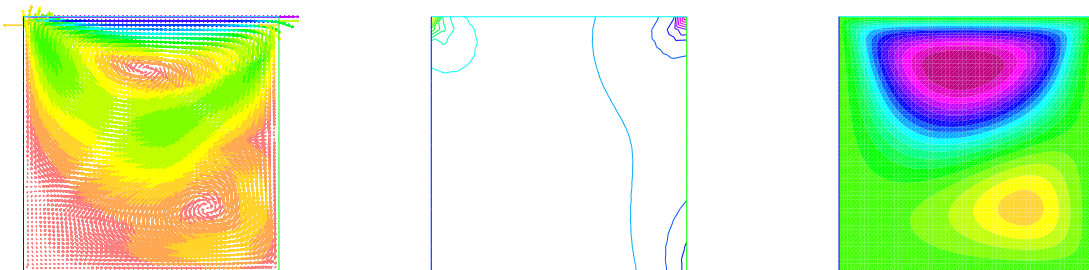


Figure 2.4: Third realization of (a) the velocity $\{\mathbf{u}_h^N\}_n$; (b) the pressure $\{p_h^N\}_n$; (c) the streamline of $\{\mathbf{u}_h^N\}_n$.

Chapter 3

High moment and pathwise error estimates for fully discrete mixed finite element approximations of stochastic Stokes equations

3.1 Introduction

In this chapter, we establish high moment and pathwise error estimates for fully discrete mixed finite element approximations of the following time-dependent stochastic Stokes (1.3) on $D = (0, L)^2 \subset \mathbb{R}^2$ represents a period of the periodic domain in $\mathbb{R}^d$.

Unlike in the deterministic case, the primary goal of the numerical analysis of SPDEs is to derive error estimates for quantities of stochastic interests of the error functions. The best known such quantities are the p th moment, $\mathbb{E}[\|u - U\|^p]$ for $2 \leq p \leq \infty$ as well as the variance $\text{Var}[\|u - U\|]$, where $\mathbb{E}[\cdot]$ and $\text{Var}[\cdot]$ stand for the expectation and variance operators, u and U denote respectively the exact and numerical solutions and $\|\cdot\|$ denotes some space-time norm. We note that when $p = \infty$, such an estimate is often called a pathwise error estimate. As expected, among these quantities of stochastic interests, the easiest one is the second moment $\mathbb{E}[\|u - U\|^2]$. This is indeed what was done in Chapter 2. Moreover, in

practice, numerical simulations for the approximate solution U are done for sample paths when the Monte Carlo method is used to compute the quantities of stochastic interests, which requires to use a large number of samples. However, to the best of our knowledge, no qualitative estimate was known for $\text{esssup}_{\omega \in \Omega} \max_{1 \leq n \leq N} \|u(t_n, \omega) - U^n(\omega)\|$ in the literature for the stochastic Stokes (and stochastic Navier-Stokes) equations. Such error estimates would provide valuable information about the quality of each computed sample path $U(\omega)$.

The goal of this chapter is to fill this gap by establishing arbitrarily high order moment and pathwise error estimates for both velocity and pressure approximations of the stochastic Stokes problem (1.3) discretized by two fully discrete mixed finite element methods. This chapter extends the results of Chapter 2 where the second moment error estimates were obtained for those mixed finite element methods.

The remainder of this chapter is organized as follows. In section 3.2, we introduce useful facts and inequalities which will be used in the entire chapter. In section 3.3, we first recall the Euler-Maruyama time-stepping scheme for problem (1.3) and then derive high moment and pathwise error estimates for the velocity and pressure approximations of the time-stepping scheme. In section 3.4, the standard mixed finite element method is introduced for the spatial discretization. The stable Taylor-Hood mixed element is chosen as a prototypical example for analysis. The highlight of this section is to derive high moment and pathwise error estimates for the velocity and pressure approximations of the mixed finite element method. In section 3.5, we consider the modified mixed method in section 2.2, Chapter 2 and obtain high moment and pathwise error estimates for this non-standard mixed finite method as well. Finally, numerical experiments are provided in section 3.6 to verify the proved error estimates and to examine the dependence/growth of the error constants on the moment order p .

3.2 Preliminaries

3.2.1 Some useful facts and inequalities

In this subsection, we collect some well-known theorems and useful facts which will be used in the later sections.

First of all, we recall the following Kolmogorov Theorem for a path-wise continuity of stochastic processes, its proof can be found in [19, Theorem 3.3] and [49, Lemma 2.1].

Theorem 3.2.1. *Let $\mathbf{X}(t), t \in [0, T]$, be a stochastic process with values in a separable Banach space E such that, for some positive constant $C > 0, \alpha > 0, \beta > 0$ and all $t, s \in [0, T]$,*

$$\mathbb{E}[\|\mathbf{X}(t) - \mathbf{X}(s)\|^\beta] \leq C|t - s|^{1+\alpha}. \quad (3.1)$$

Then for each $T > 0$, almost every ω and each $0 < \gamma < \frac{\alpha}{\beta}$ there exists a constant $K = K(\omega, \gamma, T)$ such that

$$\|\mathbf{X}(t, \omega) - \mathbf{X}(s, \omega)\| \leq K|t - s|^\gamma \quad \text{for all } t, s \in [0, T]. \quad (3.2)$$

Moreover, $\mathbb{E}[|K|^\beta] < \infty$ for all $\beta > 0$.

Next, we recall a useful inequality for martingale processes. This inequality is often referred to as the Burkholder-Davis-Gundy inequality in the literature, see [12, Theorem 2.4].

Lemma 3.2.2. *Let $\phi(t) \in L^2(D)$ be a random field for all $t \in [0, T]$. For any $q > 0$, there exists a positive constant $C_b = C_b(T, q) > 0$ such that*

$$\mathbb{E} \left[\max_{0 \leq t \leq T} \left\| \int_0^t \phi(\xi) dW(\xi) \right\|_{L^2}^q \right] \leq C_b \mathbb{E} \left[\left(\int_0^T \|\phi(\xi)\|_{L^2}^2 d\xi \right)^{q/2} \right]. \quad (3.3)$$

The next lemma recalls the well-known Itô isometry and introduces a related inequality for stochastic processes.

Lemma 3.2.3. *Let $\phi(t)$ be a stochastic process on $[0, T]$. Define $\mathbf{X}_t = \int_0^t \phi(\xi) dW(\xi)$. We have*

(i) If $\phi \in L^2(\Omega; L^2(0, T; \mathbf{L}^2(D)))$, then

$$\mathbb{E}[\|\mathbf{X}_t\|_{\mathbf{L}^2}^2] = \mathbb{E}\left[\int_0^t \|\phi(\xi)\|_{\mathbf{L}^2}^2 d\xi\right]. \quad (3.4)$$

(ii) If $\phi \in L^q(\Omega; L^q(0, T; \mathbf{L}^2(D)))$, for $q > 2$, then

$$\mathbb{E}[\|\mathbf{X}_t\|_{\mathbf{L}^2}^q] \leq C(t, q) \mathbb{E}\left[\int_0^t \|\phi(\xi)\|_{\mathbf{L}^2}^q d\xi\right], \quad (3.5)$$

where $C(t, q) = \frac{C_b}{2}(q-1)(q-2)t^{\frac{q}{2}} + (q-1)C_b$.

Proof. The proof of (3.4) can be found in [19]. Below we only give a proof for (3.5), which is based on the Itô formula and Burkholder-Davis-Gundy inequality.

By Itô's formula, we have

$$\begin{aligned} \mathbb{E}[\|\mathbf{X}_t\|_{\mathbf{L}^2}^q] &\leq q\mathbb{E}\left[\int_0^t \|\mathbf{X}_\tau\|_{\mathbf{L}^2}^{q-2}(\mathbf{X}_\tau, \phi(\tau)) dW(\tau)\right] \\ &\quad + \frac{1}{2}q(q-1)\mathbb{E}\left[\int_0^t \|\mathbf{X}_\tau\|_{\mathbf{L}^2}^{q-2}\|\phi(\tau)\|_{\mathbf{L}^2}^2 d\tau\right]. \end{aligned} \quad (3.6)$$

The expectation of the first term on the right-hand side of (3.6) vanishes due to the martingale property of Itô integrals. Therefore, we obtain

$$\begin{aligned} \mathbb{E}[\|\mathbf{X}_t\|_{\mathbf{L}^2}^q] &\leq \frac{1}{2}q(q-1) \int_0^t \mathbb{E}[\|\mathbf{X}_\tau\|_{\mathbf{L}^2}^{q-2}\|\phi(\tau)\|_{\mathbf{L}^2}^2] d\tau \\ &= \frac{1}{2}q(q-1) \int_0^t \mathbb{E}\left[\left\|\int_0^\tau \phi(\xi) dW(\xi)\right\|_{\mathbf{L}^2}^{q-2} \|\phi(\tau)\|_{\mathbf{L}^2}^2\right] d\tau \\ &\leq \frac{1}{2}q(q-1) \int_0^t \left(\mathbb{E}\left[\left\|\int_0^\tau \phi(\xi) dW(\xi)\right\|_{\mathbf{L}^2}^{\alpha(q-2)}\right]\right)^{\frac{1}{\alpha}} (\mathbb{E}[\|\phi(\tau)\|_{\mathbf{L}^2}^{2\beta}])^{\frac{1}{\beta}} d\tau \\ &\leq \frac{1}{2}q(q-1) \int_0^t \left(\mathbb{E}\left[\max_{0 \leq \tau \leq t} \left\|\int_0^\tau \phi(\xi) dW(\xi)\right\|_{\mathbf{L}^2}^{\alpha(q-2)}\right]\right)^{\frac{1}{\alpha}} (\mathbb{E}[\|\phi(\tau)\|_{\mathbf{L}^2}^{2\beta}])^{\frac{1}{\beta}} d\tau. \end{aligned} \quad (3.7)$$

We have used Hölder's inequality with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ to obtain the second inequality.

Next, applying the Burkholder-Davis-Gundy inequality to the last line of (3.7), we get

$$\mathbb{E}[\|\mathbf{X}_t\|_{\mathbf{L}^2}^q] \leq \frac{1}{2}q(q-1)C_b \int_0^t \left(\mathbb{E} \left[\left(\int_0^\tau \|\boldsymbol{\phi}(\xi)\|_{\mathbf{L}^2}^2 d\xi \right)^{\frac{\alpha(q-2)}{2}} \right] \right)^{\frac{1}{\alpha}} (\mathbb{E}[\|\boldsymbol{\phi}(\tau)\|_{\mathbf{L}^2}^{2\beta}])^{\frac{1}{\beta}} d\tau.$$

Setting $\alpha = \frac{q}{q-2}, \beta = \frac{q}{2}$ and using Young's inequality with the conjugate pair $\frac{q-2}{q}$ and $\frac{2}{q}$, we obtain

$$\begin{aligned} \mathbb{E}[\|\mathbf{X}_t\|_{\mathbf{L}^2}^q] &\leq \frac{C_b}{2}q(q-1) \int_0^t \left(\mathbb{E} \left[\left(\int_0^\tau \|\boldsymbol{\phi}(\xi)\|_{\mathbf{L}^2}^2 d\xi \right)^{\frac{q}{2}} \right] \right)^{\frac{q-2}{q}} (\mathbb{E}[\|\boldsymbol{\phi}(\tau)\|_{\mathbf{L}^2}^q])^{\frac{2}{q}} d\tau \\ &\leq \frac{1}{2}q(q-1)C_b t \mathbb{E} \left[\left(\int_0^t \|\boldsymbol{\phi}(\tau)\|_{\mathbf{L}^2}^2 d\tau \right)^{\frac{q}{2}} \right]^{\frac{q-2}{q}} \\ &\quad + \frac{1}{2}q(q-1)C_b \int_0^t \frac{\mathbb{E}[\|\boldsymbol{\phi}(\tau)\|_{\mathbf{L}^2}^q]}{q/2} d\tau \\ &\leq \left(\frac{1}{2}(q-1)(q-2)t^{\frac{q}{2}} + (q-1) \right) C_b \mathbb{E} \left[\int_0^t \|\boldsymbol{\phi}(\tau)\|_{\mathbf{L}^2}^q d\tau \right]. \end{aligned}$$

The proof is complete. $\square$

Finally, we recall the following property of the $\mathbb{R}$ -valued Wiener process:

$$\mathbb{E} \left[|W(t) - W(s)|^{2m} \right] \leq C_m |t - s|^m \quad \forall m \in \mathbb{N}. \quad (3.8)$$

When $m = 1$, the inequality becomes an equality with $C_m = 1$. We refer the reader to [46] for its generalization to infinite-dimensional Wiener processes.

We recall the following high moment stability estimate for the velocity $\mathbf{u}$ was proved in [8, 17].

Lemma 3.2.4. *Let $\mathbf{u}$ be solution defined in (2.9). Assume that $\mathbf{u}_0 \in L^r(\Omega; \mathbb{V})$ for some $r \geq 2$. Then we have*

$$\mathbb{E} \left[\left(\sup_{0 \leq t \leq T} \|\nabla \mathbf{u}(t)\|_{\mathbf{L}^2}^2 + \int_0^T \nu \|\nabla^2 \mathbf{u}(t)\|_{\mathbf{L}^2}^2 dt \right)^{\frac{r}{2}} \right] \leq C_r \mathbb{E} \left[\|\nabla \mathbf{u}_0\|_{\mathbf{L}^2}^r \right]. \quad (3.9)$$

Next, we introduce a high moment Hölder continuity estimates for the variational solution $\mathbf{u}$, a similar proof can be found in [8, 17] for the stochastic Navier-Stokes equations. we provide a proof below for completeness.

Lemma 3.2.5. Suppose $\mathbf{u}_0 \in L^q(\Omega; \mathbb{V})$ and $\mathbf{f} \in L^q(\Omega; C^{\frac{1}{2}}(0, T; \mathbf{H}^{-1}(D)))$, $\forall q \geq 2$. For $0 < \gamma < \frac{1}{2}$, there exists a constant $C \equiv C(D_T, \mathbf{u}_0) > 0$, such that the variational solution to problem (1.3) satisfies for $s, t \in [0, T]$

$$\mathbb{E}[\|\mathbf{u}(t) - \mathbf{u}(s)\|_{\mathbb{V}}^q] \leq C|t - s|^{\gamma q}. \quad (3.10)$$

Proof. Following [17, 8], we have that the mild solution of (1.3) can be represented as follow:

$$\mathbf{u}(t) = e^{-t\mathbf{A}}\mathbf{u}_0 + \int_0^t e^{(t-s)\mathbf{A}}\mathbf{P}_{\mathbb{H}}\mathbf{B}(\mathbf{u}(s)) dW(s). \quad (3.11)$$

For $t_2 < t_1$, write $\mathbf{u}(t_1) - \mathbf{u}(t_2) = \mathbf{I} + \mathbf{II}$ where

$$\begin{aligned} \mathbf{I} &= (e^{-t_1\mathbf{A}} - e^{-t_2\mathbf{A}})\mathbf{u}_0, \\ \mathbf{II} &= \int_0^{t_1} e^{(t_1-s)\mathbf{A}}\mathbf{P}_{\mathbb{H}}\mathbf{B}(\mathbf{u}(s)) dW(s) - \int_0^{t_2} e^{(t_2-s)\mathbf{A}}\mathbf{P}_{\mathbb{H}}\mathbf{B}(\mathbf{u}(s)) dW(s). \end{aligned} \quad (3.12)$$

By the standard estimates of semigroup theory, we have

$$\|\mathbf{A}^a e^{-t\mathbf{A}}\| \leq Ct^{-a}, \quad \|\mathbf{A}^{-b}(\mathbf{I} - e^{-t\mathbf{A}})\| \leq Ct^b.$$

Thus,

$$\begin{aligned} \|\mathbf{I}\|_{\mathbb{V}} &= \|e^{-t_2\mathbf{A}}(e^{-(t_1-t_2)\mathbf{A}} - \mathbf{I})\mathbf{A}^{\frac{1}{2}}\mathbf{u}_0\|_{\mathbf{L}^2} \\ &\leq C(t_1 - t_2)^{\gamma} \|\nabla \mathbf{u}_0\|_{\mathbf{L}^2}. \end{aligned} \quad (3.13)$$

Therefore, $\mathbb{E}[\|\mathbf{I}\|_{\mathbb{V}}^q] \leq C(t_1 - t_2)^{\gamma q} \mathbb{E}[\|\mathbf{u}_0\|_{\mathbb{V}}^q]$.

Next, we can write

$$\begin{aligned} \mathbf{II} &= \int_0^{t_2} (e^{-(t_1-s)\mathbf{A}} - e^{-(t_2-s)\mathbf{A}})\mathbf{B}(\mathbf{u}(s)) dW(s) \\ &\quad + \int_{t_2}^{t_1} e^{-(t_1-s)\mathbf{A}}\mathbf{B}(\mathbf{u}(s)) dW(s) =: \mathbf{II}_a + \mathbf{II}_b. \end{aligned} \quad (3.14)$$

By the Burkholder-Davis-Gundy inequality and the fact that $\|\cdot\|_{\mathbb{V}} = \|\mathbf{A}^{\frac{1}{2}}\cdot\|_{\mathbf{L}^2}$, we obtain

$$\begin{aligned}
(\mathbb{E}[\|\mathbf{II}_a\|_{\mathbb{V}}^q])^{\frac{1}{q}} &\leq C \left(\int_0^{t_2} \left(\mathbb{E} \left[\left\| (e^{-(t_1-s)\mathbf{A}} - e^{-(t_2-s)\mathbf{A}}) \mathbf{B}(\mathbf{u}(s)) \right\|_{\mathbb{V}}^q \right] \right)^{\frac{2}{q}} ds \right)^{\frac{1}{2}} \\
&\leq C \left(\int_0^{t_2} \left\| \mathbf{A}^{(1-\varepsilon)} e^{-(t_2-s)\mathbf{A}} \right\|_{\mathcal{L}(\mathbf{L}^2)}^2 \right. \\
&\quad \times \left\| \mathbf{A}^{-(1-\varepsilon)} (e^{-(t_1-t_2)\mathbf{A}} - \mathbf{I}) \right\|_{\mathcal{L}(\mathbf{L}^2)}^2 \left. \left(\mathbb{E}[\|\mathbf{u}(s)\|_{\mathbb{V}}^q] \right)^{\frac{2}{q}} ds \right)^{\frac{1}{2}} \\
&\leq C(t_1 - t_2)^{1-\varepsilon} \sup_{0 \leq s \leq T} \left(\mathbb{E}[\|\mathbf{u}(s)\|_{\mathbb{V}}^q] \right)^{\frac{1}{q}} \left(\int_0^{t_2} \frac{ds}{(t_2 - s)^{2(1-\varepsilon)}} \right)^{\frac{1}{2}} \\
&\leq C(t_1 - t_2)^{1-\varepsilon},
\end{aligned} \tag{3.15}$$

where $\frac{1}{2} < \varepsilon < 1$, and $\left(\mathbb{E}[\|\mathbf{u}(s)\|_{\mathbb{V}}^q] \right)^{\frac{1}{q}} < C_q$ by Lemma 3.2.4.

To estimate $\mathbf{II}_b$, we use Lemma 3.2.3 (ii) and then also apply Lemma 3.2.4 to obtain:

$$\begin{aligned}
\mathbb{E}[\|\mathbf{II}_b\|_{\mathbb{V}}^q] &\leq C_q \int_{t_2}^{t_1} \mathbb{E} \left[\left\| e^{-(t_1-s)\mathbf{A}} \mathbf{B}(\mathbf{u}(s)) \right\|_{\mathbb{V}}^q \right] ds \\
&\leq C_q(t_1 - t_2) \sup_{0 \leq s \leq T} \mathbb{E}[\|\mathbf{u}(s)\|_{\mathbb{V}}^q].
\end{aligned} \tag{3.16}$$

Finally, combining (3.13), (3.15) and (3.16) we obtain

$$\mathbb{E}[\|\mathbf{u}(t_1) - \mathbf{u}(t_2)\|_{\mathbb{V}}^q] \leq C(t_1 - t_2)^{\gamma q}, \tag{3.17}$$

where $0 < \gamma < \frac{1}{2}$. The proof is complete. $\square$

Remark 3.2.1. We note that due to the obstruction of nonlinearity, the estimate obtained in [8] requires higher regularity of $\mathbf{u}_0$ and $\mathbf{B} \in \mathcal{L}(\mathbf{L}_{per}^2, \mathbf{H}_{per}^2)$ to obtain the optimal order γ . On the other hand, the estimate of [17] is limited to the order $\frac{\gamma}{2}$ under the same assumptions as in Lemma 3.2.5 above.

3.3 Semi-discretization in time

In this section, we reconsider the implicit Euler-Maruyama scheme (2.2) for the time discretization of (1.3) which has been introduced in Section 2.1, Chapter 2. For the sake of completeness, we also state it below.

3.3.1 Formulation of the scheme and stability estimates

Let $I_k := \{t_n\}_{n=1}^M$ be a uniform mesh of the interval $[0, T]$ with the time step-size $k = \frac{T}{M}$. Note that $t_0 = 0$ and $t_M = T$.

Algorithm 1

Let $\mathbf{u}^0 = \mathbf{u}_0$ be a given $\mathbb{V}$ -valued random variable. Find the pair $\{\mathbf{u}^{n+1}, p^{n+1}\} \in \mathbb{V} \times L_{per}^2$ recursively such that $\mathbb{P}$ -a.s.

$$(\mathbf{u}^{n+1} - \mathbf{u}^n, \boldsymbol{\phi}) + \nu k (\nabla \mathbf{u}^{n+1}, \nabla \boldsymbol{\phi}) - k(p^{n+1}, \operatorname{div} \boldsymbol{\phi}) \quad (3.18a)$$

$$= k(\mathbf{f}^{n+1}, \boldsymbol{\phi}) + (\mathbf{B}(\mathbf{u}^n) \Delta W_{n+1}, \boldsymbol{\phi}),$$

$$(\operatorname{div} \mathbf{u}^{n+1}, \psi) = 0 \quad (3.18b)$$

for all $\boldsymbol{\phi} \in \mathbf{H}_{per}^1(D)$ and $\psi \in L_{per}^2(D)$. Where $\mathbf{f}^{n+1} := \mathbf{f}(t_{n+1})$.

The following high moment stability estimates for the velocity approximation $\{\mathbf{u}^n\}$ of Algorithm 1 were proved in in [17, Lemma 3.1].

Lemma 3.3.1. *Let $\mathbf{u}_0 \in L^{2q}(\Omega; \mathbb{V})$ for an integer $1 \leq q < \infty$ be given, such that $\mathbb{E}[\|\mathbf{u}_0\|_{\mathbb{V}}^{2q}] \leq C$. Then there exists a constant $C_{T,q} = C(T, q, \mathbf{u}_0)$ such that the following estimations hold:*

$$(i) \quad \mathbb{E} \left[\max_{1 \leq n \leq M} \|\mathbf{u}^n\|_{\mathbb{V}}^{2q} + \nu k \sum_{n=1}^M \|\mathbf{u}^n\|_{\mathbb{V}}^{2q-2} \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 \right] \leq C_{T,q}.$$

$$(ii) \quad \mathbb{E} \left[\left(\sum_{n=1}^M \|\mathbf{u}^n - \mathbf{u}^{n-1}\|_{\mathbb{V}}^2 \right)^q + \left(\nu k \sum_{n=1}^M \|\mathbf{A} \mathbf{u}^n\|_{\mathbb{V}}^2 \right)^q \right] \leq C_{T,q}.$$

Next, we want to derive some high moment stability estimates for the pressure approximation $\{p^n\}$ of Algorithm 1, which plays a crucial role in the error analysis of the full-discrete scheme later.

Lemma 3.3.2. *Let $\{(\mathbf{u}^{m+1}, p^{m+1})\}_n$ be generated by in Algorithm 1. Assume that $\mathbf{u}_0 \in L^q(\Omega; \mathbb{V})$ for $1 \leq q < \infty$. Then, there exists a constant $C > 0$ such that*

(i) *if $\mathbf{B} : \mathbf{L}^2 \rightarrow \mathbb{V}$, then*

$$\mathbb{E} \left[\left(k \sum_{n=1}^M \|\nabla p^n\|_{\mathbf{L}^2}^2 \right)^q \right] \leq C_{T,q}; \quad (3.19)$$

(ii) *if $\mathbf{B} : \mathbf{L}^2 \rightarrow \mathbf{H}_{per}^1$, then*

$$\mathbb{E} \left[\left(k \sum_{n=1}^M \|\nabla p^n\|_{\mathbf{L}^2}^2 \right)^q \right] \leq \frac{C_{T,q}}{k^q}. \quad (3.20)$$

Proof. When $q = 1$, both (3.19), (5.13) were already shown [17, Lemma 3.2]. Thus, it remains to prove them for $q > 1$.

(i) We first multiply the strong form of (3.18a) by ∇p^{n+1} and use the fact that $\mathbf{B}(\mathbf{u}) \in \mathbb{V}$, so $(\mathbf{B}(\mathbf{u}^n) \Delta W_{n+1}, \nabla p^{n+1}) = 0$ to conclude that

$$k \|\nabla p^{n+1}\|_{\mathbf{L}^2}^2 \leq Ck \|\mathbf{f}^{n+1}\|_{\mathbf{L}^2}^2. \quad (3.21)$$

Next, taking the summation over the index n followed by taking the q th power and expectation on both sides of (3.21) leads to the desired estimate.

(ii) Let $\mathbf{B} \in L^\infty(0, T; \mathbf{H}_{per}^1(D))$, then $(\mathbf{B}(\mathbf{u}^n) \Delta W_{n+1}, \nabla p^{n+1}) \neq 0$. Hence,

$$k \|\nabla p^{n+1}\|_{\mathbf{L}^2}^2 \leq Ck \|\mathbf{f}^{n+1}\|_{\mathbf{L}^2}^2 + \frac{C}{k} \|\mathbf{B}(\mathbf{u}^n) \Delta W_{n+1}\|_{\mathbf{L}^2}^2. \quad (3.22)$$

Taking the summation over the index n followed by taking the q th power and expectation on both sides of (3.22), we get

$$\begin{aligned} \mathbb{E} \left[\left(k \sum_{n=1}^M \|\nabla p^n\|_{\mathbf{L}^2}^2 \right)^q \right] &\leq C_q \mathbb{E} \left[\left(k \sum_{n=1}^M \|\mathbf{f}^n\|_{\mathbf{L}^2}^2 \right)^q \right] \\ &\quad + \frac{C_q}{k^q} \mathbb{E} \left[\left(\sum_{n=1}^M \|\mathbf{B}(\mathbf{u}^{n-1}) \Delta W_n\|_{\mathbf{L}^2}^2 \right)^q \right]. \end{aligned} \quad (3.23)$$

We now bound the last term on the right-hand side of (3.23). By the discrete Hölder inequality for summation and (1.5b), we obtain

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{n=1}^M \|\mathbf{B}(\mathbf{u}^{n-1}) \Delta W_n\|_{\mathbf{L}^2}^2 \right)^q \right] &\leq C_q \mathbb{E} \left[\left(\sum_{n=1}^M \|\mathbf{u}^{n-1}\|_{\mathbf{L}^2}^2 |\Delta W_n|^2 \right)^q \right] \\ &\leq C_q M^{q-1} \mathbb{E} \left[\sum_{n=1}^M \|\mathbf{u}^{n-1}\|_{\mathbf{L}^2}^{2q} |\Delta W_n|^{2q} \right]. \end{aligned} \quad (3.24)$$

Using the tower property of the conditional expectation, the independence of the increments of the Wiener process and (3.8), we obtain

$$\mathbb{E} [\|\mathbf{u}^{n-1}\|_{\mathbf{L}^2}^{2q} |\Delta W_n|^{2q}] \leq C_q k^q \mathbb{E} [\|\mathbf{u}^{n-1}\|_{\mathbf{L}^2}^{2q}]. \quad (3.25)$$

Substitute (3.24), (3.25) into (3.23) we obtain

$$\mathbb{E} \left[\left(k \sum_{n=1}^M \|\nabla p^n\|_{\mathbf{L}^2}^2 \right)^q \right] \leq C_q \mathbb{E} \left[\left(k \sum_{n=1}^M \|\mathbf{f}^n\|_{\mathbf{L}^2}^2 \right)^q \right] + \frac{C_q}{k^q} \mathbb{E} \left[k \sum_{n=1}^M \|\mathbf{u}^{n-1}\|_{\mathbf{L}^2}^{2q} \right]. \quad (3.26)$$

Finally, the proof is complete by using the assertion (i) of Lemma 3.3.1. $\square$

3.3.2 High moment and pathwise error estimates for the velocity approximation

In this subsection, we present the first main result of this chapter which establishes the optimal order high moment error estimates for the velocity approximation by Algorithm 1

and the sub-optimal order pathwise error estimate for the velocity approximation with the help of Theorem 3.2.1.

Theorem 3.3.3. *Let $\mathbf{u}$ be the variational solution to (1.3) and $\{\mathbf{u}^n\}_{n=1}^M$ be generated by Algorithm 1. Assume that $\mathbf{u}_0 \in L^{2q}(\Omega; \mathbb{V})$. Then there exists $C_1 = C_1(T, q, \mathbf{u}_0, \mathbf{f}) > 0$ for any integer $1 \leq q < \infty$ and real number $0 < \gamma < \frac{1}{2}$ such that*

$$\mathbb{E} \left[\max_{1 \leq n \leq M} \|\mathbf{u}(t_n) - \mathbf{u}^n\|_{\mathbf{L}^2}^{2q} \right] \leq C_1 k^{2q\gamma}. \quad (3.27)$$

Proof. When $q = 1$, the estimate was already proved in [34, 33]. Thus, it remains to show (3.27) for $q \geq 2$. We start with $q = 2$.

Let $\mathbf{e}^n = \mathbf{u}(t_n) - \mathbf{u}^n$. Integrating (2.14a) from t_n to t_{n+1} and choosing $\boldsymbol{\phi} \in \mathbb{V}$, we obtain

$$\begin{aligned} (\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n), \boldsymbol{\phi}) + \int_{t_n}^{t_{n+1}} (\nabla \mathbf{u}(s), \nabla \boldsymbol{\phi}) ds &= \int_{t_n}^{t_{n+1}} (\mathbf{f}(s), \boldsymbol{\phi}) ds \\ &+ \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)), \boldsymbol{\phi}) dW(s). \end{aligned} \quad (3.28)$$

Subtracting (3.28) from (3.18a), we obtain the following error equation for the velocity:

$$\begin{aligned} (\mathbf{e}^{n+1} - \mathbf{e}^n, \boldsymbol{\phi}) + \nu k (\nabla \mathbf{e}^{n+1}, \nabla \boldsymbol{\phi}) &= \nu \int_{t_n}^{t_{n+1}} (\nabla(\mathbf{u}(s) - \mathbf{u}(t_{n+1})), \nabla \boldsymbol{\phi}) ds \\ &+ \int_{t_n}^{t_{n+1}} (\mathbf{f}(s) - \mathbf{f}(t_{n+1}), \boldsymbol{\phi}) ds \\ &+ \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n), \boldsymbol{\phi}) dW(s). \end{aligned} \quad (3.29)$$

Choosing $\boldsymbol{\phi} = \mathbf{e}^{n+1}$ in (3.29) and using the identity $2(a - b)a = a^2 - b^2 + (a - b)^2$, then the left-hand side (LHS) and right-hand side (RHS) of (3.29) become

$$\text{LHS} = \frac{1}{2} [\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{e}^n\|_{\mathbf{L}^2}^2] + \frac{1}{2} \|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{\mathbf{L}^2}^2 + \nu k \|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2. \quad (3.30)$$

$$\begin{aligned}
\text{RHS} &= \nu \int_{t_n}^{t_{n+1}} (\nabla(\mathbf{u}(s) - \mathbf{u}(t_{n+1})), \nabla \mathbf{e}^{n+1}) ds \\
&\quad + \int_{t_n}^{t_{n+1}} (\mathbf{f}(s) - \mathbf{f}(t_{n+1}), \mathbf{e}^{n+1}) ds \\
&\quad + \left(\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s), \mathbf{e}^{n+1} - \mathbf{e}^n \right) \\
&\quad + \left(\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s), \mathbf{e}^n \right).
\end{aligned} \tag{3.31}$$

Next, multiplying (3.30) and (3.31) by $\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2$ yields

$$\begin{aligned}
\text{LHS} &= \frac{1}{4} [\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^4 - \|\mathbf{e}^n\|_{\mathbf{L}^2}^4] + \frac{1}{4} (\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{e}^n\|_{\mathbf{L}^2}^2)^2 \\
&\quad + \frac{1}{2} \|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 + \nu k \|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2.
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
\text{RHS} &= \nu \int_{t_n}^{t_{n+1}} (\nabla(\mathbf{u}(s) - \mathbf{u}(t_{n+1})), \nabla \mathbf{e}^{n+1}) ds \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \\
&\quad + \int_{t_n}^{t_{n+1}} (\mathbf{f}(s) - \mathbf{f}(t_{n+1}), \mathbf{e}^{n+1}) ds \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \\
&\quad + \left(\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s), \mathbf{e}^{n+1} - \mathbf{e}^n \right) \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \\
&\quad + \left(\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s), \mathbf{e}^n \right) \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \\
&=: \text{I} + \text{II} + \text{III} + \text{IV}.
\end{aligned} \tag{3.33}$$

Now, we estimate terms of I, II, III, IV below.

$$\begin{aligned}
\text{I} &\leq \nu \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s))\|_{\mathbf{L}^2} \|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2} \|\mathbf{e}^n\|_{\mathbf{L}^2} ds \\
&\leq \nu \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s))\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 ds \\
&\quad + \frac{\nu k}{4} \|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \\
&= \nu \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s))\|_{\mathbf{L}^2}^2 ds (\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{e}^n\|_{\mathbf{L}^2}^2) \\
&\quad + \nu \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s))\|_{\mathbf{L}^2}^2 ds \|\mathbf{e}^n\|_{\mathbf{L}^2}^2 \\
&\quad + \frac{\nu k}{4} \|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \\
&\leq 8 \left(\nu \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s))\|_{\mathbf{L}^2}^2 ds \right)^2 + \frac{1}{32} (\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{e}^n\|_{\mathbf{L}^2}^2)^2 \\
&\quad + \frac{\nu^2}{4} \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s))\|_{\mathbf{L}^2}^4 ds + k \|\mathbf{e}^n\|_{\mathbf{L}^2}^4 \\
&\quad + \frac{\nu k}{4} \|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2
\end{aligned} \tag{3.34}$$

By using (3.10), we obtain

$$\begin{aligned}
\mathbb{E}[\text{I}] &\leq C k^{1+4\gamma} + \frac{1}{32} \mathbb{E}[(\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{e}^n\|_{\mathbf{L}^2}^2)^2] \\
&\quad + \frac{\nu k}{4} \mathbb{E}[\|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2] + k \mathbb{E}[\|\mathbf{e}^n\|_{\mathbf{L}^2}^4]
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
\text{II} &\leq \int_{t_n}^{t_{n+1}} \|\mathbf{f}(s) - \mathbf{f}(t_{n+1})\|_{\mathbf{H}^{-1}} \|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2} \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 ds \\
&\leq C \int_{t_n}^{t_{n+1}} \|\mathbf{f}(s) - \mathbf{f}(t_{n+1})\|_{\mathbf{H}^{-1}}^2 \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 ds + \frac{\nu k}{4} \|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \\
&\leq C \int_{t_n}^{t_{n+1}} \|\mathbf{f}(s) - \mathbf{f}(t_{n+1})\|_{\mathbf{H}^{-1}}^4 ds + \frac{1}{32} (\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{e}^n\|_{\mathbf{L}^2}^2)^2 \\
&\quad + Ck \|\mathbf{e}^n\|_{\mathbf{L}^2}^4 + \frac{\nu k}{4} \|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2.
\end{aligned} \tag{3.36}$$

Since $\mathbf{f} \in L^{2q}(\Omega; C^{\frac{1}{2}}(0, T; \mathbf{H}^{-1}(D)))$ for $q = 2$, we have

$$\begin{aligned} \mathbb{E}[\text{II}] &\leq Ck^3 + \frac{1}{32}\mathbb{E}[(\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{e}^n\|_{\mathbf{L}^2}^2)^2] \\ &\quad + Ck\mathbb{E}[\|\mathbf{e}^n\|_{\mathbf{L}^2}^4] + \frac{\nu k}{4}\mathbb{E}[\|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2]. \end{aligned} \quad (3.37)$$

$$\begin{aligned} \text{III} &= \left(\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s), \mathbf{e}^{n+1} - \mathbf{e}^n \right) \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \\ &\leq \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \\ &\quad + \frac{1}{4} \|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \\ &= \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^2 (\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{e}^n\|_{\mathbf{L}^2}^2) \\ &\quad + \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^2 \|\mathbf{e}^n\|_{\mathbf{L}^2}^2 \\ &\quad + \frac{1}{4} \|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \\ &\leq 8 \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^4 + \frac{1}{32} (\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{e}^n\|_{\mathbf{L}^2}^2)^2 \\ &\quad + \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^2 \|\mathbf{e}^n\|_{\mathbf{L}^2}^2 \\ &\quad + \frac{1}{4} \|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2. \end{aligned} \quad (3.38)$$

$$\begin{aligned} \text{IV} &= \left(\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s), \mathbf{e}^n \right) (\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{e}^n\|_{\mathbf{L}^2}^2) \\ &\quad + \left(\int_{t_m}^{t_{m+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s), \mathbf{e}^n \right) \|\mathbf{e}^n\|_{\mathbf{L}^2}^2 \\ &\leq 8 \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^2 \|\mathbf{e}^n\|_{\mathbf{L}^2}^2 \\ &\quad + \frac{1}{32} (\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{e}^n\|_{\mathbf{L}^2}^2)^2 \\ &\quad + \left(\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s), \mathbf{e}^n \right) \|\mathbf{e}^n\|_{\mathbf{L}^2}^2. \end{aligned} \quad (3.39)$$

We note that the last term on the right-hand side of (3.39) has zero expected value because of the martingale property of the Itô integrals.

Now, substituting the above estimates for terms I, II, III, IV into RHS in (3.33) and taking expectation on both LHS and RHS, followed by absorbing the like terms of LHS in (3.32) into those of RHS in (3.33), we obtain

$$\begin{aligned}
& \frac{1}{4}\mathbb{E}[\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^4 - \|\mathbf{e}^n\|_{\mathbf{L}^2}^4] + \frac{1}{4}\mathbb{E}[\|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2] \\
& \quad + \frac{\nu k}{2}\mathbb{E}[\|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2] \\
& \leq C k^{1+4\gamma} + Ck\mathbb{E}[\|\mathbf{e}^n\|_{\mathbf{L}^2}^4] + Ck^3 \\
& \quad + C\mathbb{E}\left[\left\|\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s)\right\|_{\mathbf{L}^2}^4\right] \\
& \quad + C\mathbb{E}\left[\left\|\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s)\right\|_{\mathbf{L}^2}^2 \|\mathbf{e}^n\|_{\mathbf{L}^2}^2\right] \\
& \leq Ck^{1+4\gamma} + Ck\mathbb{E}[\|\mathbf{e}^n\|_{\mathbf{L}^2}^4] + \mathbf{V} + \mathbf{VI},
\end{aligned} \tag{3.40}$$

where

$$\begin{aligned}
\mathbf{V} &:= C\mathbb{E}\left[\left\|\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s)\right\|_{\mathbf{L}^2}^4\right], \\
\mathbf{VI} &:= C\mathbb{E}\left[\left\|\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s)\right\|_{\mathbf{L}^2}^2 \|\mathbf{e}^n\|_{\mathbf{L}^2}^2\right].
\end{aligned}$$

To estimate $\mathbf{V}$, we first use (3.5) and then use (3.10) to get

$$\begin{aligned}
\mathbf{V} &= C\mathbb{E}\left[\left\|\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s)\right\|_{\mathbf{L}^2}^4\right] \\
&\leq C\mathbb{E}\left[\int_{t_n}^{t_{n+1}} \|\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)\|_{\mathbf{L}^2}^4 ds\right] \\
&\leq C\int_{t_n}^{t_{n+1}} \mathbb{E}[\|\mathbf{u}(s) - \mathbf{u}^n\|_{\mathbf{L}^2}^4] ds \\
&\leq C\int_{t_n}^{t_{n+1}} \mathbb{E}[\|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{L}^2}^4] ds + Ck\mathbb{E}[\|\mathbf{e}^n\|_{\mathbf{L}^2}^4] \\
&\leq Ck^{1+4\gamma} + Ck\mathbb{E}[\|\mathbf{e}^n\|_{\mathbf{L}^2}^4].
\end{aligned} \tag{3.41}$$

To estimate VI, we use the Itô isometry given in (3.4) and (3.10) to get

$$\begin{aligned}
\text{VI} &= C\mathbb{E}\left[\left\|\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s)\right\|_{\mathbf{L}^2}^2 \|\mathbf{e}^n\|_{\mathbf{L}^2}^2\right] \\
&= C\mathbb{E}\left[\left\|\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) \|\mathbf{e}^n\|_{\mathbf{L}^2} dW(s)\right\|_{\mathbf{L}^2}^2\right] \\
&= C\mathbb{E}\left[\int_{t_n}^{t_{n+1}} \|\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)\|_{\mathbf{L}^2}^2 \|\mathbf{e}^n\|_{\mathbf{L}^2}^2 ds\right] \\
&\leq C \int_{t_n}^{t_{n+1}} \mathbb{E}[\|\mathbf{u}(s) - \mathbf{u}^n\|_{\mathbf{L}^2}^2 \|\mathbf{e}^n\|_{\mathbf{L}^2}^2] ds \\
&\leq C \int_{t_n}^{t_{n+1}} \mathbb{E}[\|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{L}^2}^2 \|\mathbf{e}^n\|_{\mathbf{L}^2}^2] ds + Ck\mathbb{E}[\|\mathbf{e}^n\|_{\mathbf{L}^2}^4] \\
&\leq C \int_{t_n}^{t_{n+1}} \mathbb{E}[\|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{L}^2}^4] ds + Ck\mathbb{E}[\|\mathbf{e}^n\|_{\mathbf{L}^2}^4] \\
&\leq Ck^{1+4\gamma} + Ck\mathbb{E}[\|\mathbf{e}^n\|_{\mathbf{L}^2}^4].
\end{aligned} \tag{3.42}$$

Bounding V, VI by (3.41) and (3.42) in (3.40), we obtain

$$\begin{aligned}
&\frac{1}{4}\mathbb{E}[\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^4 - \|\mathbf{e}^n\|_{\mathbf{L}^2}^4] + \frac{1}{8}\mathbb{E}[(\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{e}^n\|_{\mathbf{L}^2}^2)^2] \\
&\quad + \frac{1}{4}\mathbb{E}[\|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2] + \frac{\nu k}{2}\mathbb{E}[\|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2] \\
&\leq Ck^{1+4\gamma} + Ck\mathbb{E}[\|\mathbf{e}^n\|_{\mathbf{L}^2}^4].
\end{aligned} \tag{3.43}$$

Next, lowering the index n in (3.43) by 1 and applying the summation $\sum_{n=1}^{\ell}$ for any $1 \leq \ell \leq M$, we have

$$\begin{aligned}
&\mathbb{E}[\|\mathbf{e}^{\ell}\|_{\mathbf{L}^2}^4] + \sum_{n=1}^{\ell} \mathbb{E}[\|\mathbf{e}^n - \mathbf{e}^{n-1}\|_{\mathbf{L}^2}^2 \|\mathbf{e}^n\|_{\mathbf{L}^2}^2] + 2\nu k \sum_{n=1}^{\ell} \mathbb{E}[\|\nabla \mathbf{e}^n\|_{\mathbf{L}^2}^2 \|\mathbf{e}^n\|_{\mathbf{L}^2}^2] \\
&\leq Ck^{4\gamma} + Ck \sum_{n=1}^{\ell} \mathbb{E}[\|\mathbf{e}^{n-1}\|_{\mathbf{L}^2}^4] \\
&\leq Ck^{4\gamma} e^{Ct_{\ell}},
\end{aligned} \tag{3.44}$$

where we have used the discrete Gronwall inequality to get the last inequality.

Taking maximum over all $1 \leq \ell \leq M$ to (3.44), we conclude that

$$\max_{1 \leq \ell \leq M} \mathbb{E} [\|\mathbf{e}^\ell\|_{\mathbf{L}^2}^4] \leq Ck^{4\gamma}. \quad (3.45)$$

Since the maximum is taken outside of $\mathbb{E}[\cdot]$, hence, (3.45) is weaker than the desired estimate for $q = 2$. To show the stronger estimate, we follow the technique of Lemma 3.1 proof [14] which uses the estimate (3.45) as a bridge to obtain the desired estimate.

To the end, substituting (3.34)–(3.39) into RHS in (3.33) and equating it with LHS in (3.32) (without taking expectation), we obtain

$$\begin{aligned} & \frac{1}{4} [\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^4 - \|\mathbf{e}^n\|_{\mathbf{L}^2}^4] + \frac{1}{8} (\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{e}^n\|_{\mathbf{L}^2}^2)^2 \\ & + \frac{1}{4} \|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 + \frac{\nu k}{2} \|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \\ & \leq Ck \|\mathbf{e}^n\|_{\mathbf{L}^2}^4 + C \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s))\|_{\mathbf{L}^2}^4 ds \\ & + C \int_{t_n}^{t_{n+1}} \|\mathbf{f}(s) - \mathbf{f}(t_{n+1})\|_{\mathbf{H}^{-1}}^4 ds \\ & + C \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^4 \\ & + C \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^2 \|\mathbf{e}^n\|_{\mathbf{L}^2}^2 \\ & + \left(\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s), \mathbf{e}^n \right) \|\mathbf{e}^n\|_{\mathbf{L}^2}^2. \end{aligned} \quad (3.46)$$

Applying the summation operator $\sum_{n=1}^\ell$ followed by $\max_{1 \leq \ell \leq M}$ and taking expectation on both sides, on noting that the last term on the right side of (3.46) would not vanish anymore (which is the main difference of this new process compared with the proof of (3.45)), and by using (3.45), we have

$$\begin{aligned} & \mathbb{E} \left[\max_{1 \leq \ell \leq M} \|\mathbf{e}^\ell\|_{\mathbf{L}^2}^4 \right] \\ & \leq C \mathbb{E} \left[\max_{1 \leq \ell \leq M} \sum_{n=1}^\ell \left(\int_{t_{n-1}}^{t_n} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^{n-1})) dW(s), \mathbf{e}^{n-1} \right) \|\mathbf{e}^{n-1}\|_{\mathbf{L}^2}^2 \right] + Ck^{4\gamma}. \end{aligned} \quad (3.47)$$

To bound the first term on the right side of (3.47), we appeal to Burkholder-Davis-Gundy inequality to obtain

$$\begin{aligned}
& \mathbb{E} \left[\max_{1 \leq \ell \leq M} \sum_{n=1}^{\ell} \left(\int_{t_{n-1}}^{t_n} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^{n-1})) dW(s), \mathbf{e}^{n-1} \right) \|\mathbf{e}^{n-1}\|_{\mathbf{L}^2}^2 \right] \\
& \leq \mathbb{E} \left[\left(\sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^{n-1})\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n-1}\|_{\mathbf{L}^2}^6 ds \right)^{\frac{1}{2}} \right] \\
& \leq C \mathbb{E} \left[\left(\sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|\mathbf{u}(s) - \mathbf{u}^{n-1}\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n-1}\|_{\mathbf{L}^2}^6 ds \right)^{\frac{1}{2}} \right] \\
& \leq C \mathbb{E} \left[\max_{1 \leq \ell \leq M} \|\mathbf{e}^{\ell}\|_{\mathbf{L}^2}^2 \left(\sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|\mathbf{u}(s) - \mathbf{u}^{n-1}\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n-1}\|_{\mathbf{L}^2}^2 ds \right)^{\frac{1}{2}} \right] \\
& \leq \frac{1}{2} \mathbb{E} [\max_{1 \leq \ell \leq M} \|\mathbf{e}^{\ell}\|_{\mathbf{L}^2}^4] + C \mathbb{E} \left[\sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|\mathbf{u}(s) - \mathbf{u}^{n-1}\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n-1}\|_{\mathbf{L}^2}^2 ds \right] \\
& \leq \frac{1}{2} \mathbb{E} [\max_{1 \leq \ell \leq M} \|\mathbf{e}^{\ell}\|_{\mathbf{L}^2}^4] + C \mathbb{E} \left[\sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|\mathbf{u}(s) - \mathbf{u}(t_{n-1})\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n-1}\|_{\mathbf{L}^2}^2 ds \right] \\
& \quad + C \mathbb{E} \left[\sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|\mathbf{e}^{n-1}\|_{\mathbf{L}^2}^4 ds \right] \\
& \leq \frac{1}{2} \mathbb{E} [\max_{1 \leq \ell \leq M} \|\mathbf{e}^{\ell}\|_{\mathbf{L}^2}^4] + Ck \sum_{n=1}^M \mathbb{E} [\|\mathbf{e}^{n-1}\|_{\mathbf{L}^2}^4] \\
& \quad + C \sum_{n=1}^M \int_{t_{n-1}}^{t_n} (\mathbb{E} [\|\mathbf{u}(s) - \mathbf{u}(t_{n-1})\|_{\mathbf{L}^2}^4])^{\frac{1}{2}} (\mathbb{E} [\|\mathbf{e}^{n-1}\|_{\mathbf{L}^2}^4])^{\frac{1}{2}} ds \\
& \leq \frac{1}{2} \mathbb{E} [\max_{1 \leq \ell \leq M} \|\mathbf{e}^{\ell}\|_{\mathbf{L}^2}^4] + Ck^{4\gamma}.
\end{aligned} \tag{3.48}$$

Here, we have used (3.10) and (3.45) to obtain the last inequality of (3.48).

Combining (3.48) and (3.47) yields the desired estimate for the case $q = 2$.

To prove the general case $3 \leq q < \infty$, for the sake of notation brevity but without loss of the generality, we let $f = 0$. Our first task is to show the following inequality by induction for any $1 \leq q < \infty$: there exists a constant $c_q > 0$ such that holds $\mathbb{P}$ -a.s.

$$\begin{aligned}
\frac{1}{2^q} [\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2^q} - \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q}] &\leq c_q k \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q} + c_q \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(s) - \mathbf{u}(t_{n+1}))\|_{\mathbf{L}^2}^{2^q} ds \\
&+ c_q \sum_{j=1}^q \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^{2^j} \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q - 2^j} \\
&+ \left(\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s), \mathbf{e}^n \right) \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q - 2},
\end{aligned} \tag{3.49}$$

which has been proved to hold for $q = 2, 3$.

Suppose that (3.49) holds for any fixed integer $q(> 3)$ and we want to show it also holds for $q + 1$. To the end, multiplying (3.49) by $\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2^q}$ and use again the identity $2a(a - b) = a^2 - b^2 + (a - b)^2$ we obtain

$$\begin{aligned}
\frac{1}{2^{q+1}} [\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2^{q+1}} - \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^{q+1}}] &+ \frac{1}{2^{q+1}} (\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2^q} - \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q})^2 \\
&\leq c_q k \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q} \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2^q} + c_q \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(s) - \mathbf{u}(t_{n+1}))\|_{\mathbf{L}^2}^{2^q} ds \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2^q} \\
&+ c_q \sum_{j=1}^q \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^{2^j} \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q - 2^j} \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2^q} \\
&+ \left(\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s), \mathbf{e}^n \right) \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q - 2} \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2^q} \\
&:= \text{I} + \text{II} + \text{III} + \text{IV}.
\end{aligned} \tag{3.50}$$

For some $\delta_1, \delta_2 > 0$, we have

$$\begin{aligned}
\text{I} &= c_q k \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q} (\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2^q} - \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q}) + c_q k \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^{q+1}} \\
&\leq \frac{c_q^2 k^2}{4\delta_1} \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^{q+1}} + \delta_1 (\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2^q} - \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q})^2 + c_q k \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^{q+1}} \\
&= \left(\frac{c_q^2 k}{4\delta_1} + c_q \right) k \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^{q+1}} + \delta_1 (\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2^q} - \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q})^2.
\end{aligned} \tag{3.51}$$

$$\begin{aligned}
\text{II} &= c_q \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(s) - \mathbf{u}(t_{n+1}))\|_{\mathbf{L}^2}^{2q} ds (\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2q} - \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q}) \\
&\quad + c_q \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(s) - \mathbf{u}(t_{n+1}))\|_{\mathbf{L}^2}^{2q} ds \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q} \\
&\leq \frac{1}{4\delta_2} \left(c_q \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(s) - \mathbf{u}(t_{n+1}))\|_{\mathbf{L}^2}^{2q} ds \right)^2 + \delta_2 (\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2q} - \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q})^2 \\
&\quad + c_q \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(s) - \mathbf{u}(t_{n+1}))\|_{\mathbf{L}^2}^{2q+1} ds + c_q k \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q+1} \\
&\leq \frac{c_q^2 k}{4\delta_2} \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(s) - \mathbf{u}(t_{n+1}))\|_{\mathbf{L}^2}^{2q+1} ds + \delta_2 (\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2q} - \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q})^2 \\
&\quad + c_q \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(s) - \mathbf{u}(t_{n+1}))\|_{\mathbf{L}^2}^{2q+1} ds + c_q k \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q+1} \\
&= \left(\frac{c_q^2 k}{4\delta_2} + c_q \right) \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(s) - \mathbf{u}(t_{n+1}))\|_{\mathbf{L}^2}^{2q+1} ds \\
&\quad + \delta_2 (\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2q} - \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q})^2 + c_q k \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q+1}.
\end{aligned} \tag{3.52}$$

For $\alpha_1, \dots, \alpha_q > 0$ we have

$$\begin{aligned}
\text{III} &= c_q \sum_{j=1}^q \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^{2j} \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q-2j} (\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2q} - \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q}) \\
&\quad + c_q \sum_{j=1}^q \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^{2j} \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q-2j} \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q} \\
&\leq \sum_{j=1}^q \frac{c_q^2}{4\alpha_j} \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^{2j+1} \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q+1-2j+1} \\
&\quad + \sum_{j=1}^q \alpha_j (\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2q} - \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q})^2 \\
&\quad + c_q \sum_{j=1}^q \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^{2j} \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q+1-2j}.
\end{aligned} \tag{3.53}$$

Similarly, for some $\delta_3 > 0$ we have

$$\begin{aligned}
\text{IV} &= \left(\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s), \mathbf{e}^n \right) \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q-2} (\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2q} - \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q}) \\
&\quad + \left(\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s), \mathbf{e}^n \right) \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q-2} \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q} \\
&\leq \frac{1}{4\delta_3} \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^2 \|\mathbf{e}^n\|_{\mathbf{L}^2}^2 \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q+1-4} \\
&\quad + \delta_3 (\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2q} - \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q})^2 \\
&\quad + \left(\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s), \mathbf{e}^n \right) \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q+1-2} \\
&= \frac{1}{4\delta_3} \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^2 \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q+1-2} \\
&\quad + \delta_3 (\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2q} - \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q})^2 \\
&\quad + \left(\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s), \mathbf{e}^n \right) \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q+1-2}.
\end{aligned} \tag{3.54}$$

Substitute the estimates from (3.51)–(3.54) into (3.50) we obtain

$$\begin{aligned}
&\frac{1}{2^{q+1}} [\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2q+1} - \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q+1}] + \left(\frac{1}{2^{q+1}} - \delta_1 - \delta_2 - \delta_3 - \alpha \right) (\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2q} - \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q})^2 \\
&\leq \left(\frac{c_q^2 k}{4\delta_1} + 2c_q \right) k \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q+1} + \left(\frac{c_q^2 k}{4\delta_2} + c_q \right) \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(s) - \mathbf{u}(t_{n+1}))\|_{\mathbf{L}^2}^{2q+1} ds \\
&\quad + \sum_{j=1}^q \frac{c_q^2}{4\alpha_j} \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^{2j+1} \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q+1-2j+1} \\
&\quad + c_q \sum_{j=1}^q \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^{2j} \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q+1-2j} \\
&\quad + \frac{1}{4\delta_3} \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^2 \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q+1-2} \\
&\quad + \left(\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s), \mathbf{e}^n \right) \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2q+1-2},
\end{aligned} \tag{3.55}$$

where $\alpha = \sum_{j=1}^q \alpha_j > 0$.

Now, we choose $\delta_1, \delta_2, \delta_3, \alpha > 0$ such that $\frac{1}{2^{q+1}} - \delta_1 - \delta_2 - \delta_3 - \alpha > 0$ so that the second term on the left-hand side of (3.55) is positive and can be dropped at the end. Next, after

rearranging terms on the right-hand side, (3.55) infers that

$$\begin{aligned}
& \frac{1}{2^{q+1}} [\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2^{q+1}} - \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^{q+1}}] + \left(\frac{1}{2^{q+1}} - \delta_1 - \delta_2 - \delta_3 - \alpha \right) (\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2^q} - \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q})^2 \quad (3.56) \\
& \leq \left(\frac{c_q^2 k}{4\delta_1} + 2c_q \right) k \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^{q+1}} + \left(\frac{c_q^2 k}{4\delta_2} + c_q \right) \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(s) - \mathbf{u}(t_{n+1}))\|_{\mathbf{L}^2}^{2^{q+1}} ds \\
& \quad + \max_{1 \leq j \leq q} \frac{c_q^2}{4\alpha_j} \sum_{j=1}^q \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^{2^{j+1}} \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^{q+1}-2^{j+1}} \\
& \quad + \left(c_q + \frac{1}{4\delta_3} \right) \sum_{j=1}^q \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^{2^j} \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^{q+1}-2^j} \\
& \quad + \left(\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s), \mathbf{e}^n \right) \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^{q+1}-2} \\
& \leq \left(\frac{c_q^2 k}{4\delta_1} + 2c_q \right) k \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^{q+1}} + \left(\frac{c_q^2 k}{4\delta_2} + c_q \right) \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(s) - \mathbf{u}(t_{n+1}))\|_{\mathbf{L}^2}^{2^{q+1}} ds \\
& \quad + \max_{1 \leq j \leq q} \frac{c_q^2}{4\alpha_j} \sum_{j=1}^{q+1} \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^{2^j} \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^{q+1}-2^j} \\
& \quad + \left(c_q + \frac{1}{4\delta_3} \right) \sum_{j=1}^{q+1} \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^{2^j} \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^{q+1}-2^j} \\
& \quad + \left(\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s), \mathbf{e}^n \right) \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^{q+1}-2} \\
& = \left(\frac{c_q^2 k}{4\delta_1} + 2c_q \right) k \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^{q+1}} + \left(\frac{c_q^2 k}{4\delta_2} + c_q \right) \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(s) - \mathbf{u}(t_{n+1}))\|_{\mathbf{L}^2}^{2^{q+1}} ds \\
& \quad + \left(c_q + \frac{1}{4\delta_3} + \max_{1 \leq j \leq q} \frac{c_q^2}{4\alpha_j} \right) \sum_{j=1}^{q+1} \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^{2^j} \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^{q+1}-2^j} \\
& \quad + \left(\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s), \mathbf{e}^n \right) \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^{q+1}-2} \\
& \leq c_{q+1} k \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^{q+1}} + c_{q+1} \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(s) - \mathbf{u}(t_{n+1}))\|_{\mathbf{L}^2}^{2^{q+1}} ds \\
& \quad + c_{q+1} \sum_{j=1}^{q+1} \left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^{2^j} \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^{q+1}-2^j} \\
& \quad + \left(\int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s), \mathbf{e}^n \right) \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^{q+1}-2},
\end{aligned}$$

where

$$c_{q+1} = \max \left\{ \frac{c_q^2 k}{4\delta_1} + 2c_q, \frac{c_q^2 k}{4\delta_2} + c_q, c_q + \frac{1}{4\delta_3} + \max_{1 \leq j \leq q} \frac{c_q^2}{4\alpha_j} \right\}.$$

Hence, the proof of (3.49) is complete.

Next, we prove the statement of the theorem for the general case $3 \leq q < \infty$, which will be carried out using the same technique as that in the proof of the 4th moment (i.e. $q = 2$). Taking the expectation on (3.49) and using the martingale property of Itô integrals and the Hölder continuity in Lemma 3.2.5, we obtain

$$\begin{aligned}
\frac{1}{2^q} \mathbb{E} [\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2^q} - \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q}] &\leq c_q k \mathbb{E} [\|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q}] + c_q k^{1+2^q\gamma} \\
&\quad + c_q \sum_{j=1}^q \mathbb{E} \left[\left\| \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)) dW(s) \right\|_{\mathbf{L}^2}^{2^j} \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q-2^j} \right] \\
&\leq c_q k \mathbb{E} [\|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q}] + c_q k^{1+2^q\gamma} \\
&\quad + c_q \sum_{j=1}^q C_j \mathbb{E} \left[\int_{t_n}^{t_{n+1}} \|\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)\|_{\mathbf{L}^2}^{2^j} ds \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q-2^j} \right],
\end{aligned} \tag{3.57}$$

where the last inequality of (3.57) is obtained by using (ii) of Lemma 3.2.3. The last term on the right-hand side of (3.57) can be bounded as follows

$$\begin{aligned}
&c_q \sum_{j=1}^q C_j \mathbb{E} \left[\int_{t_n}^{t_{n+1}} \|\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^n)\|_{\mathbf{L}^2}^{2^j} ds \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q-2^j} \right] \\
&\leq c_q \sum_{j=1}^q C_j \mathbb{E} \left[\int_{t_n}^{t_{n+1}} \|\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}(t_n))\|_{\mathbf{L}^2}^{2^j} ds \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q-2^j} \right] \\
&\quad + c_q \sum_{j=1}^q C_j \mathbb{E} \left[\int_{t_n}^{t_{n+1}} \|\mathbf{B}(\mathbf{u}(t_n)) - \mathbf{B}(\mathbf{u}^n)\|_{\mathbf{L}^2}^{2^j} ds \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q-2^j} \right] \\
&\leq c_q \sum_{j=1}^q C_j \mathbb{E} \left[\int_{t_n}^{t_{n+1}} \|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{L}^2}^{2^j} ds \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q-2^j} \right] \\
&\quad + c_q \sum_{j=1}^q C_j \mathbb{E} \left[\int_{t_n}^{t_{n+1}} \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^j} ds \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q-2^j} \right] \\
&= c_q \sum_{j=1}^{q-1} C_j \mathbb{E} \left[\int_{t_n}^{t_{n+1}} \|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{L}^2}^{2^j} ds \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q-2^j} \right] \\
&\quad + \tilde{c}_q \mathbb{E} \left[\int_{t_n}^{t_{n+1}} \|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{L}^2}^{2^q} ds \right] + \tilde{c}_q k \mathbb{E} [\|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q}] \\
&\leq c_q \sum_{j=1}^{q-1} C_j \mathbb{E} \left[\int_{t_n}^{t_{n+1}} \|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{L}^2}^{2^j} ds \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q-2^j} \right] + \tilde{c}_q k^{1+2^q\gamma} + \tilde{c}_q k \mathbb{E} [\|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q}].
\end{aligned} \tag{3.58}$$

In addition, for each $1 \leq j < q$, using Young's inequality with the conjugates $a = 2^{q-j}$ and $b = \frac{2^{q-j}}{2^{q-j}-1}$ to the first term on the right side of (3.58), we get

$$\begin{aligned}
& c_q \sum_{j=1}^{q-1} C_j \mathbb{E} \left[\int_{t_n}^{t_{n+1}} \|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{L}^2}^{2^j} ds \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q-2^j} \right] \\
& \leq c_q \sum_{j=1}^{q-1} \frac{C_j}{a} \mathbb{E} \left[\int_{t_n}^{t_{n+1}} \|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{L}^2}^{a2^j} ds \right] + c_q \sum_{j=1}^{q-1} \frac{C_j}{b} k \mathbb{E} [\|\mathbf{e}^n\|_{\mathbf{L}^2}^{(2^q-2^j)b}] \\
& \leq \tilde{c}_q k^{1+2^q\gamma} + \tilde{c}_q k \mathbb{E} [\|\mathbf{e}^q\|_{\mathbf{L}^2}^{2^q}],
\end{aligned} \tag{3.59}$$

Finally, substituting (3.59) to (3.58) and then combining it with (3.57) yield

$$\frac{1}{2^q} \mathbb{E} [\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^{2^q} - \|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q}] \leq \tilde{c}_q k \mathbb{E} [\|\mathbf{e}^n\|_{\mathbf{L}^2}^{2^q}] + \tilde{c}_q k^{1+2^q\gamma}. \tag{3.60}$$

Summing (3.60) in n and then using the discrete Gronwall inequality, we get

$$\begin{aligned}
\frac{1}{2^q} \mathbb{E} [\|\mathbf{e}^\ell\|_{\mathbf{L}^2}^{2^q}] & \leq \tilde{c}_q k \sum_{n=1}^{\ell} \mathbb{E} [\|\mathbf{e}^{n-1}\|_{\mathbf{L}^2}^{2^q}] + \tilde{c}_q C_{t_\ell} k^{2^q\gamma} \\
& \leq \tilde{c}_q C_{t_\ell} k^{2^q\gamma} e^{\tilde{c}_q t_\ell}.
\end{aligned} \tag{3.61}$$

Thus,

$$\max_{1 \leq \ell \leq M} \mathbb{E} [\|\mathbf{e}^\ell\|_{\mathbf{L}^2}^{2^q}] \leq C k^{2^q\gamma}.$$

Repeating the last part of the proof of the case $q = 2$ we subsequently obtain

$$\mathbb{E} \left[\max_{1 \leq \ell \leq M} \|\mathbf{e}^\ell\|_{\mathbf{L}^2}^{2^q} \right] \leq C k^{2^q\gamma}.$$

The proof is complete. □

Corollary 3.3.4. *Under the assumptions of Theorem 3.3.3. For any real numbers $2 \leq q < \infty$ and $0 < \gamma < \frac{1}{2}$, there holds*

$$\mathbb{E} \left[\max_{1 \leq n \leq M} \|\mathbf{u}(t_n) - \mathbf{u}^n\|_{\mathbf{L}^2}^q \right] \leq C_1 k^{\gamma q}, \quad (3.62)$$

where $C_1 = C_1(T, q, \mathbf{u}_0, \mathbf{f})$.

Proof. The proof follows from using Hölder inequality and Theorem 3.3.3. $\square$

Theorem 3.3.5. *Under the assumptions of Theorem 3.3.3, there holds for $2 \leq q < \infty$ and $0 < \gamma < \frac{1}{2}$*

$$\mathbb{E} \left[\left\| \nu k \sum_{n=1}^M \nabla(\mathbf{u}(t_n) - \mathbf{u}^n) \right\|_{\mathbf{L}^2}^q \right] \leq C_1 k^{\gamma q}, \quad (3.63)$$

where $C_1 = C_1(T, q, \mathbf{u}_0, \mathbf{f})$.

Proof. For the sake of notational brevity, we set $\nu = 1$. Applying the summation operator $\sum_{n=1}^M$ to (3.29), we obtain

$$\begin{aligned} (\mathbf{e}^M, \boldsymbol{\phi}) + \left(k \sum_{n=1}^M \nabla \mathbf{e}^n, \nabla \boldsymbol{\phi} \right) &= \left(\sum_{n=1}^M \int_{t_{n-1}}^{t_n} \nabla(\mathbf{u}(s) - \mathbf{u}(t_n)) ds, \nabla \boldsymbol{\phi} \right) \\ &+ \left(\sum_{n=1}^M \int_{t_{n-1}}^{t_n} (\mathbf{f}(s) - \mathbf{f}(t_n)) ds, \boldsymbol{\phi} \right) \\ &+ \left(\sum_{n=1}^M \int_{t_{n-1}}^{t_n} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^{n-1})) dW(s), \boldsymbol{\phi} \right). \end{aligned} \quad (3.64)$$

Setting $\boldsymbol{\phi} = k \sum_{n=1}^M \mathbf{e}^n$, and using Schwarz, Young, Poincaré inequalities, we obtain

$$\begin{aligned} \left\| k \sum_{n=1}^M \nabla \mathbf{e}^n \right\|_{\mathbf{L}^2}^2 &\leq C \|\mathbf{e}^M\|_{\mathbf{L}^2}^2 + C \left\| \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \nabla(\mathbf{u}(s) - \mathbf{u}(t_n)) ds \right\|_{\mathbf{L}^2}^2 \\ &+ C \left\| \sum_{n=1}^M \int_{t_{n-1}}^{t_n} (\mathbf{f}(s) - \mathbf{f}(t_n)) ds \right\|_{\mathbf{H}^{-1}}^2 \\ &+ C \left\| \sum_{n=1}^M \int_{t_{n-1}}^{t_n} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^{n-1})) dW(s) \right\|_{\mathbf{L}^2}^2. \end{aligned} \quad (3.65)$$

Taking the $\frac{q}{2}$ -power followed by expectation on both sides of (3.65), we get

$$\begin{aligned}
\mathbb{E} \left[\left\| k \sum_{n=1}^M \nabla \mathbf{e}^n \right\|_{\mathbf{L}^2}^q \right] &\leq C_q \mathbb{E} [\|\mathbf{e}^M\|_{\mathbf{L}^2}^q] \\
&+ C_q \mathbb{E} \left[\left\| \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \nabla (\mathbf{u}(s) - \mathbf{u}(t_n)) ds \right\|_{\mathbf{L}^2}^q \right] \\
&+ C_q \mathbb{E} \left[\left\| \sum_{n=1}^M \int_{t_{n-1}}^{t_n} (\mathbf{f}(s) - \mathbf{f}(t_n)) ds \right\|_{\mathbf{H}^{-1}}^q \right] \\
&+ C_q \mathbb{E} \left[\left\| \sum_{n=1}^M \int_{t_{n-1}}^{t_n} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^{n-1})) dW(s) \right\|_{\mathbf{L}^2}^q \right] \\
&=: \text{I} + \text{II} + \text{III} + \text{IV}.
\end{aligned} \tag{3.66}$$

By using (3.27), (3.10), and the assumption on $\mathbf{f}$, we get

$$\text{I} + \text{II} + \text{III} \leq C_q k^{\gamma q}. \tag{3.67}$$

To bound IV, by (3.5) we have

$$\begin{aligned}
\text{IV} &\leq C_q \mathbb{E} \left[\sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^{n-1})\|_{\mathbf{L}^2}^q ds \right] \\
&\leq C_q \mathbb{E} \left[\sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|\mathbf{u}(s) - \mathbf{u}(t_{n-1})\|_{\mathbf{L}^2}^q ds \right] + C_q \mathbb{E} \left[\sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|\mathbf{e}^{n-1}\|_{\mathbf{L}^2}^q ds \right] \\
&\leq C_q k^{\gamma q}.
\end{aligned} \tag{3.68}$$

Here, we have used (3.10) and (3.27) to obtain the last inequality of (3.68). The proof is complete. $\square$

Remark 3.3.1. *The second-moment (i.e., $p = 2$) error estimate in the $\mathbf{H}^1$ -norm was obtained for the velocity approximation in Chapter 2. Theorem 3.3.5 proves a weak convergence of the high moments of the error in $\mathbf{H}^1$ -norm. The difficulty of obtaining the strong convergence of the high moments of the error in $\mathbf{H}^1$ -norm is explained below. After setting $\boldsymbol{\phi} = \mathbf{e}^{n+1}$ in (3.29), using the binomial formula and summing over all $0 \leq n < M$,*

we obtain a similar inequality as that in (3.66) but in strong form, namely,

$$\begin{aligned}
& \mathbb{E}[\|\mathbf{e}^M\|_{\mathbf{L}^2}^q] + \mathbb{E}\left[\left(k \sum_{n=1}^M \|\nabla \mathbf{e}^n\|_{\mathbf{L}^2}^2\right)^{\frac{q}{2}}\right] \\
& \leq C_q \mathbb{E}\left[\left(\sum_{n=1}^M \left\|\int_{t_{n-1}}^{t_n} \nabla(\mathbf{u}(s) - \mathbf{u}(t_n)) ds\right\|_{\mathbf{L}^2}^2\right)^{\frac{q}{2}}\right] \\
& \quad + C_q \mathbb{E}\left[\left(\sum_{n=1}^M \left\|\int_{t_{n-1}}^{t_n} (\mathbf{f}(s) - \mathbf{f}(t_n)) ds\right\|_{\mathbf{H}^{-1}}^2\right)^{\frac{q}{2}}\right] \\
& \quad + C_q \mathbb{E}\left[\left(\sum_{n=1}^M \left\|\int_{t_{n-1}}^{t_n} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^{n-1})) dW(s)\right\|_{\mathbf{L}^2}^2\right)^{\frac{q}{2}}\right].
\end{aligned} \tag{3.69}$$

It is unclear how to bound the noise term on the right-hand side of (3.69).

Finally, we are ready to state our first pathwise error estimate for the velocity approximation, such an estimate has not been obtained before in the literature.

Theorem 3.3.6. *Assume that the assumptions of Theorem 3.3.3 hold. Let $2 < q < \infty$ and $0 < \gamma < \frac{1}{2}$ such that $\gamma - \frac{1}{q} > 0$. Then, for $0 < \gamma_1 < \gamma - \frac{1}{q}$, there exists a random variable $K_1 = K_1(\omega; C_1)$ with $\mathbb{E}[|K_1|^q] < \infty$ such that there holds $\mathbb{P}$ -a.s.*

$$\max_{1 \leq n \leq M} \|\mathbf{u}(t_n) - \mathbf{u}^n\|_{\mathbf{L}^2} + \left\| \nu k \sum_{n=1}^M \nabla(\mathbf{u}(t_n) - \mathbf{u}^n) \right\|_{\mathbf{L}^2} \leq K_1 k^{\gamma_1}. \tag{3.70}$$

Proof. (3.70) is an immediate consequence of Corollary 3.3.4, Theorem 3.3.5 and Kolmogorov Theorem 3.2.1. $\square$

3.3.3 High moment and pathwise error estimates for the pressure approximation

In this subsection we derive high moment and pathwise error estimates for the pressure approximation generated by Algorithm 1. Once again, the pathwise error estimate is obtained by using the Kolmogorov Theorem 3.2.1 and the high moment error estimates.

Theorem 3.3.7. *Let $P(t)$ be the pressure process defined in Theorem 2.1.3 and $\{p^n\}_{n=1}^M$ be the pressure approximation generated by Algorithm 1. Assume that $\mathbf{u}_0 \in L^q(\Omega; \mathbb{V})$. Then,*

for real number $0 < \gamma < \frac{1}{2}$ and any integer $2 \leq q < \infty$, there exists a positive constant $C_2 = C_2(C_1, \beta_0)$ such that for all $1 \leq \ell \leq M$

$$\mathbb{E} \left[\left\| P(t_\ell) - k \sum_{n=1}^{\ell} p^n \right\|_{L^2}^q \right] \leq C_2 k^{\gamma q}. \quad (3.71)$$

Proof. The proof is based on the well-known inf-sup condition associated with the Stokes problem. First, let us recall the inf-sup condition at the differential level, it says that there exists $\beta_0 > 0$ such that

$$\sup_{\boldsymbol{\phi} \in \mathbf{H}_{per}^1(D)} \frac{(w, \operatorname{div} \boldsymbol{\phi})}{\|\nabla \boldsymbol{\phi}\|_{\mathbf{L}^2}} \geq \beta_0 \|w\|_{L^2} \quad \forall w \in L_{per}^2(D). \quad (3.72)$$

Now, integrating (2.14a) in t from 0 to t_ℓ for $1 \leq \ell \leq M$, we obtain

$$\begin{aligned} & (\mathbf{u}(t_\ell), \boldsymbol{\phi}) + \nu \int_0^{t_\ell} (\nabla \mathbf{u}(s), \nabla \boldsymbol{\phi}) ds - (\operatorname{div} \boldsymbol{\phi}, P(t_\ell)) \\ &= (\mathbf{u}_0, \boldsymbol{\phi}) + \int_0^{t_\ell} (\mathbf{f}(s), \boldsymbol{\phi}) ds + \int_0^{t_\ell} (\mathbf{B}(\mathbf{u}(s)), \boldsymbol{\phi}) d\mathbf{W}(s) \quad \forall \boldsymbol{\phi} \in \mathbf{H}_{per}^1(D), \end{aligned} \quad (3.73)$$

and applying $\sum_{n=1}^{\ell}$ to (3.18a), we get

$$\begin{aligned} & (\mathbf{u}^\ell, \boldsymbol{\phi}) + \nu k \sum_{n=1}^{\ell} (\nabla \mathbf{u}^n, \nabla \boldsymbol{\phi}) - k \sum_{n=1}^{\ell} (p^n, \operatorname{div} \boldsymbol{\phi}) \\ &= (\mathbf{u}^0, \boldsymbol{\phi}) + k \sum_{n=1}^{\ell} (\mathbf{f}^n, \boldsymbol{\phi}) + \sum_{n=1}^{\ell} (\mathbf{B}(\mathbf{u}^{n-1}) \Delta W_n, \boldsymbol{\phi}) \quad \forall \boldsymbol{\phi} \in \mathbf{H}_{per}^1(D). \end{aligned} \quad (3.74)$$

Let $E_P^m := P(t_m) - k \sum_{n=1}^m p^n$ and recall that $\mathbf{e}^m := \mathbf{u}(t_m) - \mathbf{u}^m$ from the proof of Theorem 3.3.3. Subtracting (3.73) from (3.74) yields

$$\begin{aligned}
(E_P^\ell, \operatorname{div} \phi) &= (\mathbf{e}^\ell, \phi) + \nu \left(\sum_{n=1}^{\ell} \int_{t_{n-1}}^{t_n} \nabla(\mathbf{u}(s) - \mathbf{u}^n) ds, \nabla \phi \right) \\
&\quad - \left(\sum_{n=1}^{\ell} \int_{t_{n-1}}^{t_n} (\mathbf{f}(s) - \mathbf{f}^n) ds, \phi \right) \\
&\quad - \left(\sum_{n=1}^{\ell} \int_{t_{n-1}}^{t_n} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^{n-1})) dW(s), \phi \right).
\end{aligned} \tag{3.75}$$

Applying Schwarz and Poincare's inequality to the right side of (3.75), we obtain

$$\begin{aligned}
\frac{(E_P^\ell, \operatorname{div} \phi)}{\|\nabla \phi\|_{\mathbf{L}^2}} &\leq C \|\mathbf{e}^\ell\|_{\mathbf{L}^2} + \nu \left\| \sum_{n=1}^{\ell} \int_{t_{n-1}}^{t_n} \nabla(\mathbf{u}(s) - \mathbf{u}^n) ds \right\|_{\mathbf{L}^2} \\
&\quad + \left\| \sum_{n=1}^{\ell} \int_{t_{n-1}}^{t_n} (\mathbf{f}(s) - \mathbf{f}^n) ds \right\|_{\mathbf{H}^{-1}} \\
&\quad + C \left\| \sum_{n=1}^{\ell} \int_{t_{n-1}}^{t_n} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^{n-1})) dW(s) \right\|_{\mathbf{L}^2}.
\end{aligned} \tag{3.76}$$

Then, it follows from applying (4.53) to the left-hand side of (3.76) that

$$\begin{aligned}
\beta_0 \|E_P^\ell\|_{L^2} &\leq C \|\mathbf{e}^\ell\|_{\mathbf{L}^2} + \nu \left\| \sum_{n=1}^{\ell} \int_{t_{n-1}}^{t_n} \nabla(\mathbf{u}(s) - \mathbf{u}^n) ds \right\|_{\mathbf{L}^2} \\
&\quad + \left\| \sum_{n=1}^{\ell} \int_{t_{n-1}}^{t_n} (\mathbf{f}(s) - \mathbf{f}^n) ds \right\|_{\mathbf{H}^{-1}} \\
&\quad + C \left\| \sum_{n=1}^{\ell} \int_{t_{n-1}}^{t_n} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^{n-1})) dW(s) \right\|_{\mathbf{L}^2}.
\end{aligned} \tag{3.77}$$

Next, taking the q th power followed by taking expectation on both sides of (3.77) yields

$$\begin{aligned}
\beta_0^q \mathbb{E}[\|E_P^\ell\|_{L^2}^q] &\leq C_q \mathbb{E}[\|\mathbf{e}^\ell\|_{\mathbf{L}^2}^q] + C_q \mathbb{E}\left[\left\|\sum_{n=1}^{\ell} \int_{t_{n-1}}^{t_n} \nabla(\mathbf{u}(s) - \mathbf{u}^n) ds\right\|_{\mathbf{L}^2}^q\right] \\
&\quad + \mathbb{E}\left[\left\|\sum_{n=1}^{\ell} \int_{t_{n-1}}^{t_n} (\mathbf{f}(s) - \mathbf{f}^n) ds\right\|_{\mathbf{H}^{-1}}^q\right] \\
&\quad + C_q \mathbb{E}\left[\left\|\sum_{n=1}^{\ell} \int_{t_{n-1}}^{t_n} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^{n-1})) dW(s)\right\|_{\mathbf{L}^2}^q\right] \\
&=: \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}.
\end{aligned} \tag{3.78}$$

We now estimate four terms on the right-side of (3.78). Using the estimates of Corollary 3.3.4, (3.63), (3.10) and the assumption $\mathbf{f} \in L^q(\Omega; C^{\frac{1}{2}}(0, T; \mathbf{H}^{-1}(D)))$, we obtain

$$\begin{aligned}
\mathbf{a} + \mathbf{b} + \mathbf{c} &\leq C_q \mathbb{E}[\|\mathbf{e}^\ell\|_{\mathbf{L}^2}^q] + C_q \mathbb{E}\left[\sum_{n=1}^{\ell} \int_{t_{n-1}}^{t_n} \|\nabla(\mathbf{u}(s) - \mathbf{u}(t_n))\|_{\mathbf{L}^2}^q ds\right] \\
&\quad + C_q \mathbb{E}\left[\left\|k \sum_{n=1}^{\ell} \nabla \mathbf{e}^n ds\right\|_{\mathbf{L}^2}^q\right] + C_q \mathbb{E}\left[\sum_{n=1}^{\ell} \int_{t_{n-1}}^{t_n} \|\mathbf{f}(s) - \mathbf{f}^n\|_{\mathbf{H}^{-1}}^q ds\right] \\
&\leq Ck^{\gamma q}.
\end{aligned}$$

Finally, to estimate term $\mathbf{d}$, using (3.5), Corollary 3.3.4 and (3.10), we get

$$\begin{aligned}
\mathbf{d} &\leq C_q \mathbb{E}\left[\sum_{n=1}^{\ell} \int_{t_{n-1}}^{t_n} \|\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{u}^{n-1})\|_{\mathbf{L}^2}^q ds\right] \\
&\leq C_q \mathbb{E}\left[\sum_{n=1}^{\ell} \int_{t_{n-1}}^{t_n} \|\mathbf{u}(s) - \mathbf{u}(t_{n-1})\|_{\mathbf{L}^2}^q ds\right] + C_q \mathbb{E}\left[\sum_{n=1}^{\ell} \int_{t_{n-1}}^{t_n} \|\mathbf{e}^{n-1}\|_{\mathbf{L}^2}^q ds\right] \\
&\leq Ck^{\gamma q}.
\end{aligned}$$

The desired estimate follows from substituting the above estimates for terms $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ into (3.78) and dividing the inequality by β_0^q . The proof is complete. $\square$

Next, we state the pathwise error estimate for the pressure approximation. For the best of our knowledge, this is the first pathwise convergence result for the pressure approximation.

Theorem 3.3.8. *Assume the assumptions of Theorem 3.3.7 hold. Let $2 < q < \infty$ and $0 < \gamma < \frac{1}{2}$ such that $\gamma - \frac{1}{q} > 0$. Then, for $0 < \gamma_1 < \gamma - \frac{1}{q}$, there exists a random variable $K_1 = K_1(\omega; C_2)$ with $\mathbb{E}[|K_1|^q] < \infty$ such that for all $1 \leq \ell \leq M$, there holds $\mathbb{P}$ -a.s.*

$$\left\| P(t_\ell) - k \sum_{n=1}^{\ell} p^n \right\|_{L^2} \leq K_1 k^{\gamma_1}. \quad (3.79)$$

Proof. The assertion follows immediately from an application of Theorem 3.2.1 based on the high moment error estimates of Theorem 3.3.7. $\square$

3.4 Fully discrete mixed finite element discretization

In this section, we formulate and analyze the spatial approximations of Algorithm 1 by using the mixed finite element method.

3.4.1 Formulation of the mixed finite element method

Let $\mathcal{T}_h$ be a quasi-uniform mesh of the domain $D \subset \mathbb{R}^2$ with mesh size $h > 0$. We introduce the following finite element spaces:

$$\begin{aligned} \mathbb{H}_h &= \{ \mathbf{v}_h \in \mathbf{C}(\overline{D}) \cap \mathbf{H}_{per}^1(D); \mathbf{v}_h \in [\mathcal{P}_i(K)]^2 \quad \forall K \in \mathcal{T}_h \}, \\ L_h &= \{ \psi_h \in C(\overline{D}) \cap L_{per}^2; \psi_h \in \mathcal{P}_j(K) \quad \forall K \in \mathcal{T}_h \}, \end{aligned}$$

where $\mathcal{P}_i(K)$ denotes the space of all polynomials on K of degree at most i . It is well-known that the mixed finite element space pair $\mathbb{H}_h$ and L_h must satisfies the Ladyzhenskaja-Babuška-Brezzi (LBB) (or inf-sup condition) which is now quoted: there exists $\beta_1 > 0$ such

that

$$\sup_{\boldsymbol{\phi}_h \in \mathbb{H}_h} \frac{(\operatorname{div} \boldsymbol{\phi}_h, \psi_h)}{\|\nabla \boldsymbol{\phi}_h\|_{\mathbf{L}^2}} \geq \beta_1 \|\psi_h\|_{L^2} \quad \forall \psi_h \in L_h, \quad (3.80)$$

where the constant β_1 is independent of h (and k).

Algorithm 2

Let $\mathbf{u}_h^0$ be a given $\mathbb{H}_h$ -valued random variable. Find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in L^2(\Omega, \mathbb{H}_h \times L_h)$ such that $\mathbb{P}$ -a.s.

$$(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n, \boldsymbol{\phi}_h) + \nu k (\nabla \mathbf{u}_h^{n+1}, \nabla \boldsymbol{\phi}_h) - k(p_h^{n+1}, \operatorname{div} \boldsymbol{\phi}_h) \quad (3.81)$$

$$= k(\mathbf{f}^{n+1}, \boldsymbol{\phi}_h) + (\mathbf{B}(\mathbf{u}_h^n) \Delta W_{n+1}, \boldsymbol{\phi}_h),$$

$$(\operatorname{div} \mathbf{u}_h^n, \psi_h) = 0, \quad (3.82)$$

for all $\boldsymbol{\phi}_h \in \mathbb{H}_h$ and $\psi_h \in L_h$.

Next, we define $\mathbb{V}_h \subset \mathbb{H}_h$ as the following space of discretely divergent-free vector fields:

$$\mathbb{V}_h = \left\{ \boldsymbol{\phi}_h \in \mathbb{H}_h; (\operatorname{div} \boldsymbol{\phi}_h, q_h) = 0 \quad \forall q_h \in L_h \right\}.$$

We notice that in general, $\mathbb{V}_h$ is not a subspace of $\mathbb{V}$.

Let $\mathbf{Q}_h : \mathbf{L}_{per}^2 \rightarrow \mathbb{V}_h$ denote the L^2 -orthogonal projection, which is defined by

$$(\mathbf{v} - \mathbf{Q}_h \mathbf{v}, \boldsymbol{\phi}_h) = 0 \quad \forall \boldsymbol{\phi}_h \in \mathbb{V}_h. \quad (3.83)$$

In addition, we recall the following well-known interpolation estimates:

$$\|\mathbf{v} - \mathbf{Q}_h \mathbf{v}\|_{\mathbf{L}^2} + h \|\nabla(\mathbf{v} - \mathbf{Q}_h \mathbf{v})\|_{\mathbf{L}^2} \leq Ch^2 \|\mathbf{A} \mathbf{v}\|_{\mathbf{L}^2} \quad \forall \mathbf{v} \in \mathbb{V} \cap \mathbf{H}^2(D), \quad (3.84)$$

$$\|\mathbf{v} - \mathbf{Q}_h \mathbf{v}\|_{\mathbf{L}^2} \leq Ch \|\nabla \mathbf{v}\|_{\mathbf{L}^2} \quad \forall \mathbf{v} \in \mathbb{V} \cap \mathbf{H}^1(D). \quad (3.85)$$

We also let $P_h : L_{per}^2 \rightarrow L_h$ denote the L^2 -orthogonal projection defined by

$$(\psi - P_h \psi, q_h) = 0 \quad \forall q_h \in L_h. \quad (3.86)$$

It is well-known that there holds

$$\|\psi - P_h\psi\|_{L^2} \leq Ch\|\nabla\psi\|_{\mathbf{L}^2} \quad \forall \psi \in L^2_{per}(D) \cap H^1(D). \quad (3.87)$$

For the sake of notation brevity, in the rest of this section, we set $\mathbf{f} = 0$.

We conclude this subsection by stating the following stability estimates for $\{\mathbf{u}_h^n\}$ which were proved in [14, Lemma 3.1].

Lemma 3.4.1. *Let $1 \leq q < \infty$ and $\mathbf{u}_h^0 \in L^{2q}(\Omega; \mathbb{H}_h)$ satisfying $\mathbb{E}[\|\mathbf{u}_h^0\|_{\mathbf{L}^2}^{2q}] \leq C$. Then, there exists a pair $\{\mathbf{u}_h^n, p_h^n\}_{n=1}^M \subset L^{2q}(\Omega; \mathbb{H}_h \times L_h)$ that solves Algorithm 2 and satisfies*

$$(i) \quad \mathbb{E} \left[\max_{1 \leq n \leq M} \|\mathbf{u}_h^n\|_{\mathbf{L}^2}^{2q} + \nu k \sum_{n=1}^M \|\mathbf{u}_h^n\|_{\mathbf{L}^2}^{2q-1} \|\nabla \mathbf{u}_h^n\|_{\mathbf{L}^2}^2 \right] \leq C_{T,q},$$

$$(ii) \quad \mathbb{E} \left[\left(k \sum_{n=1}^M \|\nabla \mathbf{u}_h^n\|_{\mathbf{L}^2}^2 \right)^{2q-1} \right] \leq C_{T,q},$$

where $C_{T,q} = C_{T,q}(D_T, q, \mathbf{u}_h^0)$.

3.4.2 High moment and pathwise error estimates for the fully discrete velocity approximation

The goal of this subsection is to establish high moment and pathwise error estimates for the fully discrete velocity approximation generated by Algorithm 2.

Theorem 3.4.2. *Let $2 \leq q < \infty$ and $\mathbf{u}_h^0 = \mathbf{Q}_h \mathbf{u}_0$. Assume that $\mathbf{u}_0 \in L^q(\Omega; \mathbb{V})$. Let $\{(\mathbf{u}^n, p^n)\}$ and $\{\mathbf{u}_h^n, p_h^n\}$ be the velocity and pressure approximations generated by Algorithm 1 and Algorithm 2, respectively. Then there holds*

$$\begin{aligned} & \mathbb{E} \left[\max_{1 \leq n \leq M} \|\mathbf{u}^n - \mathbf{u}_h^n\|_{\mathbf{L}^2}^q \right] + \mathbb{E} \left[\left(\nu k \sum_{n=1}^M \|\nabla(\mathbf{u}^n - \mathbf{u}_h^n)\|_{\mathbf{L}^2}^2 \right)^{\frac{q}{2}} \right] \\ & \leq C_3 \left(h^q + h^q \mathbb{E} \left[\left(k \sum_{n=1}^M \|\nabla p^n\|_{\mathbf{L}^2}^2 \right)^{\frac{q}{2}} \right] \right), \end{aligned} \quad (3.88)$$

where $C_3 = C_3(T, q, \mathbf{u}_0, \mathbf{f}) > 0$ is independent of k and h .

Proof. Let $\mathbf{E}^n = \mathbf{u}^n - \mathbf{u}_h^n$ for $0 \leq n \leq M-1$. Subtracting (3.18a) from (3.81) we obtain the following error equation:

$$\begin{aligned} & (\mathbf{E}^{n+1} - \mathbf{E}^n, \boldsymbol{\phi}_h) + \nu k (\nabla \mathbf{E}^{n+1}, \nabla \boldsymbol{\phi}_h) - k(p^{n+1} - p_h^{n+1}, \operatorname{div} \boldsymbol{\phi}_h) \\ & = ((\mathbf{B}(\mathbf{u}^n) - \mathbf{B}(\mathbf{u}_h^n)) \Delta W_{n+1}, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbb{H}_h. \end{aligned} \quad (3.89)$$

Since $\boldsymbol{\phi}_h = \mathbf{Q}_h \mathbf{E}^{n+1} = \mathbf{E}^{n+1} - (\mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1}) \in \mathbb{V}_h$, then $(p_h^{n+1}, \operatorname{div} \mathbf{Q}_h \mathbf{E}^{n+1}) = 0$. Thus, (3.89) becomes

$$\begin{aligned} & (\mathbf{E}^{n+1} - \mathbf{E}^n, \mathbf{Q}_h \mathbf{E}^{n+1}) + \nu k \|\nabla \mathbf{E}^{n+1}\|^2 \\ & = \nu k (\nabla \mathbf{E}^{n+1}, \nabla (\mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1})) - k(p^{n+1}, \operatorname{div} \mathbf{Q}_h \mathbf{E}^{n+1}) \\ & \quad + ((\mathbf{B}(\mathbf{u}^n) - \mathbf{B}(\mathbf{u}_h^n)) \Delta W_{n+1}, \mathbf{Q}_h \mathbf{E}^{n+1}). \end{aligned} \quad (3.90)$$

Using the orthogonality of the $\mathbf{L}^2$ -projection and the binomial $2(a, a-b) = \|a\|^2 - \|b\|^2 + \|a-b\|^2$, the left-hand side of (3.90) can be written as follows:

$$\begin{aligned} \text{LHS} &= \frac{1}{2} [\|\mathbf{Q}_h \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{Q}_h \mathbf{E}^n\|_{\mathbf{L}^2}^2] \\ & \quad + \frac{1}{2} \|\mathbf{Q}_h (\mathbf{E}^{n+1} - \mathbf{E}^n)\|_{\mathbf{L}^2}^2 + \nu k \|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 =: \text{RHS}, \end{aligned} \quad (3.91)$$

and

$$\begin{aligned} \text{RHS} &= \nu k (\nabla \mathbf{E}^{n+1}, \nabla (\mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1})) + k(p^{n+1}, \operatorname{div} \mathbf{Q}_h \mathbf{E}^{n+1}) \\ & \quad + ((\mathbf{B}(\mathbf{u}^n) - \mathbf{B}(\mathbf{u}_h^n)) \Delta W_{n+1}, \mathbf{Q}_h \mathbf{E}^{n+1}) \\ &= \nu k (\nabla \mathbf{E}^{n+1}, \nabla (\mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1})) + k(p^{n+1}, \operatorname{div} \mathbf{Q}_h \mathbf{E}^{n+1}) \\ & \quad + ((\mathbf{B}(\mathbf{u}^n) - \mathbf{B}(\mathbf{u}_h^n)) \Delta W_{n+1}, \mathbf{Q}_h \mathbf{E}^{n+1} - \mathbf{Q}_h \mathbf{E}^n) \\ & \quad + ((\mathbf{B}(\mathbf{u}^n) - \mathbf{B}(\mathbf{u}_h^n)) \Delta W_{n+1}, \mathbf{Q}_h \mathbf{E}^n) \\ &= \text{I} + \text{II} + \text{III} + \text{IV}. \end{aligned} \quad (3.92)$$

Using Cauchy-Schwarz's inequality and Young's inequality, we obtain

$$\begin{aligned} \text{I} &\leq \frac{\nu k}{4} \|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 + \nu k \|\nabla(\mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1})\|_{\mathbf{L}^2}^2, \\ &\leq \frac{\nu k}{4} \|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 + Ckh^2 \|\mathbf{A} \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 \end{aligned} \quad (3.93)$$

where the first term on the right-hand side of (3.93) will be absorbed to the left-hand side of (3.90) later. In addition,

$$\text{III} \leq \|(\mathbf{B}(\mathbf{u}^n) - \mathbf{B}(\mathbf{u}_h^n)) \Delta W_{n+1}\|_{\mathbf{L}^2}^2 + \frac{1}{4} \|\mathbf{Q}_h(\mathbf{E}^{n+1} - \mathbf{E}^n)\|_{\mathbf{L}^2}^2. \quad (3.94)$$

Moreover, using the fact that $(P_h p^{n+1}, \operatorname{div} \mathbf{Q}_h \mathbf{E}^{n+1}) = 0$, we have

$$\begin{aligned} \text{II} &= k(p^{n+1}, \operatorname{div} \mathbf{Q}_h \mathbf{E}^{n+1}) \\ &= k(p^{n+1} - P_h p^{n+1}, \operatorname{div} \mathbf{Q}_h \mathbf{E}^{n+1}) \\ &\leq \frac{\nu k}{4} \|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 + Ck \|p^{n+1} - P_h p^{n+1}\|_{L^2}^2 \\ &\leq \frac{\nu k}{4} \|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 + Ckh^2 \|\nabla p^{n+1}\|_{\mathbf{L}^2}^2. \end{aligned} \quad (3.95)$$

Substituting (3.91)–(3.95) into (3.90) yields

$$\begin{aligned} &\frac{1}{2} [\|\mathbf{Q}_h \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{Q}_h \mathbf{E}^n\|_{\mathbf{L}^2}^2] + \frac{1}{4} \|\mathbf{Q}_h(\mathbf{E}^{n+1} - \mathbf{E}^n)\|_{\mathbf{L}^2}^2 + \frac{\nu k}{2} \|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 \\ &\leq Ckh^2 \|\mathbf{A} \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 + Ckh^2 \|\nabla p^{n+1}\|_{\mathbf{L}^2}^2 + \|(\mathbf{B}(\mathbf{u}^n) - \mathbf{B}(\mathbf{u}_h^n)) \Delta W_{n+1}\|_{\mathbf{L}^2}^2 \\ &\quad + ((\mathbf{B}(\mathbf{u}^n) - \mathbf{B}(\mathbf{u}_h^n)) \Delta W_{n+1}, \mathbf{Q}_h \mathbf{E}^n). \end{aligned} \quad (3.96)$$

Lowering one index of (3.96) and applying the summation operator $\sum_{n=1}^{\ell}$ for $1 \leq \ell \leq M$, we get

$$\begin{aligned} \|\mathbf{Q}_h \mathbf{E}^{\ell}\|_{\mathbf{L}^2}^2 + \nu k \sum_{n=1}^{\ell} \|\nabla \mathbf{E}^n\|_{\mathbf{L}^2}^2 &\leq Ch^2 k \sum_{n=1}^{\ell} \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 + Ch^2 k \sum_{n=1}^{\ell} \|\nabla p^n\|_{\mathbf{L}^2}^2 \\ &\quad + 2 \sum_{n=1}^{\ell} \|(\mathbf{B}(\mathbf{u}^{n-1}) - \mathbf{B}(\mathbf{u}_h^{n-1})) \Delta W_n\|_{\mathbf{L}^2}^2 \\ &\quad + 2 \left| \sum_{n=1}^{\ell} ((\mathbf{B}(\mathbf{u}^{n-1}) - \mathbf{B}(\mathbf{u}_h^{n-1})) \Delta W_n, \mathbf{Q}_h \mathbf{E}^{n-1}) \right|. \end{aligned} \quad (3.97)$$

Next, taking maximum over all $1 \leq \ell \leq M$ and followed by taking the $\frac{q}{2}$ -power for any $2 \leq q < \infty$ and the expectation to (3.97), we obtain

$$\begin{aligned} &\mathbb{E} \left[\max_{1 \leq \ell \leq M} \|\mathbf{Q}_h \mathbf{E}^{\ell}\|_{\mathbf{L}^2}^q \right] + \mathbb{E} \left[\left(\nu k \sum_{n=1}^M \|\nabla \mathbf{E}^n\|_{\mathbf{L}^2}^2 \right)^{\frac{q}{2}} \right] \\ &\leq C_q h^q \mathbb{E} \left[\left(k \sum_{n=1}^M \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 \right)^{\frac{q}{2}} \right] + C_q h^q \mathbb{E} \left[\left(k \sum_{n=1}^M \|\nabla p^n\|_{\mathbf{L}^2}^2 \right)^{\frac{q}{2}} \right] \\ &\quad + C_q \mathbb{E} \left[\left(\sum_{n=1}^M \|(\mathbf{B}(\mathbf{u}^{n-1}) - \mathbf{B}(\mathbf{u}_h^{n-1})) \Delta W_n\|_{\mathbf{L}^2}^2 \right)^{\frac{q}{2}} \right] \\ &\quad + C_q \mathbb{E} \left[\max_{1 \leq \ell \leq M} \left| \sum_{n=1}^{\ell} ((\mathbf{B}(\mathbf{u}^{n-1}) - \mathbf{B}(\mathbf{u}_h^{n-1})) \Delta W_n, \mathbf{Q}_h \mathbf{E}^{n-1}) \right|^{\frac{q}{2}} \right]. \end{aligned} \quad (3.98)$$

We can use stability estimate (ii) in Lemma 3.3.1 to control the first term on the right-hand side of (3.98) and Lemma 3.3.2 to bound the second term. Hence, it remains to bound the last two terms on the right-hand side of (3.98). Proceeding similarly as in (3.24) and

(3.25), we obtain

$$\begin{aligned}
& \mathbb{E} \left[\left(\sum_{n=1}^M \|(\mathbf{B}(\mathbf{u}^{n-1}) - \mathbf{B}(\mathbf{u}_h^{n-1})) \Delta W_n\|_{\mathbf{L}^2}^2 \right)^{\frac{q}{2}} \right] \\
&= \mathbb{E} \left[\left(\sum_{n=1}^M \|\mathbf{B}(\mathbf{u}^{n-1}) - \mathbf{B}(\mathbf{u}_h^{n-1})\|_{\mathbf{L}^2}^2 |\Delta W_n|^2 \right)^{\frac{q}{2}} \right] \\
&\leq C_q \mathbb{E} \left[\left(\sum_{n=1}^M \|\mathbf{E}^{n-1}\|_{\mathbf{L}^2}^2 |\Delta W_n|^2 \right)^{\frac{q}{2}} \right] \\
&\leq C_q M^{\frac{q}{2}-1} \mathbb{E} \left[\sum_{n=1}^M \|\mathbf{E}^{n-1}\|_{\mathbf{L}^2}^q |\Delta W_{n+1}|^q \right] \\
&\leq C_q M^{\frac{q}{2}-1} k^{\frac{q}{2}} \sum_{n=1}^M \mathbb{E} [\|\mathbf{E}^{n-1}\|_{\mathbf{L}^2}^q] \\
&\leq C_q k \sum_{n=1}^M \mathbb{E} [\|\mathbf{Q}_h \mathbf{E}^{n-1}\|_{\mathbf{L}^2}^q] + C_q k \sum_{n=1}^M \mathbb{E} [\|\mathbf{u}^{n-1} - \mathbf{Q}_h \mathbf{u}^{n-1}\|_{\mathbf{L}^2}^q] \\
&\leq C_q k \sum_{n=1}^M \mathbb{E} [\|\mathbf{Q}_h \mathbf{E}^{n-1}\|_{\mathbf{L}^2}^q] + C_q h^q k \sum_{n=1}^M \mathbb{E} [\|\nabla \mathbf{u}^{n-1}\|_{\mathbf{L}^2}^q].
\end{aligned} \tag{3.99}$$

To bound the second term on the right-hand side of (3.98), we use the Burkholder-Davis-Gundy and Hölder inequalities to obtain

$$\begin{aligned}
& \mathbb{E} \left[\max_{1 \leq \ell \leq M} \left| \sum_{n=1}^{\ell} ((\mathbf{B}(\mathbf{u}^{n-1}) - \mathbf{B}(\mathbf{u}_h^{n-1})) \Delta W_n, \mathbf{Q}_h \mathbf{E}^{n-1}) \right|^{\frac{q}{2}} \right] \\
&\leq C_q \mathbb{E} \left[\left(\sum_{n=1}^M \|\mathbf{B}(\mathbf{u}^{n-1}) - \mathbf{B}(\mathbf{u}_h^{n-1})\|_{\mathbf{L}^2}^2 |\Delta W_n|^2 \|\mathbf{Q}_h \mathbf{E}^{n-1}\|_{\mathbf{L}^2}^2 \right)^{\frac{q}{4}} \right] \\
&\leq C_q \mathbb{E} \left[\left(\sum_{n=1}^M \|\mathbf{E}^{n-1}\|_{\mathbf{L}^2}^2 \|\mathbf{Q}_h \mathbf{E}^{n-1}\|_{\mathbf{L}^2}^2 |\Delta W_n|^2 \right)^{\frac{q}{4}} \right] \\
&\leq C_q M^{q/4-1} k^{\frac{q}{4}} \sum_{n=1}^M \mathbb{E} [\|\mathbf{E}^{n-1}\|_{\mathbf{L}^2}^{\frac{q}{2}} \|\mathbf{Q}_h \mathbf{E}^{n-1}\|_{\mathbf{L}^2}^{\frac{q}{2}}] \\
&\leq C_q k \sum_{n=1}^M \mathbb{E} [\|\mathbf{Q}_h \mathbf{E}^{n-1}\|_{\mathbf{L}^2}^q] + C_q k \sum_{n=1}^M \mathbb{E} [\|\mathbf{u}^{n-1} - \mathbf{Q}_h \mathbf{u}^{n-1}\|_{\mathbf{L}^2}^{\frac{q}{2}} \|\mathbf{Q}_h \mathbf{E}^{n-1}\|_{\mathbf{L}^2}^{\frac{q}{2}}] \\
&\leq C_q k \sum_{n=1}^M \mathbb{E} [\|\mathbf{Q}_h \mathbf{E}^{n-1}\|_{\mathbf{L}^2}^q] + C_q k \sum_{n=1}^M \mathbb{E} [\|\mathbf{u}^{n-1} - \mathbf{Q}_h \mathbf{u}^{n-1}\|_{\mathbf{L}^2}^q]
\end{aligned} \tag{3.100}$$

$$\leq C_q k \sum_{n=1}^M \mathbb{E}[\|\mathbf{Q}_h \mathbf{E}^{n-1}\|_{\mathbf{L}^2}^q] + C_q h^q k \sum_{n=1}^M \mathbb{E}[\|\nabla \mathbf{u}^{n-1}\|_{\mathbf{L}^2}^q].$$

Substituting (3.99) and (3.100) to the right-hand side of (3.98) yields

$$\begin{aligned} & \mathbb{E}[\max_{1 \leq \ell \leq M} \|\mathbf{Q}_h \mathbf{E}^\ell\|_{\mathbf{L}^2}^q] + \mathbb{E}\left[\left(\nu k \sum_{n=1}^M \|\nabla \mathbf{E}^n\|_{\mathbf{L}^2}^2\right)^{\frac{q}{2}}\right] \\ & \leq C_q h^q \mathbb{E}\left[\left(k \sum_{n=1}^M \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2\right)^{q/2}\right] + C_q h^q \mathbb{E}\left[\left(k \sum_{n=1}^M \|\nabla p^n\|_{\mathbf{L}^2}^2\right)^{\frac{q}{2}}\right] \\ & \quad + C_q h^q k \sum_{n=1}^M \mathbb{E}[\|\nabla \mathbf{u}^{n-1}\|_{\mathbf{L}^2}^q] + C_q k \sum_{n=1}^M \mathbb{E}[\|\mathbf{Q}_h \mathbf{E}^{n-1}\|_{\mathbf{L}^2}^q] \\ & \leq C_q h^q + C_q h^q \mathbb{E}\left[\left(k \sum_{n=1}^M \|\nabla p^n\|_{\mathbf{L}^2}^2\right)^{\frac{q}{2}}\right] \\ & \quad + C_q k \sum_{n=1}^{M-1} \mathbb{E}[\max_{1 \leq \ell \leq n} \|\mathbf{Q}_h \mathbf{E}^\ell\|_{\mathbf{L}^2}^q] \\ & \leq \left(C_q h^q + C_q h^q \mathbb{E}\left[\left(k \sum_{n=1}^M \|\nabla p^n\|_{\mathbf{L}^2}^2\right)^{\frac{q}{2}}\right]\right) e^{C_q T}, \end{aligned} \tag{3.101}$$

where the discrete Gronwall inequality is used to obtain the last inequality.

The proof is now completed by using the triangular inequality $\|\mathbf{E}^n\|_{\mathbf{L}^2} \leq \|\mathbf{Q}_h \mathbf{E}^n\|_{\mathbf{L}^2} + \|\mathbf{u}^n - \mathbf{Q}_h \mathbf{u}^n\|_{\mathbf{L}^2}$. $\square$

We conclude this subsection by stating a pathwise error estimate for the velocity approximation by Algorithm 2 which is a direct corollary of the Kolmogorov Theorem (cf. Theorem 3.2.1) and the high moment error estimates of Theorem 3.4.2.

Theorem 3.4.3. *Assume that the assumptions of Theorem 3.4.2 hold. Let $2 \leq q < \infty$ and $0 < \gamma_2 < 1 - \frac{1}{q}$. Then, there exists a random variable $K_2 = K_2(\omega; C_3)$ with $\mathbb{E}[|K_2|^q] < \infty$ such that there holds $\mathbb{P}$ -a.s.*

$$\max_{1 \leq n \leq M} \|\mathbf{u}^n - \mathbf{u}_h^n\|_{\mathbf{L}^2} + \left(\nu k \sum_{n=1}^M \|\nabla(\mathbf{u}^n - \mathbf{u}_h^n)\|_{\mathbf{L}^2}^2\right)^{1/2} \leq K_2 \left(h^{\gamma_2} + \left(\frac{h}{\sqrt{k}}\right)^{\gamma_2}\right), \tag{3.102}$$

3.4.3 High moment and pathwise error estimates for the fully discrete pressure approximation

In this subsection, we establish high moment and pathwise error estimates for the pressure approximation generated by Algorithm 2.

Theorem 3.4.4. *Let $2 \leq q < \infty$, under the assumptions of Theorem 3.4.2, there holds*

$$\mathbb{E} \left[\left\| k \sum_{n=1}^M (p^n - p_h^n) \right\|_{L^2}^q \right] \leq C_4 \left(h^q + h^q \mathbb{E} \left[\left(k \sum_{n=1}^M \|\nabla p^n\|_{\mathbf{L}^2}^2 \right)^{\frac{q}{2}} \right] \right), \quad (3.103)$$

where $C_4 = C_4(\beta_1, C_3)$ and independent of k, h .

Proof. The proof of (3.103) mimics that of Theorem 3.3.7. Let $\mathcal{E}_p^n = p^n - p_h^n$ and $\mathbf{E}^n = \mathbf{u}^n - \mathbf{u}_h^n$ be the same as in Theorem 3.4.2. Applying the summation operator $\sum_{n=1}^M$ to the pressure error equation, we obtain

$$\begin{aligned} \left(k \sum_{n=1}^M \mathcal{E}_p^n, \operatorname{div} \phi_h \right) &= (\mathbf{E}^M - \mathbf{E}^0, \phi_h) + \nu k \sum_{n=1}^M (\nabla \mathbf{E}^n, \nabla \phi_h) \\ &\quad - \sum_{n=1}^M ((\mathbf{B}(\mathbf{u}^{n-1}) - \mathbf{B}(\mathbf{u}_h^{n-1})) \Delta W_n, \phi_h). \end{aligned} \quad (3.104)$$

Using Schwarz inequality, we get

$$\begin{aligned} \frac{\left(k \sum_{n=1}^M \mathcal{E}_p^n, \operatorname{div} \phi_h \right)}{\|\nabla \phi_h\|_{\mathbf{L}^2}} &\leq C(\|\mathbf{E}^M\|_{\mathbf{L}^2} + \|\mathbf{E}^0\|_{\mathbf{L}^2}) + \nu k \sum_{n=1}^M \|\nabla \mathbf{E}^n\|_{\mathbf{L}^2} \\ &\quad + C \sum_{n=1}^M \|(\mathbf{B}(\mathbf{u}^{n-1}) - \mathbf{B}(\mathbf{u}_h^{n-1})) \Delta W_n\|_{\mathbf{L}^2}. \end{aligned} \quad (3.105)$$

Next, applying the discrete inf-sup condition (3.80) on the left-hand side yields

$$\begin{aligned} \beta_1 \left\| k \sum_{n=1}^M \mathcal{E}_p^n \right\|_{L^2} &\leq C(\|\mathbf{E}^M\|_{\mathbf{L}^2} + \|\mathbf{E}^0\|_{\mathbf{L}^2}) + \nu k \sum_{n=1}^M \|\nabla \mathbf{E}^n\|_{\mathbf{L}^2} \\ &\quad + C \sum_{n=1}^M \|(\mathbf{B}(\mathbf{u}^{n-1}) - \mathbf{B}(\mathbf{u}_h^{n-1})) \Delta W_n\|_{\mathbf{L}^2}. \end{aligned} \quad (3.106)$$

Taking the q -power to (3.106) followed by taking the expectations, we obtain

$$\begin{aligned}
\beta_1^q \mathbb{E} \left[\left\| k \sum_{n=1}^M \mathcal{E}_p^n \right\|_{L^2}^q \right] &\leq C_q \mathbb{E} \left[\|\mathbf{E}^M\|_{\mathbf{L}^2}^q + \|\mathbf{E}^0\|_{\mathbf{L}^2}^q \right] \\
&\quad + C_q \mathbb{E} \left[\left(\nu k \sum_{n=1}^M \|\nabla \mathbf{E}^n\|_{\mathbf{L}^2}^2 \right)^{\frac{q}{2}} \right] \\
&\quad + C_q \mathbb{E} \left[\left(\sum_{n=1}^M \|(\mathbf{B}(\mathbf{u}^{n-1}) - \mathbf{B}(\mathbf{u}_h^{n-1})) \Delta W_n\|_{\mathbf{L}^2} \right)^q \right].
\end{aligned} \tag{3.107}$$

The first three terms on the right side of (3.107) can be controlled by Theorem 3.4.2, and the noise term can be bounded similarly as in (3.24) and (3.25) and using Theorem 3.4.2. In summary, we obtain

$$\mathbb{E} \left[\left\| k \sum_{n=1}^M \mathcal{E}_p^n \right\|_{L^2}^q \right] \leq \beta_1^q C_3 \left(h^q + h^q \mathbb{E} \left[\left(k \sum_{n=1}^M \|\nabla p^n\|_{\mathbf{L}^2}^2 \right)^{\frac{q}{2}} \right] \right). \tag{3.108}$$

Hence, the proof is complete. $\square$

An immediate consequence of the above high moment error estimates is the following pathwise error estimate for the pressure approximation $\{p_h^n\}$, its proof follows from an application of Theorem 3.2.1.

Theorem 3.4.5. *Assume that the assumptions of Theorem 3.4.4 hold. Let $2 \leq q < \infty$ and $0 < \gamma_2 < 1 - \frac{1}{q}$. Then, there exists a random variable $K_2 = K_2(\omega; C_4)$ with $\mathbb{E}[|K_2|^q] < \infty$ such that there holds $\mathbb{P}$ -a.s.*

$$\left\| k \sum_{n=1}^M (p^n - p_h^n) \right\|_{L^2} \leq K_2 \left(h^{\gamma_2} + \left(\frac{h}{\sqrt{k}} \right)^{\gamma_2} \right). \tag{3.109}$$

We conclude this section by stating the global error estimates for our fully discrete numerical solution generated by Algorithm 2 by combining the above temporal and spatial error estimates.

Theorem 3.4.6. *Let $2 \leq q < \infty$ and $0 < \gamma < \frac{1}{2}$, under the assumptions of Theorem 3.3.3 and Theorem 3.4.2, there exists a constant $C_q = C(D_T, \mathbf{u}_0, q, \mathbf{f}) > 0$ such that*

$$\begin{aligned} & \left(\mathbb{E} \left[\max_{1 \leq n \leq M} \|\mathbf{u}(t_n) - \mathbf{u}_h^n\|_{\mathbf{L}^2}^q \right] \right)^{\frac{1}{q}} + \left(\mathbb{E} \left[\left\| \nu k \sum_{n=1}^M \nabla(\mathbf{u}(t_n) - \mathbf{u}_h^n) \right\|_{\mathbf{L}^2}^q \right] \right)^{\frac{1}{q}} \\ & \leq C_q \left(k^\gamma + h + \frac{h}{\sqrt{k}} \right). \end{aligned} \quad (3.110)$$

In addition, let $2 < q < \infty$ and $0 < \gamma < \frac{1}{2}$ such that $\gamma - \frac{1}{q} > 0$ and $1 - \frac{1}{q} > 0$. Then, for any $0 < \gamma_1 < \gamma - \frac{1}{q}$ and $0 < \gamma_2 < 1 - \frac{1}{q}$, there exists a random variable K with $\mathbb{E}[|K|^q] < \infty$ such that there holds $\mathbb{P} - a.s.$

$$\begin{aligned} & \max_{1 \leq n \leq M} \|\mathbf{u}(t_n) - \mathbf{u}_h^n\|_{\mathbf{L}^2} + \left\| \nu k \sum_{n=1}^M \nabla(\mathbf{u}(t_n) - \mathbf{u}_h^n) \right\|_{\mathbf{L}^2} \\ & \leq K \left(k^{\gamma_1} + h^{\gamma_2} + \left(\frac{h}{\sqrt{k}} \right)^{\gamma_2} \right). \end{aligned} \quad (3.111)$$

Theorem 3.4.7. *Let $2 \leq q < \infty$ and $0 < \gamma < \frac{1}{2}$. Under the assumptions of Theorem 3.3.7 and Theorem 3.4.4, there exists a constant $C_q = C(D_T, \mathbf{u}_0, q, \mathbf{f}, \beta_0, \beta_1) > 0$ such that for $1 \leq \ell \leq M$*

$$\left(\mathbb{E} \left[\left\| P(t_\ell) - k \sum_{n=1}^{\ell} p_h^n \right\|_{L^2}^q \right] \right)^{\frac{1}{q}} \leq C_q \left(k^\gamma + h + \frac{h}{\sqrt{k}} \right). \quad (3.112)$$

In addition, let $2 < q < \infty$ and $0 < \gamma < \frac{1}{2}$ such that $\gamma - \frac{1}{q} > 0$ and $1 - \frac{1}{q} > 0$. Then, for any $0 < \gamma_1 < \gamma - \frac{1}{q}$ and $0 < \gamma_2 < 1 - \frac{1}{q}$, there exists a random variable K with $\mathbb{E}[|K|^q] < \infty$ such that there holds $\mathbb{P} - a.s.$

$$\left\| P(t_\ell) - k \sum_{n=1}^{\ell} p_h^n \right\|_{L^2} \leq K \left(k^{\gamma_1} + h^{\gamma_2} + \left(\frac{h}{\sqrt{k}} \right)^{\gamma_2} \right). \quad (3.113)$$

Remark 3.4.1. *The error bounds for the velocity and pressure approximations contain a “bad” factor $k^{-\frac{1}{2}}$, however, the numerical tests of [34] showed that this dependence is sharp when $q = 2$ for the standard mixed finite element method in the case of general multiplicative noises. Recently, a modified mixed method was proposed in [33] which eliminates the $k^{-\frac{1}{2}}$*

factor in (3.110)–(3.113) when $q = 2$ and hence achieve optimal order error estimates. In the last section, we shall also drive high moment and pathwise error estimates for that modified mixed method.

3.5 Extension to a modified mixed finite element method

In this section, we reconsider the modified mixed formulations/methods Algorithms 1 and 2 in Chapter 2. Our goal is to obtain improved high moment and pathwise error estimates for both modified algorithms as alluded in Remark 3.4.1.

Lemma 3.5.1. *Let $\{(\mathbf{u}^{n+1}, r^{n+1})\}_n$ be generated in (2.19) by Algorithm 1, Section 2.2. Let $1 \leq q < \infty$ and assume that $\mathbf{u}_0 \in L^{2q}(\Omega; \mathbb{V})$. Then, there exists $C = C(T, q) > 0$ such that*

$$(a) \quad \mathbb{E} \left[\max_{1 \leq n \leq M} \|\nabla \mathbf{u}^n\|_{\mathbb{V}}^{2q} + \left(\nu k \sum_{n=1}^M \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 \right)^q \right] \leq C,$$

$$(b) \quad \mathbb{E} \left[\left(k \sum_{n=1}^M \|\nabla r^n\|_{\mathbf{L}^2}^2 \right)^q \right] \leq C.$$

Proof. We refer to [17, Lemma 3.1] for a proof of (a). The proof of (b) follows the same lines as the proof of (3.19) in Lemma 3.3.2. $\square$

It turns out that the improved stability estimate for the pseudo pressure approximation in Lemma 3.5.1 (b) is crucial for obtaining the optimal order high moment error estimates for $\{\mathbf{u}_h^n, p_h^n\}$ generated in (2.38), Algorithm 2, Chapter 2. We end this section by the following theorem which establishes those optimal estimates.

Theorem 3.5.2. *Let $2 \leq q < \infty$ and $0 < \gamma < \frac{1}{2}$. Assume that $\mathbf{u}_0 \in L^q(\Omega; \mathbb{V})$ and $\mathbf{u}_h^0 \in L^q(\Omega; \mathbb{H}_h)$ such that $\mathbb{E}[\|\mathbf{u}_0 - \mathbf{u}_h^0\|_{\mathbf{L}^2}^q] \leq Ch^q$. Let $(\mathbf{u}, P, R)$ be solution defined by (2.9), Theorem 2.1.3 and (2.17), respectively. Let $\{\mathbf{u}_h^n, p_h^n, r_h^n\}$ be the velocity and pressure approximation generated by Algorithm 4. Then, there exists a constant $C_q =$*

$C(D_T, \mathbf{u}_0, q, \mathbf{f}) > 0$ such that

$$\left(\mathbb{E} \left[\max_{1 \leq n \leq M} \|\mathbf{u}(t_n) - \mathbf{u}_h^n\|_{\mathbf{L}^2}^q \right] \right)^{\frac{1}{q}} + \left(\mathbb{E} \left[\left\| \nu k \sum_{n=1}^M \nabla(\mathbf{u}(t_n) - \mathbf{u}_h^n) \right\|_{\mathbf{L}^2}^q \right] \right)^{\frac{1}{q}} \leq C_q(k^\gamma + h), \quad (3.114)$$

$$\left(\mathbb{E} \left[\left\| P(t_\ell) - k \sum_{n=1}^{\ell} p_h^n \right\|_{L^2}^q \right] \right)^{\frac{1}{q}} + \left(\mathbb{E} \left[\left\| R(t_\ell) - k \sum_{n=1}^{\ell} r_h^n \right\|_{L^2}^q \right] \right)^{\frac{1}{q}} \leq C_q(k^\gamma + h), \quad (3.115)$$

where $1 \leq \ell \leq M$.

In addition, let $2 < q < \infty$ and $0 < \gamma < \frac{1}{2}$ such that $\gamma - \frac{1}{q} > 0$ and $1 - \frac{1}{q} > 0$. Then, for any $0 < \gamma_1 < \gamma - \frac{1}{q}$ and $0 < \gamma_2 < 1 - \frac{1}{q}$, there exists a random variable K with $\mathbb{E}[|K|^q] < \infty$ such that there holds $\mathbb{P}$ -a.s.

$$\max_{1 \leq n \leq M} \|\mathbf{u}(t_n) - \mathbf{u}_h^n\|_{\mathbf{L}^2} + \left\| \nu k \sum_{n=1}^M \nabla(\mathbf{u}(t_n) - \mathbf{u}_h^n) \right\|_{\mathbf{L}^2} \leq K(k^{\gamma_1} + h^{\gamma_2}), \quad (3.116)$$

$$\left\| P(t_\ell) - k \sum_{n=1}^{\ell} p_h^n \right\|_{L^2} + \left\| R(t_\ell) - k \sum_{n=1}^{\ell} r_h^n \right\|_{L^2} \leq K(k^{\gamma_1} + h^{\gamma_2}). \quad (3.117)$$

Proof. The proof of (3.114) follows the same lines as in the proofs of Theorem 3.3.3 and Theorem 3.4.2 but using instead the improved stability estimate for the pseudo pressure approximation given in Lemma 3.5.1 (b). (3.115) with $q = 2$ was proved in Theorem 2.2.2. Again, by mimicking the proofs of Theorems 3.3.7 and 3.4.4 we can obtain the desired high moment error estimates. Finally, estimates (3.116) and (3.117) are direct corollaries of Komogorov Theorem 3.2.1. $\square$

3.6 Computational experiments

In this section, we present numerical tests to validate our error estimates (3.110) and (3.112). In all the tests, we set $D = (0, 1)^2 \subset \mathbb{R}^2$, $T = 1$, $\nu = 1$, $\mathbf{f}$ given as in (2.75). In addition, we use the Taylor-Hood mixed finite element method for the spatial discretization and the

homogeneous Dirichlet boundary condition is imposed on $\mathbf{u}$. We reconsider the Q -Wiener process $\mathbf{W}$ given as in (2.73) and take $B(\mathbf{u}) = 5(u_1^2 + u_2^2 + 1)^{\frac{1}{2}}$.

Moreover, we choose $J = 4$ and $N_p = 300$ number of samples in our Monte Carlo simulations to compute the expectations in each test. We implement Algorithm 2 and evaluate the errors between the computed velocity (resp. pressure) and the reference velocity (resp. pressure) which are generated by performing Algorithm 2 on a very fine mesh with mesh sizes $k_0 = \frac{1}{1024}$, $h = \frac{1}{40}$.

Furthermore, to evaluate errors in strong norms, we use the following numerical integration formulas: for any $0 \leq m \leq M$ and any integer $2 \leq q < \infty$,

$$\begin{aligned}\mathcal{E}_{\mathbf{u},q}^m &:= \left(\mathbb{E} \left[\left\| \mathbf{u}(t_m) - \mathbf{u}_h^m(k) \right\|_{\mathbf{L}^2}^q \right] \right)^{\frac{1}{q}} \approx \left(\frac{1}{N_p} \sum_{\ell=1}^{N_p} \left\| \mathbf{u}_h^m(k_0, \omega_\ell) - \mathbf{u}_h^m(k, \omega_\ell) \right\|_{\mathbf{L}^2}^q \right)^{\frac{1}{q}}, \quad (3.118) \\ \mathcal{E}_{P,q}^m &:= \left(\mathbb{E} \left[\left\| P(t_m) - k \sum_{n=1}^m p_h^n(k) \right\|_{\mathbf{L}^2}^q \right] \right)^{\frac{1}{q}} \\ &\approx \left(\frac{1}{N_p} \sum_{\ell=1}^{N_p} \left\| k_0 \sum_{n=1}^{\frac{t_m}{k_0}} p_h^n(k_0, \omega_\ell) - k \sum_{n=1}^{\frac{t_m}{k}} p_h^n(k, \omega_\ell) \right\|_{\mathbf{L}^2}^q \right)^{\frac{1}{q}}.\end{aligned}$$

Test 1. In this test, we verify the convergence rates in the error estimates (3.110), (3.112) of Theorem 3.4.6 and 3.4.7. To the end, we compute the high moment errors $\mathcal{E}_{\mathbf{u},q}^M$ and $\mathcal{E}_{P,q}^M$ with $q = 2, 4, 8$ of the velocity and pressure approximations generated by Algorithm 2. Table 3.1 and 3.2 display the errors of the velocity and pressure approximations for the time discretization using different time step size k . The numerical results verify almost a half order convergence rate as predicted by our error estimate results.

Test 2. In this test, we would like to check numerically how the error constant C_q in Theorem 3.4.6 and 3.4.7 depends on q . To the end, we fix $h = \frac{1}{10}$ and choose the time step $k = \frac{1}{16}$ to compute the errors $\mathcal{E}_{\mathbf{u},q}^M$ and $\mathcal{E}_{P,q}^M$ for different values of q . The numerical results are given in Figure 3.1 and 3.2 (left ones). We observe that the numerical results suggest the constant C_q increases (and blows up) in q . For a close-up view, we zoom in at large q to see the behaviors of both velocity and pressure errors in Figure 3.1 and 3.2 (right ones). We still see the increasing of the errors in q although the growth becomes slower for large q . A consequence of this analysis also shows that we can not simply take limit as $p \rightarrow \infty$ in the

high moment error estimates of Theorem 3.4.6 and 3.4.7 to derive pathwise error estimates (3.111) and (3.113), and using Kolmogorov's Theorem is still the only viable approach for the job.

Test 3. In this test, we verify the pathwise error estimates (3.111) and (3.113). To the end, we select five sample paths and compute their L^2 -norm errors for both velocity and pressure approximations. The computed results are given in Figure 3.3. The numerical results indicate that the pathwise convergence of the velocity and pressure approximations is almost of order $O(k^{\frac{1}{2}})$, which matches the theoretical predication.

Table 3.1: Algorithm 2: Time discretization errors of the computed velocity $\{\mathbf{u}_h^n\}_n$

k	$\mathcal{E}_{\mathbf{u},2}^M$	order	$\mathcal{E}_{\mathbf{u},4}^M$	order	$\mathcal{E}_{\mathbf{u},8}^M$	order
1/32	0.0303112		0.0423472		0.0621948	
1/64	0.0241111	0.3302	0.0339389	0.3193	0.0511043	0.2833
1/128	0.016392	0.5567	0.0245307	0.4684	0.0409417	0.3200
1/256	0.00953818	0.7812	0.0135834	0.8527	0.0198862	1.0417

Table 3.2: Algorithm 2: Time discretization errors of the computed pressure $\{p_h^n\}_n$

k	$\mathcal{E}_{P,2}^M$	order	$\mathcal{E}_{P,4}^M$	order	$\mathcal{E}_{P,8}^M$	order
1/32	0.132024		0.138083		0.146199	
1/64	0.081971	0.6876	0.0870388	0.6658	0.0927044	0.6573
1/128	0.0542874	0.5945	0.0591613	0.5570	0.0642976	0.5278
1/256	0.0344033	0.6581	0.0370825	0.6739	0.0398915	0.6887

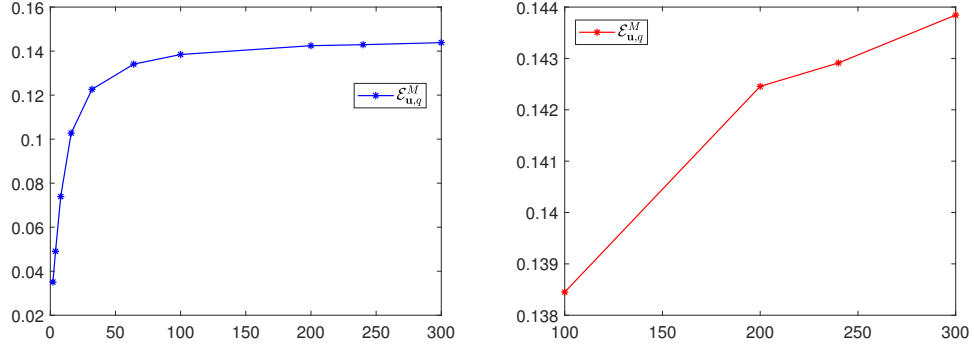


Figure 3.1: Errors of the velocity approximation (left) in $\mathcal{E}_{u,q}^M$ norm and its behavior (right) for large q .

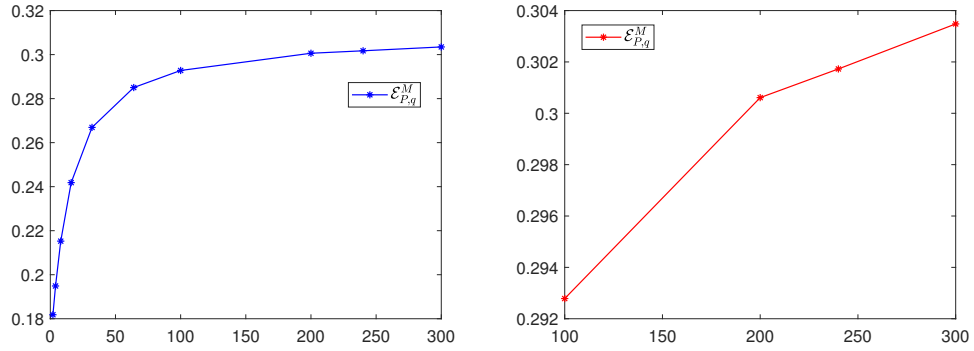


Figure 3.2: Errors of the pressure approximation (left) in $\mathcal{E}_{P,q}^M$ norm and its behavior (right) for large q .

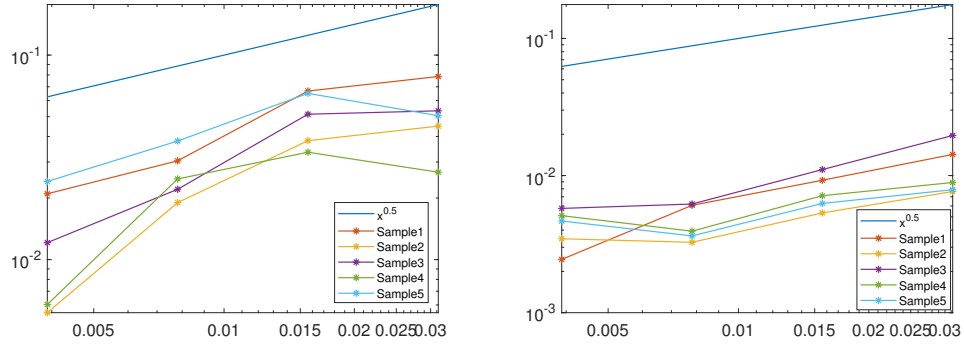


Figure 3.3: Five sample pathwise errors of the velocity (left) and pressure approximation (right) with different time steps.

Chapter 4

Chorin projection methods for stochastic Stokes equations

4.1 Introduction

This chapter is concerned with developing and analyzing Chorin-type projection finite element methods for the time-dependent stochastic Stokes equations (1.3) where $D = (0, L)^2 \subset \mathbb{R}^2$ represents a period of the periodic domain in $\mathbb{R}^2$.

In [16] the authors proposed a Chorin time-splitting finite element method for problem (1.3) and proved a suboptimal convergence rate in strong norm for the velocity approximation in the case of divergence-free noises. In [4] the authors proposed an iterative splitting scheme for stochastic Navier-Stokes equations and a strong convergence in probability was established in the 2-D case for the velocity approximation. In a recent work [5], the authors proposed another time-splitting scheme and proved its strong L^2 convergence for the velocity approximation.

Compared to the recent advances on mixed finite element methods [17, 34, 33], the numerical analysis of the well-known Chorin projection/splitting scheme for the stochastic Stokes equations lags behind. To the best of our knowledge, the only analysis result obtained in [16] is the optimal convergence in the energy norm for the velocity approximation in the case of divergence-free noises (i.e., $\mathbf{B}(\mathbf{u})$ is divergence-free). Several natural and important questions arise and must be addressed for a better understanding of the Chorin projection

scheme for problem (1.3). Among them are (i) *Does the pressure approximation converge even when the noise is divergence-free? If so, in what sense and what rate?* (ii) *Does the Chorin projection scheme converge (for both the velocity and pressure approximations) for general noises? If so, in what sense and what rate?* (iii) *Could the performance of the standard Chorin projection scheme be improved one way or another in the case of general noises?* The primary objective this chapter is to provide a positive answer to each of the above questions.

As it was shown in [16], the adaptation of the standard deterministic Chorin projection scheme to problem (1.3) is straightforward (see Algorithm 1 of Section 4.2). The idea of the Chorin scheme is to separate the computation of the velocity and pressure at each time step which is done by solving two decoupled Poisson problems and the divergence-free constraint for the velocity approximation is enforced by a Helmholtz projection technique which can be easily obtained using the solutions of the two Poisson problems. The Chorin scheme also can be compactly rewritten as a pressure stabilization scheme at each time step as follows (cf. [16]):

$$\tilde{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^n - k\nu\Delta\tilde{\mathbf{u}}^{n+1} + k\nabla p^n = k\mathbf{f}^{n+1} + \mathbf{B}(\tilde{\mathbf{u}}^n)\Delta W_{n+1} \quad \text{a.s. in } D_T, \quad (4.1a)$$

$$\operatorname{div} \tilde{\mathbf{u}}^{n+1} - k\Delta p^{n+1} = 0 \quad \text{a.s. in } D_T, \quad (4.1b)$$

$$\partial_{\mathbf{n}} p^{n+1} = 0 \quad \text{a.s. on } \partial D_T, \quad (4.1c)$$

where $\partial_{\mathbf{n}} p^{n+1}$ denotes the normal derivative of p^{n+1} and k is the time step size. One of advantages of the above Chorin scheme is that the spatial approximation spaces for $\tilde{\mathbf{u}}^{n+1}$ and p^{n+1} can be chosen independently, so unlike in the mixed finite element method, they are not required to satisfy an *inf-sup* condition. Notice that a time lag on pressure appears in equation (4.1a) which causes most of difficulties in the convergence analysis (cf. [60, 38, 58, 16]). We also note that the term $-k\Delta p^{n+1}$ in equation (4.1b) is known as a pressure stabilization term.

To improve the convergence of the standard Chorin scheme, we adopt a Helmholtz projection technique as used in [33] (also see [8, 16]). At each time step we first perform the

Helmholtz decomposition $\mathbf{B}(\tilde{\mathbf{u}}^n) = \boldsymbol{\eta}^n + \nabla \xi^n$ and then rewrite (4.1a) as

$$\tilde{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^n - k\nu \Delta \tilde{\mathbf{u}}^{n+1} + k \nabla r^n = k \mathbf{f}^{n+1} + \boldsymbol{\eta}^n \Delta W_{n+1} \quad \text{a.s. in } D_T, \quad (4.2)$$

where $r^n = p^n - k^{-1} \xi^n \Delta W_{n+1}$. Our modified Chorin scheme consists of (4.2), (4.1b)–(4.1c) and the Helmholtz decomposition $\mathbf{B}(\tilde{\mathbf{u}}^n) = \boldsymbol{\eta}^n + \nabla \xi^n$. Since $\boldsymbol{\eta}^n$ is divergence-free, it turns out that the finite element approximation of the modified Chorin scheme has better convergence properties. Notice that p^n can be recovered from r^n via the simple algebraic relation $p^n = r^n + k^{-1} \xi^n \Delta W_{n+1}$.

We establish some stability estimates for the velocity $\mathbf{u}$ and the pressure P in the following lemma.

Lemma 4.1.1. *Suppose that $\mathbf{u}_0 \in L^2(\Omega; \mathbb{V})$. Let $(\mathbf{u}, P)$ solve (2.14). Then there exists a constant $C \equiv C(D_T, \mathbf{u}_0, \mathbf{f})$ such that*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|\nabla \mathbf{u}(t)\|_{\mathbf{L}^2}^2 \right] + \mathbb{E} \left[\int_0^T \|\mathbf{A} \mathbf{u}(s)\|_{\mathbf{L}^2}^2 ds \right] \leq C, \quad (4.3)$$

$$\sup_{0 \leq t \leq T} \mathbb{E} \left[\|\nabla P(t)\|_{\mathbf{L}^2}^2 \right] \leq C. \quad (4.4)$$

Proof. The proof of (4.3) is standard by using Itô's formula and can be found in [16, Theorem 4.4 and Section 5]. To prove (4.4), using (2.13) can easily get

$$\begin{aligned} \mathbb{E} [\|\nabla P(t)\|_{\mathbf{L}^2}^2] &\leq C \left(\mathbb{E} [\|\mathbf{u}(t)\|_{\mathbf{L}^2}^2] + \mathbb{E} \left[\int_0^t \|\Delta \mathbf{u}(s)\|^2 ds \right] + \mathbb{E} \left[\int_0^t \|\mathbf{u}(s)\|_{\mathbf{L}^2}^2 ds \right] \right. \\ &\quad \left. + \mathbb{E} \left[\int_0^t \|\mathbf{f}(s)\|_{\mathbf{L}^2}^2 ds \right] \right). \end{aligned} \quad (4.5)$$

Then, the desired estimate from immediately after taking supreme over all $t \in [0, T]$ and using (4.3). $\square$

Next, we establishing the following Hölder continuity result for P , which is used for the error analysis in later sections.

Lemma 4.1.2. *Suppose that $\mathbf{u} \in L^2(\Omega; C([0, T]; \mathbb{V})) \cap L^2(\Omega; 0, T; \mathbf{H}_{per}^2(D))$, $\mathbf{f} \in L^2(\Omega; 0, T; \mathbf{L}_{per}^2(D))$ and $P \in L^2(\Omega; L^2(0, T; H_{per}^1(D/\mathbb{R})))$. Then, there holds*

$$\mathbb{E}[\|\nabla(P(s) - P(t))\|_{\mathbf{L}^2}^2] \leq C|s - t| \quad \forall s, t \in [0, T], \quad (4.6)$$

where the constant $C > 0$ depends on D_T , $\mathbf{u}_0$ and $\mathbf{f}$.

Proof. From its definition of $P(t)$, we get

$$\nabla P(t) = -\mathbf{u}(t) + \int_0^t \Delta \mathbf{u}(s) ds + \int_0^t \mathbf{f}(s) ds + \int_0^t \mathbf{B}(\mathbf{u}(s)) d\mathbf{W}(s) \quad (4.7)$$

Therefore, for any $s, t \in [0, T]$, we have

$$\begin{aligned} \nabla(P(t) - P(s)) &= -(\mathbf{u}(t) - \mathbf{u}(s)) + \int_s^t \Delta \mathbf{u}(\tau) d\tau + \int_s^t \mathbf{f}(\tau) d\tau \\ &\quad + \int_s^t \mathbf{B}(\mathbf{u}(\tau)) dW(\tau). \end{aligned} \quad (4.8)$$

Thus,

$$\begin{aligned} \mathbb{E}[\|\nabla(P(t) - P(s))\|_{\mathbf{L}^2}^2] &\leq 4\mathbb{E}[\|\mathbf{u}(t) - \mathbf{u}(s)\|_{\mathbf{L}^2}^2] + 4\mathbb{E}\left[\left\|\int_s^t \Delta \mathbf{u}(\tau) d\tau\right\|_{\mathbf{L}^2}^2\right] \\ &\quad + 4\mathbb{E}\left[\left\|\int_s^t \mathbf{f}(\tau) d\tau\right\|_{\mathbf{L}^2}^2 + \left\|\int_s^t \mathbf{B}(\mathbf{u}(\tau)) dW(\tau)\right\|_{\mathbf{L}^2}^2\right] \\ &= \text{I} + \text{II} + \text{III} + \text{IV}. \end{aligned} \quad (4.9)$$

The first term I can be controlled by using (2.11a). For II and III, by Schwarz inequality, we have

$$\text{II} + \text{III} \leq 4\mathbb{E}\left[\int_s^t \|\Delta \mathbf{u}(\tau)\|_{\mathbf{L}^2}^2 d\tau\right]|t - s| + 4\mathbb{E}\left[\int_s^t \|\mathbf{f}(\tau)\|_{\mathbf{L}^2}^2 d\tau\right]|t - s|. \quad (4.10)$$

In addition, using Itô isometry and (1.5b), we obtain

$$\text{IV} = 4\mathbb{E}\left[\int_s^t \|\mathbf{B}(\mathbf{u}(\tau))\|_{\mathbf{L}^2}^2 d\tau\right] \leq C\mathbb{E}\left[\int_s^t \|\mathbf{u}(\tau)\|_{\mathbf{L}^2}^2 d\tau\right]. \quad (4.11)$$

In summary, we have

$$\begin{aligned}
\mathbb{E}[\|\nabla(P(t) - P(s))\|_{\mathbf{L}^2}^2] &\leq C|t - s| + 4\mathbb{E}\left[\int_s^t \|\Delta \mathbf{u}(\tau)\|_{\mathbf{L}^2}^2 d\tau\right]|t - s| \\
&\quad + 4\mathbb{E}\left[\int_s^t \|\mathbf{f}(\tau)\|_{\mathbf{L}^2}^2 d\tau |t - s| + C \int_s^t \|\mathbf{u}(\tau)\|_{\mathbf{L}^2}^2 d\tau\right] \\
&\leq \left(C + 4\mathbb{E}\left[\int_s^t \|\Delta \mathbf{u}(\tau)\|_{\mathbf{L}^2}^2 d\tau + \int_s^t \|\mathbf{f}(\tau)\|_{\mathbf{L}^2}^2 d\tau\right]\right. \\
&\quad \left.+ C \max_{\tau \in [0, T]} \mathbb{E}[\|\mathbf{u}(\tau)\|_{\mathbf{L}^2}^2]\right)|t - s|.
\end{aligned} \tag{4.12}$$

Finally, the desired estimate follows from the assumptions on $\mathbf{u}$ and $\mathbf{f}$. $\square$

The remainder of this chapter is organized as follows. In section 4.2, we define the standard Chorin projection scheme as Algorithm 1 for problem (1.3) in subsection 4.2.1 and the modified Chorin scheme as Algorithm 2 in subsection 4.2.2. The highlights of this section are to prove some uniform (in k) stability estimates which are very useful for error analysis later. In section 4.3, we formulate the finite element spatial discretization for both the standard Chorin and modified Chorin schemes in Algorithm 3 and 4, respectively and prove the quasi-optimal error estimates for both algorithms as summarized above. In section 4.4, we present several numerical experiments to gauge the performance of the proposed numerical methods and to validate the sharpness of the proved error estimates.

4.2 Two Chorin-type time-stepping schemes

In this section, we first formulate two Chorin-type semi-discrete-in-time schemes for problem (1.3). The first scheme is the standard Chorin scheme and the second one is a Helmholtz decomposition enhanced nonstandard Chorin scheme. We then present a complete convergence analysis for each scheme which include establishing their stability and error estimates in strong norms for both velocity and pressure approximations.

4.2.1 Standard Chorin projection scheme

We first consider the standard Chorin scheme for (1.3), its formulation is a straightforward adaptation of the well-known scheme for the deterministic Stokes problem and is given in Algorithm 1 below. As mentioned earlier, the standard Chorin scheme for (1.3) was already studied in [16] in the special case when the noise is divergence-free and error estimates were only obtained for the velocity approximation. In contrast, here we consider the Chorin scheme for general multiplicative noise and to derive error estimates in strong norms not only for the velocity but also for pressure approximations and to achieve a full understanding about the scheme.

Let M be a (large) positive integer and $k := T/M$ be the time step size. Set $t_j = jk$ for $j = 0, 1, 2, \dots, M$, then $\{t_j\}_{j=0}^M$ forms a uniform mesh for $(0, T)$. The standard Chorin projection scheme is given as follows (cf. [16, 65, 37]):

Algorithm 1

Let $\tilde{\mathbf{u}}^0 = \mathbf{u}^0 = \mathbf{u}_0$. For $n = 0, 1, 2, \dots, M - 1$, do the following three steps.

Step 1: Given $\mathbf{u}^n \in L^2(\Omega; \mathbb{V})$ and $\tilde{\mathbf{u}}^n \in L^2(\Omega; \mathbf{H}_{per}^1(D))$, find $\tilde{\mathbf{u}}^{n+1} \in L^2(\Omega; \mathbf{H}_{per}^1(D))$ such that $\mathbb{P}$ -a.s.

$$\tilde{\mathbf{u}}^{n+1} - k\Delta\tilde{\mathbf{u}}^{n+1} = \mathbf{u}^n + k\mathbf{f}^{n+1} + \mathbf{B}(\tilde{\mathbf{u}}^n)\Delta W_{n+1} \quad \text{in } D. \quad (4.13)$$

Step 2: Find $p^{n+1} \in L^2(\Omega; H_{per}^1(D)/\mathbb{R})$ such that $\mathbb{P}$ -a.s.

$$-\Delta p^{n+1} = -\frac{1}{k}\operatorname{div} \tilde{\mathbf{u}}^{n+1} \quad \text{in } D. \quad (4.14)$$

Step 3: Define $\mathbf{u}^{n+1} \in L^2(\Omega; \mathbb{V})$ by

$$\mathbf{u}^{n+1} = \tilde{\mathbf{u}}^{n+1} - k\nabla p^{n+1}. \quad (4.15)$$

Remark 4.2.1. (a) *The above formulation is written in the way in which the scheme is implemented, it is slightly different from the traditional writing which combines Step 2 and 3 together.*

(b) It is easy to check $(\tilde{\mathbf{u}}^{n+1}, \mathbf{u}^{n+1}, p^{n+1})$ satisfies the following system:

$$\tilde{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^n - k\Delta\tilde{\mathbf{u}}^{n+1} + k\nabla p^n = k\mathbf{f}^{n+1} + \mathbf{B}(\tilde{\mathbf{u}}^n)\Delta W_{n+1} \quad \text{in } D, \quad (4.16a)$$

$$\operatorname{div} \tilde{\mathbf{u}}^{n+1} - k\Delta p^{n+1} = 0 \quad \text{in } D, \quad (4.16b)$$

where $\tilde{u}^0 = u_0$.

First, we introduce some stability estimates for Algorithm 1 in strong norms as below.

Lemma 4.2.1. *The discrete processes $\{(\tilde{\mathbf{u}}^n, p^n)\}_{n=0}^M$ defined in (4.16) satisfy*

$$\max_{0 \leq n \leq M} \mathbb{E}[\|\tilde{\mathbf{u}}^n\|_{\mathbf{L}^2}^2] + \mathbb{E}\left[\sum_{n=1}^M \|\tilde{\mathbf{u}}^n - \tilde{\mathbf{u}}^{n-1}\|_{\mathbf{L}^2}^2\right] + \mathbb{E}\left[k \sum_{n=0}^M \|\nabla \tilde{\mathbf{u}}^n\|_{\mathbf{L}^2}^2\right] \leq C, \quad (4.17a)$$

$$\mathbb{E}\left[k \sum_{n=0}^M \|\nabla p^n\|_{\mathbf{L}^2}^2\right] \leq \frac{C}{k}, \quad (4.17b)$$

$$\max_{0 \leq n \leq M} \mathbb{E}[\|\nabla \tilde{\mathbf{u}}^n\|_{\mathbf{L}^2}^2] + \mathbb{E}\left[k \sum_{n=0}^M \|\Delta \tilde{\mathbf{u}}^n\|_{\mathbf{L}^2}^2\right] \leq \frac{C}{k}, \quad (4.17c)$$

where $C > 0$ depends only on $D_T, u_0, \mathbf{f}$.

Proof. We first prove (4.17a) and (4.17b). Testing (4.16a) by $\tilde{u}^{n+1}$ and (4.16b) by p^n , and integrating by parts, we obtain

$$(\tilde{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^n, \tilde{\mathbf{u}}^{n+1}) + k\|\nabla \tilde{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2 + k(\nabla p^n, \tilde{\mathbf{u}}^{n+1}) \quad (4.18)$$

$$= k(\mathbf{f}^{n+1}, \tilde{\mathbf{u}}^{n+1}) + (\mathbf{B}(\tilde{\mathbf{u}}^n)\Delta W_{n+1}, \tilde{\mathbf{u}}^{n+1}),$$

$$(\tilde{\mathbf{u}}^{n+1}, \nabla p^n) - k(\nabla p^{n+1}, \nabla p^n) = 0. \quad (4.19)$$

Substituting (4.19) into (4.18) yields

$$(\tilde{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^n, \tilde{\mathbf{u}}^{n+1}) + k\|\nabla \tilde{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2 + k^2(\nabla p^{n+1}, \nabla p^n) \quad (4.20)$$

$$= k(\mathbf{f}^{n+1}, \tilde{\mathbf{u}}^{n+1}) + (\mathbf{B}(\tilde{\mathbf{u}}^n)\Delta W_{n+1}, \tilde{\mathbf{u}}^{n+1}).$$

Using the identity $2a(a-b) = a^2 - b^2 + (a-b)^2$ and the following identity from (4.16b)

$$k(\nabla p^{n+1}, \nabla(p^{n+1} - p^n)) = (\tilde{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^n, \nabla p^{n+1}),$$

and taking expectations on both sides of (4.20), we get

$$\begin{aligned} & \frac{1}{2} \mathbb{E}[\|\tilde{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2 - \|\tilde{\mathbf{u}}^n\|_{\mathbf{L}^2}^2 + \|\tilde{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^n\|_{\mathbf{L}^2}^2] + \mathbb{E}[k\|\nabla \tilde{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2 + k^2\|\nabla p^{n+1}\|_{\mathbf{L}^2}^2] \\ &= k\mathbb{E}[(\tilde{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^n, \nabla p^{n+1})] + k\mathbb{E}[(\mathbf{f}^{n+1}, \tilde{\mathbf{u}}^n)] \\ &+ \mathbb{E}[(\mathbf{B}(\tilde{\mathbf{u}}^n)\Delta W_{n+1}, \tilde{\mathbf{u}}^{n+1})]. \end{aligned} \quad (4.21)$$

Next, we bound each term on the right-hand side of (4.21). First, by Schwarz and Young's inequalities, we have

$$k\mathbb{E}[(\tilde{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^n, \nabla p^{n+1})] \leq \frac{3}{10} \mathbb{E}[\|\tilde{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^n\|_{\mathbf{L}^2}^2] + \frac{5}{6} k^2 \mathbb{E}[\|\nabla p^{n+1}\|_{\mathbf{L}^2}^2], \quad (4.22)$$

$$k\mathbb{E}[(\mathbf{f}^{n+1}, \tilde{\mathbf{u}}^n)] \leq Ck\mathbb{E}[\|\mathbf{f}^{n+1}\|_{L^2}^2] + \frac{k}{4} \mathbb{E}[\|\nabla \tilde{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2]. \quad (4.23)$$

In addition, by Itô isometry, we have

$$\begin{aligned} \mathbb{E}[(\mathbf{B}(\tilde{\mathbf{u}}^n)\Delta W_{n+1}, \tilde{\mathbf{u}}^{n+1})] &= \mathbb{E}[(\mathbf{B}(\tilde{\mathbf{u}}^n)\Delta W_{n+1}, \tilde{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^n)] \\ &\leq 2\mathbb{E}[\|\mathbf{B}(\tilde{\mathbf{u}}^n)\Delta W_{n+1}\|_{\mathbf{L}^2}^2] + \frac{1}{8} \mathbb{E}[\|\tilde{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^n\|_{\mathbf{L}^2}^2] \\ &= 2k\mathbb{E}[\|\mathbf{B}(\tilde{\mathbf{u}}^n)\|_{\mathbf{L}^2}^2] + \frac{1}{8} \mathbb{E}[\|\tilde{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^n\|_{\mathbf{L}^2}^2] \\ &\leq Ck\mathbb{E}[\|\tilde{\mathbf{u}}^n\|_{\mathbf{L}^2}^2] + \frac{1}{8} \mathbb{E}[\|\tilde{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^n\|_{\mathbf{L}^2}^2]. \end{aligned} \quad (4.24)$$

Substituting (4.22)–(4.24) into (4.21) gives

$$\begin{aligned} & \frac{1}{2} \mathbb{E}[\|\tilde{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2 - \|\tilde{\mathbf{u}}^n\|_{\mathbf{L}^2}^2] + \frac{1}{20} \mathbb{E}[\|\tilde{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^n\|_{\mathbf{L}^2}^2] + \frac{3k}{4} \mathbb{E}[\|\nabla \tilde{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2] \\ &+ \frac{1}{6} k^2 \mathbb{E}[\|\nabla p^{n+1}\|_{\mathbf{L}^2}^2] \leq Ck\mathbb{E}[\|\mathbf{f}^{n+1}\|_{\mathbf{L}^2}^2] + Ck\mathbb{E}[\|\tilde{\mathbf{u}}^n\|_{\mathbf{L}^2}^2]. \end{aligned} \quad (4.25)$$

Applying the summation operator $\sum_{n=0}^m$ to (4.25) for any $0 \leq m \leq M-1$ yields

$$\begin{aligned} & \frac{1}{2} \mathbb{E} [\|\tilde{\mathbf{u}}^{m+1}\|_{\mathbf{L}^2}^2 - \|\tilde{\mathbf{u}}^0\|^2] + \frac{1}{20} \sum_{n=0}^m \mathbb{E} [\|\tilde{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^n\|_{\mathbf{L}^2}^2] + k \sum_{n=0}^m \mathbb{E} [\|\nabla \tilde{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2] \\ & + \frac{1}{6} k^2 \sum_{n=0}^m \mathbb{E} [\|\nabla p^{n+1}\|_{\mathbf{L}^2}^2] \leq Ck \sum_{n=0}^m \mathbb{E} [\|\tilde{u}^n\|_{\mathbf{L}^2}^2] + Ck \sum_{n=0}^m \mathbb{E} [\|\mathbf{f}^{n+1}\|_{\mathbf{L}^2}^2]. \end{aligned} \quad (4.26)$$

Finally, by the discrete Gronwall's inequality we obtain

$$\begin{aligned} & \frac{1}{2} \max_{1 \leq n \leq M} \mathbb{E} [\|\tilde{\mathbf{u}}^n\|_{\mathbf{L}^2}^2] + \frac{1}{20} \sum_{n=1}^M \mathbb{E} [\|\tilde{\mathbf{u}}^n - \tilde{\mathbf{u}}^{n-1}\|_{\mathbf{L}^2}^2] + \frac{3k}{4} \sum_{n=1}^M \mathbb{E} [\|\nabla \tilde{\mathbf{u}}^n\|_{\mathbf{L}^2}^2] \\ & + \frac{1}{6} k^2 \sum_{n=1}^M \mathbb{E} [\|\nabla p^n\|_{\mathbf{L}^2}^2] \leq C \left(\mathbb{E} [\|\mathbf{u}_0\|_{\mathbf{L}^2}^2] + k \sum_{n=0}^m \mathbb{E} [\|\mathbf{f}^{n+1}\|_{\mathbf{L}^2}^2] \right), \end{aligned}$$

which implies the desired estimate (4.17a) and (4.17b).

It remains to show (4.17c). To the end, testing (4.16a) by $-\Delta \tilde{\mathbf{u}}^{n+1}$ we get

$$\begin{aligned} & \frac{1}{2} [\|\nabla \tilde{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2 - \|\nabla \tilde{\mathbf{u}}^n\|_{\mathbf{L}^2}^2 + \|\nabla(\tilde{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^n)\|_{\mathbf{L}^2}^2] + k \|\Delta \tilde{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2 \\ & = k(\nabla p^n, \Delta \tilde{\mathbf{u}}^{n+1}) - k(\mathbf{f}^{n+1}, \Delta \tilde{\mathbf{u}}^{n+1}) + (\nabla \mathbf{B}(\tilde{\mathbf{u}}^n) \Delta W_{n+1}, \nabla \tilde{\mathbf{u}}^{n+1}). \end{aligned} \quad (4.27)$$

Proceeding with similar arguments as used above yields

$$\begin{aligned} & \frac{1}{2} \mathbb{E} [\|\nabla \tilde{\mathbf{u}}^{m+1}\|_{\mathbf{L}^2}^2] + \frac{1}{2} k \sum_{n=0}^m \mathbb{E} [\|\Delta \tilde{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2] \\ & \leq \frac{1}{2} \mathbb{E} [\|\nabla \mathbf{u}_0\|_{\mathbf{L}^2}^2] + k \sum_{n=0}^m \mathbb{E} [\|\nabla p^n\|_{\mathbf{L}^2}^2 + \|\mathbf{f}^{n+1}\|_{\mathbf{L}^2}^2 + C \|\nabla \tilde{\mathbf{u}}^n\|_{\mathbf{L}^2}^2]. \end{aligned} \quad (4.28)$$

Then by the discrete Gronwall inequality we get

$$\begin{aligned} & \frac{1}{2} \mathbb{E} [\|\nabla \tilde{\mathbf{u}}^{m+1}\|_{\mathbf{L}^2}^2] + \frac{1}{2} k \sum_{n=0}^m \mathbb{E} [\|\Delta \tilde{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2}^2] \\ & \leq \left(\frac{1}{2} \mathbb{E} [\|\nabla \mathbf{u}_0\|_{\mathbf{L}^2}^2] + k \sum_{n=0}^m \mathbb{E} [\|\nabla p^n\|_{\mathbf{L}^2}^2] + k \sum_{n=0}^m \mathbb{E} [\|\mathbf{f}^{n+1}\|_{\mathbf{L}^2}^2] \right) \exp(Ct_m), \end{aligned}$$

which and (4.17b) immediately infer (4.17c). The proof is complete. $\square$

Next, we shall derive some error estimates for the time-discrete processes generated by Algorithm 1. To the best of our knowledge, these are the first error estimate results for the standard Chorin scheme in the case general multiplicative noises. For the sake of brevity, but without loss of the generality, we set $\mathbf{f} = 0$ in this subsection.

First, we state the following error estimate result for the velocity.

Theorem 4.2.2. *Let $\{(\tilde{\mathbf{u}}^n, p^n)\}_{n=0}^M$ be generated by Algorithm 1, then there exists a positive constant C which depends on $D_T, \mathbf{u}_0$, and $\mathbf{f}$ such that*

$$\begin{aligned} & \left(\mathbb{E} \left[k \sum_{n=0}^M \|\mathbf{u}(t_n) - \tilde{\mathbf{u}}^n\|_{\mathbf{L}^2}^2 \right] \right)^{\frac{1}{2}} \\ & + \max_{0 \leq \ell \leq M} \left(\mathbb{E} \left[\left\| k \sum_{n=0}^{\ell} \nabla(\mathbf{u}(t_n) - \tilde{\mathbf{u}}^n) \right\|_{\mathbf{L}^2}^2 \right] \right)^{\frac{1}{2}} \leq C k^{\frac{1}{4}}. \end{aligned} \quad (4.29)$$

Proof. Let $\mathbf{e}_{\tilde{\mathbf{u}}}^m = \mathbf{u}(t_m) - \tilde{\mathbf{u}}^m$ and $\mathcal{E}_p^m = P(t_m) - k \sum_{n=0}^m p^n$. Obviously, $\mathbf{e}_{\tilde{\mathbf{u}}}^m \in L^2(\Omega; \mathbf{H}_{per}^1(D))$ and $\mathcal{E}_p^m \in L^2(\Omega; H_{per}^1(D)/\mathbb{R})$. In addition, from (2.14), we have

$$\begin{aligned} & (\mathbf{u}(t_{m+1}), \mathbf{v}) + \sum_{n=0}^m \int_{t_n}^{t_{n+1}} (\nabla \mathbf{u}(s), \nabla \mathbf{v}) ds + (\mathbf{v}, \nabla P(t_{m+1})) \\ & = (\mathbf{u}_0, \mathbf{v}) + \left(\sum_{n=0}^m \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) dW(s), \mathbf{v}) \right) \quad \forall \mathbf{v} \in \mathbf{H}_{per}^1(D) \text{ a.s.} \end{aligned} \quad (4.30)$$

Applying the summation operator $\sum_{n=0}^m$ to (4.16a) yields

$$\begin{aligned} & (\tilde{\mathbf{u}}^{m+1}, \mathbf{v}) + k \sum_{n=0}^m (\nabla \tilde{\mathbf{u}}^{n+1}, \nabla \mathbf{v}) + \left(k \sum_{n=0}^m \nabla p^n, \mathbf{v} \right) \\ & = (\tilde{\mathbf{u}}_0, \mathbf{v}) + \left(\sum_{n=0}^m \int_{t_n}^{t_{n+1}} \mathbf{B}(\tilde{\mathbf{u}}^n) dW(s), \mathbf{v} \right). \end{aligned} \quad (4.31)$$

Subtracting (4.31) from (4.30) we get

$$\begin{aligned}
& (\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}, \mathbf{v}) + k \left(\sum_{n=0}^m \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{n+1}, \nabla \mathbf{v} \right) + (\nabla \mathcal{E}_p^m, \mathbf{v}) \\
&= \sum_{n=0}^m \int_{t_n}^{t_{n+1}} (\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s)), \nabla \mathbf{v}) ds - (\nabla(P(t_{m+1}) - P(t_m)), \mathbf{v}) \\
&+ \left(\sum_{n=0}^m \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\tilde{\mathbf{u}}^n)) dW(s), \mathbf{v} \right) \quad \forall \mathbf{v} \in \mathbf{H}_{per}^1(D).
\end{aligned} \tag{4.32}$$

Setting $\mathbf{v} = \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}$ in (4.32) we obtain

$$\begin{aligned}
& \|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|_{\mathbf{L}^2}^2 + k \left(\sum_{n=0}^{m+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^n, \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} \right) + (\nabla \mathcal{E}_p^m, \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}) \\
&= \sum_{n=0}^m \int_{t_n}^{t_{n+1}} (\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s)), \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}) ds \\
&- (\nabla(P(t_{m+1}) - P(t_m)), \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}) \\
&+ \left(\sum_{n=0}^m \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\tilde{\mathbf{u}}^n)) dW(s), \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} \right).
\end{aligned} \tag{4.33}$$

Similarly, by (2.14b) and (4.16b) we get

$$\operatorname{div} \mathbf{e}_{\tilde{\mathbf{u}}}^n + k \Delta p^n = 0. \tag{4.34}$$

Applying the summation $\sum_{n=0}^{m+1}$ to (4.34) and then adding $\pm \Delta P(t_{m+1})$ yield

$$\operatorname{div} \left(\sum_{n=0}^{m+1} \mathbf{e}_{\tilde{\mathbf{u}}}^n \right) - \Delta \mathcal{E}_p^{m+1} = -\Delta P(t_{m+1}).$$

Therefore,

$$\operatorname{div} \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} - \Delta(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m) = -\Delta(P(t_{m+1}) - P(t_m)). \tag{4.35}$$

Testing (4.35) by any $q \in L^2(\Omega; H_{per}^1(D)/\mathbb{R})$, we have

$$(\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}, \nabla q) = (\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m), \nabla q) - (\nabla(P(t_{m+1}) - P(t_m)), \nabla q). \quad (4.36)$$

Choosing $q = \mathcal{E}_p^m$ in (4.36) gives

$$\begin{aligned} (\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}, \nabla \mathcal{E}_p^m) &= (\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m), \nabla \mathcal{E}_p^m) - (\nabla(P(t_{m+1}) - P(t_m)), \nabla \mathcal{E}_p^m) \\ &= (\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m), \nabla \mathcal{E}_p^{m+1}) - \|\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m)\|_{\mathbf{L}^2}^2 \\ &\quad - (\nabla(P(t_{m+1}) - P(t_m)), \nabla \mathcal{E}_p^m). \end{aligned} \quad (4.37)$$

Substituting (4.37) into (4.33) we obtain

$$\begin{aligned} &\|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|_{\mathbf{L}^2}^2 + k \left(\sum_{n=0}^{m+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^n, \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} \right) + (\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m), \nabla \mathcal{E}_p^{m+1}) \\ &= \sum_{n=0}^m \int_{t_n}^{t_{n+1}} (\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s)), \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}) ds \\ &\quad + \|\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m)\|_{\mathbf{L}^2}^2 + (\nabla(P(t_{m+1}) - P(t_m)), \nabla \mathcal{E}_p^m) \\ &\quad - (\nabla(P(t_{m+1}) - P(t_m)), \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}) \\ &\quad + \left(\sum_{n=0}^m \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\tilde{\mathbf{u}}^n)) dW(s), \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} \right) \\ &\leq \sum_{n=0}^m \int_{t_n}^{t_{n+1}} (\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s)), \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}) ds \\ &\quad + \|\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m)\|_{\mathbf{L}^2}^2 + (\nabla(P(t_{m+1}) - P(t_m)), \nabla \mathcal{E}_p^m) \\ &\quad + \|\nabla(P(t_{m+1}) - P(t_m))\|_{\mathbf{L}^2}^2 + \frac{1}{4} \|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|_{\mathbf{L}^2}^2 \\ &\quad + \left\| \sum_{n=0}^m \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\tilde{\mathbf{u}}^n)) dW(s) \right\|_{\mathbf{L}^2}^2 + \frac{1}{4} \|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|_{\mathbf{L}^2}^2. \end{aligned}$$

Therefore,

$$\begin{aligned}
& \frac{1}{2} \|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|_{\mathbf{L}^2}^2 + k \left(\sum_{n=0}^{m+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^n, \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} \right) + (\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m), \nabla \mathcal{E}_p^{m+1}) \\
& \leq \sum_{n=0}^m \int_{t_n}^{t_{n+1}} (\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s)), \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}) ds \\
& \quad + \|\nabla(P(t_{m+1}) - P(t_m))\|_{\mathbf{L}^2}^2 + \|\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m)\|_{\mathbf{L}^2}^2 \\
& \quad + (\nabla(P(t_{m+1}) - P(t_m)), \nabla \mathcal{E}_p^m) \\
& \quad + \left\| \sum_{n=0}^m \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\tilde{\mathbf{u}}^n)) dW(s) \right\|_{\mathbf{L}^2}^2.
\end{aligned} \tag{4.38}$$

Using the identity $2a(a-b) = a^2 - b^2 + (a-b)^2$ in (4.38) yields

$$\begin{aligned}
& \frac{1}{2} \|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|_{\mathbf{L}^2}^2 + \frac{k}{2} \left[\left\| \sum_{n=0}^{m+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^n \right\|_{\mathbf{L}^2}^2 - \left\| \sum_{n=0}^m \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^n \right\|_{\mathbf{L}^2}^2 + \|\nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|_{\mathbf{L}^2}^2 \right] \\
& \quad + \frac{1}{2} [\|\nabla \mathcal{E}_p^{m+1}\|_{L^2}^2 - \|\nabla \mathcal{E}_p^m\|_{L^2}^2 + \|\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m)\|_{L^2}^2] \\
& \leq \sum_{n=0}^m \int_{t_n}^{t_{n+1}} (\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s)), \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}) ds \\
& \quad + \|\nabla(P(t_{m+1}) - P(t_m))\|_{L^2}^2 + \|\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m)\|_{L^2}^2 \\
& \quad + (\nabla(P(t_{m+1}) - P(t_m)), \nabla \mathcal{E}_p^m) \\
& \quad + \left\| \sum_{n=0}^m \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\tilde{\mathbf{u}}^n)) dW(s) \right\|_{\mathbf{L}^2}^2.
\end{aligned} \tag{4.39}$$

Next, we apply the summation operator $k \sum_{m=0}^\ell$ for $0 \leq \ell \leq M-1$, followed by applying the expectation operator $\mathbb{E}[\cdot]$, to (4.39) to obtain

$$\begin{aligned}
& k \sum_{m=0}^{\ell} \mathbb{E}[\|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|_{\mathbf{L}^2}^2] + \mathbb{E}\left[\left\|k \sum_{m=0}^{\ell+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^m\right\|_{\mathbf{L}^2}^2\right] + k^2 \sum_{m=0}^{\ell} \mathbb{E}[\|\nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|_{\mathbf{L}^2}^2] \\
& + k \mathbb{E}[\|\nabla \mathcal{E}_p^{\ell+1}\|_{L^2}^2] + k \sum_{m=0}^{\ell} \mathbb{E}[\|\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m)\|_{L^2}^2] \\
& \leq 2\mathbb{E}\left[k \sum_{m=0}^{\ell} \sum_{n=0}^m \int_{t_n}^{t_{n+1}} (\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s)), \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}) ds\right] \\
& + 2\mathbb{E}\left[k \sum_{m=0}^{\ell} \|\nabla(P(t_{m+1}) - P(t_m))\|_{\mathbf{L}^2}^2\right] \\
& + 2\mathbb{E}\left[k \sum_{m=0}^{\ell} \|\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m)\|_{L^2}^2\right] \\
& + 2\mathbb{E}\left[k \sum_{m=0}^{\ell} (\nabla(P(t_{m+1}) - P(t_m)), \nabla \mathcal{E}_p^m)\right] \\
& + 2\mathbb{E}\left[k \sum_{m=0}^{\ell} \left\|\sum_{n=0}^m \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\tilde{\mathbf{u}}^n)) dW(s)\right\|_{\mathbf{L}^2}^2\right] \\
& := \text{I} + \text{II} + \text{III} + \text{IV} + \text{V}.
\end{aligned} \tag{4.40}$$

Now we estimate each term on the right-hand side of (4.40) as follows.

By using (4.6) we have

$$\text{II} = 2\mathbb{E}\left[k \sum_{m=0}^{\ell} \|\nabla(P(t_{m+1}) - P(t_m))\|_{\mathbf{L}^2}^2\right] \leq Ck. \tag{4.41}$$

By using the discrete and continuous Hölder inequality estimates, (2.11b), (4.3) and (4.17a), we obtain

$$\begin{aligned}
\text{I} &= 2\mathbb{E}\left[k \sum_{m=0}^{\ell} \left(\sum_{n=0}^m \int_{t_n}^{t_{n+1}} \nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s)) ds, \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} \right)\right] \\
&\leq 2\mathbb{E}\left[k \sum_{m=0}^{\ell} \left\| \sum_{n=0}^m \int_{t_n}^{t_{n+1}} \nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s)) ds \right\|_{\mathbf{L}^2} \|\nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|_{\mathbf{L}^2} \right] \\
&\leq 2\mathbb{E}\left[\left(k \sum_{m=0}^{\ell} \left\| \sum_{n=0}^m \int_{t_n}^{t_{n+1}} \nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s)) ds \right\|_{\mathbf{L}^2}^2\right)^{\frac{1}{2}} \left(k \sum_{m=0}^{\ell} \|\nabla \mathbf{e}_{\tilde{\mathbf{u}}}^n\|_{\mathbf{L}^2}^2\right)^{\frac{1}{2}}\right] \\
&\leq C\mathbb{E}\left[\left(k \sum_{m=0}^{\ell} \sum_{n=0}^m \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s))\|_{\mathbf{L}^2}^2 ds\right)^{\frac{1}{2}} \left(k \sum_{m=0}^{\ell} \|\nabla \mathbf{e}_{\tilde{\mathbf{u}}}^n\|_{\mathbf{L}^2}^2\right)^{\frac{1}{2}}\right] \\
&\leq C\left(\mathbb{E}\left[k \sum_{m=0}^{\ell} \sum_{n=0}^m \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s))\|_{\mathbf{L}^2}^2 ds\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[k \sum_{m=0}^{\ell} \|\nabla \mathbf{e}_{\tilde{\mathbf{u}}}^n\|_{\mathbf{L}^2}^2\right]\right) \\
&\leq Ck^{\frac{1}{2}}.
\end{aligned} \tag{4.42}$$

By using (4.6) and the stability estimate (4.17b) we obtain

$$\begin{aligned}
\text{III} &= 2\mathbb{E}\left[k \sum_{m=0}^{\ell} \|\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m)\|_{L^2}^2\right] \\
&= 2\mathbb{E}\left[k \sum_{m=0}^{\ell} \|\nabla(P(t_{m+1}) - P(t_m)) - k\nabla p^{m+1}\|_{L^2}^2\right] \\
&\leq C\mathbb{E}\left[k \sum_{m=0}^{\ell} \|\nabla(P(t_{m+1}) - P(t_m))\|_{L^2}^2\right] + Ck\mathbb{E}\left[k^2 \sum_{m=0}^{\ell} \|\nabla p^{m+1}\|_{L^2}^2\right] \\
&\leq Ck.
\end{aligned} \tag{4.43}$$

It follows from the Itô isometry and (2.11a) that

$$\begin{aligned}
\text{V} &= 2k \sum_{m=0}^{\ell} \mathbb{E}\left[\left\| \sum_{n=0}^m \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\tilde{\mathbf{u}}^n)) dW(s) \right\|_{\mathbf{L}^2}^2\right] \\
&= 2k \sum_{m=0}^{\ell} \mathbb{E}\left[\sum_{n=0}^m \int_{t_n}^{t_{n+1}} \|\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\tilde{\mathbf{u}}^n)\|_{\mathbf{L}^2}^2 ds\right]
\end{aligned} \tag{4.44}$$

$$\begin{aligned}
&\leq Ck \sum_{m=0}^{\ell} \sum_{n=0}^m \int_{t_n}^{t_{n+1}} \mathbb{E}[\|\mathbf{u}(s) - \tilde{\mathbf{u}}^n\|_{\mathbf{L}^2}^2] ds \\
&\leq Ck \sum_{m=0}^{\ell} k \sum_{n=0}^m \mathbb{E}[\|\mathbf{e}_{\mathbf{u}}^n\|_{\mathbf{L}^2}^2] + Ck \sum_{m=0}^{\ell} \sum_{n=0}^m \int_{t_n}^{t_{n+1}} \mathbb{E}[\|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{L}^2}^2] ds \\
&\leq Ck \sum_{m=0}^{\ell} k \sum_{n=0}^m \mathbb{E}[\|\mathbf{e}_{\mathbf{u}}^n\|_{\mathbf{L}^2}^2] + Ck.
\end{aligned}$$

To bound term IV, we first derive its rewriting as follows:

$$\begin{aligned}
\text{IV} &= 2\mathbb{E} \left[k \sum_{m=0}^{\ell} (\nabla(P(t_{m+1}) - P(t_m)), \nabla \mathcal{E}_p^m) \right] \\
&= 2\mathbb{E} \left[k \sum_{m=0}^{\ell} (\nabla(P(t_{m+1}) - P(t_m)), \nabla \mathcal{E}_p^{m+1}) \right] \\
&\quad + 2\mathbb{E} \left[k \sum_{m=0}^{\ell} (\nabla(P(t_{m+1}) - P(t_m)), \nabla(\mathcal{E}_p^m - \mathcal{E}_p^{m+1})) \right].
\end{aligned} \tag{4.45}$$

By using the summation by parts, the first term above can be rewritten as

$$\begin{aligned}
&2\mathbb{E} \left[k \sum_{m=0}^{\ell} (\nabla(P(t_{m+1}) - P(t_m)), \nabla \mathcal{E}_p^{m+1}) \right] \\
&= 2k\mathbb{E}[(\nabla P(t_{\ell+1}), \nabla \mathcal{E}_p^{\ell+1})] - 2k\mathbb{E} \left[\sum_{m=0}^{\ell} (\nabla P(t_m), \nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m)) \right].
\end{aligned} \tag{4.46}$$

Substituting (4.46) into (4.45) yields

$$\begin{aligned}
\text{IV} &= 2k\mathbb{E}[(\nabla P(t_{\ell+1}), \nabla \mathcal{E}_p^{\ell+1})] - 2k\mathbb{E} \left[\sum_{m=0}^{\ell} (\nabla P(t_m), \nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m)) \right] \\
&\quad + 2\mathbb{E} \left[k \sum_{m=0}^{\ell} (\nabla(P(t_{m+1}) - P(t_m)), \nabla(\mathcal{E}_p^m - \mathcal{E}_p^{m+1})) \right] \\
&:= \text{IV}_1 + \text{IV}_2 + \text{IV}_3.
\end{aligned} \tag{4.47}$$

We now bound each term above. Using the stability (4.4) we get

$$\begin{aligned} \text{IV}_1 &\leq Ck\mathbb{E}[\|\nabla P(t_{\ell+1})\|_{L^2}^2] + \frac{k}{4}\mathbb{E}[\|\nabla \mathcal{E}_p^{\ell+1}\|_{L^2}^2] \\ &\leq Ck + \frac{k}{4}\mathbb{E}[\|\nabla \mathcal{E}_p^{\ell+1}\|_{L^2}^2]. \end{aligned} \quad (4.48)$$

Expectedly, the term $\frac{k}{4}\mathbb{E}[\|\nabla \mathcal{E}_p^{\ell+1}\|_{L^2}^2]$ will be absorbed to the left side of (4.40) later.

To bound term IV_2 , we reuse the estimation from III in (4.43) together with the stability of P given in (4.4) to get

$$\begin{aligned} \text{IV}_2 &\leq 2\mathbb{E}\left[k \sum_{m=0}^{\ell} \|\nabla P(t_m)\|_{L^2} \|\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m)\|_{L^2}\right] \\ &\leq C\mathbb{E}\left[\left(k \sum_{m=0}^{\ell} \|\nabla P(t_m)\|_{L^2}^2\right)^{\frac{1}{2}} \left(k \sum_{m=0}^{\ell} \|\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m)\|_{L^2}^2\right)^{\frac{1}{2}}\right] \\ &\leq C\left(\mathbb{E}\left[k \sum_{m=0}^{\ell} \|\nabla P(t_m)\|_{L^2}^2\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[k \sum_{m=0}^{\ell} \|\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m)\|_{L^2}^2\right]\right)^{\frac{1}{2}} \\ &\leq Ck^{\frac{1}{2}}. \end{aligned} \quad (4.49)$$

Using again (4.43) and (4.6) we have

$$\begin{aligned} \text{IV}_3 &\leq Ck \sum_{m=0}^{\ell} \|\nabla(P(t_{m+1}) - P(t_m))\|_{L^2}^2 + C\mathbb{E}\left[k \sum_{m=0}^{\ell} \|\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m)\|_{L^2}^2\right] \\ &\leq Ck. \end{aligned} \quad (4.50)$$

Substituting estimates (4.48)–(4.50) into (4.47) yields

$$\text{IV} \leq Ck^{\frac{1}{2}} + \frac{k}{4}\mathbb{E}[\|\nabla \mathcal{E}_p^{\ell+1}\|_{L^2}^2]. \quad (4.51)$$

Now, substituting the estimates for I, II, III, IV, V into (4.40) and using the notation $X^\ell = k \sum_{m=0}^\ell \mathbb{E}[\|\mathbf{e}_{\tilde{\mathbf{u}}}^m\|_{\mathbf{L}^2}^2]$ we obtain

$$\begin{aligned} X^{\ell+1} &+ \mathbb{E} \left[\left\| k \sum_{m=0}^{\ell+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^m \right\|_{\mathbf{L}^2}^2 \right] + k^2 \sum_{m=0}^\ell \mathbb{E} [\|\nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|_{\mathbf{L}^2}^2] \\ &+ \frac{3k}{4} \mathbb{E} [\|\nabla \mathcal{E}_p^{\ell+1}\|_{L^2}^2] + k \sum_{m=0}^\ell \mathbb{E} [\|\nabla (\mathcal{E}_p^{m+1} - \mathcal{E}_p^m)\|_{L^2}^2] \\ &\leq Ck^{\frac{1}{2}} + Ck \sum_{m=0}^\ell X^m. \end{aligned}$$

Thus, it follows from the discrete Gronwall inequality that

$$\begin{aligned} X^{\ell+1} &+ \mathbb{E} \left[\left\| k \sum_{m=0}^{\ell+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^m \right\|_{\mathbf{L}^2}^2 \right] + k^2 \sum_{m=0}^\ell \mathbb{E} [\|\nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|_{\mathbf{L}^2}^2] \\ &+ \frac{3k}{4} \mathbb{E} [\|\nabla \mathcal{E}_p^{\ell+1}\|_{L^2}^2] + k \sum_{m=0}^\ell \mathbb{E} [\|\nabla (\mathcal{E}_p^{m+1} - \mathcal{E}_p^m)\|_{L^2}^2] \\ &\leq Ck^{\frac{1}{2}} \exp(Ct_\ell), \end{aligned}$$

which yields the desired error estimate for the velocity approximation. $\square$

Next, we derive an error estimate for the pressure approximation. Our main idea is to estimate the pressure error in a time averaged fashion.

Theorem 4.2.3. *Let $\{(\tilde{\mathbf{u}}^m, p^m)\}_{m=0}^M$ be generated by Algorithm 1. Then, there exists a positive constant C which depends on $D_T, \mathbf{u}_0, \mathbf{f}$, and β such that*

$$\left(\mathbb{E} \left[k \sum_{m=0}^M \left\| P(t_m) - k \sum_{n=0}^m p^n \right\|_{L^2}^2 \right] \right)^{\frac{1}{2}} \leq Ck^{\frac{1}{4}}, \quad (4.52)$$

where β denotes the stochastic inf-sup constant (see below).

Proof. We first recall the following inf-sup condition (cf. [9]):

$$\sup_{\mathbf{v} \in \mathbf{H}_{per}^1(D)} \frac{(q, \operatorname{div} \mathbf{v})}{\|\nabla \mathbf{v}\|_{\mathbf{L}^2}} \geq \beta \|q\|_{L^2} \quad \forall q \in L_{per}^2(D)/\mathbb{R}, \quad (4.53)$$

where β is a positive constant.

Below we reuse all the notations from Theorem 4.2.2. From the error equation (4.32) we obtain for all $\mathbf{v} \in \mathbf{H}_{per}^1(D)$

$$\begin{aligned}
(\mathcal{E}_p^m, \operatorname{div} \mathbf{v}) &= (\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}, \mathbf{v}) + \left(k \sum_{n=0}^m \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{n+1}, \nabla \mathbf{v} \right) \\
&\quad - \left(\sum_{n=0}^m \int_{t_n}^{t_{n+1}} \nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s)) ds, \nabla \mathbf{v} \right) \\
&\quad + (\nabla(P(t_{m+1}) - P(t_m)), \mathbf{v}) \\
&\quad - \left(\sum_{n=0}^m \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\tilde{\mathbf{u}}^n)) dW(s), \mathbf{v} \right).
\end{aligned} \tag{4.54}$$

By using Schwarz and Poincaré inequalities on the right side of (4.54), we get

$$\begin{aligned}
(\mathcal{E}_p^m, \operatorname{div} \mathbf{v}) &\leq C \|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\| \|\nabla \mathbf{v}\|_{\mathbf{L}^2} + \left\| k \sum_{n=0}^m \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{n+1} \right\|_{\mathbf{L}^2} \|\nabla \mathbf{v}\|_{\mathbf{L}^2} \\
&\quad + \left\| \sum_{n=0}^m \int_{t_n}^{t_{n+1}} \nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s)) ds \right\|_{\mathbf{L}^2} \|\nabla \mathbf{v}\|_{\mathbf{L}^2} \\
&\quad + C \|\nabla(P(t_{m+1}) - P(t_m))\|_{\mathbf{L}^2} \|\nabla \mathbf{v}\|_{\mathbf{L}^2} \\
&\quad + C \left\| \sum_{n=0}^m \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\tilde{\mathbf{u}}^n)) dW(s) \right\|_{\mathbf{L}^2} \|\nabla \mathbf{v}\|_{\mathbf{L}^2}
\end{aligned} \tag{4.55}$$

Applying (4.53) yields

$$\begin{aligned}
\beta \|\mathcal{E}_p^m\|_{L^2} &\leq \sup_{\mathbf{v} \in \mathbf{H}_{per}^1(D)} \frac{(\mathcal{E}_p^m, \operatorname{div} \mathbf{v})}{\|\nabla \mathbf{v}\|_{\mathbf{L}^2}} \\
&\leq C \|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|_{\mathbf{L}^2} + \left\| k \sum_{n=0}^m \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{n+1} \right\|_{\mathbf{L}^2} \\
&\quad + \left\| \sum_{n=0}^m \int_{t_n}^{t_{n+1}} \nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s)) ds \right\|_{\mathbf{L}^2} \\
&\quad + C \|\nabla(P(t_{m+1}) - P(t_m))\|_{L^2} \\
&\quad + C \left\| \sum_{n=0}^m \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\tilde{\mathbf{u}}^n)) dW(s) \right\|_{\mathbf{L}^2}.
\end{aligned} \tag{4.56}$$

Next, squaring both sides of (4.56) followed by applying operators $k \sum_{m=0}^{\ell}$ and $\mathbb{E}[\cdot]$, we obtain

$$\begin{aligned}
\beta^2 \mathbb{E} \left[k \sum_{m=0}^{\ell} \|\mathcal{E}_p^m\|_{L^2}^2 \right] &\leq C \mathbb{E} \left[k \sum_{m=0}^{\ell} \|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|_{L^2}^2 \right] + Ck \sum_{m=0}^{\ell} \mathbb{E} \left[\left\| k \sum_{n=0}^m \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{n+1} \right\|_{L^2}^2 \right] \\
&\quad + Ck \sum_{m=0}^{\ell} \mathbb{E} \left[\left\| \sum_{n=0}^m \int_{t_n}^{t_{n+1}} \nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s)) ds \right\|_{L^2}^2 \right] \\
&\quad + Ck \sum_{m=0}^{\ell} \mathbb{E} [\|\nabla(P(t_{m+1}) - P(t_m))\|_{L^2}^2] \\
&\quad + Ck \sum_{m=0}^{\ell} \mathbb{E} \left[\left\| \sum_{n=0}^m \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\tilde{\mathbf{u}}^n)) dW(s) \right\|_{L^2}^2 \right] \\
&:= \text{I} + \text{II} + \text{III} + \text{IV} + \text{V}.
\end{aligned} \tag{4.57}$$

We now bound each term above as follows. Using Theorem 4.2.2 we get

$$\begin{aligned}
\text{I} + \text{II} &= C \mathbb{E} \left[k \sum_{m=0}^{\ell} \|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|_{L^2}^2 \right] + Ck \sum_{m=0}^{\ell} \mathbb{E} \left[\left\| k \sum_{n=0}^m \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{n+1} \right\|_{L^2}^2 \right] \\
&\leq Ck^{\frac{1}{2}}.
\end{aligned} \tag{4.58}$$

By the Hölder continuities given in (2.11b) and (4.6), we have

$$\begin{aligned}
\text{III} + \text{IV} &= Ck \sum_{m=0}^{\ell} \mathbb{E} \left[\left\| \sum_{n=0}^m \int_{t_n}^{t_{n+1}} \nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s)) ds \right\|_{L^2}^2 \right] \\
&\quad + Ck \sum_{m=0}^{\ell} \mathbb{E} [\|\nabla(P(t_{m+1}) - P(t_m))\|_{L^2}^2] \\
&\leq Ck \sum_{m=0}^{\ell} \mathbb{E} \left[\sum_{n=0}^m \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s))\|_{L^2}^2 ds \right] \\
&\quad + Ck \sum_{m=0}^{\ell} \mathbb{E} [\|\nabla(P(t_{m+1}) - P(t_m))\|_{L^2}^2] \\
&\leq Ck.
\end{aligned} \tag{4.59}$$

By using the Itô isometry and Theorem 4.2.2 and (2.11a), we conclude that

$$\begin{aligned}
V &= Ck \sum_{m=0}^{\ell} \mathbb{E} \left[\left\| \sum_{n=0}^m \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\tilde{\mathbf{u}}^n)) dW(s) \right\|_{\mathbf{L}^2}^2 \right] \\
&= Ck \sum_{m=0}^{\ell} \mathbb{E} \left[\sum_{n=0}^m \int_{t_n}^{t_{n+1}} \|\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\tilde{\mathbf{u}}^n)\|_{\mathbf{L}^2}^2 ds \right] \\
&\leq Ck \sum_{m=0}^{\ell} \mathbb{E} \left[\sum_{n=0}^m \int_{t_n}^{t_{n+1}} \|\mathbf{u}(s) - \tilde{\mathbf{u}}^n\|_{\mathbf{L}^2}^2 ds \right] \\
&\leq Ck \sum_{m=0}^{\ell} \mathbb{E} \left[\sum_{n=0}^m \int_{t_n}^{t_{n+1}} \|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{L}^2}^2 ds \right] + Ck \sum_{m=0}^{\ell} \mathbb{E} \left[k \sum_{n=0}^m \|\mathbf{e}_{\tilde{\mathbf{u}}}^n\|_{\mathbf{L}^2}^2 \right] \\
&\leq Ck^{\frac{1}{2}}.
\end{aligned} \tag{4.60}$$

Finally, substituting (4.58)–(4.60) into (4.57) yields

$$\beta^2 \mathbb{E} \left[k \sum_{m=0}^{\ell} \|\mathcal{E}_p^m\|_{L^2}^2 \right] \leq Ck^{\frac{1}{2}} \quad \text{for } 1 \leq \ell \leq M. \tag{4.61}$$

The proof is complete. $\square$

Remark 4.2.2. *It is interesting to point out that the above proof uses the technique from the (non-splitting) mixed method error analysis although Chorin scheme is a splitting scheme.*

We conclude this subsection by stating two stability estimates for $(\tilde{\mathbf{u}}^m, p^m)$ in high norms as immediate corollaries of the above error estimates, they will be used in the next section in deriving error estimates for a fully discrete finite element Chorin method. We note that these stability estimates improve those given in Lemma 4.2.1 and may not be obtained directly without using the above error estimates.

Corollary 4.2.4. *Under the assumptions of Theorem 4.2.2, there exists a positive constant C which depends on D_T , $\mathbf{u}_0$ and $\mathbf{f}$ such that*

$$\mathbb{E} \left[k \sum_{m=0}^M \left\| k \sum_{n=0}^m \nabla p^n \right\|_{L^2}^2 \right] \leq C, \tag{4.62}$$

$$\max_{0 \leq m \leq M} \mathbb{E} \left[\left\| k \sum_{n=0}^m \nabla \tilde{\mathbf{u}}^n \right\|_{\mathbf{L}^2}^2 \right] + \mathbb{E} \left[k \sum_{m=0}^M \left\| k \sum_{n=0}^m \Delta \tilde{\mathbf{u}}^n \right\|_{\mathbf{L}^2}^2 \right] \leq C. \tag{4.63}$$

Proof. (4.62) follows immediately from estimates (4.4) and (4.52) and (4.63)₁ follows straightforwardly from the discrete Jensen inequality and (4.17a)₃. It remains to prove (4.63)₂. To that end, testing (4.31) by $-k \sum_{n=0}^m \Delta \tilde{\mathbf{u}}^{n+1}$, we obtain

$$\begin{aligned}
\left\| k \sum_{n=0}^m \Delta \tilde{\mathbf{u}}^{n+1} \right\|_{\mathbf{L}^2}^2 &= \left(\tilde{\mathbf{u}}^{m+1} - \mathbf{u}_0, k \sum_{n=0}^m \Delta \tilde{\mathbf{u}}^{n+1} \right) \\
&+ \left(k \sum_{n=0}^m \nabla p^n, k \sum_{n=0}^m \Delta \tilde{\mathbf{u}}^{n+1} \right) \\
&- \left(\sum_{n=0}^m \int_{t_n}^{t_{n+1}} \mathbf{B}(\tilde{\mathbf{u}}^n) dW(s), k \sum_{n=0}^m \Delta \tilde{\mathbf{u}}^{n+1} \right) \\
&\leq \|\tilde{\mathbf{u}}^{m+1} - \mathbf{u}_0\|_{\mathbf{L}^2}^2 + \frac{1}{4} \left\| k \sum_{n=0}^m \Delta \tilde{\mathbf{u}}^{n+1} \right\|_{\mathbf{L}^2}^2 \\
&+ \left\| k \sum_{n=0}^m \nabla p^n \right\|_{L^2}^2 + \frac{1}{4} \left\| k \sum_{n=0}^m \Delta \tilde{\mathbf{u}}^{n+1} \right\|_{\mathbf{L}^2}^2 \\
&+ \left\| \sum_{n=0}^m \int_{t_n}^{t_{n+1}} \mathbf{B}(\tilde{\mathbf{u}}^n) dW(s) \right\|_{\mathbf{L}^2}^2 + \frac{1}{4} \left\| k \sum_{n=0}^m \Delta \tilde{\mathbf{u}}^{n+1} \right\|_{\mathbf{L}^2}^2.
\end{aligned} \tag{4.64}$$

After an rearrangement, applying operators $k \sum_{m=0}^M$ and $\mathbb{E}[\cdot]$ to (4.64) we obtain

$$\begin{aligned}
\frac{1}{4} \mathbb{E} \left[k \sum_{m=0}^M \left\| k \sum_{n=0}^m \Delta \tilde{\mathbf{u}}^n \right\|_{\mathbf{L}^2}^2 \right] &\leq \mathbb{E} \left[k \sum_{m=0}^M \left(\|\tilde{\mathbf{u}}^m - \mathbf{u}^0\|_{\mathbf{L}^2}^2 + \left\| k \sum_{n=0}^m \nabla p^n \right\|_{L^2}^2 \right) \right] \\
&+ k \sum_{m=0}^M \mathbb{E} \left[\left\| \sum_{n=0}^{m-1} \int_{t_n}^{t_{n+1}} \mathbf{B}(\tilde{\mathbf{u}}^n) dW(s) \right\|_{\mathbf{L}^2}^2 \right].
\end{aligned} \tag{4.65}$$

The first two terms can be bounded by using (4.17a) and (4.62). The third term can be controlled by using the Itô isometry and (4.17a) as follows:

$$\begin{aligned}
k \sum_{m=0}^M \mathbb{E} \left[\left\| \sum_{n=0}^{m-1} \int_{t_n}^{t_{n+1}} \mathbf{B}(\tilde{\mathbf{u}}^n) dW(s) \right\|_{\mathbf{L}^2}^2 \right] &= k \sum_{m=0}^M \mathbb{E} \left[\sum_{n=0}^{m-1} \int_{t_n}^{t_{n+1}} \|\mathbf{B}(\tilde{\mathbf{u}}^n)\|_{\mathbf{L}^2}^2 ds \right] \\
&\leq Ck \sum_{m=0}^M \mathbb{E} \left[k \sum_{n=0}^{m-1} \|\tilde{\mathbf{u}}^n\|_{\mathbf{L}^2}^2 ds \right] \leq C.
\end{aligned}$$

The proof is complete. □

4.2.2 A modified Chorin projection scheme

In this subsection, we consider a modification of Algorithm 1 which was already pointed out in [16] but did not analyze there. The modification is to perform a Helmholtz decomposition of $\mathbf{B}(\tilde{\mathbf{u}}^m)$ at each time step which allows us to eliminate the curl-free part in Step 1 of Algorithm 1, this then results in a divergent-free Helmholtz projected noise. The goal of this subsection is to present a complete convergence analysis for the modified Chorin scheme which includes deriving stronger error estimates for both velocity and pressure approximations than those for the standard Chorin scheme. We note that this Helmholtz decomposition enhancing technique was also used in Chapter 2 to improve the standard mixed finite element method for (1.3).

For ease of the presentation, we assume $W(t)$ is a real-valued Wiener process and independent of the spatial variable. The case of more general $W(t)$ can be dealt with similarly. The modified Chorin scheme is given as follows.

Algorithm 2

Set $\tilde{\mathbf{u}}^0 = \mathbf{u}^0 = \mathbf{u}_0$. For $m = 0, 1, \dots, M-1$, do the following five steps.

Step 1: Given $\tilde{\mathbf{u}}^m \in L^2(\Omega, \mathbf{H}_{per}^1(D))$, find $\xi^m \in L^2(\Omega, H_{per}^1(D)/\mathbb{R})$ such that $\mathbb{P}$ -a.s.

$$\Delta \xi^m = \operatorname{div} \mathbf{B}(\tilde{\mathbf{u}}^m) \quad \text{in } D. \quad (4.66)$$

Step 2: Set $\boldsymbol{\eta}_{\tilde{\mathbf{u}}}^m = \mathbf{B}(\tilde{\mathbf{u}}^m) - \nabla \xi^m$. Given $\mathbf{u}^m \in L^2(\Omega, \mathbb{V})$ and $\tilde{\mathbf{u}}^m \in L^2(\Omega, \mathbf{H}_{per}^1(D))$, find $\tilde{\mathbf{u}}^{m+1} \in L^2(\Omega, \mathbf{H}_{per}^1(D))$ such that $\mathbb{P}$ -a.s.

$$\tilde{\mathbf{u}}^{m+1} - k\Delta \tilde{\mathbf{u}}^{m+1} = \mathbf{u}^m + k\mathbf{f}^{m+1} + \boldsymbol{\eta}_{\tilde{\mathbf{u}}}^m \Delta W_{m+1} \quad \text{in } D. \quad (4.67)$$

Step 3: Find $r^{m+1} \in L^2(\Omega, H_{per}^1(D)/\mathbb{R})$ such that $\mathbb{P}$ -a.s.

$$-\Delta r^{m+1} = -\frac{1}{k} \operatorname{div} \tilde{\mathbf{u}}^{m+1} \quad \text{in } D. \quad (4.68a)$$

Step 4: Define $\mathbf{u}^{m+1} \in L^2(\Omega, \mathbb{V})$ as

$$\mathbf{u}^{m+1} = \tilde{\mathbf{u}}^{m+1} - k \nabla r^{m+1} \quad \text{in } D. \quad (4.69)$$

Step 5: Define the pressure approximation p^{m+1} as

$$p^{m+1} = r^{m+1} + \frac{1}{k} \xi^m \Delta W_{m+1} \quad \text{in } D. \quad (4.70)$$

Remark 4.2.3. *It follows from (1.5a) and (4.66) that the Helmholtz projection $\boldsymbol{\eta}_{\tilde{\mathbf{u}}}^m$ can be bounded in terms of $\tilde{\mathbf{u}}^m$ as follows:*

$$\|\boldsymbol{\eta}_{\tilde{\mathbf{u}}}^m\|_{\mathbf{L}^2} \leq \|\mathbf{B}(\tilde{\mathbf{u}}^m)\|_{\mathbf{L}^2} + \|\nabla \xi_{\tilde{\mathbf{u}}}^m\|_{\mathbf{L}^2} \leq 2\|\mathbf{B}(\tilde{\mathbf{u}}^m)\|_{\mathbf{L}^2} \leq C\|\tilde{\mathbf{u}}^m\|_{\mathbf{L}^2} \quad (4.71)$$

We first state some stability estimates for Algorithm 2. We then recall the Euler-Maruyama scheme for (1.3) and its stability and error estimates from Chapter 2, which will be utilized as a tool in the stability and error analysis of the modified Chorin scheme in the next subsection.

Lemma 4.2.5. *The discrete processes $\{(\tilde{\mathbf{u}}^m, r^m)\}_{m=0}^M$ defined by Algorithm 2 satisfy*

$$\max_{0 \leq m \leq M} \mathbb{E}[\|\tilde{\mathbf{u}}^m\|_{\mathbf{L}^2}^2] + \mathbb{E}\left[\sum_{m=1}^M \|\tilde{\mathbf{u}}^m - \tilde{\mathbf{u}}^{m-1}\|_{\mathbf{L}^2}^2\right] + \mathbb{E}\left[k \sum_{m=0}^M \|\nabla \tilde{\mathbf{u}}^m\|_{\mathbf{L}^2}^2\right] \leq C, \quad (4.72a)$$

$$\mathbb{E}\left[k^2 \sum_{m=0}^M \|\nabla r^m\|_{L^2}^2\right] \leq C, \quad (4.72b)$$

where C is a positive constant which depends on $D_T, \mathbf{u}_0$ and $\mathbf{f}$.

Since the proof of this lemma follows the same lines as those of Lemma 4.2.1. We omit the proof to save space.

Next, we recall the Helmholtz enhanced Euler-Maruyama scheme for (1.3) which was proposed and analyzed in Chapter 2. This scheme will be used as an auxiliary scheme in our error estimates for the velocity and pressure approximations of Algorithm 2 in the next

subsection. The Euler-Maruyama scheme reads as

$$(\mathbf{v}^{m+1} - \mathbf{v}^m) - k\Delta \mathbf{v}^{m+1} + k\nabla q^{m+1} = k\mathbf{f}^{m+1} + \boldsymbol{\eta}_{\mathbf{v}}^m \Delta W_{m+1} \quad \text{in } D, \quad (4.73a)$$

$$\operatorname{div} \mathbf{v}^{m+1} = 0 \quad \text{in } D, \quad (4.73b)$$

where $\boldsymbol{\eta}_{\mathbf{v}}^m = \mathbf{B}(\mathbf{v}^m) - \nabla \xi_{\mathbf{v}}^m$ denotes the Helmholtz projection of $\mathbf{B}(\mathbf{v}^m)$.

For the sake of completeness and also for new notations, we recall the following stability and error estimates from Lemma 2.2.1 and Theorem 2.2.2, Corollary 2.2.4 in Chapter 2 as follows.

Lemma 4.2.6. *The discrete processes $\{(\mathbf{v}^m, q^m)\}_{m=0}^M$ defined by (4.73) satisfy*

$$\max_{0 \leq m \leq M} \mathbb{E}[\|\mathbf{v}^m\|_{\mathbf{L}^2}^2] + \mathbb{E}\left[\sum_{m=1}^M \|\mathbf{v}^m - \mathbf{v}^{m-1}\|_{\mathbf{L}^2}^2\right] + \mathbb{E}\left[k \sum_{m=0}^M \|\nabla \mathbf{v}^m\|_{\mathbf{L}^2}^2\right] \leq C, \quad (4.74a)$$

$$\max_{0 \leq m \leq M} \mathbb{E}[\|\nabla \mathbf{v}^m\|_{\mathbf{L}^2}^2] + \mathbb{E}\left[\sum_{m=1}^M \left(\|\nabla(\mathbf{v}^m - \mathbf{v}^{m-1})\|_{\mathbf{L}^2}^2 + k\|\mathbf{A}\mathbf{v}^m\|_{\mathbf{L}^2}^2\right)\right] \leq C, \quad (4.74b)$$

$$\mathbb{E}\left[k \sum_{m=0}^M \|\nabla q^m\|_{L^2}^2\right] \leq C, \quad (4.74c)$$

where C is a positive constant which depends on $D_T, \mathbf{u}_0$ and $\mathbf{f}$.

Lemma 4.2.7. *There hold the the following error estimates for the discrete processes $\{(\mathbf{v}^m, q^m)\}_{m=0}^M$:*

$$\max_{0 \leq m \leq M} \left(\mathbb{E}[\|\mathbf{u}(t_m) - \mathbf{v}^m\|_{\mathbf{L}^2}^2]\right)^{\frac{1}{2}} \quad (4.75a)$$

$$\begin{aligned} & + \left(\mathbb{E}\left[k \sum_{m=0}^M \|\nabla(\mathbf{u}(t_m) - \mathbf{v}^m)\|_{\mathbf{L}^2}^2\right]\right)^{\frac{1}{2}} \leq C\sqrt{k}, \\ & \left(\mathbb{E}\left[\left\|R(t_\ell) - k \sum_{m=0}^{\ell} q^m\right\|_{L^2}^2\right]\right)^{\frac{1}{2}} \leq C\sqrt{k} \end{aligned} \quad (4.75b)$$

for $0 \leq \ell \leq M$. Where C is a positive constant which depends on $D_T, \mathbf{u}_0$ and $\mathbf{f}$.

Next, we derive error estimates for both the velocity and pressure approximations generated by Algorithm 2. The anticipated error estimates are stronger than those for the

standard Chorin scheme proved in the previous subsection. We note that our error estimate for the velocity approximation recovers the same estimate obtained in [16, Theorem 3.1] although the analysis given here is a lot simpler. On the other hand, the error estimate for the pressure approximation is apparently new. The main idea of the proofs of this subsection is to use the Euler-Maruyama scheme analyzed in Chapter 2 as an auxiliary scheme which bridges the exact solution of (1.3) and the discrete solution of Algorithm 2.

The following theorem gives an error estimate in strong norms for the velocity approximation.

Theorem 4.2.8. *Let $\{(\tilde{\mathbf{u}}^m, p^m)\}_{m=0}^M$ be the solution of Algorithm 2 and $\{(\mathbf{u}(t), P(t)); 0 \leq t \leq T\}$ be the solution of (1.3). Then there holds the following estimate:*

$$\begin{aligned} \max_{0 \leq m \leq M} \left(\mathbb{E} \left[\|\mathbf{u}(t_m) - \tilde{\mathbf{u}}^m\|_{\mathbf{L}^2}^2 \right] \right)^{\frac{1}{2}} \\ + \left(\mathbb{E} \left[k \sum_{m=0}^M \|\nabla(\mathbf{u}(t_m) - \tilde{\mathbf{u}}^m)\|_{\mathbf{L}^2}^2 \right] \right)^{\frac{1}{2}} \leq C\sqrt{k}, \end{aligned} \quad (4.76)$$

where C is a positive constant which depends on D_T , $\mathbf{u}_0$ and $\mathbf{f}$.

Proof. Let $\mathbf{e}_{\tilde{\mathbf{u}}}^m = \mathbf{v}^m - \tilde{\mathbf{u}}^m$, $e_r^m = q^m - r^m$. Then, $\mathbf{e}_{\tilde{\mathbf{u}}}^m \in L^2(\Omega, \mathbf{H}_{per}^1(D))$ and $e_r^m \in L^2(\Omega, H_{per}^1(D)/\mathbb{R})$. Subtracting (4.16) from (4.73) yields

$$(\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} - \mathbf{e}_{\tilde{\mathbf{u}}}^m) - k\Delta\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} + k\nabla e_r^m = -k(\nabla(q^{m+1} - q^m)) \quad (4.77a)$$

$$+ (\boldsymbol{\eta}_{\mathbf{v}}^m - \boldsymbol{\eta}_{\tilde{\mathbf{u}}}^m)\Delta W_{m+1} \quad \text{on } D,$$

$$\operatorname{div} \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} - k\Delta e_r^{m+1} = -k\Delta q^{m+1} \quad \text{on } D, \quad (4.77b)$$

$$\partial_{\mathbf{n}} e_r^{m+1} = \partial_{\mathbf{n}} q^{m+1} \quad \text{on } \partial D. \quad (4.77c)$$

Testing (4.77a) with $\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}$ and integrating by parts, we obtain

$$\begin{aligned} (\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} - \mathbf{e}_{\tilde{\mathbf{u}}}^m, \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}) + k\|\nabla\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|_{\mathbf{L}^2}^2 + k(\nabla e_r^m, \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}) \\ = -k(\nabla(q^{m+1} - q^m), \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}) + ((\boldsymbol{\eta}_{\mathbf{v}}^m - \boldsymbol{\eta}_{\tilde{\mathbf{u}}}^m)\Delta W_{m+1}, \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}). \end{aligned} \quad (4.78)$$

Using the algebraic identity $2ab = (a^2 - b^2) + (a - b)^2$ in the first term gives

$$\begin{aligned}
& \frac{1}{2} (\|\mathbf{e}_{\mathbf{u}}^{m+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{e}_{\mathbf{u}}^m\|_{\mathbf{L}^2}^2) + \frac{1}{2} \|\mathbf{e}_{\mathbf{u}}^{m+1} - \mathbf{e}_{\mathbf{u}}^m\|_{\mathbf{L}^2}^2 \\
& + k \|\nabla \mathbf{e}_{\mathbf{u}}^{m+1}\|_{\mathbf{L}^2}^2 + k (\nabla e_r^m, \mathbf{e}_{\mathbf{u}}^{m+1}) \\
& = -k (\nabla (q^{m+1} - q^m), \mathbf{e}_{\mathbf{u}}^{m+1}) + ((\boldsymbol{\eta}_{\mathbf{v}}^m - \boldsymbol{\eta}_{\mathbf{u}}^m) \Delta W_{m+1}, \mathbf{e}_{\mathbf{u}}^{m+1}).
\end{aligned} \tag{4.79}$$

Next, we derive a reformulation for each of the last term on the left-hand side and the first term on the right-hand side of (4.79) with a help of (4.77b). Testing (4.77b) by any $q \in L^2(\Omega, H_{per}^1(D)/\mathbb{R})$ and using (4.77c), we obtain

$$(\mathbf{e}_{\mathbf{u}}^{m+1}, \nabla q) - k (\nabla e_r^{m+1}, \nabla q) = -k (\nabla q^{m+1}, \nabla q) \tag{4.80a}$$

$$(\mathbf{e}_{\mathbf{u}}^{m+1} - \mathbf{e}_{\mathbf{u}}^m, \nabla q) - k (\nabla (e_r^{m+1} - e_r^m), \nabla q) = -k (\nabla (q^{m+1} - q^m), \nabla q). \tag{4.80b}$$

Setting $q = e_r^m$ in (4.80a), we obtain

$$\begin{aligned}
(\mathbf{e}_{\mathbf{u}}^{m+1}, \nabla e_r^m) &= k (\nabla e_r^{m+1}, \nabla e_r^m) - k (\nabla q^{m+1}, \nabla e_r^m), \\
&= k \|\nabla e_r^{m+1}\|_{L^2}^2 - k (\nabla (e_r^{m+1} - e_r^m), \nabla e_r^{m+1}) - k (\nabla q^{m+1}, \nabla e_r^m).
\end{aligned} \tag{4.81}$$

and choosing $q = e_r^{m+1}$ in (4.80b), we have

$$\begin{aligned}
k (\nabla (e_r^{m+1} - e_r^m), \nabla e_r^{m+1}) &= (\mathbf{e}_{\mathbf{u}}^{m+1} - \mathbf{e}_{\mathbf{u}}^m, \nabla e_r^{m+1}) \\
&+ k (\nabla (q^{m+1} - q^m), \nabla e_r^{m+1}).
\end{aligned} \tag{4.82}$$

Substituting (4.82) to (4.81) yields

$$\begin{aligned}
(\mathbf{e}_{\mathbf{u}}^{m+1}, \nabla e_r^m) &= k \|\nabla e_r^{m+1}\|_{L^2}^2 - (\mathbf{e}_{\mathbf{u}}^{m+1} - \mathbf{e}_{\mathbf{u}}^m, \nabla e_r^{m+1}) \\
&- k (\nabla (q^{m+1} - q^m), \nabla e_r^{m+1}) - k (\nabla q^{m+1}, \nabla e_r^m).
\end{aligned} \tag{4.83}$$

Alternatively, setting $q = q^{m+1} - q^m$ in (4.80a), we obtain

$$\begin{aligned} (\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}, \nabla(q^{m+1} - q^m)) &= k(\nabla e_r^{m+1}, \nabla(q^{m+1} - q^m)) \\ &\quad - k(\nabla q^{m+1}, \nabla(q^{m+1} - q^m)). \end{aligned} \quad (4.84)$$

Substituting (4.83) and (4.84) into (4.79) and rearranging terms, we get

$$\begin{aligned} &\frac{1}{2}(\|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{e}_{\tilde{\mathbf{u}}}^m\|_{\mathbf{L}^2}^2) + \frac{1}{2}\|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} - \mathbf{e}_{\tilde{\mathbf{u}}}^m\|_{\mathbf{L}^2}^2 + k\|\nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|_{\mathbf{L}^2}^2 + k^2\|\nabla e_r^{m+1}\|_{L^2}^2 \\ &= k(\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} - \mathbf{e}_{\tilde{\mathbf{u}}}^m, \nabla e_r^{m+1}) + k^2(\nabla q^{m+1}, \nabla e_r^m) \\ &\quad + k^2(\nabla q^{m+1}, \nabla(q^{m+1} - q^m)) + ((\boldsymbol{\eta}_{\mathbf{v}}^m - \boldsymbol{\eta}_{\tilde{\mathbf{u}}}^m)\Delta W_{m+1}, \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}). \end{aligned} \quad (4.85)$$

Finally, we bound each term on the right-hand side of (4.85). By Young's inequality, for $\delta_1, \delta_2, \delta_3 > 0$ we obtain

$$k(\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} - \mathbf{e}_{\tilde{\mathbf{u}}}^m, \nabla e_r^{m+1}) \leq \frac{1}{4\delta_1}\|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} - \mathbf{e}_{\tilde{\mathbf{u}}}^m\|_{\mathbf{L}^2}^2 + \delta_1 k^2\|\nabla e_r^{m+1}\|_{L^2}^2. \quad (4.86a)$$

$$k^2(\nabla q^{m+1}, \nabla e_r^m) \leq \frac{1}{4\delta_2}k^2\|\nabla q^{m+1}\|_{L^2}^2 + \delta_2 k^2\|\nabla e_r^m\|_{L^2}^2. \quad (4.86b)$$

$$k^2(\nabla q^{m+1}, \nabla(q^{m+1} - q^m)) \leq k^2\|\nabla q^{m+1}\|_{L^2}^2 + \frac{1}{4}k^2\|\nabla(q^{m+1} - q^m)\|_{L^2}^2. \quad (4.86c)$$

Rewriting

$$\begin{aligned} ((\boldsymbol{\eta}_{\mathbf{v}}^m - \boldsymbol{\eta}_{\tilde{\mathbf{u}}}^m)\Delta W_{m+1}, \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}) &= ((\boldsymbol{\eta}_{\mathbf{v}}^m - \boldsymbol{\eta}_{\tilde{\mathbf{u}}}^m)\Delta_{m+1}W, \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} - \mathbf{e}_{\tilde{\mathbf{u}}}^m) \\ &\quad + ((\boldsymbol{\eta}_{\mathbf{v}}^m - \boldsymbol{\eta}_{\tilde{\mathbf{u}}}^m)\Delta W_{m+1}, \mathbf{e}_{\tilde{\mathbf{u}}}^m). \end{aligned} \quad (4.87)$$

Since the expectation of the second term on the right-hand side of (4.87) vanishes due to the martingale property of Itô's integral, we only need to estimate the first term. Again, rewriting

$$\begin{aligned} ((\boldsymbol{\eta}_{\mathbf{v}}^m - \boldsymbol{\eta}_{\tilde{\mathbf{u}}}^m)\Delta_{m+1}W, \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} - \mathbf{e}_{\tilde{\mathbf{u}}}^m) &= -(\nabla(\xi_{\mathbf{v}}^m - \xi_{\tilde{\mathbf{u}}}^m)\Delta_{m+1}W, \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} - \mathbf{e}_{\tilde{\mathbf{u}}}^m) \\ &\quad + ((\mathbf{B}(\mathbf{v}^m) - \mathbf{B}(\tilde{\mathbf{u}}^m))\Delta W_{m+1}, \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} - \mathbf{e}_{\tilde{\mathbf{u}}}^m). \end{aligned} \quad (4.88)$$

Then we have

$$\mathbb{E}[(\nabla(\xi_{\mathbf{v}}^m - \xi_{\tilde{\mathbf{u}}}^m)\Delta W_{m+1}, \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} - \mathbf{e}_{\tilde{\mathbf{u}}}^m)] \quad (4.89a)$$

$$\begin{aligned} &\leq 2k\mathbb{E}[\|\nabla(\xi_{\mathbf{v}}^m - \xi_{\tilde{\mathbf{u}}}^m)\|_{L^2}^2] + \frac{1}{8}\mathbb{E}[\|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} - \mathbf{e}_{\tilde{\mathbf{u}}}^m\|_{L^2}^2] \\ &\leq 2k\mathbb{E}[\|(\mathbf{B}(\mathbf{v}^m) - \mathbf{B}(\tilde{\mathbf{u}}^m))\|_{L^2}^2] + \frac{1}{8}\mathbb{E}[\|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} - \mathbf{e}_{\tilde{\mathbf{u}}}^m\|_{L^2}^2] \\ &\leq Ck\mathbb{E}[\|\mathbf{e}_{\tilde{\mathbf{u}}}^m\|_{L^2}^2] + \frac{1}{8}\mathbb{E}[\|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} - \mathbf{e}_{\tilde{\mathbf{u}}}^m\|_{L^2}^2], \end{aligned}$$

$$\mathbb{E}[(\mathbf{B}(\mathbf{v}^m) - \mathbf{B}(\tilde{\mathbf{u}}^m))\Delta W_{m+1}, \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}] \quad (4.89b)$$

$$\begin{aligned} &\leq \frac{1}{4\delta_3}k\mathbb{E}[\|(\mathbf{B}(\mathbf{v}^m) - \mathbf{B}(\tilde{\mathbf{u}}^m))\|_{L^2}^2] + \delta_3\mathbb{E}[\|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} - \mathbf{e}_{\tilde{\mathbf{u}}}^m\|_{L^2}^2] \\ &\leq \frac{Ck}{4\delta_3}\mathbb{E}[\|\mathbf{e}_{\tilde{\mathbf{u}}}^m\|_{L^2}^2] + \delta_3\mathbb{E}[\|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} - \mathbf{e}_{\tilde{\mathbf{u}}}^m\|_{L^2}^2]. \end{aligned}$$

Now, substituting (4.86) and (4.89) into (4.85) and taking expectation on both sides, we obtain

$$\begin{aligned} &\frac{1}{2}\mathbb{E}[\|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|_{L^2}^2 - \|\mathbf{e}_{\tilde{\mathbf{u}}}^m\|_{L^2}^2] + \left(\frac{3}{8} - \frac{1}{4\delta_1} - \delta_3\right)\mathbb{E}[\|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} - \mathbf{e}_{\tilde{\mathbf{u}}}^m\|_{L^2}^2] \\ &\quad + k\mathbb{E}[\|\nabla\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|_{L^2}^2] + k^2(1 - \delta_1 - \delta_2)\mathbb{E}[\|\nabla e_r^{m+1}\|_{L^2}^2] \\ &= \frac{1}{2}k^2\mathbb{E}[\|\nabla q^{m+1}\|_{L^2}^2] + \frac{1}{2}k^2\mathbb{E}[\|\nabla(q^{m+1} - q^m)\|_{L^2}^2] + \frac{1}{4\delta_3}Ck\mathbb{E}[\|\mathbf{e}_{\tilde{\mathbf{u}}}^m\|_{L^2}^2]. \end{aligned} \quad (4.90)$$

Choosing $\delta_1 = \frac{17}{20}$, $\delta_2 = \frac{1}{10}$, $\delta_3 = \frac{1}{16}$, taking the summation for $0 \leq m \leq \ell \leq M-1$, and using (4.74c) and the discrete Gronwall's inequality, we get

$$\begin{aligned} &\frac{1}{2}\mathbb{E}[\|\mathbf{e}_{\tilde{\mathbf{u}}}^{\ell+1}\|_{L^2}^2] + \frac{5}{272}\mathbb{E}\left[\sum_{m=0}^{\ell}\|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} - \mathbf{e}_{\tilde{\mathbf{u}}}^m\|_{L^2}^2\right] + \mathbb{E}\left[k\sum_{m=0}^{\ell}\|\nabla\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|_{L^2}^2\right] \\ &\quad + \frac{1}{20}\mathbb{E}\left[k^2\sum_{m=0}^{\ell}\|\nabla e_r^{m+1}\|_{L^2}^2\right] \leq \exp(4CT)Ck. \end{aligned} \quad (4.91)$$

which and (4.75a) infer the desired estimate. The proof is complete. $\square$

An immediate corollary of the above error estimate is the following stronger stability estimates for $\{(\tilde{\mathbf{u}}^m, r^m)\}$, which may not be obtainable directly and will play an important role in the error analysis of fully discrete counterpart of Algorithm 2 in the next section.

Corollary 4.2.9. *There exists $C > 0$ which depends on $D_T, \mathbf{u}_0$ and $\mathbf{f}$ such that*

$$\max_{0 \leq m \leq M} \mathbb{E}[\|\nabla \tilde{\mathbf{u}}^m\|_{\mathbf{L}^2}^2] + \mathbb{E}\left[k \sum_{m=0}^M \|\Delta \tilde{\mathbf{u}}^m\|_{\mathbf{L}^2}^2\right] \leq C, \quad (4.92a)$$

$$\mathbb{E}\left[k \sum_{m=0}^M \|\nabla r^m\|_{L^2}^2\right] \leq C. \quad (4.92b)$$

Proof. The estimate (4.92b) follows immediately from (4.91) and the triangular inequality.

To prove (4.92a), testing (4.16a) by $-\Delta \tilde{\mathbf{u}}^{m+1}$, we obtain

$$\begin{aligned} \frac{1}{2} \mathbb{E}[\|\nabla \tilde{\mathbf{u}}^{m+1}\|_{\mathbf{L}^2}^2 - \|\nabla \tilde{\mathbf{u}}^m\|_{\mathbf{L}^2}^2] + \frac{1}{4} \mathbb{E}[\|\nabla(\tilde{\mathbf{u}}^{m+1} - \tilde{\mathbf{u}}^m)\|_{\mathbf{L}^2}^2] + \frac{1}{4} k \mathbb{E}[\|\Delta \tilde{\mathbf{u}}^{m+1}\|_{\mathbf{L}^2}^2] \\ \leq k \mathbb{E}[\|\nabla r^m\|_{L^2}^2] + k \mathbb{E}[\|\mathbf{f}^{m+1}\|_{\mathbf{L}^2}^2] + k \mathbb{E}[\|\nabla \boldsymbol{\eta}_{\tilde{\mathbf{u}}}^m\|_{\mathbf{L}^2}^2]. \end{aligned} \quad (4.93)$$

Here we have used the periodic boundary condition for $\boldsymbol{\eta}_{\tilde{\mathbf{u}}}^m$ to kill the boundary term which arises from integration by parts.

Finally, applying the operator $\sum_{m=0}^{\ell}$ to (4.93) and using the discrete Gronwall inequality and (4.92a)₃, we obtain the desired estimate. The proof is complete. $\square$

Similarly, the following estimate holds for $\{\mathbf{u}^m\}$.

Corollary 4.2.10. *There exists $C > 0$ which depends on $D_T, \mathbf{u}_0$ and $\mathbf{f}$ such that*

$$\max_{0 \leq m \leq M} (\mathbb{E}[\|\mathbf{u}(t_m) - \mathbf{u}^m\|_{\mathbf{L}^2}^2])^{\frac{1}{2}} + \left(\mathbb{E}\left[k \sum_{m=0}^M \|\mathbf{u}(t_m) - \mathbf{u}^m\|_{\mathbf{L}^2}^2\right] \right)^{\frac{1}{2}} \leq C\sqrt{k}. \quad (4.94)$$

The proof of (4.94) readily follows from (4.69) and Theorem (4.2.8) as well as the estimate (4.92b).

Next, we derive error estimates for the pressure approximations r^m and p^m generated by Algorithm 2. First, we state the following lemma.

Lemma 4.2.11. *Let $\{r^m\}_{m=0}^M$ be generated by Algorithm 2. Then, there exists a constant $C > 0$ depending on $D_T, \mathbf{u}_0, \mathbf{f}$ and β such that for $0 \leq \ell \leq M$*

$$\left(\mathbb{E}\left[\left\|k \sum_{m=1}^{\ell} (r^m - r^{m-1})\right\|_{L^2}^2\right] \right)^{\frac{1}{2}} \leq C\sqrt{k}. \quad (4.95)$$

Proof. The idea of the proof is to utilize the inf-sup condition (4.53). Testing (4.69) by any $\mathbf{v} \in L^2(\Omega; \mathbf{H}_{per}^1(D))$, we obtain

$$\begin{aligned} k(r^{m+1}, \operatorname{div} \mathbf{v}) &= (\mathbf{u}^{m+1} - \tilde{\mathbf{u}}^{m+1}, \mathbf{v}), \\ k(r^m, \operatorname{div} \mathbf{v}) &= (\mathbf{u}^m - \tilde{\mathbf{u}}^m, \mathbf{v}). \end{aligned}$$

Then, subtracting the above equations yields

$$k(r^{m+1} - r^m, \operatorname{div} \mathbf{v}) = ((\mathbf{u}^{m+1} - \mathbf{u}^m) - (\tilde{\mathbf{u}}^{m+1} - \tilde{\mathbf{u}}^m), \mathbf{v}). \quad (4.96)$$

Applying the summation operator $\sum_{m=0}^{\ell}$ for $0 \leq \ell \leq M-1$ to (4.96), we get

$$\begin{aligned} \left(k \sum_{m=0}^{\ell} (r^{m+1} - r^m), \operatorname{div} \mathbf{v} \right) &= ((\mathbf{u}^{\ell+1} - \mathbf{u}^0) - (\tilde{\mathbf{u}}^{\ell+1} - \tilde{\mathbf{u}}^0), \mathbf{v}) \\ &= (\mathbf{e}_{\mathbf{u}}^{\ell+1} - \mathbf{e}_{\tilde{\mathbf{u}}}^{\ell+1}, \mathbf{v}) \\ &\leq C(\|\mathbf{e}_{\mathbf{u}}^{\ell+1}\|_{\mathbf{L}^2} + \|\mathbf{e}_{\tilde{\mathbf{u}}}^{\ell+1}\|_{\mathbf{L}^2}) \|\nabla \mathbf{v}\|_{\mathbf{L}^2}, \end{aligned} \quad (4.97)$$

where $\mathbf{e}_{\mathbf{u}}^m$ and $\mathbf{e}_{\tilde{\mathbf{u}}}^m$ are the same as defined in the preceding subsection and we have used the fact that $\mathbf{u}^0 - \tilde{\mathbf{u}}^0 = 0$.

Finally, by using the inf-sup condition (4.53) and then taking the expectation we get

$$\beta^2 \mathbb{E} \left[\left\| k \sum_{m=0}^{\ell} (r^{m+1} - r^m) \right\|_{L^2}^2 \right] \leq C \left(\mathbb{E} [\|\mathbf{e}_{\mathbf{u}}^{\ell+1}\|_{\mathbf{L}^2}^2] + \mathbb{E} [\|\mathbf{e}_{\tilde{\mathbf{u}}}^{\ell+1}\|_{\mathbf{L}^2}^2] \right),$$

which and the estimates for $\mathbf{e}_{\mathbf{u}}^{\ell+1}$ and $\mathbf{e}_{\tilde{\mathbf{u}}}^{\ell+1}$ infer the desired estimate (4.95). The proof is complete. $\square$

We then are ready to state the following error estimate result for r^m .

Theorem 4.2.12. *Let $\{r^m\}_{m=0}^M$ be generated by Algorithm 2 and $R(t)$ be defined in (2.15). Then there exists a constant $C > 0$ depending on $D_T, \mathbf{u}_0, \mathbf{f}$ and β such that for $0 \leq \ell \leq M$*

$$\left(\mathbb{E} \left[\left\| R(t_{\ell}) - k \sum_{m=0}^{\ell} r^m \right\|_{L^2}^2 \right] \right)^{\frac{1}{2}} \leq C \sqrt{k}. \quad (4.98)$$

Proof. Subtracting (4.73) from (4.16a) and then testing the resulting equation by $\mathbf{v} \in L^2(\Omega; \mathbf{H}_{per}^1(D))$, we obtain

$$\begin{aligned} (\mathbf{e}_{\mathbf{u}}^{m+1} - \mathbf{e}_{\mathbf{u}}^m, \mathbf{v}) + k(\nabla \mathbf{e}_{\mathbf{u}}^{m+1}, \nabla \mathbf{v}) - k(q^{m+1} - r^m, \operatorname{div} \mathbf{v}) \\ = ((\boldsymbol{\eta}_{\mathbf{v}}^m - \boldsymbol{\eta}_{\mathbf{u}}^m) \Delta W_{m+1}, \mathbf{v}). \end{aligned} \quad (4.99)$$

Applying the summation operator $\sum_{m=0}^{\ell}$ to (4.99) for $0 \leq \ell \leq M-1$ yields

$$\begin{aligned} \left(k \sum_{m=0}^{\ell} (q^{m+1} - r^m), \operatorname{div} \mathbf{v} \right) &= (\mathbf{e}_{\mathbf{u}}^{\ell+1} - \mathbf{e}_{\mathbf{u}}^0, \mathbf{v}) + \left(k \sum_{m=0}^{\ell} \nabla \mathbf{e}_{\mathbf{u}}^{m+1}, \nabla \mathbf{v} \right) \\ &\quad - \left(\sum_{m=0}^{\ell} (\boldsymbol{\eta}_{\mathbf{v}}^m - \boldsymbol{\eta}_{\mathbf{u}}^m) \Delta W_{m+1}, \mathbf{v} \right) \\ &= \text{I} + \text{II} + \text{III}. \end{aligned} \quad (4.100)$$

Now, we bound each term on the right-hand side of (4.100) as follows. By Schwarz and Poincaré inequalities, we get

$$\begin{aligned} |\text{I}| + |\text{II}| &\leq C \|\mathbf{e}_{\mathbf{u}}^{\ell+1}\|_{\mathbf{L}^2} \|\nabla \mathbf{v}\|_{\mathbf{L}^2} + k \sum_{m=0}^{\ell} \|\nabla \mathbf{e}_{\mathbf{u}}^{m+1}\|_{\mathbf{L}^2} \cdot \|\nabla \mathbf{v}\|_{\mathbf{L}^2}, \\ |\text{III}| &\leq \left\| \sum_{m=0}^{\ell} (\boldsymbol{\eta}_{\mathbf{v}}^m - \boldsymbol{\eta}_{\mathbf{u}}^m) \Delta W_{m+1} \right\|_{\mathbf{L}^2} \cdot \|\nabla \mathbf{v}\|_{\mathbf{L}^2} \\ &\leq \left\| \sum_{m=0}^{\ell} (\mathbf{B}(\mathbf{v}^m) - \mathbf{B}(\tilde{\mathbf{u}}^m)) \Delta W_{m+1} \right\|_{\mathbf{L}^2} \cdot \|\nabla \mathbf{v}\|_{\mathbf{L}^2} \\ &\quad + \left\| \sum_{m=0}^{\ell} (\nabla(\xi_{\mathbf{v}}^m - \xi_{\mathbf{u}}^m)) \Delta W_{m+1} \right\|_{\mathbf{L}^2} \cdot \|\nabla \mathbf{v}\|_{\mathbf{L}^2} \\ &= (\text{III}_1 + \text{III}_2) \|\nabla \mathbf{v}\|_{\mathbf{L}^2}. \end{aligned}$$

By Itô isometry, the assumptions on $\mathbf{B}$ and Theorem 4.2.8, we obtain

$$\begin{aligned}\mathbb{E}[\mathbf{III}_1^2] &= k \sum_{m=0}^{\ell} \mathbb{E}[\|\mathbf{B}(\mathbf{v}^m) - \mathbf{B}(\tilde{\mathbf{u}}^m)\|_{\mathbf{L}^2}^2] \leq Ck \sum_{m=0}^{\ell} \mathbb{E}[\|\mathbf{e}_{\tilde{\mathbf{u}}}^m\|_{\mathbf{L}^2}^2] \leq Ck. \\ \mathbb{E}[\mathbf{III}_2^2] &= k \sum_{m=0}^{\ell} \mathbb{E}[\|\nabla(\xi_{\mathbf{v}}^m - \xi_{\tilde{\mathbf{u}}}^m)\|_{\mathbf{L}^2}^2].\end{aligned}$$

Moreover, using the Poisson equation as defined in Step 1 of Algorithm 1, we have

$$\|\nabla(\xi_{\mathbf{v}}^m - \xi_{\tilde{\mathbf{u}}}^m)\|_{\mathbf{L}^2} \leq \|\mathbf{B}(\mathbf{v}^m) - \mathbf{B}(\tilde{\mathbf{u}}^m)\|_{\mathbf{L}^2}.$$

Thus,

$$\mathbb{E}[\mathbf{III}_2^2] \leq k \sum_{m=0}^{\ell} \mathbb{E}[\|\mathbf{B}(\mathbf{v}^m) - \mathbf{B}(\tilde{\mathbf{u}}^m)\|_{\mathbf{L}^2}^2] \leq k \sum_{m=0}^{\ell} \mathbb{E}[\|\mathbf{e}_{\tilde{\mathbf{u}}}^m\|_{\mathbf{L}^2}^2] \leq Ck.$$

Finally, it follows from the inf-sup condition (4.53), Theorem 4.2.8, (4.100), and all above inequalities that

$$\left(\mathbb{E} \left[\left\| k \sum_{m=0}^{\ell} (q^{m+1} - r^m) \right\|_{L^2}^2 \right] \right)^{\frac{1}{2}} \leq C\sqrt{k}. \quad (4.101)$$

The proof is completed by applying the triangular inequality, Lemma 4.2.11 and (4.75b). $\square$

Corollary 4.2.13. *Let $\{p^m\}_{m=0}^M$ be generated by Algorithm 2. Then, there exists a constant $C > 0$ which depends on $D_T, \mathbf{u}_0, \mathbf{f}$ and β such that for $0 \leq \ell \leq M$*

$$\left(\mathbb{E} \left[\left\| P(t_{\ell}) - k \sum_{m=0}^{\ell} p^m \right\|_{L^2}^2 \right] \right)^{\frac{1}{2}} \leq C\sqrt{k}. \quad (4.102)$$

Proof. The proof is similar to that of Theorem 4.2.12. Indeed, we have

$$\begin{aligned}
\left\| P(t_m) - k \sum_{m=0}^{\ell} p^m \right\|_{L^2} &\leq \left\| R(t_m) - k \sum_{m=0}^{\ell} r^m \right\|_{L^2} \\
&\quad + \left\| \sum_{m=0}^{\ell} \int_{t_m}^{t_{m+1}} \xi(s) dW(s) - \sum_{m=0}^{\ell} \xi^m \Delta W_{m+1} \right\|_{\mathbf{L}^2} \\
&= \text{I} + \text{II}.
\end{aligned} \tag{4.103}$$

Clearly, the first term “I” can be bounded by using Theorem 4.2.12. To bound the second term “II”, we first have

$$\begin{aligned}
\text{II} &\leq \left\| \sum_{m=0}^{\ell} (\xi(t_m) - \xi^m) \Delta W_{m+1} \right\|_{\mathbf{L}^2} + \left\| \sum_{m=0}^{\ell} \int_{t_m}^{t_{m+1}} (\xi(s) - \xi(t_m)) dW(s) \right\|_{\mathbf{L}^2} \\
&= \text{II}_1 + \text{II}_2.
\end{aligned} \tag{4.104}$$

Then by Itô isometry we get

$$\begin{aligned}
\mathbb{E}[\text{II}_1]^2 &= \mathbb{E} \left[k \sum_{m=0}^{\ell} \|\xi(t_m) - \xi^m\|_{\mathbf{L}^2}^2 \right] \leq C \mathbb{E} \left[k \sum_{m=0}^{\ell} \|\mathbf{B}(\mathbf{u}(t_m)) - \mathbf{B}(\tilde{\mathbf{u}}^m)\|_{\mathbf{L}^2}^2 \right] \\
&\leq C \mathbb{E} \left[k \sum_{m=0}^{\ell} \|\mathbf{e}_{\tilde{\mathbf{u}}}^m\|_{\mathbf{L}^2}^2 \right] \leq Ck.
\end{aligned} \tag{4.105}$$

Similarly, by the Hölder continuity (2.11a) we have

$$\begin{aligned}
\mathbb{E}[\text{II}_2]^2 &= \sum_{m=0}^{\ell} \mathbb{E} \left[\int_{t_m}^{t_{m+1}} \|\xi(s) - \xi(t_m)\|_{\mathbf{L}^2}^2 ds \right] \\
&\leq \sum_{m=0}^{\ell} \mathbb{E} \left[\int_{t_m}^{t_{m+1}} \|\mathbf{u}(s) - \mathbf{u}(t_m)\|_{\mathbf{L}^2}^2 ds \right] \leq Ck,
\end{aligned} \tag{4.106}$$

Finally, substituting (4.104)–(4.106) into (4.103) and using Theorem 4.2.12 give us the desired estimate. The proof is complete. $\square$

4.3 Fully discrete finite element methods

In this section, we formulate and analyze finite element spatial discretization for Algorithm 1 and 2. To the end, let $\mathcal{T}_h$ be a quasi-uniform triangulation of the polygonal ($d = 2$) or polyhedral ($d = 3$) bounded domain D . We introduce the following two basic Lagrangian finite element spaces:

$$\mathbf{V}_h = \{\phi \in \mathbb{C}(\overline{D}); \quad \phi|_K \in \mathbf{P}_\ell(K) \quad \forall K \in \mathcal{T}_h\}, \quad (4.107)$$

$$X_h = \{\phi \in C(\overline{D}); \quad \phi|_K \in P_\ell(K) \quad \forall K \in \mathcal{T}_h\}, \quad (4.108)$$

where $P_\ell(K)$ ($\ell \geq 1$) denotes the set of polynomials of degree less than or equal to ℓ over the element $K \in \mathcal{T}_h$. The finite element spaces to be used to formulate our finite element methods are defined as follows:

$$\mathbf{H}_h = \mathbf{V}_h \cap \mathbf{H}_{per}^1(D), \quad L_h = X_h \cap L_{per}^2(D)/\mathbb{R}, \quad S_h = X_h \cap L_{per}^2(D)/\mathbb{R}. \quad (4.109)$$

In addition, we introduce spaces

$$\mathbb{V}_h = L^2(\Omega, \mathbf{H}_h), \quad \mathbb{W}_h = L^2(\Omega, L_h). \quad (4.110)$$

Recall that the L^2 -projection $\mathcal{P}_h^0 : [L_{per}^2(D)]^d \rightarrow \mathbf{H}_h$ is defined by

$$(\phi - \mathcal{P}_h^0 \phi, \xi) = 0 \quad \forall \xi \in \mathbf{H}_h \quad (4.111)$$

and the H^1 -projection $\mathcal{P}_h^1 : H_{per}^1(D)/\mathbb{R} \rightarrow L_h$ is defined by

$$((\nabla(\chi - \mathcal{P}_h^1 \chi), \nabla \eta) = 0 \quad \forall \eta \in L_h. \quad (4.112)$$

It is well known [9] that $\mathcal{P}_h^0$ and $\mathcal{P}_h^1$ satisfy following estimates:

$$\|\phi - \mathcal{P}_h^0 \phi\|_{\mathbf{L}^2} + h \|\nabla(\phi - \mathcal{P}_h^0 \phi)\|_{\mathbf{L}^2} \leq C h^2 \|\phi\|_{H^2} \quad \forall \phi \in \mathbf{H}_{per}^2(D), \quad (4.113)$$

$$\|\chi - \mathcal{P}_h^1 \chi\|_{L^2} + h \|\nabla(\chi - \mathcal{P}_h^1 \chi)\|_{L^2} \leq C h^2 \|\chi\|_{H^2} \quad \forall \chi \in H_{per}^1/\mathbb{R} \cap H_{per}^2(D). \quad (4.114)$$

For the clarity we only consider P_1 -finite element space in this section (i.e., $\ell = 1$), the results of this section can be easily extended to high order finite elements.

4.3.1 Finite element method for the standard Chorin scheme

Approximating the velocity space and pressure space respectively by the finite element spaces $\mathbf{H}_h$ and L_h in Algorithm 1, we then obtain the fully discrete finite element version of the standard Chorin scheme given below as Algorithm 3. We also note that a similar algorithm was proposed in [16].

Algorithm 3

Let $n \geq 0$. Set $\tilde{\mathbf{u}}_h^0 = \mathcal{P}_h^0 \mathbf{u}_0$. For $n = 0, 1, 2, \dots$ do the following steps:

Step 1: Given $\mathbf{u}_h^n \in L^2(\Omega, \mathbf{H}_h)$ and $\tilde{\mathbf{u}}_h^n \in L^2(\Omega, \mathbf{H}_h)$, find $\tilde{\mathbf{u}}_h^{n+1} \in L^2(\Omega, \mathbf{H}_h)$ such that $\mathbb{P}$ -a.s.

$$\begin{aligned} & (\tilde{\mathbf{u}}_h^{n+1}, \mathbf{v}_h) + k(\nabla \tilde{\mathbf{u}}_h^{n+1}, \nabla \mathbf{v}_h) \\ &= (\mathbf{u}_h^n, \mathbf{v}_h) + k(\mathbf{f}^{n+1}, \mathbf{v}_h) + (\mathbf{B}(\tilde{\mathbf{u}}_h^n) \Delta W_{n+1}, \nabla \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{H}_h. \end{aligned} \quad (4.115)$$

Step 2: Find $p_h^{n+1} \in L^2(\Omega, L_h)$ such that $\mathbb{P}$ -a.s.

$$(\nabla p_h^{n+1}, \nabla \phi_h) = \frac{1}{k}(\tilde{\mathbf{u}}_h^{n+1}, \nabla \phi_h) \quad \forall \phi_h \in L_h. \quad (4.116)$$

Step 3: Define $\mathbf{u}_h^{n+1} \in L^2(\Omega, \mathbf{H}_h)$ by

$$\mathbf{u}_h^{n+1} = \tilde{\mathbf{u}}_h^{n+1} - k \nabla p_h^{n+1}. \quad (4.117)$$

Eliminating $\mathbf{u}_h^n$ in (4.115) using (4.116), we obtain

$$(\tilde{\mathbf{u}}_h^{n+1} - \tilde{\mathbf{u}}_h^n, \mathbf{v}_h) + k(\nabla \tilde{\mathbf{u}}_h^{n+1}, \nabla \mathbf{v}_h) \quad (4.118a)$$

$$+ k(\nabla p_h^n, \mathbf{v}_h) = k(\mathbf{f}^{n+1}, \mathbf{v}_h) + (\mathbf{B}(\tilde{\mathbf{u}}_h^n) \Delta W_{n+1}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{H}_h,$$

$$(\tilde{\mathbf{u}}_h^{n+1}, \nabla \phi_h) = k(\nabla p_h^{n+1}, \nabla \phi_h) \quad \forall \phi_h \in L_h. \quad (4.118b)$$

Next, we state the stability estimates for $\{(\tilde{\mathbf{u}}_h^n, p_h^n)\}_{n=0}^M$ in the following lemma, which will be used in the fully discrete error analysis later. Since its proof follows from the same lines of that for Lemma 4.2.1, we omit it to save space.

Lemma 4.3.1. *Let $\{(\tilde{\mathbf{u}}_h^n, p_h^n)\}_{n=0}^M$ be generated by Algorithm 3, then there holds*

$$\max_{0 \leq n \leq M} \mathbb{E}[\|\tilde{\mathbf{u}}_h^n\|_{\mathbf{L}^2}^2] + \mathbb{E}\left[\sum_{n=1}^M \|\tilde{\mathbf{u}}_h^n - \tilde{\mathbf{u}}_h^{n-1}\|_{\mathbf{L}^2}^2\right] + \mathbb{E}\left[k \sum_{n=0}^M \|\nabla \tilde{\mathbf{u}}_h^n\|_{\mathbf{L}^2}^2\right] \leq C, \quad (4.119a)$$

$$\mathbb{E}\left[k \sum_{n=0}^M \|\nabla p_h^n\|_{L^2}^2\right] \leq \frac{C}{k}, \quad (4.119b)$$

where C is a positive constant depending only on D_T, u_0 , and $\mathbf{f}$.

The following theorem provides an error estimate in a strong norm for the finite element solution of Algorithm 3.

Theorem 4.3.2. *Let $\{(\tilde{\mathbf{u}}^m, p^m)\}_{m=0}^M$ and $\{(\tilde{\mathbf{u}}_h^m, p_h^m)\}_{m=0}^M$ be generated respectively by Algorithm 1 and Algorithm 3. Then under the assumptions of Lemmas 4.2.1, 4.3.1 and Corollary 4.2.4 there holds*

$$\begin{aligned} & \left(\mathbb{E}\left[k \sum_{m=0}^M \|\tilde{\mathbf{u}}^m - \tilde{\mathbf{u}}_h^m\|_{\mathbf{L}^2}^2\right] \right)^{\frac{1}{2}} \\ & + \max_{0 \leq \ell \leq M} \left(\mathbb{E}\left[\left\|k \sum_{n=1}^{\ell} \nabla(\tilde{\mathbf{u}}^n - \tilde{\mathbf{u}}_h^n)\right\|_{\mathbf{L}^2}^2\right] \right)^{\frac{1}{2}} \leq C \left(k^{\frac{1}{4}} + h k^{-\frac{1}{2}} \right), \end{aligned} \quad (4.120)$$

where $C \equiv C(D_T, \mathbf{u}_0, \mathbf{f})$ is a positive constant.

Proof. The proof is conceptually similar to that of Theorem 4.2.2. Setting $\mathbf{e}_{\mathbf{u}_h}^m =: \tilde{\mathbf{u}}^m - \tilde{\mathbf{u}}_h^m$ and $\varepsilon_{p_h}^m =: p^m - p_h^m$. Without loss of the generality, we assume $\mathbf{e}_{\mathbf{u}_h}^0 = 0$ and $\varepsilon_{p_h}^0 = 0$ because they are of high order accuracy, hence are negligible.

First, applying the summation operator $\sum_{n=0}^m$ to (4.118a), we obtain

$$\begin{aligned} & (\tilde{\mathbf{u}}_h^{m+1}, \mathbf{v}_h) + k \left(\sum_{n=0}^{m+1} \nabla \tilde{\mathbf{u}}_h^n, \nabla \mathbf{v}_h \right) + k \left(\sum_{n=0}^m \nabla p_h^n, \mathbf{v}_h \right) \\ & = (\mathbf{u}_h^0, \mathbf{v}_h) + \left(\sum_{n=0}^m \mathbf{B}(\tilde{\mathbf{u}}_h^n) \Delta W_{n+1}, \mathbf{v}_h \right) \quad \forall \mathbf{v}_h \in \mathbf{H}_h. \end{aligned} \quad (4.121)$$

Subtracting (4.31) from (4.121) yields the following error equations:

$$(\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}, \mathbf{v}_h) + k \left(\sum_{n=0}^{m+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^n, \nabla \mathbf{v}_h \right) + k \left(\sum_{n=0}^m \nabla \varepsilon_{p_h}^n, \mathbf{v}_h \right) \quad (4.122)$$

$$= \left(\sum_{n=0}^m (\mathbf{B}(\tilde{\mathbf{u}}^n) - \mathbf{B}(\tilde{\mathbf{u}}_h^n)) \Delta W_{n+1}, \mathbf{v}_h \right) \quad \forall \mathbf{v}_h \in \mathbf{H}_h,$$

$$(\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}, \nabla \phi_h) = k (\nabla \varepsilon_{p_h}^{m+1}, \nabla \phi_h) \quad \forall \phi_h \in L_h. \quad (4.123)$$

Choosing $\mathbf{v}_h = \mathcal{P}_h^0 \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1} = \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1} - \boldsymbol{\theta}^{m+1}$; $\boldsymbol{\theta}^m = \tilde{\mathbf{u}}_h^m - \mathcal{P}_h^0 \tilde{\mathbf{u}}_h^m$, then (4.122) becomes

$$\begin{aligned} \|\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|_{\mathbf{L}^2}^2 + k \left(\sum_{n=0}^{m+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^n, \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1} \right) + k \left(\sum_{n=0}^m \nabla \varepsilon_{p_h}^n, \mathcal{P}_h^0 \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1} \right) \\ = (\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}, \boldsymbol{\theta}^{m+1}) + k \left(\sum_{n=0}^{m+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^n, \nabla \boldsymbol{\theta}^{m+1} \right) \\ + \left(\sum_{n=0}^m (\mathbf{B}(\tilde{\mathbf{u}}^n) - \mathbf{B}(\tilde{\mathbf{u}}_h^n)) \Delta W_{n+1}, \mathcal{P}_h^0 \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1} \right). \end{aligned} \quad (4.124)$$

Setting $\phi_h = \sum_{n=0}^m \mathcal{P}_h^1 \varepsilon_{p_h}^n = \sum_{n=0}^m \varepsilon_{p_h}^n - \sum_{n=0}^m \xi^n$ in (4.123), where $\xi^n = p^n - \mathcal{P}_h^1 p^n$, we obtain

$$(\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}, \sum_{n=0}^m \nabla \mathcal{P}_h^1 \varepsilon_{p_h}^n) = k \left(\nabla \varepsilon_{p_h}^{m+1}, \sum_{n=0}^m \nabla \mathcal{P}_h^1 \varepsilon_{p_h}^n \right) \quad (4.125)$$

In addition, by using the properties of $\mathcal{P}_h^0$ - and $\mathcal{P}_h^1$ -projection we have

$$\begin{aligned} k \left(\sum_{n=0}^m \nabla \varepsilon_{p_h}^n, \mathcal{P}_h^0 \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1} \right) &= k \left(\sum_{n=0}^m \nabla \varepsilon_{p_h}^n, \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1} \right) - k \left(\sum_{n=0}^m \nabla \varepsilon_{p_h}^n, \boldsymbol{\theta}^{m+1} \right) \\ &= k \left(\sum_{n=0}^m \nabla \mathcal{P}_h^1 \varepsilon_{p_h}^n, \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1} \right) + k \left(\sum_{n=0}^m \nabla \xi^n, \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1} \right) - k \left(\sum_{n=0}^m \nabla \varepsilon_{p_h}^n, \boldsymbol{\theta}^{m+1} \right) \\ &= k^2 \left(\nabla \varepsilon_{p_h}^{m+1}, \sum_{n=0}^m \nabla \mathcal{P}_h^1 \varepsilon_{p_h}^n \right) + k \left(\sum_{n=0}^m \nabla \xi^n, \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1} \right) - k \left(\sum_{n=0}^m \nabla \varepsilon_{p_h}^n, \boldsymbol{\theta}^{m+1} \right) \\ &= k^2 \left(\nabla \varepsilon_{p_h}^{m+1}, \sum_{n=0}^{m+1} \nabla \mathcal{P}_h^1 \varepsilon_{p_h}^n \right) - k^2 \left(\nabla \varepsilon_{p_h}^{m+1}, \nabla \mathcal{P}_h^1 \varepsilon_{p_h}^{m+1} \right) \\ &\quad + k \left(\sum_{n=0}^m \nabla \xi^n, \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1} \right) - k \left(\sum_{n=0}^m \nabla \varepsilon_{p_h}^n, \boldsymbol{\theta}^{m+1} \right) \end{aligned} \quad (4.126)$$

$$\begin{aligned}
&= k^2 \left(\nabla \varepsilon_{p_h}^{m+1}, \sum_{n=0}^{m+1} \nabla \varepsilon_{p_h}^n \right) - k^2 \left(\nabla \varepsilon_{p_h}^{m+1}, \sum_{n=0}^{m+1} \nabla \xi^n \right) \\
&\quad - k^2 \left(\nabla \varepsilon_{p_h}^{m+1}, \nabla \mathcal{P}_h^1 \varepsilon_{p_h}^{m+1} \right) + k \left(\sum_{n=0}^m \nabla \xi^n, \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1} \right) - k \left(\sum_{n=0}^m \nabla \varepsilon_{p_h}^n, \boldsymbol{\theta}^{m+1} \right).
\end{aligned}$$

Moreover, by using the orthogonality property of $\mathcal{P}_h^1$, we have

$$\begin{aligned}
-k^2 \left(\nabla \varepsilon_{p_h}^{m+1}, \nabla \mathcal{P}_h^1 \varepsilon_{p_h}^{m+1} \right) &= -k^2 \left(\nabla (\varepsilon_{p_h}^{m+1} - \mathcal{P}_h^1 \varepsilon_{p_h}), \nabla \mathcal{P}_h^1 \varepsilon_{p_h}^{m+1} \right) \\
&\quad + k^2 \|\nabla \mathcal{P}_h^1 \varepsilon_{p_h}^{m+1}\|_{L^2}^2 = k^2 \|\nabla \mathcal{P}_h^1 \varepsilon_{p_h}^{m+1}\|_{L^2}^2,
\end{aligned} \tag{4.127}$$

which helps to reduce (4.126) into

$$\begin{aligned}
k \left(\sum_{n=0}^m \nabla \varepsilon_{p_h}^n, \mathcal{P}_h^0 \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1} \right) &= k^2 \left(\nabla \varepsilon_{p_h}^{m+1}, \sum_{n=0}^{m+1} \nabla \varepsilon_{p_h}^n \right) - k^2 \left(\nabla \varepsilon_{p_h}^{m+1}, \sum_{n=0}^{m+1} \nabla \xi^n \right) \\
&\quad + k^2 \|\nabla \mathcal{P}_h^1 \varepsilon_{p_h}^{m+1}\|_{L^2}^2 + k \left(\sum_{n=0}^m \nabla \xi^n, \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1} \right) \\
&\quad - k \left(\sum_{n=0}^m \nabla \varepsilon_{p_h}^n, \boldsymbol{\theta}^{m+1} \right).
\end{aligned} \tag{4.128}$$

Substituting (4.128) into (4.124) and rearranging terms yield

$$\begin{aligned}
&\|\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|_{\mathbf{L}^2}^2 + k \left(\sum_{n=0}^{m+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^n, \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1} \right) + k^2 \left(\nabla \varepsilon_{p_h}^{m+1}, \sum_{n=0}^{m+1} \nabla \varepsilon_{p_h}^n \right) \\
&\quad + k^2 \|\nabla \mathcal{P}_h^1 \varepsilon_{p_h}^{m+1}\|_{L^2}^2 \\
&= k^2 \left(\nabla \varepsilon_{p_h}^{m+1}, \sum_{n=0}^{m+1} \nabla \xi^n \right) + k \left(\operatorname{div} \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}, \sum_{n=0}^m \xi^n \right) \\
&\quad + k \left(\sum_{n=0}^m \nabla \varepsilon_{p_h}^n, \boldsymbol{\theta}^{m+1} \right) + (\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}, \boldsymbol{\theta}^{m+1}) + k \left(\sum_{n=0}^{m+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^n, \nabla \boldsymbol{\theta}^{m+1} \right) \\
&\quad + \left(\sum_{n=0}^m (\mathbf{B}(\tilde{\mathbf{u}}^n) - \mathbf{B}(\tilde{\mathbf{u}}_h^n)) \Delta W_{n+1}, \mathcal{P}_h^0 \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1} \right).
\end{aligned} \tag{4.129}$$

Next, we use the identity $2a(a-b) = a^2 - b^2 + (a-b)^2$ to create telescoping sums on the left-hand side of (4.129) followed by taking the expectation to get

$$\begin{aligned}
& \mathbb{E}[\|\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|_{\mathbf{L}^2}^2] + \frac{k}{2} \mathbb{E} \left[\left\| \sum_{n=0}^{m+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^n \right\|_{\mathbf{L}^2}^2 - \left\| \sum_{n=0}^m \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^n \right\|_{\mathbf{L}^2}^2 + \|\nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|_{\mathbf{L}^2}^2 \right] \\
& + \frac{k^2}{2} \mathbb{E} \left[\left\| \sum_{n=0}^{m+1} \nabla \varepsilon_{p_h}^n \right\|_{L^2}^2 - \left\| \sum_{n=0}^m \nabla \varepsilon_{p_h}^n \right\|_{L^2}^2 + \|\nabla \varepsilon_{p_h}^{m+1}\|_{L^2}^2 \right] \\
& + k^2 \mathbb{E}[\|\nabla \mathcal{P}_h^1 \varepsilon_{p_h}^{m+1}\|_{L^2}^2] \\
& = k^2 \mathbb{E} \left[\left(\nabla \varepsilon_{p_h}^{m+1}, \sum_{n=0}^{m+1} \nabla \xi^n \right) \right] + k \mathbb{E} \left[\left(\operatorname{div} \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}, \sum_{n=0}^m \xi^n \right) \right] \\
& + k \mathbb{E} \left[\left(\sum_{n=0}^m \nabla \varepsilon_{p_h}^n, \boldsymbol{\theta}^{m+1} \right) \right] + \mathbb{E} \left[\left(\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}, \boldsymbol{\theta}^{m+1} \right) \right] \\
& + k \mathbb{E} \left[\left(\sum_{n=0}^{m+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^n, \nabla \boldsymbol{\theta}^{m+1} \right) \right] \\
& + \mathbb{E} \left[\left(\sum_{n=0}^m (\mathbf{B}(\tilde{\mathbf{u}}^n) - \mathbf{B}(\tilde{\mathbf{u}}_h^n)) \Delta W_{n+1}, \mathcal{P}_h^0 \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1} \right) \right].
\end{aligned} \tag{4.130}$$

Now, applying $k \sum_{m=0}^\ell$ for $0 \leq \ell < M$ to (4.130) we obtain

$$\begin{aligned}
& k \sum_{m=0}^\ell \mathbb{E}[\|\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|_{\mathbf{L}^2}^2] + \frac{1}{2} \mathbb{E} \left[\left\| k \sum_{n=0}^{\ell+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^n \right\|_{\mathbf{L}^2}^2 \right] + \frac{k^2}{2} \sum_{m=0}^\ell \mathbb{E}[\|\nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|_{\mathbf{L}^2}^2] \\
& + \frac{k}{2} \mathbb{E} \left[\left\| k \sum_{n=0}^{\ell+1} \nabla \varepsilon_{p_h}^n \right\|_{L^2}^2 \right] + \frac{k^3}{2} \sum_{m=0}^\ell \mathbb{E}[\|\nabla \varepsilon_{p_h}^{m+1}\|_{L^2}^2 + 2\|\nabla \mathcal{P}_h^1 \varepsilon_{p_h}^{m+1}\|_{L^2}^2] \\
& = \mathbb{E} \left[k^3 \sum_{m=0}^\ell \left(\nabla \varepsilon_{p_h}^{m+1}, \sum_{n=0}^{m+1} \nabla \xi^n \right) \right] + \mathbb{E} \left[k^2 \sum_{m=0}^\ell \left(\operatorname{div} \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}, \sum_{n=0}^m \xi^n \right) \right] \\
& + \mathbb{E} \left[k^2 \sum_{m=0}^\ell \left(\sum_{n=0}^m \nabla \varepsilon_{p_h}^n, \boldsymbol{\theta}^{m+1} \right) \right] + \mathbb{E} \left[k \sum_{m=0}^\ell \left(\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}, \boldsymbol{\theta}^{m+1} \right) \right] \\
& + \mathbb{E} \left[k^2 \sum_{m=0}^\ell \left(\sum_{n=0}^{m+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^n, \nabla \boldsymbol{\theta}^{m+1} \right) \right] \\
& + k \sum_{m=0}^\ell \mathbb{E} \left[\left(\sum_{n=0}^m (\mathbf{B}(\tilde{\mathbf{u}}^n) - \mathbf{B}(\tilde{\mathbf{u}}_h^n)) \Delta W_{n+1}, \mathcal{P}_h^0 \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1} \right) \right] \\
& := \text{I} + \cdots + \text{VI}.
\end{aligned} \tag{4.131}$$

Next, we bound the right-hand side of (4.131) as follows. By using the discrete Hölder inequality we get

$$\begin{aligned}
\text{I} &= k\mathbb{E}\left[k\sum_{m=0}^{\ell}\left(\nabla\varepsilon_{p_h}^{m+1}, k\sum_{n=0}^{m+1}\nabla\xi^n\right)\right] \\
&\leq k\mathbb{E}\left[k\sum_{m=0}^{\ell}\|\nabla\varepsilon_{p_h}^{m+1}\|_{L^2}\left\|k\sum_{n=0}^{m+1}\nabla\xi^n\right\|_{\mathbf{L}^2}\right] \\
&\leq k\mathbb{E}\left[\left(k\sum_{m=0}^{\ell}\|\nabla\varepsilon_{p_h}^{m+1}\|_{L^2}^2\right)^{\frac{1}{2}}\left(k\sum_{m=0}^{\ell}\left\|k\sum_{n=0}^{m+1}\nabla\xi^n\right\|_{\mathbf{L}^2}^2\right)^{\frac{1}{2}}\right] \\
&\leq k\left(\mathbb{E}\left[k\sum_{m=0}^{\ell}\|\nabla\varepsilon_{p_h}^{m+1}\|_{L^2}^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[k\sum_{m=0}^{\ell}\left\|k\sum_{n=0}^{m+1}\nabla\xi^n\right\|_{\mathbf{L}^2}^2\right]\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.132}$$

In addition, by using the stability estimates from (4.17b) and (4.119b), we have

$$\mathbb{E}\left[k\sum_{m=0}^{\ell}\|\nabla\varepsilon_{p_h}^{m+1}\|_{L^2}^2\right] \leq \frac{C}{k},$$

Similarly, using (4.114) and the stability estimate from (4.62) we get

$$\begin{aligned}
\mathbb{E}\left[k\sum_{m=0}^{\ell}\left\|k\sum_{n=0}^{m+1}\nabla\xi^n\right\|_{\mathbf{L}^2}^2\right] &= \mathbb{E}\left[k\sum_{m=0}^{\ell}\left\|\nabla\left(k\sum_{n=0}^{m+1}p^n - \mathcal{P}_h^1\left(k\sum_{n=0}^{m+1}p^n\right)\right)\right\|_{\mathbf{L}^2}^2\right] \\
&\leq C\mathbb{E}\left[k\sum_{m=0}^{\ell}\left\|k\sum_{n=0}^{m+1}\nabla p^n\right\|_{\mathbf{L}^2}^2\right] \leq C.
\end{aligned}$$

Therefore, $\text{I} \leq Ck^{\frac{1}{2}}$.

Next, using the fact that $\left\|\sum_{n=0}^m\xi^n\right\|_{\mathbf{L}^2} \leq Ch\left\|\sum_{n=0}^m\nabla p^n\right\|_{\mathbf{L}^2}$ we have

$$\begin{aligned}
\text{II} &\leq \frac{k^2}{8}\sum_{m=0}^{\ell}\mathbb{E}[\|\nabla\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|_{\mathbf{L}^2}^2] + C\mathbb{E}\left[k^2\sum_{m=0}^{\ell}\left\|\sum_{n=0}^m\xi^n\right\|_{\mathbf{L}^2}^2\right] \\
&\leq \frac{k^2}{8}\sum_{m=0}^{\ell}\mathbb{E}[\|\nabla\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|_{\mathbf{L}^2}^2] + \frac{Ch^2}{k}\mathbb{E}\left[k\sum_{m=0}^{\ell}\left\|k\sum_{n=0}^m\nabla p^n\right\|_{\mathbf{L}^2}^2\right] \\
&\leq \frac{k^2}{8}\sum_{m=0}^{\ell}\mathbb{E}[\|\nabla\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|^2] + \frac{Ch^2}{k},
\end{aligned} \tag{4.133}$$

where (4.62) was used to obtain the last inequality. Expectedly, the first term will be absorbed to the left-hand side of (4.131) later.

Next, using summation by parts we obtain

$$\begin{aligned}
\text{III} &= \mathbb{E} \left[k^2 \sum_{m=0}^{\ell} \left(\sum_{n=0}^m \nabla \varepsilon_{p_h}^n, \boldsymbol{\theta}^{m+1} \right) \right] \\
&= \mathbb{E} \left[k^2 \left(\sum_{n=0}^{\ell+1} \boldsymbol{\theta}^n, \sum_{n=0}^{\ell+1} \nabla \varepsilon_{p_h}^n \right) \right] - \mathbb{E} \left[k^2 \sum_{m=0}^{\ell} \left(\sum_{n=0}^m \boldsymbol{\theta}^n, \nabla \varepsilon_{p_h}^m \right) \right] \\
&\leq Ck \mathbb{E} \left[\left\| \sum_{n=0}^{\ell+1} \boldsymbol{\theta}^n \right\|_{\mathbf{L}^2}^2 \right] + \frac{k^3}{8} \mathbb{E} \left[\left\| \sum_{n=0}^{\ell+1} \nabla \varepsilon_{p_h}^n \right\|_{\mathbf{L}^2}^2 \right] \\
&\quad + C \mathbb{E} \left[k \sum_{m=0}^{\ell} \left\| \sum_{n=0}^m \boldsymbol{\theta}^n \right\|_{\mathbf{L}^2}^2 \right] + \frac{k^3}{8} \sum_{m=0}^{\ell} \mathbb{E} [\| \nabla \varepsilon_{p_h}^m \|_{\mathbf{L}^2}^2].
\end{aligned} \tag{4.134}$$

In addition, we can use (4.113) and (4.63) to control the first and third terms on the right-hand side of (4.134) as follows:

$$\begin{aligned}
&Ck \mathbb{E} \left[\left\| \sum_{n=0}^{\ell+1} \boldsymbol{\theta}^n \right\|_{\mathbf{L}^2}^2 \right] + C \mathbb{E} \left[k \sum_{m=0}^{\ell} \left\| \sum_{n=0}^m \boldsymbol{\theta}^n \right\|_{\mathbf{L}^2}^2 \right] \\
&\leq Ch^4 k \mathbb{E} \left[\left\| \sum_{n=0}^{\ell+1} \Delta \tilde{\mathbf{u}}^n \right\|_{\mathbf{L}^2}^2 \right] + Ch^4 \mathbb{E} \left[k \sum_{m=0}^{\ell} \left\| \sum_{n=0}^m \Delta \tilde{\mathbf{u}}^n \right\|_{\mathbf{L}^2}^2 \right] \leq \frac{Ch^4}{k^2}.
\end{aligned} \tag{4.135}$$

Therefore,

$$\text{III} \leq \frac{Ch^4}{k^2} + \frac{k^3}{8} \mathbb{E} \left[\left\| \sum_{n=0}^{\ell+1} \nabla \varepsilon_{p_h}^n \right\|_{\mathbf{L}^2}^2 \right] + \frac{k^3}{8} \sum_{m=0}^{\ell} \mathbb{E} [\| \nabla \varepsilon_{p_h}^m \|_{\mathbf{L}^2}^2], \tag{4.136}$$

Again, expectedly, the last two terms on the right-hand side of (4.136) will be absorbed to the left-hand side of (4.131) later.

It follows from (4.113) and (4.17c) that

$$\begin{aligned}
\text{IV} &= \mathbb{E} \left[k \sum_{m=0}^{\ell} (\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}, \boldsymbol{\theta}^{m+1}) \right] \\
&\leq k \sum_{m=0}^{\ell} \mathbb{E} [\|\boldsymbol{\theta}^{m+1}\|_{\mathbf{L}^2}^2] + \frac{1}{4} k \sum_{m=0}^{\ell} \mathbb{E} [\|\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|_{\mathbf{L}^2}^2] \\
&\leq Ch^4 \mathbb{E} \left[k \sum_{m=0}^{\ell} \|\Delta \tilde{\mathbf{u}}^{m+1}\|_{\mathbf{L}^2}^2 \right] + \frac{1}{4} k \sum_{m=0}^{\ell} \mathbb{E} [\|\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|^2] \\
&\leq \frac{Ch^4}{k} + \frac{1}{4} k \sum_{m=0}^{\ell} \mathbb{E} [\|\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|_{\mathbf{L}^2}^2].
\end{aligned} \tag{4.137}$$

To estimate term V, we approach similarly as done for term III. Namely, first we use the summation by parts and then use (4.113) and (4.62).

$$\begin{aligned}
\text{V} &= \mathbb{E} \left[k^2 \sum_{m=0}^{\ell} \left(\sum_{n=0}^{m+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^n, \nabla \boldsymbol{\theta}^{m+1} \right) \right] \\
&= \mathbb{E} \left[k^2 \left(\sum_{n=0}^{\ell+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^n, \sum_{n=0}^{\ell+1} \nabla \boldsymbol{\theta}^n \right) \right] - \mathbb{E} \left[k^2 \sum_{m=0}^{\ell} \left(\sum_{n=0}^m \nabla \boldsymbol{\theta}^n, \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1} \right) \right] \\
&\leq \frac{1}{8} \mathbb{E} \left[\left\| k \sum_{n=0}^{\ell+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^n \right\|_{\mathbf{L}^2}^2 \right] + C \mathbb{E} \left[\left\| k \sum_{n=0}^{\ell+1} \nabla \boldsymbol{\theta}^n \right\|_{\mathbf{L}^2}^2 \right] \\
&\quad + \frac{1}{8} \mathbb{E} \left[k^2 \sum_{m=0}^{\ell} \|\nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|_{\mathbf{L}^2}^2 \right] + C \mathbb{E} \left[k^2 \sum_{m=0}^{\ell} \left\| \sum_{n=0}^m \nabla \boldsymbol{\theta}^n \right\|_{\mathbf{L}^2}^2 \right] \\
&\leq \frac{1}{8} \mathbb{E} \left[\left\| k \sum_{n=0}^{\ell+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^n \right\|_{\mathbf{L}^2}^2 \right] + \frac{1}{8} \mathbb{E} \left[k^2 \sum_{m=0}^{\ell} \|\nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|_{\mathbf{L}^2}^2 \right] \\
&\quad + Ch^2 \mathbb{E} \left[\left\| k \sum_{n=0}^m \Delta \tilde{\mathbf{u}}^n \right\|_{\mathbf{L}^2}^2 \right] + \frac{Ch^2}{k} \mathbb{E} \left[k \sum_{m=0}^{\ell} \left\| k \sum_{n=0}^m \Delta \tilde{\mathbf{u}}^n \right\|_{\mathbf{L}^2}^2 \right] \\
&\leq \frac{1}{8} \mathbb{E} \left[\left\| k \sum_{n=0}^{\ell+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^n \right\|_{\mathbf{L}^2}^2 \right] + \frac{1}{8} \mathbb{E} \left[k^2 \sum_{m=0}^{\ell} \|\nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|_{\mathbf{L}^2}^2 \right] + \frac{Ch^2}{k}.
\end{aligned} \tag{4.138}$$

We use the Itô isometry to handle the noise term as follows:

$$\begin{aligned}
\text{VI} &= k \sum_{m=0}^{\ell} \mathbb{E} \left[\left(\sum_{n=0}^m (\mathbf{B}(\tilde{\mathbf{u}}^n) - \mathbf{B}(\tilde{\mathbf{u}}_h^n)) \Delta W_{n+1}, \mathcal{P}_h^0 \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1} \right) \right] \\
&\leq \frac{1}{4} k \sum_{m=0}^{\ell} \mathbb{E} [\|\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|_{\mathbf{L}^2}^2] + k \sum_{m=0}^{\ell} \mathbb{E} \left[\left\| \sum_{n=0}^m (\mathbf{B}(\tilde{\mathbf{u}}^n) - \mathbf{B}(\tilde{\mathbf{u}}_h^n)) \Delta W_{n+1} \right\|_{\mathbf{L}^2}^2 \right] \\
&= \frac{1}{4} k \sum_{m=0}^{\ell} \mathbb{E} [\|\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|_{\mathbf{L}^2}^2] + k \sum_{m=0}^{\ell} \mathbb{E} \left[k \sum_{n=0}^m \|\mathbf{B}(\tilde{\mathbf{u}}^n) - \mathbf{B}(\tilde{\mathbf{u}}_h^n)\|_{\mathbf{L}^2}^2 \right] \\
&\leq \frac{1}{4} k \sum_{m=0}^{\ell} \mathbb{E} [\|\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|_{\mathbf{L}^2}^2] + Ck \sum_{m=0}^{\ell} k \sum_{n=0}^m \mathbb{E} [\|\mathbf{e}_{\tilde{\mathbf{u}}_h}^n\|_{\mathbf{L}^2}^2].
\end{aligned} \tag{4.139}$$

Finally, substituting the above estimates for terms I, II, III, IV, V, VI into (4.131) yields the following inequality for $X^\ell = k \sum_{m=0}^{\ell} \mathbb{E} [\|\mathbf{e}_{\tilde{\mathbf{u}}_h}^m\|_{\mathbf{L}^2}^2]$:

$$\begin{aligned}
&\frac{1}{2} X^{\ell+1} + \frac{3}{8} \mathbb{E} \left[\left\| k \sum_{n=0}^{\ell+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^n \right\|_{\mathbf{L}^2}^2 \right] + \frac{k^2}{4} \sum_{m=0}^{\ell} \mathbb{E} [\|\nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|_{\mathbf{L}^2}^2] \\
&\quad + \frac{3k}{8} \mathbb{E} \left[\left\| k \sum_{n=0}^{\ell+1} \nabla \varepsilon_{p_h}^n \right\|_{\mathbf{L}^2}^2 \right] + \frac{3k^3}{8} \sum_{m=0}^{\ell} \mathbb{E} [\|\nabla \varepsilon_{p_h}^{m+1}\|_{\mathbf{L}^2}^2] \\
&\quad + k^3 \sum_{m=0}^{\ell} \mathbb{E} [\|\nabla \mathcal{P}_h^1 \varepsilon_{p_h}^{m+1}\|_{\mathbf{L}^2}^2] \\
&\leq Ck^{\frac{1}{2}} + \frac{Ch^2}{k} + Ck \sum_{m=0}^{\ell} X^m.
\end{aligned} \tag{4.140}$$

The desired error estimate (4.120) then follows from an application of the discrete Gronwall inequality to (4.140). The proof is complete. $\square$

Next, we state an error estimate result for the pressure approximation generated by Algorithm 3 in a time-averaged fashion. Recall that an important advantage of Chorin-type schemes is to allow the use of a pair of independent finite element spaces which are not required to satisfy a discrete inf-sup condition, a price for this advantage is to make error estimates for the pressure approximations become more complicated even in the deterministic case. The idea for circumventing the difficulty is to utilize the following perturbed inf-sup

inequality (cf. [45]): there exists $\delta > 0$ independent of $h > 0$, such that

$$\frac{1}{\delta^2} \|q_h\|_{L^2}^2 \leq \sup_{\mathbf{v}_h \in \mathbf{H}_h} \frac{|(q_h, \operatorname{div} \mathbf{v}_h)|^2}{\|\nabla \mathbf{v}_h\|_{\mathbf{L}^2}^2} + h^2 \|\nabla q_h\|_{\mathbf{L}^2}^2 \quad \forall q_h \in S_h. \quad (4.141)$$

Theorem 4.3.3. *Under the assumptions of Theorem 4.3.2, there exists a positive constant $C \equiv C(D_T, \mathbf{u}_0, \mathbf{f}, \delta)$ such that*

$$\left(\mathbb{E} \left[k \sum_{m=0}^M \left\| k \sum_{n=1}^m (p^n - p_h^n) \right\|_{L^2}^2 \right] \right)^{\frac{1}{2}} \leq C \left(k^{\frac{1}{4}} + h k^{-\frac{1}{2}} \right), \quad (4.142)$$

Proof. We reuse all the notations from the proof of Theorem 4.3.2. First, from the error equations (4.122) we have

$$\begin{aligned} \left(k \sum_{n=0}^m \varepsilon_{p_h}^n, \operatorname{div} \mathbf{v}_h \right) &= (\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}, \mathbf{v}_h) + \left(k \sum_{n=0}^{m+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^n, \nabla \mathbf{v}_h \right) \\ &\quad - \left(\sum_{n=0}^m (\mathbf{B}(\tilde{\mathbf{u}}^n) - \mathbf{B}(\tilde{\mathbf{u}}_h^n)) \Delta W_{n+1}, \mathbf{v}_h \right) \quad \forall \mathbf{v}_h \in \mathbf{H}_h. \end{aligned} \quad (4.143)$$

Using the Schwarz inequality on the right-hand side of (4.143) yields

$$\begin{aligned} \left| \left(k \sum_{n=0}^m \varepsilon_{p_h}^n, \operatorname{div} \mathbf{v}_h \right) \right| &= C \|\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|_{\mathbf{L}^2} \|\nabla \mathbf{v}_h\|_{\mathbf{L}^2} + \left\| k \sum_{n=0}^{m+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^n \right\|_{\mathbf{L}^2} \|\nabla \mathbf{v}_h\|_{\mathbf{L}^2} \\ &\quad + C \left\| \sum_{n=0}^m (\mathbf{B}(\tilde{\mathbf{u}}^n) - \mathbf{B}(\tilde{\mathbf{u}}_h^n)) \Delta W_{n+1} \right\|_{\mathbf{L}^2} \|\nabla \mathbf{v}_h\|_{\mathbf{L}^2}. \end{aligned} \quad (4.144)$$

Next, using (4.141) we conclude that

$$\begin{aligned} \frac{1}{\delta^2} \left\| k \sum_{n=0}^m \varepsilon_{p_h}^n \right\|_{L^2}^2 - h^2 \left\| k \sum_{n=0}^m \nabla \varepsilon_{p_h}^n \right\|_{\mathbf{L}^2}^2 &\leq \sup_{\mathbf{v}_h \in \mathbf{H}_h} \frac{\left| \left(k \sum_{n=0}^m \varepsilon_{p_h}^n, \operatorname{div} \mathbf{v}_h \right) \right|^2}{\|\nabla \mathbf{v}_h\|_{\mathbf{L}^2}^2} \\ &\leq C \|\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|_{\mathbf{L}^2}^2 + C \left\| k \sum_{n=0}^{m+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^n \right\|_{\mathbf{L}^2}^2 \\ &\quad + C \left\| \sum_{n=0}^m (\mathbf{B}(\tilde{\mathbf{u}}^n) - \mathbf{B}(\tilde{\mathbf{u}}_h^n)) \Delta W_{n+1} \right\|_{\mathbf{L}^2}^2. \end{aligned} \quad (4.145)$$

Then, applying operators $k \sum_{m=0}^{\ell}$ and $\mathbb{E}[\cdot]$ on both sides we obtain

$$\begin{aligned}
\frac{1}{\delta^2} \mathbb{E} \left[k \sum_{m=0}^{\ell} \left\| k \sum_{n=0}^m \varepsilon_{p_h}^n \right\|_{\mathbf{L}^2}^2 \right] &\leq h^2 \mathbb{E} \left[k \sum_{m=0}^{\ell} \left\| k \sum_{n=0}^m \nabla \varepsilon_{p_h}^n \right\|_{\mathbf{L}^2}^2 \right] \\
&+ Ck \sum_{m=0}^{\ell} \mathbb{E} [\| \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1} \|_{\mathbf{L}^2}^2] \\
&+ Ck \sum_{m=0}^{\ell} \mathbb{E} \left[\left\| k \sum_{n=0}^{m+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^n \right\|_{\mathbf{L}^2}^2 \right] \\
&+ Ck \sum_{m=0}^{\ell} \mathbb{E} \left[\left\| \sum_{n=0}^m (\mathbf{B}(\tilde{\mathbf{u}}^n) - \mathbf{B}(\tilde{\mathbf{u}}_h^n)) \Delta W_{n+1} \right\|_{\mathbf{L}^2}^2 \right] \\
&:= \text{I} + \text{II} + \text{III} + \text{IV}.
\end{aligned} \tag{4.146}$$

We now bound each terms on the right-hand side of (4.146). By using the discrete Jensen inequality and the stability estimates from (4.17b) and (4.119b) we get

$$\text{I} = h^2 \mathbb{E} \left[k \sum_{m=0}^{\ell} \left\| k \sum_{n=0}^m \nabla \varepsilon_{p_h}^n \right\|_{\mathbf{L}^2}^2 \right] \leq h^2 \mathbb{E} \left[k \sum_{m=0}^{\ell} k \sum_{n=0}^m \|\nabla \varepsilon_{p_h}^n\|_{\mathbf{L}^2}^2 \right] \leq \frac{Ch^2}{k}. \tag{4.147}$$

Using Theorem 4.3.2, terms II and III can be bounded as follows:

$$\begin{aligned}
\text{II} + \text{III} &= Ck \sum_{m=0}^{\ell} \mathbb{E} [\| \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1} \|_{\mathbf{L}^2}^2] + Ck \sum_{m=0}^{\ell} \mathbb{E} \left[\left\| k \sum_{n=0}^{m+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^n \right\|_{\mathbf{L}^2}^2 \right] \\
&\leq C \left(k^{\frac{1}{2}} + \frac{Ch^2}{k} \right).
\end{aligned} \tag{4.148}$$

Finally, using Itô's isometry and Theorem 4.3.2 we have

$$\begin{aligned}
\text{IV} &= Ck \sum_{m=0}^{\ell} \mathbb{E} \left[\left\| \sum_{n=0}^m (\mathbf{B}(\tilde{\mathbf{u}}^n) - \mathbf{B}(\tilde{\mathbf{u}}_h^n)) \Delta W_{n+1} \right\|_{\mathbf{L}^2}^2 \right] \\
&= Ck \sum_{m=0}^{\ell} \mathbb{E} \left[k \sum_{n=0}^m \|\mathbf{B}(\tilde{\mathbf{u}}^n) - \mathbf{B}(\tilde{\mathbf{u}}_h^n)\|_{\mathbf{L}^2}^2 \right] \\
&\leq Ck \sum_{m=0}^{\ell} \mathbb{E} \left[k \sum_{n=0}^m \|\mathbf{e}_{\tilde{\mathbf{u}}_h}^n\|_{\mathbf{L}^2}^2 \right] \\
&\leq C \left(k^{\frac{1}{2}} + \frac{Ch^2}{k} \right).
\end{aligned} \tag{4.149}$$

The proof is complete after combining the above estimates. $\square$

We are now ready to state the following global error estimate theorem for Algorithm 3 which is a main result of this paper.

Theorem 4.3.4. *Under the assumptions of Theorems 4.2.2, 4.2.3 and Theorems 4.3.2 and 4.3.3, there hold the following error estimates:*

$$\left(\mathbb{E} \left[k \sum_{m=0}^M \|\mathbf{u}(t_m) - \tilde{\mathbf{u}}_h^m\|_{\mathbf{L}^2}^2 \right] \right)^{\frac{1}{2}} + \max_{0 \leq \ell \leq M} \left(\mathbb{E} \left[\left\| k \sum_{m=0}^{\ell} \nabla(\mathbf{u}(t_m) - \tilde{\mathbf{u}}_h^m) \right\|_{\mathbf{L}^2}^2 \right] \right)^{\frac{1}{2}} \quad (4.150)$$

$$\leq C \left(k^{\frac{1}{4}} + h k^{-\frac{1}{2}} \right),$$

$$\left(\mathbb{E} \left[k \sum_{m=0}^M \left\| P(t_m) - k \sum_{n=0}^m p_h^n \right\|_{L^2}^2 \right] \right)^{\frac{1}{2}} \leq C \left(k^{\frac{1}{4}} + h k^{-\frac{1}{2}} \right), \quad (4.151)$$

where $C \equiv C(D_T, \mathbf{u}_0, \mathbf{f}, \beta, \delta)$ is positive constant independent of k and h .

4.3.2 Finite element method for the modified Chorin scheme

In this subsection, we first formulate a finite element spatial discretization for Algorithm 2 and then present a complete convergence analysis by deriving error estimates which are stronger than those obtained above for the standard Chorin scheme.

Algorithm 4

Let $m \geq 0$. Set $\tilde{\mathbf{u}}_h^0 = \mathcal{P}_h^0 \mathbf{u}_0$. For $m = 0, 1, 2, \dots$ do the following steps:

Step 1: For given $\tilde{\mathbf{u}}_h^m \in L^2(\Omega, \mathbf{H}_h)$, find $\xi_h^m \in L^2(\Omega, S_h)$ by solving the following Poisson problem: for $\mathbb{P}$ -a.s.

$$(\nabla \xi_h^m, \nabla \phi_h) = (\mathbf{B}(\tilde{\mathbf{u}}_h^m), \nabla \phi_h) \quad \forall \phi_h \in S_h. \quad (4.152)$$

Step 2: Set $\boldsymbol{\eta}_{\tilde{\mathbf{u}}_h}^m = \mathbf{B}(\tilde{\mathbf{u}}_h^m) - \nabla \xi_h^m$. For given $\mathbf{u}_h^m \in L^2(\Omega, \mathbf{H}_h)$ and $\tilde{\mathbf{u}}_h^m \in L^2(\Omega, \mathbf{H}_h)$, find $\tilde{\mathbf{u}}_h^{m+1} \in L^2(\Omega, \mathbf{H}_h)$ by solving the following problem: for $\mathbb{P}$ -a.s.

$$\begin{aligned} & (\tilde{\mathbf{u}}_h^{m+1}, \mathbf{v}_h) + k(\nabla \tilde{\mathbf{u}}_h^{m+1}, \nabla \mathbf{v}_h) \\ &= (\mathbf{u}_h^m, \mathbf{v}_h) + k(\mathbf{f}^{m+1}, \mathbf{v}_h) + (\boldsymbol{\eta}_{\tilde{\mathbf{u}}_h}^m \Delta W_{m+1}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{H}_h. \end{aligned} \quad (4.153)$$

Step 3: Find $r_h^{m+1} \in L^2(\Omega, L_h)$ by solving the following Poisson problem: for $\mathbb{P}$ -a.s.

$$(\nabla r_h^{m+1}, \nabla \phi_h) = \frac{1}{k} (\tilde{\mathbf{u}}_h^{m+1}, \nabla \phi_h) \quad \forall \phi_h \in L_h. \quad (4.154)$$

Step 4: Define $\mathbf{u}_h^{m+1} \in L^2(\Omega, \mathbf{H}_h)$ by

$$\mathbf{u}_h^{m+1} = \tilde{\mathbf{u}}_h^{m+1} - k \nabla r_h^{m+1}. \quad (4.155)$$

Step 5: Define $p_h^{m+1} \in L^2(\Omega, L_h)$ by

$$p_h^{m+1} = r_h^{m+1} + \frac{1}{k} \xi_h^m \Delta W_{m+1}. \quad (4.156)$$

Since each step involves a coercive problem, hence, Algorithm 4 is well defined. The next theorem establishes a convergence rate for the finite element approximation of the velocity field. Since the proof follows the same lines as those in the proof of Theorem 4.3.2, we omit it to save space.

Theorem 4.3.5. *Let $\{\tilde{\mathbf{u}}^m\}_{m=0}^M$ and $\{\tilde{\mathbf{u}}_h^m\}_{m=0}^M$ be generated respectively by Algorithm 2 and 4. Then, there exists a constant $C \equiv C(D_T, \mathbf{u}_0, \mathbf{f}) > 0$ such that*

$$\begin{aligned} \max_{1 \leq m \leq M} \left(\mathbb{E} [\|\tilde{\mathbf{u}}^m - \tilde{\mathbf{u}}_h^m\|^2] \right)^{\frac{1}{2}} + \left(\mathbb{E} \left[k \sum_{m=1}^M \|\nabla(\tilde{\mathbf{u}}^m - \tilde{\mathbf{u}}_h^m)\|^2 \right] \right)^{\frac{1}{2}} \\ \leq C \left(\sqrt{k} + h + h^2 k^{-\frac{1}{2}} \right). \end{aligned} \quad (4.157)$$

In the next theorem, we establish an error estimate for the pressure approximation of the modified Chorin finite element method given by Algorithm 4.

Theorem 4.3.6. *Let $\{r^m\}_{m=1}^M$ and $\{r_h^m\}_{m=1}^M$ be generated respectively by Algorithm 2 and 4. Then, there exists a constant $C \equiv C(D_T, \mathbf{u}_0, \mathbf{f}, \delta) > 0$ such that*

$$\left(\mathbb{E} \left[\left\| k \sum_{m=1}^M (r^m - r_h^m) \right\|_{L^2}^2 \right] \right)^{\frac{1}{2}} \leq C \left(\sqrt{k} + h + h^2 k^{-\frac{1}{2}} \right). \quad (4.158)$$

Proof. Let $\boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^m = \tilde{\mathbf{u}}^m - \tilde{\mathbf{u}}_h^m$ and $\varepsilon_r^m = r^m - r_h^m$. It is easy to check that $(\boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^m, \varepsilon_r^m)$ satisfies the following error equation:

$$\begin{aligned} (\boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^{m+1} - \boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^m, \mathbf{v}_h) + k(\nabla \boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^{m+1}, \nabla \mathbf{v}_h) + k(\nabla \varepsilon_r^m, \mathbf{v}_h) \\ = ((\boldsymbol{\eta}_{\tilde{\mathbf{u}}}^m - \boldsymbol{\eta}_{\tilde{\mathbf{u}}_h}^m) \Delta W_{m+1}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{H}_h. \end{aligned} \quad (4.159)$$

Applying the summation operator $\sum_{m=0}^{\ell}$ ($0 \leq \ell \leq M-1$) to (4.159) yields

$$\begin{aligned} (\boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^{\ell+1} - \boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^0, \mathbf{v}_h) + \left(k \sum_{m=0}^{\ell} \nabla \boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^{m+1}, \nabla \mathbf{v}_h \right) + \left(k \sum_{m=0}^{\ell} \nabla \varepsilon_r^m, \mathbf{v}_h \right) \\ = \left(\sum_{m=0}^{\ell} (\boldsymbol{\eta}_{\tilde{\mathbf{u}}}^m - \boldsymbol{\eta}_{\tilde{\mathbf{u}}_h}^m) \Delta W_{m+1}, \mathbf{v}_h \right) \quad \forall \mathbf{v}_h \in \mathbf{H}_h. \end{aligned} \quad (4.160)$$

Thus,

$$\begin{aligned} \left(k \sum_{m=0}^{\ell} \varepsilon_r^m, \operatorname{div} \mathbf{v}_h \right) = (\boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^{\ell+1} - \boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^0, \mathbf{v}_h) + \left(k \sum_{m=0}^{\ell} \nabla \boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^{m+1}, \nabla \mathbf{v}_h \right) \\ - \left(\sum_{m=0}^{\ell} (\boldsymbol{\eta}_{\tilde{\mathbf{u}}}^m - \boldsymbol{\eta}_{\tilde{\mathbf{u}}_h}^m) \Delta W_{m+1}, \mathbf{v}_h \right) =: \text{I} + \text{II} + \text{III}. \end{aligned} \quad (4.161)$$

Using Poincaré and Schwarz inequalities, the three terms on the right-hand side of (4.161) can be bounded as follows:

$$|\text{I}| \leq C(\|\boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^{\ell+1}\|_{\mathbf{L}^2} + \|\boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^0\|_{\mathbf{L}^2}) \|\nabla \mathbf{v}_h\|_{\mathbf{L}^2}, \quad (4.162)$$

$$|\text{II}| \leq \left\| k \sum_{m=0}^{\ell} \nabla \boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^{m+1} \right\|_{\mathbf{L}^2} \|\nabla \mathbf{v}_h\|_{\mathbf{L}^2}, \quad (4.163)$$

$$|\text{III}| \leq C \left\| \sum_{m=0}^{\ell} (\boldsymbol{\eta}_{\tilde{\mathbf{u}}}^m - \boldsymbol{\eta}_{\tilde{\mathbf{u}}_h}^m) \Delta W_{m+1} \right\|_{\mathbf{L}^2} \|\nabla \mathbf{v}_h\|_{\mathbf{L}^2}. \quad (4.164)$$

Using (4.141) and (4.162)–(4.164) in (4.161), we obtain

$$\begin{aligned} \frac{1}{\delta^2} \left\| k \sum_{m=0}^{\ell} \varepsilon_r^m \right\|_{L^2}^2 - h^2 \left\| k \sum_{m=0}^{\ell} \nabla \varepsilon_r^m \right\|_{\mathbf{L}^2}^2 &\leq C \|\boldsymbol{\varepsilon}_{\mathbf{u}}^{\ell+1}\|^2 + k \sum_{m=0}^{\ell} \|\nabla \boldsymbol{\varepsilon}_{\mathbf{u}}^{m+1}\|_{\mathbf{L}^2}^2 \\ &\quad + C \left\| \sum_{m=0}^{\ell} (\boldsymbol{\eta}_{\mathbf{u}}^m - \boldsymbol{\eta}_{\mathbf{u}_h}^m) \Delta W_{m+1} \right\|_{\mathbf{L}^2}^2. \end{aligned} \quad (4.165)$$

Using itô isometry, the last term above can be bounded as

$$\begin{aligned} \mathbb{E} \left[\left\| \sum_{m=0}^{\ell} (\boldsymbol{\eta}_{\mathbf{u}}^m - \boldsymbol{\eta}_{\mathbf{u}_h}^m) \Delta_{m+1} W \right\|_{\mathbf{L}^2}^2 \right] &= \mathbb{E} \left[k \sum_{m=0}^{\ell} \|\boldsymbol{\eta}_{\mathbf{u}}^m - \boldsymbol{\eta}_{\mathbf{u}_h}^m\|_{\mathbf{L}^2}^2 \right] \\ &\leq C \mathbb{E} \left[k \sum_{m=0}^{\ell} \|\boldsymbol{\varepsilon}_{\mathbf{u}}^m\|_{\mathbf{L}^2}^2 \right] + Ch^2. \end{aligned} \quad (4.166)$$

Substituting (4.166) into (4.165) yields

$$\begin{aligned} \frac{1}{\delta^2} \mathbb{E} \left[\left\| k \sum_{m=0}^{\ell} \varepsilon_r^m \right\|_{L^2}^2 \right] &\leq C \mathbb{E} [\|\boldsymbol{\varepsilon}_{\mathbf{u}}^{\ell+1}\|_{\mathbf{L}^2}^2] + \mathbb{E} \left[k \sum_{m=0}^{\ell} \|\nabla \boldsymbol{\varepsilon}_{\mathbf{u}}^{m+1}\|_{\mathbf{L}^2}^2 \right] \\ &\quad + C \mathbb{E} \left[k \sum_{m=0}^{\ell} \|\boldsymbol{\varepsilon}_{\mathbf{u}}^m\|_{\mathbf{L}^2}^2 \right] + h^2 \mathbb{E} \left[k \sum_{m=0}^{\ell} \|\nabla \varepsilon_r^m\|_{\mathbf{L}^2}^2 \right] + Ch^2. \end{aligned} \quad (4.167)$$

Finally, the desired estimate (4.158) follows from Theorem 4.3.5 and Step 3 of Algorithm 2 and 4. The proof is complete. $\square$

Corollary 4.3.7. *Let $\{p^m\}_{m=1}^M$ and $\{p_h^m\}_{m=1}^M$ be generated respectively by Algorithm 1 and 2. Then, there exists a positive constant $C \equiv C(D_T, \mathbf{u}_0, \mathbf{f}, \delta)$ such that*

$$\left(\mathbb{E} \left[\left\| k \sum_{m=1}^M (p^m - p_h^m) \right\|^2 \right] \right)^{\frac{1}{2}} \leq C \left(\sqrt{k} + h + h^2 k^{-\frac{1}{2}} \right). \quad (4.168)$$

Proof. Since the proof follows the same lines as those of the proof for Corollary 4.2.13, we only highlight the main steps. By definition of $\{p^m\}$ and $\{p_h^m\}$, we have

$$\begin{aligned} \left\| k \sum_{m=1}^M (p^m - p_h^m) \right\|_{L^2} &\leq \left\| k \sum_{m=1}^M (r^m - r_h^m) \right\|_{L^2} + \left\| \sum_{m=1}^M (\xi_{\tilde{\mathbf{u}}}^m - \xi_{\tilde{\mathbf{u}}_h}^m) \Delta W_{m+1} \right\|_{\mathbf{L}^2} \\ &=: \text{I} + \text{II}. \end{aligned} \quad (4.169)$$

Term I can be easily bounded by using Theorem 2.3.2. To bound II, by Itô isometry we get

$$\mathbb{E}[\text{II}]^2 = \mathbb{E} \left[k \sum_{m=1}^M \|\xi^m - \xi_h^m\|_{\mathbf{L}^2}^2 \right].$$

In addition, by (4.66) and (4.152), we get

$$\begin{aligned} \|\nabla(\xi_{\tilde{\mathbf{u}}}^m - \xi_{\tilde{\mathbf{u}}_h}^m)\|_{\mathbf{L}^2}^2 &\leq C \|\mathbf{B}(\tilde{\mathbf{u}}^m) - \mathbf{B}(\tilde{\mathbf{u}}_h^m)\|_{\mathbf{L}^2}^2 + Ch^2 \|\xi_{\tilde{\mathbf{u}}}^m\|_{H^2}^2 \\ &\leq C \|\mathbf{B}(\tilde{\mathbf{u}}^m) - \mathbf{B}(\tilde{\mathbf{u}}_h^m)\|_{\mathbf{L}^2}^2 + Ch^2 \|\operatorname{div}(\mathbf{B}(\tilde{\mathbf{u}}^m))\|_{\mathbf{L}^2}^2 \\ &\leq C \|\mathbf{B}(\tilde{\mathbf{u}}^m) - \mathbf{B}(\tilde{\mathbf{u}}_h^m)\|_{\mathbf{L}^2}^2 + Ch^2 \|\nabla(\mathbf{B}(\tilde{\mathbf{u}}^m))\|_{\mathbf{L}^2}^2 \\ &\leq C \|\mathbf{B}(\tilde{\mathbf{u}}^m) - \mathbf{B}(\tilde{\mathbf{u}}_h^m)\|_{\mathbf{L}^2}^2 + Ch^2 \|\nabla \tilde{\mathbf{u}}^m\|_{\mathbf{L}^2}^2. \end{aligned} \quad (4.170)$$

Finally, by Poincaré inequality, Lipschitz continuity of $\mathbf{B}$, Theorem 4.3.5 and (4.170), we obtain

$$\mathbb{E}[\text{II}]^2 = \mathbb{E} \left[k \sum_{m=1}^M \|\xi^m - \xi_h^m\|_{\mathbf{L}^2}^2 \right] \leq C \left(k + h^2 + \frac{h^4}{k} \right).$$

The proof is complete. □

We conclude this section by stating the following global error estimate theorem for Algorithm 4, which is another main result of this paper.

Theorem 4.3.8. *Let $(\mathbf{u}, P)$ be the solution of (1.3) and $\{(\tilde{\mathbf{u}}_h^m, r_h^m, p_h^m)\}_{m=1}^M$ be the solution of Algorithm 4. Then, there exists a constant $C \equiv C(D_T, \mathbf{u}_0, \mathbf{f}, \beta, \delta) > 0$ such that*

$$\begin{aligned} & \max_{1 \leq m \leq M} \left(\mathbb{E} \left[\left\| \mathbf{u}(t_m) - \tilde{\mathbf{u}}_h^m \right\|_{\mathbf{L}^2}^2 \right] \right)^{\frac{1}{2}} + \left(\mathbb{E} \left[k \sum_{m=1}^M \left\| \nabla(\mathbf{u}(t_m) - \tilde{\mathbf{u}}_h^m) \right\|_{\mathbf{L}^2}^2 \right] \right)^{\frac{1}{2}} \\ & \leq C \left(\sqrt{k} + h + h^2 k^{-\frac{1}{2}} \right), \\ & \left(\mathbb{E} \left[\left\| R(t_m) - k \sum_{n=1}^m r_h^n \right\|_{L^2}^2 \right] \right)^{\frac{1}{2}} + \left(\mathbb{E} \left[\left\| P(t_m) - k \sum_{n=1}^m p_h^n \right\|_{L^2}^2 \right] \right)^{\frac{1}{2}} \\ & \leq C \left(\sqrt{k} + h + h^2 k^{-\frac{1}{2}} \right). \end{aligned}$$

Remark 4.3.1. *The above error estimates are of the same nature as those obtained in [34] for the standard Euler-Maruyama mixed finite element method. On the other hand, the error estimates obtained in Chapter 2 for the Helmholtz enhanced Euler-Maruyama mixed finite element method do not have the growth term $h^2 k^{-\frac{1}{2}}$. Hence, in the case of general multiplication noise, the Helmholtz enhanced Euler-Maruyama mixed finite element method performs better than the Helmholtz enhanced Chorin finite element method in terms of rates of convergence.*

4.4 Computational experiments

In this section, we present two two-dimensional numerical tests to gauge the performance of the proposed numerical methods/algorithms. The first test is to verify the convergent rates proved in Theorem 4.3.4 for Algorithm 3 while the second test is to validate the convergent rates proved in Theorem 4.3.8. In both tests the computational domain is chosen as $D = (0, 1) \times (0, 1)$, the linear finite element spaces are used for spatial discretization, the terminal time is $T = 1$, the fine time and space mesh sizes $k_0 = \frac{1}{256}$ and $h = \frac{1}{40}$ are used to compute the numerical true solution, and the number of realizations is set as $N_p = 300$. The body force $\mathbf{f}$ is considered as in (2.75). Moreover, to evaluate errors in strong norms,

we use the following numerical integration formulas: for any $0 \leq m \leq M$

$$\begin{aligned}
\mathcal{E}_{\mathbf{u}}^m &:= \left(\mathbb{E} \left[\|\mathbf{u}(t_m) - \mathbf{u}_h^m(k)\|_{vL^2}^2 \right] \right)^{\frac{1}{2}} \approx \left(\frac{1}{N_p} \sum_{\ell=1}^{N_p} \|\mathbf{u}_h^m(k_0, \omega_\ell) - \mathbf{u}_h^m(k, \omega_\ell)\|_{\mathbf{L}^2}^2 \right)^{\frac{1}{2}}, \\
\mathcal{E}_{\mathbf{u},av}^M &:= \left(\mathbb{E} \left[k \sum_{m=0}^M \|\mathbf{u}(t_m) - \mathbf{u}_h^m(k)\|_{\mathbf{L}^2}^2 \right] \right)^{\frac{1}{2}} \\
&\approx \left(\frac{1}{N_p} \sum_{\ell=1}^{N_p} \left(k \sum_{m=0}^M \|\mathbf{u}_h^m(k_0, \omega_\ell) - \mathbf{u}_h^m(k, \omega_\ell)\|_{\mathbf{L}^2}^2 \right) \right)^{\frac{1}{2}}, \\
\mathcal{E}_{p,av}^M &:= \left(\mathbb{E} \left[k \sum_{m=0}^M \left\| P(t_m) - k \sum_{n=0}^m p_h^n(k) \right\|_{L^2}^2 \right] \right)^{\frac{1}{2}} \\
&\approx \left(\frac{1}{N_p} \sum_{\ell=1}^{N_p} \left(k \sum_{m=0}^M \left\| k_0 \sum_{n=1}^{\frac{tm}{k_0}} p_h^n(k_0, \omega_\ell) - k \sum_{n=1}^{\frac{tm}{k}} p_h^n(k, \omega_\ell) \right\|_{L^2}^2 \right) \right)^{\frac{1}{2}}.
\end{aligned}$$

Test 1. In this test, the nonlinear multiplicative noise function B is chosen as $B(\mathbf{u}) = (u_1^2 + u_2^2 + 1)^{\frac{1}{2}}$ and the initial value $\mathbf{u}_0 = (0, 0)^\top$. Moreover, we choose $\mathbb{R}^J$ -valued Wiener process $\mathbf{W}$ with increments satisfying

$$\Delta \mathbf{W}_{m+1} = \mathbf{W}^J(t_{m+1}, \mathbf{x}) - \mathbf{W}^J(t_m, \mathbf{x}) = \sqrt{k_0} \sum_{j,\ell=0}^J \sqrt{\lambda_{j,\ell}} \mathbf{e}_{j,\ell}(\mathbf{x}) \beta_{j,\ell}^m, \quad (4.171)$$

where $\mathbf{x} = (x_1, x_2) \in D$, $\beta_{j,\ell}^m \sim N(0, 1)$ and $\{\mathbf{e}_{j,\ell}(\mathbf{x})\}_{j,\ell}$ are orthonormal functions defined by

$$\mathbf{e}_{j,\ell}(x_1, x_2) = (\sin(j\pi x_1) \sin(\ell\pi x_2), \sin(j\pi x_1) \sin(\ell\pi x_2)) \quad (4.172)$$

and $\lambda_{j,\ell} = \frac{1}{10(j+\ell)^2}$. In this test, we set $J = 4$, $\nu = 1$.

Figure 4.1 displays the convergence rates of the time discretization produced by Algorithm 3 (and Algorithm 1) using different time step size k . The left figure shows the convergence rate $O(k^{\frac{1}{4}})$ in the $\mathcal{E}_{\mathbf{u},av}^M$ -norm for the velocity approximation, while the right graph shows the same convergence rate in the $\mathcal{E}_{p,av}^M$ -norm for the pressure approximation, both match the theoretical rates proved in our theoretical error estimates.

To verify the sharpness of the error bounds on the factor $k^{-\frac{1}{2}}$, we implement Algorithm 3 using different pairs (k, h) , which satisfy the relation $h \approx k$, and display the numerical

results in Figure 4.2. We observe $\frac{1}{4}$ order convergence rate for both the velocity and pressure approximations as predicted in Theorem 4.3.4.

Test 2. We use the same test problem as in **Test 1** to validate the theoretical error estimates for our modified Chorin scheme given by Algorithm 4. It should be noted that a similar numerical experiment was done in [16]. However, only the velocity approximation was analyzed and tested, no convergent rate for the pressure approximation was proved or tested in [16]. Here we want to emphasize the optimal convergence rate for the pressure approximation in the time-averaged norm.

Figure 4.3 displays the $\frac{1}{2}$ order convergence rate in time for both the velocity and pressure approximations by Algorithm 4 as predicted by Theorem 4.3.8. We note that the velocity error is measured in the strong norm and the pressure error is measured in a time-averaged norm.

Similar to **Test 1**, we want to test whether the dependence of the error bounds on the factor $k^{-\frac{1}{2}}$ is valid and sharp. To the end, we use the same strategy as we did in **Test 1**, namely, we fix mesh size $h = \frac{1}{4}$ and decrease time step size k . As expected, we observe that the errors blow up as shown in Figure 4.4.

Finally, Figure 4.5 shows the $\frac{1}{2}$ order convergence rate for both the velocity and pressure approximations by Algorithm 4 when the time step size k and the space mesh size h satisfy the balancing condition $h \approx \sqrt{k}$, which verifies the sharpness of the dependence of the error bounds on the factor $k^{-\frac{1}{2}}$ as predicted by Theorem 4.3.8.

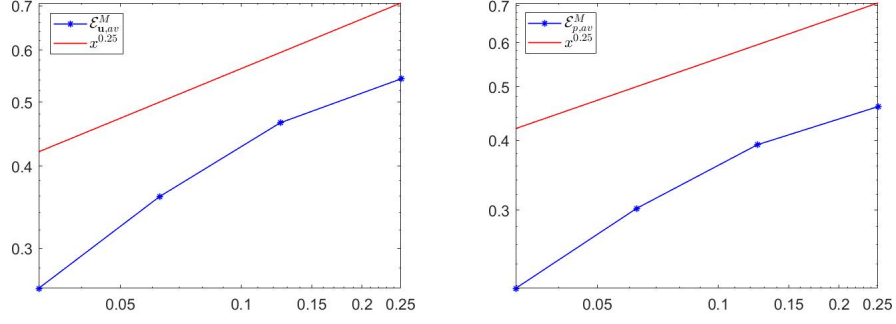


Figure 4.1: Convergence rates of the time discretization for the velocity (left) and pressure (right) approximations by Algorithm 3 in the $\mathcal{E}_{u,av}^M$ norm and $\mathcal{E}_{p,av}^M$ norm respectively.

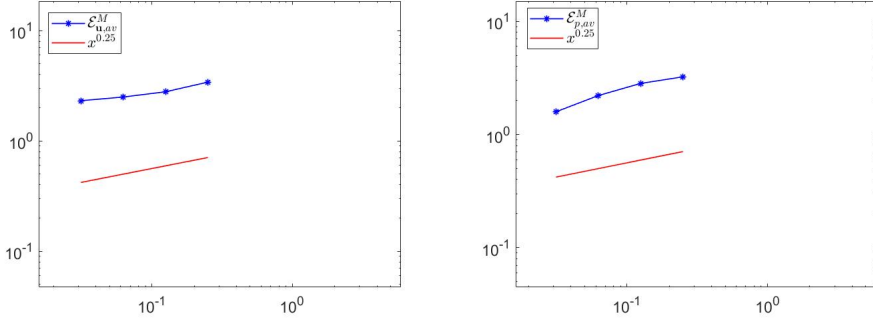


Figure 4.2: Convergence rates in the $\mathcal{E}_{u,av}^M$ norm for the velocity (left) approximation and the $\mathcal{E}_{p,av}^M$ norm for the pressure (right) approximation by Algorithm 3 under the mesh condition $h \approx k$.

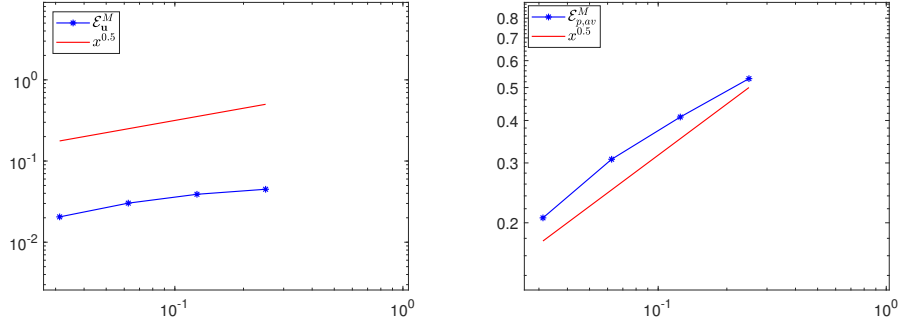


Figure 4.3: Convergence rates of the time discretization for the velocity in strong norm (left) and pressure in time-averaged norm (right) by Algorithm 4.

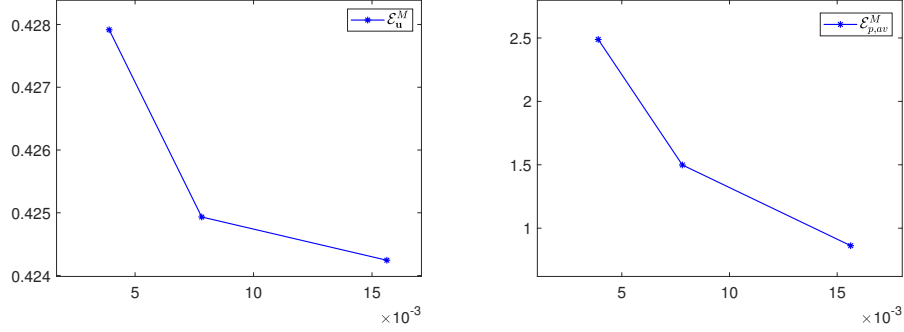


Figure 4.4: Errors for the velocity approximation in strong norm (left) and pressure approximation in time-averaged norm (right) by Algorithm 4.

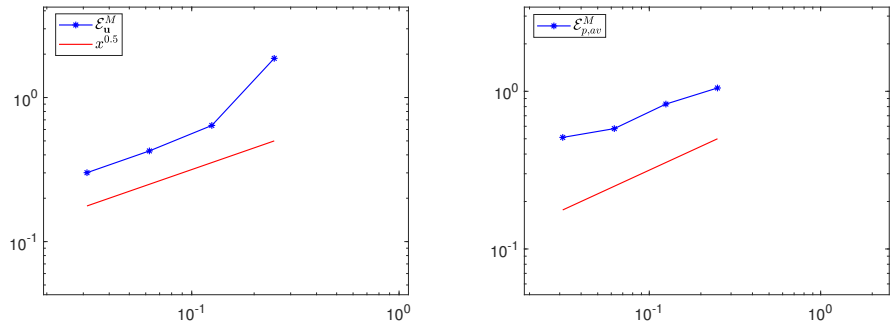


Figure 4.5: Convergence rates for the velocity approximation in strong norm (left) and pressure approximation in time-averaged norm (right) under the mesh condition $h \approx \sqrt{k}$.

Chapter 5

Mixed finite element methods for stochastic Navier-Stokes equations

5.1 Introduction

In this chapter, we consider the time-dependent stochastic Navier-Stokes equations (1.2) with additive noise, i.e., $\mathbf{B}(\mathbf{u}(t)) = \mathbf{g}(t)$ for all $t \in [0, T]$ (see Subsection 5.2.2 for definition of $\mathbf{g}$).

The first paper which considers numerical analysis of (1.2) is [14], where the authors established stability and convergence of the standard mixed finite element method (FEM) of (1.2). Later on, Carelli and Prohl [17] proved the rates of convergence in probability for the mixed finite approximation of (1.2). The main difficulty for establishing a strong convergence for any numerical approximation is the interplay between the nonlinearity of the Navier-Stokes equations and its stochastic noise term. To compute quantities of stochastic interest such as the expectation, all the norms must have another layer of the integration which is the main reason why the classical Gronwall inequality argument fails. To overcome this difficulty, the authors in [17] defined a sequence of sub-sample spaces which converges to the sample space under the probability measure. However, the error estimates on these sub-sample spaces are computed with partial expectations and are considered as weak convergence. Bessaih and Millet [5, 6] later showed the strong L^2 -convergence of the mixed finite element method for (1.2) by estimating the error on the complements of these sub-sample spaces.

They have obtained the strong convergence but the convergence order is only logarithmic rate. Recently, Bessaih and Millet [7] has established a strong convergence with polynomial rate for the mixed finite element method for (1.2) in the case of a divergence-free additive noise by using exponential stability estimates. The authors' idea is to apply the Gronwall inequality to derive a master inequality before taking the expectation to extract a convergence rate later. This idea was used earlier in [15, section 5.3.3] to overcome the same difficulty.

To the best of our knowledge, all the error estimates of (1.2) in the literature until now are second moment estimates and these estimates are only for the velocity approximation and no pressure error estimate was known. The primary goals of this chapter are to derive high moment error estimates for both the velocity and pressure approximations of the standard mixed finite element method for (1.2) in the case of general additive noise and then obtain pathwise error estimates by using Kolmogorov Theorem 3.2.1. The high moment error estimate in this chapter generalizes all the results stated above for additive noise. Moreover, high moment error estimate for the pressure approximation in a time-averaged norm is obtained by the velocity error estimate and the stochastic inf-sup condition and Kolmogorov Theorem 3.2.1 infers a pathwise error estimate for the pressure approximation.

The remainder of this chapter is organized as follows. In section 5.2, we give the precise definition of the variational solutions to (1.2). In addition, the assumption on the diffusion function $\mathbf{g}$ is specified. In section 5.3, we present the time discretization in Algorithm 1 and state some stability estimates for its solution, including an important exponential stability estimate which plays an important role in error analysis for the velocity approximation in Theorems 5.3.4, and 5.3.7, and the error estimates for the pressure approximation in Theorems 5.3.8 and 5.3.9. In section 5.4, we formulate a fully discrete mixed finite element method in Algorithm 2. High moment and pathwise error estimates are obtained for both fully discrete velocity and pressure approximations. Finally, we state two global error estimates in Theorems 5.4.8 and 5.4.9.

For the sake of presentation, we let $\mathbf{f} = 0$. In addition, all the constants in this chapter are independent of the mesh parameter h and k .

5.2 Preliminaries

5.2.1 Some useful facts and inequalities

In this subsection, we quote a few known and useful facts which will be used in our proofs later.

Lemma 5.2.1 (Gronwall inequality). *(a) Let y and g be nonnegative integrable functions and c a nonnegative constant. If*

$$y(t) \leq c + \int_0^t g(s)y(s) ds \quad \forall t \geq 0,$$

then

$$y(t) \leq c \exp\left(\int_0^t g(s) ds\right) \text{ for all } t \geq 0.$$

(b) Let $\{y_n\}$ and $\{g_n\}$ be nonnegative sequences and c a nonnegative constant. If

$$y_n \leq c + \sum_{0 \leq j < n} g_j y_j \quad \forall n \geq 0,$$

then

$$y_n \leq c \prod_{0 \leq j < n} (1 + g_j) \leq c \exp\left(\sum_{0 \leq j < n} g_j\right) \text{ for all } n \geq 0.$$

Lemma 5.2.2 (Hoeffding's lemma). *Let X be a real-valued random variable with $\mathbb{E}[X] = \mu$, such that $a \leq X \leq b$ a.s., for some $a, b \in \mathbb{R}$. Then, for all $\lambda \in \mathbb{R}$,*

$$\mathbb{E}[e^{\lambda X}] \leq \exp\left(\lambda\mu + \frac{\lambda^2}{8}(b-a)^2\right). \quad (5.1)$$

Finally, we recall the following property of the $\mathbb{R}$ -valued Wiener process, whose proof can be found in [57, Corollary 1.20]

There exists a deterministic constant $C > 0$ such that:

$$|W(t) - W(s)| \leq C|t - s|^\alpha, \quad \mathbb{P} - a.s., \quad (5.2)$$

where $0 < \alpha < \frac{1}{2}$.

5.2.2 Variational solutions

Let $\mathbf{g}(\cdot, \cdot) : D \times [0, T] \rightarrow \mathbb{R}$ and $\mathbf{g} \in L^\infty(0, T; \mathbf{H}^1(D))$. We make the following additional assumptions on $\mathbf{g}$:

(G1) $\mathbf{g}$ is Lipchitz continuous in time, that is, there is a constants $C_{\mathbf{g}} > 0$ such that

$$\|\mathbf{g}(t) - \mathbf{g}(s)\|_{\mathbf{L}^2} \leq C_{\mathbf{g}}|t - s| \quad \forall t, s \in [0, T].$$

(G2) Let $\sigma > 0$. Assume that there exists a constant $0 < K_0 \leq \frac{\nu}{8C_L\sigma}$ such that

$$\|\mathbf{g}\|_{L^\infty(0, T; \mathbf{H}^1)} \leq K_0, \text{ where } C_L \text{ is the constant from the inequality (5.5).}$$

Remark 5.2.1. *The assumption (G2) will be crucially used to obtain the exponential stability estimates in Lemma 5.3.3. On the other hand, both (G1) and (G2) will be needed to derive the order of convergence in Theorem 5.3.4 and Theorem 5.4.2.*

Next, we define the variational solution concept for (1.2) and refer the reader to [18, 19] for a proof of its existence and uniqueness.

Definition 5.2.3. *Given $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, let W be an $\mathbb{R}$ -valued Wiener process on it. Suppose $\mathbf{u}_0 \in L^2(\Omega, \mathbb{V})$. An $\{\mathcal{F}_t\}$ -adapted stochastic process $\{\mathbf{u}(t); 0 \leq t \leq T\}$ is called a variational solution of (1.2) if $\mathbf{u} \in L^2(\Omega; C([0, T]; \mathbb{V})) \cap L^2(\Omega; 0, T; \mathbf{H}_{per}^2(D))$, and satisfies $\mathbb{P}$ -a.s. for all $t \in (0, T]$*

$$\begin{aligned} (\mathbf{u}(t), \mathbf{v}) + \int_0^t \nu(\nabla \mathbf{u}(s), \nabla \mathbf{v}) ds + \int_0^t (\mathbf{u}(s) \cdot \nabla \mathbf{u}(s), \mathbf{v}) ds \\ = (\mathbf{u}_0, \mathbf{v}) + \left(\int_0^t \mathbf{g}(s) dW(s), \mathbf{v} \right) \quad \forall \mathbf{v} \in \mathbb{V}. \end{aligned} \quad (5.3)$$

We also recall some properties of the convection term. Define

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) := (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}_{per}^1.$$

It is easy to check that $b(\mathbf{u}, \cdot, \cdot)$ is skew-symmetric for $\mathbf{u} \in \mathbb{V}$, that is,

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}) \quad \forall \mathbf{v}, \mathbf{w} \in \mathbf{H}_{per}^1 \quad (5.4)$$

and

$$b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{H}_{per}^1.$$

In addition, we also recall the following interpolation inequality for $\mathbf{H}^1$ -functions:

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq \|\mathbf{u}\|_{\mathbf{L}^4} \|\nabla \mathbf{v}\|_{\mathbf{L}^2} \|\mathbf{w}\|_{\mathbf{L}^4} \leq C_L \|\mathbf{u}\|_{\mathbf{L}^2}^{1/2} \|\nabla \mathbf{u}\|_{\mathbf{L}^2}^{1/2} \|\nabla \mathbf{v}\|_{\mathbf{L}^2} \|\mathbf{w}\|_{\mathbf{L}^2}^{1/2} \|\nabla \mathbf{w}\|_{\mathbf{L}^2}^{1/2}. \quad (5.5)$$

Definition 5.2.3 only defines the velocity $\mathbf{u}$ for (1.2), its associated pressure p is subtle to define. We do so in the following theorem.

Theorem 5.2.4. *Let $\{\mathbf{u}(t); 0 \leq t \leq T\}$ be the variational solution of (1.2). There exists a unique adapted process $P \in L^2(\Omega, L^2(0, T; H_{per}^1(D)))$ such that $(\mathbf{u}, P)$ satisfies $\mathbb{P}$ -a.s. for all $t \in (0, T]$*

$$(\mathbf{u}(t), \mathbf{v}) + \nu \int_0^t (\nabla \mathbf{u}(s), \nabla \mathbf{v}) ds + \int_0^t (\mathbf{u}(s) \cdot \nabla \mathbf{u}(s), \mathbf{v}) ds - (\operatorname{div} \mathbf{v}, P(t)) \quad (5.6a)$$

$$= (\mathbf{u}_0, \mathbf{v}) + \int_0^t (\mathbf{f}(s), \mathbf{v}) ds + \int_0^t (\mathbf{B}(\mathbf{u}(s)), \mathbf{v}) dW(s) \quad \forall \mathbf{v} \in \mathbf{H}_{per}^1(D),$$

$$(\operatorname{div} \mathbf{u}, q) = 0 \quad \forall q \in L_0^2(D) := \{q \in L_{per}^2(D) : (q, 1) = 0\}. \quad (5.6b)$$

Proof. The proof is similar to Theorem 2.1.3 of Chapter 2 with a few modifications. $\square$

Lemma 5.2.5 ([8, 17]). *Let $\mathbf{u}$ be solution defined in Definition 5.2.3. Then we have*

(a) Assume that $\mathbf{u}_0 \in L^r(\Omega, \mathbb{V})$ for some $r \geq 2$. Then we have

$$\mathbb{E} \left[\left(\sup_{0 \leq t \leq T} \|\nabla \mathbf{u}(t)\|_{\mathbf{L}^2}^2 + \int_0^T \nu \|\nabla^2 \mathbf{u}(t)\|_{\mathbf{L}^2}^2 dt \right)^{\frac{r}{2}} \right] \leq C_r \mathbb{E} \left[\|\nabla \mathbf{u}_0\|_{\mathbf{L}^2}^r \right]. \quad (5.7)$$

(b) Assume that $\mathbf{u}_0 \in L^r(\Omega, \mathbf{H}^2(\operatorname{div}; D)) \cap L^{5r}(\Omega, \mathbb{V})$ for some $r \geq 2$. Then we have

$$\mathbb{E} \left[\left(\sup_{0 \leq t \leq T} \|\nabla^2 \mathbf{u}(t)\|_{\mathbf{L}^2}^2 + \int_0^T \nu \|\nabla^3 \mathbf{u}(t)\|_{\mathbf{L}^2}^2 dt \right)^{\frac{r}{2}} \right] \leq C_r \mathbb{E} \left[\left(\|\mathbf{u}_0\|_{\mathbf{H}^2}^2 + \|\nabla \mathbf{u}_0\|_{\mathbf{L}^2}^{10} \right)^{\frac{r}{2}} \right]. \quad (5.8)$$

5.2.3 Hölder continuity of the variational solution

We cite the following high moment Hölder continuity estimate for the variational solution whose proof can be found in [8, 16].

Lemma 5.2.6 ([8, 16]). *Suppose $\mathbf{u}_0$ satisfies the assumptions in Lemma 5.2.5 (b) for some $r > 2$ and $\mathbf{g} \in L^2(\Omega, L^\infty(0, T; \mathbf{H}^2(D)))$. Then there exists a constant $C \equiv C(D_T, \mathbf{u}_0, \mathbf{g}, r) > 0$, such that the variational solution to problem (1.2) satisfies for $s, t \in [0, T]$*

$$\mathbb{E} \left[\|\mathbf{u}(t) - \mathbf{u}(s)\|_{\mathbb{V}}^r \right] \leq C |t - s|^{r\gamma} \quad \forall \gamma \in \left(0, \frac{1}{2}\right). \quad (5.9)$$

5.3 Discretization in time

In this section, we consider the time discretization of (5.3) that is based on the fully implicit Euler-Maruyama method.

5.3.1 Formulation and stability of the time discretization

Algorithm 1. Let $\mathbf{u}^0 = \mathbf{u}_0$ be a given $\mathbb{V}$ -valued random variable. Find $(\mathbf{u}^{n+1}, p^{n+1}) \in L^2(\Omega, \mathbb{V} \times L^2_{per}(D)/\mathbb{R})$ such that $\mathbb{P}$ -a.s.

$$\begin{aligned} (\mathbf{u}^{n+1} - \mathbf{u}^n, \boldsymbol{\phi}) + \nu k (\nabla \mathbf{u}^{n+1}, \nabla \boldsymbol{\phi}) + k (\mathbf{u}^{n+1} \cdot \nabla \mathbf{u}^{n+1}, \boldsymbol{\phi}) - k (p^{n+1}, \operatorname{div} \boldsymbol{\phi}) \\ = (\mathbf{g}^n \Delta W_{n+1}, \boldsymbol{\phi}), \end{aligned} \quad (5.10)$$

$$(\operatorname{div} \mathbf{u}^{n+1}, \psi) = 0 \quad (5.11)$$

for all $\boldsymbol{\phi} \in \mathbf{H}^1_{per}(D)$ and $\psi \in L^2_{per}(D)$. Where $\mathbf{g}^n = \mathbf{g}(t_n)$ and $\Delta W_{n+1} = W(t_{n+1}) - W(t_n)$.

If $\boldsymbol{\phi} \in \mathbb{V}$, then (5.10) reduces to

$$(\mathbf{u}^{n+1} - \mathbf{u}^n, \boldsymbol{\phi}) + \nu k (\nabla \mathbf{u}^{n+1}, \nabla \boldsymbol{\phi}) + k (\mathbf{u}^{n+1} \cdot \nabla \mathbf{u}^{n+1}, \boldsymbol{\phi}) = (\mathbf{g}^n \Delta W_{n+1}, \boldsymbol{\phi}). \quad (5.12)$$

We quote the following stability estimates for $\{\mathbf{u}^n\}$ given in [14].

Lemma 5.3.1. *Let $\mathbf{u}_0 \in L^{2q}(\Omega, \mathbb{V})$ for an integer $1 \leq q < \infty$ be given, such that $\mathbb{E}[\|\mathbf{u}_0\|_{\mathbb{V}}^{2q}] \leq C$. Then there exists a constant $C_{T,q} = C(T, q, \mathbf{u}_0)$ such that the following estimations hold:*

$$(i) \quad \mathbb{E} \left[\max_{1 \leq n \leq M} \|\mathbf{u}^n\|_{\mathbb{V}}^{2q} + \nu k \sum_{n=1}^M \|\mathbf{u}^n\|_{\mathbb{V}}^{2q-2} \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 \right] \leq C_{T,q}.$$

$$(ii) \quad \mathbb{E} \left[\sum_{n=1}^M \|\mathbf{u}^n - \mathbf{u}^{n-1}\|_{\mathbb{V}}^2 \|\mathbf{u}^n\|_{\mathbb{V}}^2 \right] \leq C_{T,2}.$$

$$(iii) \quad \mathbb{E} \left[\left(\sum_{n=1}^M \|\mathbf{u}^n - \mathbf{u}^{n-1}\|_{\mathbb{V}}^2 \right)^q + \left(\nu k \sum_{n=1}^M \|\mathbf{A} \mathbf{u}^n\|_{\mathbb{V}}^2 \right)^q \right] \leq C_{T,q}.$$

Stability estimates for the pressure approximation $\{p^n\}$ can also be obtained accordingly.

Lemma 5.3.2. *Let $\{p^n\}_{n=1}^M$ be the pressure approximation from Algorithm 1. Under the assumptions of Lemma 5.3.1 and for $1 \leq q < \infty$, we have*

$$\mathbb{E} \left[\left(k \sum_{n=1}^M \|\nabla p^n\|_{\mathbf{L}^2}^2 \right)^q \right] \leq \frac{C_{T,q}}{k^q}. \quad (5.13)$$

Next, we present an exponential stability estimate for the velocity approximation from Algorithm 1. Such estimate can be found in [6, Theorem 8.3]. However, we provide a new proof which allows us to obtain the exponential stability estimate of the semi-discrete solution $\{\mathbf{u}^n\}$ without requiring M to be sufficiently large. This estimate will be used in the proofs of Theorem 5.3.4 and Theorem 5.4.2.

Lemma 5.3.3. *Assume that $\mathbb{E}[\exp(4\sigma\|\nabla\mathbf{u}_0\|_{\mathbf{L}^2}^2)] \leq C$ for some $\sigma > 0$. Let $\{\mathbf{u}^n\}$ be generated by Algorithm 1 and assume that $\mathbf{g}$ satisfies (G2). Then, there holds*

$$\mathbb{E}\left[\exp\left(\sigma \max_{1 \leq \ell \leq M} \|\nabla \mathbf{u}^\ell\|_{\mathbf{L}^2}^2 + \sigma \nu k \sum_{n=1}^M \|\mathbf{A}\mathbf{u}^n\|_{\mathbf{L}^2}^2\right)\right] \leq C_2, \quad (5.14)$$

where $C_2 = C_2(\sigma, \nu, T, \mathbb{E}[\exp(4\sigma\|\nabla\mathbf{u}_0\|_{\mathbf{L}^2}^2)])$.

Proof. Choosing $\phi = \mathbf{A}\mathbf{u}^{n+1}$ in (5.12) and using integration by parts, the binomial formula $2(a, a - b) = \|a\|^2 - \|b\|^2 + \|a - b\|^2$, we obtain

$$\begin{aligned} & \|\nabla \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 - \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 + \|\nabla(\mathbf{u}^{n+1} - \mathbf{u}^n)\|_{\mathbf{L}^2}^2 + 2\nu k \|\mathbf{A}\mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 \\ &= 2(\nabla \mathbf{g}^n \Delta W_{n+1}, \nabla(\mathbf{u}^{n+1} - \mathbf{u}^n)) + 2(\nabla \mathbf{g}^n \Delta W_{n+1}, \nabla \mathbf{u}^n) \\ &\leq \|\nabla \mathbf{g}^n \Delta W_{n+1}\|_{\mathbf{L}^2}^2 + \|\nabla(\mathbf{u}^{n+1} - \mathbf{u}^n)\|_{\mathbf{L}^2}^2 \\ &\quad + 2(\nabla \mathbf{g}^n \Delta W_{n+1}, \nabla \mathbf{u}^n). \end{aligned} \quad (5.15)$$

Therefore,

$$\begin{aligned} & \|\nabla \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 - \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 + 2\nu k \|\mathbf{A}\mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 \\ &\leq \|\nabla \mathbf{g}^n \Delta W_{n+1}\|_{\mathbf{L}^2}^2 + 2(\nabla \mathbf{g}^n \Delta W_{n+1}, \nabla \mathbf{u}^n). \end{aligned} \quad (5.16)$$

Next, we lower the index by one and apply the summation $\sum_{n=1}^{\ell}$ for any $1 \leq \ell \leq M$ to get

$$\begin{aligned} \|\nabla \mathbf{u}^{\ell}\|_{\mathbf{L}^2}^2 + 2\nu k \sum_{n=1}^{\ell} \|\mathbf{A}\mathbf{u}^n\|_{\mathbf{L}^2}^2 &\leq \|\nabla \mathbf{u}^0\|_{\mathbf{L}^2}^2 + \sum_{n=1}^{\ell} \|\nabla \mathbf{g}^{n-1} \Delta W_n\|_{\mathbf{L}^2}^2 \\ &\quad + 2 \sum_{n=1}^{\ell} (\nabla \mathbf{g}^n \Delta W_{n+1}, \nabla \mathbf{u}^n). \end{aligned} \quad (5.17)$$

Taking $\max_{1 \leq \ell \leq M}$ on both sides of (5.17) leads to

$$\begin{aligned} \max_{1 \leq \ell \leq M} \|\nabla \mathbf{u}^{\ell}\|_{\mathbf{L}^2}^2 + 2\nu k \sum_{n=1}^M \|\mathbf{A}\mathbf{u}^n\|_{\mathbf{L}^2}^2 &\leq \|\nabla \mathbf{u}^0\|_{\mathbf{L}^2}^2 + \sum_{n=1}^M \|\nabla \mathbf{g}^{n-1} \Delta W_n\|_{\mathbf{L}^2}^2 \\ &\quad + 2 \max_{1 \leq \ell \leq M} \sum_{n=1}^{\ell} (\nabla \mathbf{g}^{n-1} \Delta W_n, \nabla \mathbf{u}^{n-1}) \\ &\leq 2\|\nabla \mathbf{u}^0\|_{\mathbf{L}^2}^2 + 2K_0 \sum_{n=1}^M |\Delta W_n|^2 \\ &\quad + 2 \max_{2 \leq \ell \leq M} \sum_{n=2}^{\ell} (\nabla \mathbf{g}^{n-1} \Delta W_n, \nabla \mathbf{u}^{n-1}). \end{aligned} \quad (5.18)$$

For any $\sigma > 0$, denote $Z_{\ell} = 2\sigma \sum_{n=2}^{\ell} (\nabla \mathbf{g}^{n-1} \Delta W_n, \nabla \mathbf{u}^{n-1})$, for $2 \leq \ell \leq M$. Define the following martingale by using a piece-wise linear function that defines as follow: for $s \in [t_n, t_{n+1})$, with $n = 1, 2, \dots, M-1$, set $\underline{s} = t_n$, $\mathbf{u}^{\underline{s}} = \mathbf{u}^n$. Then $Z_{\ell} = \tilde{Z}_{t_{\ell}}$, where

$$\tilde{Z}_t = 2\sigma \int_{t_1}^t (\nabla \mathbf{g}^{\underline{s}} dW(s), \nabla \mathbf{u}^{\underline{s}}) \quad \forall t \in [t_1, T]. \quad (5.19)$$

Then, $\tilde{Z}_t$ is a $\mathcal{F}_t$ -martingale for $t \in [t_1, T]$ and its quadratic variation satisfies

$$\begin{aligned} \langle \tilde{Z} \rangle_{t_{\ell}} &\leq 4\sigma^2 \int_{t_1}^{t_{\ell}} \|\nabla \mathbf{g}^{\underline{s}}\|_{\mathbf{L}^2}^2 \|\nabla \mathbf{u}^{\underline{s}}\|_{\mathbf{L}^2}^2 ds \\ &\leq 4C_L \sigma^2 K_0 k \sum_{n=1}^{\ell} \|\mathbf{A}\mathbf{u}^n\|_{\mathbf{L}^2}^2, \end{aligned} \quad (5.20)$$

where C_L is the constant from the inequality $\|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \leq C_L \|\mathbf{A}\mathbf{u}^n\|_{\mathbf{L}^2}^2$.

We multiply (5.18) by $\sigma > 0$ and then take the exponential on both sides to get

$$\begin{aligned}
& \exp\left(\sigma \max_{1 \leq \ell \leq M} \|\nabla \mathbf{u}^\ell\|_{\mathbf{L}^2}^2 + \sigma \nu k \sum_{n=1}^M \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2\right) \\
& \leq \exp(2\sigma \|\nabla \mathbf{u}^0\|_{\mathbf{L}^2}^2) \times \exp\left(2K_0\sigma \sum_{n=1}^M |\Delta W_n|^2\right) \\
& \quad \times \exp\left(\max_{2 \leq \ell \leq M} [Z_\ell - 2\langle \tilde{Z} \rangle_{t_\ell}]\right) \\
& \quad \times \exp\left(2 \max_{2 \leq \ell \leq M} \langle \tilde{Z} \rangle_{t_\ell} - \sigma \nu k \sum_{n=1}^M \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2\right).
\end{aligned} \tag{5.21}$$

Moreover, from (5.20) we have

$$\begin{aligned}
& 2 \max_{2 \leq \ell \leq M} \langle \tilde{Z} \rangle_{t_\ell} - \sigma \nu k \sum_{n=1}^M \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 \\
& \leq 8C_L\sigma^2 K_0 k \sum_{n=1}^\ell \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 - \sigma \nu k \sum_{n=1}^M \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 \\
& \leq (8K_0C_L\sigma^2 - \nu\sigma)k \sum_{n=1}^M \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2.
\end{aligned} \tag{5.22}$$

Now, by using (G2), we have $8K_0C_L\sigma^2 - \nu\sigma \leq 0$, which also implies that

$$\exp\left(2 \max_{2 \leq \ell \leq M} \langle \tilde{Z} \rangle_{t_\ell} - \sigma \nu k \sum_{n=1}^M \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2\right) \leq 1 \quad \mathbb{P} - a.s. \tag{5.23}$$

Next, taking the expectation on (5.21) and using (5.23) we obtain

$$\begin{aligned}
& \mathbb{E}\left[\exp\left(\sigma \max_{1 \leq \ell \leq M} \|\nabla \mathbf{u}^\ell\|_{\mathbf{L}^2}^2 + \sigma \nu k \sum_{n=1}^M \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2\right)\right] \\
& \leq \mathbb{E}\left[\exp(2\sigma \|\nabla \mathbf{u}^0\|_{\mathbf{L}^2}^2) \times \exp\left(2K_0\sigma \sum_{n=1}^M |\Delta W_n|^2\right) \right. \\
& \quad \left. \times \exp\left(\max_{2 \leq \ell \leq M} [\tilde{Z}_{t_\ell} - 2\langle \tilde{Z} \rangle_{t_\ell}]\right)\right].
\end{aligned} \tag{5.24}$$

Using the Hölder inequality to separate the product on the right-hand side of (5.24) yields

$$\begin{aligned}
& \mathbb{E} \left[\exp \left(\sigma \max_{1 \leq \ell \leq M} \|\nabla \mathbf{u}^\ell\|_{\mathbf{L}^2}^2 + \sigma \nu k \sum_{n=1}^M \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 \right) \right] \\
& \leq \left(\mathbb{E} \left[\exp(4\sigma \|\nabla \mathbf{u}^0\|_{\mathbf{L}^2}^2) \right] \right)^{1/2} \\
& \quad \times \left(\mathbb{E} \left[\exp \left(8K_0 \sigma \sum_{n=1}^M |\Delta W_n|^2 \right) \right] \right)^{1/4} \\
& \quad \times \left(\mathbb{E} \left[\exp \left(\max_{2 \leq \ell \leq M} \left[4\tilde{Z}_{t_\ell} - \frac{1}{2} \langle 4\tilde{Z} \rangle_{t_\ell} \right] \right) \right] \right)^{1/4}.
\end{aligned} \tag{5.25}$$

Since $\left\{ \exp \left(4\tilde{Z}_t - \frac{1}{2} \langle 4\tilde{Z} \rangle_t \right) \right\}$ is an exponential martingale on $[t_1, T]$, so it suffices to show that $\mathbb{E} \left[\exp \left(8K_0 \sigma \sum_{n=1}^M |\Delta W_n|^2 \right) \right]$ is bounded. To the end, let $\beta = 8K_0 \sigma$ and $X_n = |\Delta W_n|^2, 1 \leq n \leq M$. Then, by using the independence of the increments of $W(t)$, we can rewrite

$$\mathbb{E} \left[\exp \left(8K_0 \sigma \sum_{n=1}^M |\Delta W_n|^2 \right) \right] = \mathbb{E} \left[\exp \left(\beta \sum_{n=1}^M X_n \right) \right] = \prod_{n=1}^M \mathbb{E} [\exp(\beta X_n)]. \tag{5.26}$$

In addition, by using (3.8), we have that for each $1 \leq n \leq M$, $\mathbb{E}[X_n] = \mathbb{E}[|\Delta W_n|^2] = k$ and then by (5.2), we also have $0 \leq X_n \leq Ck^\alpha$ a.s. $\forall \alpha \in (0, 1)$.

Next, we use Hoeffding's lemma to conclude that

$$\mathbb{E} [\exp(\beta X_n)] \leq \exp \left(\beta \mathbb{E}[X_n] + \frac{C^2 \beta^2}{8} k^{2\alpha} \right) = \exp \left(\beta k + \frac{C^2 \beta^2}{8} k^{2\alpha} \right). \tag{5.27}$$

Substituting (5.27) into the right-hand side of (5.26), we arrive at

$$\mathbb{E} \left[\exp \left(8K_0 \sigma \sum_{n=1}^M |\Delta W_n|^2 \right) \right] \leq \exp \left(\beta M k + \frac{C^2 \beta^2}{8} M k^{2\alpha} \right) = \exp \left(\beta T + \frac{C^2 \beta^2}{8} T k^{2\alpha-1} \right). \tag{5.28}$$

The proof is complete by choosing $\frac{1}{2} \leq \alpha < 1$.

□

5.3.2 High moment and pathwise error estimates for the velocity approximation

In this subsection, we present the first main result of this chapter which establishes the optimal order high moment and sub-optimal order pathwise error estimates for the velocity approximation by Algorithm 1.

Theorem 5.3.4. *Let $\mathbf{u}$ be the variational solution to (5.6) and $\{\mathbf{u}^n\}_{n=1}^M$ be generated by Algorithm 1. Assume that $\mathbf{u}_0 \in L^q(\Omega; \mathbb{V})$ and . Then there exists $C_q = C(T, q, \mathbf{u}_0, \mathbf{f}) > 0$ for any integer $2 \leq q < \infty$ and real number $0 < \gamma < \frac{1}{2}$ such that*

$$\left(\mathbb{E} \left[\max_{1 \leq n \leq M} \|\mathbf{u}(t_n) - \mathbf{u}^n\|_{\mathbf{L}^2}^q \right] \right)^{\frac{1}{q}} + \left(\mathbb{E} \left[\left(\nu k \sum_{n=1}^M \|\nabla(\mathbf{u}(t_n) - \mathbf{u}^n)\|_{\mathbf{L}^2}^2 \right)^{q/2} \right] \right)^{\frac{1}{q}} \leq C_q k^{\frac{1}{2}-\gamma}. \quad (5.29)$$

Define $\mathbf{e}^n = \mathbf{u}(t_n) - \mathbf{u}^n$. The proof of Theorem 5.3.4 has several steps and is accomplished through Lemma 5.3.5 and Lemma 5.3.6 below.

Lemma 5.3.5. *Under the assumptions of Theorem 5.3.4, the following inequality holds $\mathbb{P}$ -a.s.*

$$\max_{1 \leq \ell \leq M} \|\mathbf{e}^\ell\|_{\mathbf{L}^2}^2 + \nu k \sum_{n=1}^M \|\nabla \mathbf{e}^n\|_{\mathbf{L}^2}^2 \leq (A + B) \exp \left(k \sum_{n=1}^M D^n \right), \quad (5.30)$$

where

$$\begin{aligned} A &= 2\nu \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|\nabla(\mathbf{u}(t_n) - \mathbf{u}(s))\|_{\mathbf{L}^2}^2 ds \\ &\quad + \frac{4}{\nu} \sum_{n=1}^M \int_{t_{n-1}}^{t_n} C_L^2 \|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{H}^1}^2 \|\nabla \mathbf{u}(s)\|_{\mathbf{L}^2}^2 ds \\ &\quad + \frac{4}{\nu} \sum_{n=1}^M \int_{t_{n-1}}^{t_n} C_L^2 \|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{H}^1}^2 \|\nabla \mathbf{u}(t_n)\|_{\mathbf{L}^2}^2 ds \\ &\quad + \frac{2C_L^2}{\nu} k \max_{1 \leq \ell \leq M} \|\mathbf{e}^\ell\|_{\mathbf{L}^2}^2 \|\nabla \mathbf{u}^\ell\|_{\mathbf{L}^2}^2, \end{aligned}$$

$$\begin{aligned}
B &= 2 \sum_{n=1}^M \left\| \int_{t_{n-1}}^{t_n} (\mathbf{g}(s) - \mathbf{g}(t_{n-1})) dW(s) \right\|_{\mathbf{L}^2}^2 \\
&\quad + 2 \max_{1 \leq \ell \leq M} \left| \sum_{n=1}^{\ell} \left(\int_{t_{n-1}}^{t_n} (\mathbf{g}(s) - \mathbf{g}(t_{n-1})) dW(s), \mathbf{e}^{n-1} \right) \right|, \\
D^n &= \frac{2C_L^2}{\nu} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \quad \forall 1 \leq n \leq M.
\end{aligned}$$

Proof. Subtracting (5.12) from (5.3) we obtain

$$\begin{aligned}
(\mathbf{e}^{n+1} - \mathbf{e}^n, \mathbf{v}) + \nu k (\nabla \mathbf{e}^{n+1}, \nabla \mathbf{v}) &= \nu \int_{t_n}^{t_{n+1}} (\nabla(\mathbf{u}(s) - \mathbf{u}(t_{n+1})), \nabla \mathbf{v}) ds \\
&\quad + \int_{t_n}^{t_{n+1}} (\mathbf{u}^{n+1} \cdot \nabla \mathbf{u}^{n+1} - \mathbf{u}(s) \cdot \nabla \mathbf{u}(s), \mathbf{v}) ds \\
&\quad + \left(\int_{t_n}^{t_{n+1}} (\mathbf{g}(s) - \mathbf{g}(t_n)) dW(s), \mathbf{v} \right).
\end{aligned} \tag{5.31}$$

Choosing $\mathbf{v} = \mathbf{e}^{n+1}$ yields

$$\begin{aligned}
(\mathbf{e}^{n+1} - \mathbf{e}^n, \mathbf{e}^{n+1}) + \nu k \|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 &= \nu \int_{t_n}^{t_{n+1}} (\nabla(\mathbf{u}(s) - \mathbf{u}(t_{n+1})), \nabla \mathbf{e}^{n+1}) ds \\
&\quad + \int_{t_n}^{t_{n+1}} (\mathbf{u}^{n+1} \cdot \nabla \mathbf{u}^{n+1} - \mathbf{u}(s) \cdot \nabla \mathbf{u}(s), \mathbf{e}^{n+1}) ds \\
&\quad + \left(\int_{t_n}^{t_{n+1}} (\mathbf{g}(s) - \mathbf{g}(t_n)) dW(s), \mathbf{e}^{n+1} \right) \\
&= \text{I} + \text{II} + \text{III}.
\end{aligned} \tag{5.32}$$

The left-hand side of (5.32) can be easily controlled by using the formula $2(a, a - b) = \|a\|^2 - \|b\|^2 + \|a - b\|^2$. The right-hand side of (5.32) can be bounded as follows.

Using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\text{I} &\leq \nu \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s))\|_{\mathbf{L}^2} \|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2} ds \\
&\leq \nu \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s))\|_{\mathbf{L}^2}^2 ds + \frac{\nu k}{4} \|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2.
\end{aligned} \tag{5.33}$$

We rewrite the second term as

$$\begin{aligned}
\text{II} &= \int_{t_n}^{t_{n+1}} ((\mathbf{u}(s) - \mathbf{u}(t_{n+1})) \cdot \nabla \mathbf{u}(s), \mathbf{e}^{n+1}) \, ds \\
&\quad + \int_{t_n}^{t_{n+1}} (\mathbf{u}(t_{n+1}) \cdot \nabla (\mathbf{u}(s) - \mathbf{u}(t_{n+1})), \mathbf{e}^{n+1}) \, ds \\
&\quad + \int_{t_n}^{t_{n+1}} (\mathbf{e}^{n+1} \cdot \nabla \mathbf{u}^{n+1}, \mathbf{e}^{n+1}) \, ds \\
&\quad + \int_{t_n}^{t_{n+1}} (\mathbf{u}^{n+1} \cdot \nabla \mathbf{e}^{n+1}, \mathbf{e}^{n+1}) \, ds \\
&= \int_{t_n}^{t_{n+1}} ((\mathbf{u}(s) - \mathbf{u}(t_{n+1})) \cdot \nabla \mathbf{u}(s), \mathbf{e}^{n+1}) \, ds \\
&\quad + \int_{t_n}^{t_{n+1}} (\mathbf{u}(t_{n+1}) \cdot \nabla (\mathbf{u}(s) - \mathbf{u}(t_{n+1})), \mathbf{e}^{n+1}) \, ds \\
&\quad + \int_{t_n}^{t_{n+1}} (\mathbf{e}^{n+1} \cdot \nabla \mathbf{u}^{n+1}, \mathbf{e}^{n+1}) \, ds \\
&= \text{II}_1 + \text{II}_2 + \text{II}_3.
\end{aligned} \tag{5.34}$$

Here we have used the facts that $(\mathbf{u}^{n+1} \cdot \nabla \mathbf{e}^{n+1}, \mathbf{e}^{n+1}) = 0$ and $(\mathbf{e}^{n+1} \cdot \nabla \mathbf{u}^{n+1}, \mathbf{e}^{n+1}) = (\mathbf{e}^{n+1} \cdot \nabla \mathbf{u}(t_{n+1}), \mathbf{e}^{n+1})$. Each term can be bounded as follows.

In addition,

$$\begin{aligned}
\text{II}_1 &\leq \int_{t_n}^{t_{n+1}} \|\mathbf{u}(s) - \mathbf{u}(t_{n+1})\|_{\mathbf{L}^4} \|\mathbf{u}(s)\|_{\mathbf{L}^4} \|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2} \, ds \\
&\leq \frac{2}{\nu} \int_{t_n}^{t_{n+1}} \|\mathbf{u}(s) - \mathbf{u}(t_{n+1})\|_{\mathbf{L}^4}^2 \|\mathbf{u}(s)\|_{\mathbf{L}^4}^2 \, ds + \frac{\nu k}{8} \|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \\
&\leq \frac{2}{\nu} \int_{t_n}^{t_{n+1}} C_L^2 \|\mathbf{u}(s) - \mathbf{u}(t_{n+1})\|_{\mathbf{H}^1}^2 \|\nabla \mathbf{u}(s)\|_{\mathbf{L}^2}^2 \, ds + \frac{\nu k}{8} \|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2;
\end{aligned}$$

$$\text{II}_2 \leq \frac{2}{\nu} \int_{t_n}^{t_{n+1}} C_L^2 \|\mathbf{u}(s) - \mathbf{u}(t_{n+1})\|_{\mathbf{H}^1}^2 \|\nabla \mathbf{u}(t_{n+1})\|_{\mathbf{L}^2}^2 \, ds + \frac{\nu k}{8} \|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2;$$

$$\begin{aligned}
\text{II}_3 &\leq k \|\mathbf{e}^{n+1}\|_{\mathbf{L}^4}^2 \|\nabla \mathbf{u}(t_{n+1})\|_{\mathbf{L}^2} \\
&\leq C_L k \|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2} \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2} \|\nabla \mathbf{u}(t_{n+1})\|_{\mathbf{L}^2} \\
&\leq \frac{C_L^2}{\nu} k \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \|\nabla \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 + \frac{\nu k}{4} \|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\text{II} &\leq \frac{2}{\nu} \int_{t_n}^{t_{n+1}} C_L^2 \|\mathbf{u}(s) - \mathbf{u}(t_{n+1})\|_{\mathbf{H}^1}^2 \|\nabla \mathbf{u}(t_{n+1})\|_{\mathbf{L}^2}^2 ds \\
&\quad + \frac{2}{\nu} \int_{t_n}^{t_{n+1}} C_L^2 \|\mathbf{u}(s) - \mathbf{u}(t_{n+1})\|_{\mathbf{H}^1}^2 \|\nabla \mathbf{u}(t_{n+1})\|_{\mathbf{L}^2}^2 ds \\
&\quad + \frac{\nu k}{2} \|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 + \frac{C_L^2}{\nu} k \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \|\nabla \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2.
\end{aligned} \tag{5.35}$$

Now, we estimate the noise term as below

$$\begin{aligned}
\text{III} &= \left(\int_{t_n}^{t_{n+1}} (\mathbf{g}(s) - \mathbf{g}(t_n)) dW(s), \mathbf{e}^{n+1} - \mathbf{e}^n \right) \\
&\quad + \left(\int_{t_n}^{t_{n+1}} (\mathbf{g}(s) - \mathbf{g}(t_n)) dW(s), \mathbf{e}^n \right) \\
&\leq \left\| \int_{t_n}^{t_{n+1}} (\mathbf{g}(s) - \mathbf{g}(t_n)) dW(s) \right\|_{\mathbf{L}^2}^2 + \frac{1}{4} \|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{\mathbf{L}^2}^2 \\
&\quad + \left(\int_{t_n}^{t_{n+1}} (\mathbf{g}(s) - \mathbf{g}(t_n)) dW(s), \mathbf{e}^n \right).
\end{aligned} \tag{5.36}$$

Substituting (5.33), (5.35) and (5.36) into (5.32), we obtain

$$\begin{aligned}
&\frac{1}{2} [\|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{e}^n\|_{\mathbf{L}^2}^2] + \frac{1}{4} \|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{\mathbf{L}^2}^2 + \frac{\nu k}{4} \|\nabla \mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \\
&\leq \nu \int_{t_n}^{t_{n+1}} \|\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s))\|_{\mathbf{L}^2}^2 ds \\
&\quad + \frac{2}{\nu} \int_{t_n}^{t_{n+1}} C_L^2 \|\mathbf{u}(s) - \mathbf{u}(t_{n+1})\|_{\mathbf{H}^1}^2 \|\nabla \mathbf{u}(s)\|_{\mathbf{L}^2}^2 ds \\
&\quad + \frac{2}{\nu} \int_{t_n}^{t_{n+1}} C_L^2 \|\mathbf{u}(s) - \mathbf{u}(t_{n+1})\|_{\mathbf{H}^1}^2 \|\nabla \mathbf{u}(t_{n+1})\|_{\mathbf{L}^2}^2 ds \\
&\quad + \frac{C_L^2}{\nu} k \|\mathbf{e}^{n+1}\|_{\mathbf{L}^2}^2 \|\nabla \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 \\
&\quad + \left\| \int_{t_n}^{t_{n+1}} (\mathbf{g}(s) - \mathbf{g}(t_n)) dW(s) \right\|_{\mathbf{L}^2}^2 \\
&\quad + \left(\int_{t_n}^{t_{n+1}} (\mathbf{g}(s) - \mathbf{g}(t_n)) dW(s), \mathbf{e}^n \right).
\end{aligned} \tag{5.37}$$

Next, lowering the index n in (5.37) by 1 and applying the summation operator $\sum_{n=1}^{\ell}$ to both sides of (5.37) for any $1 \leq \ell \leq M$, we obtain

$$\begin{aligned}
& \|\mathbf{e}^{\ell}\|_{\mathbf{L}^2}^2 + \frac{1}{2} \sum_{n=1}^{\ell} \|\mathbf{e}^n - \mathbf{e}^{n-1}\|_{\mathbf{L}^2}^2 + \frac{\nu k}{2} \sum_{n=1}^{\ell} \|\nabla \mathbf{e}^n\|_{\mathbf{L}^2}^2 \\
& \leq 2\nu \sum_{n=1}^{\ell} \int_{t_{n-1}}^{t_n} \|\nabla(\mathbf{u}(t_n) - \mathbf{u}(s))\|_{\mathbf{L}^2}^2 ds \\
& \quad + \frac{4}{\nu} \sum_{n=1}^{\ell} \int_{t_{n-1}}^{t_n} C_L^2 \|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{H}^1}^2 \|\nabla \mathbf{u}(s)\|_{\mathbf{L}^2}^2 ds \\
& \quad + \frac{4}{\nu} \sum_{n=1}^{\ell} \int_{t_{n-1}}^{t_n} C_L^2 \|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{H}^1}^2 \|\nabla \mathbf{u}(t_n)\|_{\mathbf{L}^2}^2 ds \\
& \quad + \frac{2C_L^2}{\nu} k \sum_{n=1}^{\ell} \|\mathbf{e}^n\|_{\mathbf{L}^2}^2 \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \\
& \quad + 2 \sum_{n=1}^{\ell} \left\| \int_{t_{n-1}}^{t_n} (\mathbf{g}(s) - \mathbf{g}(t_{n-1})) dW(s) \right\|_{\mathbf{L}^2}^2 \\
& \quad + 2 \sum_{n=1}^{\ell} \left(\int_{t_{n-1}}^{t_n} (\mathbf{g}(s) - \mathbf{g}(t_{n-1})) dW(s), \mathbf{e}^{n-1} \right) \\
& \leq 2\nu \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|\nabla(\mathbf{u}(t_n) - \mathbf{u}(s))\|_{\mathbf{L}^2}^2 ds \\
& \quad + \frac{4}{\nu} \sum_{n=1}^M \int_{t_{n-1}}^{t_n} C_L^2 \|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{H}^1}^2 \|\nabla \mathbf{u}(s)\|_{\mathbf{L}^2}^2 ds \\
& \quad + \frac{4}{\nu} \sum_{n=1}^M \int_{t_{n-1}}^{t_n} C_L^2 \|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{H}^1}^2 \|\nabla \mathbf{u}(t_n)\|_{\mathbf{L}^2}^2 ds \\
& \quad + \frac{2C_L^2}{\nu} k \sum_{n=1}^{\ell-1} \|\mathbf{e}^n\|_{\mathbf{L}^2}^2 \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 + \frac{2C_L^2 k}{\nu} \max_{1 \leq \ell \leq M} \|\mathbf{e}^{\ell}\|_{\mathbf{L}^2}^2 \|\nabla \mathbf{u}^{\ell}\|_{\mathbf{L}^2}^2 \\
& \quad + 2 \sum_{n=1}^M \left\| \int_{t_{n-1}}^{t_n} (\mathbf{g}(s) - \mathbf{g}(t_{n-1})) dW(s) \right\|_{\mathbf{L}^2}^2 \\
& \quad + 2 \max_{1 \leq \ell \leq M} \left| \sum_{n=1}^{\ell} \left(\int_{t_{n-1}}^{t_n} (\mathbf{g}(s) - \mathbf{g}(t_{n-1})) dW(s), \mathbf{e}^{n-1} \right) \right| \\
& = A + B + k \sum_{n=1}^{\ell-1} D^n \|\mathbf{e}^n\|_{\mathbf{L}^2}^2,
\end{aligned} \tag{5.38}$$

where A, B are constants in ℓ and are defined by

$$\begin{aligned}
A &:= 2\nu \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|\nabla(\mathbf{u}(t_n) - \mathbf{u}(s))\|_{\mathbf{L}^2}^2 ds \\
&\quad + \frac{4}{\nu} \sum_{n=1}^M \int_{t_{n-1}}^{t_n} C_L^2 \|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{H}^1}^2 \|\nabla \mathbf{u}(s)\|_{\mathbf{L}^2}^2 ds \\
&\quad + \frac{4}{\nu} \sum_{n=1}^M \int_{t_{n-1}}^{t_n} C_L^2 \|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{H}^1}^2 \|\nabla \mathbf{u}(t_n)\|_{\mathbf{L}^2}^2 ds \\
&\quad + \frac{2C_L^2}{\nu} k \max_{1 \leq \ell \leq M} \|\mathbf{e}^\ell\|_{\mathbf{L}^2}^2 \|\nabla \mathbf{u}^\ell\|_{\mathbf{L}^2}^2, \\
B &:= 2 \sum_{n=1}^M \left\| \int_{t_{n-1}}^{t_n} (\mathbf{g}(s) - \mathbf{g}(t_{n-1})) dW(s) \right\|_{\mathbf{L}^2}^2 \\
&\quad + 2 \max_{1 \leq \ell \leq M} \left| \sum_{n=1}^{\ell} \left(\int_{t_{n-1}}^{t_n} (\mathbf{g}(s) - \mathbf{g}(t_{n-1})) dW(s), \mathbf{e}^{n-1} \right) \right|, \\
D^n &:= \frac{2C_L^2}{\nu} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \quad \forall 1 \leq n \leq M.
\end{aligned} \tag{5.39}$$

Finally, applying the discrete Gronwall inequality to (5.38), we obtain

$$\begin{aligned}
&\|\mathbf{e}^\ell\|_{\mathbf{L}^2}^2 + \sum_{n=1}^M \|\mathbf{e}^n - \mathbf{e}^{n-1}\|_{\mathbf{L}^2}^2 + \nu k \sum_{n=1}^M \|\nabla \mathbf{e}^n\|_{\mathbf{L}^2}^2 \\
&\leq (A + B) \exp\left(k \sum_{n=1}^{\ell-1} D^n\right) \quad \mathbb{P} - \text{a.s.}
\end{aligned} \tag{5.40}$$

The proof is complete by taking $\max_{1 \leq \ell \leq M}$ on both sides. $\square$

Next, we estimate $\mathbb{E}\left[(A + B)^{\frac{q}{2}} \exp\left(\frac{kq}{2} \sum_{n=1}^M D^n\right)\right]$ in the following lemma to extract the convergent rates as stated in Theorem 5.3.4.

Lemma 5.3.6. *Assume that (G2) satisfies with $\sigma = \frac{2C_L^2}{\nu}$. Then, under the assumptions of Theorem 5.3.4 and Lemma 5.3.5, there holds*

$$\left(\mathbb{E}\left[(A + B)^{\frac{q}{2}} \exp\left(\frac{kq}{2} \sum_{n=1}^M D^n\right)\right]\right)^{\frac{1}{q}} \leq C_q k^{\frac{1}{2}-\gamma}. \tag{5.41}$$

Here, A, B, D^n are defined as in (5.39) and $0 < \gamma < \frac{1}{2}$. C_q is the same constant as in Theorem 5.3.4 which depends on $\nu, q, \mathbf{g}, D_T, \mathbf{u}_0$.

Proof. By using the simple inequality $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ for $1 \leq p < \infty$ we have

$$\begin{aligned}
\mathbb{E} \left[(A + B)^{\frac{q}{2}} \exp \left(\frac{kq}{2} \sum_{n=1}^M D^n \right) \right] &\leq C_q \mathbb{E} \left[A^{\frac{q}{2}} \exp \left(\frac{kq}{2} \sum_{n=1}^M D^n \right) \right] \\
&\quad + C_q \mathbb{E} \left[B^{\frac{q}{2}} \exp \left(\frac{kq}{2} \sum_{n=1}^M D^n \right) \right] \\
&\leq C_q (\mathbb{E}[A^q])^{\frac{1}{2}} \left(\mathbb{E} \left[\exp \left(qk \sum_{n=1}^M D^n \right) \right] \right)^{\frac{1}{2}} \\
&\quad + C_q (\mathbb{E}[B^q])^{\frac{1}{2}} \left(\mathbb{E} \left[\exp \left(qk \sum_{n=1}^M D^n \right) \right] \right)^{\frac{1}{2}}.
\end{aligned} \tag{5.42}$$

Therefore, there are three factors in (5.42) which we need to estimate, namely, $\mathbb{E}[A^q]$, $\mathbb{E}[B^q]$ and $\mathbb{E} \left[\exp \left(qk \sum_{n=1}^M D^n \right) \right]$.

(a) To estimate $\mathbb{E}[A^q]$, by (5.39) and the discrete Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\mathbb{E}[A^q] &= \mathbb{E} \left[\left(2\nu \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|\nabla(\mathbf{u}(t_n) - \mathbf{u}(s))\|_{\mathbf{L}^2}^2 ds \right. \right. \\
&\quad + \frac{4}{\nu} \sum_{n=1}^M \int_{t_{n-1}}^{t_n} C_L^2 \|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{H}^1}^2 \|\nabla \mathbf{u}(s)\|_{\mathbf{L}^2}^2 ds \\
&\quad + \frac{4}{\nu} \sum_{n=1}^M \int_{t_{n-1}}^{t_n} C_L^2 \|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{H}^1}^2 \|\nabla \mathbf{u}(t_n)\|_{\mathbf{L}^2}^2 ds \\
&\quad \left. \left. + \frac{2C_L^2}{\nu} k \max_{1 \leq \ell \leq M} \|\mathbf{e}^\ell\|_{\mathbf{L}^2}^2 \|\nabla \mathbf{u}^\ell\|_{\mathbf{L}^2}^2 \right)^q \right] \\
&\leq C_q \mathbb{E} \left[\left(\sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|\nabla(\mathbf{u}(t_n) - \mathbf{u}(s))\|_{\mathbf{L}^2}^2 ds \right)^q \right] \\
&\quad + C_q \mathbb{E} \left[\left(\sup_{s \in [0, T]} \|\nabla \mathbf{u}(s)\|_{\mathbf{L}^2}^2 \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{H}^1}^2 ds \right)^q \right] \\
&\quad + C_q k^q \mathbb{E} \left[\max_{1 \leq n \leq M} \|\mathbf{e}^n\|_{\mathbf{L}^2}^q \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^q \right] \\
&= A_1 + A_2 + A_3.
\end{aligned}$$

By using Cauchy-Schwarz's inequality and (5.9), we have

$$\begin{aligned}
A_1 &= C_q \mathbb{E} \left[\left(\sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|\nabla(\mathbf{u}(t_n) - \mathbf{u}(s))\|_{\mathbf{L}^2}^2 ds \right)^q \right] \\
&\leq C_q \mathbb{E} \left[\sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|\nabla(\mathbf{u}(t_n) - \mathbf{u}(s))\|_{\mathbf{L}^2}^{2q} ds \right] \\
&\leq C_q k^{2q(\frac{1}{2}-\gamma)}.
\end{aligned} \tag{5.43}$$

By using Cauchy-Schwarz's inequality, (5.3.1) and (5.9), we also obtain

$$\begin{aligned}
A_2 &= C_q \mathbb{E} \left[\left(\sup_{s \in [0, T]} \|\nabla \mathbf{u}(s)\|_{\mathbf{L}^2}^2 \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{H}^1}^2 ds \right)^q \right] \\
&\leq C_q \mathbb{E} \left[\sup_{s \in [0, T]} \|\nabla \mathbf{u}(s)\|_{\mathbf{L}^2}^{2q} \sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{H}^1}^{2q} ds \right] \\
&\leq C_q \left(\mathbb{E} \left[\sup_{s \in [0, T]} \|\nabla \mathbf{u}(s)\|_{\mathbf{L}^2}^{4q} \right] \right)^{\frac{1}{2}} \left(\mathbb{E} \left[\sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{H}^1}^{4q} ds \right] \right)^{\frac{1}{2}} \\
&\leq C_q k^{2q(\frac{1}{2}-\gamma)}.
\end{aligned} \tag{5.44}$$

In addition, by using (5.3.1), $A_3 \leq C_q k^q$. In summary, $\mathbb{E}[A^q] \leq C_q k^{2q(\frac{1}{2}-\gamma)}$.

(b) To estimate $\mathbb{E}[B^q]$, by (5.39), we have

$$\begin{aligned}
\mathbb{E}[B^q] &= \mathbb{E} \left[\left(2 \sum_{n=1}^M \left\| \int_{t_{n-1}}^{t_n} (\mathbf{g}(s) - \mathbf{g}(t_{n-1})) dW(s) \right\|_{\mathbf{L}^2}^2 \right. \right. \\
&\quad \left. \left. + 2 \max_{1 \leq \ell \leq M} \left| \sum_{n=1}^{\ell} \left(\int_{t_{n-1}}^{t_n} (\mathbf{g}(s) - \mathbf{g}(t_{n-1})) dW(s), \mathbf{e}^{n-1} \right) \right| \right) \right)^q \right] \\
&\leq C_q \mathbb{E} \left[\left(\sum_{n=1}^M \left\| \int_{t_{n-1}}^{t_n} (\mathbf{g}(s) - \mathbf{g}(t_{n-1})) dW(s) \right\|_{\mathbf{L}^2}^2 \right)^q \right] \\
&\quad + C_q \mathbb{E} \left[\max_{1 \leq \ell \leq M} \left| \sum_{n=1}^{\ell} \left(\int_{t_{n-1}}^{t_n} (\mathbf{g}(s) - \mathbf{g}(t_{n-1})) dW(s), \mathbf{e}^{n-1} \right) \right|^q \right] \\
&= B_1 + B_2.
\end{aligned}$$

Next, by using the discrete Hölder inequality, (3.5) and (G1) we have

$$\begin{aligned}
B_1 &= C_q \mathbb{E} \left[\left(\sum_{n=1}^M \left\| \int_{t_{n-1}}^{t_n} (\mathbf{g}(s) - \mathbf{g}(t_{n-1})) dW(s) \right\|_{\mathbf{L}^2}^2 \right)^q \right] \\
&\leq C_q M^{q-1} \mathbb{E} \left[\sum_{n=1}^M \left\| \int_{t_{n-1}}^{t_n} (\mathbf{g}(s) - \mathbf{g}(t_{n-1})) dW(s) \right\|_{\mathbf{L}^2}^{2q} \right] \\
&\leq C_q M^{q-1} \mathbb{E} \left[\sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|\mathbf{g}(s) - \mathbf{g}(t_{n-1})\|_{\mathbf{L}^2}^{2q} ds \right] \\
&\leq C_q k^{q+1}.
\end{aligned}$$

To estimate B_2 , we use the Burkholder–Davis–Gundy inequality and (G1) to get

$$\begin{aligned}
B_2 &= C_q \mathbb{E} \left[\max_{1 \leq \ell \leq M} \left| \sum_{n=1}^{\ell} \left(\int_{t_{n-1}}^{t_n} (\mathbf{g}(s) - \mathbf{g}(t_{n-1})) dW(s), \mathbf{e}^{n-1} \right) \right|^q \right] \\
&\leq C_q \mathbb{E} \left[\left(\sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|\mathbf{g}(s) - \mathbf{g}(t_{n-1})\|_{\mathbf{L}^2}^2 \|\mathbf{e}^{n-1}\|_{\mathbf{L}^2}^2 ds \right)^{q/2} \right] \\
&\leq C_q k^q \mathbb{E} \left[\left(k \sum_{n=1}^M \|\mathbf{e}^{n-1}\|_{\mathbf{L}^2}^2 \right)^{q/2} \right] \leq C_q k^q.
\end{aligned}$$

In summary,

$$\mathbb{E}[B^q] \leq C_q k^q.$$

(c) To estimate $\mathbb{E} \left[\exp \left(qk \sum_{n=1}^M D^n \right) \right]$ by (5.39) and the discrete Jensen inequality, we have

$$\begin{aligned}
\mathbb{E} \left[\exp \left(qk \sum_{n=1}^M D^n \right) \right] &\leq \mathbb{E} \left[\exp \left(\frac{2qC_L^2}{\nu} k \sum_{n=0}^M \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \right) \right] \\
&\leq \mathbb{E} \left[k \sum_{n=0}^M \exp \left(\frac{2qC_L^2}{\nu} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \right) \right] \\
&\leq k \sum_{n=0}^M \mathbb{E} \left[\max_{0 \leq n \leq M} \exp \left(\frac{2qC_L^2}{\nu} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \right) \right] \leq CT. \tag{5.45}
\end{aligned}$$

Here, we have used Lemma 5.3.3 with $\sigma = \frac{2qC_L^2}{\nu}$, the last inequality of (5.45) to obtain

Finally, combining (a), (b), (c) and (5.42), we get

$$\mathbb{E} \left[(A + B)^{\frac{q}{2}} \exp \left(\frac{kq}{2} \sum_{n=1}^M D^n \right) \right] \leq C_q k^{(\frac{1}{2}-\gamma)q}. \quad (5.46)$$

The proof is complete. $\square$

We are ready to state our first pathwise error estimate for the velocity approximation, such an estimate has not been obtained before in the literature.

Theorem 5.3.7. *Assume that the assumptions of Theorem 5.3.4 hold. Let $2 < q < \infty$ and $0 < \gamma < \frac{1}{2}$ such that $\frac{1}{2} - \gamma - \frac{1}{q} > 0$. Then, for $0 < \gamma_1 < \frac{1}{2} - \gamma - \frac{1}{q}$, there exists a random variable $K = K(\omega)$ with $\mathbb{E}[|K|^q] < \infty$ such that there holds $\mathbb{P}$ -a.s.*

$$\max_{1 \leq n \leq M} \|\mathbf{u}(t_n) - \mathbf{u}^n\|_{\mathbf{L}^2} + \left(\nu k \sum_{n=1}^M \|\nabla(\mathbf{u}(t_n) - \mathbf{u}^n)\|_{\mathbf{L}^2}^2 \right)^{\frac{1}{2}} \leq K k^{\gamma_1}. \quad (5.47)$$

Proof. (5.47) is an immediate consequence of Theorem 5.3.4 and Kolmogorov Theorem 3.2.1. $\square$

5.3.3 High moment and pathwise error estimates for the pressure approximation

We state and prove the following pathwise error estimate for the pressure approximation which is the first of such results.

Theorem 5.3.8. *Assume that the assumptions of Theorem 5.3.4 hold. Then we have*

$$\left(\mathbb{E} \left[\left\| P(t_m) - k \sum_{n=1}^m p^n \right\|_{\mathbf{L}^2}^q \right] \right)^{\frac{1}{q}} \leq C_q k^{\frac{1}{2}-\gamma}. \quad (5.48)$$

Proof. The proof is based on the well-known inf-sup (LBB) condition which is quoted below. There exists a constant $\beta > 0$ such that

$$\beta \|\xi\|_{L^2} \leq \sup_{\phi \in \mathbf{H}_{per}^1(D)} \frac{(\xi, \operatorname{div} \phi)}{\|\phi\|_{L^2}} \quad \forall \xi \in L_0^2(D). \quad (5.49)$$

Set

$$\mathbf{e}^m := \mathbf{u}(t_m) - \mathbf{u}^m \text{ and } \mathcal{E}^m := P(t_m) - P^m, \text{ where } P^m := k \sum_{n=1}^m p^n.$$

Summing (5.10) from 1 to m we obtain

$$\begin{aligned} (\mathbf{u}^m, \phi) + \nu k \sum_{n=1}^m (\nabla \mathbf{u}^n, \nabla \phi) + k \sum_{n=1}^m (\mathbf{u}^n \cdot \nabla \mathbf{u}^n, \phi) - (P^m, \operatorname{div} \phi) \\ = (\mathbf{u}^0, \phi) + \sum_{n=1}^m (\mathbf{g}^{n-1} \Delta W_n, \phi) \quad \forall \phi \in \mathbf{H}_{per}^1(D). \end{aligned} \quad (5.50)$$

Subtracting (5.50) from (5.6a), we obtain the following error equations:

$$\begin{aligned} (\mathbf{e}^m, \phi) + \left(\nu k \sum_{n=1}^m \nabla \mathbf{e}^n, \nabla \phi \right) + \nu k \sum_{n=1}^m \int_{t_{n-1}}^{t_n} (\nabla(\mathbf{u}(s) - \mathbf{u}(t_n)), \nabla \phi) ds \\ + k \sum_{n=1}^m (\mathbf{u}(t_n) \cdot \nabla \mathbf{u}(t_n) - \mathbf{u}^n \cdot \nabla \mathbf{u}^n, \phi) \\ + \sum_{n=1}^m \int_{t_{n-1}}^{t_n} (\mathbf{u}(s) \cdot \nabla \mathbf{u}(s) - \mathbf{u}(t_n) \cdot \nabla \mathbf{u}(t_n), \phi) ds - (\mathcal{E}^m, \operatorname{div} \phi) \\ = \left(\sum_{n=1}^m \int_{t_{n-1}}^{t_n} (\mathbf{g}(s) - \mathbf{g}^{n-1}) dW(s), \phi \right). \end{aligned} \quad (5.51)$$

Therefore,

$$\begin{aligned}
(\mathcal{E}^m, \operatorname{div} \phi) &= (\mathbf{e}^m, \phi) + \left(\nu k \sum_{n=1}^m \nabla \mathbf{e}^n, \nabla \phi \right) + \nu \sum_{n=1}^m \int_{t_{n-1}}^{t_n} (\nabla(\mathbf{u}(s) - \mathbf{u}(t_n)), \nabla \phi) ds \quad (5.52) \\
&+ k \sum_{n=1}^m (\mathbf{u}(t_n) \cdot \nabla \mathbf{u}(t_n) - \mathbf{u}^n \cdot \nabla \mathbf{u}^n, \phi) \\
&+ \sum_{n=1}^m \int_{t_{n-1}}^{t_n} (\mathbf{u}(s) \cdot \nabla \mathbf{u}(s) - \mathbf{u}(t_n) \cdot \nabla \mathbf{u}(t_n), \phi) ds \\
&+ \left(\sum_{n=1}^m \int_{t_{n-1}}^{t_n} (\mathbf{g}(t_{n-1}) - \mathbf{g}(s)) dW(s), \phi \right) \\
&= \text{I} + \text{II} + \text{III} + \text{IV} + \text{V} + \text{VI}.
\end{aligned}$$

Next, we use Cauchy-Schwarz and Poincaré inequalities to estimate each term on the right-hand side of (5.52).

$$\begin{aligned}
\text{I} + \text{II} &\leq C \|\mathbf{e}^m\|_{\mathbf{L}^2} \|\nabla \phi\|_{\mathbf{L}^2} + \nu k \sum_{n=1}^m \|\nabla \mathbf{e}^n\|_{\mathbf{L}^2} \|\nabla \phi\|_{\mathbf{L}^2}. \quad (5.53) \\
\text{III} &\leq \nu \sum_{n=1}^m \int_{t_{n-1}}^{t_n} \|\nabla(\mathbf{u}(s) - \mathbf{u}(t_n))\|_{\mathbf{L}^2} ds \|\nabla \phi\|_{\mathbf{L}^2}. \\
\text{IV} &= k \sum_{n=1}^m (\mathbf{u}(t_n) \cdot \nabla \mathbf{u}(t_n) - \mathbf{u}^n \cdot \nabla \mathbf{u}^n, \phi) \\
&= k \sum_{n=1}^m (\mathbf{u}(t_n) \cdot \nabla \mathbf{u}(t_n) - \mathbf{u}^n \cdot \nabla \mathbf{u}(t_n), \phi) \\
&\quad + k \sum_{n=1}^m (\mathbf{u}^n \cdot \nabla \mathbf{u}(t_n) - \mathbf{u}^n \cdot \nabla \mathbf{u}^n, \phi) \\
&= k \sum_{n=1}^m (\mathbf{e}^n \cdot \nabla \mathbf{u}(t_n), \phi) + k \sum_{n=1}^m (\mathbf{u}^n \cdot \nabla \mathbf{e}^n, \phi) \\
&\leq k \sum_{n=1}^m \|\mathbf{e}^n\|_{\mathbf{L}^4} \|\mathbf{u}(t_n)\|_{\mathbf{L}^4} \|\nabla \phi\|_{\mathbf{L}^2} + k \sum_{n=1}^m \|\mathbf{u}^n\|_{\mathbf{L}^4} \|\mathbf{e}^n\|_{\mathbf{L}^4} \|\nabla \phi\|_{\mathbf{L}^2}.
\end{aligned}$$

$$\begin{aligned}
V &= \sum_{n=1}^m \int_{t_{n-1}}^{t_n} (\mathbf{u}(s) \cdot \nabla \mathbf{u}(s) - \mathbf{u}(t_n) \cdot \nabla \mathbf{u}(s), \phi) ds \\
&\quad + \sum_{n=1}^m \int_{t_{n-1}}^{t_n} (\mathbf{u}(t_n) \cdot \nabla \mathbf{u}(s) - \mathbf{u}(t_n) \cdot \nabla \mathbf{u}(t_n), \phi) ds \\
&\leq \sum_{n=1}^m \int_{t_{n-1}}^{t_n} \|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{L}^4} \|\mathbf{u}(s)\|_{\mathbf{L}^4} \|\nabla \phi\|_{\mathbf{L}^2} ds \\
&\quad + \sum_{n=1}^m \int_{t_{n-1}}^{t_n} \|\mathbf{u}(t_n)\|_{\mathbf{L}^4} \|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{L}^4} \|\nabla \phi\|_{\mathbf{L}^2} ds. \\
VI &\leq C \left\| \sum_{n=1}^m \int_{t_{n-1}}^{t_n} (\mathbf{g}(s) - \mathbf{g}(t_{n-1})) dW(s) \right\|_{\mathbf{L}^2} \|\nabla \phi\|_{\mathbf{L}^2}
\end{aligned} \tag{5.54}$$

Substituting the above estimates into (5.52) and using the inf-sup condition yield

$$\begin{aligned}
\beta \|\mathcal{E}^m\|_{L^2} &\leq \sup_{\phi \in \mathbf{H}_{per}^1(D)} \frac{(\mathcal{E}^m, \operatorname{div} \phi)}{\|\nabla \phi\|_{\mathbf{L}^2}} \\
&\leq C \|\mathbf{e}^m\|_{\mathbf{L}^2} + \nu k \sum_{n=1}^m \|\nabla \mathbf{e}^n\|_{\mathbf{L}^2} + \nu \sum_{n=1}^m \int_{t_{n-1}}^{t_n} \|\nabla(\mathbf{u}(s) - \mathbf{u}(t_n))\|_{\mathbf{L}^2} ds \\
&\quad + k \sum_{n=1}^m \|\mathbf{e}^n\|_{\mathbf{L}^4} (\|\mathbf{u}(t_n)\|_{\mathbf{L}^4} + \|\mathbf{u}^n\|_{\mathbf{L}^4}) \\
&\quad + \sum_{n=1}^m \int_{t_{n-1}}^{t_n} \|\mathbf{u}(s) - \mathbf{u}(t_n)\|_{\mathbf{L}^4} (\|\mathbf{u}(s)\|_{\mathbf{L}^4} + \|\mathbf{u}(t_n)\|_{\mathbf{L}^4}) ds \\
&\quad + C \left\| \sum_{n=1}^m \int_{t_{n-1}}^{t_n} (\mathbf{g}(s) - \mathbf{g}(t_{n-1})) dW(s) \right\|_{\mathbf{L}^2}.
\end{aligned} \tag{5.55}$$

Taking the expectation on (5.55) and then using Theorem 5.3.4, Lemmas 5.2.6–5.3.1 and Itô isometry, we obtain

$$\begin{aligned}
\beta^q \mathbb{E}[\|\mathcal{E}^m\|_{L^2}^q] &\leq C_q \mathbb{E}[\|\mathbf{e}^m\|_{\mathbf{L}^2}^q] + C_q \mathbb{E} \left[\left(\nu k \sum_{n=1}^m \|\nabla \mathbf{e}^n\|_{\mathbf{L}^2}^2 \right)^{\frac{q}{2}} \right] \\
&\quad + C_q \mathbb{E} \left[\sum_{n=1}^m \int_{t_{n-1}}^{t_n} \|\nabla(\mathbf{u}(s) - \mathbf{u}(t_n))\|_{\mathbf{L}^2}^q \right] \\
&\quad + C_q \mathbb{E} \left[\left(k \sum_{n=1}^m \|\nabla \mathbf{e}^n\|_{\mathbf{L}^2} (\|\nabla \mathbf{u}(t_n)\|_{\mathbf{L}^2} + \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}) \right)^q \right]
\end{aligned} \tag{5.56}$$

$$\begin{aligned}
& + C_q \mathbb{E} \left[\sum_{n=1}^m \int_{t_{n-1}}^{t_n} \|\nabla(\mathbf{u}(s) - \mathbf{u}(t_n))\|_{\mathbf{L}^2}^q (\|\nabla \mathbf{u}(s)\|_{\mathbf{L}^2}^q + \|\nabla \mathbf{u}(t_n)\|_{\mathbf{L}^2}^q) ds \right] \\
& + C_q \mathbb{E} \left[\left(\sum_{n=1}^M \int_{t_{n-1}}^{t_n} \|\mathbf{g}(s) - \mathbf{g}(t_{n-1})\|_{\mathbf{L}^2}^2 ds \right)^{\frac{q}{2}} \right] \\
& \leq C_q k^{(\frac{1}{2}-\gamma)q} \\
& + C_q \mathbb{E} \left[\left(k \sum_{n=1}^m \|\nabla \mathbf{e}^n\|_{\mathbf{L}^2}^2 \right)^{\frac{q}{2}} \left(k \sum_{n=1}^m (\|\nabla \mathbf{u}(t_n)\|_{\mathbf{L}^2} + \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2})^2 \right)^{\frac{q}{2}} \right] \\
& + C_q \left(\mathbb{E} \left[\max_{s \in [0, T]} \|\nabla \mathbf{u}(s)\|_{\mathbf{L}^2}^{2q} \right] \right)^{\frac{1}{2}} \sum_{n=1}^m \int_{t_{n-1}}^{t_n} (\mathbb{E} [\|\nabla(\mathbf{u}(s) - \mathbf{u}(t_n))\|_{\mathbf{L}^2}^{2q}])^{1/2} ds \\
& \leq C_q k^{(\frac{1}{2}-\gamma)q} \\
& + C_q \left(\mathbb{E} \left[\left(k \sum_{n=1}^m \|\nabla \mathbf{e}^n\|_{\mathbf{L}^2}^2 \right)^q \right] \right)^{\frac{1}{2}} \left(\mathbb{E} \left[\left(k \sum_{n=1}^m (\|\nabla \mathbf{u}(t_n)\|_{\mathbf{L}^2} + \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2})^2 \right)^q \right] \right)^{\frac{1}{2}} \\
& \leq C_q k^{(\frac{1}{2}-\gamma)q}.
\end{aligned}$$

Finally, the proof is complete by dividing β^q on both sides. $\square$

Next, we state a pathwise error estimate for the pressure approximation. To the best of our knowledge, this is the first pathwise convergence result for the pressure approximation.

Theorem 5.3.9. *Assume the assumptions of Theorem 5.3.8 hold. Let $2 < q < \infty$ and $0 < \gamma < \frac{1}{2}$ such that $\frac{1}{2} - \gamma - \frac{1}{q} > 0$. Then, for $0 < \gamma_1 < \frac{1}{2} - \gamma - \frac{1}{q}$, there exists a random variable $K = K(\omega)$ with $\mathbb{E}[|K|^q] < \infty$ such that for all $1 \leq \ell \leq M$, there holds $\mathbb{P}$ -a.s.*

$$\left\| P(t_\ell) - k \sum_{n=1}^{\ell} p^n \right\|_{L^2} \leq K k^{\gamma_1}. \quad (5.57)$$

Proof. The assertion follows immediately from an application of Theorem 3.2.1 using the high moment error estimate of Theorem 5.3.8. $\square$

5.4 Discretization in space

In this section, we consider the spatial approximation for the Algorithm 1 by using finite element method. Let $\mathcal{T}_h$ be a quasi-uniform triangular mesh of the given domain $D \subset \mathbb{R}^2$ with mesh size $h > 0$. We introduce the following finite element spaces notation

$$\begin{aligned}\mathbb{H}_h &:= \{\mathbf{v}_h \in C(\overline{D}) \cap \mathbf{H}_{per}^1(D); \mathbf{v}_h \in \mathcal{P}_i(K) \forall K \in \mathcal{T}_h\}, \\ L_h &:= \{q_h \in L_{per}^2; q_h \in \mathcal{P}_j(K) \forall K \in \mathcal{T}_h\},\end{aligned}$$

where $\mathcal{P}_i(K)$ denotes the space of all polynomials on K of degree at most i . It's well-known [10] that the choice of the pair $(\mathbb{H}_h, L_h)$ must satisfy the Ladyzhenskaja-Babuška-Brezzi (LBB) condition, which is stated below, to ensure the stabilization of the method.

$$\sup_{\boldsymbol{\phi}_h \in \mathbb{H}_h} \frac{(\operatorname{div} \boldsymbol{\phi}_h, q_h)}{\|\nabla \boldsymbol{\phi}_h\|_{\mathbf{L}^2}} \geq C \|q_h\|_{L^2} \quad \forall q_h \in L_h, \quad (5.58)$$

where the constant $C > 0$ is independent of h and k .

Next, we define the space of weakly divergent-free velocity field as follows:

$$\mathbb{V}_h := \{\boldsymbol{\phi}_h \in \mathbb{H}_h; (\operatorname{div} \boldsymbol{\phi}_h, q_h) = 0 \quad \forall q_h \in L_h\}.$$

In general, $\mathbb{V}_h$ is not a subspace of $\mathbb{V}$. Let $\mathbf{Q}_h : \mathbf{L}^2 \rightarrow \mathbb{V}_h$ denote the L^2 -orthogonal projection, which is defined by

$$(\mathbf{v} - \mathbf{Q}_h \mathbf{v}, \boldsymbol{\phi}_h) = 0 \quad \forall \boldsymbol{\phi}_h \in \mathbb{V}_h. \quad (5.59)$$

It is well-known [37] that $\mathbf{Q}_h$ satisfies the following estimates:

$$\|\mathbf{v} - \mathbf{Q}_h \mathbf{v}\|_{\mathbf{L}^2} + h \|\nabla(\mathbf{v} - \mathbf{Q}_h \mathbf{v})\|_{\mathbf{L}^2} \leq Ch^2 \|\mathbf{A} \mathbf{v}\|_{\mathbf{L}^2} \quad \forall \mathbf{v} \in \mathbb{V} \cap \mathbf{H}^2(D), \quad (5.60)$$

$$\|\mathbf{v} - \mathbf{Q}_h \mathbf{v}\|_{\mathbf{L}^2} \leq Ch \|\nabla \mathbf{v}\|_{\mathbf{L}^2} \quad \forall \mathbf{v} \in \mathbb{V} \cap \mathbf{H}^1(D). \quad (5.61)$$

Similarly, let $P_h : L_{per}^2(D) \rightarrow L_h$ be the scalar L^2 -orthogonal projection, defined by

$$(\psi - P_h \psi, q_h) = 0 \quad \forall q_h \in L_h, \quad (5.62)$$

then there also holds

$$\|\psi - P_h \psi\|_{L^2} \leq Ch \|\nabla \psi\|_{\mathbf{L}^2} \quad \forall \psi \in L_{per}^2(D) \cap H^1(D). \quad (5.63)$$

Next, we also introduce the well-known trilinear form [65]

$$\tilde{b}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) + \frac{1}{2}([\operatorname{div} \mathbf{u}] \mathbf{v}, \mathbf{w}) \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(D), \quad (5.64)$$

which is anti-symmetric in the sense that

$$\tilde{b}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -\tilde{b}(\mathbf{u}, \mathbf{w}, \mathbf{v}) \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(D). \quad (5.65)$$

Therefore,

$$\tilde{b}(\mathbf{u}, \phi, \phi) = 0 \quad \forall \mathbf{u}, \phi \in \mathbf{H}^1(D).$$

Algorithm 2. Let $\mathbf{u}_h^0$ be a given $\mathbb{H}_h$ -valued random variable. Find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in L^2(\Omega, \mathbb{H}_h \times L_h)$ such that there holds $\mathbb{P}$ -a.s.

$$(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n, \phi_h) + \nu k(\nabla \mathbf{u}_h^{n+1}, \nabla \phi_h) + k(\mathbf{u}_h^{n+1} \cdot \nabla \mathbf{u}_h^{n+1}, \phi_h) \quad (5.66)$$

$$+ \frac{k}{2}([\operatorname{div} \mathbf{u}_h^{n+1}] \mathbf{u}_h^{n+1}, \phi_h) - k(p_h^{n+1}, \operatorname{div} \phi_h) = (\mathbf{g}_h^n \Delta W_{n+1}, \phi_h),$$

$$(\operatorname{div} \mathbf{u}_h^n, \psi_h) = 0, \quad (5.67)$$

for all $\phi_h \in \mathbb{H}_h$ and $\psi_h \in L_h$. $\mathbf{g}_h^n$ denotes an approximation of $\mathbf{g}^n$, which satisfies $\|\mathbf{g}^n - \mathbf{g}_h^n\|_{\mathbf{L}^2} \leq Ch^2$.

Next, we recall the following stability estimates for $\{\mathbf{u}_h^n\}$ from [14, Lemma 3.1].

Lemma 5.4.1. *Let $1 \leq q < \infty$ and $\mathbf{u}_h^0 \in L^{2q}(\Omega; \mathbb{H}_h)$ such that $\mathbb{E}[\|\mathbf{u}_h^0\|_{\mathbf{L}^2}^{2q}] \leq C$. Then, there exists a pair $\{(\mathbf{u}_h^n, p_h^n)\}_{n=1}^M \subset L^2(\Omega; \mathbb{H}_h \times L_h)$ that solves Algorithm 2 and satisfies*

$$(i) \mathbb{E} \left[\max_{1 \leq n \leq M} \|\mathbf{u}_h^n\|_{\mathbf{L}^2}^{2^q} + \nu k \sum_{n=1}^M \|\mathbf{u}_h^n\|_{\mathbf{L}^2}^{2^q-2} \|\nabla \mathbf{u}_h^n\|_{\mathbf{L}^2}^2 \right] \leq C_{T,q},$$

$$(ii) \mathbb{E} \left[\left(k \sum_{n=1}^M \|\nabla \mathbf{u}_h^n\|_{\mathbf{L}^2}^2 \right)^{2^{q-1}} \right] \leq C_{T,q},$$

where $C_{T,q} = C_{T,q}(D_T, q, \mathbf{u}_h^0)$.

5.4.1 High moment and pathwise error estimates for the fully discrete velocity approximation

In this subsection, we state and prove the second main result of this chapter.

Theorem 5.4.2. *Let $2 \leq q < \infty$ and $\mathbf{u}_h^0 = \mathbf{Q}_h \mathbf{u}_0$. Assume that $\mathbf{u}_0 \in L^q(\Omega; \mathbb{V})$. Let $\{(\mathbf{u}^n, p^n)\}$ and $\{\mathbf{u}_h^n, p_h^n\}$ be the velocity and pressure approximations generated by Algorithm 1 and Algorithm 2, respectively. Then there holds*

$$\begin{aligned} & \left(\mathbb{E} \left[\max_{1 \leq n \leq M} \|\mathbf{u}^n - \mathbf{u}_h^n\|_{\mathbf{L}^2}^q \right] \right)^{\frac{1}{q}} + \left(\mathbb{E} \left[\left(\nu k \sum_{n=1}^M \|\nabla(\mathbf{u}^n - \mathbf{u}_h^n)\|_{\mathbf{L}^2}^2 \right)^{\frac{q}{2}} \right] \right)^{\frac{1}{q}} \\ & \leq C_q \left(k^{\frac{1}{2}} + h + k^{-\frac{1}{2}} h \right), \end{aligned} \quad (5.68)$$

where $C_q = C(T, q, \mathbf{u}_0, \mathbf{f}) > 0$ is independent of k and h .

The proof of Theorem 5.4.2 has several steps and will be accomplished through the following two lemmas.

Lemma 5.4.3. *Under the assumptions of Theorem 5.4.2, there holds $\mathbb{P}$ -a.s.*

$$\max_{1 \leq \ell \leq M} \left[\|\mathbf{Q}_h(\mathbf{u}^\ell - \mathbf{u}_h^\ell)\|_{\mathbf{L}^2}^2 + k \sum_{n=1}^{\ell} \|\nabla(\mathbf{u}^n - \mathbf{u}_h^n)\|_{\mathbf{L}^2}^2 \right] \leq (A_h + B_h) \exp \left(k \sum_{n=1}^M D_h^n \right), \quad (5.69)$$

where

$$\begin{aligned}
A_h &:= \|\mathbf{Q}_h \mathbf{E}^0\|_{\mathbf{L}^2}^2 + Ch^2k \sum_{n=1}^M \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 + Ch^3k \sum_{n=1}^M \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 \\
&\quad + Ch^2k \sum_{n=1}^M \|\mathbf{u}_h^n\|_{\mathbf{L}^2}^2 \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 \\
&\quad + Ck \max_{1 \leq n \leq M} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^4 \left(k \sum_{n=1}^M \|\nabla \mathbf{u}_h^n\|_{\mathbf{L}^2}^2 + k \sum_{n=1}^M \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \right) \\
&\quad + \frac{16C_L^2k}{\nu} \max_{1 \leq n \leq M} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \|\mathbf{Q}_h \mathbf{E}^n\|_{\mathbf{L}^2}^2 + Ch^2k \sum_{n=1}^M \|\nabla p^n\|_{\mathbf{L}^2}^2, \\
B_h &:= \sum_{n=1}^M \|(\mathbf{g}^{n-1} - \mathbf{g}_h^{n-1}) \Delta W_n\|_{\mathbf{L}^2}^2 + \max_{1 \leq \ell \leq M} \left| \sum_{n=1}^{\ell} ((\mathbf{g}^{n-1} - \mathbf{g}_h^{n-1}) \Delta W_n, \mathbf{Q}_h \mathbf{E}^{n-1}) \right|, \\
D_h^n &:= \frac{16C_L^2}{\nu} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 + \frac{2\tilde{C}}{\nu} \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2.
\end{aligned} \tag{5.70}$$

Proof. Let $\mathbf{E}^n := \mathbf{u}^n - \mathbf{u}_h^n$ for all $0 \leq n \leq M$. Subtracting (5.10) from (5.66), we obtain the following error equation

$$\begin{aligned}
&(\mathbf{E}^{n+1} - \mathbf{E}^n, \phi_h) + \nu k (\nabla \mathbf{E}^{n+1}, \nabla \phi_h) + k \tilde{b}(\mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \phi_h) \\
&\quad - k \tilde{b}(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}, \phi_h) - k(p^{n+1} - p_h^{n+1}, \operatorname{div} \phi_h) \\
&= ((\mathbf{g}^n - \mathbf{g}_h^n) \Delta W_{n+1}, \phi_h) \quad \forall \phi_h \in \mathbb{H}_h.
\end{aligned} \tag{5.71}$$

Setting $\phi_h = \mathbf{Q}_h \mathbf{E}^{n+1} \in \mathbb{V}_h$ in (5.71), using the orthogonality of the $\mathbf{L}^2$ -projection and the fact that $(p_h^{n+1}, \operatorname{div} \mathbf{Q}_h \mathbf{E}^{n+1}) = 0$, we obtain

$$\begin{aligned}
&(\mathbf{Q}_h \mathbf{E}^{n+1} - \mathbf{Q}_h \mathbf{E}^n, \mathbf{Q}_h \mathbf{E}^{n+1}) + \nu k \|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 \\
&= \nu k (\nabla \mathbf{E}^{n+1}, \nabla (\mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1})) - k \tilde{b}(\mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \mathbf{Q}_h \mathbf{E}^{n+1}) \\
&\quad + k \tilde{b}(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}, \mathbf{Q}_h \mathbf{E}^{n+1}) + k(p^{n+1}, \operatorname{div} \mathbf{Q}_h \mathbf{E}^{n+1}) \\
&\quad + ((\mathbf{g}^n - \mathbf{g}_h^n) \Delta W_{n+1}, \mathbf{Q}_h \mathbf{E}^{n+1}).
\end{aligned} \tag{5.72}$$

We use the identity $2(a, a - b) = \|a\|^2 - \|b\|^2 + \|a - b\|^2$ to rewrite the left-hand side of (5.72) as

$$\begin{aligned}
& \frac{1}{2} [\|\mathbf{Q}_h \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{Q}_h \mathbf{E}^n\|_{\mathbf{L}^2}^2 + \|\mathbf{Q}_h(\mathbf{E}^{n+1} - \mathbf{E}^n)\|_{\mathbf{L}^2}^2] + \nu k \|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 \\
&= \nu k (\nabla \mathbf{E}^{n+1}, \nabla(\mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1})) - [k \tilde{b}(\mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \mathbf{Q}_h \mathbf{E}^{n+1}) \\
&\quad - k \tilde{b}(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}, \mathbf{Q}_h \mathbf{E}^{n+1})] + k(p^{n+1}, \operatorname{div} \mathbf{Q}_h \mathbf{E}^{n+1}) \\
&\quad + ((\mathbf{g}^n - \mathbf{g}_h^n) \Delta W_{n+1}, \mathbf{Q}_h \mathbf{E}^{n+1}) \\
&= \text{I} + \text{II} + \text{III} + \text{IV}.
\end{aligned} \tag{5.73}$$

We note that the term II needs a special treatment but other terms can be easily controlled as follows:

$$\text{I} \leq \frac{\nu k}{16} \|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 + Ckh^2 \|\mathbf{A} \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2. \tag{5.74}$$

On noting that $(P_h p^{n+1}, \operatorname{div} \mathbf{Q}_h \mathbf{E}^{n+1}) = 0$, we have

$$\begin{aligned}
\text{III} &= k(p^{n+1}, \operatorname{div} \mathbf{Q}_h \mathbf{E}^{n+1}) \\
&= k(p^{n+1} - P_h p^{n+1}, \operatorname{div} \mathbf{Q}_h \mathbf{E}^{n+1}) \\
&\leq \frac{\nu k}{16} \|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 + Ckh^2 \|\nabla p^{n+1}\|_{\mathbf{L}^2}^2.
\end{aligned} \tag{5.75}$$

$$\begin{aligned}
\text{IV} &= ((\mathbf{g}^n - \mathbf{g}_h^n) \Delta W_{n+1}, \mathbf{Q}_h(\mathbf{E}^{n+1} - \mathbf{E}^n)) + ((\mathbf{g}^n - \mathbf{g}_h^n) \Delta W_{n+1}, \mathbf{Q}_h \mathbf{E}^n) \\
&\leq 2\|(\mathbf{g}^n - \mathbf{g}_h^n) \Delta W_{n+1}\|_{\mathbf{L}^2}^2 + \frac{1}{8} \|\mathbf{Q}_h(\mathbf{E}^{n+1} - \mathbf{E}^n)\|_{\mathbf{L}^2}^2 \\
&\quad + ((\mathbf{g}^n - \mathbf{g}_h^n) \Delta W_{n+1}, \mathbf{Q}_h \mathbf{E}^n)
\end{aligned} \tag{5.76}$$

To bound the term II , we first rewrite it as

$$\begin{aligned}
\text{II} &= -k[\tilde{b}(\mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \mathbf{Q}_h \mathbf{E}^{n+1}) - \tilde{b}(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}, \mathbf{Q}_h \mathbf{E}^{n+1})] \\
&= -k\tilde{b}(\mathbf{u}^{n+1}, \mathbf{E}^{n+1}, \mathbf{Q}_h \mathbf{E}^{n+1}) - k\tilde{b}(\mathbf{E}^{n+1}, \mathbf{u}_h^{n+1}, \mathbf{Q}_h \mathbf{E}^{n+1}) \\
&= -k\tilde{b}(\mathbf{u}^{n+1}, \mathbf{E}^{n+1}, \mathbf{E}^{n+1}) + k\tilde{b}(\mathbf{u}^{n+1}, \mathbf{E}^{n+1}, \mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1}) \\
&\quad + k\tilde{b}(\mathbf{E}^{n+1}, \mathbf{E}^{n+1}, \mathbf{Q}_h \mathbf{E}^{n+1}) - k\tilde{b}(\mathbf{E}^{n+1}, \mathbf{u}^{n+1}, \mathbf{Q}_h \mathbf{E}^{n+1}) \\
&= -k\tilde{b}(\mathbf{u}^{n+1}, \mathbf{E}^{n+1}, \mathbf{E}^{n+1}) + k\tilde{b}(\mathbf{u}^{n+1}, \mathbf{E}^{n+1}, \mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1}) \\
&\quad + k\tilde{b}(\mathbf{E}^{n+1}, \mathbf{E}^{n+1}, \mathbf{E}^{n+1}) - k\tilde{b}(\mathbf{E}^{n+1}, \mathbf{E}^{n+1}, \mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1}) \\
&\quad - k\tilde{b}(\mathbf{E}^{n+1}, \mathbf{u}^{n+1}, \mathbf{Q}_h \mathbf{E}^{n+1}) \\
&= k\tilde{b}(\mathbf{u}^{n+1}, \mathbf{E}^{n+1}, \mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1}) - k\tilde{b}(\mathbf{E}^{n+1}, \mathbf{E}^{n+1}, \mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1}) \\
&\quad - k\tilde{b}(\mathbf{E}^{n+1}, \mathbf{u}^{n+1}, \mathbf{Q}_h \mathbf{E}^{n+1}) \\
&:= \text{II}_1 + \text{II}_2 + \text{II}_3.
\end{aligned} \tag{5.77}$$

Here, in order to obtain the last line of (5.77), we have used the facts that $\tilde{b}(\mathbf{u}^{n+1}, \mathbf{E}^{n+1}, \mathbf{E}^{n+1}) = 0$ and $\tilde{b}(\mathbf{E}^{n+1}, \mathbf{E}^{n+1}, \mathbf{E}^{n+1}) = 0$. Then we have

$$\begin{aligned}
\text{II}_1 &= k\tilde{b}(\mathbf{u}^{n+1}, \mathbf{E}^{n+1}, \mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1}) \\
&= k(\mathbf{u}^{n+1} \cdot \nabla \mathbf{E}^{n+1}, \mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1}) + \frac{1}{2}k([\text{div } \mathbf{u}^{n+1}] \mathbf{E}^{n+1}, \mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1}) \\
&\leq k\|\mathbf{u}^{n+1}\|_{\mathbf{L}^4} \|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2} \|\mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1}\|_{\mathbf{L}^4} \\
&\leq \frac{\nu k}{16} \|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 + Ck\|\mathbf{u}^{n+1}\|_{\mathbf{L}^4}^2 \|\mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1}\|_{\mathbf{L}^4}^2 \\
&\leq \frac{\nu k}{16} \|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 + Ckh^3 \|\nabla \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 \|\mathbf{A} \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2.
\end{aligned} \tag{5.78}$$

$$\begin{aligned}
\text{II}_2 &= -k \tilde{b}(\mathbf{E}^{n+1}, \mathbf{E}^{n+1}, \mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1}) \\
&= -k(\mathbf{E}^{n+1} \cdot \nabla \mathbf{E}^{n+1}, \mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1}) - \frac{k}{2}([\text{div } \mathbf{E}^{n+1}] \mathbf{E}^{n+1}, \mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1}) \\
&\leq 2k \|\mathbf{E}^{n+1}\|_{\mathbf{L}^4} \|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2} \|\mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1}\|_{\mathbf{L}^4} \\
&\leq \frac{\nu k}{16} \|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 + Ck \|\mathbf{E}^{n+1}\|_{\mathbf{L}^4}^2 \|\mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1}\|_{\mathbf{L}^4}^2 \\
&\leq \frac{\nu k}{16} \|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 + Ckh^3 \|\mathbf{E}^{n+1}\|_{\mathbf{L}^4}^2 \|\mathbf{A} \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 \\
&\leq \frac{\nu k}{16} \|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 + Ckh^3 \|\mathbf{u}^{n+1}\|_{\mathbf{L}^4}^2 \|\mathbf{A} \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 + Ckh^3 \|\mathbf{u}_h^{n+1}\|_{\mathbf{L}^4}^2 \|\mathbf{A} \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 \\
&\leq \frac{\nu k}{16} \|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 + Ckh^3 \|\nabla \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 \|\mathbf{A} \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 + Ckh^2 \|\mathbf{u}_h^{n+1}\|_{\mathbf{L}^2}^2 \|\mathbf{A} \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2.
\end{aligned}$$

Here, we have used the inverse inequality $\|\mathbf{u}_h^{n+1}\|_{\mathbf{L}^4} \leq Ch^{-\frac{1}{2}} \|\mathbf{u}_h^{n+1}\|_{\mathbf{L}^2}$ to get the last inequality.

$$\begin{aligned}
\text{II}_3 &= -k \tilde{b}(\mathbf{E}^{n+1}, \mathbf{u}^{n+1}, \mathbf{Q}_h \mathbf{E}^{n+1}) \tag{5.79} \\
&= -k(\mathbf{E}^{n+1} \cdot \nabla \mathbf{u}^{n+1}, \mathbf{Q}_h \mathbf{E}^{n+1}) - \frac{k}{2}([\text{div } \mathbf{E}^{n+1}] \mathbf{u}^{n+1}, \mathbf{Q}_h \mathbf{E}^{n+1}) \\
&= -k(\mathbf{E}^{n+1} \cdot \nabla \mathbf{u}^{n+1}, \mathbf{Q}_h \mathbf{E}^{n+1}) - \frac{k}{2}([\text{div } \mathbf{E}^{n+1}] \mathbf{u}^{n+1}, \mathbf{Q}_h (\mathbf{E}^{n+1} - \mathbf{E}^n)) \\
&\quad + \frac{k}{2}([\text{div } \mathbf{E}^{n+1}] \mathbf{u}^{n+1}, \mathbf{Q}_h \mathbf{E}^n) \\
&= \text{II}_{3a} + \text{II}_{3b} + \text{II}_{3c}.
\end{aligned}$$

We can bound each term on the right-hand side of II_3 as follows.

$$\begin{aligned}
\text{II}_{3a} &= -k(\mathbf{Q}_h \mathbf{E}^{n+1} \cdot \nabla \mathbf{u}^{n+1}, \mathbf{Q}_h \mathbf{E}^{n+1}) - k((\mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1}) \cdot \nabla \mathbf{u}^{n+1}, \mathbf{Q}_h \mathbf{E}^{n+1}) \tag{5.80} \\
&\leq k \|\mathbf{Q}_h \mathbf{E}^{n+1}\|_{\mathbf{L}^4}^2 \|\nabla \mathbf{u}^{n+1}\|_{\mathbf{L}^2} + k \|\mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1}\|_{\mathbf{L}^4} \|\nabla \mathbf{u}^{n+1}\|_{\mathbf{L}^2} \|\mathbf{Q}_h \mathbf{E}^{n+1}\|_{\mathbf{L}^4} \\
&\leq C_L k \|\mathbf{Q}_h \mathbf{E}^{n+1}\|_{\mathbf{L}^2} \|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2} \|\nabla \mathbf{u}^{n+1}\|_{\mathbf{L}^2} \\
&\quad + Ck \|\mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1}\|_{\mathbf{L}^4} \|\nabla \mathbf{u}^{n+1}\|_{\mathbf{L}^2} \|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2} \\
&\leq \frac{8C_L^2}{\nu} k \|\nabla \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 \|\mathbf{Q}_h \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 \\
&\quad + Ck \|\mathbf{u}^{n+1} - \mathbf{Q}_h \mathbf{u}^{n+1}\|_{\mathbf{L}^4}^2 \|\nabla \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 + \frac{\nu k}{16} \|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 \\
&\leq \frac{8C_L^2}{\nu} k \|\nabla \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 \|\mathbf{Q}_h \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 + Ckh^3 \|\nabla \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 \|\mathbf{A} \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 + \frac{\nu k}{16} \|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2.
\end{aligned}$$

$$\begin{aligned}
\text{II}_{3b} &= -\frac{k}{2}([\text{div } \mathbf{E}^{n+1}]\mathbf{u}^{n+1}, \mathbf{Q}_h(\mathbf{E}^{n+1} - \mathbf{E}^n)) \\
&\leq \frac{k}{2}\|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}\|\mathbf{u}^{n+1}\|_{\mathbf{L}^4}\|\mathbf{Q}_h(\mathbf{E}^{n+1} - \mathbf{E}^n)\|_{\mathbf{L}^4} \\
&\leq Ck\|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}\|\nabla \mathbf{u}^{n+1}\|_{\mathbf{L}^2}\|\mathbf{Q}_h(\mathbf{E}^{n+1} - \mathbf{E}^n)\|_{\mathbf{L}^2}^{1/2}\|\nabla(\mathbf{E}^{n+1} - \mathbf{E}^n)\|_{\mathbf{L}^2}^{1/2} \\
&\leq \frac{\nu k}{16}\|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 + Ck\|\nabla \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2\|\mathbf{Q}_h(\mathbf{E}^{n+1} - \mathbf{E}^n)\|_{\mathbf{L}^2}\|\nabla(\mathbf{E}^{n+1} - \mathbf{E}^n)\|_{\mathbf{L}^2} \\
&\leq \frac{\nu k}{16}\|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 + \frac{1}{8}\|\mathbf{Q}_h(\mathbf{E}^{n+1} - \mathbf{E}^n)\|_{\mathbf{L}^2}^2 \\
&\quad + Ck^2\|\nabla \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^4\|\nabla(\mathbf{E}^{n+1} - \mathbf{E}^n)\|_{\mathbf{L}^2}^2. \\
\text{II}_{3c} &= \frac{k}{2}([\text{div } \mathbf{E}^{n+1}]\mathbf{u}^{n+1}, \mathbf{Q}_h \mathbf{E}^n) \\
&\leq \frac{k}{2}\|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}\|\mathbf{u}^{n+1} \cdot \mathbf{Q}_h \mathbf{E}^n\|_{\mathbf{L}^2} \\
&\leq \frac{k}{2}\|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}\|\mathbf{u}^{n+1}\|_{\mathbf{L}^\infty}\|\mathbf{Q}_h \mathbf{E}^n\|_{\mathbf{L}^2} \\
&\leq \frac{\nu k}{16}\|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 + \frac{k}{\nu}\|\mathbf{u}^{n+1}\|_{\mathbf{L}^\infty}^2\|\mathbf{Q}_h \mathbf{E}^n\|_{\mathbf{L}^2}^2 \\
&\leq \frac{\nu k}{16}\|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 + \frac{\tilde{C}}{\nu}k\|\mathbf{A}\mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2\|\mathbf{Q}_h \mathbf{E}^n\|_{\mathbf{L}^2}^2,
\end{aligned}$$

where the last inequality was obtained by using the Sobolev embedding: $\|\mathbf{u}^{n+1}\|_{\mathbf{L}^\infty}^2 \leq \tilde{C}\|\mathbf{A}\mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2$ for some constant $\tilde{C} > 0$.

Now, substituting the above estimates for the terms I–IV into the right-hand side of the right-hand side of (5.73), we obtain

$$\begin{aligned}
&\frac{1}{2}[\|\mathbf{Q}_h \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{Q}_h \mathbf{E}^n\|_{\mathbf{L}^2}^2] + \frac{1}{4}\|\mathbf{Q}_h(\mathbf{E}^{n+1} - \mathbf{Q}_h \mathbf{E}^n)\|_{\mathbf{L}^2}^2 \\
&\quad + \frac{9\nu k}{16}\|\nabla \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 \\
&\leq Ckh^2\|\mathbf{A}\mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 + Ckh^3\|\nabla \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2\|\mathbf{A}\mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 \\
&\quad + Ckh^2\|\mathbf{u}_h^{n+1}\|_{\mathbf{L}^2}^2\|\mathbf{A}\mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 + \frac{8C_L^2}{\nu}k\|\nabla \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2\|\mathbf{Q}_h \mathbf{E}^{n+1}\|_{\mathbf{L}^2}^2 \\
&\quad + Ck^2\|\nabla \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^4\|\nabla(\mathbf{E}^{n+1} - \mathbf{E}^n)\|_{\mathbf{L}^2}^2 \\
&\quad + \frac{\tilde{C}k}{\nu}\|\mathbf{A}\mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2\|\mathbf{Q}_h \mathbf{E}^n\|_{\mathbf{L}^2}^2 + Ckh^2\|\nabla p^{n+1}\|_{\mathbf{L}^2}^2 \\
&\quad + \|(\mathbf{g}^n - \mathbf{g}_h^n)\Delta W_{n+1}\|_{\mathbf{L}^2}^2 + \|(\mathbf{g}^n - \mathbf{g}_h^n)\Delta W_{n+1}, \mathbf{Q}_h \mathbf{E}^n\|.
\end{aligned} \tag{5.81}$$

Lowering the index n in (5.81) by 1 and taking the summation operator $\sum_{n=1}^{\ell}$ on both sides yield

$$\begin{aligned}
& \frac{1}{2} \|\mathbf{Q}_h \mathbf{E}^\ell\|_{\mathbf{L}^2}^2 + \frac{1}{4} \sum_{n=1}^{\ell} \|\mathbf{Q}_h (\mathbf{E}^n - \mathbf{E}^{n-1})\|_{\mathbf{L}^2}^2 + \frac{9\nu k}{16} \sum_{n=1}^{\ell} \|\nabla \mathbf{E}^n\|_{\mathbf{L}^2}^2 \\
& \leq \frac{1}{2} \|\mathbf{Q}_h \mathbf{E}^0\|_{\mathbf{L}^2}^2 + Ch^2 k \sum_{n=1}^{\ell} \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 + Ch^3 k \sum_{n=1}^{\ell} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 \\
& \quad + Ch^2 k \sum_{n=1}^{\ell} \|\mathbf{u}_h^n\|_{\mathbf{L}^2}^2 \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 + \frac{8C_L^2}{\nu} k \sum_{n=1}^{\ell} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \|\mathbf{Q}_h \mathbf{E}^n\|_{\mathbf{L}^2}^2 \\
& \quad + Ck^2 \sum_{n=1}^{\ell} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^4 (\|\nabla (\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\|_{\mathbf{L}^2}^2 + \|\nabla (\mathbf{u}^n - \mathbf{u}^{n-1})\|_{\mathbf{L}^2}^2) \\
& \quad + \frac{\tilde{C}}{\nu} k \sum_{n=1}^{\ell} \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 \|\mathbf{Q}_h \mathbf{E}^{n-1}\|_{\mathbf{L}^2}^2 + Ch^2 k \sum_{n=1}^{\ell} \|\nabla p^n\|_{\mathbf{L}^2}^2 \\
& \quad + \sum_{n=1}^{\ell} \|(\mathbf{g}^{n-1} - \mathbf{g}_h^{n-1}) \Delta W_n\|_{\mathbf{L}^2}^2 + \sum_{n=1}^{\ell} ((\mathbf{g}^{n-1} - \mathbf{g}_h^{n-1}) \Delta W_n, \mathbf{Q}_h \mathbf{E}^{n-1}).
\end{aligned} \tag{5.82}$$

Bounding the sixth term on the right-hand side of (5.82) as follows

$$\begin{aligned}
& Ck^2 \sum_{n=1}^{\ell} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^4 (\|\nabla (\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\|_{\mathbf{L}^2}^2 + \|\nabla (\mathbf{u}^n - \mathbf{u}^{n-1})\|_{\mathbf{L}^2}^2) \\
& \leq Ck \max_{1 \leq n \leq M} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^4 \left(k \sum_{n=1}^M \|\nabla \mathbf{u}_h^n\|_{\mathbf{L}^2}^2 + k \sum_{n=1}^M \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \right),
\end{aligned}$$

then, (5.82) becomes

$$\begin{aligned}
& \frac{1}{2} \|\mathbf{Q}_h \mathbf{E}^\ell\|_{\mathbf{L}^2}^2 + \frac{1}{4} \sum_{n=1}^{\ell} \|\mathbf{Q}_h (\mathbf{E}^n - \mathbf{E}^{n-1})\|_{\mathbf{L}^2}^2 + \frac{9\nu k}{16} \sum_{n=1}^{\ell} \|\nabla \mathbf{E}^n\|_{\mathbf{L}^2}^2 \\
& \leq \frac{1}{2} \|\mathbf{Q}_h \mathbf{E}^0\|_{\mathbf{L}^2}^2 + Ch^2 k \sum_{n=1}^{\ell} \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 + Ch^3 k \sum_{n=1}^{\ell} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 \\
& \quad + Ch^2 k \sum_{n=1}^{\ell} \|\mathbf{u}_h^n\|_{\mathbf{L}^2}^2 \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 + \frac{8C_L^2}{\nu} k \sum_{n=1}^{\ell} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \|\mathbf{Q}_h \mathbf{E}^n\|_{\mathbf{L}^2}^2
\end{aligned} \tag{5.83}$$

$$\begin{aligned}
& + Ck \max_{1 \leq n \leq M} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^4 \left(k \sum_{n=1}^M \|\nabla \mathbf{u}_h^n\|_{\mathbf{L}^2}^2 + k \sum_{n=1}^M \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \right) \\
& + \frac{\tilde{C}}{\nu} k \sum_{n=1}^{\ell} \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 \|\mathbf{Q}_h \mathbf{E}^{n-1}\|_{\mathbf{L}^2}^2 + Ch^2 k \sum_{n=1}^{\ell} \|\nabla p^n\|_{\mathbf{L}^2}^2 \\
& + \sum_{n=1}^{\ell} \|(\mathbf{g}^{n-1} - \mathbf{g}_h^{n-1}) \Delta W_n\|_{\mathbf{L}^2}^2 + \sum_{n=1}^{\ell} ((\mathbf{g}^{n-1} - \mathbf{g}_h^{n-1}) \Delta W_n, \mathbf{Q}_h \mathbf{E}^{n-1}) \\
& \leq \frac{1}{2} \|\mathbf{Q}_h \mathbf{E}^0\|_{\mathbf{L}^2}^2 + Ch^2 k \sum_{n=1}^{\ell} \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 + Ch^3 k \sum_{n=1}^{\ell} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 \\
& + Ch^2 k \sum_{n=1}^{\ell} \|\mathbf{u}_h^n\|_{\mathbf{L}^2}^2 \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 \\
& + k \sum_{n=1}^{\ell-1} \left(\frac{8C_L^2}{\nu} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 + \frac{\tilde{C}}{\nu} \|\mathbf{A} \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 \right) \|\mathbf{Q}_h \mathbf{E}^n\|_{\mathbf{L}^2}^2 \\
& + Ck \max_{1 \leq n \leq M} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^4 \left(k \sum_{n=1}^M \|\nabla \mathbf{u}_h^n\|_{\mathbf{L}^2}^2 + k \sum_{n=1}^M \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \right) \\
& + \frac{8C_L^2 k}{\nu} \max_{1 \leq n \leq M} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \|\mathbf{Q}_h \mathbf{E}^n\|_{\mathbf{L}^2}^2 + Ch^2 k \sum_{n=1}^{\ell} \|\nabla p^n\|_{\mathbf{L}^2}^2 \\
& + \sum_{n=1}^{\ell} \|(\mathbf{g}^{n-1} - \mathbf{g}_h^{n-1}) \Delta W_n\|_{\mathbf{L}^2}^2 + \sum_{n=1}^{\ell} ((\mathbf{g}^{n-1} - \mathbf{g}_h^{n-1}) \Delta W_n, \mathbf{Q}_h \mathbf{E}^{n-1}) \\
& \leq \frac{1}{2} \|\mathbf{Q}_h \mathbf{E}^0\|_{\mathbf{L}^2}^2 + Ch^2 k \sum_{n=1}^M \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 + Ch^3 k \sum_{n=1}^M \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 \\
& + Ch^2 k \sum_{n=1}^M \|\mathbf{u}_h^n\|_{\mathbf{L}^2}^2 \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 \\
& + k \sum_{n=1}^{\ell-1} \left(\frac{8C_L^2}{\nu} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 + \frac{\tilde{C}}{\nu} \|\mathbf{A} \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 \right) \|\mathbf{Q}_h \mathbf{E}^n\|_{\mathbf{L}^2}^2 \\
& + Ck \max_{1 \leq n \leq M} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^4 \left(k \sum_{n=1}^M \|\nabla \mathbf{u}_h^n\|_{\mathbf{L}^2}^2 + k \sum_{n=1}^M \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \right) \\
& + \frac{8C_L^2 k}{\nu} \max_{1 \leq n \leq M} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \|\mathbf{Q}_h \mathbf{E}^n\|_{\mathbf{L}^2}^2 + Ch^2 k \sum_{n=1}^M \|\nabla p^n\|_{\mathbf{L}^2}^2 \\
& + \sum_{n=1}^M \|(\mathbf{g}^{n-1} - \mathbf{g}_h^{n-1}) \Delta W_n\|_{\mathbf{L}^2}^2 \\
& + \max_{1 \leq \ell \leq M} \left| \sum_{n=1}^{\ell} ((\mathbf{g}^{n-1} - \mathbf{g}_h^{n-1}) \Delta W_n, \mathbf{Q}_h \mathbf{E}^{n-1}) \right|,
\end{aligned}$$

which is equivalent to (see the notations defined in (5.73))

$$\begin{aligned}
& \|\mathbf{Q}_h \mathbf{E}^\ell\|_{\mathbf{L}^2}^2 + \sum_{n=1}^{\ell} \|\mathbf{Q}_h(\mathbf{E}^n - \mathbf{E}^{n-1})\|_{\mathbf{L}^2}^2 + \nu k \sum_{n=1}^{\ell} \|\nabla \mathbf{E}^n\|_{\mathbf{L}^2}^2 \\
& \leq A_h + B_h + k \sum_{n=1}^{\ell-1} D_h^n \max_{1 \leq \ell \leq n} \|\mathbf{Q}_h \mathbf{E}^\ell\|_{\mathbf{L}^2}^2 \\
& \leq (A_h + B_h) \exp\left(k \sum_{n=1}^{m-1} D_h^n\right).
\end{aligned} \tag{5.84}$$

The proof is completed by replacing m by M in (5.84). $\square$

Lemma 5.4.4. *Under the assumptions of Theorem 5.4.2 and Lemma 5.4.3, there holds*

$$\left(\mathbb{E} \left[(A_h + B_h)^{\frac{q}{2}} \exp\left(\frac{qk}{2} \sum_{n=1}^M D_h^n\right) \right] \right)^{\frac{1}{q}} \leq C_q \left(k^{\frac{1}{2}} + h + k^{-\frac{1}{2}} h \right). \tag{5.85}$$

Proof. First, by using Cauchy-Schwarz inequality, we have:

$$\begin{aligned}
& \mathbb{E} \left[(A_h + B_h)^{\frac{q}{2}} \exp\left(\frac{qk}{2} \sum_{n=1}^M D_h^n\right) \right] \\
& \leq C_q (\mathbb{E}[A_h^q])^{\frac{1}{2}} \left(\mathbb{E} \left[\exp\left(qk \sum_{n=1}^M D_h^n\right) \right] \right)^{\frac{1}{2}} \\
& \quad + C_q (\mathbb{E}[B_h^q])^{\frac{1}{2}} \left(\mathbb{E} \left[\exp\left(qk \sum_{n=1}^M D_h^n\right) \right] \right)^{\frac{1}{2}}.
\end{aligned} \tag{5.86}$$

By using Lemma 5.3.3, we can easily control the factor $\left(\mathbb{E} \left[\exp\left(qk \sum_{n=1}^M D_h^n\right) \right] \right)^{\frac{1}{2}}$. In fact,

$$\begin{aligned}
\mathbb{E} \left[\exp\left(qk \sum_{n=1}^M D_h^n\right) \right] &= \mathbb{E} \left[\exp\left(k \sum_{n=1}^M \frac{32qC_L^2}{\nu} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 + k \sum_{n=1}^M \frac{4q\tilde{C}}{\nu} \|\mathbf{A} \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 \right) \right] \\
&\leq \mathbb{E} \left[\exp\left(\frac{32qC_L^2 T}{\nu} \max_{1 \leq n \leq M} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 + k \sum_{n=1}^M \frac{4q\tilde{C}}{\nu} \|\mathbf{A} \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 \right) \right] \\
&\leq \mathbb{E} \left[\exp\left(\sigma \max_{1 \leq n \leq M} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 + \sigma k \sum_{n=1}^M \|\mathbf{A} \mathbf{u}^{n+1}\|_{\mathbf{L}^2}^2 \right) \right] \leq C_2,
\end{aligned} \tag{5.87}$$

where $\sigma_q = \max\left\{\frac{32qC_L^2T}{\nu}, \frac{4q\tilde{C}}{\nu}\right\}$.

Therefore, it remains to bound $\mathbb{E}[A_h^q]$ and $\mathbb{E}[B_h^q]$. By the assumptions on $\mathbf{g}$, we have

$$\begin{aligned}
\mathbb{E}[B_h^q] &= \mathbb{E}\left[\left(\sum_{n=1}^M \|(\mathbf{g}^{n-1} - \mathbf{g}_h^{n-1})\Delta W_n\|_{\mathbf{L}^2}^2 + \max_{1 \leq \ell \leq M} \left|\sum_{n=1}^{\ell} ((\mathbf{g}^n - \mathbf{g}_h^n)\Delta W_{n+1}, \mathbf{Q}_h \mathbf{E}^n)\right|\right)^q\right] \quad (5.88) \\
&\leq C_q \mathbb{E}\left[\left(\sum_{n=1}^M \|\mathbf{g}^{n-1} - \mathbf{g}_h^{n-1}\|_{\mathbf{L}^2}^2 |\Delta W_n|^2\right)^q\right] \\
&\quad + C_q \mathbb{E}\left[\max_{1 \leq \ell \leq M} \left|\sum_{n=1}^{\ell} ((\mathbf{g}^n - \mathbf{g}_h^n)\Delta W_{n+1}, \mathbf{Q}_h \mathbf{E}^n)\right|^q\right] \\
&\leq C_q h^{2q} \mathbb{E}\left[\left(\sum_{n=1}^M |\Delta W_n|^2\right)^q\right] + C_q \mathbb{E}\left[\max_{1 \leq \ell \leq M} \left|\sum_{n=1}^{\ell} ((\mathbf{g}^{n-1} - \mathbf{g}_h^{n-1})\Delta W_n, \mathbf{Q}_h \mathbf{E}^{n-1})\right|^q\right] \\
&= B_{h,1} + B_{h,2}.
\end{aligned}$$

It follows from the discrete Hölder inequality and (3.8) that

$$\begin{aligned}
B_{h,1} &= C_q h^{2q} \mathbb{E}\left[\left(\sum_{n=1}^M |\Delta W_n|^2\right)^q\right] \quad (5.89) \\
&\leq C_q h^{2q} M^{q-1} \sum_{n=1}^M \mathbb{E}[|\Delta W_n|^{2q}] \leq C_q h^{2q} M^q k^q \leq C_q h^{2q}.
\end{aligned}$$

To estimate $B_{h,2}$, we use the Burkholder–Davis–Gundy inequality and the assumptions on $\mathbf{g}$ to obtain

$$\begin{aligned}
B_{h,2} &= C_q \mathbb{E}\left[\max_{1 \leq \ell \leq M} \left|\sum_{n=1}^{\ell} ((\mathbf{g}^n - \mathbf{g}_h^n)\Delta W_n, \mathbf{Q}_h \mathbf{E}^{n-1})\right|^q\right] \quad (5.90) \\
&\leq C_q \mathbb{E}\left[\left(k \sum_{n=1}^M \|\mathbf{g}^{n-1} - \mathbf{g}_h^{n-1}\|_{\mathbf{L}^2}^2 \|\mathbf{Q}_h \mathbf{E}^{n-1}\|_{\mathbf{L}^2}^2\right)^{\frac{q}{2}}\right] \\
&\leq C_q h^{2q} \mathbb{E}\left[\left(k \sum_{n=1}^M \|\mathbf{E}^{n-1}\|_{\mathbf{L}^2}^2\right)^{\frac{q}{2}}\right] \leq C_q h^{2q}.
\end{aligned}$$

In addition, using the stability estimates from Lemmas 5.3.1, 5.4.1 and 5.3.2 and the assumption $\mathbb{E}[\|\mathbf{Q}_h \mathbf{E}^0\|_{\mathbf{L}^2}^q] \leq Ch^q$, we obtain

$$\begin{aligned}
\mathbb{E}[A_h^q] &= \mathbb{E} \left[\left(\|\mathbf{Q}_h \mathbf{E}^0\|_{\mathbf{L}^2}^2 + Ch^2k \sum_{n=1}^M \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 + Ch^3k \sum_{n=1}^M \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 \right. \right. \\
&\quad + Ch^2k \sum_{n=1}^M \|\mathbf{u}_h^n\|_{\mathbf{L}^2}^2 \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 \\
&\quad + Ck \max_{1 \leq n \leq M} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^4 \left(k \sum_{n=1}^M \|\nabla \mathbf{u}_h^n\|_{\mathbf{L}^2}^2 + k \sum_{n=1}^M \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \right) \\
&\quad \left. + \frac{16C_L^2k}{\nu} \max_{1 \leq n \leq M} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \|\mathbf{Q}_h \mathbf{E}^n\|_{\mathbf{L}^2}^2 + Ch^2k \sum_{n=1}^M \|\nabla p^n\|_{\mathbf{L}^2}^2 \right)^q \Big] \\
&\leq C_q h^{2q} + C_q h^{2q} \mathbb{E} \left[\left(k \sum_{n=1}^M \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 \right)^q \right] \\
&\quad + Ch^{3q} \mathbb{E} \left[\left(k \sum_{n=1}^M \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 \right)^q \right] \\
&\quad + C_q h^{2q} \mathbb{E} \left[\max_{1 \leq n \leq M} \|\mathbf{u}_h^n\|_{\mathbf{L}^2}^{2q} \left(k \sum_{n=1}^M \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 \right)^q \right] \\
&\quad + C_q k^q \mathbb{E} \left[\max_{1 \leq n \leq M} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^{4q} \left(k \sum_{n=1}^M \|\nabla \mathbf{u}_h^n\|_{\mathbf{L}^2}^2 \right)^q \right] \\
&\quad + C_q k^q \mathbb{E} \left[\max_{1 \leq n \leq M} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^{4q} \left(k \sum_{n=1}^M \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \right)^q \right] \\
&\quad + C_q k^q \mathbb{E} \left[\max_{1 \leq n \leq M} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^{2q} \|\mathbf{E}^n\|_{\mathbf{L}^2}^{2q} \right] + C_q h^{2q} \mathbb{E} \left[\left(k \sum_{n=1}^M \|\nabla p^n\|_{\mathbf{L}^2}^2 \right)^q \right] \\
&\leq C_q h^{2q} + C_q h^{2q} \left(\mathbb{E} \left[\max_{1 \leq n \leq M} \|\mathbf{u}_h^n\|_{\mathbf{L}^2}^{4q} \right] \right)^{\frac{1}{2}} \left(\mathbb{E} \left[\left(k \sum_{n=1}^M \|\mathbf{A} \mathbf{u}^n\|_{\mathbf{L}^2}^2 \right)^{2q} \right] \right)^{\frac{1}{2}} \\
&\quad + C_q k^q \left(\mathbb{E} \left[\max_{1 \leq n \leq M} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^{8q} \right] \right)^{\frac{1}{2}} \left(\mathbb{E} \left[\left(k \sum_{n=1}^M \|\nabla \mathbf{u}_h^n\|_{\mathbf{L}^2}^2 \right)^{2q} \right] \right)^{\frac{1}{2}} \\
&\quad + C_q k^q \left(\mathbb{E} \left[\max_{1 \leq n \leq M} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^{8q} \right] \right)^{\frac{1}{2}} \left(\mathbb{E} \left[\left(k \sum_{n=1}^M \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 \right)^{2q} \right] \right)^{\frac{1}{2}} \\
&\quad + C_q k^q \mathbb{E} \left[\max_{1 \leq n \leq M} \|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^{2q} (\|\mathbf{u}^n\|_{\mathbf{L}^2}^{2q} + \|\mathbf{u}_h^n\|_{\mathbf{L}^2}^{2q}) \right] \\
&\quad + C_q h^{2q} \mathbb{E} \left[\left(k \sum_{n=1}^M \|\nabla p^n\|_{\mathbf{L}^2}^2 \right)^q \right] \leq C_q h^{2q} + C_q k^q + C_q k^{-q} h^{2q}.
\end{aligned} \tag{5.91}$$

We note that the last term $C_q k^{-q} h^{2q}$ is a consequence of allowing general noises which clearly impact on the stability estimate of the time-semidiscrete pressure approximation as seen from Lemma 5.3.2.

In summary, substituting the estimates for A_h and B_h into (5.86), we obtain

$$\mathbb{E} \left[(A_h + B_h)^{\frac{q}{2}} \exp \left(\frac{qk}{2} \sum_{n=1}^M D_h^n \right) \right] \leq C_q \left(k^{\frac{q}{2}} + h^q + k^{-\frac{q}{2}} h^q \right). \quad (5.92)$$

The proof is complete. $\square$

We conclude this section by stating a pathwise error estimate for the velocity approximation by Algorithm 2 which is a direct corollary of the Kolmogorov Theorem (cf. Theorem 3.2.1) and the high moment error estimates of Theorem 5.4.2.

Theorem 5.4.5. *Assume that the assumptions of Theorem 5.4.2 hold. Let $2 \leq q < \infty$ and $0 < \gamma_2 < 1 - \frac{1}{q}$. Then, there exists a random variable $K = K(\omega)$ with $\mathbb{E}[|K|^q] < \infty$ such that there holds $\mathbb{P}$ -a.s.*

$$\begin{aligned} \max_{1 \leq n \leq M} \|\mathbf{u}^n - \mathbf{u}_h^n\|_{\mathbf{L}^2} + \left(\nu k \sum_{n=1}^M \|\nabla(\mathbf{u}^n - \mathbf{u}_h^n)\|_{\mathbf{L}^2}^2 \right)^{1/2} \\ \leq K \left(k^{\frac{\gamma_2}{2}} + h^{\gamma_2} + k^{-\frac{\gamma_2}{2}} h^{\gamma_2} \right). \end{aligned} \quad (5.93)$$

5.4.2 High moment and pathwise error estimates for the fully discrete pressure approximation

In this subsection, we establish high moment and pathwise error estimates for the pressure approximation generated by Algorithm 2.

Theorem 5.4.6. *Let $2 \leq q < \infty$, under the assumptions of Theorem 5.4.2, there holds*

$$\left(\mathbb{E} \left[\left\| k \sum_{n=1}^M (p^n - p_h^n) \right\|_{L^2}^q \right] \right)^{\frac{1}{q}} \leq C_q (k^{\frac{1}{2}} + h + k^{-\frac{1}{2}} h), \quad (5.94)$$

where $C_q = C(\beta_1, q, D_T, \mathbf{f}, \mathbf{u}_0) > 0$ and independent of k, h .

Proof. The proof of this theorem is similar to that of Theorem 5.3.8. We recall the discrete inf-sup (LBB) condition: There exists a $\beta_1 > 0$ such that

$$\beta_1 \|\xi_h\|_{L^2} \leq \sup_{\phi_h \in \mathbb{H}_h} \frac{(\xi_h, \operatorname{div} \phi_h)}{\|\nabla \phi_h\|_{\mathbf{L}^2}} \quad \forall \xi_h \in L_h. \quad (5.95)$$

Set $\mathbf{E}^m := \mathbf{u}^m - \mathbf{u}_h^m$ and $E_p^m := k \sum_{n=1}^m (p^n - p_h^n)$.

Summing (5.71) from 1 to M , we obtain

$$\begin{aligned} (\mathbf{E}^M - \mathbf{E}^0, \phi_h) + \nu k \sum_{n=1}^M (\nabla \mathbf{E}^n, \nabla \phi_h) + k \sum_{n=1}^M (\mathbf{u}^n \cdot \nabla \mathbf{u}^n - \mathbf{u}_h^n \cdot \nabla \mathbf{u}_h^n, \phi_h) \\ - \frac{k}{2} \sum_{n=1}^M ([\operatorname{div} \mathbf{u}_h^n] \mathbf{u}_h^n, \phi_h) - (E_p^M, \operatorname{div} \phi_h) \\ = \left(\sum_{n=1}^M (\mathbf{g}^{n-1} - \mathbf{g}_h^{n-1}) \Delta W_n, \phi_h \right) \quad \forall \phi_h \in \mathbb{H}_h. \end{aligned} \quad (5.96)$$

Thus,

$$\begin{aligned} (E_p^M, \operatorname{div} \phi_h) &= (\mathbf{E}^M - \mathbf{E}^0, \phi_h) + \nu k \sum_{n=1}^M (\nabla \mathbf{E}^n, \nabla \phi_h) \\ &\quad + k \sum_{n=1}^M (\mathbf{u}^n \cdot \nabla \mathbf{u}^n - \mathbf{u}_h^n \cdot \nabla \mathbf{u}_h^n, \phi_h) \\ &\quad - \frac{k}{2} \sum_{n=1}^M ([\operatorname{div} \mathbf{u}_h^n] \mathbf{u}_h^n, \phi_h) - \left(\sum_{n=1}^M (\mathbf{g}^{n-1} - \mathbf{g}_h^{n-1}) \Delta W_n, \phi_h \right) \\ &= (\mathbf{E}^M - \mathbf{E}^0, \phi_h) + \nu k \sum_{n=1}^M (\nabla \mathbf{E}^n, \nabla \phi_h) \\ &\quad + k \sum_{n=1}^M (\mathbf{E}^n \cdot \nabla \mathbf{u}^n + \mathbf{u}_h^n \cdot \nabla \mathbf{E}^n, \phi_h) \\ &\quad + \frac{k}{2} \sum_{n=1}^M ([\operatorname{div} \mathbf{E}^n] \mathbf{u}_h^n, \phi_h) - \left(\sum_{n=1}^M (\mathbf{g}^{n-1} - \mathbf{g}_h^{n-1}) \Delta W_n, \phi_h \right) \end{aligned} \quad (5.97)$$

$$\begin{aligned}
&\leq C(\|\mathbf{E}^M\|_{\mathbf{L}^2} + \|\mathbf{E}^0\|_{\mathbf{L}^2})\|\nabla\phi_h\|_{\mathbf{L}^2} + \nu k \sum_{n=1}^M \|\nabla\mathbf{E}^n\|_{\mathbf{L}^2} \|\nabla\phi_h\|_{\mathbf{L}^2} \\
&\quad + k \sum_{n=1}^M (\|\mathbf{E}^n\|_{\mathbf{L}^4} \|\nabla\mathbf{u}^n\|_{\mathbf{L}^2} + \|\mathbf{u}_h^n\|_{\mathbf{L}^4} \|\nabla\mathbf{E}^n\|_{\mathbf{L}^2}) \|\phi_h\|_{\mathbf{L}^4} \\
&\quad + \frac{k}{2} \sum_{n=1}^M \|\nabla\mathbf{E}^n\|_{\mathbf{L}^2} \|\mathbf{u}_h^n\|_{\mathbf{L}^4} \|\phi_h\|_{\mathbf{L}^4} \\
&\quad + C \left\| \sum_{n=1}^M (\mathbf{g}^{n-1} - \mathbf{g}_h^{n-1}) \Delta W_n \right\|_{\mathbf{L}^2} \|\nabla\phi_h\|_{\mathbf{L}^2} \\
&\leq C(\|\mathbf{E}^M\|_{\mathbf{L}^2} + \|\mathbf{E}^0\|_{\mathbf{L}^2})\|\nabla\phi_h\|_{\mathbf{L}^2} + \nu k \sum_{n=1}^M \|\nabla\mathbf{E}^n\|_{\mathbf{L}^2} \|\nabla\phi_h\|_{\mathbf{L}^2} \\
&\quad + Ck \sum_{n=1}^M (\|\nabla\mathbf{E}^n\|_{\mathbf{L}^2} \|\nabla\mathbf{u}^n\|_{\mathbf{L}^2} + \|\nabla\mathbf{u}_h^n\|_{\mathbf{L}^2} \|\nabla\mathbf{E}^n\|_{\mathbf{L}^2}) \|\nabla\phi_h\|_{\mathbf{L}^2} \\
&\quad + \frac{Ck}{2} \sum_{n=1}^M \|\nabla\mathbf{E}^n\|_{\mathbf{L}^2} \|\nabla\mathbf{u}_h^n\|_{\mathbf{L}^2} \|\nabla\phi_h\|_{\mathbf{L}^2} \\
&\quad + C \left\| \sum_{n=1}^M (\mathbf{g}^{n-1} - \mathbf{g}_h^{n-1}) \Delta W_n \right\|_{\mathbf{L}^2} \|\nabla\phi_h\|_{\mathbf{L}^2}.
\end{aligned}$$

Using the discrete inf-sup condition, we obtain

$$\begin{aligned}
\beta_1 \|E_p^M\|_{L^2} &\leq \sup_{\phi_h \in \mathbb{H}_h} \frac{(E_p^M, \operatorname{div} \phi_h)}{\|\nabla\phi_h\|_{\mathbf{L}^2}} \tag{5.98} \\
&\leq C(\|\mathbf{E}^M\|_{\mathbf{L}^2} + \|\mathbf{E}^0\|_{\mathbf{L}^2}) + \nu k \sum_{n=1}^M \|\nabla\mathbf{E}^n\|_{\mathbf{L}^2} \\
&\quad + Ck \sum_{n=1}^M \|\nabla\mathbf{E}^n\|_{\mathbf{L}^2} (\|\nabla\mathbf{u}^n\|_{\mathbf{L}^2} + \|\nabla\mathbf{u}_h^n\|_{\mathbf{L}^2}) \\
&\quad + \frac{Ck}{2} \sum_{n=1}^M \|\nabla\mathbf{E}^n\|_{\mathbf{L}^2} \|\nabla\mathbf{u}_h^n\|_{\mathbf{L}^2} + C \left\| \sum_{n=1}^M (\mathbf{g}^{n-1} - \mathbf{g}_h^{n-1}) \Delta W_n \right\|_{\mathbf{L}^2}.
\end{aligned}$$

Taking the q th-power, expectation, and then using Theorem 5.4.2 and the stability estimates of Lemmas 5.3.1 and 5.4.1, we obtain

$$\begin{aligned}
\beta_1 \mathbb{E}[\|E_p^M\|_{L^2}^q] &\leq C_q \left(k^{\frac{q}{2}} + h^q + k^{-\frac{q}{2}} h^q \right) \\
&\quad + C_q \mathbb{E} \left[\left(k \sum_{n=1}^M \|\nabla \mathbf{E}^n\|_{\mathbf{L}^2}^2 \right)^{\frac{q}{2}} \left(k \sum_{n=1}^M (\|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 + \|\nabla \mathbf{u}_h^n\|_{\mathbf{L}^2}^2) \right)^{\frac{q}{2}} \right] \\
&\quad + C_q \mathbb{E} \left[\left(k \sum_{n=1}^M \|\nabla \mathbf{E}^n\|_{\mathbf{L}^2}^2 \right)^{\frac{q}{2}} \left(k \sum_{n=1}^M \|\nabla \mathbf{u}_h^n\|_{\mathbf{L}^2}^2 \right)^{\frac{q}{2}} \right] \\
&\quad + C_q \mathbb{E} \left[\left(k \sum_{n=1}^M \|\mathbf{g}^{n-1} - \mathbf{g}_h^{n-1}\|_{\mathbf{L}^2}^2 \right)^{\frac{q}{2}} \right] \\
&\leq C_q \left(k^{\frac{q}{2}} + h^q + k^{-\frac{q}{2}} h^q \right) \\
&\quad + C_q \left(\mathbb{E} \left[\left(k \sum_{n=1}^M \|\nabla \mathbf{E}^n\|_{\mathbf{L}^2}^2 \right)^q \right] \right)^{\frac{1}{2}} \left(\mathbb{E} \left[\left(k \sum_{n=1}^M (\|\nabla \mathbf{u}^n\|_{\mathbf{L}^2}^2 + \|\nabla \mathbf{u}_h^n\|_{\mathbf{L}^2}^2) \right)^q \right] \right)^{\frac{1}{2}} \\
&\quad + C_q \left(\mathbb{E} \left[\left(k \sum_{n=1}^M \|\nabla \mathbf{E}^n\|_{\mathbf{L}^2}^2 \right)^q \right] \right)^{\frac{1}{2}} \left(\mathbb{E} \left[\left(k \sum_{n=1}^M \|\nabla \mathbf{u}_h^n\|_{\mathbf{L}^2}^2 \right)^q \right] \right)^{\frac{1}{2}} \\
&\quad + C_q \mathbb{E} \left[\left(k \sum_{n=1}^M \|\mathbf{g}^{n-1} - \mathbf{g}_h^{n-1}\|_{\mathbf{L}^2}^2 \right)^{\frac{1}{2}} \right] \\
&\leq C_q \left(k^{\frac{q}{2}} + h^q + k^{-\frac{q}{2}} h^q \right).
\end{aligned} \tag{5.99}$$

Finally, the proof is complete after dividing both sides by β_1 . $\square$

An immediate consequence of the above high moment error estimates is the following pathwise error estimate for the pressure approximation $\{p_h^n\}$, it follows immediately from an application of Theorem 3.2.1.

Theorem 5.4.7. *Assume that the assumptions of Theorem 5.4.6 hold. Let $2 \leq q < \infty$ and $0 < \gamma_2 < 1 - \frac{1}{q}$. Then, there exists a random variable $K = K(\omega)$ with $\mathbb{E}[|K|^q] < \infty$ such that there holds $\mathbb{P}$ -a.s.*

$$\left\| k \sum_{n=1}^M (p^n - p_h^n) \right\|_{L^2} \leq K \left(k^{\frac{\gamma_2}{2}} + h^{\gamma_2} + k^{-\frac{\gamma_2}{2}} h^{\gamma_2} \right). \tag{5.100}$$

We conclude this section by stating the global error estimates for our fully discrete numerical solution generated by Algorithm 2, which follows from combining the above temporal and spatial error estimates.

Theorem 5.4.8. *Let $2 \leq q < \infty$ and $0 < \gamma < \frac{1}{2}$, under the assumptions of Theorem 5.3.4 and Theorem 5.4.2, there exists a constant $C_q = C(D_T, \mathbf{u}_0, q, \mathbf{f}) > 0$ such that*

$$\begin{aligned} & \left(\mathbb{E} \left[\max_{1 \leq n \leq M} \|\mathbf{u}(t_n) - \mathbf{u}_h^n\|_{\mathbf{L}^2}^q \right] \right)^{\frac{1}{q}} + \left(\mathbb{E} \left[\left(\nu k \sum_{n=1}^M \|\nabla(\mathbf{u}(t_n) - \mathbf{u}_h^n)\|_{\mathbf{L}^2}^2 \right)^{\frac{q}{2}} \right] \right)^{\frac{1}{q}} \\ & \leq C_q \left(k^{\frac{1}{2}-\gamma} + h + k^{-\frac{1}{2}}h \right). \end{aligned} \quad (5.101)$$

In addition, let $2 < q < \infty$ and $0 < \gamma < \frac{1}{2}$ such that $\frac{1}{2} - \gamma - \frac{1}{q} > 0$. Then, for any $0 < \gamma_1 < \frac{1}{2} - \gamma - \frac{1}{q}$ and $0 < \gamma_2 < 1 - \frac{1}{q}$, there exists a random variable K with $\mathbb{E}[|K|^q] < \infty$ such that there holds $\mathbb{P} - a.s.$

$$\begin{aligned} & \max_{1 \leq n \leq M} \|\mathbf{u}(t_n) - \mathbf{u}_h^n\|_{\mathbf{L}^2} + \left(\nu k \sum_{n=1}^M \|\nabla(\mathbf{u}(t_n) - \mathbf{u}_h^n)\|_{\mathbf{L}^2}^2 \right)^{\frac{1}{2}} \\ & \leq K \left(k^{\gamma_1} + h^{\gamma_2} + k^{-\frac{\gamma_2}{2}} h^{\gamma_2} \right). \end{aligned} \quad (5.102)$$

Theorem 5.4.9. *Let $2 \leq q < \infty$ and $0 < \gamma < \frac{1}{2}$. Under the assumptions of Theorem 5.3.8 and Theorem 5.4.6, there exists a constant $C_q = C(D_T, \mathbf{u}_0, q, \mathbf{f}, \beta_0, \beta_1) > 0$ such that for $1 \leq \ell \leq M$*

$$\left(\mathbb{E} \left[\left\| P(t_\ell) - k \sum_{n=1}^{\ell} p_h^n \right\|_{L^2}^q \right] \right)^{\frac{1}{q}} \leq C_q \left(k^{\frac{1}{2}-\gamma} + h + k^{-\frac{1}{2}}h \right), \quad (5.103)$$

In addition, let $2 < q < \infty$ and $0 < \gamma < \frac{1}{2}$ such that $\frac{1}{2} - \gamma - \frac{1}{q} > 0$. Then, for any $0 < \gamma_1 < \frac{1}{2} - \gamma - \frac{1}{q}$ and $0 < \gamma_2 < 1 - \frac{1}{q}$, there exists a random variable K with $\mathbb{E}[|K|^q] < \infty$ such that there holds $\mathbb{P} - a.s.$

$$\left\| P(t_\ell) - k \sum_{n=1}^{\ell} p_h^n \right\|_{L^2} \leq K \left(k^{\gamma_1} + h^{\gamma_2} + k^{-\frac{\gamma_2}{2}} h^{\gamma_2} \right). \quad (5.104)$$

5.5 Computational experiments

In this section, we present numerical tests to validate our error estimates in (5.101)–(5.104). In all the tests, we set $D = (0, 1)^2 \subset \mathbb{R}^2$, $T = 1$, $\nu = 1$, $\mathbf{f}$ as in (2.75). In addition, we use the Taylor-Hood mixed finite element method for the spatial discretization and the homogeneous Dirichlet boundary condition is imposed on $\mathbf{u}$. We reconsider the Q -Wiener process $\mathbf{W}$ given in (2.73) and take $g \equiv 10$.

Moreover, we choose $J = 4$ and $N_p = 300$ as number of samples in our Monte Carlo simulations to compute the expectations in each test. We implement Algorithm 2 and evaluate the errors between the computed velocity (resp. pressure) and the reference velocity (resp. pressure) which are generated by Algorithm 2 on a very fine mesh with mesh sizes $k_0 = \frac{1}{1024}$, $h = \frac{1}{40}$. Furthermore, to evaluate errors in strong norms, we use the numerical integration formulas as in (3.118).

In addition, in order to compute the quantities of stochastic interest, we want to generate as many samples as possible via the Monte Carlo method. However, this approach is very expensive if Newton's method is used at each time step to solve the nonlinear system in Algorithm 2. Therefore, we introduce a cheaper fixed point iteration in Algorithm 3 below to solve the nonlinear system.

Algorithm 3. Given $\mathbf{u}_h^{n+1, \ell-1} = \mathbf{u}_h^n$ for $n = 0, 1, \dots, M-1$; $\ell = 1, \dots, L$. Find $(\mathbf{u}_h^{n+1, \ell}, p_h^{n+1, \ell}) \in L^2(\Omega, \mathbb{H}_h \times L_h)$ such that $\mathbb{P}$ -a.s.

$$\begin{aligned} & (\mathbf{u}_h^{n+1, \ell}, \boldsymbol{\phi}_h) + \nu k(\nabla \mathbf{u}_h^{n+1, \ell}, \boldsymbol{\phi}_h) - k(p_h^{n+1, \ell}, \operatorname{div} \boldsymbol{\phi}_h) \\ & = (\mathbf{u}_h^n, \boldsymbol{\phi}_h) - k\tilde{b}(\mathbf{u}_h^{n+1, \ell-1}, \mathbf{u}_h^{n+1, \ell-1}, \boldsymbol{\phi}_h) + (g_h^n \Delta \mathbf{W}_{n+1}, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbb{H}_h, \\ & (\operatorname{div} \mathbf{u}_h^{n+1, \ell}, \psi_h) = 0 \quad \forall \psi_h \in L_h. \end{aligned}$$

Test 1. In this test, we want to verify the convergence rates in the error estimates (5.101), (5.103). To do that, the high moment errors $\mathcal{E}_{\mathbf{u}, q}^M$ and $\mathcal{E}_{P, q}^M$ with $q = 2, 4, 8$ of the velocity and pressure approximations generated by Algorithm 2 are computed and the numerical results are shown in Table 5.1 and 5.2. The numerical results verify almost a half order convergence rate as predicted by our error estimate results.

Test 2. The constant C_q in Theorem 5.4.8 and Theorem 5.4.9 depends on q . However, no analytical formula for C_q is known. The goal of this test is to find out numerically how the error constant C_q depends on q . To the end, we fix $h = \frac{1}{40}$ and choose the time step $k = \frac{1}{16}$ to compute the errors $\mathcal{E}_{\mathbf{u},q}^M$ and $\mathcal{E}_{P,q}^M$ for different values of q . The numerical results are given in Figure 5.1 and 5.2 (left ones). We observe that the numerical results suggest the constant C_q is increasing (and blows up) in q . For a close-up view, we zoom in at large q to see the behaviors of both velocity and pressure errors in Figure 5.1 and 5.2 (right ones). We still see the increasing of the errors in q although the growth becomes slower for large q . A consequence of this analysis also shows that we can not simply take limit as $p \rightarrow \infty$ in the high moment error estimates of Theorem 5.4.8 and 5.4.9 to derive pathwise error estimates (5.102) and (5.104), and using Kolmogorov's Theorem is still the only viable approach for obtaining pathwise error estimates.

Test 3. In this test, we verify the pathwise error estimates (5.102) and (5.104). We choose five sample paths and compute their L^2 -norm errors for both velocity and pressure approximations. The numerical results are displayed in Figure 5.3 which verify that the pathwise convergence of the velocity and pressure approximations is almost of order $O(k^{\frac{1}{2}})$ as predicted by our pathwise error estimates.

Table 5.1: Algorithm 2: Errors of the computed velocity $\{\mathbf{u}_h^n\}_n$

k	$\mathcal{E}_{\mathbf{u},2}^M$	order	$\mathcal{E}_{\mathbf{u},4}^M$	order	$\mathcal{E}_{\mathbf{u},8}^M$	order
1/32	0.02673		0.03446		0.04635	
1/64	0.02161	0.3061	0.02819	0.3001	0.03842	0.2881
1/128	0.01500	0.5274	0.02012	0.4957	0.02940	0.3899
1/256	0.00884	0.7633	0.01141	0.8186	0.01530	0.9424

Table 5.2: Algorithm 2: Errors of the computed pressure $\{p_h^n\}_n$

k	$\mathcal{E}_{P,2}^M$	order	$\mathcal{E}_{P,4}^M$	order	$\mathcal{E}_{P,8}^M$	order
1/32	0.18883		0.22143		0.25376	
1/64	0.117758	0.6764	0.13742	0.6883	0.15686	0.6940
1/128	0.07205	0.7078	0.08382	0.7101	0.09545	0.7159
1/256	0.03911	0.8823	0.04543	0.8835	0.05172	0.8843

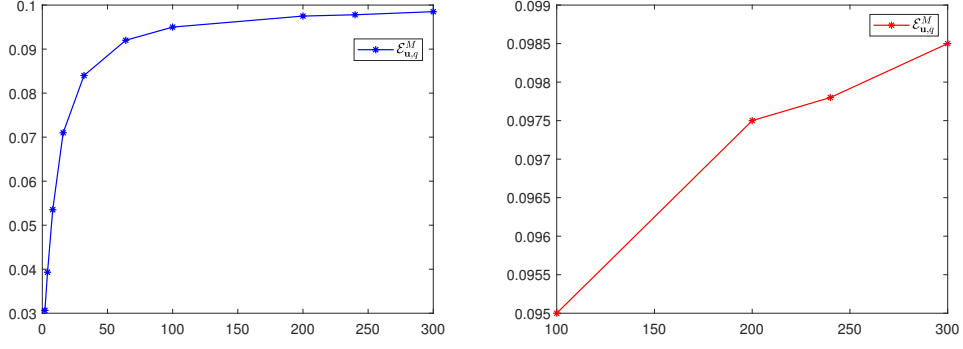


Figure 5.1: Errors of the velocity approximation (left) in $\mathcal{E}_{u,q}^M$ norm and its behavior (right) for large q .

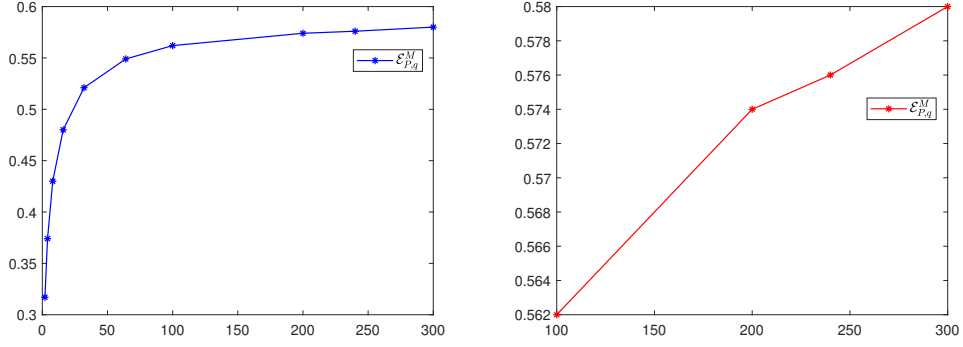


Figure 5.2: Errors of the pressure approximation (left) in $\mathcal{E}_{P,q}^M$ norm and its behavior (right) for large q .

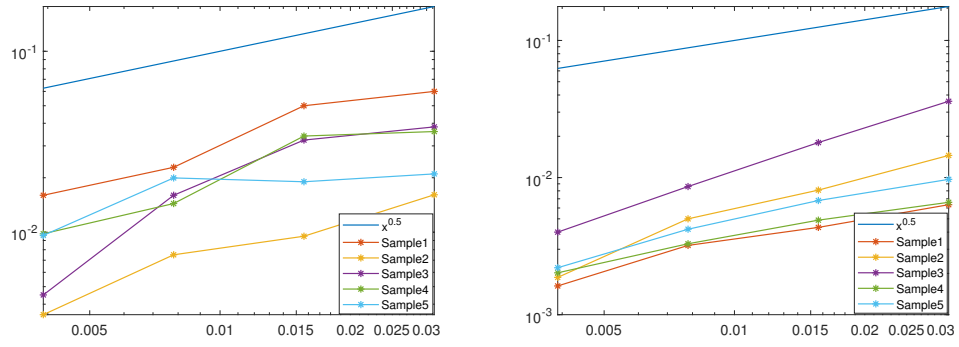


Figure 5.3: Five sample pathwise errors of the velocity (left) and pressure approximations (right).

Chapter 6

An iterative framework for parameter-dependent and random convection-diffusion problems

6.1 Introduction

Parameter-dependent problems arise from many engineering and scientific applications such as fluid mechanics, porous media flow, wave propagation and various biological models (cf. [62] and the references therein). The parameter could be deterministic (such as viscosity, diffusivity, permeability, frequency, and birth/death rate) or random, they appear as parameters in their respective mathematical (i.e., PDE) models. As the solution of such a model depends on the parameter, usually in a nonlinear manner even the PDE is linear, to compute approximate solutions, one is forced to solve the same problem multiple times for a given set of parameters. Such a direct (or brute force) approach may still be feasible if the parameter set is small even it is not efficient, however, it becomes too expensive to use if the parameter set is large due to the sheer amount of computation required to solve a complicated PDE problem tens, hundreds even thousands of times.

In addition, after a stochastic PDE is discretized in time, regardless which time-stepping method is used, the resulting problem at each time step is a random PDE (i.e., its coefficients

and source terms are random variables). A standard and brute force approach is to solve the random PDE using a Monte Carlo method. However, such an approach is computationally very expensive because it requires solving the underlying deterministic PDE problem (such as the Navier-Stokes) hundreds even thousands of times. Solving the resulting random PDEs is the bottle-neck of the overall numerical solution method for stochastic PDEs.

To overcome the computational challenge, several approaches have been proposed in literature including variants of the model order reduction (MOR) method [43, 59, 1, 11], the ensemble method [47, 40, 39, 55, 53, 54] and the multi-modes method [27, 28, 29]. The MOR methods aim to lower the computational complexity of the underlying model/problem by approximating the solution manifold using a handful of degrees of freedom, a low-dimensional surrogate to the original model is then built which is usually cheaper to simulate and often referred to as a reduced order model. Therefore, the computational saving is due to the dimensionality reduction. On the other hand, noticing the similarity of solving the same type problems with different parameters, a natural idea is to solve those problems together as a group and to reuse the computation as much as possible, so the computational saving is due to reusing a major portion of the computation for a set of parameters and requires designing better algorithms. This is exactly the idea used by the ensemble and multi-modes methods, and is also the approach adopted in this paper.

Inspired by the ensemble and multi-modes methods, the primary objective of this chapter is to develop and analyze a more general and unified iterative framework for solving parameter-dependent (including the case of random parameters) convection-diffusion problems in an abstract variational setting. Our main idea is to reformulate a given parameter-dependent problem into another problem with parameter-independent convection and diffusion coefficients and a parameter-dependent (and solution-dependent) right-hand side, its solution is then computed using a fixed-point iteration algorithm. Since the convection and diffusion coefficients are parameter-independent and iterate-independent, while the right-hand side term is parameter-dependent and iterate-dependent, this presents an ideal set-up for us to employ an efficient direct solver and a block Krylov subspace iterative solver at each iteration for solving a linear system with multiple right-hand vectors. It is the reuse of the LU matrix factorization and efficiently to compute matrix-matrix multiplication

for all parameters that results in significant computation saving, hence, leads to an efficient overall numerical method.

The remainder of this chapter is organized as follows. In section 6.2 we present the abstract variational setting for the parameter-dependent convection-diffusion problems to be considered in this paper. Structure conditions are stated to ensure the well-posedness of the variational problem. Such an abstract setting allows the wider applicability of the method and results of this chapter, it also permits a clean and concise presentation of the convergence analysis. In section 6.3 we first present our main algorithm in the variational continuous level and then establish the rate of convergence of the algorithm under some structure conditions. Section 6.4 presents finite element approximations of both the variational problem and the iterative algorithm. The highlight of this section is to derive the rate of convergence for the discretized algorithm. It worths noting that the finite element approximations are chosen for the preciseness and ease of presentation, the convergence results are also valid for other Galerkin-type approximations such as discontinuous Galerkin and spectral approximations. section 6.5 is devoted to computational complexity analysis of the proposed algorithm and to discussing an implementation strategy of the algorithm based on a CVT (centroidal Voronoi tessellations) clustering method. Finally, in section 6.6 we present three sets of numerical tests, the first one solves a parameter-dependent diffusion problem and the second solves a random diffusion problem, and the third computes a random convection-dominated convection-diffusion problem which is a randomized double-glazing problem [64]. Both one-dimensional and two-dimensional numerical test results are presented to demonstrate the efficiency of the proposed iterative method and to validate the theoretical convergence results. In particular, the last two tests show that the proposed iterative method can successfully solve some strongly random diffusion and convection-diffusion problems which the multi-modes method fails to do.

6.2 Variational problem setting

Let H be a Hilbert space and $V \subset H$ be a Banach space. V^* denotes the dual space of V , the space of bounded linear functionals on V . We consider the following variational problem

of seeking $u \in V$ such that

$$A(\omega; u, v) = F(\omega; v) \quad \forall v \in V, \quad (6.1)$$

where $F(\omega; \cdot) \in V^*$ and $A(\omega; \cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ depends on a parameter ω , which belongs to a (given) parameter space Ω and satisfies the following properties:

- (i) *Bi-linearity*: A is bilinear mapping on $V \times V$.
- (ii) *Boundedness*: There exists an ω -independent constant $\Lambda > 0$ such that

$$|A(\omega; v, w)| \leq \Lambda \|v\|_V \|w\|_V \quad \forall v, w \in V.$$

- (iii) *Coercivity*: There is an ω -independent constant $\lambda > 0$ such that

$$A(\omega; v, v) \geq \lambda \|v\|_V^2 \quad \forall v \in V.$$

Remark 6.2.1. (a) Since $A(\omega, \cdot, \cdot)$ may be non-symmetric, then problem (6.1) contains both coercive diffusion and coercive convection-diffusion problems which could be convection-dominated, see Section 6.6.3-6.6.4 for more discussions.

(b) The parameter space Ω can be a deterministic or probability space, in the latter case, equation (6.1) becomes a random PDE. Moreover, Ω can be finite or infinite set.

(c) Applicable examples to be considered later are (i) parameter-dependent PDEs; (ii) random PDE.

It is well known [9] that under the assumptions (i)–(iii), there holds the following Lax-Milgram Theorem.

Proposition 6.2.1. Under the above assumptions, problem (6.1) has a unique solution $u(\omega) \in V$ for each $\omega \in \Omega$ and there holds

$$\|u(\omega)\|_V \leq \frac{\|F(\omega)\|_{V^*}}{\lambda} \quad \forall \omega \in \Omega. \quad (6.2)$$

Remark 6.2.2. We note that $\|F(\omega)\|_{V^*} < \infty$ for each $\omega \in \Omega$, however, the bound may depend on ω . In the subsequent sections, unless stated otherwise, we shall assume that there exists an ω -independent constant $C_F > 0$ such that $\|F(\omega)\|_{V^*} \leq C_F$ for all $\omega \in \Omega$, that is, the operator norm of $F(\omega)$ is uniformly bounded in Ω . This condition will be relaxed to $E[\|F(\cdot)\|_{V^*}] < \infty$ for random diffusion and convection-diffusion applications in Section 6.6, namely, the expected value of the operator norm of $F(\omega)$ is bounded.

As mentioned earlier, the aim of this chapter is to develop efficient numerical methods/algorithms for computing the solution set $u(\omega)$ for all $\omega \in \Omega$.

6.3 The proposed iterative framework

The goal of this section is to formulate our main iterative algorithm for solving problem (6.1) and to establish the rate of convergence for the algorithm.

6.3.1 Formulation of the main algorithm

Let $A_0(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ be an ω -independent bilinear form, and set $A_1 = A - A_0$. Trivially

$$A(\omega; \cdot, \cdot) = A_0(\cdot, \cdot) + A_1(\omega; \cdot, \cdot) \quad (6.3)$$

is also a bilinear form on $V \times V$. The equation (6.1) can be rewritten as

$$A_0(u, v) = F(\omega; v) - A_1(\omega; u, v). \quad (6.4)$$

We now are ready to state our main iterative algorithm for solving (6.1).

Algorithm 1: Main Algorithm in Variational Setting

Step 1: Find $U_0 \in V$ by solving the following problem:

$$A_0(U_0, v) = F(\omega; v) \quad \forall v \in V. \quad (6.5)$$

Step 2: For each $\omega \in \Omega$, determine $\{U_n = U_n(\omega)\}_{n \geq 1} \subset V$ recursively by solving

$$A_0(U_n, v) = F(\omega; v) - A_1(\omega; U_{n-1}, v) \quad \forall v \in V. \quad (6.6)$$

To complete the construction of Algorithm 1, we need to specify the bilinear form A_0 . As expected, it must be problem-dependent, hence, we will do so in Section 6.6 when we apply Algorithm 1 to specific application problems. We conclude this subsection with the following remarks.

Remark 6.3.1. *The main advantage of Algorithm 1 is that the left hand-side bilinear forms in (6.5) and (6.6) are the same and independent of the parameter ω which allows to design fast solvers for computing the solutions in both steps of Algorithm 1.*

Remark 6.3.2. (a) *To recover the multi-modes method of [28] for random diffusion problems with the diffusion coefficients of the form $a(\omega, x) = a_0(x) + \eta(\omega, x)$ (see Section 6.6.2 for the details), let $a_0(x)$ be the expected value of $a(\omega, x)$ and we further assume that $\eta = \varepsilon\xi$, then define the n th mode function by*

$$u_0 = U_0, \quad u_n := \frac{1}{\varepsilon^n} (U_n - U_{n-1}) \quad \text{for } n \geq 1, \quad (6.7)$$

which implies that

$$U_n = U_{n-1} + \varepsilon^n u_n = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 \cdots + \varepsilon^n u_n.$$

Thus, the multi-modes method of [28] is recovered. We note that the multi-modes method of [28] does compute the mode functions $\{u_j\}_{j=1}^n$ in order to form the final approximate U_n . On the other hand, Algorithm 1 does not compute the mode functions, instead, it computes the approximate U_n directly. We also note that the parameter space Ω does not need to be given a priori but can be sampled on fly in simulations.

(b) To recover the ensemble method of [53] for parameter-dependent problems with the leading-term coefficients $a(\omega_j, x)$ and the parameter set $\Omega = \{\omega_j\}_{j=1}^J$ which are a priori given (see Section 6.6.1 for the details), we simply choose a_0 to be

$$a_0(x) = \frac{1}{J} \sum_{j=1}^J a(\omega_j, x) \quad \text{or} \quad a_0(x) = \max_{1 \leq j \leq J} a(\omega_j, x) \quad \forall x \in D,$$

Algorithm 1 then leads to the ensemble method.

6.3.2 Convergence analysis

To ensure the convergence of Algorithm 1, we impose the following criterion (and practical guideline) for choosing A_0 which would ensure the convergence of the algorithm.

Criterion for selecting A_0

(a) A_0 must satisfy (i), (ii), (iii) from definition of A . That is,

(i) A_0 is bilinear on $V \times V$.

(ii) There exists a constant $\Lambda_0 > 0$ such that

$$|A_0(v, w)| \leq \Lambda_0 \|v\|_V \|w\|_V \quad \forall v, w \in V.$$

(iii) There is a constant $\lambda_0 > 0$ such that

$$A_0(v, v) \geq \lambda_0 \|v\|_V^2 \quad \forall v \in V.$$

(b) *Relative dominance*: There exists a $\rho \in (0, 1)$ such that

$$\frac{\|A_1(\omega)\|_{\mathcal{L}(V \times V, \mathbb{R})}}{\lambda_0} < \rho \quad \forall \omega \in \Omega. \quad (6.8)$$

Under the above assumptions, we have the following convergence theorem.

Theorem 6.3.1. *Let $u(\omega)$ be the solution of (6.1) and $\{U_n(\omega)\}_{n \geq 0}$ be generated by Algorithm 1. Suppose the conditions of the above criterion hold. Then there exists a constant $C > 0$ independent of Ω such that*

$$\|u(\omega) - U_n(\omega)\|_V \leq C\rho^{n+1} \quad \forall \omega \in \Omega. \quad (6.9)$$

Hence, $U_n(\omega)$ converges to $u(\omega)$ strongly in V as $n \rightarrow \infty$ for every $\omega \in \Omega$.

Proof. For each fixed $\omega \in \Omega$, let $r_n = r_n(\omega) = u(\omega) - U_n(\omega) \in V$ for $n \geq 0$. Then, subtracting equation (6.5) from (6.4) yields the following error equation for $n = 0$:

$$A_0(r_0, v) = -A_1(\omega; u, v) \quad \forall v \in V. \quad (6.10)$$

Similarly, for any $n \geq 1$ subtracting (6.6) from (6.4) gives the error equation

$$A_0(r_n, v) = -A_1(\omega; r_{n-1}, v) \quad \forall v \in V. \quad (6.11)$$

Choose $v = r_0$ in (6.10) we obtain:

$$\lambda_0 \|r_0\|_V^2 \leq A_0(r_0, r_0) = -A_1(\omega; u, r_0) \leq \|A_1(\omega)\|_{\mathcal{L}(V \times V, \mathbb{R})} \|u\|_V \|r_0\|_V. \quad (6.12)$$

This implies that

$$\|r_0\|_V \leq \frac{\|A_1(\omega)\|_{\mathcal{L}(V \times V, \mathbb{R})}}{\lambda_0} \|u\|_V \leq \rho \|u\|_V. \quad (6.13)$$

We also choose $v = r_n$ in (6.11) to obtain:

$$\begin{aligned} \lambda_0 \|r_n\|_V^2 &\leq A_0(r_n, r_n) = -A_1(\omega; r_{n-1}, r_n) \\ &\leq \|A_1(\omega)\|_{\mathcal{L}(V \times V, \mathbb{R})} \|r_{n-1}\|_V \|r_n\|_V. \end{aligned} \quad (6.14)$$

Therefore,

$$\|r_n\|_V \leq \frac{\|A_1(\omega)\|_{\mathcal{L}(V \times V, \mathbb{R})}}{\lambda_0} \|r_{n-1}\|_V \leq \rho \|r_{n-1}\|_V. \quad (6.15)$$

By induction we have

$$\|r_n\|_V \leq \rho^{n+1} \|u\|_V \leq \rho^{n+1} \frac{1}{\lambda} \|F\|_{V^*}. \quad (6.16)$$

The proof is complete. $\square$

6.4 Finite element approximation

In this section, we first formulate the finite element Galerkin approximation of (6.1) and then present a discrete analogue of Algorithm 1 as a fast solver for computing the finite element solutions for all $\omega \in \Omega$. To the end, let $\mathcal{T}_h$ be a quasi-uniform triangulation of a bounded domain $D \subset \mathbb{R}^d$ ($d = 1, 2, 3$) with mesh size $h \in (0, 1)$. Let V_r^h ($r \geq 1$) denote the finite element space consisting of continuous piecewise r th order polynomials associated with $\mathcal{T}_h$. We assume that V_r^h is a subspace of V , which implicitly assumes that $V \subset W^{1,1}(D)$ but $V \not\subset W^{2,1}(D)$.

For each $\omega \in \Omega$, the standard finite element Galerkin approximation of (6.1) is defined as seeking $u_h = u_h(\omega) \in V_r^h$ such that

$$A(\omega; u_h, v_h) = F(\omega; v_h) \quad \forall v_h \in V_r^h. \quad (6.17)$$

It is easy to show that (6.17) has a unique solution u_h which also satisfies the stability estimate (6.2). Moreover, there holds the following error estimate.

Proposition 6.4.1. *Let $u(\omega)$ and $u_h(\omega)$ denote the solutions of (6.1) and (6.17) respectively. Then, there exists a constant $C = C(\omega) > 0$ independent of h and a positive integer $\ell(\leq r)$ such that*

$$\|u(\omega) - u_h(\omega)\|_V \leq C h^\ell \quad \forall \omega \in \Omega. \quad (6.18)$$

Remark 6.4.1. *The dependence of C on ω is through the norm $\|u(\omega)\|_{H^{r+1}}$ and ℓ depends on the space V . For example, if $V = H_0^1(D)$, then $\ell = r$.*

Using the definition of A_0 and A_1 , we can rewrite (6.17) as follows:

$$A_0(u_h, v_h) = F(\omega; v_h) - A_1(\omega; u_h, v_h) \quad \forall v_h \in V_r^h. \quad (6.19)$$

We then can easily formulate the discrete counterpart of Algorithm 1 for computing the solution of (6.17) below.

Algorithm 2: Main Algorithm in Discrete Setting

Step 1: Find $U_0^h \in V_r^h$ by solving the following problem:

$$A_0(U_0^h, v_h) = F(\omega; v_h) \quad \forall v_h \in V_r^h. \quad (6.20)$$

Step 2: For each $\omega \in \Omega$, determine $\{U_n^h = U_n^h(\omega)\}_{n \geq 1} \subset V_r^h$ recursively by solving

$$A_0(U_n^h, v_h) = F(\omega; v_h) - A_1(\omega; U_{n-1}^h, v_h) \quad \forall v_h \in V_r^h. \quad (6.21)$$

Remark 6.4.2. *We note that the main advantage of Algorithm 2 is that the left hand-side bilinear forms (or stiffness matrices) in (6.20) and (6.21) are the same and independent of the parameter ω which allows to design fast solvers for computing the solutions in both steps of Algorithm 2.*

Similarly, there also holds the following convergence result for Algorithm 2.

Theorem 6.4.2. *Let $u_h(\omega)$ be solution of (6.19) and $\{U_n^h(\omega)\}_{n \geq 0}$ be generated by Algorithm 2. Then there exists a constant $C > 0$ independent of ω such that*

$$\|u_h(\omega) - U_n^h(\omega)\|_V \leq C\rho^{n+1} \quad \forall \omega \in \Omega. \quad (6.22)$$

Hence, $U_n^h(\omega)$ converges to $u^h(\omega)$ strongly in V as $n \rightarrow \infty$ for every $\omega \in \Omega$.

Proof. For each fixed $\omega \in \Omega$, let $e_n^h = u_h(\omega) - U_n^h(\omega) \in V_r^h$ for $n \geq 0$. Then, subtracting equation (6.20) from (6.19) yields the following error equation for $n = 0$:

$$A_0(e_0^h, v_h) = -A_1(\omega; u_h, v_h) \quad \forall v_h \in V_r^h. \quad (6.23)$$

Similarly, by subtracting (6.21) from (6.19), we obtain the following error equation for $n \geq 1$:

$$A_0(e_n^h, v_h) = -A_1(\omega; e_{n-1}^h, v_h) \quad \forall v_h \in V_r^h. \quad (6.24)$$

Choosing $v_h = e_0^h \in V_r^h$ in (6.23), we have

$$\lambda_0 \|e_0^h\|_V^2 \leq A_0(e_0^h, e_0^h) = -A_1(\omega; u_h, e_0^h) \leq \|A_1(\omega)\|_{\mathcal{L}(V \times V, \mathbb{R})} \|u_h\|_V \|e_0^h\|_V. \quad (6.25)$$

This implies that

$$\|e_0^h\|_V \leq \frac{\|A_1(\omega)\|_{\mathcal{L}(V \times V, \mathbb{R})}}{\lambda_0} \|u_h\|_V \leq \rho \|u_h\|_V. \quad (6.26)$$

In addition, setting $v_h = e_n^h \in V_r^h$ in (6.24) yields

$$\begin{aligned} \lambda_0 \|e_n^h\|_V^2 &\leq A_0(e_n^h, e_n^h) = -A_1(\omega; e_{n-1}^h, e_n^h) \\ &\leq \|A_1(\omega)\|_{\mathcal{L}(V \times V, \mathbb{R})} \|e_{n-1}^h\|_V \|e_n^h\|_V. \end{aligned} \quad (6.27)$$

Therefore,

$$\|e_n^h\|_V \leq \frac{\|A_1(\omega)\|_{\mathcal{L}(V \times V, \mathbb{R})}}{\lambda_0} \|e_{n-1}^h\|_V \leq \rho \|e_{n-1}^h\|_V. \quad (6.28)$$

By induction we have

$$\|e_n^h\|_V \leq \rho^{n+1} \|u_h\|_V \leq \rho^{n+1} \frac{1}{\lambda} \|F\|_{V^*}. \quad (6.29)$$

The proof is complete. □

An immediate corollary is the following global error estimate.

Theorem 6.4.3. *Let $u(\omega)$ be solution of (6.1) and $\{U_n^h(\omega)\}_{n \geq 0}$ be generated by Algorithm 2. Then, there holds*

$$\|u(\omega) - U_n^h(\omega)\|_V \leq C(h^\ell + \rho^{n+1}) \quad \forall \omega \in \Omega. \quad (6.30)$$

Proof. The proof follows from Proposition 6.4.1, Theorem 6.4.2 and an application of the triangular inequality to the decomposition $u - U_n^h = (u - u_h) + (u_h - U_n^h)$. $\square$

6.5 Computational complexity and implementation strategies

6.5.1 Linear solvers

Steps in Algorithm 2 solve linear systems with a coefficient matrix independent of the parameter ω , which is a highly appealing feature when a group of problems, associated to different ω 's, needs to be solved. Since their discrete systems would share a common coefficient matrix, one can solve them simultaneously from a single system with multiple right-hand-side (RHS) vectors. Suppose J problems are considered corresponding to $\{\omega_1, \dots, \omega_J\}$. For j th problem, at the algebraic level, Algorithm 2 finds $\mathbf{u}_n^j$ such that: for $n \geq 0$,

$$\mathbf{A}_0 \mathbf{u}_n^j = \mathbf{b}_n^j = \begin{cases} \mathbf{f}^j & \text{if } n = 0, \\ \mathbf{f}^j - \mathbf{A}_1^j \mathbf{u}_{n-1}^j & \text{otherwise,} \end{cases} \quad (6.31)$$

where $\mathbf{A}_0$ is an $N \times N$ matrix. The set of systems can be recast into a matrix form:

$$\mathbf{A}_0 \mathbb{U}_n = \mathbf{B}_n, \quad (6.32)$$

where $\mathbb{U}_n = [\mathbf{u}_n^1, \dots, \mathbf{u}_n^J]$ is the solution matrix and $\mathbf{B}_n = [\mathbf{b}_n^1, \dots, \mathbf{b}_n^J]$ contains RHS information from individual problems.

Based on the structure and size of the coefficient matrix $\mathbf{A}_0$, a direct solver or an iterative one can be used to find the solutions. Generally speaking, Algorithm 2 has several computational advantages: (i) $\mathbf{A}_0$ only needs to be accessed once for all the problems; when

accessing or generating $\mathbf{A}_0$ represents a major bottleneck of a linear solver, this leads to a significant computational advantage [3]; (ii) A factorization of $\mathbf{A}_0$, if used, only needs to be implemented once in solving all the problems; (iii) The multiple RHS vectors in $\mathbf{B}_n$ lead to more efficient matrix-matrix products than matrix-vector products [23, Section 5.6]. Moreover, if a block Krylov subspace algorithm is used in solving (6.32), the multiple RHS vectors would enlarge the search space for minimizing residuals and thus could accelerate the convergence. We consider the direct solver in the paper and refer the readers to [48] and reference therein for discussions of block iterative solvers.

A direct solver contains the build phase and the solve phase, in which the former normally has higher complexity than the later. For a dense linear system, the factorization takes $\mathcal{O}(N^3)$ while the solve phase requires $\mathcal{O}(N^2)$. Thus, using Algorithm 2 is more efficient than solving problems separately as factorization is needed only once. Since $\mathbf{A}_0$ is sparse and SPD in the finite element discretization, fast direct solvers are available: using the nested dissection methods [36] or multi-frontal methods, the complexity for building the LU/Cholesky factorization is $\mathcal{O}(N^{3/2})$ in 2-D problem and $\mathcal{O}(N^2)$ in 3-D (see, for instance, [23, Section 9.3] and [20, Section 7.6]). Solving the resulting triangular systems has the complexity $\mathcal{O}(N \log N)$ in 2-D and $\mathcal{O}(N^{4/3})$ in 3-D. We next denote the cost of the build stage by $\mathcal{O}(N^p)$ and that of the solve stage by $\mathcal{O}(C_s)$, and analyze the total complexity of Algorithm 2 for solving J problems. Suppose the algorithm converges after $K - 1$ iterations, the total computation complexity is $\mathcal{O}(N^p + KJC_s)$. For which, one factorization of $\mathbf{A}_0$ costs $\mathcal{O}(N^p)$ and KJ times solves of two triangular systems cost $\mathcal{O}(KJC_s)$. But solving J problems separately has the complexity $\mathcal{O}(J(N^p + C_s))$. The ratio of the latter to the former provides a speedup factor

$$S_f = \frac{J(N^p + C_s)}{N^p + KJC_s} = \frac{1 + C_s N^{-p}}{J^{-1} + KC_s N^{-p}}.$$

Thus, given a finite K , for a fixed N , if J is big enough, the factor is on the order of $\frac{N^p}{KC_s}$ that scales as $\mathcal{O}(\sqrt{N}(\log N)^{-1}K^{-1})$ in 2-D and $\mathcal{O}(N^{\frac{2}{3}}K^{-1})$ in 3-D; while for a fixed J , as N becomes sufficiently large, the factor scales as J in both 2-D and 3-D.

6.5.2 Grouping

Given a group of problems, the convergence of the iterative algorithm could be slow if ρ is close to 1. In such a case, we propose to divide the problems into smaller groups to ensure a small ρ for each group. Motivated by the Centroidal Voronoi tessellations (CVT) method [22], we introduce the following algorithm.

Consider a set $W \subset \mathbb{R}^{\hat{d}}$. A set $\{V_i\}_{i=1}^{n_c}$ is a tessellation of W if $V_i \cap V_j = \emptyset$ for $i \neq j$ and $\cup_{i=1}^{n_c} V_i = W$. Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^N$ and, for any v with magnitude greater than 0, define

$$r(x, v) = \frac{|x - v|}{|v|}.$$

Given a set of points $Z = \{z_i\}_{i=1}^{n_c}$, define the set V_i by

$$V_i = \{x \in W : r(x, z_i) \leq r(x, z_j) \text{ for } j = 1, \dots, k, j \neq i\},$$

where the equality holds only for $i < j$. For any region $V \subset \mathbb{R}^{\hat{d}}$, z^* denotes the center of the region. In a discrete setting, suppose there are n_s elements $\{x_i\}_{i=1}^{n_s} \in V$, we have

$$z^* = \frac{1}{n_s} \sum_{i=1}^{n_s} x_i.$$

Given a set of samples, we need to determine the regions V_i and centers z_i . Thus, we design an algorithm, presented in Algorithm 3, that uses iterations to find them. First, it initializes the regions (also referred to be clusters) and centers (also referred to be cluster generators). For each data point $x \in W$, the algorithm finds the nearest generator $z \in Z$. If that does not match the cluster to which the data point is currently assigned, the data point is “transferred” to the cluster associated with the nearest generator. When all data points have been considered, the cluster generators z are replaced by centers of the clustered data points. The process ends when there is no further “transfers” occurred.

Algorithm 3: Grouping algorithm

Input: Sample set W , number of samples n_s , number of centers n_c , maximum number of iterations $iter_{max}$.

Output: Centers Z , Regions $V_1, \dots, V_{n_c}$.

Initialize centers $Z = \{z_1, \dots, z_{n_c}\}$ uniformly spaced in the set;

Initialize regions $\{V_1, \dots, V_{n_c}\}$ associated to each z_j using the empty set, for $j = 1, \dots, n_c$;

for $k = 1 : iter_{max}$ **do**

for $i = 1 : n_s$ **do**

 Take i -th sample $x_i = W(i)$;

 Determine $z_j = \arg \min_{v \in Z} r(x_i, v)$, where $r(x_i, v) = \frac{|x_i - v|}{|v|}$;

 Include x_i into the j -th region V_j controlled by z_j ;

end

 Update z_j by the center of V_j , for $j = 1, \dots, n_c$;

if *no transfers occurred* **then**

 Exit the loop

end

end

6.6 Applications

In this section, we conduct three sets of numerical tests on three application problems. The first one solves a parameter-dependent diffusion problem, the second solves a random diffusion problem, and the third simulates a random convection-dominated convection-diffusion problem which is a randomized double-glazing problem [64]. Both 1-D and 2-D numerical test results are presented to demonstrate the efficiency of the proposed iterative method and to validate the theoretical convergence results. Particularly, the last two tests show that the proposed iterative method can successfully solve some strongly random diffusion and convection-diffusion problems which the multi-modes method fails to do.

6.6.1 Parameter-dependent diffusion problems

For a finite set of parameters $\{\omega_j\}_{j=1}^J$, let $a_j(x) = a(\omega_j, x)$, $f_j = f(\omega_j, x)$ and consider the following parameter-dependent diffusion problems:

$$-\nabla \cdot (a_j \nabla u_j) = f_j \quad \text{in } D, \quad (6.33a)$$

$$u_j = 0 \quad \text{on } \partial D, \quad (6.33b)$$

where D is a bounded Lipschitz domain in $\mathbb{R}^d$ for $d = 1, 2, 3$.

We assume the above PDE is uniformly elliptic, that is, there exist two constants $0 < \lambda < \Lambda$ such that the diffusion coefficient $\{a_j(x)\}_{j=1}^J$ satisfies

$$\lambda \leq a_j(x) \leq \Lambda \quad \forall x \in D, j = 1, 2, \dots, J. \quad (6.34)$$

In addition, assume that $a_j(x) = a_0(x) + \eta_j(x)$ with $\eta_j(x) = \eta(\omega_j, x)$ and $a_0(x)$ satisfies

(i) Uniform ellipticity: $0 < \underline{a}_0 \leq a_0(x) \leq \bar{a}_0$.

(ii) Relative dominance: there exists a number $\rho \in (0, 1)$ such that

$$\frac{\|a_j - a_0\|_{L^\infty}}{\underline{a}_0} = \frac{\|\eta_j\|_{L^\infty}}{\underline{a}_0} < \rho \quad \forall j = 1, \dots, J. \quad (6.35)$$

There are a couple of options to choose $a_0(x)$ which are given below (cf. [53]).

(a) $a_0(x)$ is chosen as the arithmetic average of $\{a_j(x)\}_{j=1}^J$, that is,

$$a_0(x) := \frac{1}{J} \left(a_1(x) + a_2(x) + \dots + a_J(x) \right). \quad (6.36)$$

(b) $a_0(x)$ is chosen as the largest value of $\{a_j(x)\}_{j=1}^J$, that is,

$$a_0(x) := \max_{1 \leq j \leq J} a_j(x). \quad (6.37)$$

To fit the abstract framework, we set $V = H_0^1(D)$ and define the bilinear forms A, A_0, A_1 and the linear functional F respectively as

$$\begin{aligned} A(\omega; u, v) &= (a(\omega, \cdot) \nabla u, \nabla v), & F(\omega; v) &= (f(\omega, \cdot), v), \\ A_0(u, v) &= (a_0 \nabla u, \nabla v), & A_1(\omega; u, v) &= (\eta(\omega, \cdot) \nabla u, \nabla v). \end{aligned}$$

It is easy to verify that A, A_0, A_1 and F satisfy the convergence criteria laid out in Sections 6.3 and 6.4. Hence, Theorems 6.3.1, 6.4.2 and 6.4.3 apply to this problem with $\ell = r$.

Two test cases are considered in this section: one is a 1-D diffusion with an analytic solution, the other is a 2-D diffusion with more degrees of freedom but no analytic solution. We use the former to illustrate the theoretical results and the latter for checking the numerical efficiency. To investigate the numerical performance of the iterative algorithm, we consider two metrics:

$$\mathcal{E}_j = \|u_j - u_{j,n}^h\|_{H^1} \quad \text{and} \quad \mathcal{E}_j^h = \|u_j^h - u_{j,n}^h\|_{H^1},$$

where u_j is the exact solution, u_j^h is the finite element solution from individual simulations and $u_{j,n}^h$ is the finite element solution at n -th iteration of the iterative algorithm.

Test 1. Consider a 1-D diffusion problem (with homogeneous boundary condition) on $D = [0, 1]$ with the diffusion coefficient

$$a_j(x) = 1 + x + \epsilon_j \sin(x).$$

The exact solution is given by $u(x) = x(x - 1) + 0.5 \sin(20\pi x) + \epsilon_j \sin(40\pi x)$, and the source term is determined by plugging the exact solution in (6.33). For investigating the iterative algorithm, we consider $J = 5$ problems with the choices of $\epsilon_j \in \{0.1035, 0.0727, -0.0303, 0.0294, -0.0787\}$.

For the spatial discretization, the quadratic conforming finite elements ($\ell = 2$) are used on a uniform mesh with the size h . In the iterative algorithm, we set the stopping criterion to be $\max_j \|u_{j,n+1}^h - u_{j,n}^h\|_{H^1} < \text{tol} = 10^{-4}$, that is, the maximum H^1 norm of the differences between numerical solutions of adjacent iterations is less than **tol**.

Firstly, we choose $a_0(x) = 1 + x + \bar{\epsilon} \sin(x)$ with $\bar{\epsilon} := \frac{1}{5} \sum_{j=1}^5 \epsilon_j = 0.0193$. The iterative algorithm takes 4 iterations to complete for meshes with different resolutions. The numerical errors $\mathcal{E}_j$, for $j = 1, \dots, 5$, are listed in Table 6.1, which shows for each of the five problems, the approximation error decays at the second order as mesh is uniformly refined. The convergence history of $\mathcal{E}_j^h$ for the five problems is shown in Figure 6.1 by taking the $h = 2^{-10}$ case for example. Applying regressions on the results shows $\mathcal{E}_1^h \sim \mathcal{O}(0.032^{n+1})$ in problem 1, $\mathcal{E}_2^h \sim \mathcal{O}(0.021^{n+1})$ in problem 2, $\mathcal{E}_3^h \sim \mathcal{O}(0.019^{n+1})$ in problem 3, $\mathcal{E}_4^h \sim \mathcal{O}(0.0039^{n+1})$ in problem 4, and $\mathcal{E}_5^h \sim \mathcal{O}(0.038^{n+1})$ in problem 5. The ratios $\rho_j = \frac{\|a_j - a_0\|_{L^\infty}}{a_0}$ for these five problems are 0.071, 0.045, 0.042, 0.009, and 0.083, respectively, which are close and

proportional to the regression rates. We found a better ratio that matches the regression analysis result can be defined by $\hat{\rho}_j = \left\| \frac{a_j - a_0}{a_0} \right\|_{L^\infty}$, whose values are respectively 0.035, 0.022, 0.021, 0.004 and 0.041 for these problems. Further analysis in this aspect will be performed in a future work.

Secondly, we choose $a_0 = a_\infty := \max_j \max_{x \in D} |a_j(x)| = 2.0871$ while keeping the same computational setting as the previous case. The iterative algorithm achieves the same errors listed in Table 6.1, which is not surprising because the same stopping criterion is used in both tests. However, the total number of iterations increases to 16 in this case. The evolution of $\mathcal{E}_j^h$ for the five problems during the iteration when $h = 2^{-10}$ is plotted in Figure 6.2. Regressions on the data shows $\mathcal{E}_j^h \sim \mathcal{O}(0.49^{n+1})$ in these 5 problems. Note that the ratios $\rho_j = \hat{\rho}_j$ in this case because a_0 is a constant. For the five problems, the ratios are about 0.52 that are close to the regression results.

It is observed that the numerical results match the theoretical analysis in Theorems 6.4.2 - 6.4.3. In particular, $\mathcal{E}_j$ is dominated by the finite element discretization error, that is of the order $\mathcal{O}(h^\ell)$ with $\ell = 2$; and $\mathcal{E}_j^h$ is of the order $\mathcal{O}(\rho_j^{n+1})$, for $j = 1, \dots, 5$.

Table 6.1: Errors $\mathcal{E}_j$, for $j = 1, \dots, 5$, of the iterative algorithm solutions at different h .

h	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_4$	$\mathcal{E}_5$
$\frac{1}{2^7}$	3.82×10^{-1}	3.03×10^{-1}	2.21×10^{-1}	2.19×10^{-1}	3.18×10^{-1}
$\frac{1}{2^8}$	9.62×10^{-2}	7.63×10^{-2}	5.54×10^{-2}	5.50×10^{-2}	8.00×10^{-2}
$\frac{1}{2^9}$	2.41×10^{-2}	1.91×10^{-2}	1.39×10^{-2}	1.38×10^{-2}	2.00×10^{-2}
$\frac{1}{2^{10}}$	6.03×10^{-3}	4.78×10^{-3}	3.46×10^{-3}	3.44×10^{-3}	5.01×10^{-3}

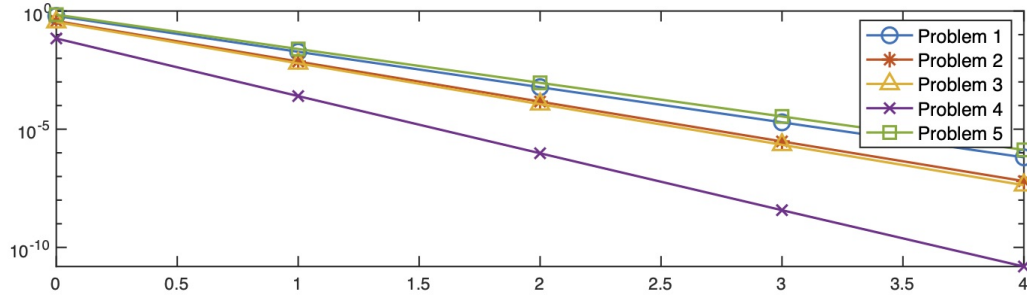


Figure 6.1: Evolution of $\mathcal{E}_j^h$, for $j = 1, \dots, 5$, during the iteration.

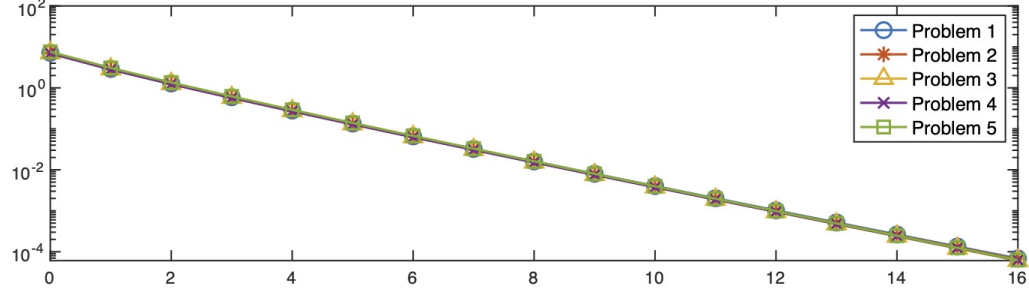


Figure 6.2: Evolution of $\mathcal{E}_j^h$, for $j = 1, \dots, 5$, during the iteration.

Test 2. We consider a diffusion problem on the domain $D = [-1, 1]^2$. Let D_1 be a disk centered at the origin of radius 0.5 and $D_0 = D \setminus \overline{D_1}$. The conductivity $a(x)$ is a piecewise constant on D :

$$a(x)|_{D_1} = \mu^{[1]} \quad \text{and} \quad a(x)|_{D_0} = 1.$$

The problem has zero forcing and is associated to a Dirichlet boundary condition on the top edge and Neumann boundary conditions on the other sides. In particular,

$$\begin{aligned} u &= 0 \text{ on } \Gamma_{top} = [-1, 1] \times \{1\}, \\ a \nabla u \cdot n &= 0 \text{ on } \Gamma_{side} = \{\pm 1\} \times (-1, 1), \\ a \nabla u \cdot n &= \mu^{[2]} \text{ on } \Gamma_{bottom} = [-1, 1] \times \{-1\}. \end{aligned}$$

Denote the parameter vector by $\boldsymbol{\mu} = (\mu^{[1]}, \mu^{[2]})$, whose ranges is specified by

$$\Omega := \{\boldsymbol{\mu} \in \mathbb{R}^2; \ 0.1 \leq \mu^{[1]} \leq 10, \ -1 \leq \mu^{[2]} \leq 1\}.$$

In this case, the bilinear forms A, A_0, A_1 and the linear functional F have the following form:

$$\begin{aligned} A(\boldsymbol{\mu}; u, v) &= (\mu^{[1]} \nabla u, \nabla v)_{D_1} + (\nabla u, \nabla v)_{D_0}, \quad F(\boldsymbol{\mu}; v) = (f(\boldsymbol{\mu}), v) + \langle \mu^{[2]}, v \rangle_{\Gamma_{bottom}}, \\ A_0(u, v) &= (\mu_0^{[1]} \nabla u, \nabla v)_{D_1} + (\nabla u, \nabla v)_{D_0}, \quad A_1(\boldsymbol{\mu}; u, v) = ((\mu^{[1]} - \mu_0^{[1]}) \nabla u, \nabla v)_{D_1}, \end{aligned}$$

where we use $a_0|_{D_1} := \mu_0^{[1]}$ and $a_0|_{D_0} := 1$. Since these forms have affine parameter dependence, the assembly of the associated matrices and vectors can be greatly simplified in computation.

We use the quadratic conforming finite element method for spatial discretization on a triangulation of D with the mesh size h . For instance, the triangulation of $h = 1/8$ is shown in Figure 6.3 (left), which contains 496 elements in total. In the iterative algorithm, the stopping criterion $\text{tol} = 10^{-4}$ is selected.

There are n_s $\boldsymbol{\mu}$ -samples $\{\boldsymbol{\mu}_j = (\mu_j^{[1]}, \mu_j^{[2]})\}_{j=1}^{n_s}$ randomly chosen in the parameter domain, e.g., $n_s = 500$ randomly selected samples are shown in Figure 6.3 (right). Note that only $\mu_j^{[1]}$ appears in the bilinear form, the ratio ρ_j only depends on $\mu_j^{[1]}$. In such a case, choosing

$\mu_0^{[1]} := \max_j \mu_j^{[1]}$ requires many iterations as shown in previous test, but choosing $\mu_0^{[1]} := \frac{1}{j} \sum_{j=1}^{n_s} \mu_j^{[1]}$ also will lead to a big ρ_j for some problems and require many iterations as well. Therefore, we first divide the sample set into multiple subgroups using Algorithm 3 and then apply the iterative algorithm to each of them.

We use a mesh of size $h = \frac{1}{32}$ for spatial discretization, which contains 8,136 elements and 16,529 nodes. For the same set including $n_s = 500$ samples, Algorithm 3 is fed with $W := \{\mu_1^{[1]}, \dots, \mu_{n_s}^{[1]}\}$ and finds $n_c = 10$ regions and associated centers. The evolution of centers during the process is shown in Figure 6.4 (left), which shows the sample points stop transferring after 78 steps. Each sample from the set is assigned to a region V_k with the center z_k , for $k = 1, \dots, n_c$. The ratio $r(\mu_i^{[1]}, z_k)$ for each μ_i is shown in Figure 6.4 (right). It is seen that all these ratios are below 0.3. The iterative algorithm is sequentially performed in the n_c groups with $a_0|_{D_1} := z_k$ for each group. The entire simulation is completed in 29.48 seconds and the maximum number of iterations for each group is 5. The detailed information for all the groups is listed in Table 6.2, where ρ represents the maximum value of $r(\mu_i^{[1]}, z_j)$ for elements in j -th group. The maximum errors between the iterative solution and the finite element solution, $\max_j \mathcal{E}_j^h$, in each group are plotted in Figure 6.5.

Next, we vary the number of samples and compare the iterative algorithm with the individual simulations in terms of wall-clock times. Since the exact solution is unknown, we measure the maximum difference in H^1 norm between iterative algorithm solutions and individual solutions. The results are listed in Table 6.3, where T_{it} represents the time elapsed for integration in the iterative algorithm and T_{ind} is that for integrating individual systems. It is observed that as the size of sample set increases, the efficiency of iterative algorithm improves. When $n_s = 2500$, the iterative algorithm saves about 50% simulation time.

Finally, we fix $n_s = 2500$ while doubling the number of centers n_c from 5 to 160. The wall-clock times T_{it} are listed in Table 6.4, which shows the simulation time first decreases then increases. The least computational time is achieved when $n_c = 80$, which saves over 60% simulation time compared to the individual simulations. It is also observed that as the number of groups increases, the maximum difference between iterative algorithm and individual simulation solutions, $\max_j \mathcal{E}_j^h$, decreases, which is because ρ shrinks as the problems are divided into more groups.

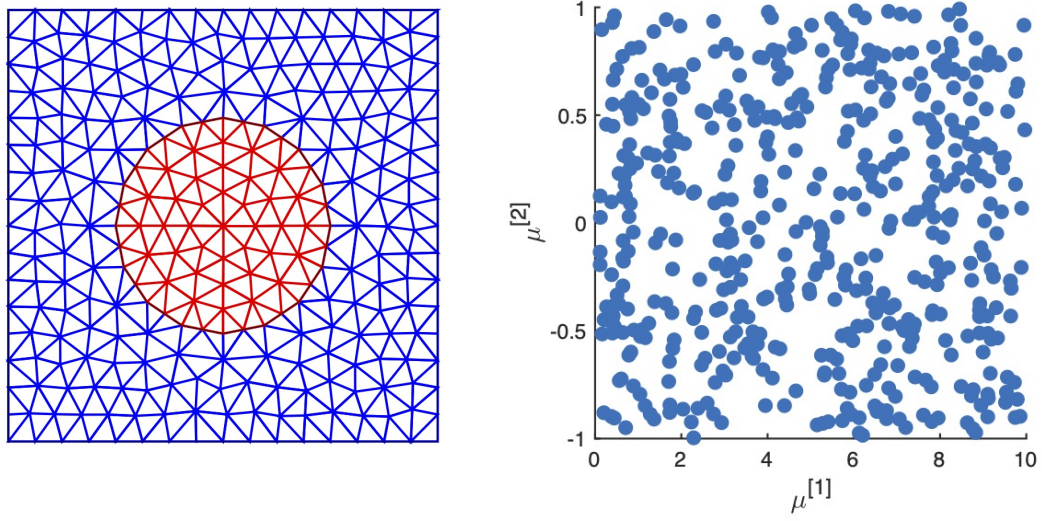


Figure 6.3: (Left) A finite element mesh with 496 elements; (Right) μ -samples.

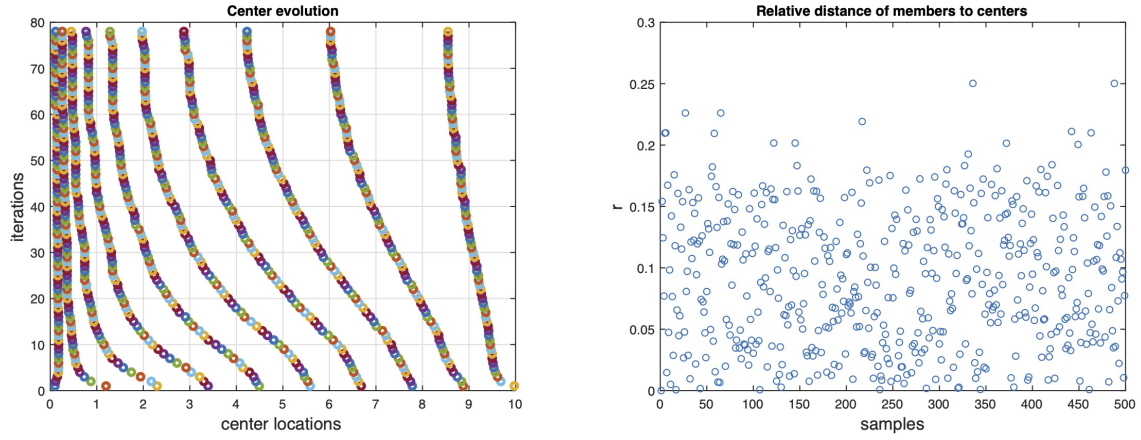


Figure 6.4: (Left) Evolution of centers of 10 subgroups; (Right) $r(\mu_i^{[1]}, z_j)$ for each $\mu_i \in W$ belongs to V_j and z_j is the center of V_j .

Table 6.2: Information of subgroups: size, region, center and maximum ρ ; performance of the iterative algorithm including number of iterations, maximum errors and wall-clock time for simulations.

group	size	$\mu^{[1]}$ -region	center	ρ	iterations	$\max \mathcal{E}_j^h$	time
1	4	[0.10, 0.13]	0.12	0.12	3	2.15×10^{-7}	0.49
2	15	[0.19, 0.32]	0.26	0.25	4	3.92×10^{-6}	1.06
3	14	[0.36, 0.54]	0.45	0.20	4	4.99×10^{-6}	1.05
4	17	[0.59, 0.94]	0.77	0.23	5	3.81×10^{-6}	1.37
5	33	[1.01, 1.55]	1.28	0.21	5	5.48×10^{-6}	2.18
6	32	[1.61, 2.33]	1.97	0.18	5	3.05×10^{-6}	2.12
7	50	[2.36, 3.39]	2.87	0.18	5	4.55×10^{-6}	3.07
8	68	[3.49, 4.97]	4.23	0.17	5	5.08×10^{-6}	3.90
9	111	[4.99, 7.06]	6.03	0.17	5	2.78×10^{-6}	6.15
10	156	[7.12, 9.97]	8.55	0.17	5	3.72×10^{-6}	8.09

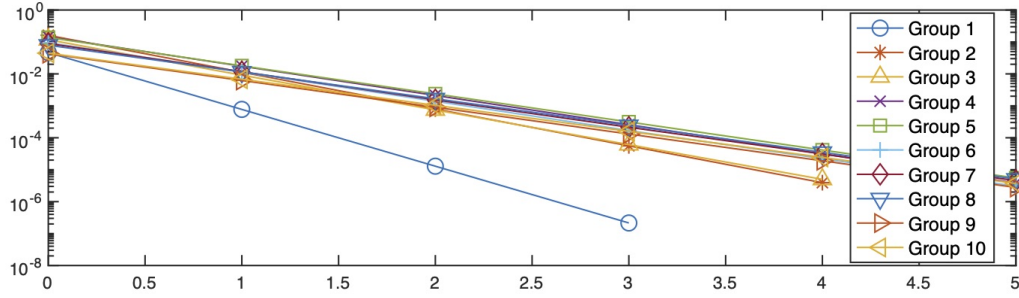


Figure 6.5: Evolution of $\max_j \mathcal{E}_j^h$ for each group during the iteration.

Table 6.3: Comparison of the iterative algorithm with individual simulations at $h = 1/32$: wall-clock times and maximum values of $\mathcal{E}_j^h$ for $j = 1, \dots, n_s$.

n_s	T_{it}	T_{ind}	$\max_j \mathcal{E}_j^h$
100	8.09	8.74	7.2×10^{-6}
500	29.48	43.14	5.48×10^{-6}
2500	116.10	233.23	1.27×10^{-6}

Table 6.4: Wall-clock times and maximum values of $\mathcal{E}_j^h$ in the iterative algorithm for 2500 samples that are divided into n_c groups.

n_c	5	10	20	40	80	160
T_{it}	187.5	116.1	102.64	93.01	89.69	111.75
$\max_j \mathcal{E}_j^h$	0.0000551	0.0000127	0.000005	0.00000327	0.00000328	0.00000167

6.6.2 Random diffusion equations

In this subsection we consider the following random diffusion test problem:

$$-\nabla \cdot (a(\omega, \cdot) \nabla u) = f(\omega, \cdot) \quad \text{in } D, \quad \mathbb{P}\text{-a.s.} \quad (6.38a)$$

$$u = 0 \quad \text{on } \partial D, \quad \mathbb{P}\text{-a.s.} \quad (6.38b)$$

Where D is a bounded Lipschitz domain in $\mathbb{R}^d$ for $d = 1, 2, 3$. $a(\omega, x)$ and $f(\omega, x)$ are random fields on the probability space $(\Omega, \mathbb{P}, \mathcal{F})$ and they are assume to have continuous and bounded covariance function. $\mathbb{E}[\cdot]$ denotes the expectation operator. In addition, a satisfies $\mathbb{P}$ -a.s. the following condition:

$$0 < \lambda \leq a(\omega, x) \leq \Lambda \quad \forall x \in D. \quad (6.39)$$

Moreover, we assume that $\mathbb{E}[\|f\|_{L^2(D)}] < \infty$. Below we propose two approaches for utilizing the abstract framework for problem (6.38). The first approach mimics the multi-modes method of [28] while the second one mimics the ensemble method of [53].

Approach #1

Suppose that there exists following decomposition of a :

$$a(\omega, x) = a_0(x) + \eta(\omega, x) \quad \text{a.s. } \forall x \in D, \quad (6.40)$$

where a_0 is independent of ω and also satisfies

(i) *Uniform ellipticity*: $0 < \underline{a}_0 \leq a_0(x) \leq \bar{a}_0$ for all $x \in D$.

(ii) *Relative dominance*: there exists a number $\rho \in (0, 1)$ such that

$$\mathbb{P}\{\omega \in \Omega; \|\eta(\omega, \cdot)\|_{L^\infty(D)} \leq \rho \underline{a}_0\} = 1. \quad (6.41)$$

To apply the abstract framework, we define

$$\begin{aligned} A(\omega; u, v) &= (a(\omega, \cdot) \nabla u, \nabla v), & F(\omega; v) &= (f(\omega, \cdot), v), \\ A_0(u, v) &= (a_0 \nabla u, \nabla v), & A_1(\omega; u, v) &= (\eta(\omega, \cdot) \nabla u, \nabla v). \end{aligned}$$

It is easy to verify that A, A_0, A_1 and F satisfy $\mathbb{P}$ -a.s. the convergent criteria laid out in Sections 6.3 and 6.4. Hence, Theorems 6.3.1, 6.4.2 and 6.4.3 apply to this problem with $\ell = r$. In particular, there holds

$$\mathbb{E}[\|u - U_n^h\|_{H^1}] \leq C(h^r + \rho^{n+1}). \quad (6.42)$$

It should be noted that $C > 0$ is independent of ω because $\mathbb{E}[\|f\|_{L^2(D)}] < \infty$.

We remark that in this first approach we do not specify how to discretize the stochastic variable ω . However, since the abstract framework suits best with the random sampling method, in practice, the Monte Carlo method would be used to discretize the right-hand sides in Step 1 and 2 of Algorithm 2.

Finally, we note that the choice of a_0 depends on the structure of a (cf. [28]).

To verify the convergence rates in (6.42) of this approach, we consider the following random coefficient boundary value problem [28] for our numerical tests:

$$\begin{aligned} -\frac{d}{dx} \left((1 + \varepsilon X(\omega)) \frac{du(\omega, x)}{dx} \right) &= X(\omega), & 0 < x < 1, \\ u(\omega, 0) &= 0, & u(\omega, 1) &= 0, \end{aligned} \quad (6.43)$$

where $X(\omega)$ is a uniformly distributed random variable defined in a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, where the sample space $\Omega = [0, 1]$, $\mathcal{B}$ denotes the σ -algebra of the Borel sets and $\mathbb{P}$ denotes the Lebesgue probability measure. In addition, we assume that $\varepsilon > 0$. The true solution of (6.43) is given by

$$u(\omega, x) = \frac{X(\omega)}{2(1 + \varepsilon X(\omega))} (x - x^2). \quad (6.44)$$

We have that $a(\omega, x) = 1 + \varepsilon X(\omega)$ satisfies the elliptic condition with $\lambda = 1, \Lambda = 1 + \varepsilon$. In addition, $a(\omega, x) = a_0(x) + \eta(\omega, x)$, where $a_0(x) = 1$ and $\eta(\omega, x) = \varepsilon X(\omega)$ satisfies the conditions (i) with $\underline{a}_0 = 1$ and (ii) with $\rho = \varepsilon$. In order to fit the abstract framework, in this example, we define $V = H_0^1(0, 1)$ and

$$\begin{aligned} A(\omega; u, v) &= ((1 + \varepsilon X(\omega))u', v'), & F(\omega; v) &= (X(\omega), v), \\ A_0(u, v) &= (u', v'), & A_1(\omega; u, v) &= (\varepsilon X(\omega)u', v'). \end{aligned}$$

Let V_1^h denote the standard linear finite element subspace of V which is used for the spatial discretization of (6.43). Let $\{U_n^h\}_{n \geq 0}$ be the approximate solution from Algorithm 2, where n is the number of iterations. Let

$$\mathcal{E}_{L^2}^n := \mathbb{E}[\|u - U_n^h\|_{L^2}], \quad \mathcal{E}_{H^1}^n := \mathbb{E}[\|u - U_n^h\|_{H^1}]. \quad (6.45)$$

As mentioned above, we use the Monte Carlo method to discretize the stochastic variable ω on the right-hand side in Step 1 and 2 of Algorithm 2. The number of the Monte Carlo samples is chosen to be $J = 10^4$. To test the validity and accuracy of the proposed iterative method, we set $h = 0.01$ for spatial discretization and then vary the number of iterations n . Table 6.5 displays the H^1 -norm errors of the iterative solutions for different values of n and ε . As expected, the computed solutions have smaller errors for smaller $\rho = \varepsilon$, and the method converges as long as $\varepsilon < 1$ which is predicted by our convergence theorem. In addition, our numerical tests show that the method ceases to converge for $\rho = \varepsilon > 1$, this indicates that our convergence result is sharp.

To test the convergence orders for the spatial discretization, we fix $\rho = \varepsilon = 0.1$ and the number of iterations $n = 10$, then vary the mesh size h . Table 6.6 shows the H^1 - and L^2 -errors of the computed solution by Algorithm 2.

We observe that an $O(h)$ order convergent rate in the H^1 -norm and an $O(h^2)$ order rate in the L^2 -norm are obtained for the numerical solution U_n^h which is consistent with the theoretical error estimate in (6.42).

Next, we verify the dependence of the errors on the parameter $\rho = \varepsilon$. To the end, we fix $\rho = 0.4, h = 0.01$ and then vary the number of iterations n . We also use the stopping criteria $\mathbb{E}[\|U_{n+1}^h - U_n^h\|_{H^1}] < \text{tol} := 10^{-4}$. The H^1 - and L^2 -norm errors of the computed solutions are shown in Figure 6.6. It is clear that the errors are decreasing as the number of iterations increases and the convergence order $O(\rho^{n+1})$ is indeed observed as predicted in (6.42). In addition, Figure 6.6 also shows that only 6 iterations is needed to trigger the stopping criterion.

Furthermore, we plot the errors of the first five Monte Carlo samples in Figure 6.7 to show that the errors for different samples are different and they are also different from the expected value which is presented in Figure 6.6, but they all are decreasing as n increases provided that ε is small.

Finally, to check the dependence of the iterative method on the parameter ε , we fix $n = 10, h = 0.01$ and then consider $\varepsilon = 2, 2.5, 3, 3.5$, which all are larger than 1. We note that in these cases, the relative dominant condition (ii) is violated, hence, our convergence results do not apply anymore. Figure 6.8 shows evidently that the H^1 - and L^2 -norm errors increase significantly as ε becomes larger, which clearly indicates that the iterative method may diverge when $\varepsilon > 1$ and also shows that the relative dominant condition (ii) is sharp.

Table 6.5: Approach #1: Errors $\mathcal{E}_{H^1}^n$ of the computed solutions with different ρ and n .

ρ	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
0.4	0.093	0.0033	0.0016	0.00123	0.00115	0.00114
0.6	0.0184	0.0089	0.0047	0.0027	0.0018	0.0014
0.8	0.0295	0.0187	0.0124	0.0086	0.0062	0.0046
0.9	0.0356	0.0253	0.188	0.0145	0.0115	0.0093

Table 6.6: Errors $\mathcal{E}_{H^1}^n$ and $\mathcal{E}_{L^2}^n$, for $n = 10$, of the iterative algorithm solutions at different h .

h	$\mathcal{E}_{H^1}^n$	Order	$\mathcal{E}_{L^2}^n$	Order
0.2	2.43×10^{-2}		1.5000×10^{-3}	
0.1	1.29×10^{-2}	0.92	4.0659×10^{-4}	1.88
0.05	6.60×10^{-3}	0.97	1.0443×10^{-4}	1.96
0.025	3.30×10^{-3}	1.00	2.6450×10^{-5}	1.98

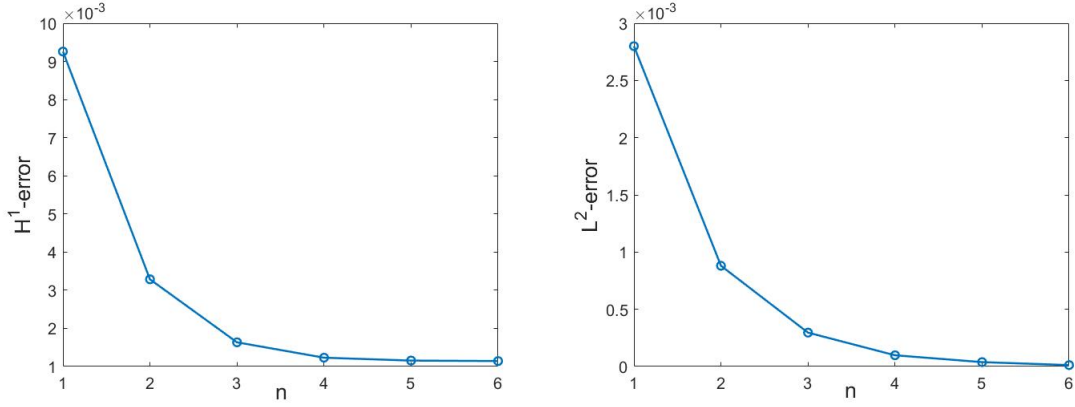


Figure 6.6: Approach #1: Errors $\mathcal{E}_{H^1}^n$ (left) and $\mathcal{E}_{L^2}^n$ (right) for $\rho = \varepsilon = 0.4$, $h = 0.01$, $\text{tol} = 10^{-4}$ and different n .

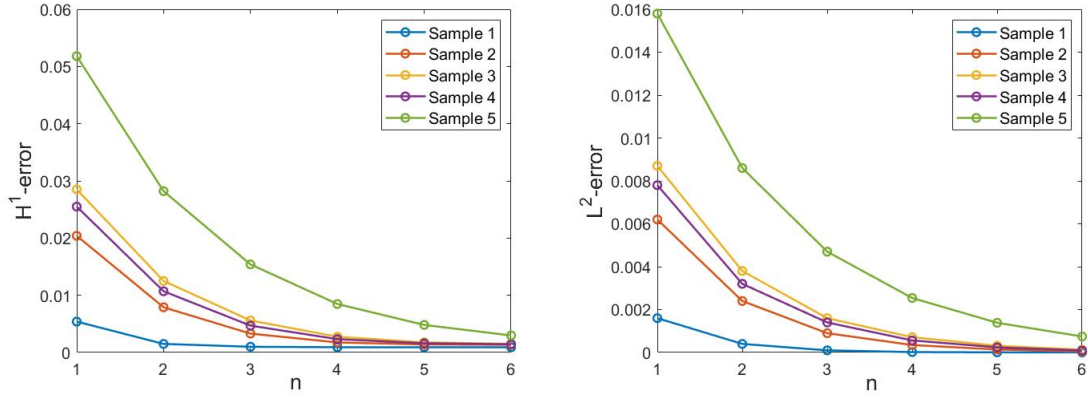


Figure 6.7: Approach #1: Errors $\mathcal{E}_{H^1}^n$ (left) and $\mathcal{E}_{L^2}^n$ (right) decay for the first 5 Monte Carlo samples with $\rho = \varepsilon = 0.4$, $h = 0.01$ and different n .

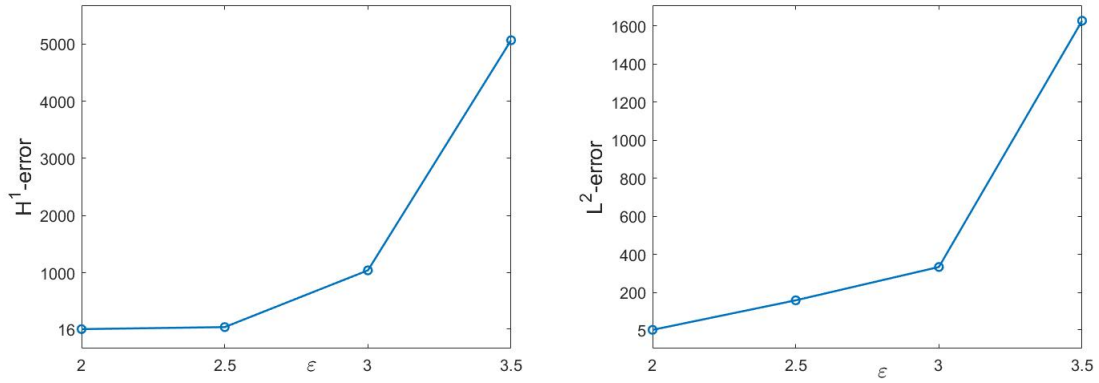


Figure 6.8: Approach #1: Errors $\mathcal{E}_{H^1}^n$ (left) and $\mathcal{E}_{L^2}^n$ (right) for different $\varepsilon > 1$ and $n = 10$.

Approach #2

In this subsection, we present a different approach to utilize the proposed iterative method for solving the same random diffusion problem (6.38) based on the Monte Carlo finite element discretization. To solve (6.38) by the Monte Carlo method, let $\{\omega_j\}_{j=1}^J$ be J samples from the sample space Ω , we then consider (deterministic) problems for $j = 1, 2, \dots, J$

$$-\nabla \cdot (a(\omega_j, \cdot) \nabla u) = f(\omega_j, \cdot) \quad \text{in } D, \quad (6.46a)$$

$$u = 0 \quad \text{on } \partial D. \quad (6.46b)$$

Hence, the Monte Carlo method converts the random diffusion problem (6.38) into the parameter-dependent problem (6.46) which is of the same type as problem (6.33). As a result, the method described in Section 6.6.1 readily applies to problem (6.46). The only extra step is to bound the expected value of the error as follows:

$$\begin{aligned} \|\mathbb{E}[u] - \mathbb{E}_{mc}[U_n^h]\|_V &\leq \|\mathbb{E}[u] - \mathbb{E}[U_n^h]\|_V + \|\mathbb{E}[U_n^h] - \mathbb{E}_{mc}[U_n^h]\|_V \\ &\leq C(\rho^{n+1} + h^r) + CJ^{-\frac{1}{2}}, \end{aligned} \quad (6.47)$$

where $\mathbb{E}_{mc}[\cdot]$ denotes the Monte Carlo approximation of the expected value. In addition, unlike Approach #1 above, which is equivalent to the multimodes method of [28] and imposes the restrictive condition $0 < \varepsilon < 1$ for convergence because $\rho = \varepsilon$, we like to show below that using Approach #2 the iterative method also converges for $\varepsilon > 1$ because we can have $\rho < 1$ even $\varepsilon > 1$ in this approach.

We reconsider the random diffusion test problem (6.43) whose exact solution is given by (6.44). It is easy to check that

$$\mathbb{E}[u] = \frac{1}{2} \left(\frac{1}{\varepsilon} - \frac{1}{\varepsilon^2} \ln(1 + \varepsilon) \right) (x - x^2).$$

The parameter-dependent problem (6.46) now becomes

$$\begin{aligned} -\frac{d}{dx} \left((1 + \varepsilon X(\omega_j)) \frac{du(\omega_j, x)}{dx} \right) &= X(\omega_j), \quad 0 < x < 1, \\ u(\omega_j, 0) &= 0, \quad u(\omega_j, 1) = 0, \end{aligned}$$

for $j = 1, \dots, J$.

To fit the setup of Section 6.6.1, we also define the following bilinear forms:

$$\begin{aligned} A(\omega_j; u, v) &= (a(\omega_j, \cdot) u', v'), \quad F(\omega_j; v) = (X(\omega_j), v), \\ A_0(u, v) &= (a_0 u', v'), \quad A_1(\omega_j; u, v) = (\eta(\omega_j, \cdot) u', v'), \end{aligned}$$

where $a(\omega_j, x) = 1 + \varepsilon X(\omega_j)$; a_0 can be chosen as in Section 6.6.1. For example, we can take $a_0(x) = \max_{1 \leq j \leq J} a(\omega_j, x)$ or $a_0(x) = \mathbb{E}_{mc}[a(\cdot, x)]$. Then, $\eta(\omega_j, x) = a(\omega_j, x) - a_0(x)$ will automatically satisfy the relative dominant condition (ii) with

$$\rho = \frac{\max_{1 \leq j \leq J} \max_{x \in D} |\eta(\omega_j, x)|}{\min_{x \in D} a_0(x)} < 1.$$

We also introduce the following error functions:

$$\mathcal{E}_{L^2}^n = \|\mathbb{E}[u] - \mathbb{E}_{mc}[U_n^h]\|_{L^2}, \quad \mathcal{E}_{H^1}^n = \|\mathbb{E}[u] - \mathbb{E}_{mc}[U_n^h]\|_{H^1}.$$

We now want to verify the convergence rate given in (6.47) for the iterative method. Notice that there are three terms in the error estimate (6.47). Since the error (i.e., the term $CJ^{-\frac{1}{2}}$) due to the Monte Carlo method is standard and we omit it here. In order to neglect this Monte Carlo error, we choose a large sample number $J = 10^6$ in all the numerical tests below.

To verify the finite element method error term $O(h)$, we consider $\varepsilon = 2.0$ and fix $n = 10$ then choose different values of the mesh size h . It is easy to check that $\rho = 0.5003$ when $a_0(x) = \mathbb{E}_{mc}[a(\cdot, x)]$ and $\rho = 0.6667$ when $a_0(x) = \max_{1 \leq j \leq J} a(\omega_j, x)$. In both cases, $\rho < 1$, which satisfies the relative dominant assumption in the convergence theorem. Table 6.7

displays the H^1 - and L^2 -norm errors for the first choice of a_0 and Table 6.8 shows the errors for the second choice of a_0 . We observe a convergence rate $O(h)$ for the H^1 -norm errors in both cases which is consistent with our error estimate in (6.47). In addition, we observe that the L^2 -norm errors exhibit an $O(h^2)$ order of the convergence.

To verify the iteration error term $O(\rho^{n+1})$, we fix $h = 0.01$, $\varepsilon = 2.0$ and also consider two cases of a_0 as in the previous test. A stopping criteria $\|\mathbb{E}_{mc}[U_{n+1}^h - U_n^h]\|_{H^1} < \text{tol} := 10^{-4}$ is used to terminate the iteration. Figure 6.9 displays the H^1 - and L^2 -norm errors of the computed solutions. We observe that the errors are decreasing as the number of iterations increases. Figure 6.9 also shows that only 6 iterations are needed to trigger the stopping criterion.

Finally, we plot the errors of the first five Monte Carlo samples in Figure 6.10 for $\varepsilon = 2.0$, $a_0(x) = \mathbb{E}_{mc}[a(\cdot, x)]$, $h = 0.01$ and various n . As expected, the errors are different for different samples and they all are also different from the expected value in Figure 6.9, but they all are decreasing as the number of iterations n increases.

Table 6.7: Approach #2: Errors $\mathcal{E}_{H^1}^n$ and $\mathcal{E}_{L^2}^n$, for $\varepsilon = 2.0$, $a_0 = \mathbb{E}_{mc}[a] = 2.0011$, $\rho = 0.5003$, $n = 10$ and different h .

h	$\mathcal{E}_{H^1}^n$	Order	$\mathcal{E}_{L^2}^n$	Order
0.2	1.17×10^{-2}		7.2850×10^{-4}	
0.1	6.20×10^{-3}	0.93	1.8724×10^{-4}	1.96
0.05	3.20×10^{-3}	0.96	4.2524×10^{-5}	2.14
0.025	1.60×10^{-3}	1.00	6.9804×10^{-6}	2.61

Table 6.8: Approach #2: Errors $\mathcal{E}_{H^1}^n$ and $\mathcal{E}_{L^2}^n$, for $\varepsilon = 2.0$, $a_0(x) = \max_{1 \leq j \leq J} a(\omega_j, x) = 3.0$, $\rho = 0.6667$, $n = 10$ and different h .

h	$\mathcal{E}_{H^1}^n$	Order	$\mathcal{E}_{L^2}^n$	Order
0.2	1.17×10^{-2}		7.3323×10^{-4}	
0.1	6.20×10^{-3}	0.93	1.9227×10^{-4}	1.93
0.05	3.20×10^{-3}	0.96	4.7262×10^{-5}	2.02
0.025	1.60×10^{-3}	1.00	9.9898×10^{-6}	2.24

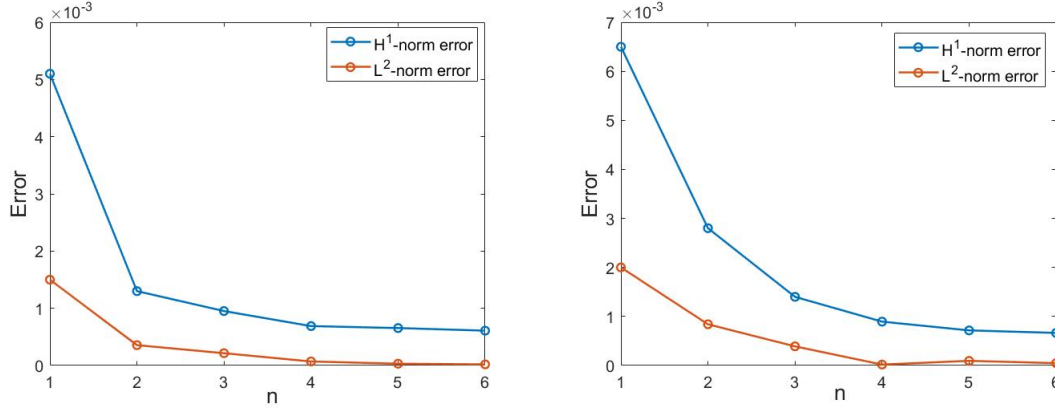


Figure 6.9: Approach #2: Errors $\mathcal{E}_{H^1}^n$ and $\mathcal{E}_{L^2}^n$ with $a_0(x) = \mathbb{E}_{mc}[a(\cdot, x)] = 2.0011$ (left) and $a_0(x) = \max_{1 \leq j \leq J} a(\omega_j, x) = 3.0$ (right) for $h = 0.01, \varepsilon = 2.0, \text{tol} = 10^{-4}$ and various n .

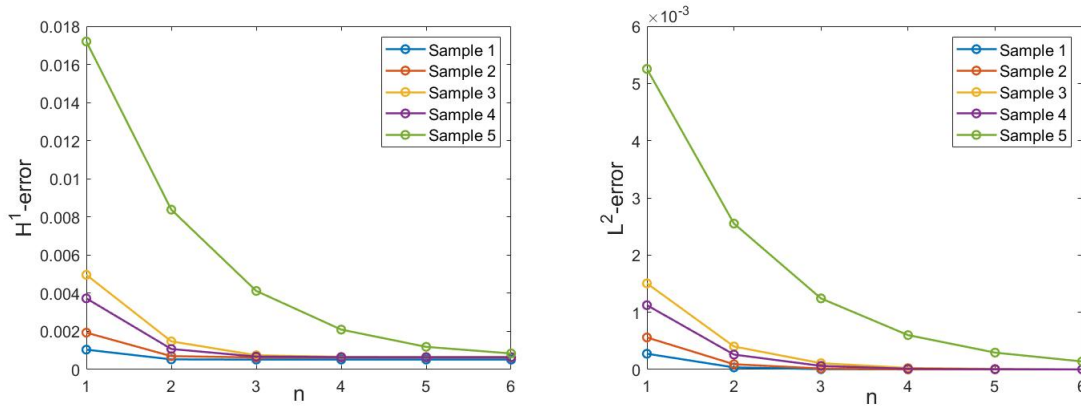


Figure 6.10: Approach #2: Errors $\mathcal{E}_{H^1}^n$ (left) and $\mathcal{E}_{L^2}^n$ (right) decays for first 5 samples with $\varepsilon = 2.0, a_0(x) = \mathbb{E}_{mc}[a(\cdot, x)] = 2.001, \rho = 0.5003, h = 0.01$ and various n .

6.6.3 1-D random convection-diffusion problems

In this subsection, we consider the following 1-D randomized double-glazing problem [64]:

$$\begin{aligned} -\frac{d}{dx} \left((1 + \varepsilon X(\omega)) \frac{du(\omega, x)}{dx} \right) + 100(1 + \varepsilon X(\omega)) \frac{du(\omega, x)}{dx} &= f(\omega, x), \\ u(\omega, 0) &= 0, \quad u(\omega, 1) = 0, \end{aligned} \quad (6.48)$$

where $0 \leq x \leq 1$, $\varepsilon > 0$ and $X(\omega)$ is a uniformly distributed random variable on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ with $\mathcal{B}$ is the Borel σ -algebra. In addition, we choose $f(\omega, x) = X(\omega)(51 - 100x)$ so that the exact solution is again given by

$$u(\omega, x) = \frac{X(\omega)}{2(1 + \varepsilon X(\omega))} (x - x^2). \quad (6.49)$$

Its expected value is already given in previous subsection.

To apply the iterative method, we adopt Approach #2 proposed in Section 6.6.1 to (6.48). Namely, we consider the following parameter-dependent problem:

$$\begin{aligned} -\frac{d}{dx} \left((1 + \varepsilon X(\omega_j)) \frac{du(\omega_j, x)}{dx} \right) + 100(1 + \varepsilon X(\omega_j)) \frac{du(\omega_j, x)}{dx} &= f(\omega_j, x), \\ u(\omega_j, 0) &= 0, \quad u(\omega_j, 1) = 0, \end{aligned} \quad (6.50)$$

for $j = 1, \dots, J$. To fit the setup of Section 6.6.1, we introduce the following coefficient functions:

$$\begin{aligned} a(\omega_j, x) &= 1 + \varepsilon X(\omega_j), & b(\omega_j, x) &= 100(1 + \varepsilon X(\omega_j)) \\ a_0(x) &= \mathbb{E}_{mc}[a(\cdot, x)], & b_0(x) &= \mathbb{E}_{mc}[b(\cdot, x)], \\ \eta_a(\omega_j, x) &= a(\omega_j, x) - a_0(x), & \eta_b(\omega_j, x) &= b(\omega_j, x) - b_0(x), \end{aligned}$$

which result in the following bilinear forms:

$$\begin{aligned} A(\omega_j; u, v) &= (a(\omega_j, \cdot)u', v') + (b(\omega_j, \cdot)u', v), & F(\omega_j; v) &= (f(\omega_j, \cdot), v), \\ A_0(u, v) &= (a_0 u', v') + (b_0 u', v), & A_1(\omega_j; u, v) &= (\eta_a(\omega_j, \cdot)u', v') + (\eta_b(\omega_j, \cdot)u', v). \end{aligned}$$

Then, the relative dominant number ρ in the convergence criteria of Section 6.3 can be estimated by

$$\rho \approx \frac{\max_{1 \leq j \leq J} \max_{x \in D} (|\eta_a(\omega_j, x)| + |\eta_b(\omega_j, x)|)}{\min_{x \in D} a_0(x)}.$$

In this test, we use $J = 10^4$ number of samples for the Monte Carlo method, the mesh size $h = 0.01$ and the maximum number of iterations $n = 10$. We also select $\varepsilon = 0.2, 0.005$ in (6.50).

Figure 6.11 shows the expected values $\mathbb{E}_{mc}[U_n^h]$ and $\mathbb{E}[u]$ of the computed and exact solutions and Table 6.9 displays the expected values of the H^1 - and L^2 -norm errors. We observe from Figure 6.11 and Table 6.9 that the proposed iterative method performs well for problem (6.48) when ε is sufficiently small, but the errors become larger as ε increases, which is expected. In addition, due to the contribution of the convection term in (6.48), the relative dominance parameter $\rho > 1$ when $\varepsilon = 0.2$, which explains the poor performance of the iterative method in this case.

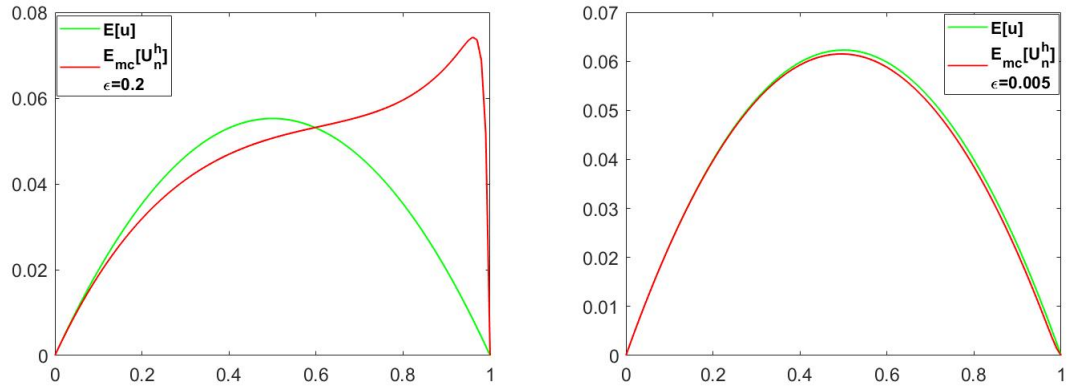


Figure 6.11: 1-D random convection-diffusion problem: Plots of $\mathbb{E}[u]$ and $\mathbb{E}_{mc}[U_n^h]$ with $h = 0.01, n = 10$ and $\varepsilon = 0.2$ (left) and $\varepsilon = 0.005$ (right).

Table 6.9: 1-D random convection-diffusion problem: Errors $\mathcal{E}_{H^1}^n$ and $\mathcal{E}_{L^2}^n$ with $h = 0.01, n = 10$ and various ε .

ε	ρ	$\mathcal{E}_{H^1}^n = \ \mathbb{E}[u] - \mathbb{E}_{mc}[U_n^h]\ _{H^1}$	$\mathcal{E}_{L^2}^n = \ \mathbb{E}[u] - \mathbb{E}_{mc}[U_n^h]\ _{L^2}$
0.2	9.1912	0.5328	2.2100×10^{-2}
0.005	0.2522	0.0117	9.6119×10^{-4}

6.6.4 2-D randomized double-glazing problems

In this subsection, we consider the random perturbation of the double-glazing problem given in [64]. Specifically, we consider

$$-\delta\Delta u + \mathbf{b} \cdot \nabla u = f(\omega, x, y) \quad \text{in } D = (0, 1)^2 \quad (6.51)$$

$$u(\omega, \cdot) = 0 \quad \text{on } \partial D, \quad (6.52)$$

where $\mathbf{b}(\omega, x, y) := (1 + \varepsilon X(\omega))(2y(1 - x^2), -2x(1 - y^2))$ and $0 < \delta \ll |\mathbf{b}|$, $\varepsilon > 0$. X is a uniformly distributed random variable on $[0, 1]$. It is easy to check that $\text{div } \mathbf{b} = 0$. We set the right-hand side force function

$$\begin{aligned} f(\omega; x, y) = & \frac{\delta X(\omega)}{1 + \varepsilon X(\omega)}(y - y^2 + x - x^2) + X(\omega)y(y - y^2)(1 - x^2)(1 - 2x) \\ & - X(\omega)x(x - x^2)(1 - y^2)(1 - 2y) \end{aligned}$$

so that the exact solution is given by

$$u(\omega; x, y) = \frac{X(\omega)}{2(1 + \varepsilon X(\omega))}(x - x^2)(y - y^2).$$

Thus,

$$\mathbb{E}[u] = \frac{1}{2} \left(\frac{1}{\varepsilon} - \frac{1}{\varepsilon^2} \ln(1 + \varepsilon) \right) (x - x^2)(y - y^2).$$

We again adopt Approach #2 of Section 6.6.1 to solve (6.51) with $\delta = 0.1$ using the proposed iterative method with $J = 10^4$ number of Monte Carlo samples. Consequently, we need to solve the following parameter-dependent problem: for $1 \leq j \leq J$

$$\begin{aligned} -\delta\Delta u(\omega_j, x, y) + \mathbf{b}(\omega_j, x, y) \cdot \nabla u(\omega_j, x, y) &= f(\omega_j, x, y) \quad \text{in } D = (0, 1)^2 \\ u(\omega_j, \cdot) &= 0 \quad \text{on } \partial D. \end{aligned} \quad (6.53)$$

To fit the setup of Section 6.6.1, we define

$$\begin{aligned}\mathbf{b}_0(x, y) &:= \mathbb{E}_{mc}[\mathbf{b}(\cdot, x, y)] = 2.001(2y(1 - x^2), -2x(1 - y^2)), \\ \boldsymbol{\eta}(\omega_j, x, y) &:= \mathbf{b}(\omega_j, x, y) - \mathbf{b}_0(x, y),\end{aligned}$$

and the following bilinear forms and functional:

$$\begin{aligned}A(\omega_j; u, v) &= \delta(\nabla u, \nabla v) + (\mathbf{b} \cdot \nabla u, v), \quad F(\omega_j; v) = (f, v), \\ A_0(u, v) &= \delta(\nabla u, \nabla v) + (\mathbf{b}_0 \cdot \nabla u, v), \quad A_1(\omega_j; u, v) = (\boldsymbol{\eta} \cdot \nabla u, v).\end{aligned}$$

Table 6.10 and Figure 6.12 show respectively the convergence orders of the spatial discretization error and the iteration errors in both H^1 - and L^2 -norm. We observe that the proposed iterative method performs well for the random convection-dominated convection-diffusion problem (6.51). From Table 6.10 we see that the optimal convergence orders are achieved for the linear finite element method. In addition, Figure 6.12 shows that the H^1 - and L^2 -norm errors of the iterative solution rapidly decrease as the number of iterations increases.

Table 6.10: 2-D random convection-diffusion problem: Errors $\mathcal{E}_{H^1}^n$ and $\mathcal{E}_{L^2}^n$ with $\varepsilon = 2.0$, $n = 10$ and various h .

h	$\mathcal{E}_{H^1}^n$	Order	$\mathcal{E}_{L^2}^n$	Order
0.2	5.4020×10^{-3}		3.4360×10^{-4}	
0.1	2.7339×10^{-3}	0.9825	8.2969×10^{-5}	2.0501
0.05	1.3709×10^{-3}	0.9958	1.7666×10^{-5}	2.2316
0.025	6.8635×10^{-4}	0.9806	3.8150×10^{-6}	2.2112

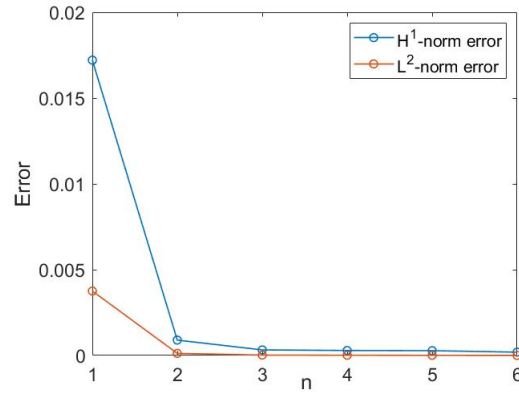


Figure 6.12: 2-D convection-diffusion problem: Errors $\mathcal{E}_{H^1}^n$ and $\mathcal{E}_{L^2}^n$ with $h = 0.01$, $\varepsilon = 2.0$, $\text{tol} = 10^{-4}$ and various n .

Chapter 7

Future work

Based on my research experience and results, I am currently working on and/or plan to pursue the following research projects.

- *Mixed finite element methods for the stochastic Navier-Stokes equations with general multiplicative noise.* I am currently working on this project. We found that using the same approach as in [7] for the stochastic Navier-Stokes equations with additive noises to treat the multiplicative noise case does not work. Therefore, we are exploiting other new approaches in this project including using the linearized Navier-Stokes operator and fixed point argument which are inspired by [30]. In addition, numerical experiments using multiplicative noises show that the standard mixed FEM is working well in approximating the equations. We are expecting to obtain some strong convergence rates for the mixed FEM in this project.
- *Splitting methods for the stochastic Navier-Stokes equations with general multiplicative noise.* We plan to extend the results of the Chorin projection method of the stochastic Stokes equations in [35] to the Navier-Stokes equations. The difference between the two models is obviously the nonlinear term in the Navier-Stokes equations. In [4], the authors reused the idea of sub-sample spaces from [16] to deal with the nonlinearity, but they have only obtained a rate of convergence in probability. We are expecting to obtain some strong convergence with a polynomial order of convergence for the Chorin projection method in this project.

- *Convergence rates for finite element (FE) approximations of the two-phase fluid flows of the stochastic Navier-Stokes-Cahn-Hilliard system.* This project is inspired by the deterministic counterpart [26]. Recently, the convergence of the fully discrete FEM for this system was obtained in [21]. We are expecting to obtain strong rates of convergence for the FE approximations of this model.
- *Convergence rates of the stochastic total variation flow with multiplicative noises.* We are currently working on this project. We extend the deterministic counterpart [32], [31] to this stochastic version. The PDE solution concept has been established by [63], so in this project we would like to focus on numerical part. Namely, we are expecting to obtain the convergence rates of the FE approximations of the stochastic total variation flow.
- *High order time discretization schemes for stochastic PDEs.* So far, the implicit Euler-Maruyama (EM) scheme has been used for time discretization in all my above mentioned works, which only gives $\frac{1}{2}$ -order of convergence. We plan to extend our EM scheme results to potentially high order methods such as Milstein and Runge-Kutta time-stepping schemes.
- *Discontinuous Galerkin (DG) methods for the stochastic Stokes and Navier-Stokes equations.* We plan to extend our finite element results for the stochastic Stokes and Navier-Stokes equations to DG spatial discretization methods. We expect that the extension will be very interesting and plausible.
- *Numerical methods for stochastic homogenization.* This is a very interesting research topic to me. We plan to extend our iterative framework discussed in Chapter 6 to stochastic homogenization problems. Our main idea is to develop more efficient methods and algorithm for approximating the homogenized coefficients which is very expensive to do using current known approaches.

Moreover, I am also open-minded and like to learn new research topics and to enter new research areas, such as deep neural networks and machine learning. Especially, I look forward to use my mathematical and computational knowledge to tackle new exciting projects.

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Appendix

A Mixed finite element method for stochastic heat equation

In this appendix, we present error estimates for mixed finite element approximations of the following stochastic heat equation with general multiplicative noise:

$$\frac{du}{dt} = \Delta u + B(u) \frac{dW(t)}{dt} \quad \text{in } D_T := (0, T) \times D, \quad (1a)$$

$$u = 0 \quad \text{on } \partial D_T := (0, T) \times \partial D, \quad (1b)$$

$$u(0) = u_0 \quad \text{in } D, \quad (1c)$$

where D is a bounded domain in $\mathbb{R}^d$ for $d = 1, 2, 3$. $\{W(t); t > 0\}$ is a real-valued Wiener process.

The mixed formulation of (1) introduces a vector-valued stochastic process $\mathbf{p} = -\nabla u$ to convert the underlying equation into a system of first-order linear equations as follows:

$$\mathbf{p} = -\nabla u \quad \text{in } D_T, \quad (2a)$$

$$\frac{du}{dt} = -\operatorname{div} \mathbf{p} + B(u) \frac{dW(t)}{dt} \quad \text{in } D_T, \quad (2b)$$

$$u = 0 \quad \text{on } \partial D_T, \quad (2c)$$

$$u(0) = u_0 \quad \text{in } D. \quad (2d)$$

A.1 Mixed variational formulation

We define the following space

$$\mathbf{H}(\operatorname{div}; D) := \{\mathbf{v} \in \mathbf{L}^2(D); \operatorname{div} \mathbf{v} \in L^2(D)\}.$$

The variational formulation of (2) is defined as follows: Find $(\mathbf{p}, u) \in L^2(\Omega, \mathbf{H}(\text{div}; D)) \times L^2(\Omega, L^2(D))$ such that $\mathbb{P}$ -a.s.

$$(\mathbf{p}, \mathbf{v}) - (\text{div } \mathbf{v}, u) = 0 \quad \forall \mathbf{v} \in \mathbf{H}(\text{div}; D), \quad (3)$$

$$(u(t), \phi) + \int_0^t (\text{div } \mathbf{p}(s), \phi) ds = (u_0, \phi) + \int_0^t (B(u(s)), \phi) dW(s) \quad \forall \phi \in L^2(D). \quad (4)$$

A.2 Discretization in time

Let $N > 0$, define a uniform mesh that covers $[0, T]$ with time step $k = \frac{T}{N}$ and $t_n = nk$ for $n = 0, 1, \dots, N$. Set $u^0 = u_0$, the Euler-Maruyama method for (3)-(4) is defined by seeking $(\mathbf{p}^{n+1}, u^{n+1}) \in L^2(\Omega, \mathbf{H}(\text{div}; D)) \times L^2(\Omega, L^2(D))$ such that $\mathbb{P}$ -a.s.

$$(\mathbf{p}^{n+1}, \mathbf{v}) - (\text{div } \mathbf{v}, u^{n+1}) = 0 \quad \forall \mathbf{v} \in \mathbf{H}(\text{div}; D), \quad (5)$$

$$(u^{n+1}, \phi) + k(\text{div } \mathbf{p}^{n+1}, \phi) = (u^n, \phi) + (B(u^n)\Delta W_{n+1}, \phi) \quad \forall \phi \in L^2(D), \quad (6)$$

where $\Delta W_{n+1} := W(t_{n+1}) - W(t_n) \sim N(0, k)$.

We also can write (5), (6) in the following strong form:

$$\mathbf{p}^{n+1} = -\nabla u^{n+1}, \quad (7)$$

$$u^{n+1} + k \text{div } \mathbf{p}^{n+1} = u^n + B(u^n)\Delta W_{n+1}. \quad (8)$$

Substituting (7) into (8) we obtain the Euler-Maruyama method for (1)

$$u^{n+1} - k\Delta u^{n+1} = u^n + B(u^n)\Delta W_{n+1}. \quad (9)$$

Therefore, (5)-(6) are equivalent to (9). We quote the following error and stability estimates from [41] for the time discretization.

Lemma A.2.1. *Let $\{u^n\}$ be the solution of (9). Then there exists a constant $C = C(u_0, D_T) > 0$ such that*

$$\max_{0 \leq n \leq N} \mathbb{E}[\|u^n\|_{H^1}^2] + \mathbb{E}\left[k \sum_{n=1}^N \|u^n\|_{H^2}^2\right] \leq C. \quad (10)$$

Remark A.2.1. *Since $\mathbf{p}^n = -\nabla u^n$, (10) immediately infers the following estimate for $\mathbf{p}^n$:*

$$\max_{0 \leq n \leq N} \mathbb{E}[\|\mathbf{p}^n\|_{\mathbf{L}^2}^2] + \mathbb{E}\left[k \sum_{n=1}^N \|\mathbf{p}^n\|_{\mathbf{H}^1}^2\right] \leq C. \quad (11)$$

Theorem A.2.2. *Let $(\mathbf{p}, u)$ and $(\mathbf{p}^n, u^n)$ be the solutions of (3)-(4) and (5)-(6) respectively. Then there exists a constant $C = C(u_0, D_T) > 0$ such that*

$$\max_{1 \leq n \leq N} \mathbb{E}[\|u(t_n) - u^n\|_{L^2}^2] + \mathbb{E}\left[k \sum_{n=1}^N \|\mathbf{p}(t_n) - \mathbf{p}^n\|_{\mathbf{L}^2}^2\right] \leq Ck. \quad (12)$$

A.3 Discretization in space

Let $\mathcal{T}_h$ be a quasi-uniform triangular mesh for D with mesh size h . Define

$$\mathbf{V}_h = \{\mathbf{v}_h \in \mathbb{C}(\overline{D}); \mathbf{v}_{h|_K} \in RT_1(K) \ \forall K \in \mathcal{T}_h\}, \quad (13)$$

$$Q_h = \{\phi_h \in L^2(D); \phi_{h|_K} \in P_1(K) \ \forall K \in \mathcal{T}_h\}, \quad (14)$$

where $RT_1(K) = \mathbf{P}_1(K) \oplus \mathbf{x}P_1(K)$, and $P_1(K)$ is the set of polynomials of degree at most 1. We recall the following Raviart-Thomas [61] finite element spaces

$$\mathbf{M}_h = \mathbf{V}_h \cap \mathbf{H}(\text{div}; D) \text{ and } X_h = Q_h. \quad (15)$$

It is known that $(\mathbf{M}_h, X_h)$ satisfies the discrete inf-sup condition [9, 10].

The mixed finite element method for (5)-(6) (or (9)) is defined by seeking $(\mathbf{p}_h^{n+1}, u_h^{n+1}) \in L^2(\Omega, \mathbf{M}_h \times X_h)$ such that $\mathbb{P}$ -a.s.

$$(\mathbf{p}_h^{n+1}, \mathbf{v}_h) - (\operatorname{div} \mathbf{v}_h, u_h^{n+1}) = 0 \quad \forall \mathbf{v}_h \in \mathbf{M}_h, \quad (16)$$

$$(u_h^{n+1}, \phi_h) + k(\operatorname{div} \mathbf{p}_h^{n+1}, \phi_h) = (u_h^n, \phi_h) + (B(u_h^n) \Delta W_{n+1}, \phi_h) \quad \forall \phi_h \in X_h, \quad (17)$$

where $\Delta W_{n+1} = W(t_{n+1}) - W(t_n)$.

Recall that the L^2 -projection $P_h : L^2(D) \rightarrow X_h$ is defined by $\forall \psi \in L^2(D)$

$$(\psi - P_h \psi, \phi_h) = 0 \quad \forall \phi_h \in X_h. \quad (18)$$

In addition, we also recall the Fortin operator [10], $\pi_h : \mathbf{H}(\operatorname{div}; D) \rightarrow \mathbf{M}_h$ by

$$(\operatorname{div}(\pi_h \mathbf{v} - \mathbf{v}), \psi_h) = 0 \quad \forall \psi_h \in X_h. \quad (19)$$

Finally, we recall the following standard inequalities [10]

$$\|\mathbf{v} - \pi_h \mathbf{v}\|_{\mathbf{L}^2} \leq Ch^m \|\mathbf{v}\|_{\mathbf{H}^m} \quad \forall \mathbf{v} \in \mathbf{H}(\operatorname{div}; D), \quad (20)$$

$$\|\phi - P_h \phi\|_{L^2} + h \|\nabla(\phi - P_h \phi)\|_{L^2} \leq Ch^2 \|\phi\|_{H^2} \quad \forall \phi \in L^2(D), \quad (21)$$

where $m = 1, 2$.

Next we derive error estimates between $\{(\mathbf{p}^n, u^n)\}$ and $\{\mathbf{p}_h^n, u_h^n\}$ in the following theorem.

Theorem A.3.1. *Let $\{(\mathbf{p}^n, u^n)\}$ and $\{(\mathbf{p}_h^n, u_h^n)\}$ be the solutions of (3)-(4) and (16)-(17) respectively. Then there holds*

$$\max_{1 \leq n \leq N} \mathbb{E}[\|u^n - u_h^n\|_{L^2}^2] + \mathbb{E}\left[k \sum_{n=1}^N \|\mathbf{p}^n - \mathbf{p}_h^n\|_{\mathbf{L}^2}^2\right] \leq C h^2, \quad (22)$$

where $C = C(u_0, D_T) > 0$.

Proof. Define $e^n := u^n - u_h^n$ and $\boldsymbol{\tau}^n := \mathbf{p}^n - \mathbf{p}_h^n$ which can be decomposed as follows:

$$e^n = (u^n - P_h u^n) + (P_h u^n - u_h^n) = \theta^n + \varepsilon^n, \quad (23)$$

$$\boldsymbol{\tau}^n = (\mathbf{p}^n - \pi_h \mathbf{p}^n) + (\pi_h \mathbf{p}^n - \mathbf{p}_h^n) = \boldsymbol{\eta}^n + \boldsymbol{\xi}^n. \quad (24)$$

Then, we obtain the following error equations by subtracting (3)-(4) from (16)-(17)

$$(e^{n+1} - e^n, \phi_h) + k(\operatorname{div} \boldsymbol{\tau}^{n+1}, \phi_h) = ((B(u^n) - B(u_h^n))\Delta W_{n+1}, \phi_h) \quad \forall \phi_h \in X_h, \quad (25)$$

$$(\boldsymbol{\tau}^{n+1}, \mathbf{v}_h) = (\operatorname{div} \mathbf{v}_h, e^{n+1}) \quad \forall \mathbf{v}_h \in \mathbf{M}_h. \quad (26)$$

Choosing $\phi_h = \varepsilon^{n+1} \in X_h$ in (25) and $\mathbf{v}_h = \boldsymbol{\xi}^{n+1} \in \mathbf{M}_h$ in (26) we obtain

$$(e^{n+1} - e^n, \varepsilon^{n+1}) + k(\operatorname{div} \boldsymbol{\tau}^{n+1}, \varepsilon^{n+1}) = ((B(u^n) - B(u_h^n))\Delta W_{n+1}, \varepsilon^{n+1}), \quad (27)$$

$$(\boldsymbol{\tau}^{n+1}, \boldsymbol{\xi}^{n+1}) = (\operatorname{div} \boldsymbol{\xi}^{n+1}, e^{n+1}). \quad (28)$$

Moreover, by using (18) and (19), we can rewrite the right-hand side of (27) as follows:

$$\begin{aligned} (e^{n+1} - e^n, \varepsilon^{n+1}) &= (\theta^{n+1} - \theta^n, \varepsilon^{n+1}) + (\varepsilon^{n+1} - \varepsilon^n, \varepsilon^{n+1}) \\ &= (\varepsilon^{n+1} - \varepsilon^n, \varepsilon^{n+1}) \\ &= \frac{1}{2}[\|\varepsilon^{n+1}\|_{L^2}^2 - \|\varepsilon^n\|_{L^2}^2 + \|\varepsilon^{n+1} - \varepsilon^n\|_{L^2}^2]. \end{aligned} \quad (29)$$

$$\begin{aligned} (\operatorname{div} \boldsymbol{\tau}^{n+1}, \varepsilon^{n+1}) &= (\operatorname{div} \boldsymbol{\eta}^{n+1}, \varepsilon^{n+1}) + (\operatorname{div} \boldsymbol{\xi}^{n+1}, \varepsilon^{n+1}) \\ &= (\operatorname{div} \boldsymbol{\xi}^{n+1}, \varepsilon^{n+1}). \end{aligned} \quad (30)$$

From equation (28), we get

$$\begin{aligned} (\operatorname{div} \boldsymbol{\xi}^{n+1}, \varepsilon^{n+1}) &= -(\operatorname{div} \boldsymbol{\xi}^{n+1}, \theta^{n+1}) + \|\boldsymbol{\xi}^{n+1}\|_{\mathbf{L}^2}^2 + (\boldsymbol{\eta}^{n+1}, \boldsymbol{\xi}^{n+1}) \\ &= (\boldsymbol{\xi}^{n+1}, \nabla \theta^{n+1}) + \|\boldsymbol{\xi}^{n+1}\|_{\mathbf{L}^2}^2 + (\boldsymbol{\eta}^{n+1}, \boldsymbol{\xi}^{n+1}). \end{aligned} \quad (31)$$

Substituting (31) into (30) we obtain

$$(\operatorname{div} \boldsymbol{\tau}^{n+1}, \varepsilon^{n+1}) = (\boldsymbol{\xi}^{n+1}, \nabla \theta^{n+1}) + \|\boldsymbol{\xi}^{n+1}\|_{\mathbf{L}^2}^2 + (\boldsymbol{\eta}^{n+1}, \boldsymbol{\xi}^{n+1}). \quad (32)$$

Next, substituting (29) and (32) into (4.34) yields

$$\begin{aligned} & \frac{1}{2} [\|\varepsilon^{n+1}\|_{L^2}^2 - \|\varepsilon^n\|_{L^2}^2 + \|\varepsilon^{n+1} - \varepsilon^n\|_{L^2}^2] + k \|\boldsymbol{\xi}^{n+1}\|_{\mathbf{L}^2}^2 \\ &= -k(\boldsymbol{\xi}^{n+1}, \nabla \theta^{n+1}) - k(\boldsymbol{\eta}^{n+1}, \boldsymbol{\xi}^{n+1}) + ((B(u^n) - B(u_h^n))\Delta W_{n+1}, \varepsilon^{n+1}) \\ &:= \text{I} + \text{II} + \text{III}. \end{aligned} \quad (33)$$

By using Cauchy-Schwarz inequality, we can bound the right-hand side of (33) as follows.

$$\begin{aligned} \text{I} + \text{II} &\leq \frac{k}{2} \|\boldsymbol{\xi}^{n+1}\|_{\mathbf{L}^2}^2 + k \|\nabla \theta^{n+1}\|_{\mathbf{L}^2}^2 + k \|\boldsymbol{\eta}^{n+1}\|_{\mathbf{L}^2}^2. \\ \text{III} &\leq \|(B(u^n) - B(u_h^n))\Delta W_{n+1}\|_{L^2}^2 + \frac{1}{4} \|\varepsilon^{n+1} - \varepsilon^n\|_{L^2}^2 \\ &\quad + ((B(u^n) - B(u_h^n))\Delta W_{n+1}, \varepsilon^n). \end{aligned} \quad (34)$$

Substituting (34) into (33), taking the expectation, and absorbing the like-terms from the left-hand side to the right-hand side we obtain

$$\begin{aligned} & \frac{1}{2} \mathbb{E} [\|\varepsilon^{n+1}\|_{L^2}^2 - \|\varepsilon^n\|_{L^2}^2] + \frac{1}{4} \mathbb{E} [\|\varepsilon^{n+1} - \varepsilon^n\|_{L^2}^2] + \frac{k}{2} \mathbb{E} [\|\boldsymbol{\xi}^{n+1}\|_{\mathbf{L}^2}^2] \\ &\leq k \mathbb{E} [\|\nabla \theta^{n+1}\|_{\mathbf{L}^2}^2 + \|\boldsymbol{\eta}^{n+1}\|_{\mathbf{L}^2}^2] + \mathbb{E} [\|(B(u^n) - B(u_h^n))\Delta W_{n+1}\|_{L^2}^2] \\ &\leq Ckh^2 \mathbb{E} [\|u^{n+1}\|_{H^2}^2] + Ckh^2 \mathbb{E} [\|\mathbf{p}^{n+1}\|_{\mathbf{H}^1}^2] + Ck \mathbb{E} [\|\varepsilon^n\|_{L^2}^2], \end{aligned} \quad (35)$$

where we obtain the first inequality of (35) by using the fact that $\mathbb{E}[(B(u^n) - B(u_h^n))\Delta W_{n+1}, \varepsilon^n] = 0$, and the second inequality by using (20) and (21).

Now, summing over $0 \leq n \leq \ell$ for $0 \leq \ell \leq N - 1$ on both sides of (35) we obtain

$$\begin{aligned} & \mathbb{E}[\|\varepsilon^{\ell+1}\|] + k \sum_{n=0}^{\ell} \mathbb{E}[\|\boldsymbol{\xi}^{n+1}\|_{\mathbf{L}^2}^2] \\ & \leq Ch^2 \mathbb{E}\left[k \sum_{n=0}^{N-1} \|u^{n+1}\|_{H^2}^2 + \|\mathbf{p}^{n+1}\|_{\mathbf{H}^1}^2\right] + Ck \sum_{n=0}^{\ell} \mathbb{E}[\|\varepsilon^n\|_{L^2}^2]. \end{aligned} \quad (36)$$

It follows from the discrete Gronwall inequality and the stability estimate (10) that

$$\mathbb{E}[\|\varepsilon^{\ell+1}\|] + k \sum_{n=0}^{\ell} \mathbb{E}[\|\boldsymbol{\xi}^{n+1}\|_{\mathbf{L}^2}^2] \leq Ch^2 \mathbb{E}\left[k \sum_{n=0}^{N-1} \|u^{n+1}\|_{H^2}^2 + \|\mathbf{p}^{n+1}\|_{\mathbf{H}^1}^2\right] e^{CT} \leq Ch^2. \quad (37)$$

Finally, the desired inequality is obtained by taking maximum over $0 \leq \ell \leq N - 1$ and then use triangular inequality on the decompositions (23) and (24). The proof is complete. $\square$

Now, we are ready to give a global error estimate in the following theorem.

Theorem A.3.2. *Let $\{(\mathbf{p}, u)\}$ and $\{(\mathbf{p}_h^n, u_h^n)\}$ be the solutions of (3)-(4) and (16)-(17) respectively. Then there holds*

$$\max_{1 \leq n \leq N} \mathbb{E}[\|u(t_n) - u_h^n\|_{L^2}^2] + \mathbb{E}\left[k \sum_{n=1}^N \|\mathbf{p}(t_n) - \mathbf{p}_h^n\|_{\mathbf{L}^2}^2\right] \leq C(k + h^2), \quad (38)$$

where $C = C(u_0, D_T) > 0$.

Proof. The proof is followed directly by applying the triangular inequality and Theorem A.2.2, Theorem A.3.1. $\square$

Remark A.3.1. *Theorem A.3.2 shows that unlike the stochastic Stokes and Navier-Stokes equations, the mixed finite element method works well for the stochastic heat equation because the error bound in (38) does not contain the bad factor $k^{-\frac{1}{2}}$ that appears in the error estimates for the stochastic Stokes and Navier-Stokes equations.*

Vita

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