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FAILURE AND FRACTURE ANALYSIS OF BRITTLE AND DUCTILE
COMPOSITES USING THE COMPUTER SIMULATION METHOD

A Dissertation
Presented for the
Doctor of Philosophy
Degree
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Kamalaksha Sarkar

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DEDICATION

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In the glorious memory of my parents, Kamal.

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ABSTRACT

A computer simulation method was developed to study the failure and fracture behavior of both brittle and ductile composite materials.

Unidirectional and cross-ply glass fiber-reinforced epoxy (GFRE) composites with approximately 50% reinforcement were investigated to predict their unnotched strengths. Relations were also developed to predict the notched strengths and the critical stress intensity factors for off-axis composites. The agreement between the predicted and the experimental results was very good. It was found that there was a critical region beyond which the singular, continuum solution was not valid. For brittle GFRE composites, they were reasonably constant. The typical magnitude of the critical region was 10^{-2} in. for these composites.

Considering the importance of the changing order of the singularity of the stresses at the crack tip, a gradually graded finite element method was developed to study the failure of ductile composites. It was shown that the simple method took a reasonable amount of computer time to solve elasto-plastic problems using conventional elements. The Tsai-Wu failure theory of brittle composites was used to predict the initial yielding of ductile composites. An anisotropic plasticity analysis was done using isotropic-hardening and associated flow rules. The material was assumed to be work-hardening.

A fracture specimen was monotonically loaded until failure was reached in at least one of its elements. The corresponding fracture parameters were assumed to be the critical fracture parameters for crack initiation.

A relationship was developed between the J-integral and the crack-opening displacement (COD) using small-scale deformation in a strain-hardening material. Using this relation and the failure loads predicted from the computer simulation, it was concluded that the critical J, being dependent on fiber orientations (unidirectional) and crack lengths (cross-ply), was not a material parameter for B-A1 composites. Conversely, the value of COD at failure was reasonably constant for all the fiber orientations (unidirectional) and crack lengths (cross-ply) investigated; thus, it could be considered to be a material parameter.

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LIST OF SYMBOLS

A	Area under load displacement curve
$\bar{A}$	Slope of uniaxial stress versus plastic strain
a	Semi-crack length for center cracked tension specimen; also crack length for compact tension (CT) specimen
a_e	Effective crack length
a_{ij}	Functions of elastic moduli
B	Thickness
$[B]$	Strain-displacement matrix
b	Semi-bandwidth
C	Compliance
C'	Compliance per unit thickness
$[C]$	Displacement matrix to relate elemental and nodal values
$\bar{C}$	Distance of crack tip from neutral axis in a CT specimen
c	Constant
$[D]$	Stress-strain matrix
$[D]_{ep}$	Elasto-plastic matrix
d	Fiber diameter
$E, \bar{E}$	Elastic modulus
E'	Total number of elements
E_p, E_s	Primary and secondary elastic moduli
F	Generalized yield surface
F_i, F_{ij}	Strength parameter
f	Tsai-Wu failure surface
f_i	Functions of elastic moduli

G	Shear modulus
G	Strain energy release rate
g_i	Functions of elastic moduli
H'	Number of hybrid elements
h_i	Size of a finite element
$\bar{I}$	Moment of inertia
I_n	Numerical constant
I_m	Imaginary part of a complex function
i	Number of iterations
J	J-integral
K	Stress intensity factor
$[K]$	Stiffness matrix
k	Material constant
L	Debonded length
L_c	Critical length
M	Constant
$2m$	Order of the differential equation
N	Number of nodes
n	Strain-hardening parameter
P	Load
$\{Q\}$	External force vector
$\{\Delta Q_0\}_i^{j+1}$	Residual nodal forces
$\{q\}$	Nodal displacement vector
Re	Real part of a complex function
S	Shear strength
S	Strain energy density

S_{ij}	Deviatoric stress
s	Arc length
s_i	Complex roots
T_i	Surface traction
U	Potential energy
u_i	Deformations
V	Volume
W	Strain energy density
X, Y	Normal strengths
x, y, r, θ	Coordinates
CFD	Crack face displacement
COD	Crack-opening displacement
LLD	Load line deflection
LPD	Load point deflection
σ	Normal stress
τ	Shear stress
ϵ	Strain
α	Strain-hardening function
β	Order of singularity
ψ	Functions of angular coordinates
θ	Fiber orientation with respect to the crack axis
λ	Constant
ϕ	Contact angle
δ	Crack-opening displacement (COD)
d, δ, Δ	Small variations of a parameter
γ	Plastic work done for unit crack area

Subscripts

C,c	Critical value
D	Debonding
e	Elastic or effective value
f	Fiber or failure value
h	Horizontal component
i,j	Directional value
L	Longitudinal value
LT	Planar value
MPO	Matrix pull out
m	Matrix
notch	Notched value
PO	Pull out
p	Plastic
r	Residual value
SR	Stress relaxation
T	Transverse value
v	Vertical component
x,y,xy	Directional value
θ	Direction of fiber reinforcement
I,II	Modes I and II
0	Applied value or fiber orientation
80,90	Fiber orientation
-,+	Negative and positive values
-,=,~	Variation of a given function

CHAPTER I

INTRODUCTION

Making composites is an old art but a new science. The concept of using two or more elemental materials to form a "new material" is age-old [1].¹ It was recognized even thousands of years back that the new materials exhibit a combination of properties which cannot be achieved by simply adding the elemental properties. This type of man-made "new material" has been identified [2] as "composite material." The main load-carrying elements of the composite materials considered in this investigation are filaments or fibers. The other element which binds the fibers and transfers the load to them is known as the matrix material.

The development of glass fibers [3] in the early 1940's allowed high specific strength composite materials to be fabricated which found their first applications in the aerospace industry. In the 1960's, the subsequent introduction of new fibers like carbon, boron, Kevlar, SiC, alumina, and BN made further impact by allowing both high specific strength and stiffness composite materials to be fabricated. These types of composite materials are known as "advanced composite materials." The advanced composites made it possible to use metal (aluminum, magnesium, lead, etc.) as the matrix material, in addition to the earlier matrix materials such as epoxies, phenolics, polyesters,

¹Numbers in brackets refer to similarly numbered references in the List of References.

polyimides, and so forth. Industries are now innovating new means to use composite materials to save weight, and hence, energy. Composites are now used by almost every industry, including aerospace, automobile, chemical, and recreational, to name a few. The glass fiber-reinforced plastics are the most widely used composite system. Recently, advanced composites [4] are also being used in many industrial applications.

The major advantages of advanced composite materials include high specific modulus, high specific strength, and tailor-made properties. By choosing a proper combination of fiber and matrix materials, their percent volumes and orientation of fibers, a wide range of material properties can be achieved along any desired direction. This is not possible with any conventional structural material. This choice of properties has drawn the material scientists and structural designers together for the first time to work in close collaboration.

Although there are a large number of composite systems available (by varying matrix or fiber, or both), they can be classified broadly as brittle composite materials and ductile composite materials. Brittle composites fail in a brittle manner (i.e., the stress strain curve is effectively linear until failure). A typical example is longitudinal glass fiber-reinforced epoxy (GFRE), where the fiber direction coincides with the direction of loading. On the other hand, many composites fail in a ductile manner (i.e., they exhibit a considerable amount of nonlinearity in their stress-strain responses). A typical example is the unidirectional boron-aluminum (B-Al) composite, where the boron fibers are at 45° with the loading direction.

Interestingly, along with the rapid growth in the use of advanced composites, the last two decades also saw a similar growth in the science of fracture mechanics. In fracture mechanics the quantitative effect of the presence of a crack or discontinuity on the strength of a structure is studied. The success of fracture mechanics to predict the strength of a notched, brittle, isotropic, homogeneous material has made its acceptance in the structural design in a routine way. This linear treatment of fracture mechanics is popularly known as linear elastic fracture mechanics (LEFM). For these materials the stresses ahead of a sharp crack are singular in nature and they can be described by a material constant known as a stress intensity factor (K) [5]. At failure, the critical value of this constant is a material parameter known as fracture toughness (K_{IC}). Depending upon the nature of loading, an isotropic, homogeneous material may fail in three possible modes; these are shown in Fig. 1.² Accordingly, the stress intensity factor assumes three possible values. Standard experimental methods [6] have been established to determine the fracture toughness of metallic materials in Mode I (so-called crack-opening mode, shown in Fig. 1) under plane strain condition (K_{IC}). Under the testing conditions, the failure is brittle and the crack propagates unstably without any appreciable plastic deformation at the tip of the crack. For metallic materials, the crack extends in a self-similar [7] manner, meaning the crack extends in the same plane of the crack without changing the direction of crack growth. Since in the fracture mechanics approach, the crack is assumed to be extremely sharp, and in

²All figures appear in Appendix A.

actual designs, all the cutouts and discontinuities are rounded off, designs based on critical fracture toughness, in general, are conservative. However, fatigue cracks in metallic materials, and debonding or fiber cracks in composite materials, may lead to sharp cracks.

Another important fracture parameter is known popularly as the strain energy release rate (unfortunately, a misnomer). This is really the rate of potential energy per unit thickness needed to extend a crack by unit length (G). For brittle fracture the strain energy release rate (G) is related directly to the stress intensity factor (K). The critical value of G , and hence, K , for which an existing crack grows unstably under loading, is a material constant (G_c or K_{Ic}).

As the stress-strain relationship deviates from the linearity, the degree of plasticity at the crack tip becomes important in determining the fracture toughness (K_{Ic}). Although it is not yet established if the plastic work done per unit increase in crack area (γ_p) is a material constant [9], a modified relation has been developed to predict the failure of an isotropic, homogeneous material which exhibits considerable plastic deformation at the crack tip before failure [10]. Rice [11] has postulated a path-independent, two-dimensional energy integral (J-integral) for nonlinear elasticity, which has been used to predict fracture of materials in the plastic range [12, 13]. The critical value of J that is obtained at fracture is regarded as a material constant.

Another popular approach to predict failure of notched, metallic materials in the nonlinear range is known as the critical value of crack-opening displacement, or COD_c [8]. It is postulated

that, for a given temperature, strain rate, and thickness, the unstable fracture of a notched specimen occurs when the deformation at the crack tip (popularly known as crack-opening displacement, or COD), reaches a critical value. This is proposed to be a constant, regardless of the specimen size and degree of yield at the crack tip for a given thickness.

It is interesting to note that, even without conclusive theoretical justification [9, 14], both the J-integral and the COD approaches are quite popular [15 through 16], even in the general plastic regime, at least for predicting the failure of isotropic, homogeneous materials. It is also believed that fracture toughness (K_{IC}) can be used to predict the strength of a notched specimen of brittle composites, since the linearity is effectively preserved in the stress-strain relationship [19, 31, 63, 66, 70, 71].

Care has to be taken in applying the fracture mechanistic concepts developed for isotropic, homogeneous materials to multiphase, composite materials. A material model is necessary to use these concepts. Composite materials are fabricated from more than one phase. Thus, each and every individual phase influences the overall fracture properties of the composite system. The fracture is further influenced by the fabrication process needed to make the composite. The failure in a composite may initiate from a broken fiber, a flaw in the matrix, a debonded interface, or a combination of these. Thus, failure and fracture in a composite material are complex functions of the constituents, their state of combination, combination process, and the mode of loading. Any attempt to include all the factors in an analytical model is overwhelming. Also, the basic purpose of any

scientific investigation is to incorporate maximum information in a rational way without foregoing the overall, operational simplicity [7]. Thus, any failure and fracture analysis of a composite material should reflect the overall response of the material in a simple manner. While the proposed analysis need not account for all of the failure mechanisms individually, it should reflect the overall failure behavior, at least on a macroscopic level. This is what the designers of composite materials want [18]. They need the ability to predict the strength of a composite structure routinely and reliably in an operationally simple way.

To achieve the goal, the composite material is, generally, modeled in a microscopic or macroscopic manner. In microscopic modeling, the composite material is assumed to be an isotropic, heterogeneous, multiphase material. Unfortunately, the method cannot be used except for very simple cases, such as unidirectional fiber orientation [19] or regular arrays [20 through 22]. Macroscopic modeling is, basically, a phenomenological theory. Consequently, it has the advantage of being used for an arbitrary composite system [23 through 26], once the basic aggregate properties are identified from experimental results for a given composite material.

In this investigation, the macroscopic material modeling was used in order to study the off-axis, anisotropic fracture and failure behavior of unidirectionally reinforced and $0^\circ/90^\circ$ cross-ply composites. For unidirectional composites the anisotropy arose from the change of orientation of the fibers with respect to the loading direction, which was always perpendicular to the crack axis (Figs. 2 and 3). Two

typical composite systems were chosen to predict failure in the presence of a crack.

The brittle composite systems studied in this investigation were unidirectional and $0^\circ/90^\circ$ cross-ply glass fiber-reinforced epoxy (GFRE). The notched strength and critical stress intensity factors were calculated and compared with the experimental results of Kulkarni [19] for different orientations of the fibers and the crack lengths.

Unidirectional and $0^\circ/90^\circ$ cross-ply B-Al composites were investigated to study the effects of plasticity of ductile composite systems. Two popular fracture specimens studied in this investigation were center-cracked tension (CCT) and compact tension (CT) specimens (Figs. 2 and 3).

For center-cracked tension (CCT) specimens of unidirectional B-Al composites, two crack lengths ($2a$) were chosen as 0.2 in. and 0.4 in. for 1-in.-wide and 3-in.-long specimens. Six fiber orientations ($\theta = 90^\circ, 0^\circ, 30^\circ, 45^\circ, 60^\circ, \text{ and } 75^\circ$) were chosen to study the effect of anisotropy.

Compact tension specimens were used to study the fracture behavior of $0^\circ/90^\circ$ cross-ply B-Al composites. The results were compared with the experimental results of Jollay [66]. Four different crack lengths (crack lengths to width ratio = 0.2, 0.4, 0.6, and 0.8) were chosen to study the effect of the crack length on the failure load and fracture parameters like J-integral and crack-opening displacement (COD). Further studies were made to investigate the effects of the stiffness and partial contact of the pin on the fracture behavior of the material. In all of these cases, the composite specimens were

loaded along the longitudinal strength direction ($\theta = 90^\circ$) and the crack was oriented parallel to the transverse strength direction ($\theta = 90^\circ$).

In all of these analyses, the plain stress, finite element method was used as the solution technique. An anisotropic failure theory, as proposed by Tsai and Wu [29], was used to predict failure and fracture of both the ductile and the brittle composite systems investigated here.

CHAPTER II

REVIEW OF RELATED LITERATURE

In this chapter, only the related literature relevant to the work carried out is reviewed critically. For this purpose, the chapter is divided into five sections. In Section A, the two material models (microscopic and macroscopic) are discussed, along with their respective advantages and disadvantages. In Section B, the pertinent failure theories of composite materials are outlined. The state-of-the-art of the plasticity theory is discussed in Section C. The prevailing fracture mechanics concepts of composite materials and their limitations are reviewed in Section D. The finite element methods, as applied for singular stress and nonlinear analyses, are discussed in Section E.

A. Composite Materials: Microscopic and Macroscopic Approaches

If a center-cracked tension or a compact tension specimen of metallic material is loaded in tension perpendicular to the crack axis, as shown in Figs. 2 and 3, the crack will extend along its own axis. This is the well-known crack-opening mode, or Mode I failure. Conversely, depending upon the degree of anisotropy, a crack in an anisotropic material may propagate in any direction when stressed under identical conditions [18, 31 through 33]. Unidirectionally reinforced composites are anisotropic materials that also fail by a number of additional modes. These include fiber pull out, debonding, matrix

failure, fiber fracture, fiber splitting, and a combination of these. Figure 4 shows the schematic diagrams for some of these failure modes. The exact operating mechanism of fracture in a composite material depends on factors such as the relative strengths of the filaments, the matrix, and the interface. Additional influencing factors include processing history, porosity, fiber shape, size, and spacing.

Unidirectional filamentary composites may be idealized in a microscopic or macroscopic manner. In microscopic analysis [19 through 22] the unidirectional, orthotropic composites are analyzed as isotropic, heterogeneous, multiphase material. In macroscopic analysis [23 through 28] the composite material is idealized as an anisotropic, homogeneous material.

In an earlier investigation conducted by Kulkarni [19], the microscopic analysis was used since the filamentary composite was treated as heterogeneous, bi-phase, isotropic material. From this investigation he concluded that the fracture behavior of the B-A1 composite was a strong function of the fiber orientation and the matrix material. However, the analysis was restricted to longitudinal and transverse composite systems only, since the requirement of the computer memory became too large to be handled for any off-axis orientation of the fiber in the microscopic model. Consequently, to investigate further the effect of fiber orientation with respect to the loading axis, it was decided to use the macroscopic material modeling, which can be used for any fiber orientation without needing any extra memory.

In the macroscopic approach, it is assumed that a composite material can be characterized by an equivalent anisotropic, homogeneous

material. Experimental results obtained using this approach have been correlated very successfully in typical areas of strength [29] or stress-strain response [28]. It may be further noted that due to the limitation in computer memory again, the microscopic material model cannot be used for $0^\circ/90^\circ$ cross-ply composites, as has been done in this investigation using a macroscopic material model.

B. Failure Theories for Composite Materials

For the purpose of characterization and design of materials, an operationally simple failure theory is essential for composite materials. Those theories most widely considered to describe the failure of composite materials include [37] the maximum stress theory, the maximum strain theory, the Tsai-Hill theory, and the Tsai-Wu theory. In all of these theories the composite material is assumed to be homogeneous and to exhibit two principal material directions (for the planar case).

The maximum stress and strain theories are very similar. If the maximum stress (or strain) value exceeds the critical level along any principal material direction, the composite is assumed to have failed. In application of the maximum stress (or strain) criterion, the stresses (or strains) in the body under consideration must be transformed to stresses in the principal material directions. Consequently, for off-axis composites, the applied stress (σ_0) is to be transformed as follows:

$$\sigma_L = \sigma_0 \sin^2 \theta$$

$$\sigma_T = \sigma_0 \cos^2 \theta \quad (1a)$$

$$\tau_{LT} = -\sigma_0 \sin \theta \cos \theta ,$$

where,

θ = off-axis fiber orientation (Fig. 2)

L,T = principal material directions .

The transformed stresses are then compared to the corresponding failure stresses (strains). Explicitly, the maximum stress failure theory for the off-axis composite becomes

$$\sigma_0 < \frac{X}{\sin^2 \theta}$$

$$\sigma_0 < \frac{Y}{\cos^2 \theta} \quad (1b)$$

$$\sigma_0 < \frac{S}{\sin \theta \cos \theta} ,$$

where,

X = normal failure stress along direction L

Y = normal failure stress along direction T

S = in-plane failure stress in shear .

Similarly, the maximum strain theory is

$$\begin{aligned}\sigma_0 &< \frac{X}{\sin^2\theta - \nu_{LT} \cos^2\theta} \\ \sigma_0 &< \frac{Y}{\cos^2\theta - \nu_{TL} \sin^2\theta} \\ \sigma_0 &< \frac{S}{\sin\theta \cos\theta} .\end{aligned}\tag{1c}$$

It is noted that in both of the theories, there is no interaction between the modes of failures. There are actually three criteria in each theory.

Conversely, the Tsai-Hill [37, 39, 40] theory is one single failure envelope. Hill's failure criterion for anisotropic materials may be written as [39]

$$\frac{\sigma_L^2}{X^2} + \frac{\sigma_T^2}{Y^2} - \frac{\sigma_L \sigma_T}{X^2} + \frac{\tau_{LT}^2}{S^2} = 1 .\tag{1d}$$

For off-axis composites, substitution of the stress transformation equations, Eq. (1a), yields the Tsai-Hill failure theory as

$$\frac{\sin^4\theta}{X^2} + \left(\frac{1}{S^2} - \frac{1}{X^2}\right) \cos^2\theta \sin^2\theta + \frac{\cos^4\theta}{Y^2} = \frac{1}{\sigma_0^2} .\tag{1e}$$

Unlike the maximum stress or the maximum strain failure theory, this failure theory gives a smooth failure envelope. However, the interaction term, $\cos^2\theta \sin^2\theta$, does not depend on the transverse strength Y , which is unexpected.

Modifying the strength theory of Goldenblat and Koponov [38] for glass fiber-reinforced plastics (GFRE), Tsai and Wu [29] proposed a failure surface in the stress space in the following scalar form

$$F \equiv - F_i \sigma_i + F_{ij} \sigma_i \sigma_j - 1 = 0 , \quad (2a)$$

with the following constraints:

$$F_{ii} F_{jj} - F_{ij}^2 > 0 \text{ (no sum)} , \quad (2b)$$

$$F_{ii} > 0 \text{ (no sum)} , \quad (2c)$$

$$F_{ij} = F_{ji} , \quad (2d)$$

$$i, j = 1 \text{ to } 6 .$$

F_i and F_{ij} are second- and fourth-rank tensors, respectively.

For plane stress condition the failure surface becomes

$$\begin{aligned} f \equiv & F_{11}^2 \sigma_1 + F_{22}^2 \sigma_2 + F_{66}^2 \sigma_6 + 2 (F_{12} \sigma_1 \sigma_2 + F_{16} \sigma_1 \sigma_6 + F_{26} \sigma_2 \sigma_6) \\ & - F_1 \sigma_1 - F_2 \sigma_2 - F_6 \sigma_6 = 0 . \end{aligned} \quad (3)$$

The linear terms (F_i) in σ_i account for the Bauchinger effect or the internal stress in the composite material. This accommodation of the Bauchinger effect makes this theory more realistic for composite materials [40]. Moreover, this theory admits another experimentally observed fact of composite materials; namely, failure under hydrostatic pressure [42, 44]. However, if failure is not allowed under hydrostatic pressure, it can be shown that [24]

$$F_1 + F_2 = 0 , \quad (4a)$$

$$F_{11} + F_{22} + 2F_{12} = 0 . \quad (4b)$$

If it is further assumed that there is no internal stress ($F_i = 0$), then the Tsai-Wu theory degenerates to the von Mises theory [29].

The constant terms of the Tsai-Wu theory can be determined from the experimental values of strengths along the principal material directions:

$$F_i = \frac{1}{X_i^-} - \frac{1}{X_i^+} , \quad (5)$$

$$F_{ii} = \frac{1}{X_i^- X_i^+} \text{ (no sum) } . \quad (6)$$

The superscripts, - and +, refer to negative and positive failure stresses, respectively.

The interaction terms (F_{ij} , $i \neq j$) are not defined precisely, but limited by Eq. (2b). Experimental methods have been suggested under combined loading to evaluate the F_{ij} terms [29]. For composite materials under a plane stress situation, the interaction terms are F_{12} , F_{16} , and F_{26} . Since the unidirectional and $0^\circ/90^\circ$ cross-ply composite materials have the same shear (positive or negative) strength in the principal material directions (independent of differences in tensile and compressive behaviors of the composite) [37], the interaction terms, F_{16} and F_{26} , become zero [29]. Thus, for plane stress problems concerning unidirectional or $0^\circ/90^\circ$ cross-ply filamentary composites, F_{12} is the only interaction term to be determined when the

principal material directions coincide with the axes of the load and the crack. Unfortunately, determination of F_{12} is very difficult. It has been shown [29, 45, 46] that F_{12} is highly sensitive to small errors in the measured stresses. Consequently, even careful experiments may, sometimes, lead to unusable results [45]. Optimal experimental measurements [42] have been suggested for the purpose. On the other extreme, it has been suggested to eliminate the term altogether [37]. However, the interaction term F_{12} does influence the failure surface [29]. So, based on numerical data alone [47], it is not justified to drop the term. An approximate value of F_{12} can be derived from Eq. (2b) as

$$F_{12} = c (F_{11}F_{22})^{1/2} , \quad (7)$$

where,

$$c \leq 1 .$$

Experimental verification of these failure theories has been done [37]. A typical example is shown from the results reported by Pipes and Cole [45] in Fig. 5. It is clear from the figure that the Tsai-Wu theory shows excellent agreement for the boron-epoxy (AVCO 5505) composite. The Tsai-Hill theory also shows good agreement with the experimental data. But the agreement of the experimental data is not good with the maximum stress theory. The three independent, intersecting failure criteria of the maximum stress theory are clear in the figure. This is an unwanted feature of both the maximum stress and the maximum strain failure theories. The small disagreement of the Tsai-Hill failure theory with the experimental data may be attributed

to the absence of the linear terms in Eq. (1d) and the dependence of the interaction term on the longitudinal strength alone (see Eqs. (1d) and (3)).

Except for the determination (experimental or otherwise) of the F_{12} term, the Tsai-Wu theory has great advantages over the existing theories discussed previously [37], namely,

1. It is a scalar equation and hence, invariant in any Cartesian coordinate system.
2. The strength components are tensors so that they can be transformed easily from one coordinate system to another coordinate system.
3. It accommodates most known failure phenomena of the composites in the macroscopic level.

To exploit the above advantages, the strength constants (F values) can be determined for the principal material directions, and then re-evaluated for any arbitrary off-axis configuration (θ) of the unidirectional filamentary composites using the standard tensor transformations shown in Table 1.¹ For the analysis of off-axis composites, these advantages made the Tsai-Wu theory very attractive. Hence, this theory was used in this investigation.

C. Theory of Plasticity

Plastic deformation of a material is characterized by a non-unique relationship between stress and strain. Consequently, the

¹All tables appear in Appendix B.

nonlinear behavior alone does not signify plastic deformation. Indeed, one definition of plasticity is the presence of irrecoverable deformation on removal of load.

It is realized that plasticity (like elasticity) is a mathematical modeling of the behavior of a material under the influence of a certain level of load under certain conditions. Unlike the theory of elasticity, the theory of plasticity is not well-developed. Consequently, exact agreement with experimental data is not expected, even for metallic materials [48]. The situation is more pessimistic [49] for the anisotropic materials. In this investigation, the plastic analysis is limited to isothermal, time-independent static problems in which small displacement assumptions are applicable.

The actual nonlinear behavior of a material beyond the linear elastic zone is normally idealized (Fig. 6) using one of these models (i.e., ideal plastic; linear hardening; or power law hardening).

Ramberg-Osgood [50] proposed the following expression to describe a power law hardening behavior:

$$\bar{\epsilon} = \frac{\bar{\sigma}}{E} + k \frac{\bar{\sigma}_y}{E} \left(\frac{\bar{\sigma}}{\bar{\sigma}_y} \right)^n, \quad (8)$$

where,

$\bar{\sigma}$ = applied uniaxial stress

$\bar{\epsilon}$ = corresponding total strain in the direction of load

E = elastic modulus in the direction of load

$\bar{\sigma}_y$ = uniaxial yield stress in the direction of load

n = strain-hardening parameter

k = material constant .

The values of k and n are the numerical best fit for the experimental data with the proposed equation. If the elastic strain part is negligible compared to the plastic strain, Eq. (8) can be simplified as

$$\bar{\epsilon} = c \bar{\sigma}^n , \quad (9)$$

where,

$$c = \frac{k}{E \bar{\sigma}_y^{n-1}} , \quad (9a)$$

by neglecting the first term.

It is quite generally postulated, as an experimental fact, that a material starts behaving plastically (beyond the linear range), when the combination of stresses applied to the material satisfies a certain mathematical criterion known as the yield criterion. The surface so generated in the n -dimensional hyperspace of stress is known as the yield surface. The general equation of the yield surface is given by

$$F(\{\sigma\}, \{\epsilon\}, \alpha) = 0 , \quad (10)$$

where,

$\{\sigma\}$ = stress state

$\{\epsilon\}$ = strain state

α = material dependent function of
strain-hardening parameter n .

Depending upon the theory, the yield surface may be singular (Tresca) or regular (von Mises). The anisotropic plasticity theory of Tsai-Wu leads to a regular surface in the hyperspace of stress.

Similar to the strain energy density function, it is postulated that, there exists a scalar function $Q(\{\sigma\}, \{\epsilon\}, \alpha)$ from which the plastic strain increment can be obtained as follows:

$$\{d\epsilon_p\} = d\lambda \frac{\partial Q}{\partial \{\sigma\}} . \quad (11)$$

The scalar function $Q(\{\sigma\}, \{\epsilon\}, \alpha)$ is known as the plastic potential. Equation (11) is known as the normality principal, since it requires that the plastic strain increment is normal to the plastic potential in the stress space. The term $d\lambda$ represents a non-negative scalar quantity known as the plastic multiplier, which depends on the stress-strain history the material has undergone. Instead of using the plastic potential hypothesis, if the yield surface $F(\{\sigma\}, \{\epsilon\}, \lambda)$ is used to determine the incremental plastic strain for a given stress state beyond the linear elastic region (i.e., yield point), Eq. (11) becomes

$$\{d\epsilon_p\} = d\lambda \frac{\partial F}{\partial \{\epsilon\}} . \quad (12)$$

Equation (11) or Eq. (12) is known as the flow rule. Equation (12) gives rise to different flow rules associated with each yield criterion, and hence, is known as the associated flow rule. Equation (11), on the other hand, is known as the nonassociated flow rule. It has been proven, however, that the plastic potential and the yield surface are the same function for metals [55].

To obtain the total plastic strain, Eq. (12) is to be integrated over the whole history of loading. This makes it necessary to store the total stress history in the incremental plastic analysis,

where Eq. (12) is used without further simplification. However, if it is assumed that the plastic multiplier $d\lambda$ depends only on the current state of stress and not on the whole history of deformation, considerable simplicity can be achieved in calculating the plastic strain.

This simplified theory is known as the deformation theory of plasticity.

This can be demonstrated if the von Mises failure criterion is used so that

$$F = 1/6 \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 + 6\tau_{12}^2 + 6\tau_{12}^2 + 6\tau_{23}^2 + 6\tau_{31}^2 \right] \quad (12a)$$

and

$$\frac{\partial F}{\partial \sigma_{ij}} = \sigma_{ij} - \delta_{ij} (\sigma_{kk}/3) = \sigma_{ij} - \delta_{ij} \sigma_m = S_{ij} , \quad (12b)$$

where,

σ_m = mean stress

S_{ij} = deviatoric stress .

The two commonly used invariant functions associated with plastic analysis are effective or significant stress and strain, defined as

$$\bar{\sigma}_e = \frac{1}{\sqrt{2}} \sqrt{S_{ij} S_{ij}} ,$$

$$\bar{\epsilon}_e = \frac{\sqrt{2}}{3} \sqrt{\epsilon_{ij}^e \epsilon_{ij}^e} , \quad (12c)$$

and

$$\frac{P}{d\epsilon} = \frac{\sqrt{2}}{3} \sqrt{d\epsilon_{ij}^P d\epsilon_{ij}^P} .$$

It is noted that independent of the type of test, both effective stress and effective strain reduce to the axial normal component of stress and strain for a tensile test.

Using Eqs. (12) and (12b) in Eq. (12c), it can be shown, on simplification, that

$$d\lambda = \frac{3}{2} \frac{d\bar{\epsilon}_p}{\bar{\sigma}_e} , \quad (12d)$$

so that,

$$d\epsilon_{ij}^P = \frac{3}{2} \frac{d\bar{\epsilon}_p}{\bar{\sigma}_e} S_{ij} . \quad (12e)$$

If the deformation theory of plasticity is used (i.e., $d\bar{\epsilon}_p$ is assumed to be independent of loading history), Eq. (12e) can be integrated directly to calculate the plastic strain as a function of current stress, namely,

$$\epsilon_{ij}^P = \frac{3}{2} \frac{\bar{\epsilon}_p}{\bar{\sigma}_e} S_{ij} . \quad (12f)$$

However, the incremental theory of plasticity does not allow Eq. (12f) to be used directly. Instead, the load is added incrementally and the corresponding plastic strain increment is calculated from Eq. (12). At the end of each load, the plastic strain is updated by adding the incremental plastic strain to the existing strain. Unfortunately, except for radial or proportional loading when all of the stresses increase in a definite ratio, the deformation theory of plasticity is not, in general, justified [55].

Another necessary concept in the plastic analysis is known as the hardening rule, which has two parts. First, the hardening rule prescribes the material strain-hardening at any point of the loading history by relating this point to an equivalent point of a simple uniaxial test. If the equivalency depends on the plastic strain alone, the material is called a strain-hardening material. On the other hand, if it depends on both stress and plastic strain, the material is known as a work-hardening material. Under the work-hardening hypothesis,

$$d\alpha = \bar{\sigma}_y d\bar{\epsilon}_p, \quad (13)$$

where,

$$\bar{\epsilon}_y = \sigma_y(\alpha) = \text{uniaxial stress at yield}$$

$$d\bar{\epsilon}_p = \text{uniaxial plastic strain increment}.$$

In addition to prescribing material strain-hardening, the hardening rule also provides the modification in the initial yield surface as loading continues.

In the isotropic-hardening rule the subsequent yield surface expands uniformly about the origin of stress space, maintaining its original shape and orientation (Fig. 7). If the initial yield surface is an ellipse in stress space, the subsequent yield surface will be a similar, but larger, ellipse, the size of which can be obtained by relating the values of effective stresses in the two cases. Hence, all subsequent yield surfaces are simple inflated versions of the initial yield surface, as shown in Fig. 7.

In contrast, the kinematic-hardening rule postulates that the subsequent yield surface retains the initial shape, size, and orientation, but translates as a rigid body during plastic deformation. According to this model, if the initial yield surface is an ellipse, the subsequent loading will not inflate the yield surface. The subsequent yield surface will be the same size ellipse displaced from its original position to other points in the stress space, depending upon the load applied and the loading history, as shown in Fig. 7. Obviously, the determination of the displaced position of the yield surface is very complex.

In a recent analysis [48] it is pointed out that the kinematic-hardening model is the most complex, but the results are consistently inferior to those obtained assuming isotropic-hardening. Although more complex theories have recently been proposed, isotropic-hardening is still the most widely used hardening rule [55, 56].

For the plastic analysis carried out in this investigation, an isotropic-hardening, work-hardening material was assumed. The associated flow rule and the incremental flow theory as described previously

were applied. Under these assumptions, it can be shown [49, 57]

$$\{\mathrm{d}\sigma\} = [D]_{\mathrm{ep}} \{\mathrm{d}\epsilon\} , \quad (14)$$

$$\{\mathrm{d}\epsilon\} = \{\mathrm{d}\epsilon_{\mathrm{e}}\} + \{\mathrm{d}\epsilon_{\mathrm{p}}\} , \quad (15)$$

and

$$[D]_{\mathrm{ep}} = [D_{\theta}] - [D_{\theta}] \left(\frac{\partial F}{\partial \{\sigma\}} \right) \left(\frac{\partial F}{\partial \{\sigma\}} \right)^{\mathrm{T}} \\ [D_{\theta}] \times \left[\bar{A} + \left(\frac{\partial F}{\partial \{\sigma\}} \right) [D_{\theta}] \left(\frac{\partial F}{\partial \{\sigma\}} \right)^{\mathrm{T}} \right]^{-1} , \quad (16)$$

where,

$\{\mathrm{d}\sigma\}$ = stress increment

$\{\mathrm{d}\epsilon\}$ = total strain increment

$\{\mathrm{d}\epsilon_{\mathrm{e}}\}$ = total elastic strain increment

$\{\mathrm{d}\epsilon_{\mathrm{p}}\}$ = total plastic strain increment

$[D_{\theta}]$ = elastic stress-strain matrix in the
direction of loading

$[D]_{\mathrm{ep}}$ = elasto-plastic matrix

F = general yield surface

$\bar{A}$ = slope of uniaxial stress vs plastic
strain curve .

For an elastic, perfectly plastic (flat top) stress-strain curve,

$\bar{A} = 0$. For a bi-linear stress-strain curve, $\bar{A}$ is given by

$$\bar{A} = \frac{E_s}{1 - (E_s/E_p)} , \quad (16a)$$

where subscripts s and p denote the secondary and primary moduli of elasticity. The calculation of the elastic matrix for the off-axis, unidirectional composites, $[D_\theta]$, is shown in Addendum I, Appendix C.

Using the deformation theory of plasticity and the approximate Ramberg-Osgood relation shown in this work as Eq. (9), Hutchinson [51], and Rice and Rosengren [52] showed that for small-scale yielding in a plane strain condition, the stress near a sharp crack remains singular and has the form

$$\sigma_{ij}(r, \theta) \rightarrow r^{\frac{-n}{1+n}} \tilde{\sigma}_{ij}(\theta) , \quad (17a)$$

$$\epsilon_{ij}(r, \theta) \rightarrow r^{\frac{-1}{n+1}} \tilde{\epsilon}_{ij}(\theta) , \quad (17b)$$

where,

n = strain-hardening coefficient of the material

r, θ = polar coordinates of the point of interest

$\tilde{\sigma}_{ij}(\theta)$, $\tilde{\epsilon}_{ij}(\theta)$ = functions of θ for the given strain-hardening coefficient of the material .

This is known as a Hutchinson-Rice-Rosengren, or HRR, singularity. Using a path-independent, energy integral J , introduced by Rice [11] for small-scale yielding (to be discussed later), the above equations can be written [53, 54] as

$$\sigma_{ij}(r, \theta) = \bar{\sigma}_y \left[\frac{J}{\bar{\sigma}_y I_n} \right]^{\frac{n}{n+1}} r^{\frac{-n}{n+1}} \tilde{\sigma}_{ij}(\theta), \quad (18a)$$

$$\varepsilon_{ij}(r, \theta) = \left[\frac{J}{\bar{\sigma}_y I_n} \right]^{\frac{1}{n+1}} r^{\frac{-1}{n+1}} \tilde{\varepsilon}_{ij}(\theta), \quad (18b)$$

where,

I_n = numerical constant determined from a singularity analysis .

It is shown [11, 53] that the constant J depends on the remotely applied load, the crack length, and the failure mode. On the other hand, the numerical constant I_n depends [51, 52] on the strain-hardening coefficient and the stress state, as shown in Table II. Since neither J nor I_n depends on r and θ , the singular nature of the stress state ahead of the crack tip is explicit in Eq. (18a), provided the deformation theory of plasticity is valid for the case in question. It is further realized that this solution, being singular, is valid only when r is larger than some critical value (r_c , say), since, for any $r < r_c$, the above solution leads to unrealistic values of stresses which the material cannot withstand.

Using this solution, the effective stress (see Eq. (12c)) and displacement for any point (r, θ) ahead of the crack tip can be shown [53, 55] to be

$$\sigma_e(r, \theta) = \bar{\sigma}_y \left(\frac{J}{\bar{\sigma}_y I_n} \right)^{\frac{n}{n+1}} r^{\frac{-n}{n+1}} \tilde{\sigma}_e(\theta) , \quad (19a)$$

$$u_i(r, \theta) = \left(\frac{J}{\bar{\sigma}_y I_n} \right)^{\frac{1}{n+1}} r^{\frac{n}{n+1}} \tilde{u}_i(\theta) , \quad (19b)$$

where,

$\tilde{\sigma}_e(\theta)$, $\tilde{u}_i(\theta)$ = functions of θ for a given material .

D. Fracture of Composite Materials

The fracture of composite materials has been studied from different angles. In the purely analytical approach [8, 13, 15, 16, 72, 75, 78, 80], the composite material is assumed to be a homogeneous, anisotropic material, and relevant fracture criteria are developed by modifying or extending the existing fracture concepts of isotropic, homogeneous material. In the macromechanical approach [33, 62, 63, 66 through 68, 70, 71], experimental verifications of these concepts have been attempted by correlating the fracture data of composite materials. This is in contrast to the earlier micromechanical approach [31, 58 through 62, 64] of semi-empirical nature. Recently, macromechanical fracture theories [23, 74, 81] have been proposed to correlate the experimental results of composite material in a semi-analytical manner.

Analytical Approach

Stress intensity factor. Wu [72] has shown that the stress state ahead of a sharp crack in an anisotropic, homogeneous material

can be written as

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re} \left\{ \frac{s_1 s_2}{(s_1 - s_2)} \left[\frac{s_2}{\sqrt{\psi_2}} - \frac{s_1}{\sqrt{\psi_1}} \right] \right\} + \frac{K_{II}}{\sqrt{2\pi r}} \operatorname{Re} \left\{ \frac{1}{(s_1 - s_2)} \left[\frac{s_2^2}{\sqrt{\psi_2}} - \frac{s_1^2}{\sqrt{\psi_1}} \right] \right\}, \quad (20a)$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re} \left\{ \frac{1}{(s_1 - s_2)} \left[\frac{s_1}{\sqrt{\psi_2}} - \frac{s_2}{\sqrt{\psi_1}} \right] \right\} + \frac{K_{II}}{\sqrt{2\pi r}} \operatorname{Re} \left\{ \frac{1}{(s_1 - s_2)} \left[\frac{1}{\sqrt{\psi_2}} - \frac{1}{\sqrt{\psi_1}} \right] \right\}, \quad (20b)$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \operatorname{Re} \left\{ \frac{s_1 s_2}{(s_1 - s_2)} \left[\frac{1}{\sqrt{\psi_1}} - \frac{1}{\sqrt{\psi_2}} \right] \right\} + \frac{K_{II}}{\sqrt{2\pi r}} \operatorname{Re} \left\{ \frac{1}{(s_1 - s_2)} \left[\frac{s_1}{\sqrt{\psi_1}} - \frac{s_2}{\sqrt{\psi_2}} \right] \right\}, \quad (20c)$$

where,

$$\psi_1 = \cos\theta + s_1 \sin\theta, \quad (20d)$$

$$\psi_2 = \cos\theta + s_2 \sin\theta,$$

s_1 and s_2 are complex roots of

$$\alpha_{11} s^4 - 2\alpha_{16} s^3 + (2\alpha_{12} + \alpha_{66}) s^2 - 2\alpha_{26} s + \alpha_{22} = 0. \quad (21a)$$

The terms α_{ij} ($i, j = 1, 2$ and 6) are strain-stress coefficients given by

$$\epsilon_i = \alpha_{ij} \sigma_j . \quad (21b)$$

The terms K_I and K_{II} are known as stress intensity factors for Modes I and II (Fig. 1) defined as

$$K_I = \sigma_0 \sqrt{\pi a} , \quad (22a)$$

$$K_{II} = \tau_0 \sqrt{\pi a} ,$$

where, σ_0 and τ_0 are the applied normal and shear stresses for an infinitely wide specimen with a sharp crack of length $2a$, as indicated in Fig. 2.

Wu [72] has shown that unidirectionally reinforced brittle composite specimens satisfy the following conditions; thus, anisotropic fracture mechanics as developed above can be used directly for such materials. The conditions are:

1. The orientation of the flaw with respect to the principal axis of symmetry is fixed.
2. The stress intensity factors defined for anisotropic cases are consistent with the isotropic counterpart.
2. The critical orientation of the crack coincides with one of the principal directions of the elastic symmetry.

Equations (20a) through (20c) may also be used to find the stress distribution ahead of a crack tip under combined loading for an isotropic, homogeneous material. In this case, Eqs. (20a) through (20c) degenerate to the following familiar expressions:

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left[1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right] + \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \left[2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right] \quad (23a)$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left[1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right] + \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} , \quad (23b)$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2} + \frac{K_{II}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left[1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right] . \quad (23c)$$

It is noted that Eqs. (20) and (23) are obtained by superimposing the corresponding solutions for Modes I and II. It may be realized that this superimposition technique is not valid for nonlinear plastic deformations. Equations (20) or (23) may be written in contracted form as

$$\sigma_i = \frac{K_I}{\sqrt{2\pi r}} f_i(\theta, s) + \frac{K_{II}}{\sqrt{2\pi r}} g_i(\theta, s) , \quad (24)$$

where, s is a function of the complex roots s_1, s_2 or equivalently a function of elastic properties.

It is interesting to note that the stress distribution ahead of a sharp crack tip is determined completely for any position (r, θ) in an infinitely wide plate, provided K_I and K_{II} are known. A width correction factor must be incorporated into the expression if the stress distribution is required for a finite width plate. From Eq. (24) it is also clear that the stress state ahead of the crack tip is singular in nature, so that as the points nearer the crack tip are approached, the stress state, and hence, the effective stress, increase rapidly. For many brittle materials (composite or otherwise), when a critical value of the stress intensity factor K_c is reached, the crack propagates unstably. This critical stress intensity factor K_c has been shown to be a material property for many brittle composites [7, 19, 25, 63, 65, 72].

Strain energy release rate. The Griffith-Irwin theory [8] proposes that the potential energy release rate per unit thickness associated with an unstable crack extension of ∂a is equal to the energy required to close the crack by the same amount. Sih [72] defines this energy release rate as

$$G_j = \lim_{\partial a \rightarrow 0} \frac{1}{\partial a} \int_0^{\partial a} \sigma_u(\partial a - r, 0) u_j(r, \pi) dr . \quad (25a)$$

For combined fracture modes in an anisotropic media, Eq. (25a) reduces to

$$G = \frac{K_I}{2} \alpha_{22} I_m \left[\frac{K_I (S_1 + S_2) + K_{II}}{S_1 S_2} \right] + \frac{K_{II}}{2} \alpha_{11} I_m \left[K_{II} (S_1 + S_2) + K_I S_1 S_2 \right] . \quad (25b)$$

For isotropic, homogeneous material, the strain energy release rate can be further simplified to

$$G = \frac{\mu + 1}{8G} (K_I^2 + K_{II}^2) , \quad (25c)$$

where,

$$\mu = 1 - 4 \text{ for plane strain ; } \frac{3 - \nu}{1 + \nu} \text{ for plane stress}$$

ν = Poisson's ratio

G = shear modulus .

Mixed mode fracture. For plane problems, the stresses at the tip of a line crack in an isotropic, homogeneous material may be written in polar coordinates [75] as

$$\sigma_r = \frac{1}{\sqrt{2\pi r}} \cdot \frac{1}{2} \cdot \cos \frac{\theta}{2} \left[K_I (1 + \cos \theta) - 3K_{II} \sin \theta \right] , \quad (26a)$$

$$\sigma_\theta = \frac{1}{\sqrt{2\pi r}} \cdot \frac{1}{2} \cdot \cos \frac{\theta}{2} \left[K_I \sin \theta + K_{II} (3 \cos \theta - 1) \right] . \quad (26b)$$

As the applied load increases to an appropriate level, the crack will begin to propagate from the existing crack tip in an arbitrary radial direction $\theta = \theta_0$, as shown in Fig. 8. Denoting all quantities

associated with the branch crack by placing a bar over those quantities, it is clear that at the tip of the branch crack the stresses are of the form of Eq. (26), with r , θ , K_I , and K_{II} replaced by $\bar{r}$, θ , $\bar{K}_I$, and $\bar{K}_{II}$, respectively. It should be noted that the stress intensity factors associated with the original crack, K_I and K_{II} , are in general, different from those of the branch crack, $\bar{K}_I$ and $\bar{K}_{II}$, with the latter being dependent on the branch geometry. The strain energy release rate for the branch crack is, then, given by

$$\bar{G} = \frac{\mu + 1}{8G} (\bar{K}_I^2 + \bar{K}_{II}^2) , \quad (26c)$$

If it is assumed that in the limit as the branch $\bar{a}$ shrinks to zero, the stress field at the tip of the branch crack approaches the stress field at the tip of the original crack before propagation begins, it is found that

$$\bar{K}_{I_0} = \lim_{\bar{a} \rightarrow 0^+} \bar{K}_I = \frac{1}{2} \cos \frac{\theta_0}{2} \left[K_I (1 + \cos \theta_0) - 3K_{II} \sin \theta_0 \right] , \quad (26d)$$

$$\bar{K}_{II_0} = \lim_{\bar{a} \rightarrow 0^+} \bar{K}_{II} = \frac{1}{2} \cos \frac{\theta_0}{2} \left[K_I \sin \theta_0 + K_{II} (3 \cos \theta_0 - 1) \right] . \quad (26e)$$

Nuismer [75] further postulates that, (1) the crack will propagate in the direction which causes the maximum energy release, and (2) the fracture will initiate when the release rate in that direction reaches a critical value. Application of the first postulate ($\partial G / \partial \theta = 0$) in conjunction with the second one shows that the crack will branch along a direction along which the circumferential stress, σ_θ , vanishes at the

crack tip. Under this situation, the critical strain energy release rate, Eq. (26c), reduces to

$$\bar{G}_c = \frac{\mu + 1}{8G} \bar{K}_{Ic}^2, \quad (26f)$$

so that the critical stress intensity factor for the initiation of a radially branched crack may be written as

$$\bar{K}_{I_0c} = K_{\theta_0c} = \frac{1}{2} \cos \frac{\theta_0}{2} \left[K_I (1 \cos \theta_0) - 3K_{II} \sin \theta_0 \right], \quad (27)$$

where,

θ_0 = radial direction along which the crack branches

c = critical value .

It is realized that since the direction of crack propagation is the direction along which the circumferential stress vanishes, the radially branched crack will propagate in Mode I manner, although in reality it is a mixed mode fracture, as is evidenced from Eq. (27).

Since the assumptions in the derivation of Eq. (27) are equally valid in an anisotropic, homogeneous composite material and the equation of the stresses ahead of the crack tip (see Eqs. (20) and (23)) has the same form for both isotropic and anisotropic materials, it appears reasonable to use it for anisotropic, homogeneous, brittle composite materials, provided the stress intensity factors are consistently defined [72] and the crack propagates radially.

As a phenomenological approach, Wu [79] has suggested the following expression which can predict failure of brittle composite materials under mixed mode fracture:

$$\left(\frac{K_I}{K_{Ic}} \right)^x + \left(\frac{K_{II}}{K_{IIc}} \right)^y = 1 , \quad (28a)$$

where,

x, y = numerical fits of experimental data.

As a special case, this reduces to

$$\left(\frac{K_I}{K_{Ic}} \right)^2 + \left(\frac{K_{II}}{K_{IIc}} \right)^2 = 1 , \quad (28b)$$

which has shown good agreement for glass fiber-reinforced composites [80].

Ductile fracture.

a. Plastic zone correction

Equations (22a) and (22b) allow the stress intensity factors K_I and K_{II} to be evaluated under conditions of linear elasticity. In order to calculate corresponding stress intensity factors when small-scale yielding occurs at the crack tip subjected to plain stress conditions, Irwin [8] suggested that the initial crack length, a , be replaced in Eq. (22) by the effective crack length, a_e , defined as

$$a_e = a + \frac{1}{2\pi} (K_I/\sigma_y)^2 , \quad (29)$$

where the second term on the right-hand side of Eq. (29) represents the length of the crack tip plastic zone in a direction coplanar with the crack,

b. J-integral

Using nonlinear elasticity, Rice [11] suggested a path-independent, two-dimensional energy integral defined as

$$J = \int_{\Gamma} \left(W \, dy + T_i \frac{\partial u_i}{\partial x} \, ds \right), \quad (30a)$$

where,

W = strain energy density

T_i = surface traction along direction i

u_i = deformation along direction i

Γ = any contour around the crack .

The J-integral can also be determined by an energy interpretation [11], so that

$$J = - \left. \frac{dU}{da} \right|_{\Delta = \text{constant}}, \quad (30b)$$

where,

U = potential energy

a = crack length

Δ = load point displacement .

For linear elasticity, Eq. (30b) becomes identical with the definition of strain energy release rate (G), so that $J = G$. It may be further

recalled that the expressions for crack tip stress distribution in the linear case in terms of stress intensity factors (Eq. (23)), and in the nonlinear case in terms of the J-integral (Eq. (18a)) are very similar. All of these observations lead to the popular proposition that the J-integral at fracture is a material constant, even if the material behaves nonlinearly [13, 15, 16]. Considering the complexity of evaluating the J-integral in the form of Eq. (30a), an approximate formulation for calculating the J-integral has been proposed by Rice, et al., [12] as

$$J = G + \frac{1}{B\ell} \left[2 \int_0^{\Delta_p} P \, d\Delta_p - P\Delta_p \right] \quad (31a)$$

for center-cracked tension specimens (Fig. 2), and for the compact tension specimen (Fig. 3)

$$J = \frac{2A}{B\ell}, \quad (31b)$$

where,

P = applied load

B = thickness of the specimen

ℓ = uncracked length of the ligament

Δ_p = displacement of the load point due to plasticity

A = area under load displacement curve .

The reported success of the critical J-integral [13, 15, 16] as a material parameter, even for metallic material, is an attractive

proposition since it correlates macroscopic load deflection history to the microscopic phenomena leading to unstable fracture.

However, it may be recalled that the J-integral is theoretically justified only as a path-independent integral and not as a material constant [9]. Even the path-independency is valid only if the deformation theory of plasticity is applicable. Considering the limitation of the deformation theory of plasticity, questions have been raised against the J-integral approach to characterize fracture [9, 14]. This doubt is further substantiated by recent research reports [76 through 78]. Yoder and Griffis [76] found significant variation in the critical value of the J-integral (J_{IC}) from Ti-6Al-4V alloy specimens machined from different positions through the plate thickness. In addition, J_{IC} has been shown to be dependent on yield stress and Young's modulus for tool steel, D6, 2024-T351 aluminum alloy and Ti-6Al-4V [78]. Furthermore, using the finite element method, McMeeking [77] has recently shown that the J-integral is not path-independent if the path is chosen immediately adjacent to a blunted crack tip and large-scale deformation at the crack tip is accounted for.

The applicability of the J-integral to describe failure of anisotropic composite materials appears less reliable when it is realized that the following assumptions of the deformation theory of plasticity, which are necessary conditions for valid J-integral determinations, are violated.

1. Material isotropy is maintained.
2. Plastic deformation is incompressible.

3. Plastic strains are uniquely determined by the final state of stress (i.e., plastic strains are path-independent).

c. Crack-opening displacement (COD)

Like the J-integral, crack-opening displacement (COD) is another popular approach to characterize ductile fracture. The J-integral may be viewed as a macroscopic approach to predict unstable fracture of a material in the presence of a plastic deformation at the tip of the crack. The COD approach, on the other hand, is a microscopic approach to predict crack growth. The COD hypothesis can be stated as follows. Regardless of the degree of yield at the crack tip and the size of the specimen, a crack advances when the average displacement at the crack tip reaches a critical level (COD_C). From careful experiments, it has been shown [8] that, the critical crack-opening displacement for crack initiation is actually a material constant for isotropic homogeneous materials. Except for very thin specimens, critical COD has been shown to be independent of thickness [8], provided the slot width is the same in each case. Expectedly, critical COD varies linearly with the slot width [8]. Because of the lack of a suitable analytical result, the strength of this approach is based on experimental results [9, 14] obtained with metallic materials. Consequently, further experimental results are necessary to establish this approach as an acceptable fracture criterion, particularly for composite materials.

It may be noted that due to the symmetry condition, the deformation at the tip of a sharp crack ($\pm a$) or a crack-opening displacement (δ) in a center-cracked tension specimen is zero for an isotropic, homogeneous, linear, elastic material. However, the formation of a plastic zone at the tip of the crack effectively increases the length

of the crack (Eq. (29)). Consequently, COD (δ) attains a finite value at the tip of the physical crack, which can be calculated using a simplified model of ideal plasticity. It is shown [8] that under these conditions

$$\delta = \frac{8}{\pi} \cdot \frac{\sigma_y}{E} a \log \left[\sec \frac{\pi \sigma_0}{2 \sigma_y} \right], \quad (31c)$$

where,

$$\sigma_0 = \text{applied stress}.$$

For low applied stress ($\sigma_0/\sigma_y \ll 1$), the right-hand side may be expanded to get

$$\delta = \frac{8}{\pi} \frac{\sigma_y}{E} \cdot a \cdot \left[\frac{1}{2} \left(\frac{\pi \sigma_0}{2 \sigma_y} \right)^2 + \frac{1}{12} \left(\frac{\pi \sigma_0}{2 \sigma_y} \right)^4 + \dots \right]$$

$$\approx \frac{\pi \sigma_0^2 a}{E \sigma_y} = \frac{G}{\sigma_y}$$

$$\therefore G = \delta \sigma. \quad (31d)$$

Michromechanical Approach

Olster and Jones [58] suggested the following relation to calculate the fracture toughness of a unidirectionally reinforced composite in terms of its components:

$$(G_c)_{\text{composite}} = V_f (G_c)_{\text{fiber}} + (1-V_f)(G_c)_{\text{matrix}}. \quad (32)$$

It is obvious that the fracture toughness of the composite is a function of fiber volume fraction (V_f). In this proposed relation, the critical value of the toughness was chosen because it was independent of the crack size and the specimen geometry [58].

However, the actual fracture energy must be more than the simple addition of the fracture energies of the individual components [58 through 62]. For instance, typical fracture processes which can occur in a composite include fiber pull out, fiber failure, matrix failure, stress relaxation (or redistribution), interface failure, and a combination of these processes (Fig. 4). Unfortunately, however, the degree to which each of these mechanisms determine the overall fracture is not known for a given composite material [64].

In the micromechanical approach, simple models have been proposed to calculate the energy associated with some of these processes [31, 64] as given by

$$G_D = \frac{V_f \sigma_f^2 L}{4E_f}, \quad (33)$$

$$G_{PO} = \frac{V_f \sigma_f L_c}{12}, \quad (34)$$

$$G_{SR} = \frac{V_f \sigma_f^2 L}{2E_f}, \quad (35)$$

$$G_{MPD} = \frac{(1 - V_f)^2}{V_f} \frac{\sigma_m^3 d}{2\tau E_m}, \quad (36)$$

where,

L = debonded length

L_e = critical length over which stress is transferred by the matrix to the fiber from the broken end, and calculated as $(\sigma_f d / 2\tau)$

d = fiber diameter

σ, τ = normal and shear strength

E = elastic modulus ;

subscripts D, P0, SR, MPD, m, and f represent debonding, pulling out, stress relaxation, matrix plastic deformation, matrix, and fiber, respectively.

It should be realized that the above expressions are approximate in nature. Depending upon the model chosen, slight variations in these equations are found [58 through 62]. But the observations to be made from these equations are as follows:

1. Microfracture processes depend on fiber geometry.
2. The stiffness and strength of both the phases play important roles in the fracture process.
3. The fiber-matrix interface controls the fracture toughness by controlling the critical length.

Macromechanical Approach

The macroscopic approach to the fracture of composites avoids considering the basic mechanisms of failure. Thus, some investigators [33, 62, 63, 68 through 70, 73] have attempted to explain fracture data

using the macroscopic fracture mechanics theories developed previously for metallic materials.

Konish, et al. [66] investigated single- and angle-ply ($\pm 45^\circ, 0/\pm 45^\circ/90^\circ$) graphite epoxy systems using a three-point bend test. They concluded that, the fracture of this composite can be characterized using theories of linear elastic fracture mechanics (LEFM) developed for metallic materials. They further concluded that the heterogeneity of the composite has a negligible effect on this conclusion.

Zimmer [70] studied $0^\circ/90^\circ/0^\circ$ graphite polyimide composites to conclude that the critical stress intensity factor was, indeed, a material constant. He, however, recommended to verify this conclusion for different fiber orientations and constituents.

Wright and Iannuzzi [33] showed that, if anisotropy is properly accounted for, the principles of LEFM can be used to describe the failure of carbon-reinforced epoxy specimens, irrespective of the failure mode.

Holdsworth and Owen [71] concluded from their study of glass fiber chopped strand mat and plain weave fabric-reinforced polyester resin laminates, that LEFM is a valid technique to predict fracture for these materials, provided small-scale yielding is properly accounted for.

As a result of the above work, it became clear that a number of parameters became important when the failure of an anisotropic, heterogeneous, brittle composite material was described using theories developed for isotropic, homogeneous solids. The most important

parameters seem to be anisotropy, mixed mode fracture, and small-scale yielding.

Using a finite element method, Mendell, et al. [67] showed that the degree of anisotropy had a significant effect on the stress intensity factor in some cases, with differing effects for different specimen shapes. They further showed that the effects of anisotropy are relatively constant for varying crack lengths of a given shape.

The effect of mixed mode fracture (Modes I and II) was studied by Lauraitis [80] and Saxena [83]. Both investigators concluded that, for brittle fracture, a single parameter can be evolved using LEFM in order to predict fracture. Sih [7] proposed a new theory to correlate the experimental data of unidirectional composites [25] under mixed mode failure. This will be discussed in detail later.

An additional complication that affects the fracture of metal matrix composites comes from the nonlinear behavior of the composite in the plastic range. It may be recalled that fracture characterization in such cases is complex, even for metallic materials. The two fracture concepts currently used for this purpose are J-integral and COD. The most widely studied ductile composite material is boron-aluminum.

Earlier experimental work on the fracture of boron-aluminum composites has been carried out by Olster and Jones [60], Kulkarni [19], Welch [69], and Jollay [68]. Olster and Jones [60] emphasized the importance of plastic zone size for unidirectionally reinforced B-Al composites. However, their estimated critical strain energy release rate was unexplainably higher than the predicted value. In other experimental works [19, 68, 69], attempts were made to generate

critical values of the stress intensity factors from the so-called R, or resistance curve, an early technique [8] to characterize ductile fracture. J-integral values were also evaluated using Eq. (30a), or from the load point deflections, using Eq. (31b). It was concluded that values of the critical stress intensity factor, K_{IC} , depended on the specimen size. The R-curve's criteria were shown [69] to be of limited utility due to matrix plasticity. However, definite conclusions were not made concerning the J-integral [19, 68, 69]. Conclusions could not be made for the COD technique either, since specific measurements could not be done [68, 69] to monitor crack tip displacements until fracture occurred.

Fracture Theories

It may be recalled that both the J-integral and the COD approaches attempt to characterize the unstable fracture from a materials point of view without explicitly caring for the actual state of stress near the crack tip. In recently developed macromechanical fracture theories [23, 74, 81], on the other hand, continuum solutions are used to predict unstable fracture. Both new theories discussed here are based on linear elastic continuum solutions, and avoid the unrealistic singularity ahead of the crack tip (see Eq. (28)).

Strain energy density. In the first theory, Sih [8] proposed the strain energy density concept to predict both the limiting condition of load and the direction of crack growth from a single parameter, strain energy density, defined as

$$S = a_{11} K_I^2 + 2a_{12} K_I K_{II} + a_{22} K_{II}^2, \quad (37)$$

where,

a_{ij}^S = functions of elastic moduli and Poisson's ratio .

This S_c theory is based upon the following assumptions:

1. The continuum solution (Eq. (28)) is valid only beyond a core region r (Fig. 9a), which is approximately the same order of magnitude of the crack tip radius.
2. Crack initiation takes place in a direction determined by the stationary value of the strain energy density factor, as given by

$$\frac{\delta S}{\delta \theta} = 0 \text{ at } \theta = \theta_0 .$$

3. Crack extension occurs when the strain energy density S a critical value, S_c , namely

$$S_c = S (K_I, K_{II}) \text{ for } \theta = \theta_0 .$$

This S_c theory showed good agreement [25] for unidirectional Scotchply 1002 composites, except for small fiber orientations. An average value of the radius of the core region was estimated as 0.02 in.

Critical volume concept. In the second theory [23], Wu proposed the existence of a characteristic volume (defined by r_c , Fig. 9b), which fully encapsulated each of many randomly distributed microscopic flaws. He further assumed that the stress was singular within the characteristic volume only, and the stress state outside the critical

volume was bounded and given by a known continuum solution. He finally proposed that failure or fracture occurred when the known stress state outside the characteristic volume satisfied a suitable strength criterion for the particular material. Since the stress state in a material (cracked or uncracked) can be determined independently using the continuum theory or numerical analysis, an appropriate phenomenological failure theory should be able to predict failure for a given material. It is interesting to note that failure and fracture were integrated in his theory. Using this theory, Wu [82] attempted to justify the statistical variation in unnotched strengths of composites. Wu has obtained excellent agreement with experimental results of notched strength of the unidirectional Scotchply 1002 brittle composite for mixed mode loading with cracks parallel to the fibers. He further showed that K_{IC} and K_{IIC} were, indeed, material constants for this composite material, as was critical volume (0.076 in.). Recently, he [82] proposed that the critical volume of brittle composites were given by the characteristic size (r_c) as

$$r_c \approx \frac{1}{2\pi} \left(\frac{K_c}{\sigma_f} \right)^2. \quad (38)$$

Compared to the Sih theory, the Wu theory is more useful for brittle composite materials where LEFM is valid. The advantages of the theory include

1. Both fracture and failure is characterized.

2. The theory can be used directly for anisotropic, homogeneous composite systems, at least in the linear range.
3. The crack need not be mathematical.

On the other hand, the singularity of stress is still allowed in the critical volume. Conceptually, it is not clear then, why the critical volume itself will not break down first for any nonzero loading. In other words, in this hypothesis, singularity is pushed in the back stage of critical volume. Secondly, the analysis has been limited to the linear, elastic case only. Nevertheless, the basic approach of this theory to unify fracture and failure theories is extremely interesting.

E. Finite Element Method

General Principles

The finite element method is a well-established numerical method used routinely by many practicing engineers. The method has been widely generalized during the last two decades to cover almost all of the major disciplines of engineering. Excellent texts [57, 84, 85] are available on this subject. The method is based upon idealizing a continuum structure with infinite degrees of freedom by a finite number of small subregions (known as finite elements) interconnected by a finite number of joints (known as nodal points), each with a predefined number of degrees of freedom. The choice of element shape and of the form of displacement or stress, or both, within the element leaves much to the ingenuity and skill of the engineer. The convergence of this approximate method is assured by minimizing the appropriate functional

of energy of the system in terms of the function (displacement or stress) chosen within the element.

In the displacement finite element the energy functional is the potential energy, and the chosen function within the element is the displacement. In the hybrid element [95] an equilibrating stress field is chosen within the element and a compatible displacement field is chosen along the element boundary. The complementary energy functional is used to ensure convergence of the assumed solution. Another popular finite element is known as the isoparametric element [57], where both the shape of the element and the assumed function in the element, namely, displacement, are evaluated at the same nodal points and with the same function of local coordinates. The various element specifications are shown in Fig. 10.

In the displacement finite element method, the displacement function (u) in a finite element is approximated as

$$\{\bar{u}\}^e = [C] \{q\}^e, \quad (39)$$

where,

$\{q\}^e$ = nodal displacement vector

$[C]$ = assumed relation between the nodal displacement and the displacement at any point in the element .

Using strain-displacement relations the approximate strains can be calculated from Eq. (39) as

$$\{\bar{\epsilon}\}^e = [B] \{q\}^e. \quad (40)$$

To make the assumed nodal forces statically equivalent to the actual distributed forces in the structure, infinitesimal nodal displacements, $\delta\{q\}^e$, are imposed without disturbing the equilibrium. This is known as virtual displacement. Now the work done by the external and internal forces is equated. Work done by the external forces, $\{Q\}^e$, is

$$(\delta\{q\}^e)^T \{Q\}^e . \quad (41a)$$

Work done per unit volume by the internal stresses is

$$(\delta\{\bar{\epsilon}\}^T \{\bar{\sigma}\}^e = (\delta\{\epsilon\}^3)^T) [D] \{\bar{\epsilon}\}^e , \quad (41b)$$

where, D is the elastic stress-strain matrix, so that the total work on the element by the internal stress equals

$$\begin{aligned} & \int_{V^e} (\delta\{\epsilon\}^e)^T [D] \{\bar{\epsilon}\}^e dV \\ &= \int_{V^e} ([B] \delta\{q\}^e)^T [D] [B] \{q\}^e dV \\ &= (\delta\{q\}^2)^T \left(\int_{V^e} [B]^T [D] [B] dV \right) \{q\}^e . \quad (41c) \end{aligned}$$

Equating Eqs. (41a) and (41c) gives the expression

$$\{Q\}^e = [K]^e \{q\}^e , \quad (42)$$

where,

$$[K]^e = \int_{V^e} [B]^T [D] [B] dV . \quad (42a)$$

It is noted that this relation is valid for any stress-strain relation, provided there is no initial strain or initial stress of body force. The term $[K]^2$ is known as the element stiffness matrix. Once the element stiffnesses are known, they can be assembled routinely for a given structure to get the total or master stiffness matrix. The solution is obtained by solving a set of equations similar to Eq. (42) by using a standard matrix inversion method used for a large number of symmetric, banded algebraic equations.

In Eq. (39), if the assumed displacement is linear in a triangular finite element, it is known as a constant strain triangle, or CST element. This is the most popular finite element for its simplicity. Although this element is very effective for solving regular structural problems, its effectiveness diminishes rapidly if there is a sharp crack in the structure, since this type of discontinuity leads to a singularity in the stress field at the crack tip.

Analysis of a Singular Stress Field

It is recalled that, for an isotropic, homogeneous material, a sharp crack introduces an inverse square root singularity in the stress ahead of the crack tip (see Eq. (28)), provided the material behaves linearly. It should be noted that, any deviation from any of the four following conditions changes the stress state ahead of the crack tip. Nonlinearity introduces the well-known HRR singularity (see Eqs. (17) and (35)), the severity of which depends on the strain-hardening parameter. Anisotropy [86] increases the singular stress value further, depending upon the severity of anisotropy (see Eq. (20)). The geometry of the crack tip also influences the stress field ahead of the

geometric discontinuity [87]. The crack tip stress singularity changes from an inverse square root to an inverse fractional power [35, 36] as the crack approaches a material discontinuity at a phase boundary, or a geometric discontinuity at an interface. These influences on the high stress gradient near the crack tip are important considerations in the analysis of fracture using the finite element method. To ensure dependable fracture analysis, it is necessary to ensure that the stress field near the crack tip is described accurately by the numerical method.

To solve the problem, special elements have been introduced in which the singularity is deliberately embedded in the element, provided the singularity is known.

Perhaps the most popular singular element is a quarter-point isoparametric element [89 through 91]. Here, the midside nodes of the adjacent sides of a regular quadrilateral isoparametric element situated at the crack tip are brought to the quarter points to incorporate the inverse square root singularity of the stress within the element. The major reason for the popularity is associated with the fact that this element already exists in most general purpose finite element programs. Additionally, any fixed inverse fractional power can be achieved with this element by suitably adjusting the position of the midside node, provided the singularity of the stress is known a priori. The need to know about the exact nature of the singularity prior to obtaining a solution restricts the utility of the element. It is also known that these types of elements cannot handle problems in which the material behaves nonlinearly [89, 91], or those types of problems where

the nature of the singularity changes [35, 36]. Although the compatibility of these elements with regular isoparametric elements is assured, it has been suggested recently [92] that transition elements should be used between these two sets of elements.

Recently, singular hybrid elements have been proposed [93 through 95] for solving fracture problems of isotropic or anisotropic material in both the linear and nonlinear ranges. In this element, the known singular stress solution is used to describe the stress distribution ahead of the crack tip. While this development is interesting and appears to be more accurate compared to other singular elements, this element can accommodate only inverse square root singularity of the stress field. Ensuring the compatibility of these elements with regular elements is difficult and the optimization of the size of these elements is not routine. Consequently, these elements are still restricted in makeshift research codes, rather than general purpose programs. Additionally, like most singular elements, these elements are "super elements" (i.e., their size is sufficiently large to account completely for the singular effect in the adjacent regular elements). Consequently, the geometrical details of the discontinuity cannot, in general, be reproduced very faithfully in the analysis. Finally, it may be noted that due to the inherent complexity in the formulation of these elements, the computer time in implementing them is expected to be large.

Recently, another method has been suggested to solve singular problems. In this method, the mesh is refined near the singularity in a perpetual manner. In one approach [96], the consequent internal

degrees of freedom are eliminated in order to reduce the computer time. In a second approach, a limiting condition is set in order to extract the solution without increasing the computer time considerably. However, this leads to a secondary eigenvalue problem [97]. Consequently, neither approach considers regions immediately adjacent to the singularity. This does save time, but in turn, misses the vital considerations of the problems considered here; namely, the spatial dependence of the element properties.

Plasticity Analysis

Apart from the singular stress behavior, any nonlinearity of the material must be considered in the finite element analysis of many fracture problems. This is particularly true for a ductile composite material like the B-Al composite.

There are several methods to implement the effects of plasticity in a structure using the finite element method [30, 57, 84].

In the variable stiffness method [30, 57] the total load is applied and the stresses and strains are determined elastically. Using an effective stress, the total load is scaled to a fractional load for which yielding occurs in a few elements only at some critical effective stress. The rest of the load is applied in small increments. As more elements yield, the stiffness of each yielded element is replaced using the tangent elastic modulus from the effective stress-effective strain curve, and the value of the Poisson ratio is assumed equal to 0.5 (Fig. 11a).

In the initial strain method [30, 57], the solution of an elasto-plastic problem is based on the original elastic stiffness

matrix. But the effect of plastic strains is taken into account as initial strains resulting in fictitious nodal forces. The problem is solved elastically for every iteration using a fractional load (as in the variable stiffness method) and the fictitious nodal forces. For the same load increment, a prescribed number of iterations are carried out with updated fictitious nodal forces to ensure convergence (Fig. 11b).

The initial stress method [30, 57, 84] is similar to the initial strain approach within a load increment. However, unlike the initial strain method, the plastic strain itself is not used to calculate the fictitious nodal forces in this method. Instead, the residual stresses are used to calculate the fictitious nodal forces. The residual stress is calculated as the difference of the elastic stress calculated for the load increment applied and the actual stress that will be reached using the elasto-plastic stiffness matrix for the particular load increment (Fig. 12).

In this analysis the initial stress method [30] is used. This method has the advantage [57] of handling nearly flat stress-strain curves, unlike the initial strain method, which can only handle materials exhibiting marked strain-hardening. By optimizing the inversion of the total stiffness matrix, a quick convergence is ensured without much loss in computer time, unlike the case with the variable stiffness method [57], where the total stiffness matrix is updated as any element yields and is then inverted, leading to a large requirement of computing time.

Basically, the initial stress method is a mixed method in which the stiffness matrix is modified for each increment of load, but is kept constant for the iterations within the increment. For the i^{th} increment of the load, the iterative procedure is

$$[K]_i \{\Delta q\}_i^j = \{\Delta Q\}_i + \{\Delta Q_0\}_i^j, \quad (43)$$

where,

$$\{\Delta Q_0\}_i^1 = \{0\}$$

i = a load increment

j = one cycle of iteration with the
load increment i .

The elastic stresses $\{\Delta \sigma_e\}_i^j$ and strains $\{\Delta \epsilon_e\}_i^j$ are calculated for each element. From the elastic stresses, it can be determined if the element has exceeded the yield condition. If the stress state is in the elastic range, the next load increment $i = i+1$ is implemented. If the stress state exceeds the yield surface, the stress state is interpolated to bring it on the yield surface. The effect of yielding for an element implies that the calculated stress state has over-estimated the actual stress state. The actual stress increment, $\{\Delta \sigma_c\}_i^j$, can be calculated from Eqs. (14) through (16). The difference between these two increments of stresses are redistributed as the nodal equilibrating forces of the element, as is done with thermal stresses. This is done as follows:

$$\{\Delta Q_0\}_i^{j+1} = \int_V [B]^T (\{\Delta \sigma_e\}_i^j - \{\Delta \sigma_c\}_i^j) dV, \quad (44)$$

where, $[B]$ is the strain nodal displacement matrix. This iterative process is carried until convergence is achieved or a pre-fixed number of iterations is accomplished. If the residual nodal forces, as calculated using Eq. (44), are still appreciable, they are, then, carried over to the next increment of the load.

The process is continued until the total load is applied incrementally. If, at the end of the last incremental load, the iterative procedure of Eq. (43) leads to approximately zero, the desired solution has been reached. Otherwise, it is assumed that the collapse condition has been reached.

CHAPTER III

PRESENT ANALYSIS

In this chapter the details of the analysis used in this investigation are presented. For this purpose the chapter is divided into three sections. In Section A, modifications to the existing concepts of fracture and failure of composite materials are suggested. Necessary relations have been developed to integrate the concepts of existing fracture parameters. In Section B, the material properties of the composite specimens analyzed in this investigation are described. In Section C, a clear, step-by-step methodology is suggested to predict the fracture and failure behaviors of a composite material using the computer simulation technique.

A. Modifications

Gradual Grading Method for Crack Analysis

If the fracture behavior of a composite material (brittle or ductile) is investigated using a finite element method, a number of flexibilities have to be incorporated in the analysis. They include:

1. Flexibility on the geometry of the crack (i.e., no "super element").
2. Flexibility on the material behavior (i.e., orthotropy or anisotropy).
3. Flexibility on the stress-strain law (i.e., elastic or plastic analysis).

In addition, computing time should be reasonable. Recalling the problems associated with the singular elements (particularly, the inflexibility in the order of singularity), it was decided to use the graded finite element mesh in this investigation.

It was predicted [85] that the same order of accuracy could be achieved for a singular as for a regular solution using the finite element method, provided the singularity was accounted for properly by the use of either a singular element or suitably graded mesh. Therefore, if constant strain triangle (CST) elements are used to solve an elastic crack problem with an inverse square root singularity of stress, the error in the estimated displacement field would be proportional to the average size of the gradually reducing elements ahead of the crack tip [85]. But the error would be much higher (proportional to the square root of the size of the element) if a uniform mesh was used [85]. For grading of the element mesh for one-dimensional problems with a singularity that varies as x^β at the origin, Strang and Fix [85] suggested the following:

$$h_i^{k-m+1/2} x_i^{\beta-k} \approx \text{constant} , \quad (45)$$

where,

h_i = size of the i^{th} element

x_i = distance of the i^{th} element from the origin

$(k-1)$ = degree of the polynomial approximation

$2m$ = order of the differential equation to be solved .

It is noted from Eq. (45) that the elements nearer the origin are smaller in size. The actual reduction in size needed to achieve a desired degree of accuracy obviously depends on the order of the singularity β for a given problem and the approximation of the finite element. Typically, to solve an elastic crack problem ($m = 1$, $\beta = -0.5$) with linear elements ($k = 2$), the element size restriction becomes $h_{i+1} \approx h_i \times (x_i + 1/x_i)^{1.7}$, so that the reduction in the sizes of the element has to be very drastic as the crack tip is approached.

However, for two-dimensional problems, continual reduction in element size, as the crack tip is approached, may lead to a very large bandwidth (defined by the maximum difference in the numbers of nodal points in an element of the mesh), as clarified in Figs. 13 and 14. This is an unsatisfactory feature, since an increase in bandwidth causes a rapid increase in the computing time. From a number of experimental runs with different mesh sizes, it was found that for large mesh sizes, the computer time estimate on an IBM 360/65 computer was given by

$$T(\text{sec}) = 0.2' + 0.1H' + 4.31 \times 10^{-5} Nb^2, \quad (46)$$

where,

E' = total number of elements

N = total number of nodes

H' = number of seven-term hybrid rectangular element

b = semi-bandwidth .

It is noted that this estimation was based on the program used in this investigation. From Eq. (46) it is clear that regular hybrid

elements consume considerably more time, compared to a CST element. Further, it may be observed that the computer time increases very rapidly as the semi-bandwidth increases. Reduction in semi-bandwidth, and hence, computer time, may be achieved by suitably introducing "slender" (implying length/width ≥ 2) elements. But, on the other hand, it should be realized that, if the ratio of length over width becomes large (≥ 5) in an element (known as "needle element"), it may lead to poor numerical results [85] due to the large difference in the elements of the stiffness matrix so generated. Consequently, care has to be exerted in using a graded finite element mesh to solve a singular problem for a reasonable computer time. The method is detailed in Figs. 13 and 14 and is described below.

In Fig. 13, RXYZ is the region to be analyzed with the crack shown as RO. The terms RX and RZ are the lines of symmetry. Region OMNP is assumed to be the region over which the effect of the crack is dominant in the solution. Normally, this is a circular region of radius equaling three to four times the crack length. Consequently, finite element grading is done only in this region (OMNP). This is chosen as a square region for the convenience of using triangular and rectangular elements. Another square region, OJKL, is now inserted into the region OMNP, where J, L, and K are midpoints on the sides OM, OP, and the diagonal ON, respectively. Similarly, another square region, OGHI, is introduced into the square region OJKL. The process is continued until the smallest square insert OABC at the crack tip is small enough to achieve a subsequently computed result of the desired accuracy. The determination of the optimum size of this square OABC

may need some trial runs, the number of which can be reduced considerably from the knowledge of the problem and past experience. It is noted that the element size within any resulting polygon (JKLPNM or GHILKJ or OABC) is approximately the same. To reduce the bandwidth and eliminate "needle elements," the size of the elements adjacent to the refined region (region OPQR) might have to be adjusted.

The details of the resulting finite element meshes are shown in Fig. 14. As the element size is reduced near the crack tip O (from Fig. 14a and b), the possibility of increased bandwidth and/or generation of slender elements is shown in Fig. 14b with the help of dotted lines. Using the mesh generated by the dotted lines (Fig. 14b), both objectives of reducing the semi-bandwidth and elimination of "needle elements" might be achieved. Of course, this leads to the increase in the number of elements. But it is clear from Eq. (46) that this method still reduces considerably the total computing time for a comparable accuracy. The elimination of "needle elements" also ensures the numerical stability of the method. Figure 14c shows a typical finite element mesh finally generated using this method. The repetitive nature in this method is evidenced from Fig. 14a, b, and c.

Investigation of J- δ Relationship

The initiation of unstable fracture in a ductile specimen may be characterized by the critical value of either the J-integral or the COD (δ). Since both J and COD can be used to characterize the same fracture phenomena, it was expected that they should be related in a simple way [8, 16, 101, 102]. A number of investigators attempted to

obtain an expression that would correlate these two parameters, at least, for isotropic, homogeneous metallic materials. It was indicated that, for small-scale yielding the following empirical relation should exist [101],

$$\frac{J}{\delta\sigma_y} = M . \quad (47)$$

Unfortunately, the value of M varied, depending on the strain-hardening coefficient [16], state of stress [102], and the specimen geometry [101]. In general, however, the value of M , reportedly, varied [101] between 1.0 and 2.98. Consequently, M is not a constant, as was expected from the proposed relation (Eq. (47)). From the experimental data of the three-point, single-edge, notched bend specimen of BS 4360 grade 50C steel [101], as shown in Table 3, it was found that the value of M depended on the specimen geometry and also on the stress level. Since the above empirical relation did not reflect the dependence of M on factors such as specimen size, strain-hardening coefficient, and stress level, it was decided to investigate a more precise relationship between the J-integral and COD.

It is recalled that for small-scale yielding, the effective stress and displacement ahead of a sharp crack in an infinitely large plate are given by [51 through 54] as

$$\sigma_e(r, \theta) = \bar{\sigma}_y \left(\frac{J}{\bar{\sigma}_y I_n} \right)^{\frac{n}{n+1}} r^{-\frac{n}{n+1}} \bar{\sigma}_e(\theta) , \quad (19a)$$

$$u_i(r, \theta) = \left(\frac{J}{\bar{\sigma}_y I_n} \right)^{\frac{1}{n+1}} r^{\frac{n}{n+1}} \tilde{u}_i(\theta) . \quad (19b)$$

It is noted from Eq. (18a) or Eq. (19a) that, the stress state or effective stress ahead of a sharp crack tip becomes unrealistic at points very close to the crack tip. In other words, these mathematically idealized equations are valid only beyond a limiting value of r , say r_c . This critical value of r ($=r_c$) can be calculated from the equation of effective stress, if it is assumed that the effective stress cannot exceed the failure stress, σ_f , of the material. For this purpose, it is convenient to express $\tilde{\sigma}_e(\theta)$ and $\tilde{u}_i(\theta)$ as dimensionless quantities with maximum values being unity. Rearranging Eqs. (19a) and (19b), Sih [53] showed that they can be written equivalently as

$$\sigma_e(r, \theta) = \bar{\sigma}_y \left(\frac{J\bar{E}}{k I_n \bar{\sigma}_y^2} \right)^{\frac{1}{n+1}} r^{\frac{n}{n+1}} \bar{\sigma}_e(\theta) \quad (48a)$$

$$u_i = \frac{k \bar{\sigma}_y}{\bar{E}} \left(\frac{J\bar{E}}{I_n k \bar{\sigma}_y^2} \right)^{\frac{n}{n+1}} r^{\frac{1}{n+1}} \bar{u}_i(\theta) \quad (48b)$$

It is recalled that k is the Ramberg-Osgood constant, and $\bar{E}$ is Young's modulus of a material that exhibits a power law strain-hardening.

Denoting θ_c as the direction along which the effective stress reaches

the maximum value of failure stress, σ_f , the critical values of r ($=r_c$) are then found from Eq. (47) as the maximum value r , so that $\sigma_e(\theta_c) = 1$, and

$$r_c = \left(\frac{\bar{\sigma}_y}{\sigma_f} \right)^{n+1} \cdot \frac{J_c \bar{E}}{k I_n \bar{\sigma}_y^2} . \quad (49)$$

Since the effective stress is equal to the failure stress, the J-integral reaches the critical value J_c in this equation. If J_c is a material constant for a particular material, the right-hand side of this equation contains parameters which are specific only to the material investigated. In other words, r_c is a material constant, as has been anticipated by many investigators [23, 74, 81]. For the linear elastic case, $n = 1$, $k = 1$, and if the failure is of the Mode I type, it is found that $I_n \approx 6.3$ [53]. Substituting these values and $J_{Ic} = K_{Ic}^2 (1 - \nu^2) / \bar{E}$ into Eq. (49), it is found that

$$\sigma_f^2 r_c \approx \frac{K_{Ic}^2}{2\pi} . \quad (50)$$

It may be recalled that Wu [82], using a generalized Weibull distribution, derived a similar expression from his random crack hypothesis [23], namely,

$$\sigma_f^2 r_c \approx \frac{K_{Ic}^2}{2\pi} . \quad (38)$$

If the failure is of a Mode II type for a metallic material, $I_n = 1.7$, so that r_c is given by

$$\tau_f^2 r_c \approx \frac{2}{\pi} K_{IIc}^2 . \quad (51)$$

Since for many brittle composites, K_{IC} and K_{IIIC} are material parameters, it may be concluded that, indeed, there is a "critical region," the boundary of which is defined by r_c . In general, the stress distribution in the critical region is similar to that developed for the nonlinear behavior of a material in the presence of a crack. In other words, this critical region is a mathematical modeling of an "intrinsic crack."

Since the effective stress reaches the maximum value on the boundary of r_c and the mathematical modeling is not valid beyond this region, it is reasonable to assume that this effective stress remains constant within ($r < r_c$), the critical region. With this assumption, it is possible to calculate the complete boundary of the critical region using Eqs. (47) and (49). However, the typical value of r_c is of the order of 10^{-2} in. or less; thus, the shape of the critical region has little practical significance. Since the stress, and hence, strain state do not change within the critical region, it is possible to correlate the effective stress at the crack tip, i.e., $\sigma_e(r_c, \theta_0)$, and the crack-opening displacement δ , i.e., $u_2(r_c, \theta_0)$, measured along the direction θ_0 , using Eqs. (48a) and (48b). On multiplication of these two equations, it is found

$$\begin{aligned} \sigma_{e_{\max}} \cdot \delta &= \sigma_e(r_c, \theta_0) u_2(r_c, \theta_0) \\ &= \frac{J}{I_n} \bar{\sigma}_e(\theta_0) \bar{u}_2(\theta_0) . \end{aligned} \quad (52)$$

It is noted that the uniform strain state in the critical region allows the COD to be measured at the crack tip. However, this is true

provided the crack width is at least of a comparable size to the critical region. The rearrangement of Eq. (52) leads to

$$\frac{J}{\delta\sigma_y} = \left(\frac{\sigma_{e_{\max}}}{\sigma_y} \right) \left[\frac{I_n}{\bar{\sigma}_e(\theta_0) \bar{u}_2(\theta_0)} \right]. \quad (53)$$

It is recalled that, I_n depends on the strain-hardening coefficient (Table 2) and failure mode [53]. On the other hand, the maximum effective stress, $\sigma_{e_{\max}}$, depends on the applied stress and the specimen geometry. It is apparent from Eqs. (47) and (53) that, M is, indeed, a function of the material, specimen geometry, and applied stress. Recalling that I_n , $\bar{\sigma}_e(\theta_0)$, and $\bar{u}_2(\theta_0)$ are numerical constants to be evaluated for the particular case, the following empirical relation is suggested for experimental purposes:

$$\frac{J}{\delta\sigma_y} = M \left(\frac{\sigma}{\sigma_f} \right)^\lambda. \quad (54)$$

In this equation, M and λ account for the geometric factors and strain-hardening coefficient, respectively. From one single experiment, this equation can be plotted on a log-log paper to calculate both M and λ .

Fracture of Notched Brittle Composites

It is recalled that if a crack propagates radially without changing the direction (Fig. 8), the propagation of the branched crack may be treated in a Mode I manner, even if it is a mixed mode failure [75]. Consequently, the critical stress intensity factor can be

calculated as follows [75] for a center-cracked tension specimen (Fig. 2):

$$K_{\theta_c} = \sigma_{\theta \text{notch}} \sqrt{\pi a \sec \frac{\pi a}{w}} , \quad (55)$$

where,

θ = radial direction of crack propagation

$\sigma_{\theta \text{notch}}$ = applied stress to cause failure

$\sqrt{\sec \frac{\pi a}{w}}$ = width correction factor .

Since the problem can be treated in a Mode I manner, Eq. (50) can be used, provided the material behaves linearly until failure, so that

$$\frac{K_{\theta_c}^2}{\sigma_{\theta_f}^2} = 2\pi r_c , \quad (56)$$

where,

σ_{θ_f} = failure stress of the material along the direction θ .

For an anisotropic, homogeneous material or a composite material which can be treated so, the above equations can be used to predict the notched strength from the knowledge of the unnotched strength, provided the material deforms linearly when loaded. For a longitudinally or transversely reinforced composite, Eq. (56) can be written as

$$\frac{K_{0c}^2}{\sigma_{0f}^2} = 2\pi r_c, \quad (57a)$$

$$\frac{K_{90c}^2}{\sigma_{90f}^2} = 2\pi r_c, \quad (57b)$$

where,

K_{0c}, K_{90c} = critical stress intensity factors

σ_0, σ_{90f} = unnotched strengths of the composites .

Similarly, for an off-axis, unidirectional composite, Eq. (56) is written as

$$\frac{K_{\theta c}^2}{\sigma_{\theta f}^2} = 2\pi r_c. \quad (57c)$$

Combining Eqs. (56a) or (56b) with (57c) it is seen that

$$K_c = \frac{\sigma_{\theta f}}{\sigma_{0f}} K_{0c}, \quad (58a)$$

$$K_c = \frac{\sigma_{\theta f}}{\sigma_{90f}} K_{90c}. \quad (58b)$$

Equations (58a) and (58b) imply that the critical stress intensity factor can be determined for an off-axis composite from the knowledge of its unnotched strength, provided the unnotched strength and critical stress intensity factor are known for at least one other fiber orientation of the composite.

It is noted from Eqs. (57) and (58) that, the critical stress intensity factors for a unidirectional, filamentary composite are not independent of each other for two fiber orientations. They are dependent on their unnotched strengths, which in turn, depend on the strengths of the composite along the principal material directions.

If the crack lengths are identical for two fiber orientations, application of Eq. (55) in Eq. (58a) or Eq. (58b) shows that

$$\sigma_{\theta\text{notch}} = \frac{\sigma_{\theta f}}{\sigma_{0f}} \sigma_{0\text{notch}} , \quad (59a)$$

$$\sigma_{\theta\text{notch}} = \frac{\sigma_{\theta f}}{\sigma_{90f}} \sigma_{90\text{notch}} , \quad (59b)$$

where, the subscript "notch" implies the notched strength of the composite with identical crack length.

Thus, the notched strength of a brittle, unidirectional composite material can be predicted from the unnotched strength if the notched and unnotched strengths are known for at least one other orientation of the fiber.

Although the equations are referred to for unidirectionally reinforced brittle composites, they are also valid for other brittle composites, provided the composite can be treated as an anisotropic, homogeneous material with two principal material directions, and the crack propagates in a radial direction at failure.

Interaction Term in the Tsai-Wu Theory

The Tsai-Wu theory may be used to predict the unnotched filamentary strength for any fiber orientation, as discussed previously. It is also, however, realized that one of the problems of the Tsai-Wu theory is the determination of the interaction term F_{12} . In the absence of the Bauehinger effect ($F_i = 0$) and inadmissibility of failure under hydrostatic pressure, the Tsai-Wu theory for anisotropic composite materials reduces to the von Mises theory for isotropic, homogeneous material. In other words, the Tsai-Wu theory is an extension of the von Mises theory. Consequently, it seems a reasonable choice for c in Eq. (7) to be $-\frac{1}{2}$, so that

$$F_{12} = -\frac{1}{2} (F_{11}F_{22})^{1/2} . \quad (60)$$

This definition of the interaction term is independent of the admissibility of failure under hydrostatic pressure. The effectiveness of this modification of F_{12} is checked with the values reported by Pipes and Cole [45]. From careful experiments of multiaxial loadings of boron epoxy (AVCO 5505) composites, F_{12} values were determined between $-0.58 \times 10^{-10} \text{ in.}^2/\text{lb}^2$ to $+29.50 \times 10^{-10} \text{ in.}^4/\text{lb}^2$. Using the reported material properties [45] in Eqs. (7) and (60), the value of F_{12} was calculated as $-0.953 \times 10^{-10} \text{ in.}^4/\text{lb}^2$. The value of F_{12} which was finally chosen by Pipes and Cole [45] was $-0.58 \times 10^{-10} \text{ in.}^4/\text{lb}^2$. Consequently, it is believed that Eq. (60) is reasonable. Thus, in this work, the value of c in Eq. (7) was given as $-\frac{1}{2}$. With this simplification, the failure theory of Tsai-Wu becomes

$$f \equiv F_{11} \sigma_x^2 + F_{22} \sigma_y^2 + F_{66} \tau_{xy}^2$$

$$- \sqrt{F_{11}F_{22}} \sigma_x \sigma_y - F_1 \sigma_x - F_2 \sigma_y - 1 = 0 , \quad (61)$$

when the axes of the load and crack coincide with the material axes in plane stress. The major advantage of this simplification is the elimination of complex and difficult multiaxial loading experiments necessary to determine F_{12} .

Generalized Definition of Crack-Opening Displacement

Definition of the term crack-opening displacement (COD) for anisotropic, homogeneous material is difficult, since, due to anisotropy, the deformation at the crack tip has both the horizontal and vertical components (Fig. 15). Unfortunately, the horizontal slip at the crack tip cannot be monitored for a composite specimen using conventional clip gages. But this horizontal deformation forms an integral part of the total deformation at the crack tip, which will be accepted here as a more general definition of crack-opening displacement (COD). Thus, the crack-opening displacement ($COD = \delta$) is defined as (Fig. 15)

$$\delta = \left(\delta_h^2 + \delta_v^2 \right)^{1/2} , \quad (62)$$

where,

δ_h = horizontal deformation at the crack tip

δ_v = vertical deformation at the crack tip

δ = total crack-opening displacement .

Experimental measurement of both of the components or the total COD might not be easy. Since this investigation was a numerical experiment, δ was defined by Eq. (62). It might be of interest to point out that δ measured at the tip of the crack might be significantly different from the deformation at the center of the crack for an anisotropic composite material. It is expected that the points away from the crack tip need not necessarily bear the direct signature of the singularity, and hence, initiation, of the crack. Consequently, as a fracture criterion, COD was measured only at the tip of the crack and nowhere else. This becomes particularly critical for anisotropic materials, as was found during the investigation.

B. Material Properties

In this investigation, the stress fields generated in center-cracked tension (Fig. 2) and compact tension (Fig. 3) the specimens were subjected to in plane stresses, were computed. Glass fiber-reinforced epoxy (GFRE) and boron-aluminum (B-Al) composites were chosen as the brittle and ductile composite materials, respectively. Both unidirectional and $0^\circ/90^\circ$ cross-ply composites were studied in each case. The approximate fiber volume fraction in the composites was 0.5. The effects of orientations and cracks lengths were studied for both of the composite systems. The details of the specimen geometries are shown in Figs. 2 and 3.

Glass Fiber-Reinforced Epoxy (GFRE)

Experimental results for GFRE were obtained by Kulkarni [19] for both unidirectional and $0^\circ/90^\circ$ cross-ply composites. Experimental

values of longitudinal and transverse strengths were obtained. However, other parameters necessary for the computation were estimated from the data compiled by Lubin [3]. Specific values of the properties of the unidirectional and $0^\circ/90^\circ$ GFRE composites used in this investigation are shown in Tables 4 and 5. It is noted that the shear strengths and moduli are the same for both unidirectional and $0^\circ/90^\circ$ composites, because the planar shear properties are identical for 0° or 90° or $0^\circ/90^\circ$ composites.

Boron-Aluminum (B-Al) Composites

The $0^\circ/90^\circ$ cross-ply B-Al composite. Experimental results for $0^\circ/90^\circ$ B-Al composites were obtained by Jollay [68]. For the composite with 13 layers (seven longitudinal and six transverse composite layers), he found the complete stress-strain curve along the longitudinal (0°) direction. Counting the number of transverse layers, the transverse strength was scaled down from the longitudinal strength by a factor of $6/7$. The consequent stress-strain curve for the $0^\circ/90^\circ$ B-Al composite used in this investigation is shown in Fig. 16. Since shear properties are identical for 0° or 90° or $0^\circ/90^\circ$ laminates, they were taken from the reported values [104 through 107]. Kreider [104] reported that, depending upon the experimental method used, the shear modulus of the B-Al composite (with 50% fiber) varied from 7.5 to 9.5 MSI. The shear modulus was selected as 8.0 MSI in this analysis for both $0^\circ/90^\circ$ and unidirectional composites. The inplane shear strength was reported by Davis, et al. [107] as 9.3 to 15.3 KSI for 50% fiber-reinforced unidirectional composites, and 14 KSI to 16.3 KSI

for 45% fiber-reinforced $0^\circ/90^\circ$ cross-ply composites. Since the estimated fiber volume percent was about 44% [68] for the $0^\circ/90^\circ$ composite, shear strength was selected as 16 KSI. For the 50% volume fiber-reinforced unidirectional composite, the shear strength was chosen as 12 KSI. The material properties of the $0^\circ/90^\circ$ B-Al composite used in this investigation are shown in Table 5.

Unidirectional B-Al composite. The longitudinal and transverse properties of unidirectionally reinforced boron 6061-aluminum composites were selected from the experimental work of Welch [69], who used 50% volume 5.6 mil boron fibers. The mechanical properties of off-axis (30° , 45° , and 60°) composites were based mostly on the experimental work of Kreider and Prewo [104, 105, 108], who used 4 mil Borsic fiber in the 6061 aluminum matrix. Table 6 shows the material properties of unidirectionally reinforced B-Al composites used in this investigation. Figure 17 shows the complete stress-strain curves for all the composites as a function of fiber orientation (θ) measured with respect to the crack axis, which was always perpendicular to the loading axis (Fig. 2) in this investigation. It was assumed that the stress would be proportional to the strain up to the yield point for any composite. Except for the longitudinally reinforced specimen ($\theta = 90^\circ$), all composites were assumed to deform in a totally plastic manner after the yield point (i.e., no work-hardening was assumed for transverse and off-axis composites).

The secondary modulus and the ultimate strength of the longitudinal composite ($\theta = 90^\circ$) were chosen as 30 MSI and 200 KSI, respectively, from the experimental work of Welch [69] and White [109].

The failure strain was chosen as 0.0065 from the reported values [69, 106]. Using these values and the assumed bilinear stress-strain relation, the yield stress was calculated as 35 KSI. The primary modulus used in this calculation of the yield strength was used as 35 MSI from the rule of mixture (implying proportional contribution of each constituent).

For the transverse composite ($\theta = 0^\circ$), the ultimate strength and the primary modulus were chosen as 30 KSI and 17 MSI, respectively, from the reported experimental results [69, 104 through 197, 109]. It is noted that, the small flat top portion of the stress-strain diagram obtained from the transversely reinforced composite is due to the ductile aluminum matrix (failure strain $\sim 20\%$). As the composite reached its failure stress, one of the matrix layers between two fibers would flow to failure, leading to the failure of the total composite. Thus, the actual fracture strain appeared to have a slight dependence on the length of the specimen. However, the estimated failure strain of 0.002 of the transverse composite of the 3-in. specimen compared favorably against the reported failure strains of 2.3 to 2.4×10^{-3} for the 5.6 mil B-A1 (55% of fiber) composite [108].

It is noted that, the primary moduli and the yield strengths of the off-axis composites can be predicted from the corresponding values of the longitudinal and transverse composites using the Tsai-Wu theory. From the experimental results of Kreider and Prewo [104], as shown in the insert of Fig. 17, the stress-strain curves of the off-axis composite were assumed to be a flat top beyond the yield stresses. The failure strains parallel to the loading direction were calculated on

the assumption that the failure occurred when the component of the strain along a direction parallel to the fiber axis had reached a critical value. An approximate experimental value of 0.11 failure strain of the 45° off-axis composite [105] was used as the basis of calculating the failure strain of the 30° and 60° off-axis composites. The failure strains thus calculated and used in this investigation were 0.052, 0.11, and 0.127 for $\theta = 30^\circ$, 45° , and 60° , respectively. The corresponding experimental values obtained by Kreider and Prewé [105] were 0.5, 0.107, and 0.12, respectively, for the B-Al composites.

Considering the variations in the reported values of other strength data (longitudinal or transverse or shear), the agreement between the experimental and estimated failure strains of the off-axis composites seems reasonable. Comparisons were also made for the off-axis composite strengths. The data sheet of Composite Technology, Incorporated [106] showed that the ultimate strengths for 55° and 45° off-axis composites were 21 to 28 KSI and 17 to 23 KSI, respectively. The estimated ultimate strengths used in this investigation for 60° and 45° were 24.6 KSI and 23.6 KSI, respectively.

The complete stress-strain curves in tension which were used in this investigation are shown in Fig. 17, along with the experimental results of Kreider and Prewé [105] for Borsic-aluminum. They further reported that for B-Al composites the stress-strain curves were approximately the same in both tension and compression [105]. In this investigation also, the same stress-strain curves were used in tension and compression for both the unidirectional and the 0°/90° composites.

C. Methodology

A finite element mesh was constructed using the following sequence:

1. A crude mesh was identified that could be used to determine the stress field within the structure, assuming there was no singularity.
2. A sufficiently large square (or circular) region was identified in which the solution would be dominated by the singularity (Fig. 13).
3. Refinement was done by dividing the square block in a systematic manner such that, in every resulting polygonal region the size of the elements was approximately a quarter of the elements in the next outer polygonal region, and approximately four times the elements in the next inner polygonal region (Fig. 14). It was noted that the largest element in the outermost polygonal region was approximately equal to the size of the initial crude mesh. The refinement procedure was actually based on the simple geometric construction of reducing the size of the square by halving the diagonal first and then constructing the next series of squares.
4. Introduction of new nodes to eliminate "needle elements" was optimized in such a way that a balance was reached between the semi-bandwidth and the number of extra nodes added.

Since the method is repetitive, the total number of nodes (n), semi-bandwidth (b), slenderness ratio, and size of the smallest element can be predicted before actually drawing the detailed mesh for computational purposes. Using Eq. (46) the estimate of the computer time can also be made. Depending upon the difference between the estimated and spendable computer time, the number of nodes in the refined zone, and hence, the size of the smallest element at the crack tip can be controlled. It may be recalled that the size of the smallest element at the crack tip [85] is an indicator of convergence of the singular problem using graded refinement. Another advantage of this method is that, even if further refinement is necessary near the crack tip, the refinement can be done locally without disturbing the rest of the mesh. However, the real advantage of this gradual grading method comes from the fact that any spatial material property can be assigned for each and every individual element. This is particularly important in the fracture and failure analysis of composite materials.

Once the finite element mesh is decided, the failure analysis can be done for the given composite material. However, for the failure analysis it is necessary to know if the composite material is brittle or ductile (i.e., the stress-strain response of the unnotched specimen is linear or nonlinear).

The failure of a brittle composite (GFRE) was computed in the following manner:

1. The strengths and moduli of the composite material along the principal material directions were estimated from the experimental or reported data.

2. The Tsai-Wu theory with the suggested modification for the interaction term F_{12} was used to predict the unnotched strength of the composite for any orientation of the fibers with respect to the loading axis.
3. Using the graded finite element mesh generated earlier, the notched strength of the composite was calculated for the given crack length and any fiber orientation (principal or off-axis). In this analysis, an element was assumed to have failed if it satisfied the Tsai-Wu failure theory. The failure of one or more element indicated the failure of the specimen.
4. If the notched strength was calculated directly for the off-axis orientation of interest, Eq. (55) was used to calculate the critical fracture toughness.
5. If the notched strength was calculated for any principal material direction (0° or 90°), the off-axis notched strength and critical stress intensity factors were calculated using Eqs. (58) and (59).

For ductile composites a bilinear analysis was conducted as follows:

1. The total load was applied to the structure, and the elastic deformations were calculated (Fig. 12). The effective stress was calculated for each element (Addendum II, Appendix C). The ratio between the uniaxial yield stress and the maximum effective stress was denoted as the load factor. For a load factor greater

than unity only, the following steps were followed. Otherwise, the loading process being elastic, further iteration was not required.

2. Initial load increment, P_0 , was taken as the total load divided by the load factor. The rest of the load was applied incrementally in a predefined number of equal increments (ΔP).
3. The load increment, ΔP , was applied and the elastic increments in the stresses $\{\Delta\sigma_{ei}^{(j)}\}$ and the strains $\{\Delta\varepsilon_{ei}^{(j)}\}$ were calculated.
4. The elastic increments in the stresses were added to the existing stresses (for the first cycle it was the stresses calculated and consequently reduced by the load factor) to find the final stress state $\{\sigma'\}$. Using the Tsai-Wu theory and the final state of stress, it was calculated if the stress system was in the elastic ($f < 0$) or plastic ($f \geq 0$) range. If the process was elastic, no more iteration was needed for the element.
5. If $f > 0$, the corresponding stress system was adjusted suitably to bring it on the yield surface to make $f = 0$.
6. If two stress states were found in the consecutive iterations, so that $f\{\sigma_0\} = 0$ and $f\{\sigma'\} \geq 0$, the nonlinear stresses were calculated using Eqs. (14) through (16).
7. The difference between the linear and nonlinear stress increments was treated as residual stress:

$$\left\{ \Delta\sigma_{r,i}^{(j+1)} \right\} = \left\{ \Delta\sigma_{e,i}^{(j)} - \Delta\sigma_{c,i}^{(j)} \right\}. \quad (63)$$

These were used to calculate the fictitious nodal forces using Eq. (44).

8. The stresses and strains were updated as follows:

$$\text{current stress} = \{\sigma\} = \{\sigma'\} - \left\{ \Delta\sigma_{r,i}^{(j+1)} \right\}, \quad (64a)$$

$$\text{current strain} = \{\epsilon\} = \{\epsilon'\} + \left\{ \Delta\epsilon_{e,i}^{(j)} \right\}. \quad (64b)$$

9. Equation (43) was re-used with the updated fictitious nodal forces. The new elastic solution was used to follow steps 4 through 8.
10. If the nodal forces became approximately zero, a new load increment was applied and steps 3 through 9 were followed.
11. If the nodal forces were not zero, but the predefined number of iterations (say, eight) for solving Eq. (43) were performed, the excess nodal forces were added to a new incremental load and steps 3 through 9 were followed.
12. Solution was achieved if the full load was applied and a corresponding iterative solution led to approximately zero nodal forces. Otherwise, the structure had supposedly collapsed.

CHAPTER IV

RESULTS AND DISCUSSION

This chapter is divided into three sections. In Section A, the elastic stress analysis is presented for a $0^\circ/90^\circ/\pm 45^\circ$ cross-ply boron-aluminum (B-Al) composite, assuming it to be an isotropic, homogeneous material. Compared to the uniform mesh method, the effectiveness of the gradual method to predict the singular stress behavior of the material in the presence of a crack is described.

In Section B, comparisons are made between the predicted and the experimentally observed values of notched and unnotched strengths of unidirectional and $0^\circ/90^\circ$ cross-ply glass fiber-reinforced epoxy (GFRE) composites. Using the relation developed in Chapter III, the sizes of the "critical regions" were estimated for different fiber orientations of these brittle composites.

In the last section, the computer-simulated test results of center-cracked tension specimens of unidirectionally reinforced B-Al composites are presented. The effects of fiber orientation and crack length on the material response and fracture behavior of the composite are also presented in Section C. Finally, the results of the nonlinear analysis of $0^\circ/90^\circ$ cross-ply and unidirectional B-Al composites are presented. The influences of the stiffness and partial contact of the pin on the overall fracture process of a ductile composite using a compact tension specimen are also discussed. Comparisons are made between different fracture parameters obtained from the laboratory and the computer-simulated tests using compact tension specimens (Fig. 3)

of the $0^\circ/90^\circ$ cross-ply B-Al composite. The empirical relation between the J-integral and the crack-opening displacement (COD), as developed in Chapter III, is verified here. In this section, the relative effectiveness of the J-integral and the COD in characterizing the initiation of unstable fracture of a ductile composite material is discussed.

A. Analysis of a Crack Using the Gradual Grading Method

A center-cracked specimen (Fig. 2) subjected to a tensile load was investigated using both the uniform mesh and the gradual grading method. In order to determine the superiority, if any, of the gradual grading method over the uniform mesh method, the analysis was restricted to the linear range. The material properties were chosen as

$$E = 14 \text{ MSI} \quad G = 6.36 \text{ MSI} \quad \nu = 0.1 .$$

These were believed to be the representative material properties for the $0^\circ/90^\circ/\pm 45^\circ$ cross-ply B-Al composite [110]. The crack length-to-width ratio was chosen as 0.2.

Quarter Plate Analysis

From symmetry conditions, a quarter of the center-cracked tension (CCT) specimen was studied. For comparison, two uniform and two gradually graded meshes were studied. The basic crude mesh for the finite element analysis of the problem is shown in Fig. 18. In uniform grading (Fig. 19), the elements are refined uniformly near the crack tip region without any special attention for the singular point, as is

done in gradual grading. In gradual grading (Figs. 20 and 21), the sizes of the elements are reduced progressively as they approach the singular point.

Using the method of gradual grading described in Chapter III, the first graded mesh for the quarter plate was made with 234 nodes (Fig. 20). The final graded mesh was made with 246 nodes and 288 elements (Fig. 21). The two uniform meshes studied here for comparison had 860 nodes (Fig. 19) and 2,931 nodes, respectively. The details of the meshes and the computing time for the elastic solution of the quarter plate crack problem are shown in Table 7. It is noted that the smallest element in the 2,931-node uniform mesh was $6.25 \times 10^{-4} a^2$, where a is the semi-crack length. In comparison, the final graded mesh with 246 nodes had the smallest element size of $3.05 \times 10^{-5} a^2$. Expectedly, the resulting solution with the graded mesh was considered to be more accurate than the fine uniform mesh. This was reflected in the stress concentration factor (SCF), defined as the maximum stress generated near the crack tip divided by the average stress applied remotely to the specimen. Since the analysis was linearly elastic, the SCF at the crack tip should go to infinity. However, in reality, the SCF is bounded, since the size of the element at the crack tip is finite in size. Consequently, a higher SCF from a linear elastic analysis is an indication of a more accurate finite element solution. The SCF was 7.72 for the graded mesh in comparison with an SCF of 6.46 for the uniform mesh. Obviously, the difference in computing time was tremendous (more than one order of magnitude!) with the gradual grading method compared to the uniform mesh method, as is evidenced from Table 7.

Even though the size of the elements becomes extremely small near the crack tip, any spatial property can be assigned to each element of a gradually graded mesh. This is very important for nonlinear analysis where the singularity changes its order across the elasto-plastic boundary, the unknown location of which is a function of the applied load.

Analysis of a Full Plate

For off-axis composites, the asymmetry of the material properties about the loading axis makes it necessary to analyze the complete plate, rather than the quarter plate. Analysis of the full plate implies the inclusion of two singular points. Consequently, the number of nodal points is quadrupled and the semi-bandwidth is doubled. The tremendous increase in computer time is obvious from Eq. (46). It may be realized that for nonlinear analysis the computing time will be N times more than the time needed for the elastic solution, where N is the number of iterations needed to do the nonlinear analysis. In this investigation, eight iterations were used for all of the nonlinear solutions. The tremendous computing time needed for one single nonlinear solution of an off-axis composite is obvious. The use of a graded finite element makes the savings in computing time extremely rewarding in the case of nonlinear analysis of a full plate, compared to the elastic analysis of a quarter plate.

Consequently, two gradually graded finite element meshes were used for solving the elastic crack problem using the full plate. The fine mesh had 708 elements and 666 nodes (Fig. 22). The second mesh had 592 elements and 512 nodes. The largest elements in both the

meshes were a^2 in size. The sizes of the smallest elements in the two meshes were $7.8 \times 10^{-5} a^2$ and $1.25 \times 10^{-3} a^2$, respectively. The stress concentration factors (SCF) obtained using these meshes were 6.0 and 3.75, respectively. The computing times were 346 and 260 sec for the two cases. Comparing these results with the corresponding results of uniform mesh, the savings in computational time for both the meshes were obvious. It may be realized that even for the second graded mesh with 512 nodes, the actual time saving is approximately 15 times, compared to the uniform mesh with 860 nodes, since in the second case, only a quarter of the plate was used.

Further comparisons were made with the experimental results of Jollay [110], as shown in Table 8. For a load of 40 KSI, the deflection at the load face was measured for a center-cracked tension specimen of $0^\circ/90^\circ/\pm 45^\circ$ cross-ply B-Al composite with a crack-to-width ratio of 0.2. The linear solution proposed by Tada [87] is also shown along with the experimental data and finite element results. The full plate solution of the finely graded mesh showed the closest agreement with the experimental data, as expected. The error was only 2.5%. Although the second graded mesh had a slightly inferior result compared to the uniform meshes, the large savings in computer time still made it attractive for a nonlinear analysis. Consequently, this second graded mesh with 512 nodes and 592 elements was used throughout this investigation for the nonlinear analysis.

From these analyses, it was clear that the gradual grading method is an extremely effective method. As discussed already, it has all the necessary flexibilities expected for analyzing the fracture

problem of a composite material. The computing time was very reasonable.

As a final remark, it is noted that it is always advisable to use triangular displacement elements near the singular point rather than a quadrilateral displacement element formed by the static condensation technique. This is due to the rapid increase of stress near the singularity. The averaging method of quadrilateral elements misses the fingerprint of the singularity.

B, Analysis of Brittle GFRE Composites

A computer-simulated test was done to predict the off-axis strengths of unnotched unidirectional and cross-ply ($0^\circ/90^\circ$) GFRE composites. Using the equations (Eqs. (58) and (59)) developed in Chapter III, the notched strength and critical stress intensity factors were predicted for these materials. These results were compared with the experimental results. Since these materials behaved linearly, the size of the "critical region" r_c was calculated using Eq. (50) for each fiber orientation.

Experimental Method

The experimental work for these materials (uniaxial and cross-ply GFRE) was done by Kulkarni [19]. The experimental procedure he followed is described below.

Center-cracked specimens (Fig. 2), 1.0-in.-wide, 0.045-in.-thick, and 10.0-in.-long, were cut from a master panel of a four-ply unidirectional GFRE composite using a band saw. Similar specimens were

cut from cross-ply material reinforced in the 0° and 90° directions. However, the thickness of this seven-ply material was 0.070 in. Center notches of various lengths were cut into each specimen using a 0.062-in.-diam milling cutter. In order to protect the gripped section and to distribute the load evenly, fiberglass tabs were bonded to the faces of the unnotched specimens using Eccobond adhesive. Each specimen was loaded to failure in the Instron Tensile Test Machine using a cross-head speed of 0.02-in./min.

The strength of unidirectionally reinforced specimens, with and without notches, was measured using a series of experiments in which the orientation of the fibers, θ , with respect to the crack axis (Fig. 2), was varied between 90° and 0° . Similar tests were run for the cross-ply composite. Both sets of results are shown in Tables 9 and 10, respectively. It was noted that the stress necessary to cause failure of any of the specimens decreased as the fiber orientation angle (θ) decreased or crack length increased. This trend was true for both unidirectional and cross-ply composites.

The failure mode of unidirectional specimens was identical and independent of the orientation of the reinforcement. In all the cases, failure occurred by splitting of the matrix. It was noted that failure resulting from cracks propagating through the fibers was not observed.

Inspection of the failure surfaces of cross-ply composites indicated that a shear failure or splitting of the matrix had occurred in each layer. Thus, the actual failure mode was identical to that observed in the unidirectional material. However, the onset of this

type of failure had been delayed to higher stress values, since the failure of the cross-ply composite implied the failure of the strong fibers of one of the plies.

Computer Simulation Method

Like laboratory tests, both notched and unnotched specimens can be analyzed in a computer using the finite element method. With the suggested modification for the interaction term F_{12} (Chapter III), the Tsai-Wu theory was used to predict failure of the specimens. Using the appropriate material properties in Eqs. (5) and (6), the strength coefficients were calculated for both the unidirectional and cross-ply composites when the principal material axes coincided with the loading and crack axes (Fig. 2). It may be recalled that for this situation, $F_3 = F_{16} = F_{26} = 0$. The only interaction term to be determined in the Tsai-Wu theory was F_{12} . This was calculated from Eq. (60), as suggested earlier:

$$F_{12} = -\frac{1}{2} F_{11} F_{22} . \quad (60)$$

Thus, an additional and complex test was avoided to calculate the interaction term, F_{12} . This is particularly important if it is realized that the parameters needed to calculate other strength coefficients of the Tsai-Wu theory are generally reported in the literature. The resulting strength coefficients of both of the composites are shown in Table 11. Typically, then, the Tsai-Wu failure theory for cross-ply GFRE became:

$$f = 1.74 \times 10^{-6} (\sigma_x + \sigma_y) + 10^{-10} \left[5.814 \left(\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y \right) + 123.46 \tau_{xy}^2 \right] - 1 . \quad (65)$$

However, if the loading and crack axes did not coincide with the principal material axes, as were the cases with the off-axis composites, the strength coefficients were transformed along the loading and crack axes. This was done using the transformations of Table 1. Consequently, for off-axis composites, the Tsai-Wu theory took the general form of Eq. (3).

In computer-simulated tests, a finite element mesh was generated for the specimens (center-notched or compact tension) to be tested. For the notched specimens, the finite element mesh was chosen carefully using the gradual grading method described in the preceding section of this chapter. Typical mesh for the center-cracked tension was similar to Fig. 22. The specimen was then loaded incrementally, the level of which varied in each case. To make the computer time reasonable, eight increments of load were applied in each case.

When the combination of the stresses in an element was such that it satisfied the modified Tsai-Wu theory, Eq. (60), the element was assumed to have failed. Once an element failed, further loading was stopped. The final load was described as the load for crack initiation. Since no dynamic effect of the crack propagation was considered, the analysis was restricted to crack initiation only. It may be noted that the failure zone obtained by this method was

typically of the order of 0.001 in.². For fast, brittle fracture, once a crack is initiated, it almost immediately leads to total failure. Consequently, the failure load identified by this computer simulation method should be the actual failure load too. This may be compared with the failure load of ductile composites.

For ductile composites, the initiation of the crack does not mean total failure, implying physical separation of the test piece into two parts. In these materials, the crack initiation is followed by the crack growth when the large stresses at the tip of the crack are partially relieved by a small advancement of the crack. Since the crack growth mechanism tries to blunt the crack tip, and hence, reduce the stress singularity effect, the crack advances further as the load is increased further. The crack growth process continues until the crack is large enough to cause final failure through the net section. Consequently, the crack initiation load in laboratory tests will be the load for which the crack growth can be monitored by some experimental method. Normally, large growths are monitored in laboratory tests and then extrapolated in order to determine the actual load for crack initiation. The corresponding load and deflection are used to calculate the fracture parameters at crack initiation.

Comparison of Results

Unnotched strengths. Figures 23 and 24 show the results of both the unidirectional and 0°/90° cross-ply composites as obtained using laboratory and computer-simulated tests. Tables 12 and 13 show the strength values and corresponding errors in the predicted values of unidirectional composites. The agreement between the experimental

data and the modified Tsai-Wu theory was very good. There were, however, some discrepancies for large angles. This was not unexpected, since there was a possibility of error in measuring the angle, particularly for large angles. A small error in measuring an angle leads to a considerable error in the strength, as was obvious from the nature of precipitous drop in strength with decrease in angle, particularly near the large angle side. This may be realized from an illustration in Fig. 23. If an error of only 4° is made in measuring the 75° angle, the unnotched strength value falls right on the predicted strength for unidirectional GFRE. Another possible source of error in the experimental determination of unnotched strength might come from the possible fabrication defect leading to small voids or delamination of the order of 0.1 in., which was not unexpected. Considering these factors, the agreement between predicted and experimental values was reasonably good.

Cross-plying of laminates effectively eliminates the first source of error, namely, the effect of error in measuring the orientation. This may be understood if it is realized that the decrease in strength due to the erroneous decrease in fiber orientation will be approximately compensated by the corresponding increase in strength in the cross-ply. Therefore, the agreement in predicted and experimental unnotched strength of cross-plyed ($0^\circ/90^\circ$) GFRE was expected to be better than that of unidirectional GFRE. This was really the case, as is clear from Fig. 24 or Table 13. This agreement of results was also a pointer for the effectiveness of the suggested modification in the Tsai-Wu theory for the interaction term F_{12} . This

simple modification eliminates completely the necessity of delicate and complex biaxial test methods.

Notched strengths. The notched strengths of both composites were calculated using Eq. (59). For comparison, experimental and predicted data for these materials are shown in Figs. 23 and 24. Like unnotched strengths, the notched strengths also showed very good agreement between predicted and experimental values. The small discrepancies may be explained along the same arguments as stated above. The predicted strengths for all the crack lengths are shown in Tables 12 and 13.

Critical stress intensity factor. The critical stress intensity factors for brittle GFRE composites were calculated using

$$K_{\theta c} = \sigma_{\theta} \sqrt{\pi a \sec \frac{\pi a}{w}} , \quad (55)$$

where,

$$\sqrt{\sec \frac{\pi a}{w}} = \text{width correction factor}$$

σ_{θ} = failure stress of the notched composite

$2a$ = crack length

w = width of the specimen .

As mentioned earlier, this is valid even for mixed mode failure if the failure is radial so that the peripheral stress is zero. The experimental values of the critical stress intensity factors and their mean are shown in Tables 14 and 15 for unidirectional and $0^{\circ}/90^{\circ}$ cross-ply GFRE. The variation in the experimental values was about 10%.

The critical stress intensity factors were calculated for these materials using the experimental failure stress values in Eq. (58). Figure 25 shows these values compared to the experimental values for both materials. Like strength values, critical stress values were equally well predicted by Eqs. (58) and (59) developed herein.

Critical region. The effectiveness of Eqs. (58) and (59) to predict the notched strength and critical stress intensity factors of brittle composites indicated the existence of the "critical region," as expressed by Eq. (50). Using the experimental values, the critical region sizes at fracture were calculated and are shown in Table 16. Interestingly, the calculated values were fairly constant, irrespective of the fiber orientations; however, they varied with material. The critical region sizes for unidirectional and cross-plyed GFRE were about 0.2 and 0.3 in., respectively.

It was noted that this critical region was different from the plastic zone size as postulated for ductile materials earlier [8] by Irwin. The plastic zone is not defined for a brittle material. But, a critical region is defined for both brittle and ductile materials. It is of interest to note that the critical region size was of the same order of magnitude as was the crack tip radius normally used in cracked specimens. Since this region is the limiting size for the validity of the continuum solution, it is reasonable to assume that the stress state in this critical region is constant and equal to the limiting stress state achieved at the periphery of the region. This argument, then, gives credence to the COD approach, although it is measured at the crack tip instead of the critical region. Due to limitation in the

measurement of COD values, this variation (measuring displacement at the crack tip instead of the critical region) does not introduce a large error. Thus, the COD approach seems to be a practical approach to monitor initiation of unstable failure. However, this argument does not, in any way, deny the fact that COD is a purely empirical approach.

Another implication of this "critical region" concept is that, in a numerical analysis like the finite element method, the smallest elements ahead of the crack tip need not be smaller than some critical size calculated for the material. Additionally, the failure of this element is synonymous with the fracture process of the notched specimen. Thus, the critical region concept attempts to unify failure and fracture of a material.

C. Analysis of Ductile B-Al Composites

The computer-simulated tests were done to predict the fracture behavior of ductile B-Al composites. For unidirectional B-Al composites, center-cracked tension (CCT) specimens (Fig. 2) were used. Compact tension (CT) specimens (Fig. 3) were used for the $0^\circ/90^\circ$ B-Al composites. Both the J-integral and COD parameters were used to predict the initiation of unstable fracture. The verification of the J- δ relation, developed in Chapter III, was also done for small-scale yielding. This relation was finally used to identify the appropriate fracture parameter for this ductile composite material.

Computer Simulation Method

In a computer-simulated test, a finite element mesh was chosen first for the specimen to be tested. As already mentioned, a gradual grading method was used in finalizing the finite element mesh of the center-cracked tension specimens used in this investigation (Fig. 22). The mesh had 512 nodes and 592 elements. Similarly, for the compact tension specimens, a finite element mesh with 623 nodes and 666 elements was chosen (Fig. 26). The test specimen was then subjected to an incremental loading, and appropriate deformation at points of interest were measured for each load. These loads and deformations were then used to calculate the fracture parameters of interest, namely, J-integral and COD.

Method of loading. In the computer-simulated test, the specimen (Fig. 22 or Fig. 26) was loaded incrementally until failure was reached in at least one element. The incrementally applied load might be written as:

$$P_i = P_0 + \Delta P(i-1) , \quad (66)$$

where,

P_0 = load corresponding to the minimum load leading
to initial yielding of at least one element

ΔP = subsequent increment of the load

P_i = load for the i^{th} increment

i = iteration number which varied between 6 and 8
for initiation of failure .

By limiting the number of iterations to between six and eight, the computer time was reduced to 10 minutes for the new computer (about three times faster than the old one on which the computer time estimate was made) located at The University of Tennessee, Knoxville (IBM-370/3031) computer center. However, the number of iterations was large enough to identify the behavior of fracture parameters with increasing load.

Identification of failure load. As the load was increased with every additional iteration, new elements yielded and the old yielded elements either strain-hardened or deformed further, depending upon the stress-strain behavior (Figs. 16 and 17) of the material. For all of the elements, the effective stress (Eq. (12c)) and strain (Eq. (12d)) were calculated for every applied load. For a strain-hardening material such as the $0^\circ/90^\circ$ B-Al composite, an element was assumed to have failed when its effective stress was equal to the failure stress of the material. For an ideally plastic material (flat top stress-strain curve) as in off-axis unidirectional B-Al composites, failure was assumed to have occurred in an element when its effective strain was equal to the failure strain of the material. Failure of one or more elements in a specimen was assumed to be the indication of fracture initiation. Fracture initiation was indicated by a typical failure area of about 10^{-2} in.². All critical values of the fracture parameters (J_c and COD_c) reported in this investigation were the values of those parameters corresponding to the failure load at crack initiation.

Unidirectional B-Al Composite

Using an incremental loading (Eq. (66)), center-cracked tension specimens (Fig. 2) with 512 nodes and 592 elements (Fig. 22) were investigated to study the fracture behavior of the unidirectional B-Al composite. Two specimens with $2a/w = 0.2$ and 0.4 were tested for each of the six fiber orientations (Fig. 2); namely, 90° , 0° , 30° , 45° , 60° , and 75° . The general behavior of this material as predicted by the computer simulation method is discussed in the following sections.

Load versus deflection. Figures 27 through 31 show the load versus load point displacement curves for the unidirectional center-notched B-Al composite specimens when the orientation of the fibers with respect to the crack axis varied as 90° , 0° , 30° , 45° , and 60° .

The load versus load point deflection curves were essentially linear for longitudinal, transverse, and 30° off-axis specimens (Figs. 27 through 29). The 45° and 60° off-axis specimens (Figs. 30 and 31) showed considerable nonlinearity for both specimens ($2a/w = 0.2$ and 0.4) at higher loads. As expected, the failure stress decreased in each case as the crack length was increased to 0.4 in. from 0.2 in. for 1-in.-wide specimens. The specimens with larger cracks also showed a consistent decrease in the stiffness.

Unlike the isotropic or orthotropic specimens, the deformations at the load points of the off-axis specimens varied from point to point. This is seen in Figs. 27 through 31. For any level of load, the longitudinal (Fig. 27) and transverse (Fig. 28) specimens had uniform extensions at the load points. For off-axis specimens (Figs. 29 through 31), however, the load points extended to different levels,

depending upon the fiber orientation and the level of load. For all of the off-axis specimens, this range of deformations increased as the load increased. This range of deformation of the loading points made it difficult to apply Eq. (31b) directly to calculate the J-integral for any level of load, as will be discussed later.

Notched strength. Figure 32 shows the variation of notched strength of unidirectional B-A1 composites as a function of fiber orientation with respect to the crack axis. It is noted from the figure that for any crack length ($2a/w = 0.2$ or 0.4), the failure strength decreases continually as the orientation of the unidirectional fibers with respect to the crack axis decreases to 0° from 90° . For any fiber orientation, the notched strength of the composite specimen decreases as the length of the crack increases. This trend may be compared with Figs. 23 and 24, where the variations of notched strength of brittle GFRE (unidirectional and cross-ply) composites are shown. Interestingly, the trends of strength decrease with the fiber orientation, and the crack lengths appear to be the same, even though the former material was brittle.

The critical stress intensity factors were calculated using Eq. (22a), namely,

$$K_{\theta c} = \sigma_{\theta \text{ notch}} \sqrt{\pi a} . \quad (22a)$$

Table 17 shows those values, calculated from both crack lengths. The average critical stress intensity factors are also shown in the table. Figure 33 shows that, like the failure stress, the critical

stress intensity factor of the unidirectional B-Al composite decreases continually as the fiber orientation with the crack axis decreases continually. This trend of the K_{Ic} versus θ curve of the ductile B-Al composite is similar to that of the brittle $0^\circ/90^\circ$ GFRE composite (Fig. 25b). It may be noted that the 75° off-axis specimen has been added in the analysis in order to identify the variation of both failure stress and critical stress intensity factors as a function of fiber orientation.

Plastic zone. Figures 34 through 43 show schematically the shape of the growing plastic zones near the crack tip as the load was increased for each of the 10 cases (five fiber orientations with two crack lengths in each case). The general observations may be drawn as follows:

- 1 The plastic zone developed and grew along the direction of the fiber orientation.
2. The shape of the initial plastic zone that was developed did not depend on the crack length. It depended on the orientation of the fibers alone.
3. The size of the plastic zone depended on the applied load.
4. The symmetry (orthotropic cases) or antisymmetry (off-axis cases) of the plastic zones was maintained in each case.
5. The size of the plastic zone appeared to be increasing as some power of the applied load. The exact relation could not be determined.

6. At failure, general yielding occurred for each of the specimens with $2a/w = 0.2$, except the transverse specimen. But the yielding was restricted to the midsection of the specimens for all filament orientations for $2a/w = 0.4$, except for the longitudinal and transverse specimens.

For large fiber orientations ($\theta = 60^\circ$, 75° , and 90°), the plastic zone also developed at the center of the specimen for higher loads for both crack lengths (Figs. 34, 35, 42, and 43). This spreading of the plastic zone from the center of the crack, in general, implied a tendency of the specimen to buckle. Table 18 summarizes the spread of the plastic zones for all 12 specimens.

It was of interest to note how the plastic zone spread all the way to the grip for longitudinal specimens, including the specimen with a large crack ($2a/w = 0.4$). For off-axis specimens, the plastic zones spread through the specimen width. As the plastic zones in the off-axis specimens reached the crack width, the elastic constraint imposed earlier by the unyielded elements on the yielded elements were removed. For flat top stress-strain curves, this situation led to quick failure, depending upon the failure strain and applied load. It was also noted for these off-axis composites, that any further loading after initial failures led to exhaustive failure, implying the failure region is of the order 0.1 in.^2 , for the same reason as stated above. However, for the longitudinal specimens ($\theta = 90^\circ$), strain-hardening did not allow exhaustive failure, even if the load applied was higher than the load needed for initial failure.

The initiation of failure from the crack was always restricted in the initial plastic zone for all the cases. The failure load

reported in this investigation was always the load corresponding to which the crack initiation was restricted to a small region of the order of 10^{-2} in.².

It was observed through a number of initial experimental runs, that the failure load did not change even if the lower grip was partially or fully restrained. In all of these cases, load was applied vertically at the upper grip and the grip points were freed at the lower end. Horizontal movement was restricted for all the grip points (six on each end). However, depending upon the type of restraint at the lower grip, the shape of the plastic zone was slightly different, particularly when it grew towards the lower grip at higher load. In this investigation, all of the results are based on the fully restrained lower grip.

Crack-opening displacement. Figures 44 through 50 show the computed load versus crack-opening displacement (COD) curves obtained from center-notched specimens containing different fiber orientations. A general nonlinear trend is noted in each configuration. Figure 44 shows a comparison between the results of Welch [69], who used center-crack tension specimens with longitudinal fibers. The test specimens were fabricated from nominal 50% volume 0.0057-in. boron fibers in 6061 aluminum. The longitudinal strength of the composite was 216.2 KSI, compared to 200 KSI assumed in this analysis. The predicted and the experimental results showed the same trend of nonlinearity. The overall agreement of the results seemed reasonable for the given situation with $2a/w = 0.4$. It is interesting to note the similarity between the critical values of COD, irrespective of whether they were

obtained experimentally (3.3×10^{-4} in.) in the laboratory, or numerically (3.6×10^{-4} in.), using a computer. In all of the laboratory tests [69], COD was measured by 0.031-in.-wide gages at a distance of 0.0155-in. ahead of the crack tip. Unfortunately, all of the gages failed prior to the failure of the specimen. Welch [69] reported the COD values prior to the strain gage failure. From his results, it appeared that all of the strain gages failed when COD reached a critical value of about 4×10^{-4} in.

Recalling the definition of fracture initiation in this investigation, it appeared that the failure of the strain gages was an indication of the fracture initiation which ultimately led to failure at a higher load. From the experimental [69] and simulated test data, the critical COD was found to be 4.0×10^{-4} in. (Fig. 44) within the experimental error. If the actual failure load (implying physical separation of the specimen) is higher than the failure load at crack initiation (as is defined here), the corresponding critical COD at failure might be estimated, in some cases at least, by extrapolating the load versus COD curve up to that failure load. However, in this investigation, critical COD was consistently estimated at crack initiation, as mentioned earlier. The most interesting point in Fig. 44 appears to be the approximate constant value of critical COD (4.0×10^{-4} in.) for a wide range of specimens. Considering the limitation of measurement, it was seen that the transverse specimens ($\theta = 0^\circ$) also had a critical COD as 4.0×10^{-4} in. (Fig. 45).

Figures 46 through 50 show the load versus COD curves obtained from off-axis composites. The highly nonlinear nature of the curves is

evident. However, it may be recalled that the value of COD was computed from Eq. (63) using both the horizontal and vertical components of the crack tip movement. To emphasize the importance of the horizontal component of COD, both components were plotted in Fig. 47 for $\theta = 30^\circ$ and $2a/w = 0.2$. Comparing Figs. 46 and 47, it was seen that all three components--horizontal, vertical, and total--of COD showed the similar trend of nonlinearity. From Fig. 47, it was also seen that the horizontal component of COD was not negligible. Since this was anticipated, all of the total COD values were computed using both components.

It was also noticed (Fig. 48) that the computed COD had a slightly different numerical value, depending on which tip of the crack, left or right, was being used. However, the difference was too small to be measured. Moreover, this effect might be due to the constraining of the lower end of the CCT specimen. For computational convenience, zero displacement was assigned to the lower end of the CCT specimen.

Since the load into the specimens was applied in finite increments, the total number of elements that failed (i.e., failure region) ahead of the crack tip depended on the difference between the estimated and actual failure loads rather than on the absolute value of the estimated failure load. It was observed that the final failure load led to the failure of most of the elements surrounding the crack tip for all of the off-axis specimens except the 30° ones. This, in turn, led to large COD values for these specimens, as is evidenced from the asymptotic nature of the load versus COD curves (Figs. 48 through 50).

Consequently, the critical COD had to be estimated for these specimens. It was noted from the off-axis results (Figs. 46 through 50), that exhaustive failure occurred when the COD exceeded the critical value of 4.0×10^{-4} in. Recalling the limitation of measurement capability, it was concluded that the critical COD was 4.0×10^{-4} in. for all of the specimens, irrespective of the fiber orientation and the crack length. The critical COD being anticipated as a material property, this observation was not unexpected. Since the material was the same, namely, the unidirectional B-A1 composite, the orientation of the loading axis with respect to the fiber axis alone was not expected to change a one-parameter material property, such as COD, particularly because it was measured as a total deformation, implying its independence from the arbitrarily chosen coordinate system.

The critical COD being constant for the unidirectional B-A1 composite, its estimated value of 4.0×10^{-4} in. was, in turn, used to predict the failure load from the load versus COD curves (Figs. 44 through 50). The results are shown in Table 19. The estimated values were plotted in Fig. 32 to compare with the results obtained directly from the computer simulation method. It is seen from the figure that the trend is predicted correctly by the critical COD estimation. The error was within approximately $\pm 10\%$, as shown in Table 19. Considering the variation in material properties (modulus and strength), as seen in Fig. 17, and two crack lengths ($2a/w = 0.2$ and 0.4), the agreement appeared to be very good. This is further realized if it is noted that, the strength varied by a factor of more than four for this wide range of 12 specimens. Thus, it appeared that the critical COD is a

material property for the unidirectional B-Al composite. Its estimated value is 4.0×10^{-4} in. for the unidirectionally reinforced B-Al composite with 50% reinforcement.

As a final remark, it is pointed out that the estimated critical COD was related to the overall failure phenomena near the crack tip. But it was not related directly to the failure strain of the specimen, as might be anticipated for a numerical method like the finite element method using constant strain triangle elements. This may be realized by recalling that the failure strains of unidirectional B-Al composites varied from 0.002 for $\theta = 0^\circ$ to 0.0127 for $\theta = 60^\circ$, while the crack width was equal to 0.2 in. in each case. Despite a sixfold increase in the failure strain, the constant value of the critical COD (4.0×10^{-4} in.) indicated that it did signal the formation of a crack tip failure environment which was typical to a material, irrespective of the fiber orientation, crack length, and extent of plasticity in the specimen. This is the proposition of the crack-opening displacement (COD) hypothesis and it appeared to be true for the unidirectional B-Al composite.

J-integral. J-integral values of unidirectional B-Al composite specimens were calculated using the approximate formula proposed by Rice, et al. [12], as shown in Eq. (31a):

$$J = G + \frac{1}{Bb} \left[2 \int_0^{\Delta p} P \, d\Delta p - d\Delta p \right] . \quad (31a)$$

However, complications arose in the evaluation of the J-integral whenever the deformation at the load points varied from point to point for any given load. This variation of load point deformations was shown as the range of deformation in Figs. 3 and 31. Consequently, the J-integral values were calculated for longitudinal (90°), transverse (0°), and 30° off-axis specimens for which the load point deformations did not vary from point to point for any given load (Figs. 27 through 29). Figures 51 through 53 show the J versus COD (δ) curves for these specimens. It is clear from the figures that the J- δ relation is linear for $\theta = 0^\circ$ and nonlinear for the two other cases ($\theta = 90^\circ$ and 30°). No trend was found in this transition of linear to nonlinear relationship between J and COD values. Moreover, the critical values of J ($= J_c$) at failure load varied from 8.5 in.-lb/in.^2 for $\theta = 0^\circ$, to 26 in.-lb/in.^2 for $\theta = 90^\circ$. For $\theta = 30^\circ$, J_c value was 8.9 in.-lb/in.^2 . Thus, it was obvious that J_c was not a constant value for the unidirectional B-A1 composite. Hence, a critical value of the J-integral did not appear to be a material parameter for this material. The lack of a theoretical justification of using J_c as a material parameter for composites was discussed earlier. These numerical results showing the dependence of J_c on fiber orientation seem to support this doubt. The reason lies partly in the fact that the linear load versus load point deflection relation neglects totally the effect of the large plastic regions developed (Figures 34 through 43) at the crack tip, as is obvious from Eq. (31a).

In any case, the threefold increase in J_c values as the fiber orientation changed to 90° from 0° is particularly interesting if it

is recalled that in contrast, the critical COD was constant (4.0×10^{-4} in.) for the 12 specimens investigated.

0°/90° Cross-Ply B-Al Composite

Experimental method. The experimental work for the 0°/90° cross-ply B-Al composite was done by Jollay [68] using compact tension (CT) specimens (Fig. 3). The lay-up was

$$0^\circ/90^\circ/0^\circ/90^\circ/0^\circ/90^\circ/0^\circ/90^\circ/0^\circ/90^\circ/0^\circ/90^\circ/0^\circ,$$

meaning seven 0° layers and six 90° layers in the total of 13 layers. The nominal diameter of the fibers was 0.0056 in., and the matrix material was 6061-T0 aluminum. The total thickness of the lay-up was approximately 0.1 in. The material was fabricated by DWA Associates using a diffusion bonded technique as delivered in 12-in. square panels. Specimens were cut from these panels using electric discharge machining (EDM). The experimental procedure followed by Jollay [68] is described below.

The CT specimens had the geometry as shown in Fig. 3, with a width of 4 in. and a height of 4.8 in. The diameter of the steel bolts used was 0.5 in. Four crack length-to-width ratios (a/w) were used as 0.2, 0.4, 0.6, and 0.8. Two specimens were tested for each a/w ratio. All specimens were tested in an MTS 810 System machine using a constant displacement rate of 0.02 in./min. Clip gages were used to measure load point and crack-opening displacements. However, Jollay measured the crack-opening displacements on the edge of the specimens. The two points of gage attachment were 0.2-in. apart about the center. So, in

this analysis, this displacement is referred to as crack face displacement (CFD) to differentiate it from the actual crack-opening displacement (COD) as shown in Fig. 3.

All of the tests were performed using an MTS machine in the 5,000-lb, full-scale range, with an accuracy of ± 25 lb. Both of the accuracies of the clip gages at the load point and crack faces were $\pm 5 \times 10^{-4}$ in. This limit of accuracy is particularly important for the measurement of crack-opening displacements.

Results of computer simulation method. To characterize the fracture behavior of the $0^\circ/90^\circ$ B-A1 composite, 4-in.-wide compact tension specimens (Fig. 3) were investigated using four different crack lengths ($a/w = 0.2, 0.4, 0.6, \text{ and } 0.8$). Using the symmetry conditions, only upper halves of the specimens were investigated. A finite element mesh with 666 elements and 623 nodes was used for this purpose (Fig. 26). The specimens were loaded incrementally (Eq. (66)) through the pin until failure was reached in at least one element. For all compact tension specimens, deflections were measured at four different locations (Figs. 3 and 26) for each load. Load point deflection (LPD) was measured at the center of the pin. Load line deflection (LLD) was measured on the intersecting point of the crack and the line joining the centers of the pins. Crack face deflection (CFD) was measured on the free edge of the specimen at a point 0.25 in. above the crack line. Crack-opening displacement (COD) was measured at a point nearest to the crack tip (Figs. 3 and 26).

Application of pins to transfer the load into a compact tension specimen has two important consequences. The first one comes from the

partial contact of the pin with the specimen. This may arise due to a noncircular shape of the pin or hole, or both. The partial contact may be defined by contact angle ϕ . For the ideal case, ϕ is 180° , implying the full contact of the appropriate half of the pin (upper half for the upper pin and lower half for the lower pin). The second consequence of using a pin to transfer the load comes into the picture due to the elastic or elasto-plastic deformation of the pin. Depending upon the stiffness and strength of the pin and the level of load, the pin will deform elastically or elasto-plastically to transfer the load into the specimen. It was anticipated that, without the knowledge of the experimentalist, the partial contact ($\phi < 180^\circ$) of mechanical properties of the pin might influence the deflections at all four points, and hence, the estimated fracture properties. Therefore, it was decided to study these two effects on a typical compact tension specimen with $w = 4$ in. and $a/w = 0.4$.

a. Partial contact of the pin

To study the effect of partial contact of the pin on the load deflection curves of the specimens, three contact angles (ϕ) were chosen (180° , 150° , and 120°). It was found that no appreciable difference in any of the four deflections (LPD, LLD, CFD, and COD) could be measured. This is typically shown in Tables 20 and 21. It was concluded that the partial contact of the pin with the specimen did not influence the estimation of fracture parameters like J-integral and COD. Newman [111] also made a similar conclusion for a similarly shaped metal specimen. However, his analysis did not include the effects of pin geometry and material nonlinearity, as was done in this investigation.

b. Effect of pin material

To study the effect of pin material on the estimation of fracture parameters (J and COD), three types of pins were chosen. They were aluminum, steel, and $0^\circ/90^\circ$ B-Al composite with 50% reinforcement (i.e., the same material as the specimen itself). Aluminum ($E = 10$ MSI) was chosen as a soft material. The B-Al composite was chosen to study the effects of anisotropy and nonlinearity. The computed deflections at the four points discussed earlier and the estimated J and COD values of these two specimens with aluminum and B-Al composite pins were compared with an identical compact tension specimen ($w = 4$ in., $a/w = 0.4$) with steel pins.

Figure 54 shows the load versus deflection curves obtained from compact tension composite specimens using all three pin materials, namely, steel, aluminum, and B-Al composite. It is clear from the figure that the pin material did influence the load point deflections, although the failure load and the nonlinear trend of the curves were the same. The load point deflection at failure load was 0.021 in. for the standard specimen with the steel pin. The corresponding deflections for the other two specimens with aluminum and B-Al composite pins differed by $\pm 1.0 \times 10^{-3}$ in.

For the corresponding failure loads of the specimens, it was noted that the steel pin was in the elastic range. But the aluminum pin was in the nonlinear range, the effect of which was not included in the analysis. The first yielding in the B-Al composite pin occurred at 45.2% of the fracture load, P_f , compared to a 23.7% of P_f for the first yielding in the B-Al composite specimen. At 56.2% of P_f , the upper

half of the bolt yielded completely. During the subsequent loading, the upper half of the pin strain-hardened only. Throughout the loading period no yielding occurred in the lower half of the pin.

To study further the effect of this difference in load point deflections, the critical J values corresponding to the failure load were calculated for all the specimens using Eq. (31b), namely,

$$J = \frac{2A}{B\ell} . \quad (31b)$$

The three critical J ($= J_{IC}$) values for the three specimens with aluminum, steel, and B-Al composite pins were 177, 187, and 199 in.-lb/in.², respectively. The variation was about $\pm 6\%$ from the expected value of 187 in.-lb/in.² to be obtained from a 4-in.-wide compact tension specimen ($a/w = 0.4$) using steel pins. Obviously, the difference in J_{IC} values was not significant.

The critical COD values at failure were 0.0032, 0.0034, and 0.0035 in. for the specimens with aluminum, steel, and B-Al composite pin, respectively. The variation in critical COD was of the order of 10^{-4} in. If critical COD is estimated as 0.0034 in. (corresponding to the steel pin), the variation is less than $\pm 6\%$ for the different pin materials investigated here. Considering the possible errors in actual laboratory experiments, this variation also did not appear to be significant.

From this initial investigation it was concluded that, the estimation of fracture parameters (J and COD), and hence, fracture behavior, of a compact tension specimen is influenced by the mechanical

properties of the pin used. So, care should be taken in using and reporting the pin material if a compact tension specimen is used to characterize the fracture behavior of a material. However, this influence did not appear to be significant for the $0^\circ/90^\circ$ B-Al composite material.

c. Load versus deflection curves

Since it was determined that the partial contact and the mechanical behavior of the loading system did not influence significantly the fracture behavior of $0^\circ/90^\circ$ B-Al composite material, a series of computer-simulated tests were done to study the fracture behavior of the material using 4-in.-wide compact tension specimens with four different crack lengths, namely, $a/w = 0.2, 0.4, 0.6,$ and 0.8 . Steel pins ($E = 30 \text{ MIS}$) were used in each case to compare some of the computer-simulated test results with the laboratory test results of Jollay [68].

Figures 55 through 58 show the computed load versus deflection curves for the compact tension specimens investigated. The deflections include load point deflection, load line deflection, and crack face deflections, as defined earlier (Figs. 3 and 26).

Except for a crack-to-width ratio of as low as 0.2 , the deflections at the load point (LPD) and load line (LLD) were effectively the same as is seen from Figs. 55 through 58. Consequently, the load point deflections may be monitored at the load line (Fig. 3) if the crack is not very small. The nonlinear nature of the load versus load point (or line) deflection is very clear from these figures, except for the case of $a/w = 0.4$. However, the deviation from linearity is not so large that an assumed linear relation between the load and deflection

will introduce a considerable error in the calculation of the J-integral using Eq. (31b). On the other hand, the nonlinearity of these curves indicates that the fracture process was surely not a brittle one. The existence of the plastic zone was thus reflected indirectly in these curves.

It may be mentioned that, using a simple beam theory the local stress ahead of the crack may be calculated from the applied load. Even without the singular effect of the crack, this local stress varies from tension at the crack tip to compression at the edge of the specimen. This inplane bending effect is typical for compact tension specimens. Because of this inplane bending effect, the crack tip experiences the maximum tensile local stress leading to fracture if the load is sufficiently large. The local stress may be calculated using the simple beam theory,

$$\sigma_0 = \frac{P}{B(w-a)} + \frac{\bar{M}\bar{C}}{\bar{I}}, \quad (67a)$$

where,

P = applied load

B = thickness of the specimen

W = width of the specimen

a = crack length

$\bar{M}$ = bending moment generated due to the applied load

$\bar{C}$ = distance of the crack tip from the neutral axis
under pure inplane bending

$\bar{I}$ = moment of inertia of the section subjected to the
bending moment .

It is noted that,

$$\bar{C} = \frac{w - a}{2} , \quad (67b)$$

$$\bar{M} = P(\bar{C} + a) , \quad (67c)$$

$$\bar{I} = \frac{B(2\bar{C})^3}{12} . \quad (67d)$$

Using these equations in Eq. (67a), the local stress at the crack tip can be calculated from

$$\sigma_0 = \frac{2P(2w + a)}{B(w - a)^2} . \quad (67e)$$

Since the deflections of points are geometry dependent, they (LPD, LLD, and CFD) were plotted against the values of load, rather than stress, as is customary. These deflection curves, in general, indicated the nonlinear behavior of the material. Jollay [68] reported very similar curves from his experimental results with a 2-in.-wide specimen.

d. Compliance curves

When a body is subjected to a load P , the compliance of any point in the body is defined as

$$C = \frac{\Delta}{P} , \quad (68)$$

where, Δ is the deformation of the point under load P . However, to include the effect of the thickness B , it is often rewritten as

$$C' = \frac{\Delta \cdot B}{P} . \quad (68a)$$

For linear load-deflection curves, compliance is obviously the same for a particular point in a specimen for any level of load, Eqs. (68) or (68a). Consequently, the compliance method has been used successfully to characterize fracture of a brittle material by calculating the strain energy release rate ($G = \frac{1}{2} P^2 \frac{\delta C}{\delta a}$) from the compliance versus crack length curves [8]. However, for ductile composites, the compliance has to be corrected for both anisotropy [68] and nonlinearity [8].

To compare the results of computer simulation tests with the experimental results of Jollay [68], compliance curves were calculated using displacements measured at the load point and the crack face. Table 22 shows the compliance values at the load point and crack face as a function of load. As a typical sample, compliance values at the load line also are shown in the table for $a/w = 0.4$. Since the load point and load line deflections were effectively the same, the corresponding compliances were also effectively the same in this case, as was expected. However, the compliance values increased slightly for all points (load line, load point, and crack face) as the load increased (Table 22), with the exception of the specimens with $a/w = 0.2$ and 0.8 . For the specimen with $a/w = 0.2$, the compliance values were the same throughout the loading. On the other hand, for the specimen with $a/w = 0.8$, the variation in compliance values with increasing load was significant (Table 22). This situation will be discussed later in further detail.

The slight variation in compliance values for $a/w = 0.4$ and 0.6 was anticipated as the effect of nonlinearity developed into the

specimen as a result of plastic deformation. This was evidenced in larger variation of the crack face compliance compared to the load point compliance for a given range of load, as was typified in the specimen with $a/w = 0.6$ (Table 22). The reason was believed to be the effect of larger plastic deformation at the crack tip compared to the load point for a given load.

Since the variation in compliance values was not significant (with the noted exception of $a/w = 0.8$), and Jollay [68] did not report the loads at which compliances were measured, the compliance versus crack width curves were plotted using failure loads. Those compliance curves are shown in Figs. 59 and 60.

The comparison of compliances is shown graphically in Figs. 59 and 60. The agreement between the simulated and laboratory tests of the load point compliances is excellent. Although the nature of the crack face compliances agrees very well for these two cases, the experimental crack face compliances were consistently higher than the computed ones. It was noted that the experimental compliance curves were computed from the results of 2-in.-wide compact tension specimens [68], compared to a 4-in.-wide specimen used in the simulated compliance curves. On the other hand, the computer-simulated tests were limited to small-scale deformation theory. Consequently, large-scale deformation at the crack tip might have affected the compliance values at the crack tip, in particular. Considering these factors, the agreement between the experimental and computed crack face compliance values appeared reasonably good.

In any case, the influence of nonlinearity on the compliance values was evident from these plots. Therefore, it was of interest to plot the plastic zones developed in a typical compact tension specimen.

e. Plastic zone

The variation in the compliance values and nonlinear behavior of the load point (LPD) and crack face (CFD) deflections suggested the formation of the plastic zone in the specimen. Thus, it was of interest to determine the extent of this plasticity and to plot the shape of the plastic zones which were formed in typical specimens. Figure 61 shows the plastic zones developed in a compact tension specimen for $a/w = 0.4$. The plastic zone sizes correspond to those generated upon application of the failure load. The figure also indicates the small region at the crack tip which failed first on the application of this load.

The elongation of the plastic zone also delineated along the direction of the fracture path. The failure was obviously expected to be of the Mode I type (Fig. 1). The slenderness of the plastic zone can be explained in terms of stresses ahead of the crack tip. Typical stresses are shown in Figs. 62 and 63, calculated using a local applied stress of 47 KSI at the crack tip. Figure 62 shows only the nondimensionalized normal stresses ahead of the crack tip as a function of distance x , since the shear stress was effectively zero near the crack tip. It may be noted that, due to the finite size of the elements ahead of the crack tip, these stresses were measured along approximately $y = 0.015$ in. The neutral axis (meaning zero δ_y stress) was noted to be at 1.1 in. ahead of the crack tip.

The interesting point in Fig. 62 is the intersecting point of σ_x and σ_y stresses at $x = 0.1$ in. ahead of the crack tip. This implied a situation of hydrostatic pressure in a two-dimensional analysis. Since generation of hydrostatic pressure limits the yielding of the material, the plastic zone was expected to be limited to this point, as was the actual case (Fig. 61). The normal stresses (σ_x and σ_y) across the crack were never identical, since σ_y was always greater than σ_x . This effect is shown in Fig. 63 for varying y along $x = 2.6$ in. This allowed the plastic zone (and hence, the crack) to propagate across the crack in a Mode I manner. The shear stress was not significant in this section either.

f. J-integral

Using Eq. (31b), J-integral values were calculated for different loads for the four specimens with $a/w = 0.2, 0.4, 0.6,$ and 0.8 . The variations of the J-integral with load for different crack lengths are shown in Fig. 64. The nonlinear relation between load and the J-integral was evident in the curves, as expected. It is recalled that, the critical value of the J-integral for crack initiation (J_{IC}) was defined as the J value corresponding to the load needed to initiate failure of a typically 10^{-2} in.² area ahead of the crack tip. From Fig. 64 it is seen that J_{IC} values were estimated as 134, 187, 270, and 876 for $a/w = 0.2, 0.4, 0.6,$ and 0.8 , respectively (all units were in.-lb/in.²). Since the values of J_{IC} varied significantly with the length of the crack in the specimen, it was evident that J_{IC} is not a material constant for the $0^\circ/90^\circ$ B-Al composite. This is true even if the specimen with a crack length of $a/w = 0.8$ was excluded in the discussion.

The above-noted observation was supported by the experimental results of Jollay [68]. Although he suggested an average value of J_{IC} as 344 in.-lb/in.², his estimated values for identical specimens varied from 280 in.-lb/in.² for $a/w = 0.4$ and 0.6 , to 455 in.-lb/in.² for $a/w = 0.2$. He observed a variation of J_{IC} from 280 in.-lb/in.² to 360 in.-lb/in.² for two identical specimens with $a/w = 0.6$. Because of further inconsistent results, he did not report the J_{IC} values for $a/w = 0.8$ and 0.9 .

The J_{IC} values calculated from the computer simulation tests were somewhat lower than the experimentally observed values. Apart from the observed scatter in experimental data, the other possible reason seemed to lie in the definition of failure, as discussed earlier. The failure region used to characterize initiation in this investigation was too small to be detected by ordinary experimental methods. On the other hand, the correctness of the extrapolated J_{IC} value depended largely on the scatter and total number of experimental data used in the extrapolation.

The important point to note, however, is the general observation in both the experimental and the simulation data. The significant dependence of the J_{IC} values on the crack lengths (even excluding the case with $a/w = 0.8$), reinforce the doubt [9, 14] that it was not a material parameter implying its independence from the specimen geometry. It may be recalled that the validity of J_{IC} as a material parameter to describe the fracture of ductile composite materials was questioned on theoretical background in Chapter II. The experimental observations tend to support the view.

It is again emphasized that the results obtained from specimens containing a large crack length ($a/w = 0.8$) were not included to arrive at the above conclusion.

g. Crack-opening displacement (COD)

The load versus crack-opening displacement (COD) curves for different crack lengths are shown in Fig. 65. The nonlinear nature of the curves is evident. The critical values of COD were estimated as the values of COD corresponding to the failure load of the specimen (Fig. 65). The interesting point to note is that, the critical COD (COD_c) is approximately 3.5×10^{-3} in. in each case (except for $a/w = 0.8$, for which $COD_c = 6.5 \times 10^{-3}$), even though the failure load decreased from 32 KSI for $a/w = 0.2$, to 13 KSI for $a/w = 0.6$. Unlike J_{Ic} , COD_c seems to be a material constant for this material, with the noted exception. Before exploring the reason for this deviation, one important fact is re-emphasized concerning the location for measurement of COD. It is, perhaps, experimentally convenient to measure crack face deflection, CFD (Fig. 3), but its value at fracture need not be a material constant. For this reason, only CFD was differentiated from COD to discuss the result of Jollay [68], who measured CFD and designated it as COD. The point is further clarified from Figs. 55 through 58. These figures show the load versus crack face displacement for different crack lengths. It is apparent from these figures that the critical CFD depended on crack length. Typically, critical CFD were 0.2 and 0.45 in. for $a/w = 0.2$ and 0.6, respectively.

The large inconsistent values of critical J and COD for $a/w = 0.8$ may be explained as follows. For a crack as large as

$a/w = 0.8$, the local stress at the crack tip is magnified many times for a given load P . This may be observed from Eq. (67a). For $w = 4$ in., $B = 1$ in., and $a = 3.2$ in., the local stress at the crack tip, σ_0 , is 35 times the applied load. Consequently, the failure load of 9 KIPS generates a local stress of 315 KSI at the crack tip. It may be noted that in this calculation, the simple beam theory was used without accounting for the singular effect of the crack. It is further recalled that the failure stress for the material is 115.9 KSI. This implied that due to the inplane bending effect, the specimen actually failed by net section failure, rather than by the expected fracture failure under the influence of the crack. This explains why Jollay [68] obtained inconsistent J_{IC} values for $a/w = 0.8$ and 0.9 . Since both critical J and COD concepts are used to explain fracture processes, they are not expected to be applicable in gross failure processes like these. This explanation seems to justify the fact that critical COD is independent of the crack length in the specimen, and hence, is a material constant. The critical COD for crack initiation was estimated as 3.5×10^{-3} for the $0^\circ/90^\circ$ cross-ply B-Al composite.

h. J - δ relation

Considering the importance of the conclusions that, for $0^\circ/90^\circ$ B-Al composites the critical COD is a material constant and the critical value of the J -integral is not, further verifications to support those conclusions were needed.

In Chapter III, it was shown that the J -integral and the crack-opening displacement (δ) may be related in an empirical manner as

$$\frac{J}{\delta\sigma_y} = M \left(\frac{\sigma_0}{\sigma_f} \right)^\lambda, \quad (54)$$

where,

σ_0 = nominal stress

σ_y = yield stress

σ_f = fracture stress

M, λ = numerical constants ,

provided the small-scale deformation theory is applicable. It was decided to verify this equation for $0^\circ/90^\circ$ B-Al composites.

Figure 66 shows the log-log plots of $(J/\delta\sigma_y)$ versus (σ_0/σ_f) for different crack lengths. It is interesting to note that, the plots did verify the validity of the proposed relation. From the data points it was found that

for $\frac{a}{w} = 0.2$,

$$\frac{J}{\delta\sigma_y} = 0.79 \left(\frac{\sigma_0}{\sigma_f} \right)^{0.88}; \quad (69a)$$

for $\frac{a}{w} = 0.4$,

$$\frac{J}{\delta\sigma_y} = 0.94 \left(\frac{\sigma_0}{\sigma_f} \right)^{0.83}; \quad (69b)$$

for $\frac{a}{w} = 0.6$,

$$\frac{J}{\delta\sigma_y} = 0.90 \left(\frac{\sigma_0}{\sigma_f} \right)^{0.82} . \quad (69c)$$

Since the data points for $a/w = 0.8$ showed nonlinearity, no equation was derived. This was not unexpected, due to anticipated net section failure for this specimen, as discussed earlier. Under this condition, the small-scale deformation theory was not expected to be valid. The deviation of $a/w = 0.8$ test data from linearity reflected the view that these results were inapplicable for fracture analysis. This explained the inconsistent values of critical J values as found by Jollay [68]. This seemed to further explain the deviation of the critical COD value for this specimen from the critical COD value (3.6×10^{-3} in.) of all other specimens for this material. In other words, critical COD seemed to be independent of crack length, and hence, a material constant for the $0^\circ/90^\circ$ B-AI composite. Consequently, this specimen ($a/w = 0.8$) is excluded in the subsequent discussion of fracture of this material.

A closer look into the $\log (J/\delta\sigma_y)$ versus $\log (\sigma_0/\sigma_f)$ curves revealed that the slopes (λ) of all three lines, corresponding to $a/w = 0.2, 0.4$, and 0.6 , are approximately the same. The value of slope λ showed less variation than the intercept M.

The validity of Eq. (54) for this material confirmed one important conclusion, namely, critical J is not a material property for ductile $0^\circ/90^\circ$ cross-ply B-AI composites. A careful look into the expression (Eq. (54)) confirmed this. For a given material, σ_y and σ_f are constants. Further, it was demonstrated that critical COD was

constant for this material (Fig. 36). But the right-hand side of the equation is a function of the notched failure stress when the critical values are inserted into the expression. Since notch strength depended upon the crack length, critical J (J_{Ic}) should also depend on the crack length. This implied that J_{Ic} was not a material constant, at least for this material, as was demonstrated in Fig. 64. In fact, the invalidation of J_{Ic} as a material property will be established whenever a material shows a behavior of the form of Eq. (54) and critical COD is a material constant.

i. Critical J and COD as fracture criterion

To finally conclude that the critical COD is, indeed, a material property for the $0^\circ/90^\circ$ B-Al composite, and that critical J is not, Eqs. (69a) through (69c) were rearranged in terms of critical fracture parameters and corresponding failure loads of the specimens, so that

$$\text{for } \frac{a}{w} = 0.2 ,$$

$$\sigma_c = \frac{J_{Ic}}{0.79 \sigma_y} \left(\frac{\sigma_f}{\sigma_{0f}} \right)^{0.88} ; \quad (70a)$$

$$\text{for } \frac{a}{w} = 0.4 ,$$

$$\sigma_c = \frac{J_{Ic}}{0.94 \sigma_y} \left(\frac{\sigma_f}{\sigma_{0f}} \right)^{0.83} ; \quad (70b)$$

for $\frac{a}{w} = 0.6$,

$$\sigma_c = \frac{J_{IC}}{0.90 \sigma_y} \left(\frac{\sigma_f}{\sigma_{0f}} \right)^{0.82} , \quad (70c)$$

where,

σ_{0f} = notched strength of the particular specimen .

Using the unnotched strength (σ_f) of 115.9 KSI and the computed values of J_{IC} and σ_{0f} , the critical COD (σ_c) were calculated from the above equations for each specimen. The critical values were calculated as 3.61×10^{-3} in. for $a/w = 0.2$; 3.33×10^{-3} in. for $a/w = 0.4$; and 3.35×10^{-3} in. for $a/w = 0.6$. It is recalled that independent of the crack length, the constant value of critical COD was found to be 3.5×10^{-3} in. from the finite element results discussed earlier. Thus, it was finally concluded that the critical COD is, indeed, a material constant for the $0^\circ/90^\circ$ B-A1 composite.

Once the critical COD was established as a material constant, the invalidation of the critical J-integral as a material constant could be further shown from Eqs. (70a) through (70c). Treating δ_c as a material constant like unnotched strength (σ_f), it can be eliminated from Eqs. (70a) and (70b) or (70b) and (70c) to estimate J_{IC} for $a/w = 0.2$ or $a/w = 0.6$, respectively. Using $J_{IC} = 134$ in.-lb/in.² for $a/w = 0.4$, it was calculated from these equations that $J_{IC} = 187$ in.-lb/in.² for $a/w = 0.2$; and $J_{IC} = 277$ in.-lb/in.² for $a/w = 0.6$. The corresponding values of the critical J-integral were computed directly from finite element results as 187 in.-lb/in.² for

$a/w = 0.2$; and 270 in.-lb/in.^2 for $a/w = 0.6$. The errors were less than 8%. Thus, variations in the critical values of the J-integral from 134 in.-lb/in.^2 for $a/w = 0.4$, to 270 in.-lb/in.^2 for $a/w = 0.6$ were due to their dependence on the crack length. The correct estimation of J_{Ic} from one crack length ($a/w = 0.4$) to another crack length ($a/w = 0.2$ and 0.6) also verified another fact, that the critical COD was not put implicitly as a failure criterion in the finite element analysis. Hence, it was further concluded that the critical J was not a material parameter for the $0^\circ/90^\circ$ B-Al composite.

CHAPTER V

CONCLUSIONS

1. Gradual grading of the finite element method is a useful procedure for analyzing structures with singularities. The method is particularly effective for fracture analysis of composite structures.
2. The Tsai-Wu failure theory predicted very satisfactorily the failure and fracture of both brittle GFRE and ductile B-Al composites.
3. The necessity of performing complex and careful tests to determine the interaction term F_{12} has been eliminated by treating the theory as a generalization of the von Mises theory. This is done by using

$$F_{12} = -\frac{1}{2} F_{11}F_{22} .$$

4. A "critical region" concept has been developed using the results of the deformation theory of plasticity for small-scale yielding. The size of the region has been estimated as 0.02 and 0.03 in. for unidirectional and 0°/90° cross-ply GFRE composites, respectively. Since the size of the region depends on J_{IC} , its size has been estimated as approximately 0.05 in. for the 0°/90° cross-ply B-Al composite.

5. A correlation between notched and unnotched strengths of brittle composites has been developed and verified for GFRE composites.
6. A similar correlation has been developed and verified for critical stress intensity factors of brittle composites.
7. The critical stress intensity factors for unidirectional and $0^\circ/90^\circ$ cross-ply GFRE composites depend on the orientation of the fibers with respect to the crack axis.
8. A similar dependence on fiber orientation has been found for the critical stress intensity factors of ductile B-Al composites with unidirectional reinforcements.
9. A J-COD correlation is developed for small-scale yielding. This is verified with the $0^\circ/90^\circ$ cross-ply B-Al composite.
10. If a compact tension specimen is used, the fracture parameters (J or COD) are not affected by the partial contact of the pin, at least for $a/w = 0.4$. The partial contact may arise due to the plastic deformation of the specimen around the pin.
11. In a compact tension specimen the pin material influences the fracture parameters. However, the influence is not very large for the cases studied here, namely, steel, aluminum, and $0^\circ/90^\circ$ cross-ply B-Al composites.
12. The critical value of the J-integral at fracture depends on the crack length of $0^\circ/90^\circ$ B-Al composite specimens. For unidirectional B-Al composites, J_{IC} varies with fiber orientation. Thus, it is found that J_{IC} is not a material property for ductile B-Al composites.

13. The definition of crack-opening displacement has been generalized to include the analysis of anisotropic materials. Critical crack-opening displacement (COD) appears to be independent of fiber orientation of the unidirectional B-Al composite. It is also independent of crack length for the $0^\circ/90^\circ$ B-Al composite. The values of critical COD are 3.5×10^{-3} in. and 4.0×10^{-4} in. for $0^\circ/90^\circ$ cross-ply and unidirectional B-Al composites, respectively. Thus, critical COD seems to be a material property for ductile B-Al composites.

CHAPTER VI

RECOMMENDATIONS

1. The effectiveness of the gradual grading method should be studied in further detail. Specific areas should include convergence and savings in computer time compared to singular elements.
2. Further verification of the suggested relation for the interaction term F_{12} of the Tsai-Wu theory is suggested. This can be done by direct experimental verification. Otherwise, comparisons can be made with the predicted and the experimentally observed values of failure stress on an anisotropic composite.
3. Further experimental verification of the "critical region" concept is needed for small-scale yielding.
4. The relations developed to predict notched strength and critical stress intensity factors need experimental verifications.
5. The empirical relation between the J-integral and the COD should be verified for small-scale yielding of both metallic and composite materials.
6. The verification of critical COD as a material parameter is suggested, using different ductile composite systems. However, the general definition of COD, suggested in this

investigation, has to be used for both numerical and experimental verifications.

7. Experimental methods to measure total COD are emphasized. Recommendations are also made to measure COD to the accuracy of, at least, $\pm 2.5 \times 10^{-5}$ in.
8. The effect of slow crack growth should be studied by allowing unloading during crack growth.

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APPENDIXES

APPENDIX A

FIGURES

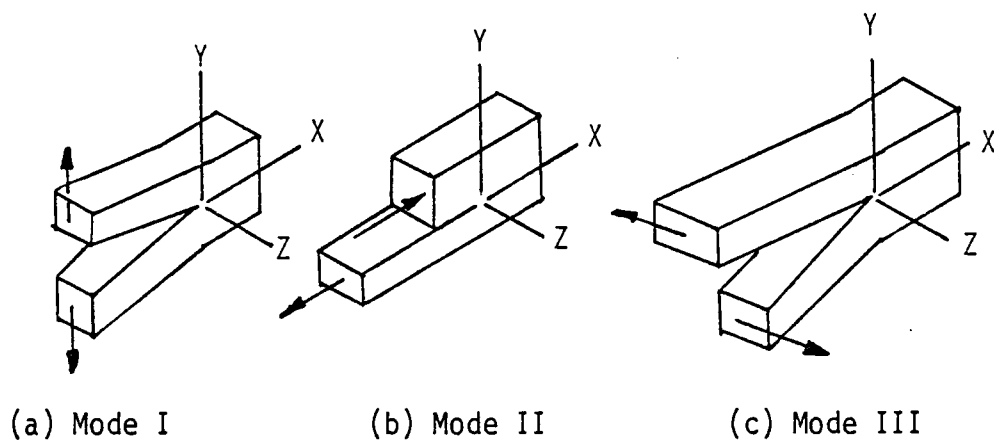


Figure 1. Basic modes of crack surface displacement.

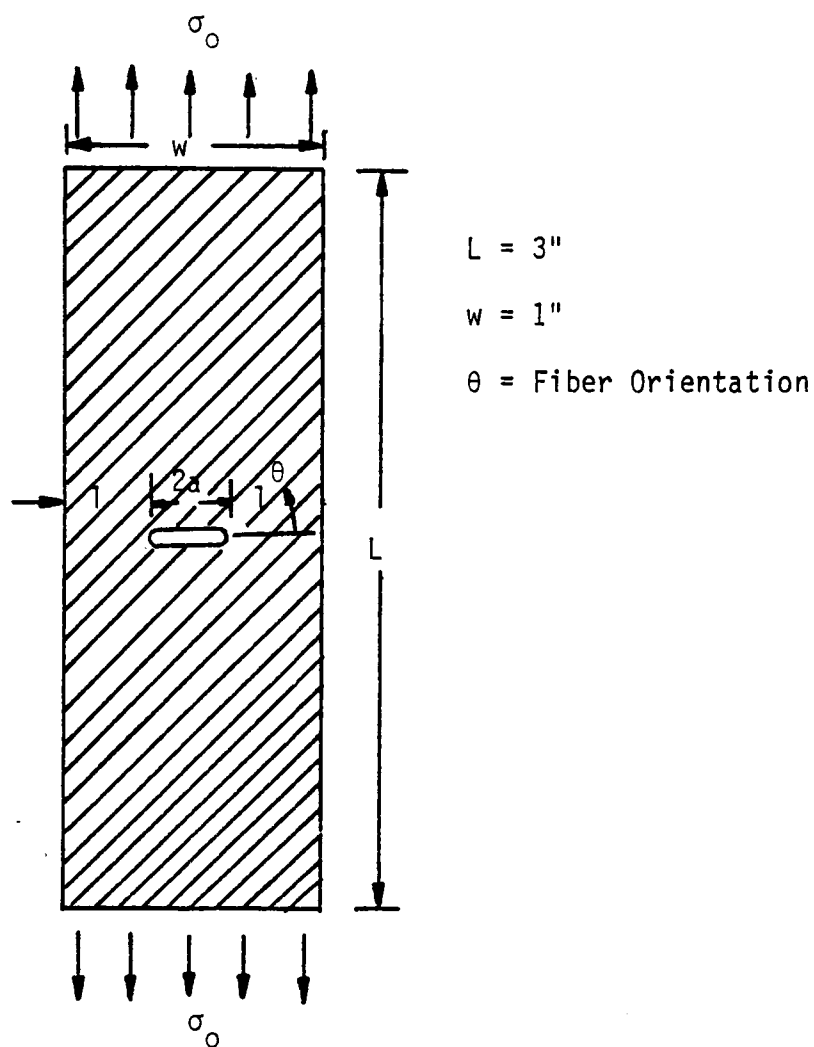


Figure 2. Schematic diagram of a center-cracked tension (CCT) specimen.

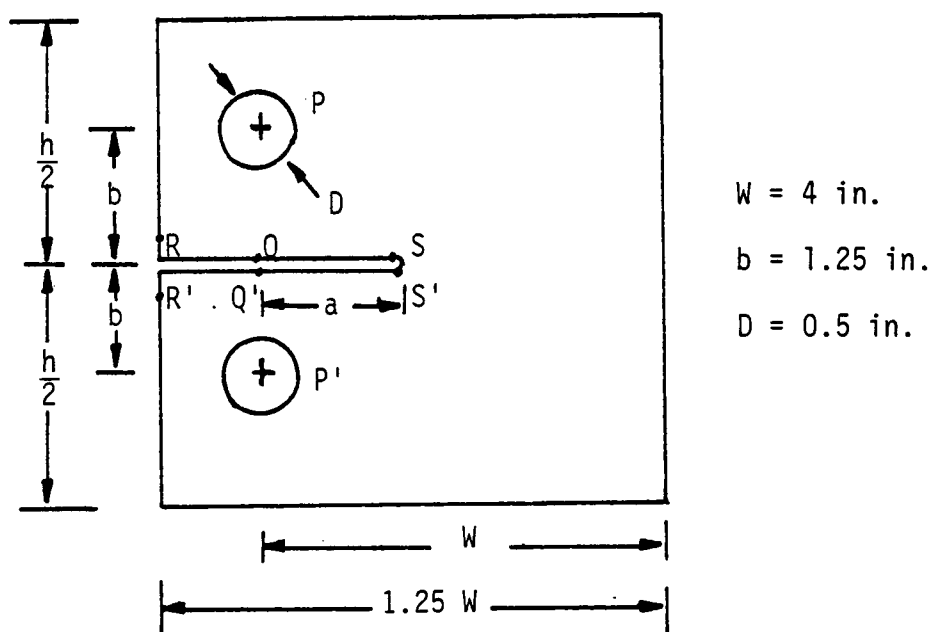


Figure 3. Schematic diagram of a compact tension specimen showing dimensions and points of deflection measurements.

Load point deflection (LPD) is measured between PP' ; load line deflection (LLD) is measured between QQ' ; crack face deflection (CFD) is measured between RR' ; crack-opening displacement (COD) is measured between crack tip points SS' .

Figure 4. Schematic diagrams of typical fracture modes of unidirectional composites.

(a) Fiber Splitting [59]; (b) Matrix Failure [19]; (c) Interface Failure [19]; (d) Fiber Pull Outs and Debonding [31]; (e) Crack Bridging [7].

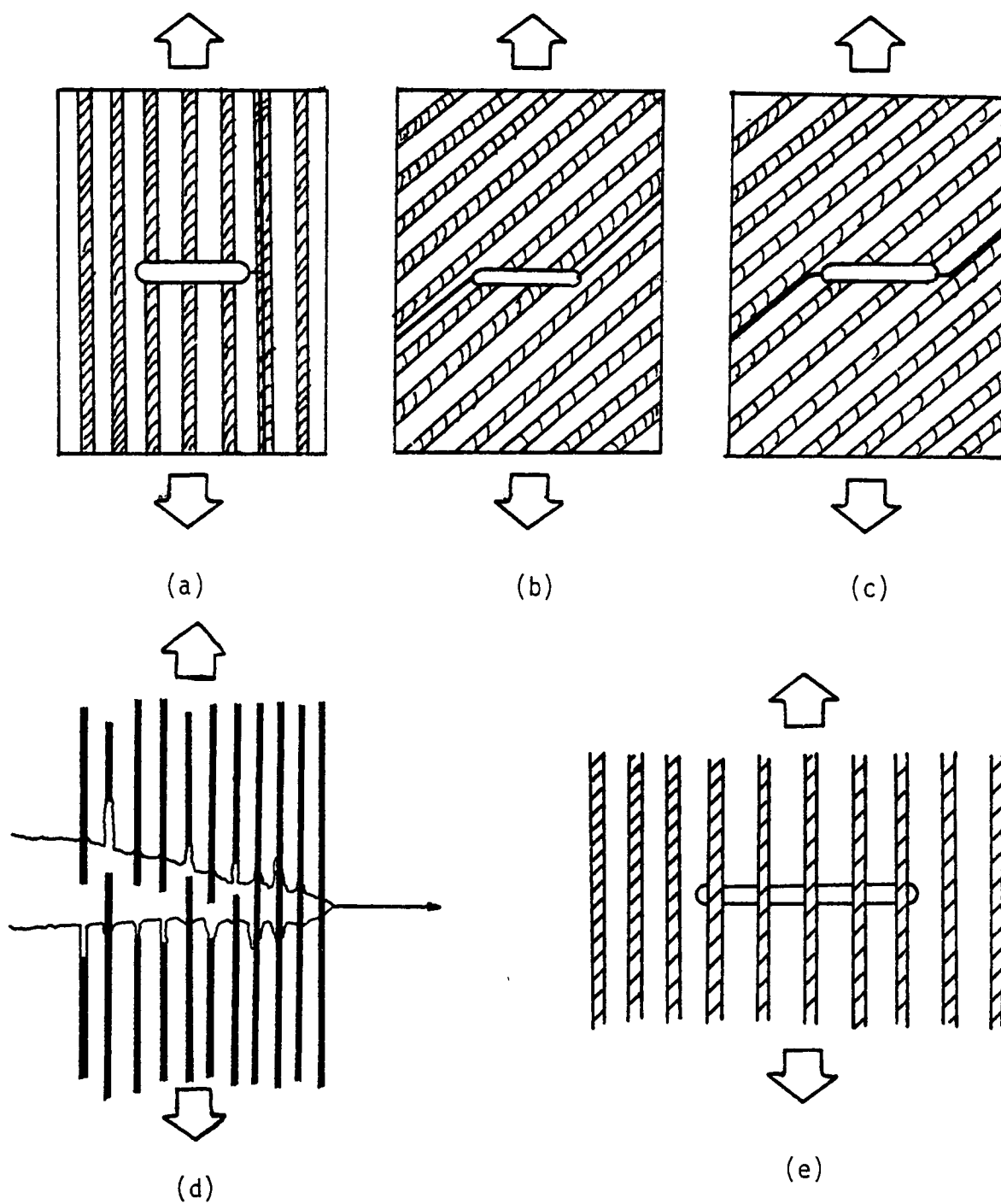


Figure 4.

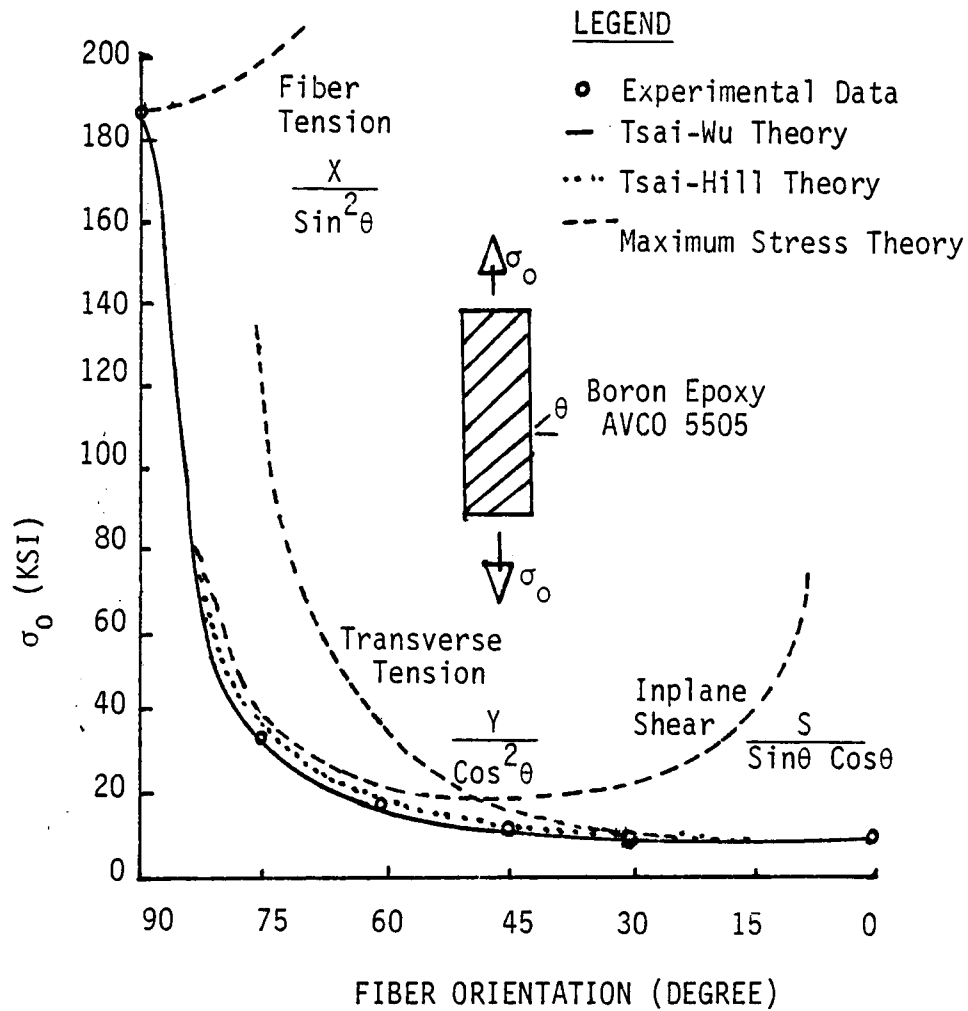


Figure 5. Comparison of different failure theories with the experimental data of uniaxial boron epoxy (AVCO 5505) composite.

Figure 6. Uniaxial behavior of materials.

(a) Nonlinear Elastic and Plastic; (b) Ideal Plasticity (Flat Top Stress-Strain Curve); (c) Power-Law Hardening (Ramberg-Osgood Relation); (d) Linear Strain-Hardening.

Slope of the uniaxial stress versus plastic strain ($\bar{A}$) is shown in each case.

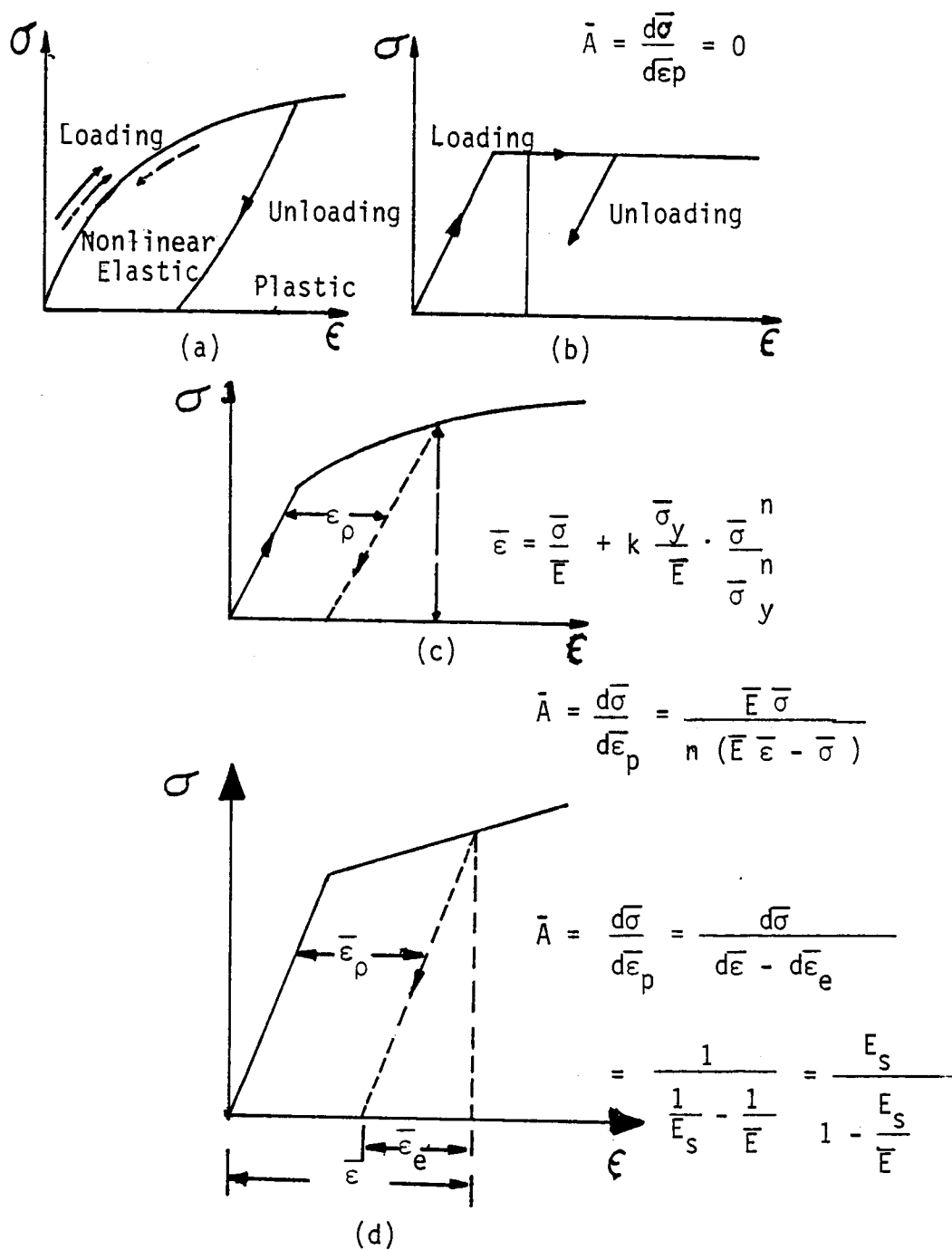


Figure 6.

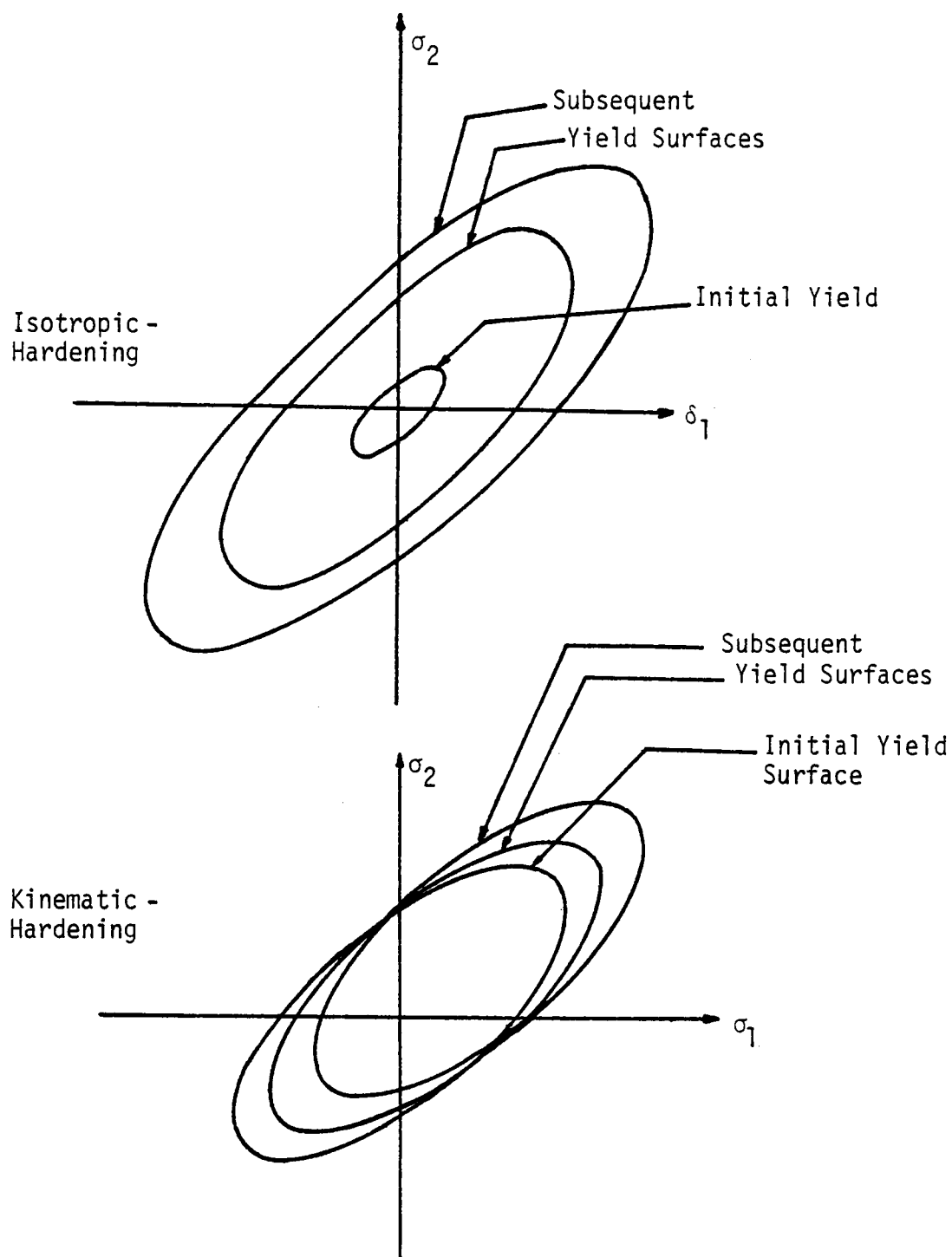


Figure 7. Schematic diagrams of isotropic- and kinematic-hardening rules in the plane of principal stresses.

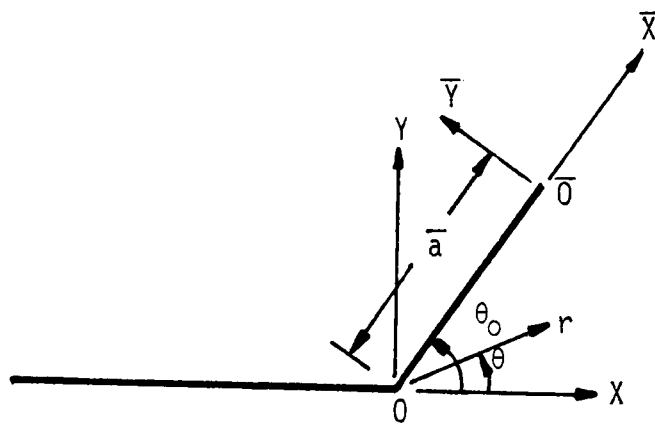
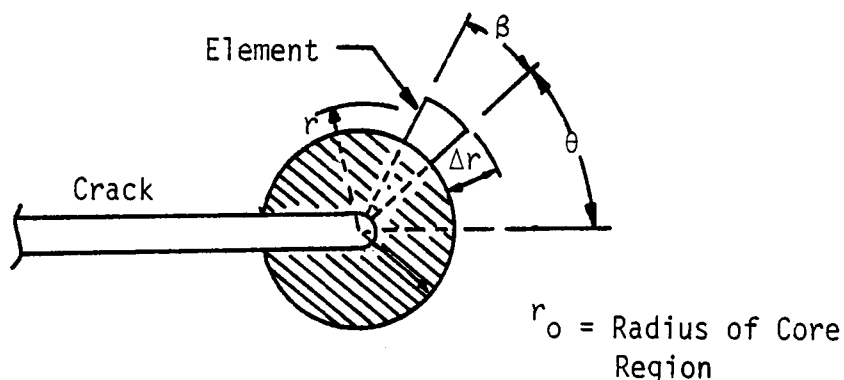
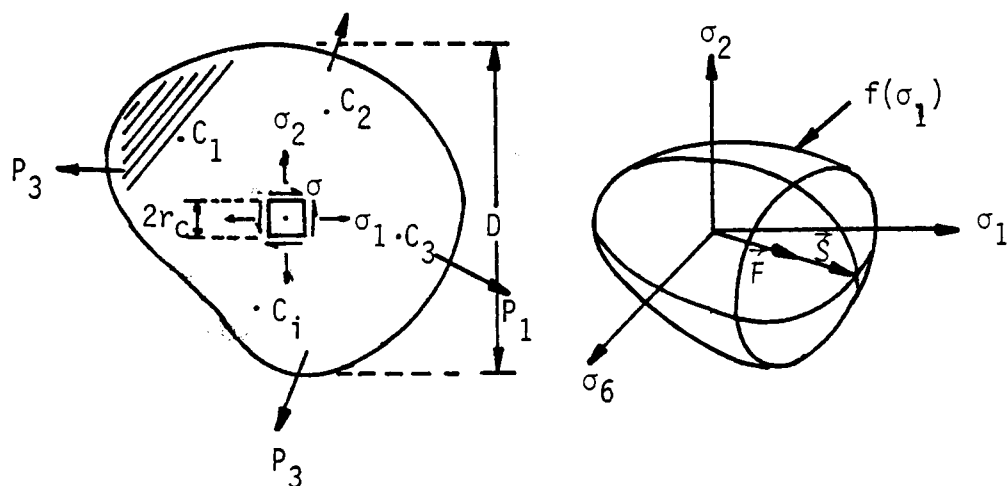


Figure 8. Geometry and coordinate systems for the branched crack.

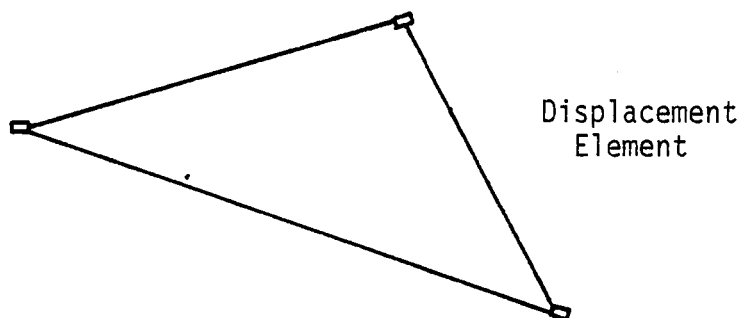


(a) Nonsingular "Core" Region and an Element in the Singular Region (Sih [81])

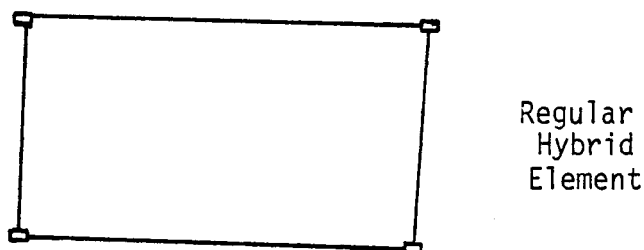


(b) Randomly Distributed Microscopic Flaws ($C_1, C_2, \dots$) along with the Stress Vector ($\vec{F}$), Strength Vector ($\vec{S}$), and Failure Surface $f(\sigma_i)$ (Wu [23])

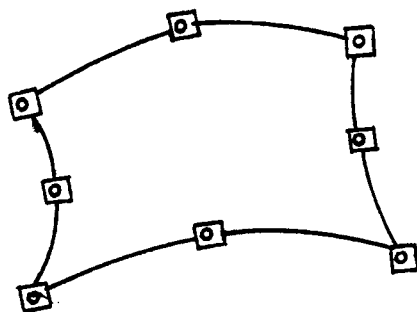
Figure 9. Schematic diagrams of macromechanical fracture theories.



(a) Displacement field is assumed throughout the element

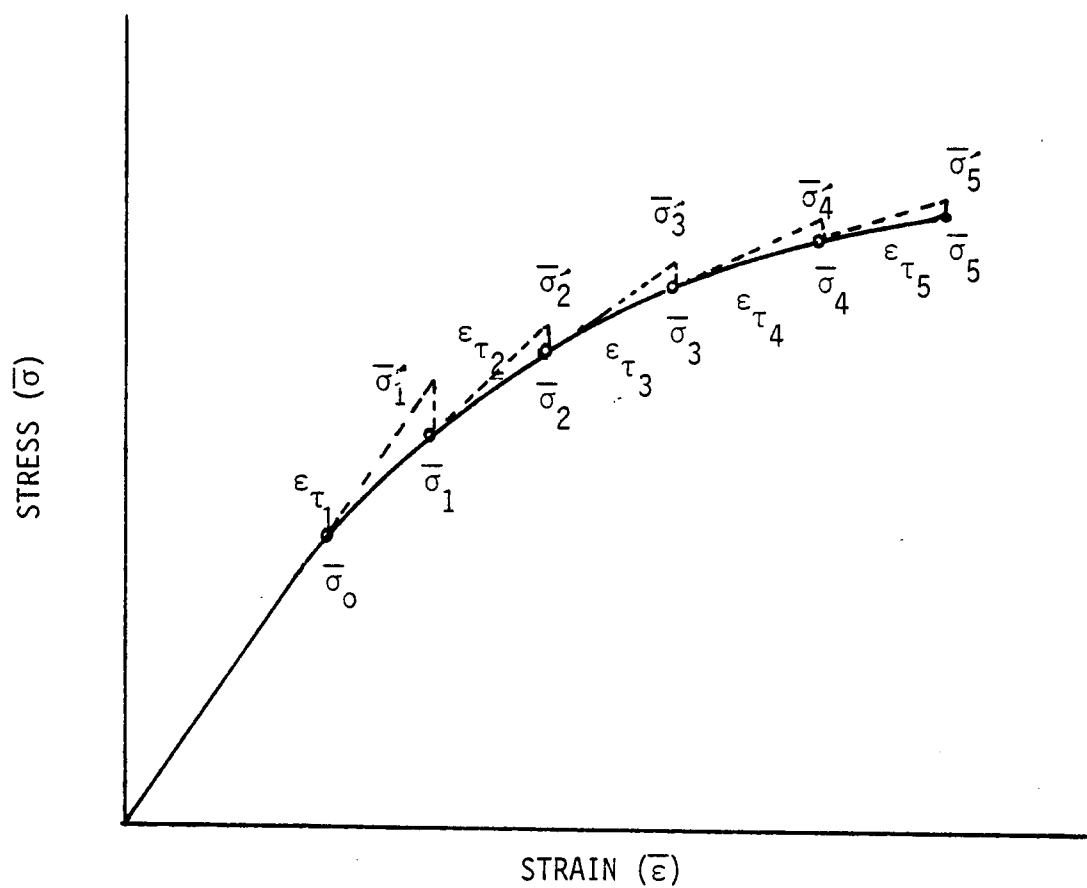


(b) Equilibrating stress field is assumed within the element and compatible displacement field is assumed along the element boundary

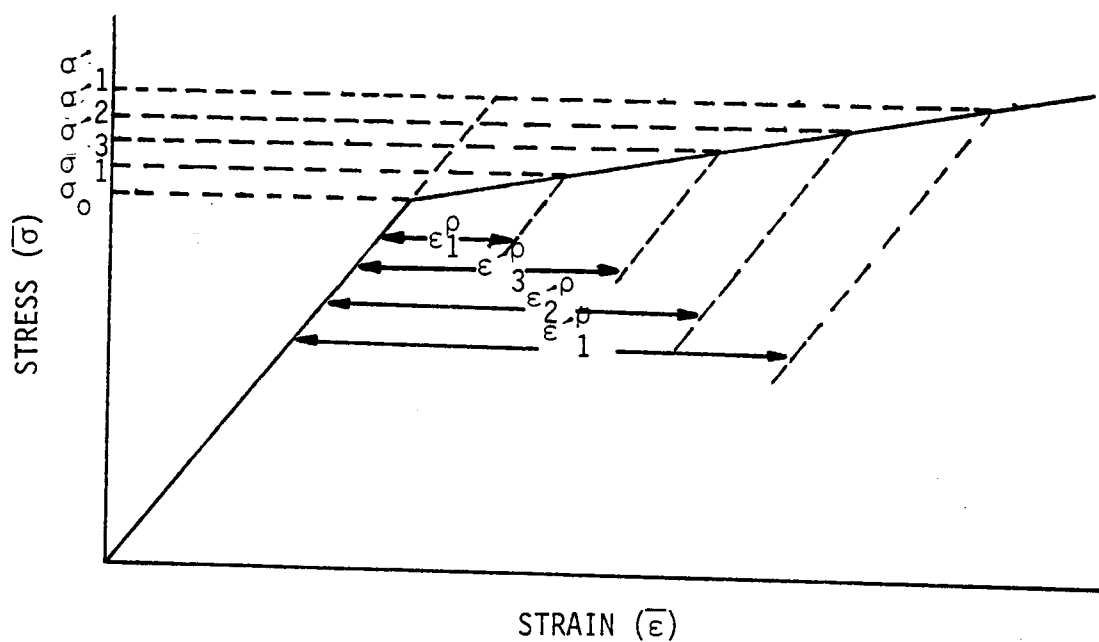


- (c)
- Point at which element coordinate is specified
 - Point at which displacement is specified

Figure 10. Specifications of commonly used finite elements.



(a) Variable Stiffness



(b) Initial Strain Methods of Plasticity

Figure 11. Schematic diagrams of variable stiffness and initial strain methods of plasticity.

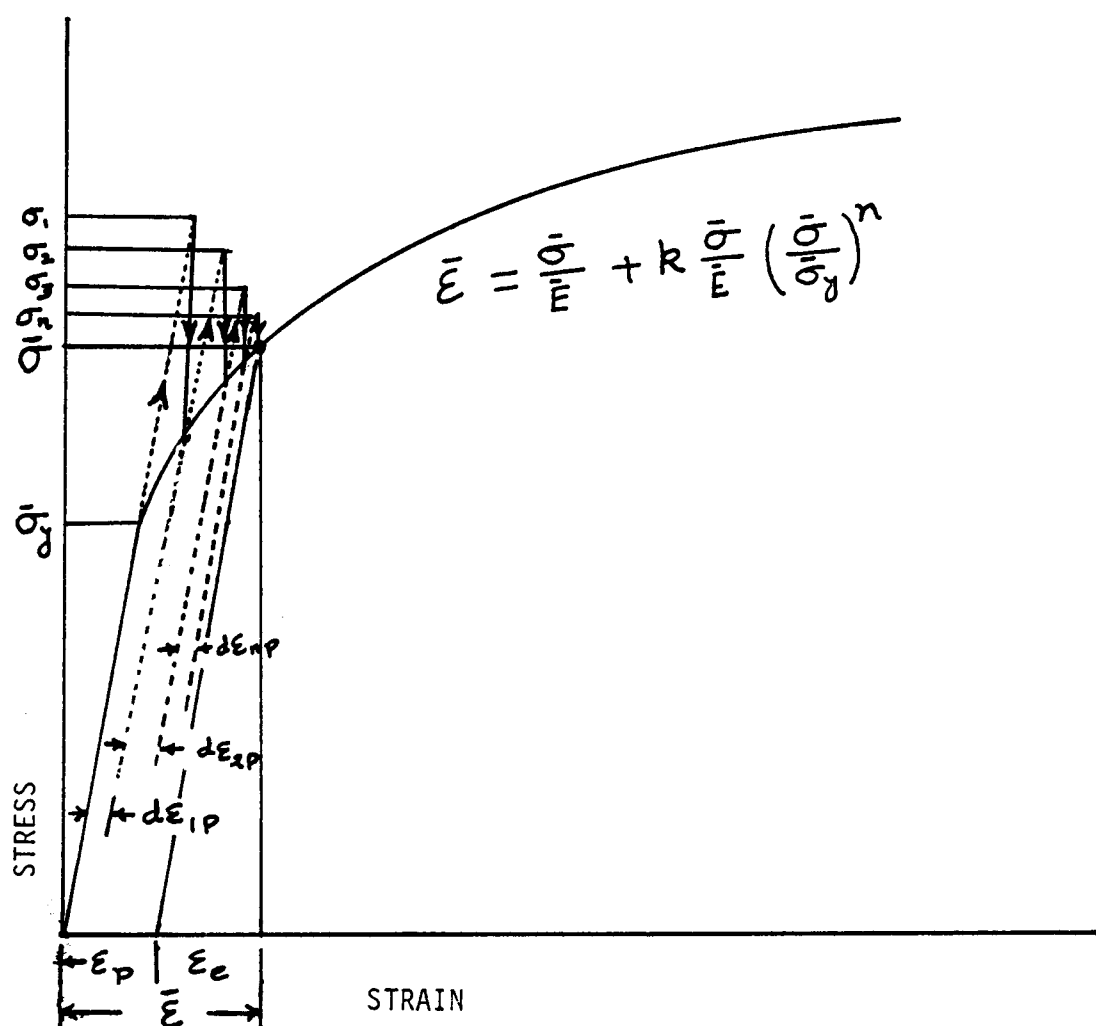


Figure 12. Graphical representation of initial stress method in the effective stress-effective strain plane.

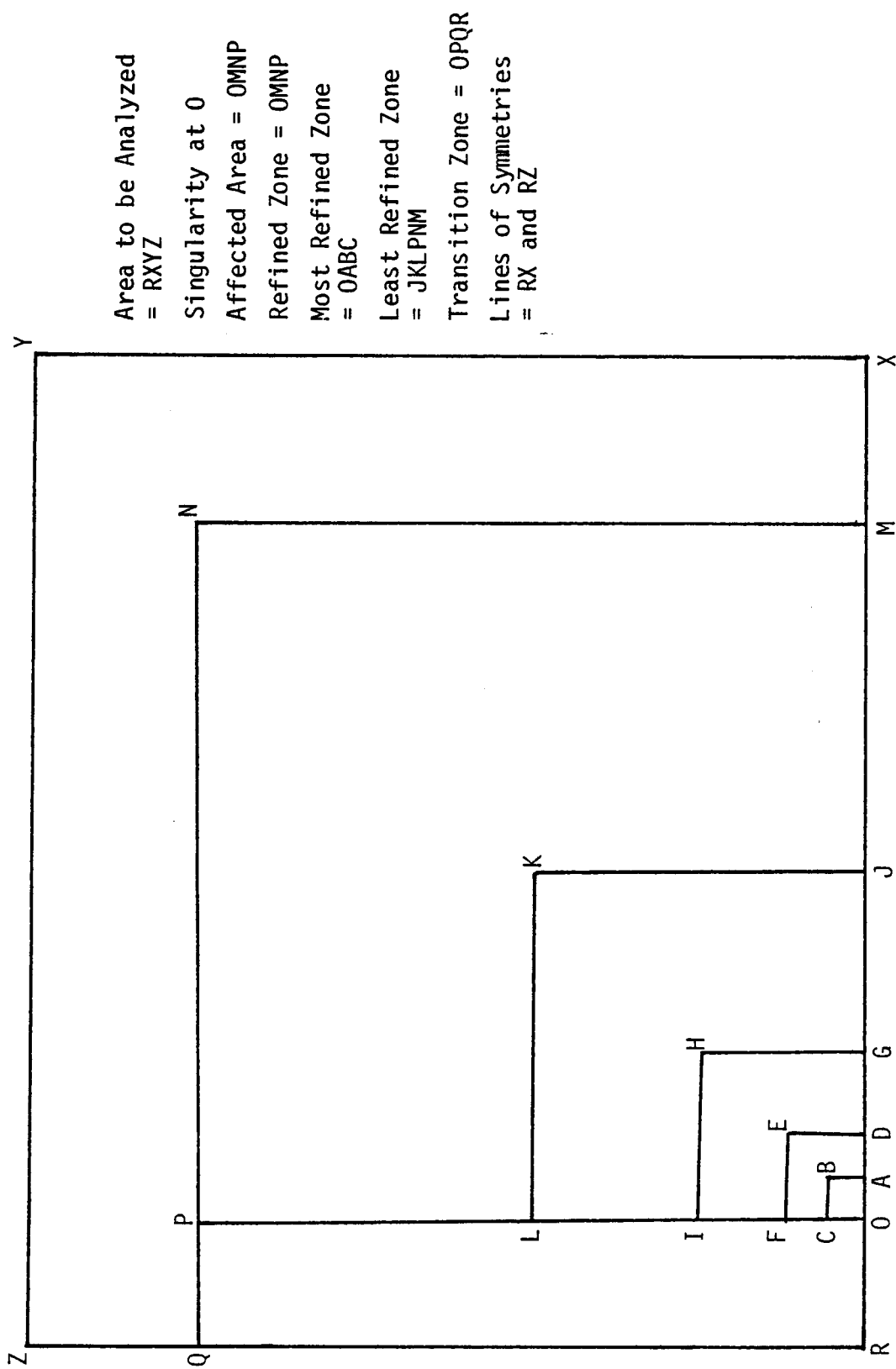
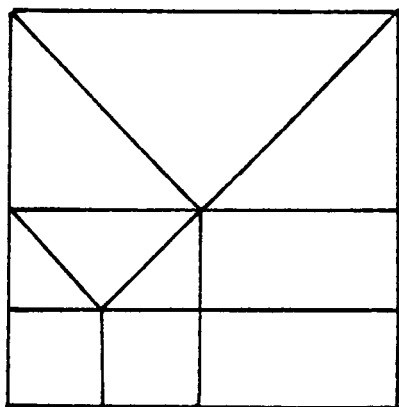
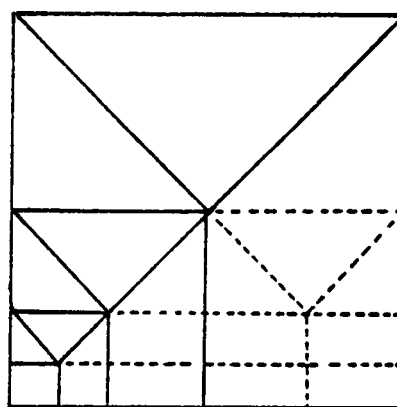


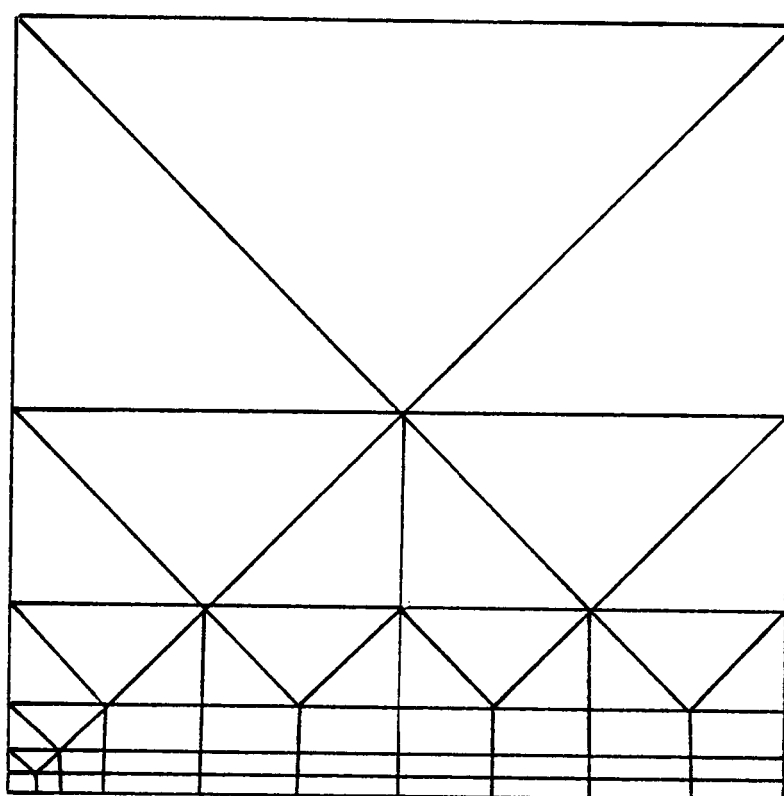
Figure 13. Proposed method to gradually grade the finite element mesh to treat a singularity.



(a) $n = 13$ $p = 2$ $r = 2$
 $b = 10$



(b) $n = 22$ $p = 3$ $r = 2$
 $b = 22$



(c) $n = 51$ $p = 5$ $r = 4$ $b = 26$

n = Number of Nodal Points

p = Number of Nodes on the Diagonal

r = Slenderness Ratio

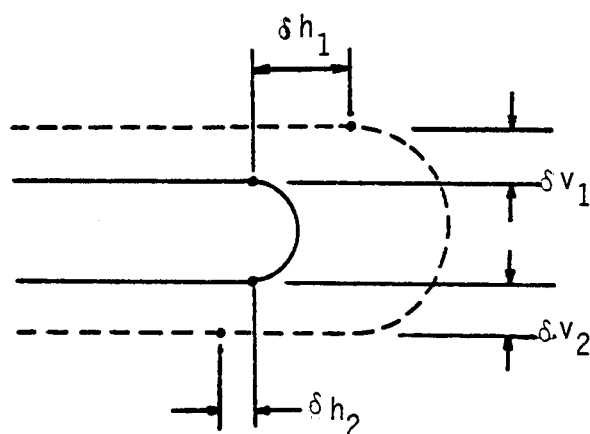
b = Semi-bandwidth

Figure 14. Proposed refinement procedure near a singularity.

LEGEND

— Undeformed Crack

--- Deformed Crack



δ_v = Vertical Deformation of the Crack Tip = $\delta v_1 + \delta v_2$

δ_h = Horizontal Deformation of the Crack Tip = $\delta h_1 + \delta h_2$

δ = Crack-Opening Displacement = COD = $(\delta_h^2 + \delta_v^2)^{1/2}$

Figure 15. Calculation of the crack-opening displacement for a general anisotropic material.

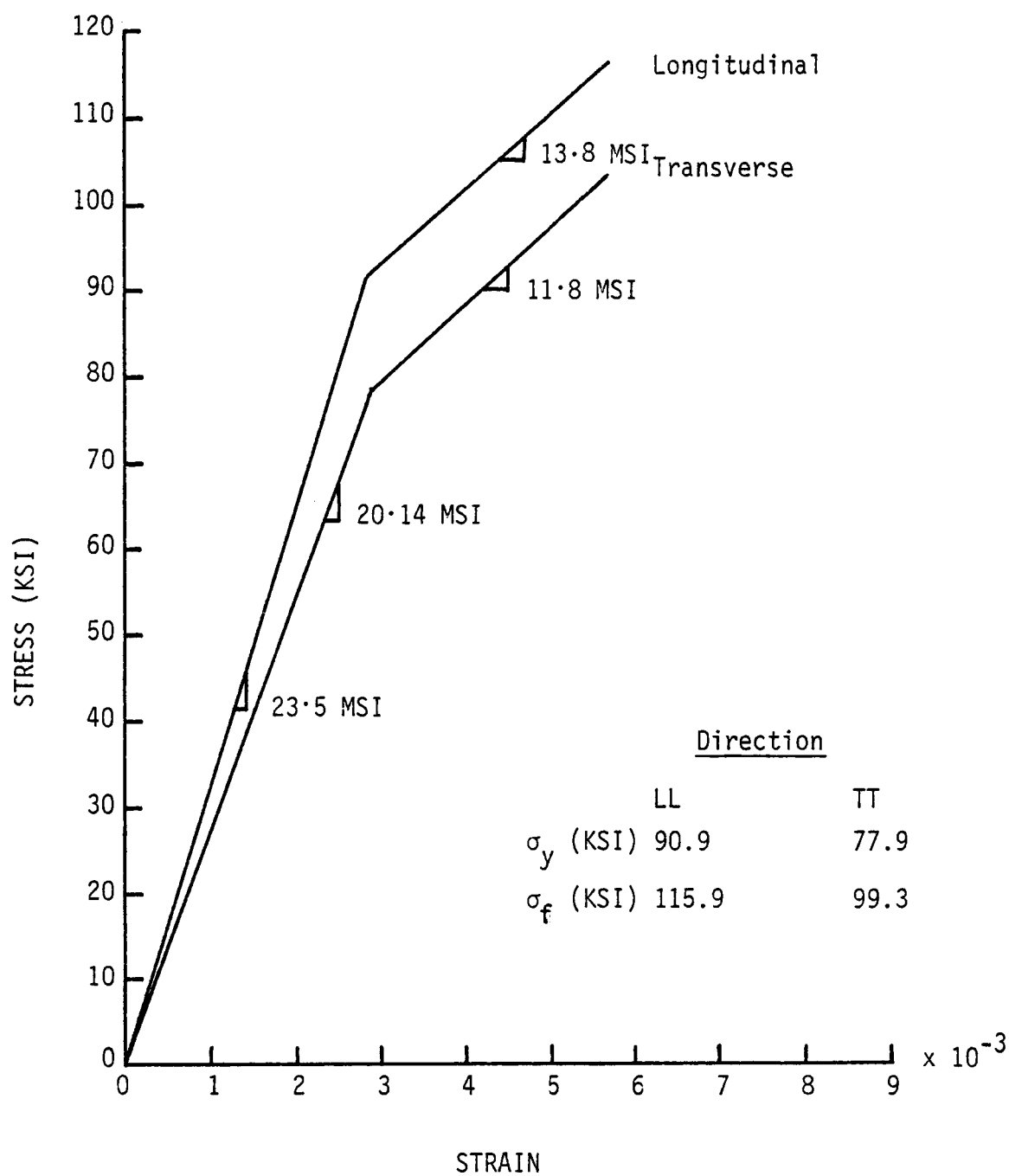


Figure 16. Stress-strain curves for 0°/90° cross-ply B-Al composite.

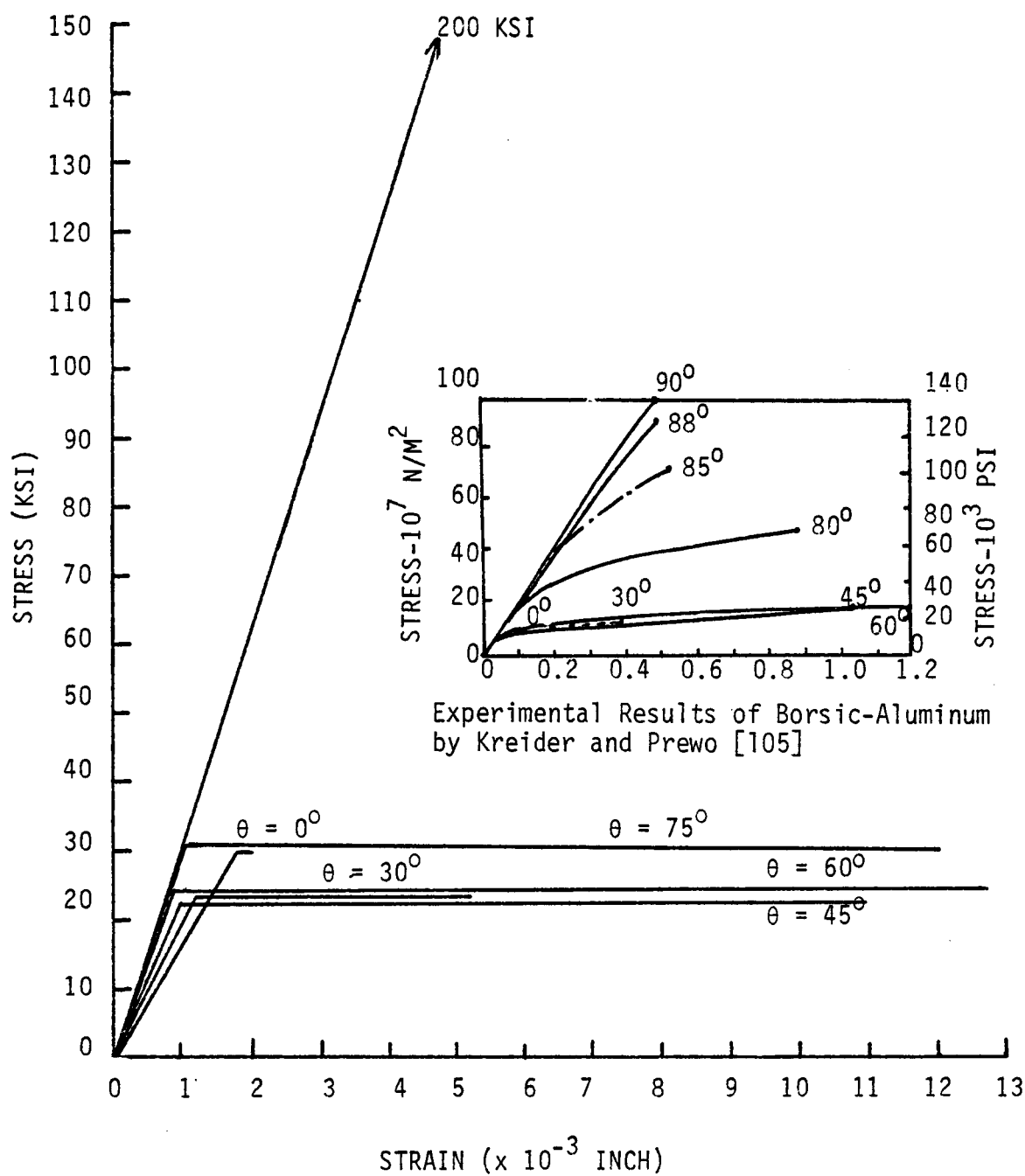


Figure 17. Idealized stress-strain curves for B-Al composites for different fiber orientation.

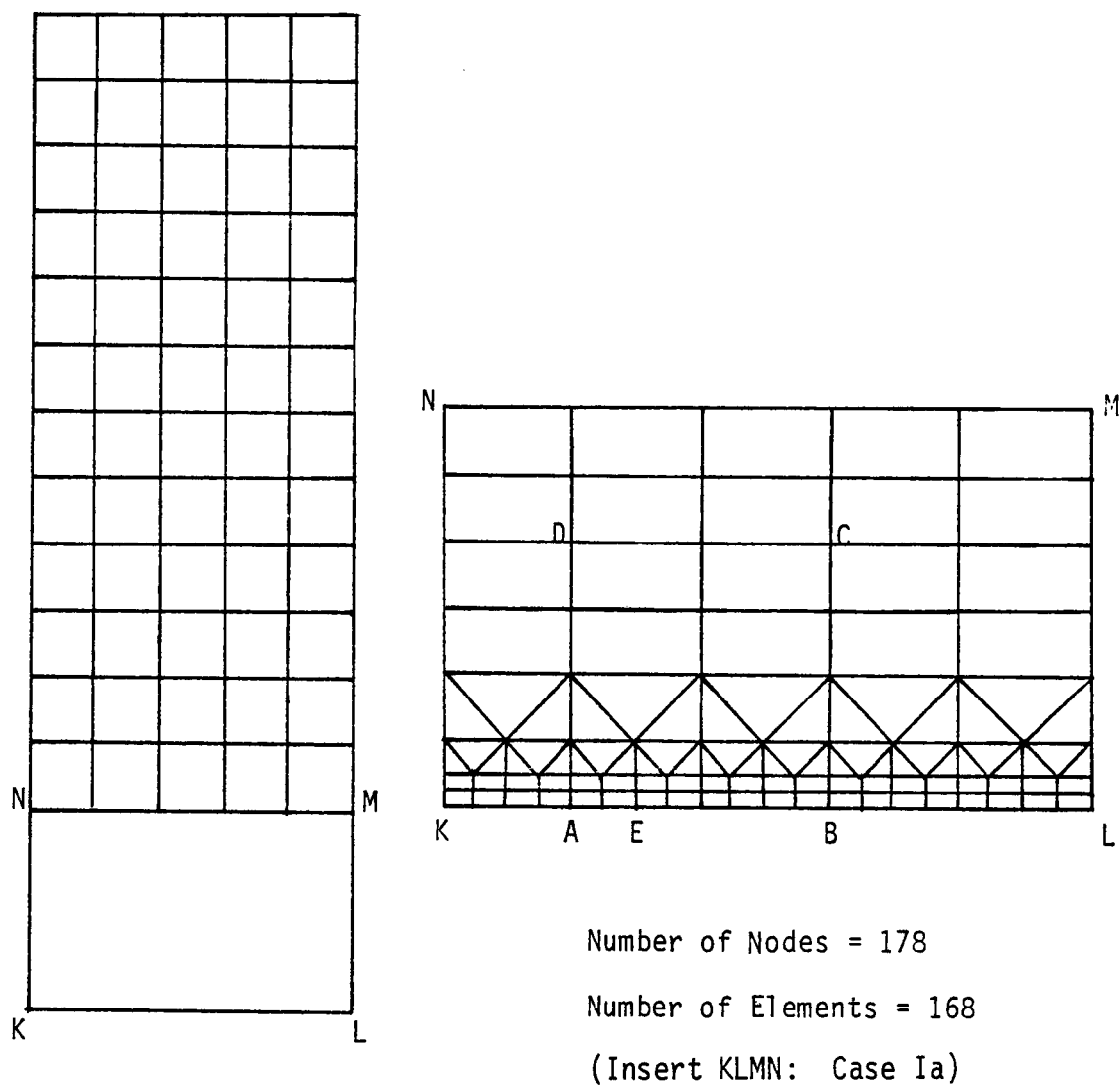
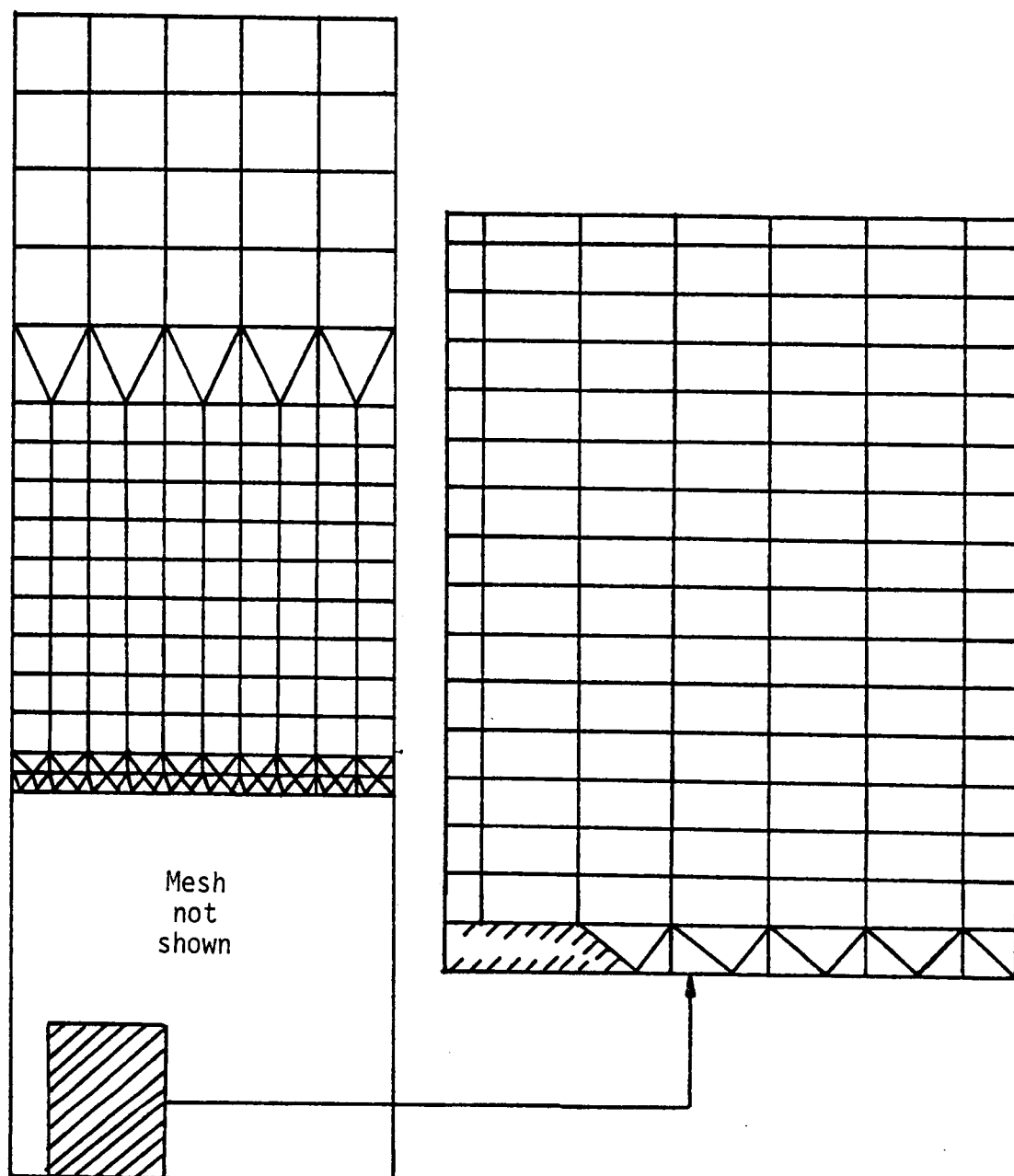


Figure 18. Finite element mesh for the quarter plate of a center-cracked tension specimen using gradual grading method.



Magnification x 4

Number of Nodes = 860

Number of Elements = 853

Figure 19. Uniform mesh finite elements for the quarter plate of a center-cracked tension specimen.

Number of Nodes = 234

Number of Elements = 230

(Insert ABCD: Case Ib)

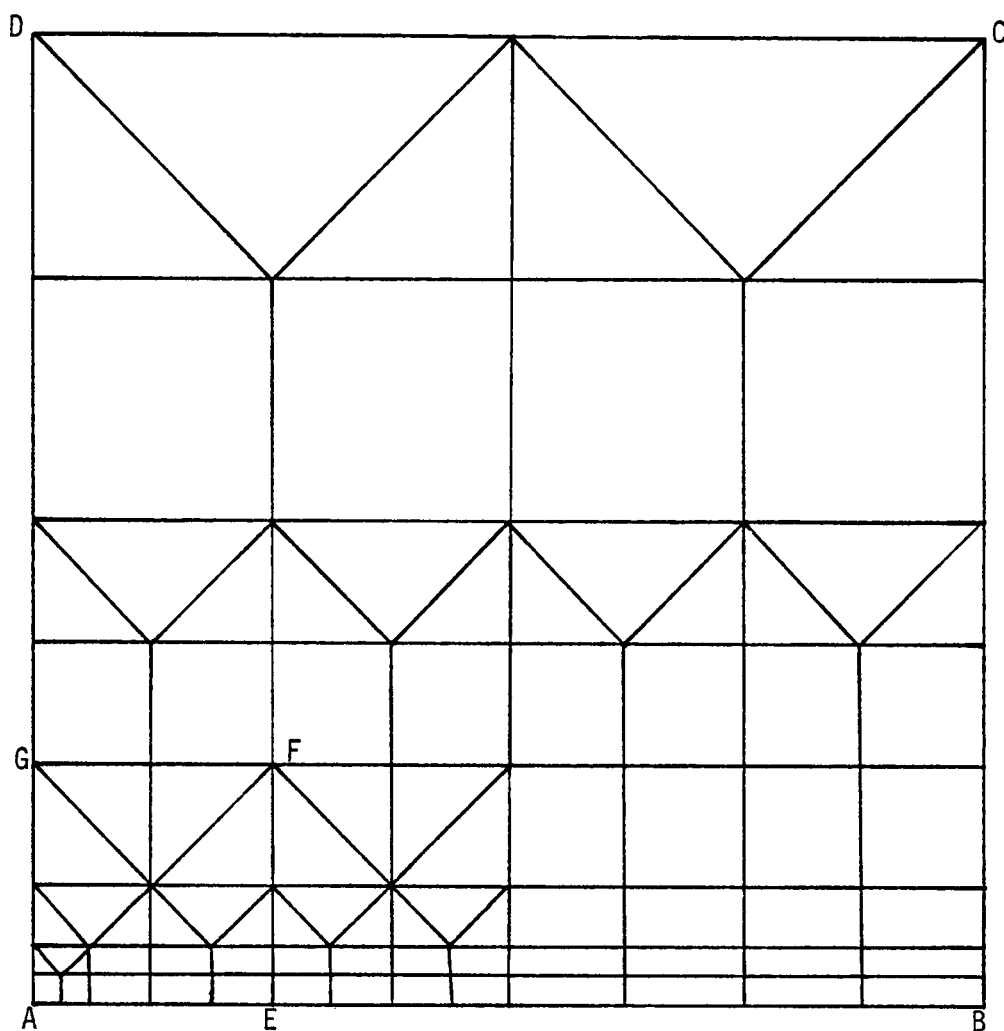


Figure 20. Gradually graded finite element mesh for the quarter plate of a center-cracked tension specimen (Case Ib).

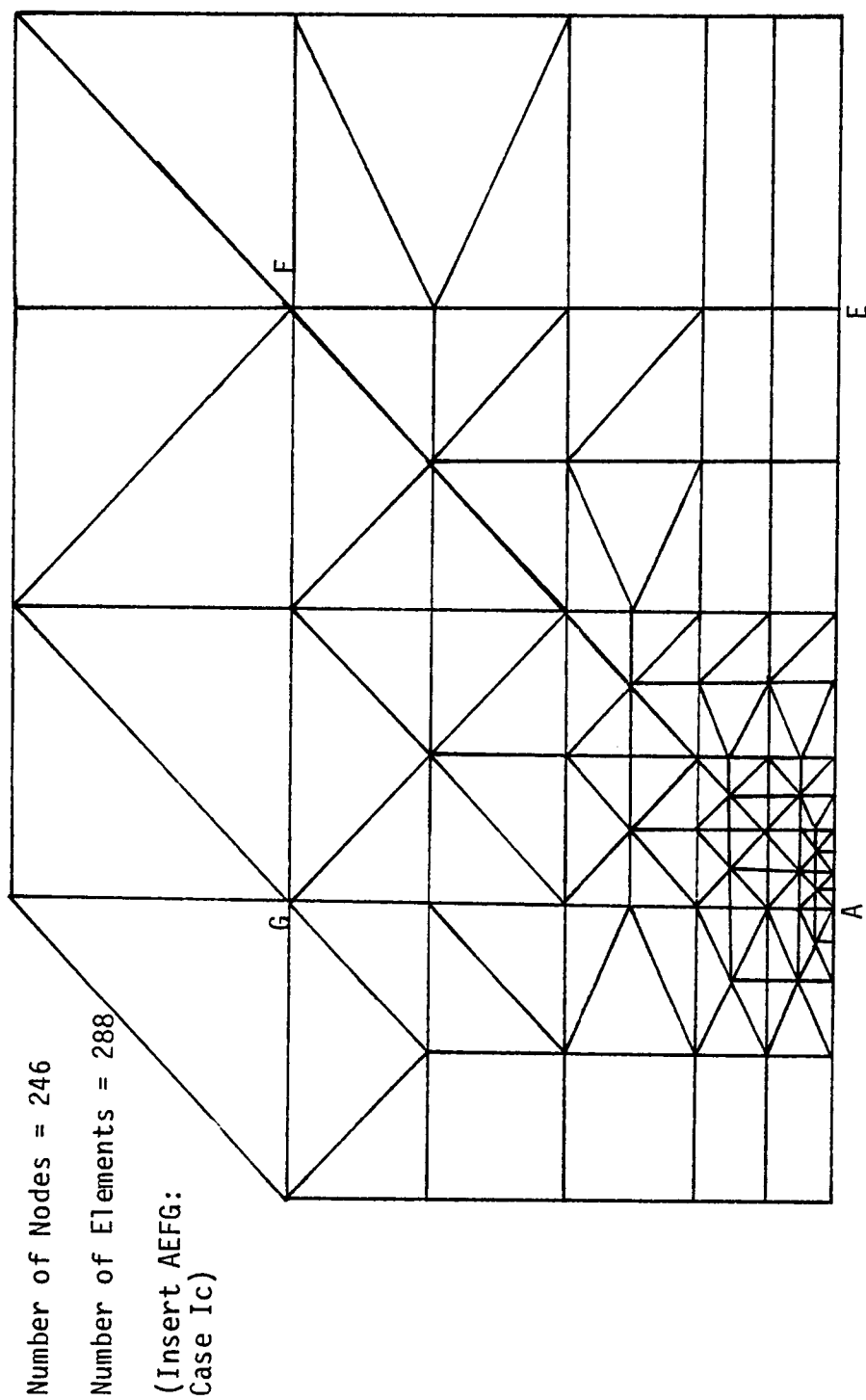
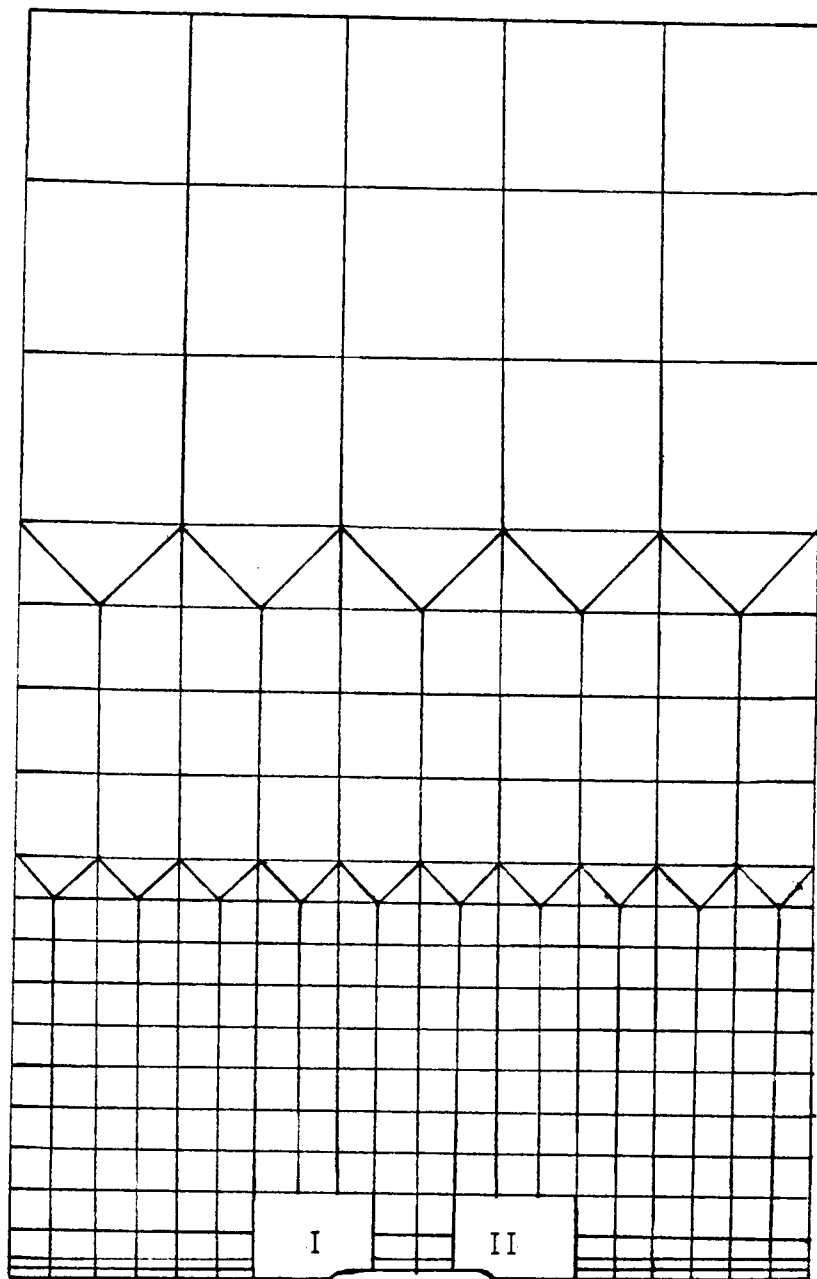


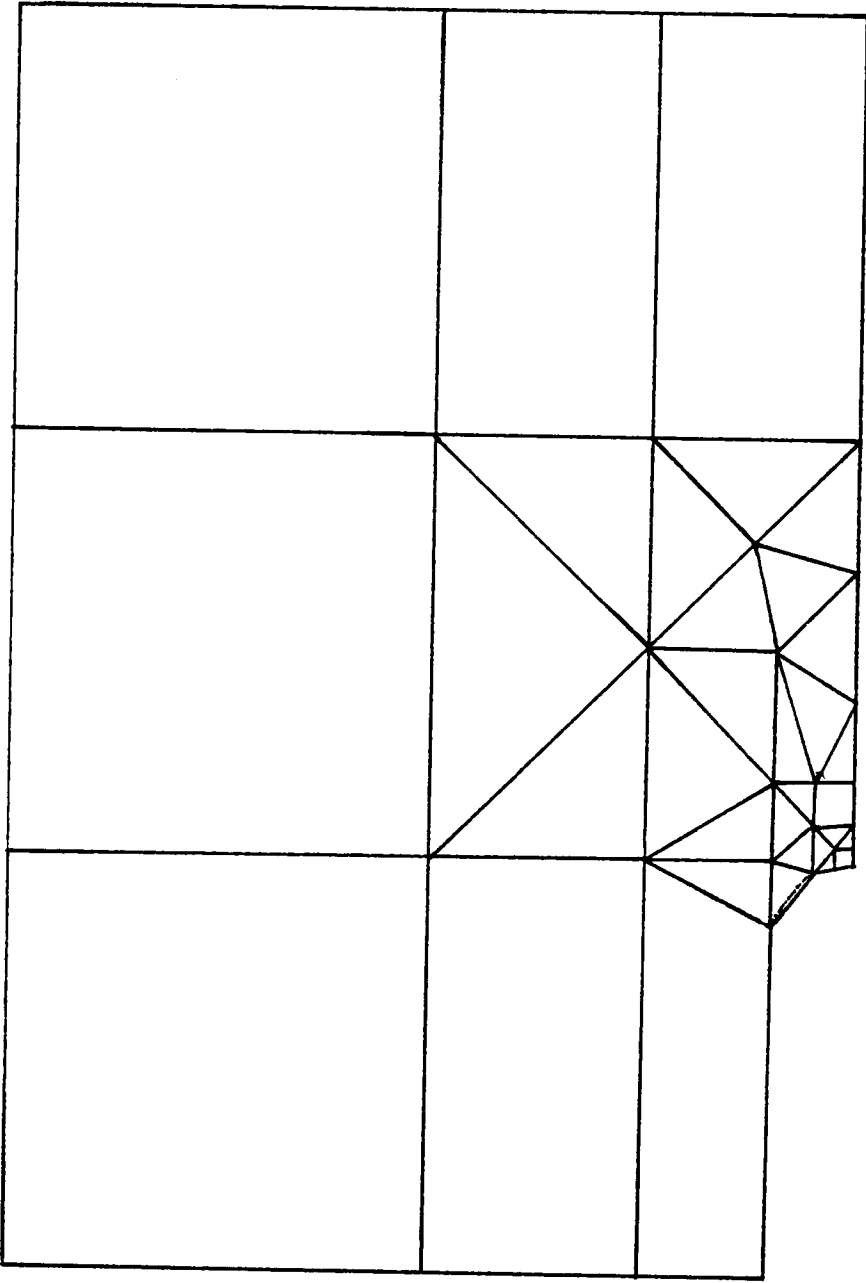
Figure 21. Gradually graded finite element mesh for the quarter plate of f a center-cracked tension specimen (Case Ic).



Number of Nodes = 666 Number of Elements = 708

(a) Finite element mesh for the region away from the crack

Figure 22. Half of the finite element mesh for a center-cracked tension specimen.



(b) Exploded view for the region near the crack

Figure 22. (continued)

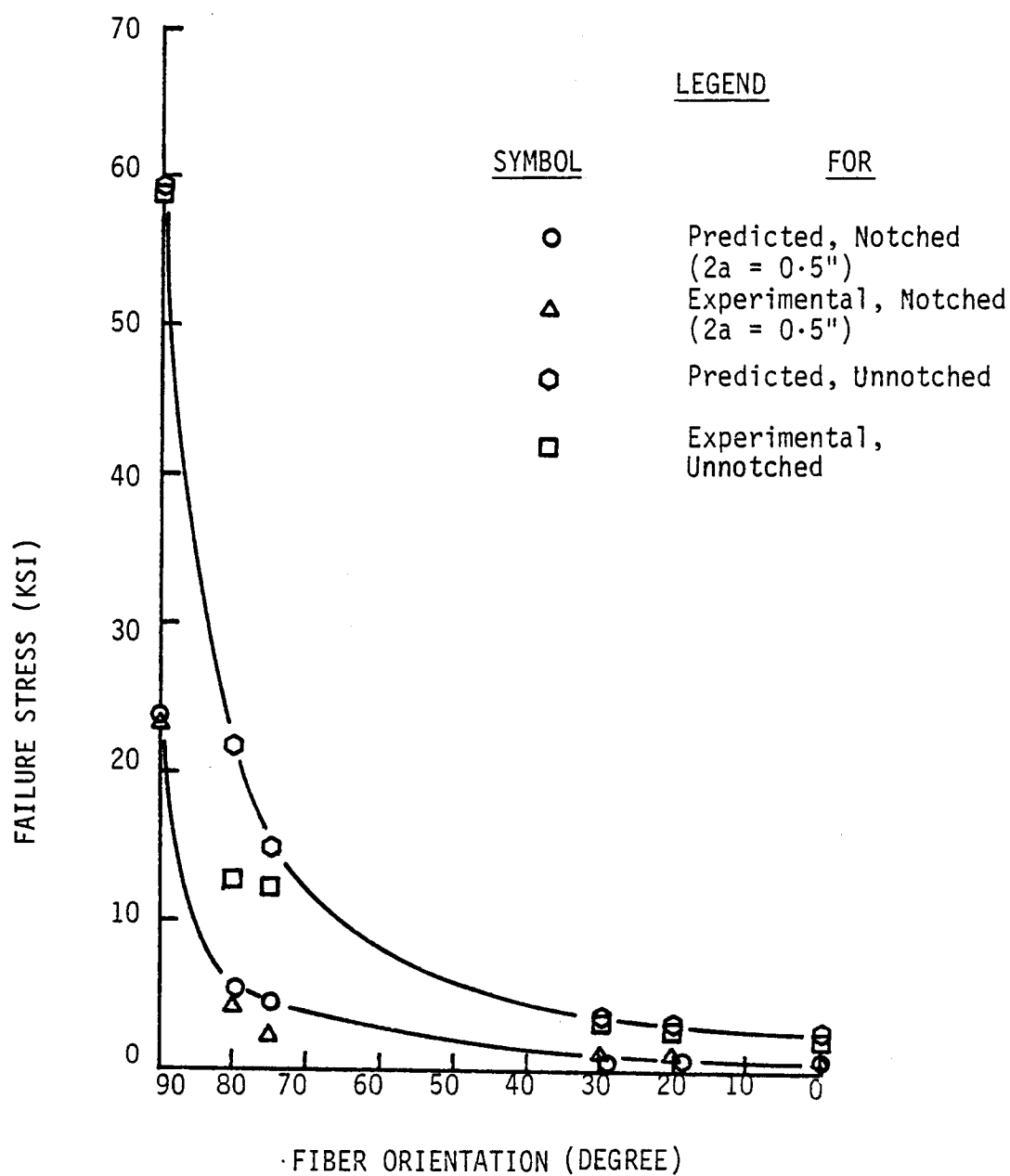


Figure 23. Comparison of predicted and experimental strengths of unidirectional GFRE as the function of fiber orientation.

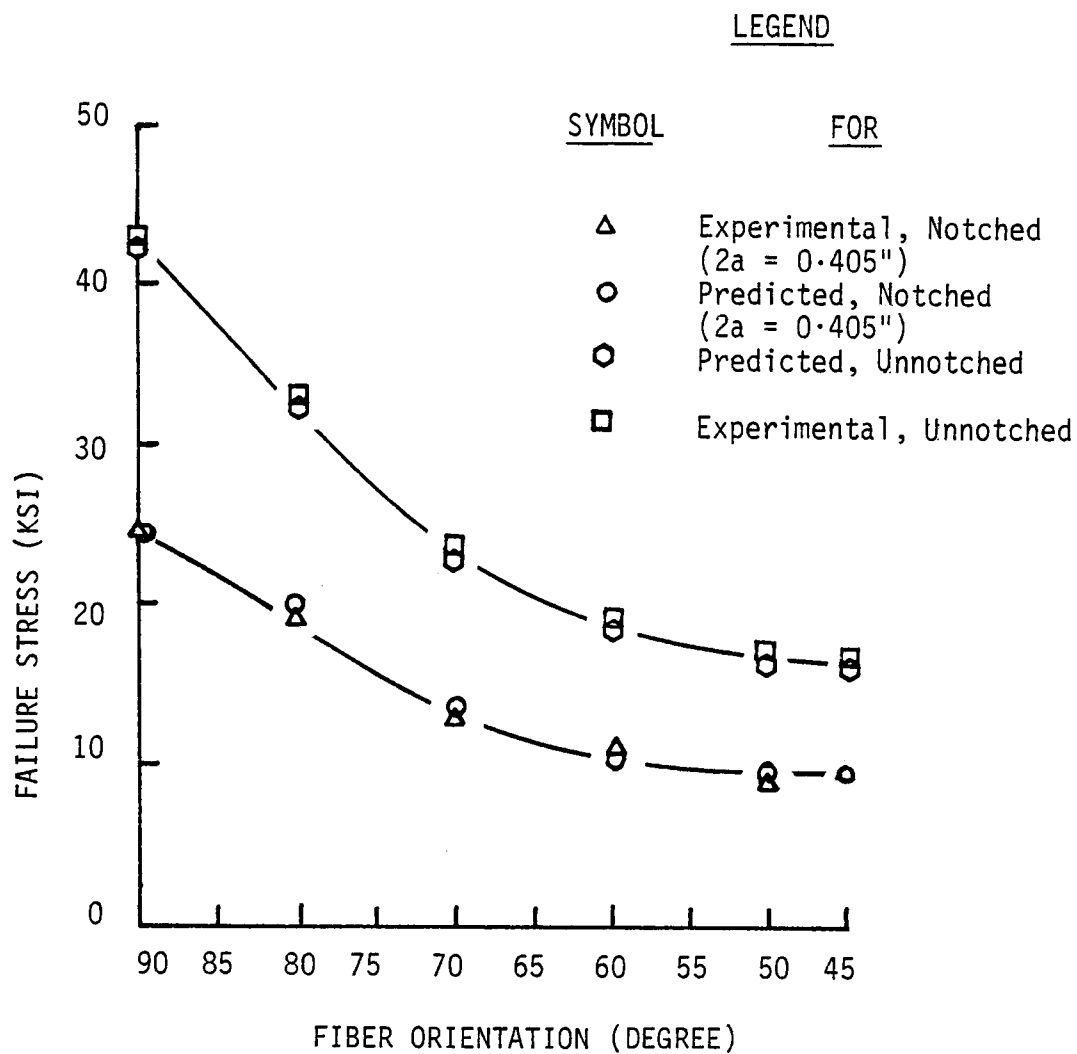
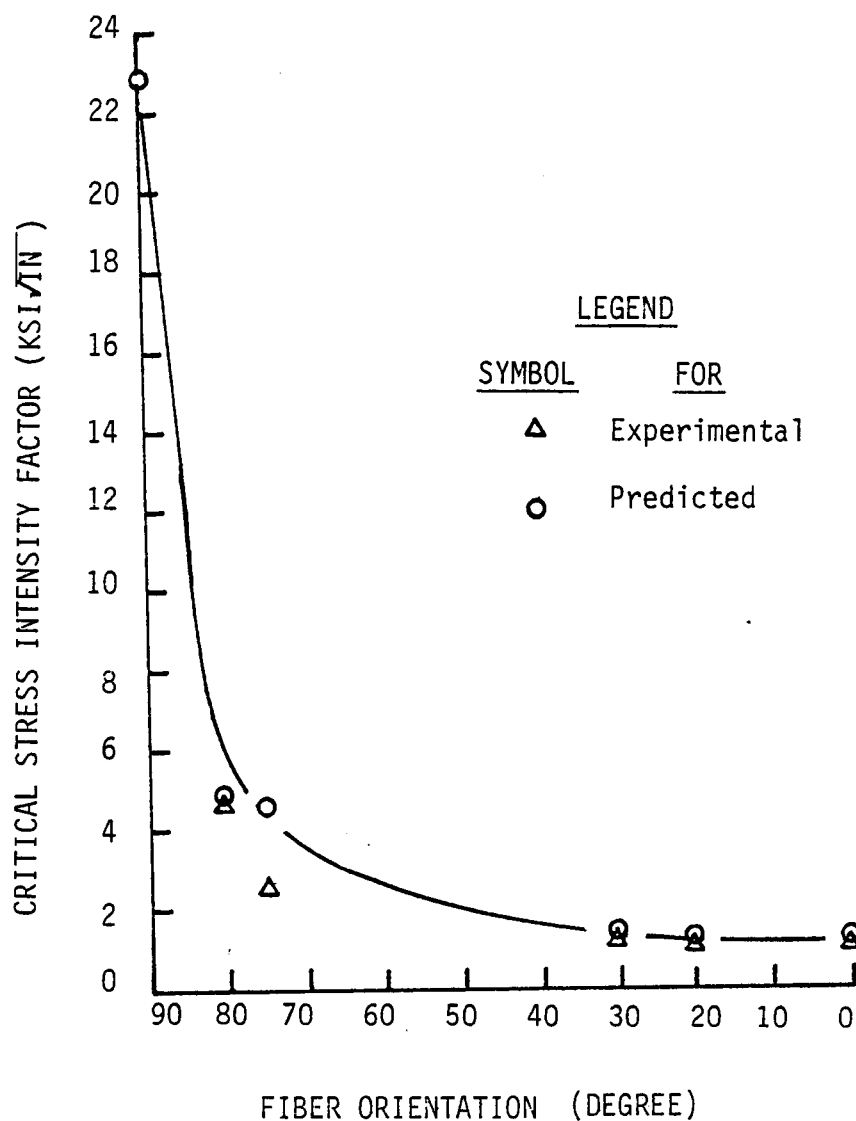
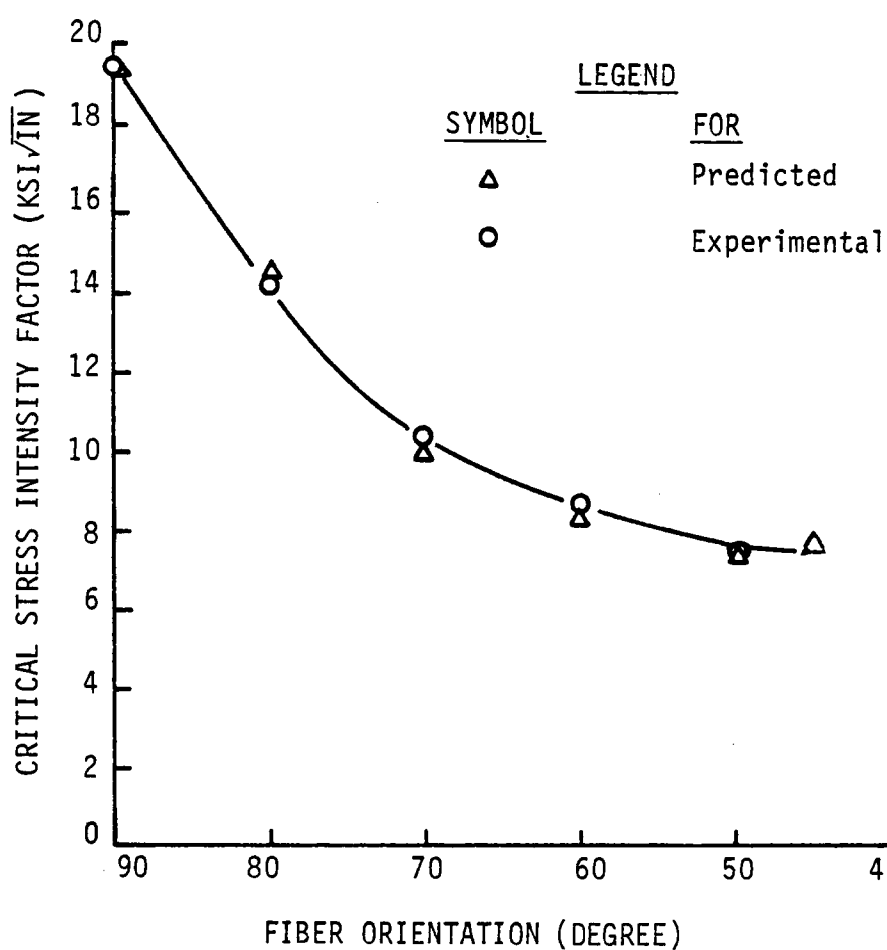


Figure 24. Comparison of predicted and experimental strengths for cross-plyed GFRE as a function of fiber orientation.



(a) Comparison between predicted and experimental critical stress intensity factors for unidirectional GFRE as a function of fiber orientation

Figure 25. Effect of the fiber orientation on the critical stress intensity factor of GFRE composites.



(b) Comparison of predicted and experimental critical stress intensity factors (K_{Ic}) for cross-ply ($0^\circ/90^\circ$) GFRE as a function of fiber orientation

Figure 25. (continued)

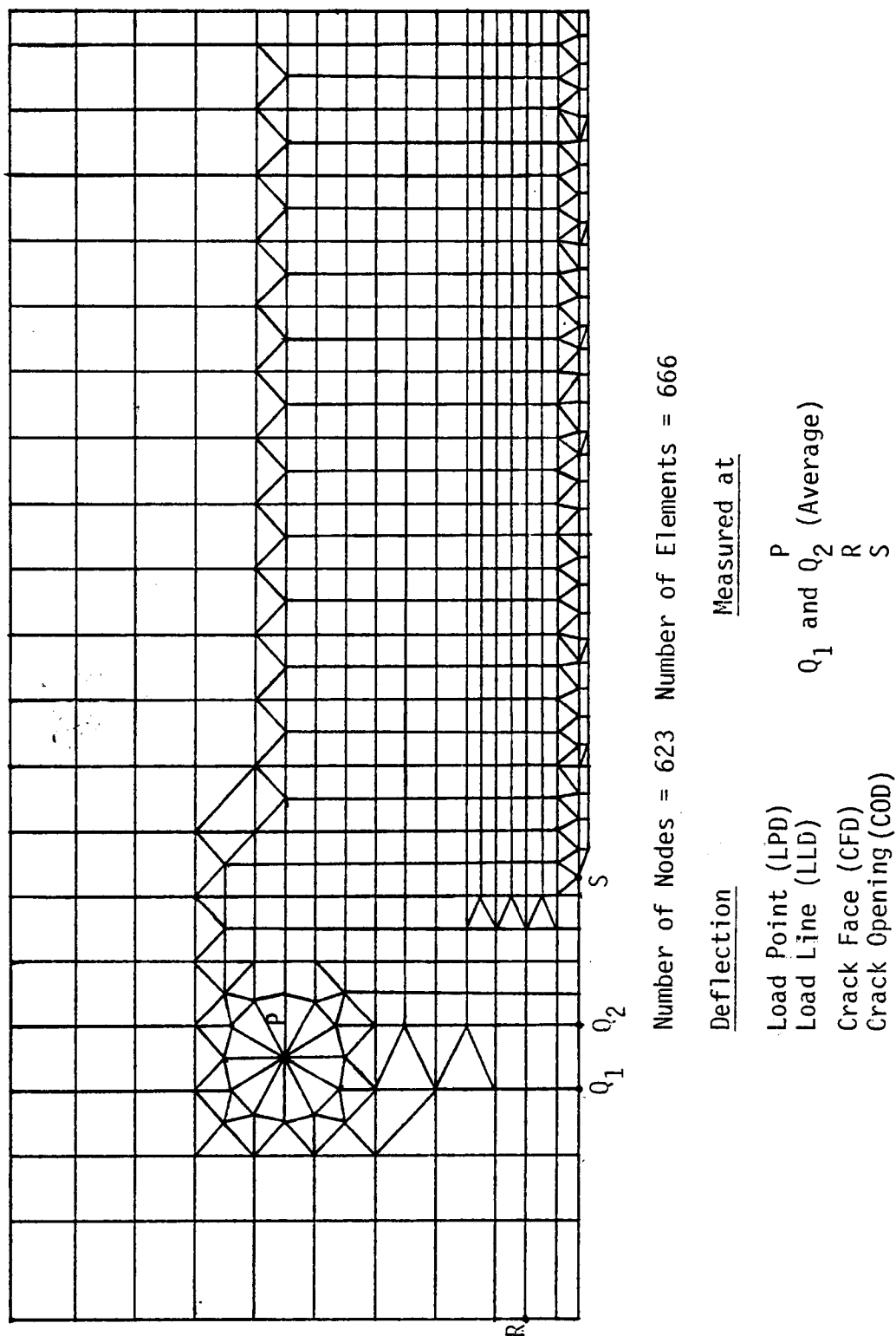


Figure 26.. General finite element mesh for a compact tension specimen.

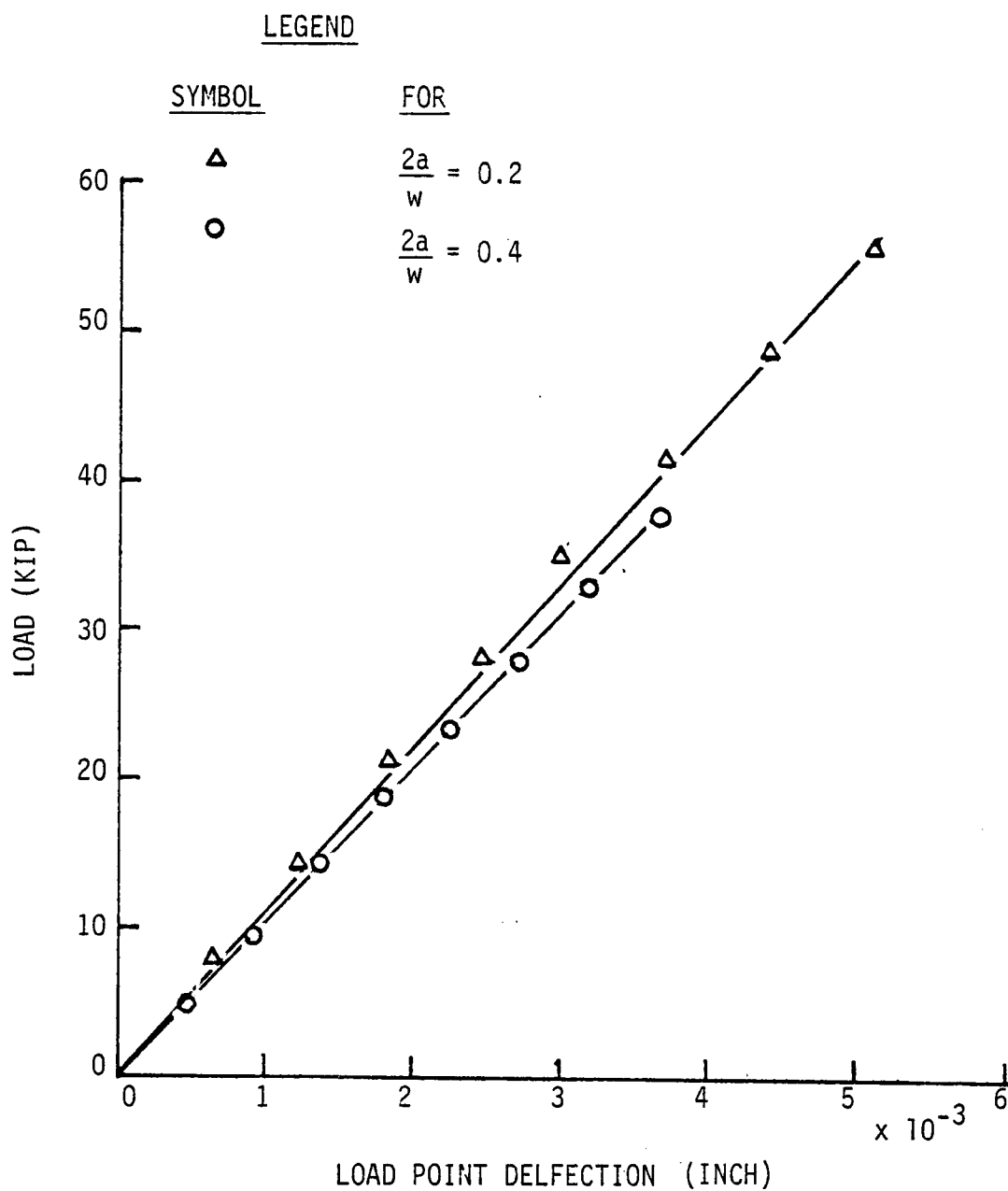


Figure 27. Variation of load point deflection with load for unidirectional B-A1 composite for $\theta = 90^\circ$.

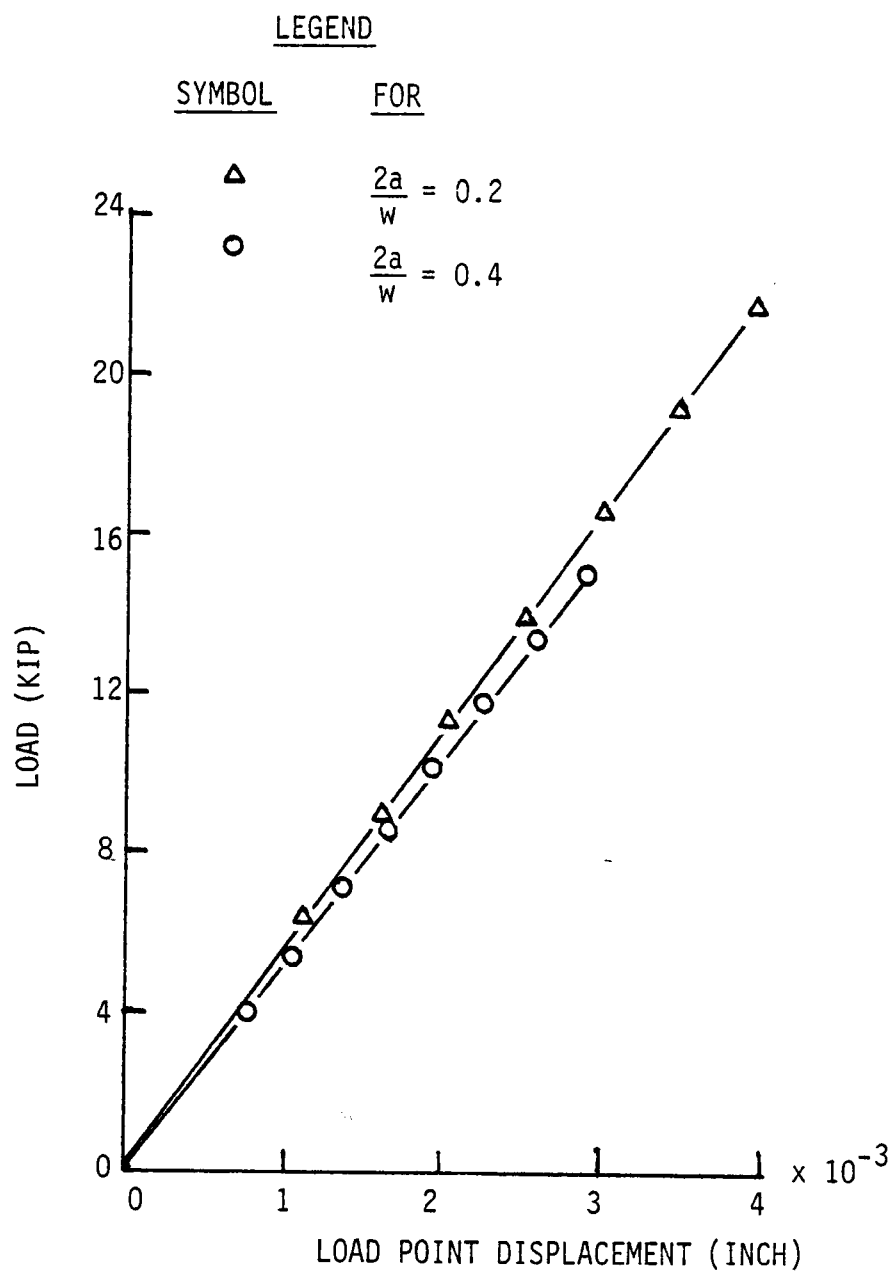


Figure 28. Variation of load point deflection with load for unidirectional B-A1 composite for $\theta = 0^\circ$.

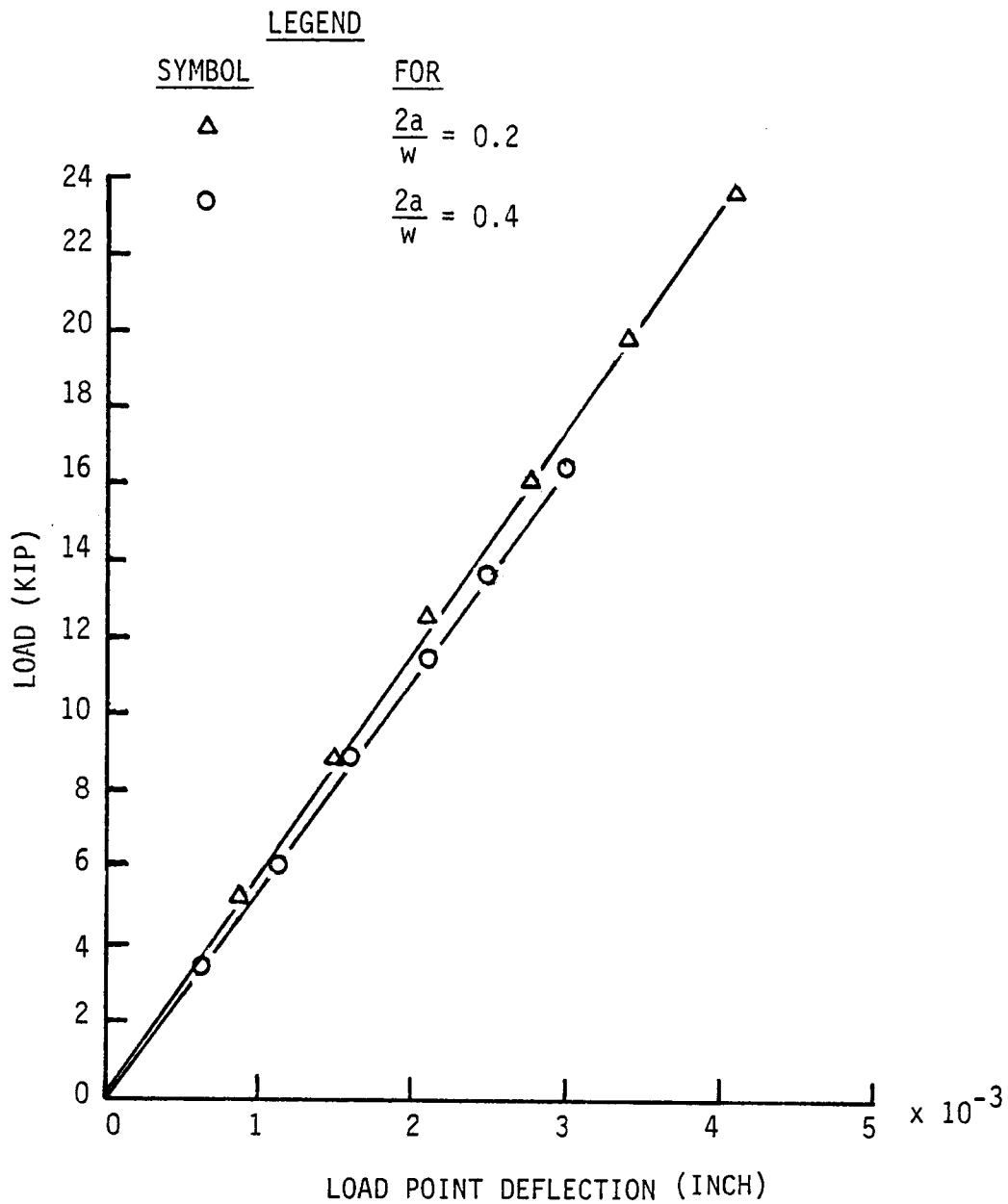


Figure 29. Variation of load point deflection with load for unidirectional B-Al composite for $\theta = 30^\circ$.

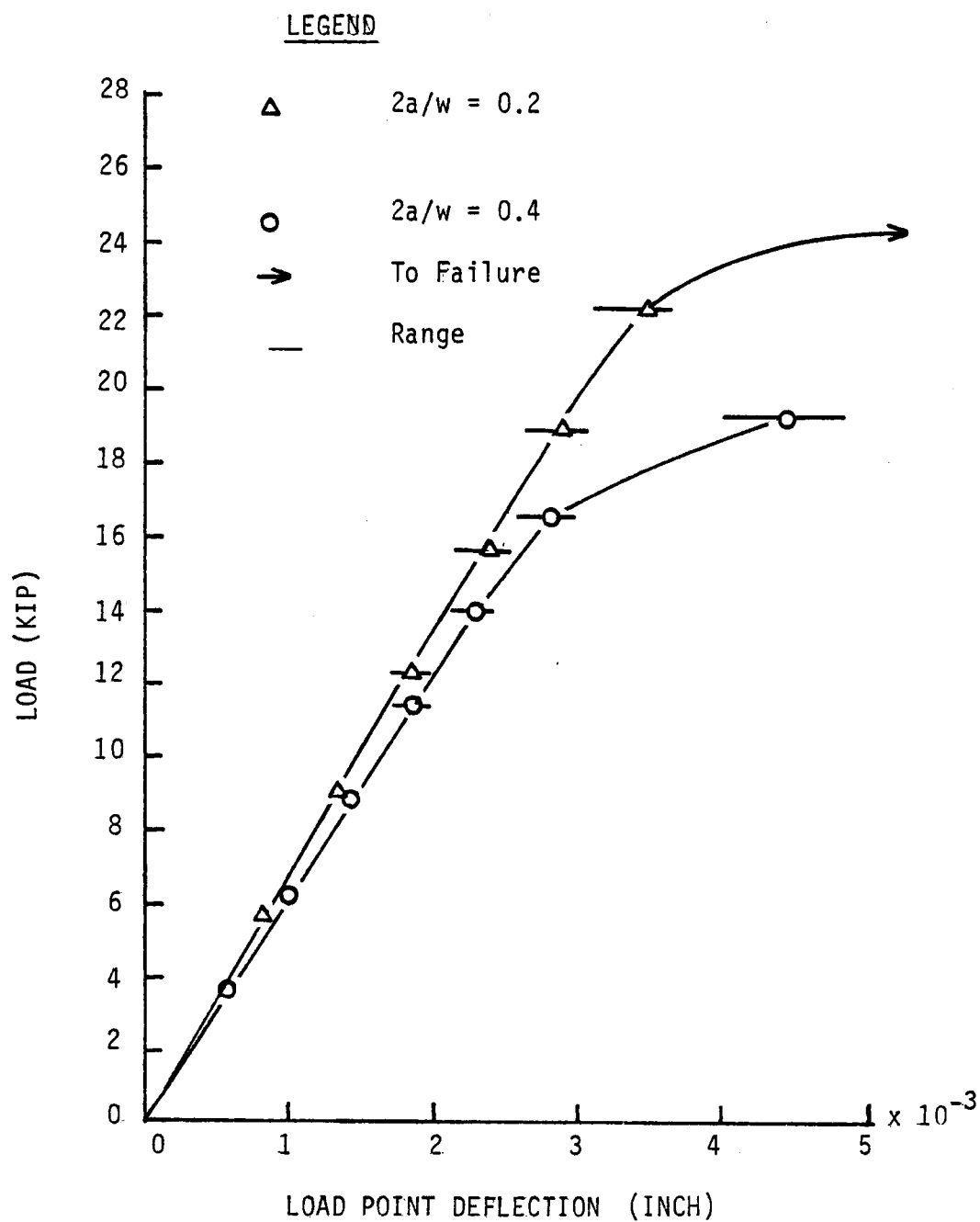


Figure 30. Variation of load point deflection with load for unidirectional B-Al composite for $\theta = 45^\circ$.

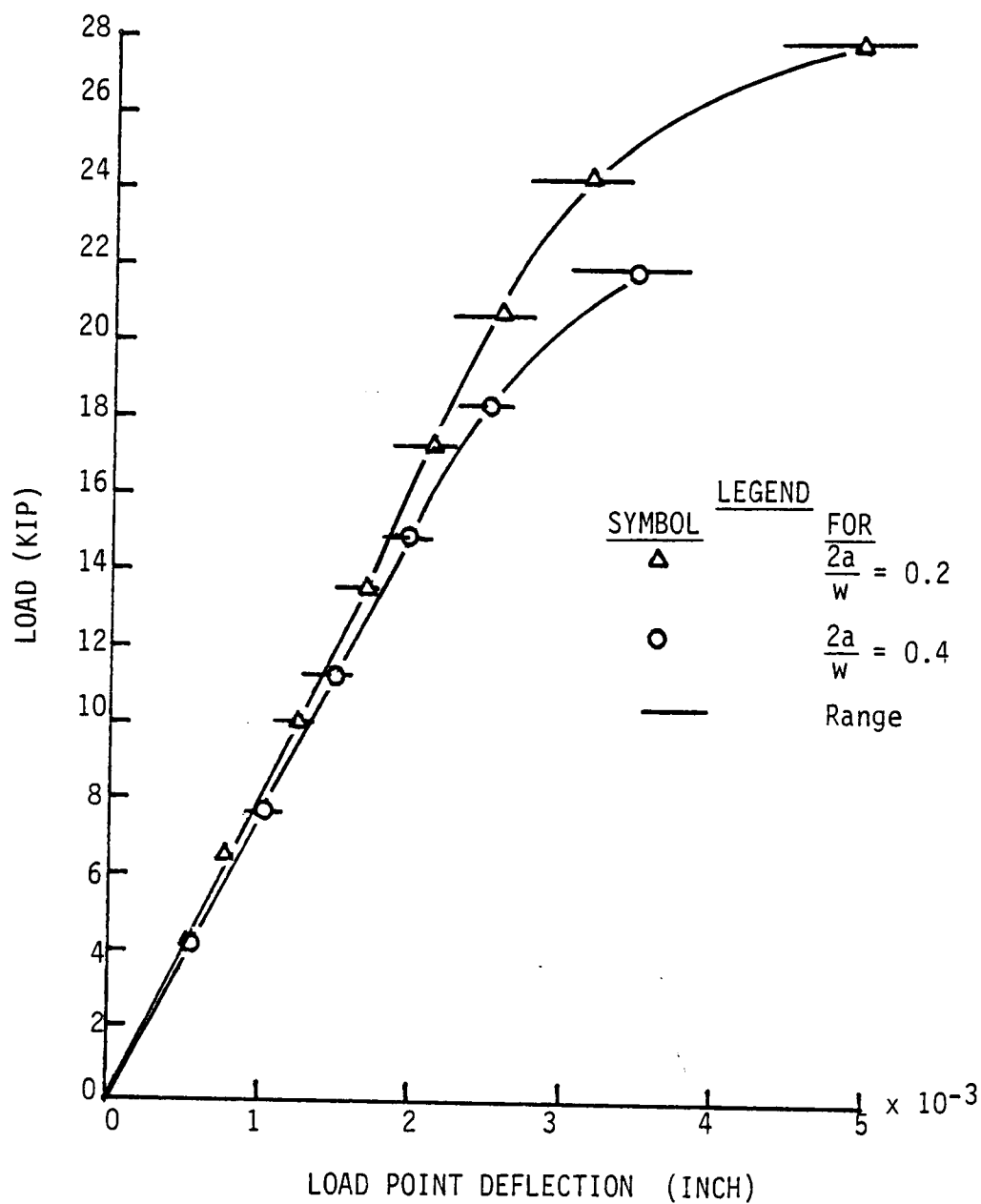


Figure 31. Variation of load point deflection with load for unidirectional B-Al composite for $\theta = 60^\circ$.

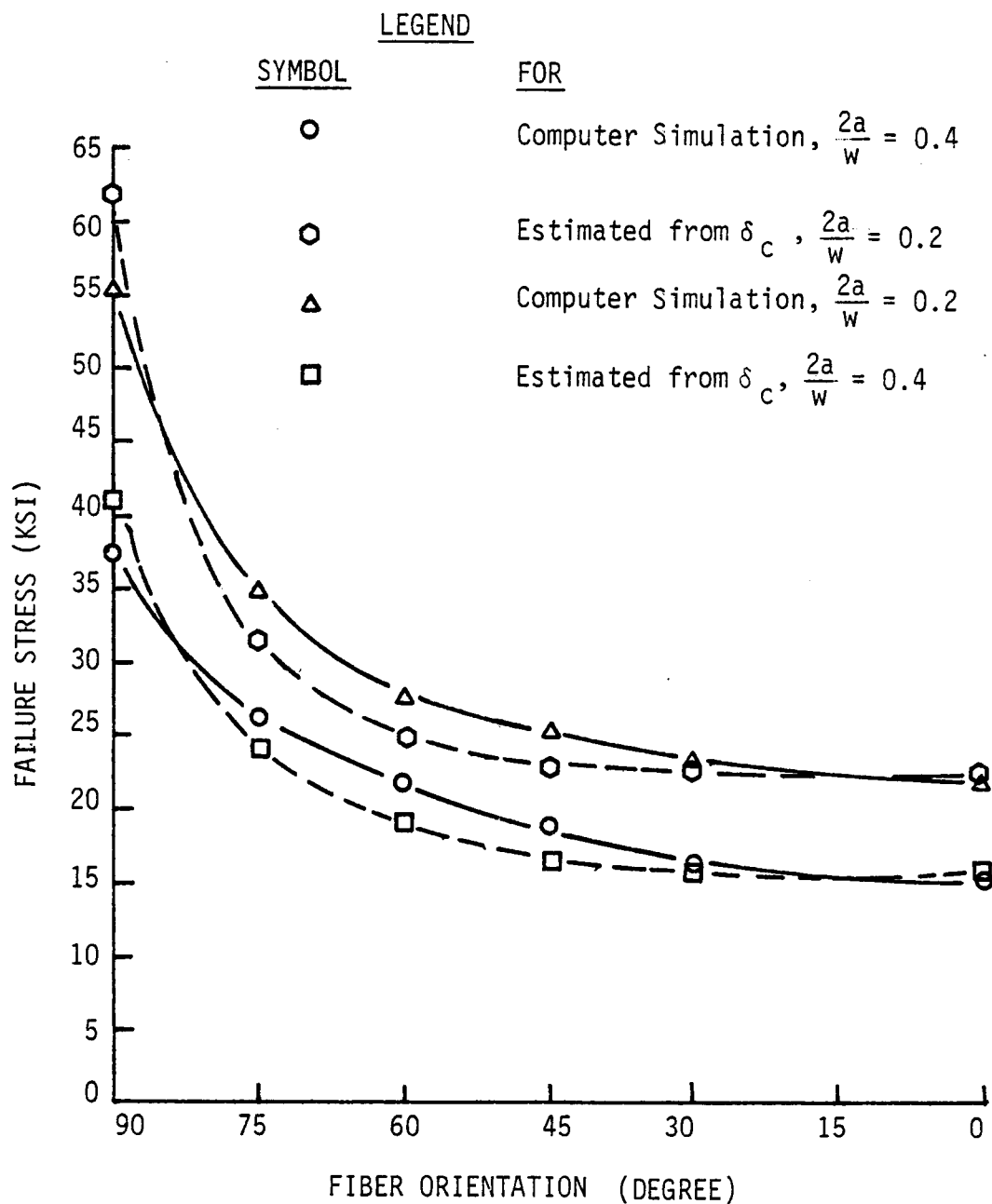


Figure 32. Effect of fiber orientation on the failure stress of unidirectional B-Al composite.

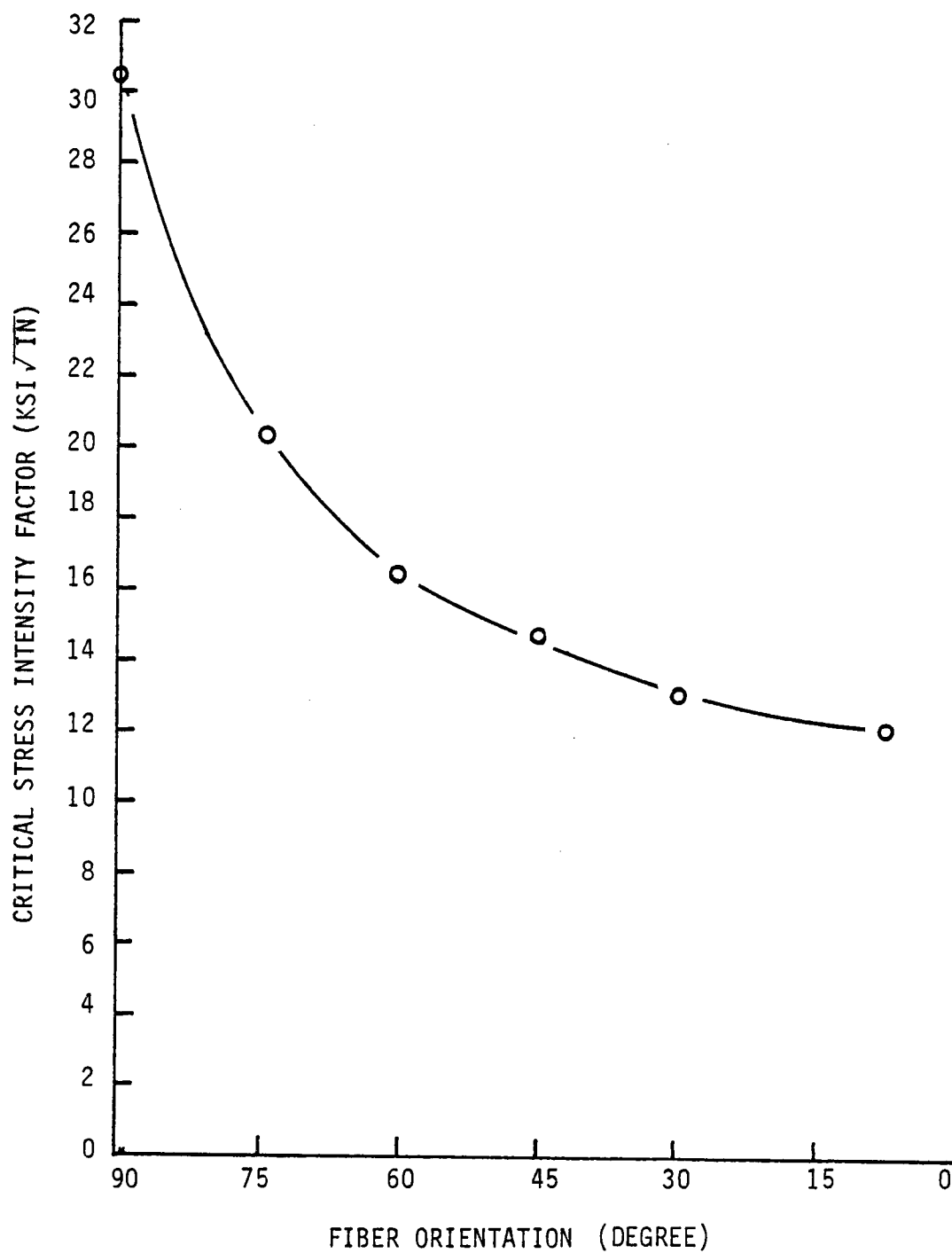


Figure 33. Effect of fiber orientation on the critical stress intensity factor of unidirectional B-Al composite.

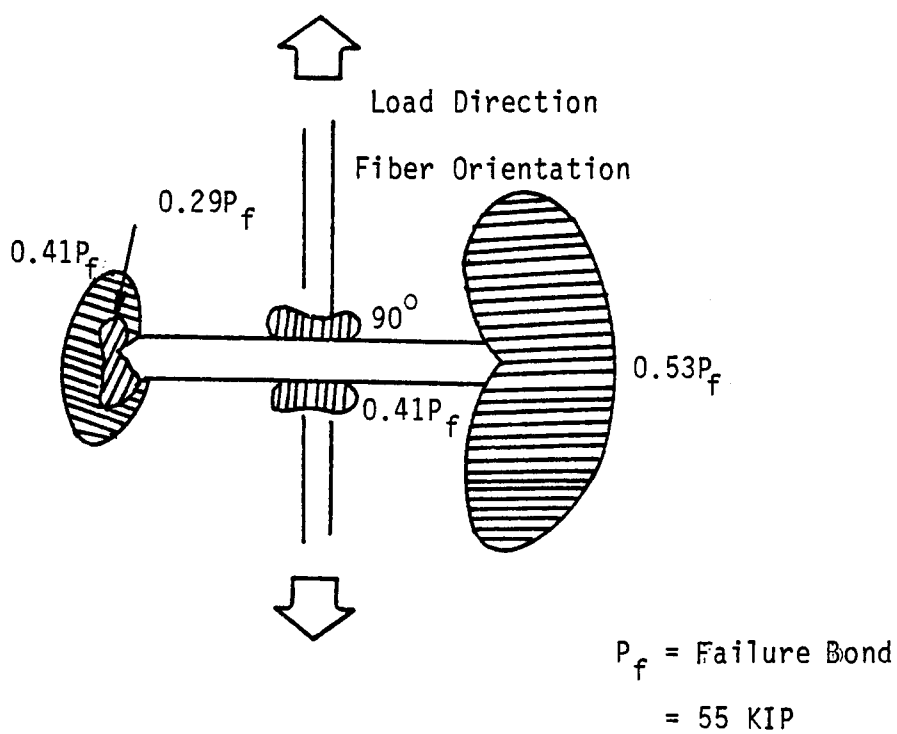
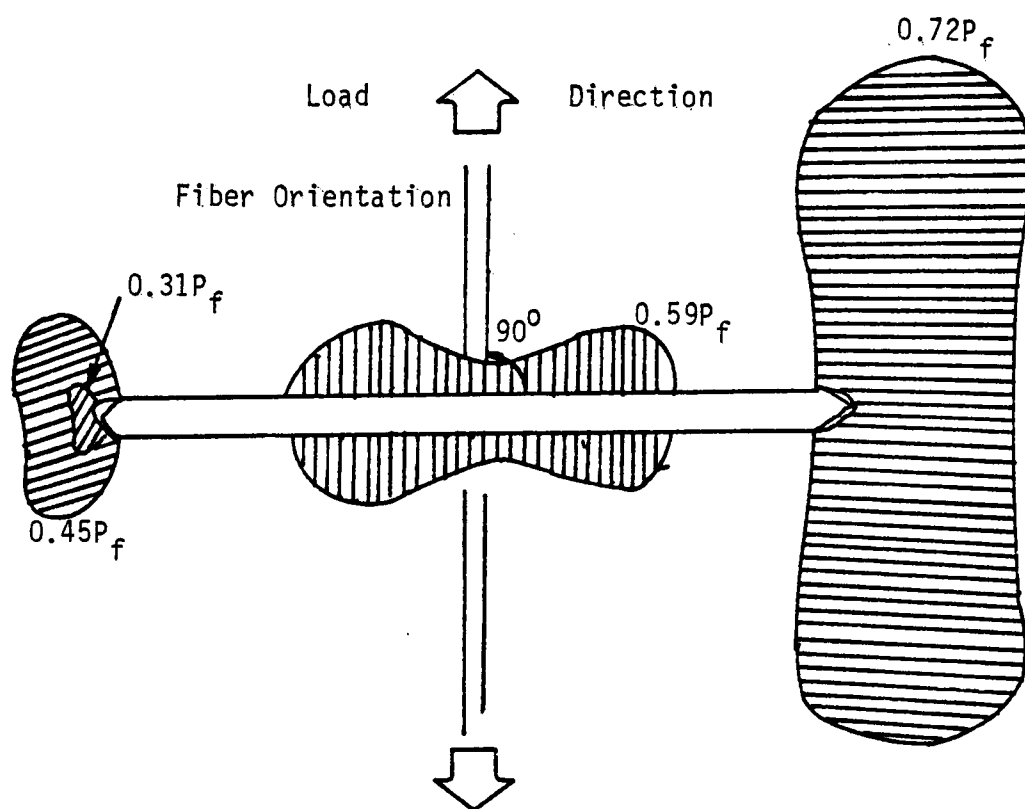
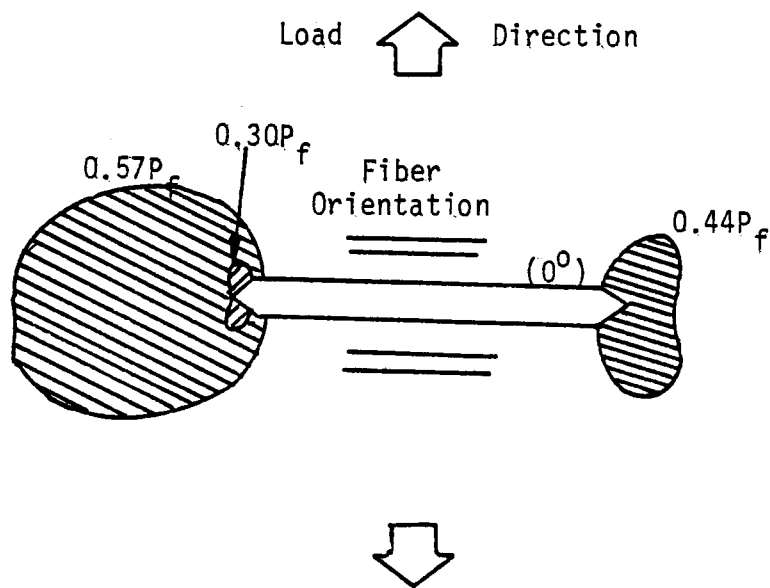


Figure 34. Schematic diagram of plastic zone and its growth in unidirectional B-Al composite for $2a/w = 0.2$ and $\theta = 90^\circ$.



P_f = Failure Load = 37.5 KIP

Figure 35. Schematic diagram of plastic zone and its growth in unidirectional B-Al composite for $2a/w = 0.4$ and $\theta = 90^\circ$.



P_f = Failure Load = 22 KIP

Figure 36. Schematic diagram of plastic zone and its growth in unidirectional B-Al composite for $2a/w = 0.2$ and $\theta = 0^\circ$.

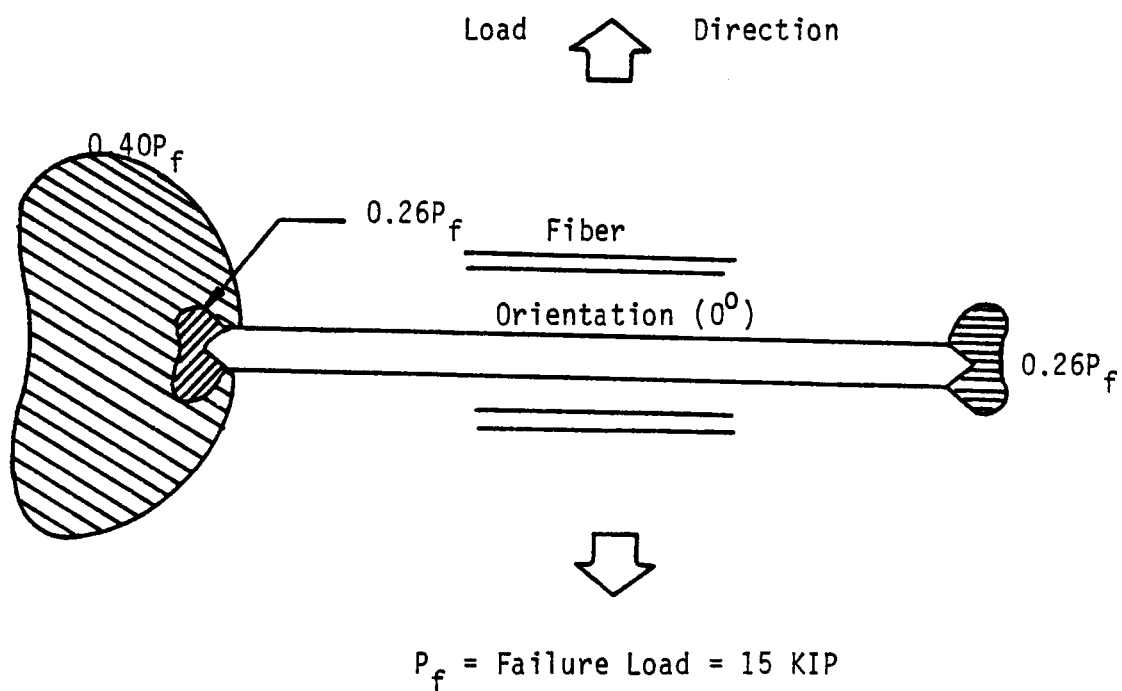


Figure 37. Schematic diagram of plastic zone and its growth in unidirectional B-Al composite for $2a/w = 0.4$ and $\theta = 0^\circ$.

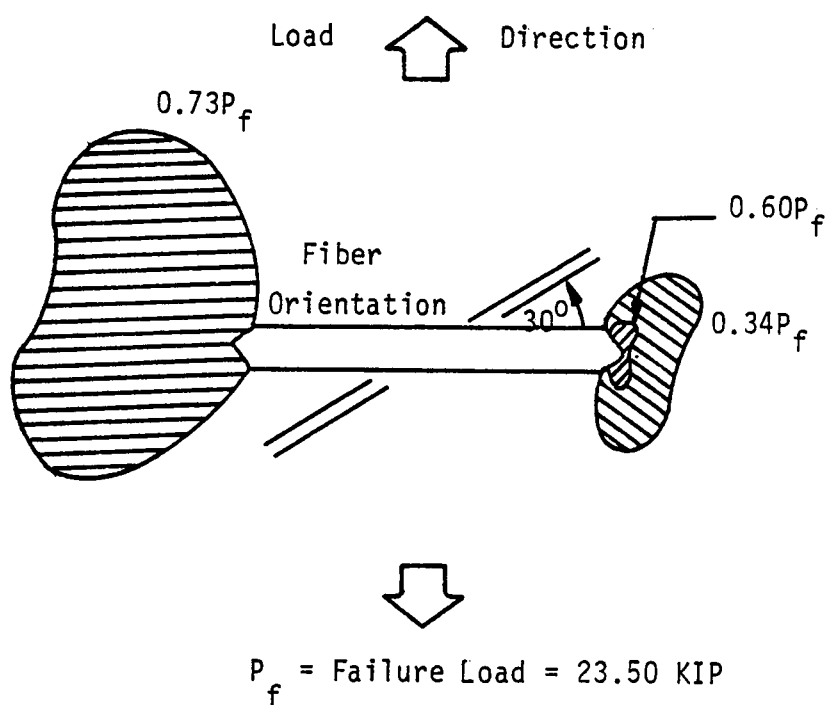


Figure 38. Schematic diagram of plastic zone and its growth in unidirectional B-Al composite for $2a/w = 0.2$ and $\theta = 30^\circ$.

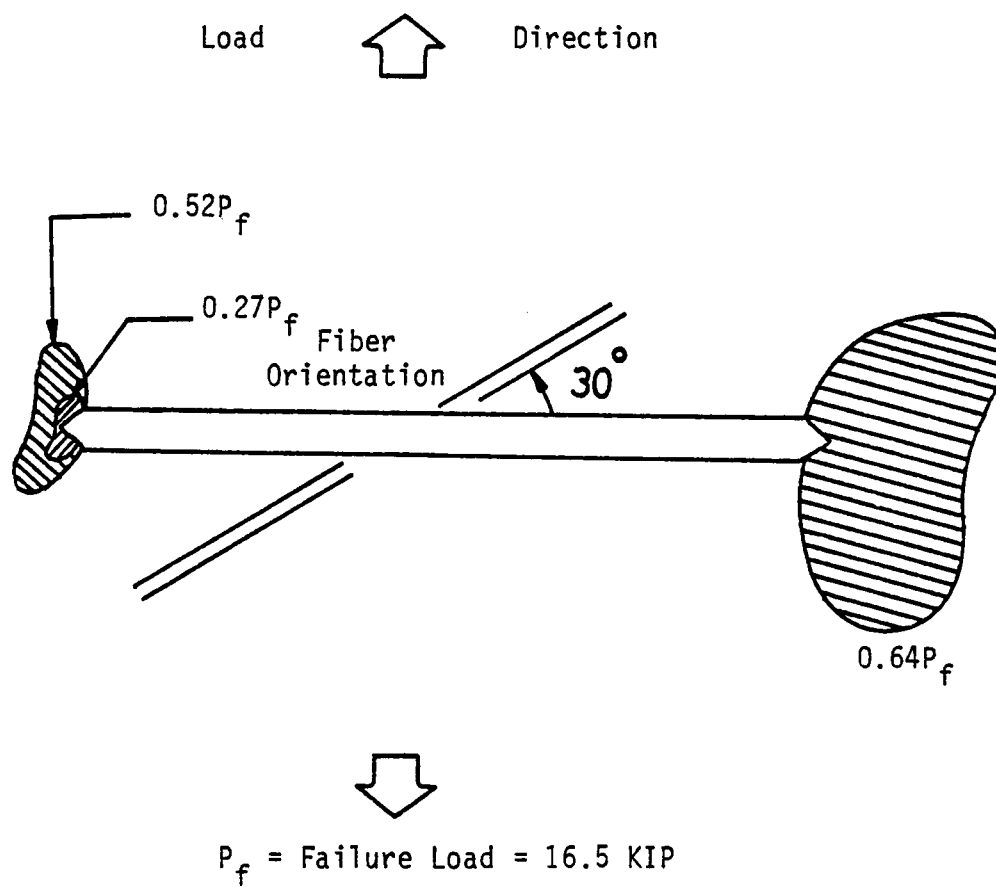


Figure 39. Schematic diagram of plastic zone and its growth in unidirectional B-Al composite for $2a/w = 0.4$ and $\theta = 30^\circ$.

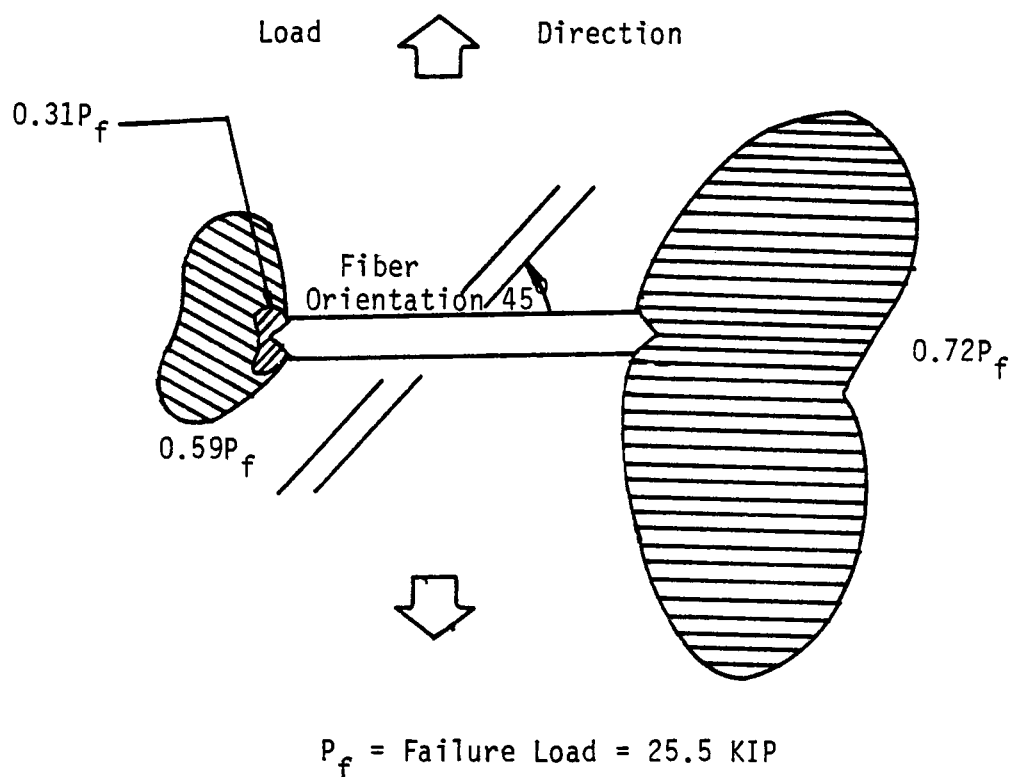


Figure 40. Schematic diagram of plastic zone and its growth in unidirectional B-Al composite for $2a/w = 0.2$ and $\theta = 45^\circ$.

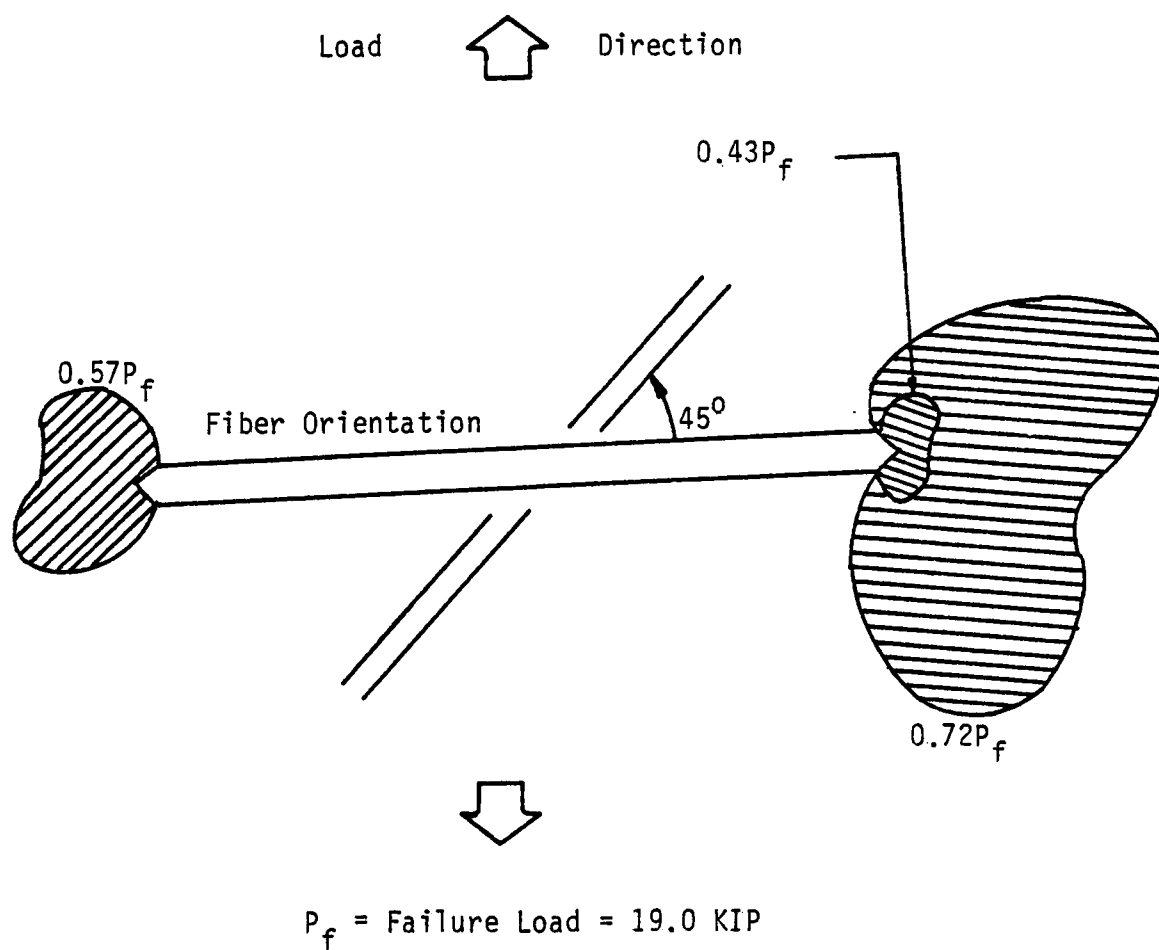


Figure 41. Schematic diagram of plastic zone and its growth in unidirectional B-Al composite for $2a/w = 0.4$ and $\theta = 45^\circ$.

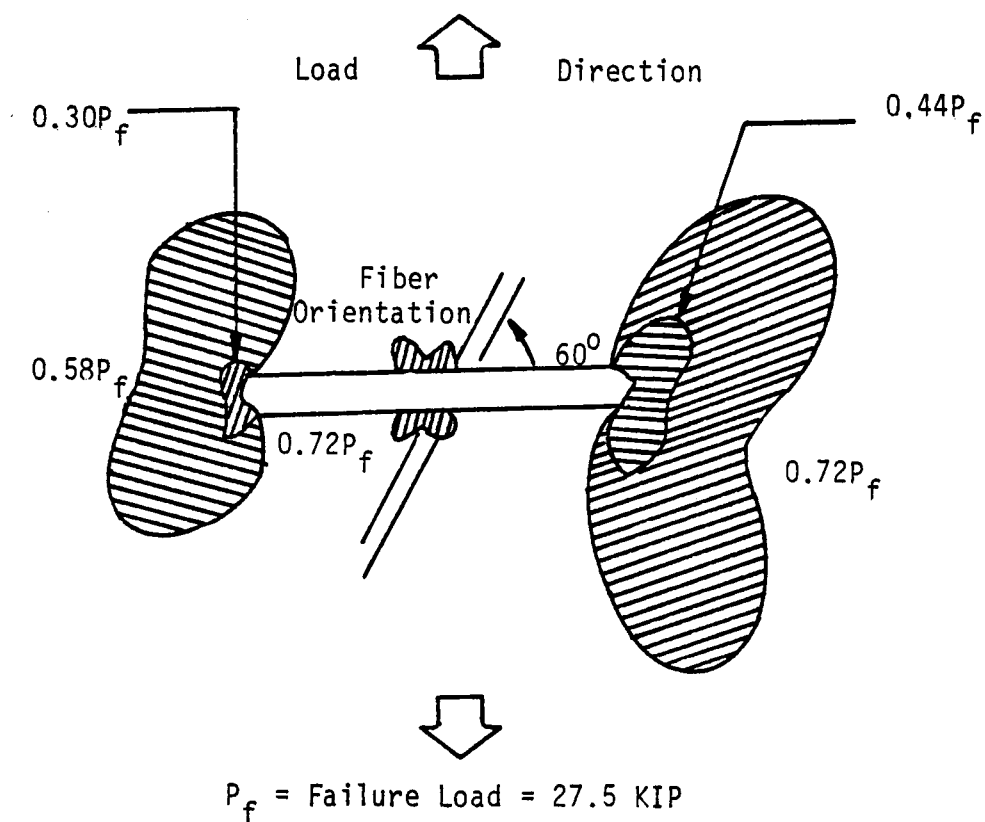


Figure 42. Schematic diagram of plastic zone and its growth in unidirectional B-Al composite for $2a/w = 0.2$ and $\theta = 60^\circ$.

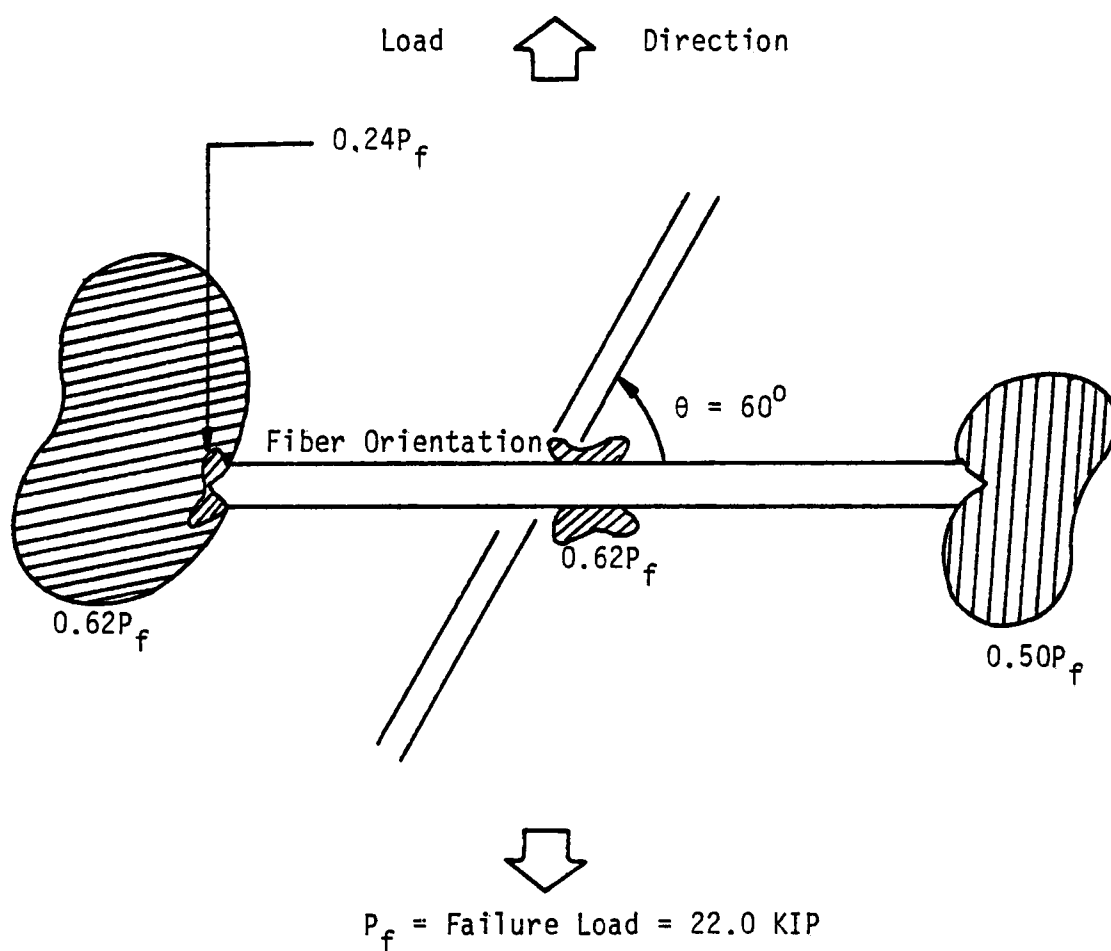


Figure 43. Schematic diagram of plastic zone and its growth in unidirectional B-Al composite for $2a/w = 0.4$ and $\theta = 60^\circ$.

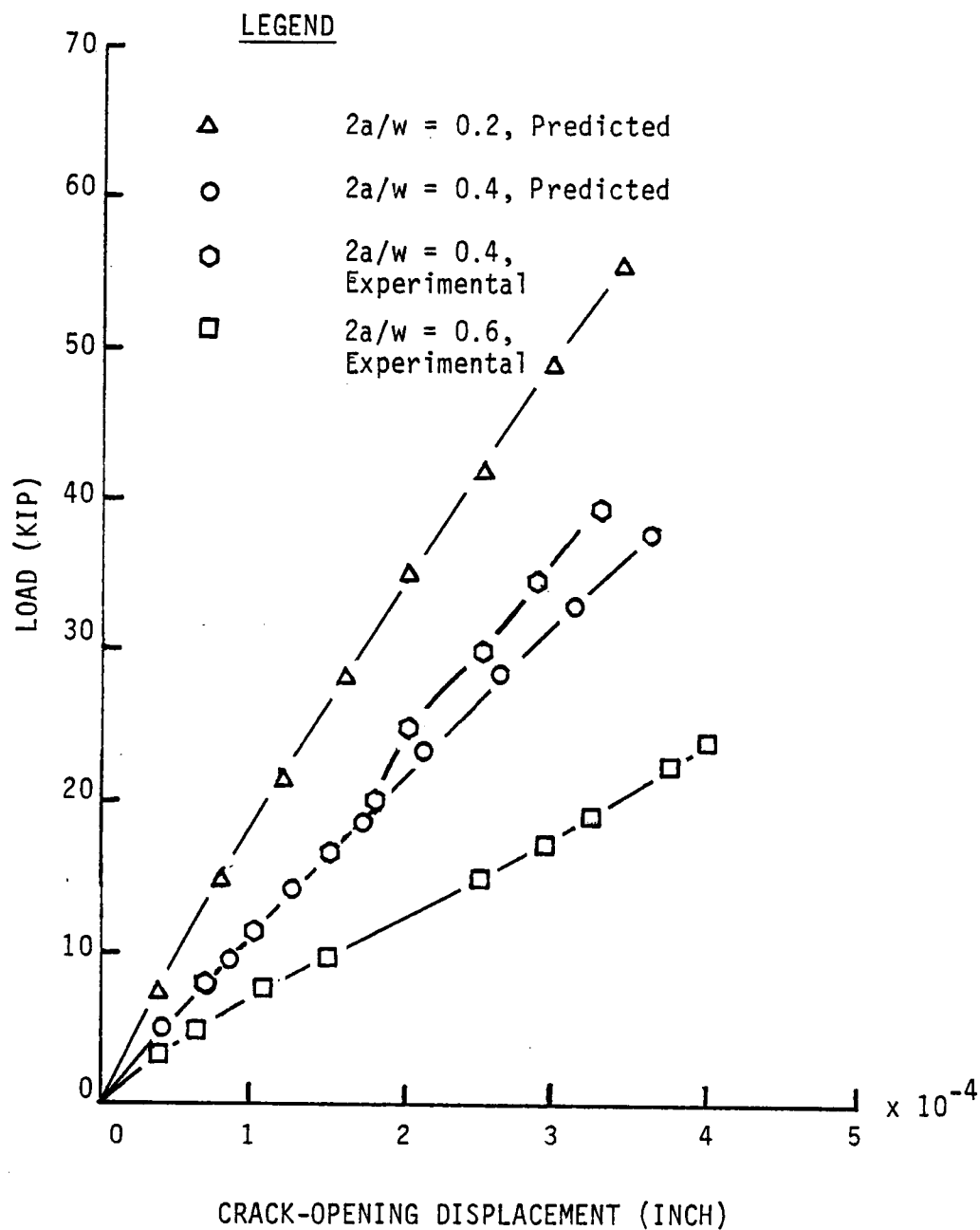


Figure 44. Variation of crack-opening displacement with load for unidirectional B-Al composite with $\theta = 90^\circ$.

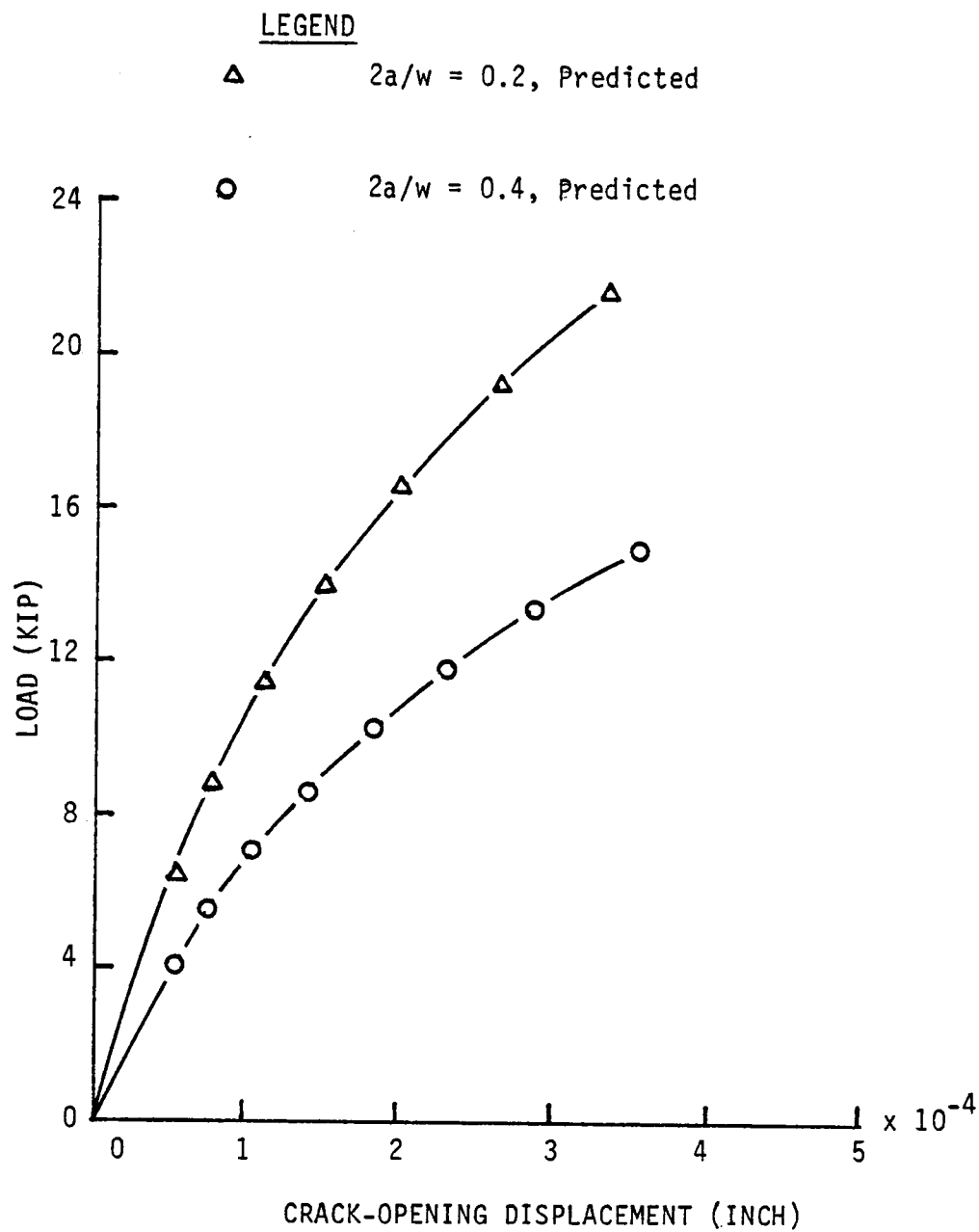


Figure 45. Variation of crack-opening displacement with load for unidirectional B-Al composite with $\theta = 0^\circ$.

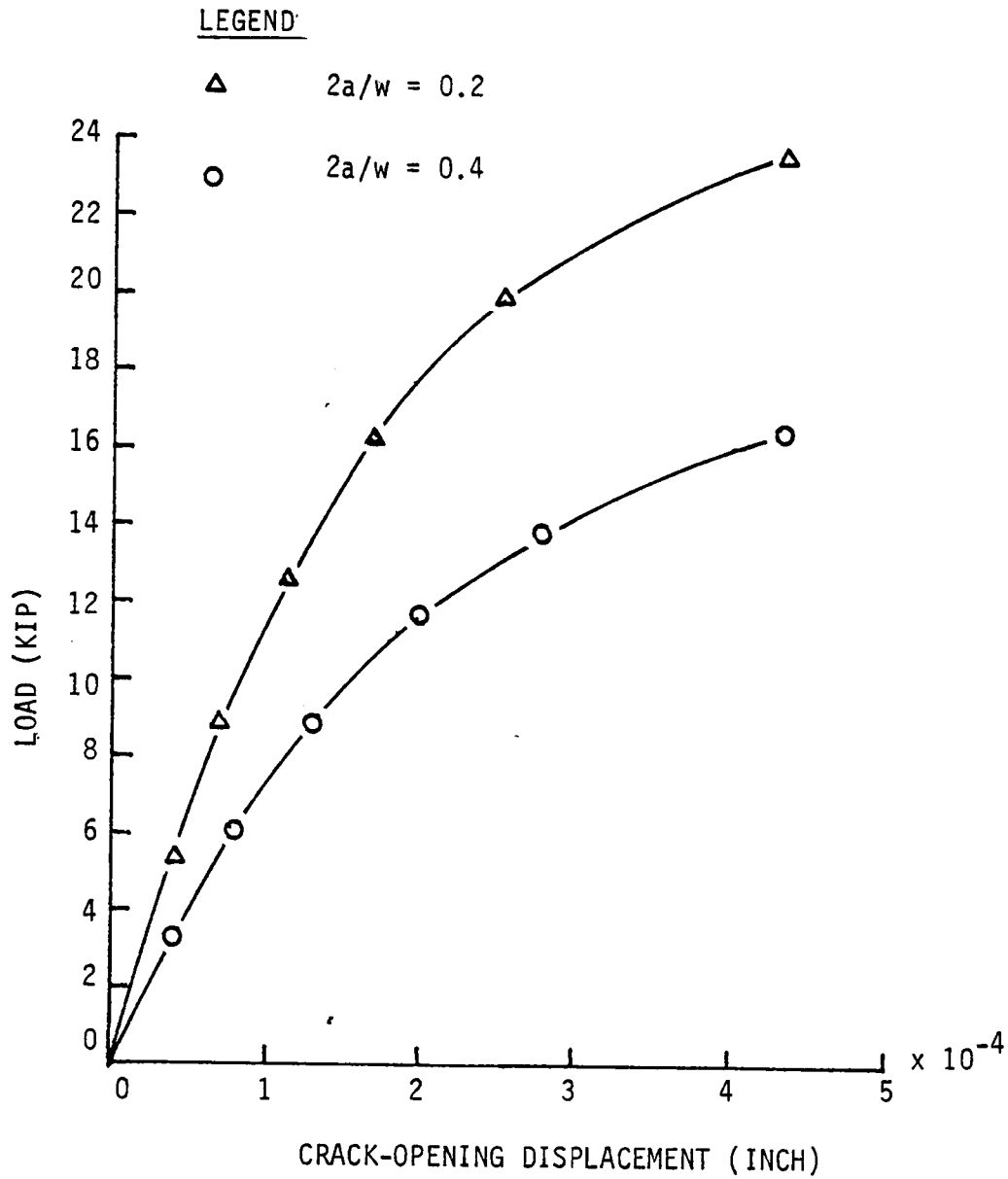


Figure 46. Variation of crack-opening displacement with load for unidirectional B-Al composite with $\theta = 30^\circ$.

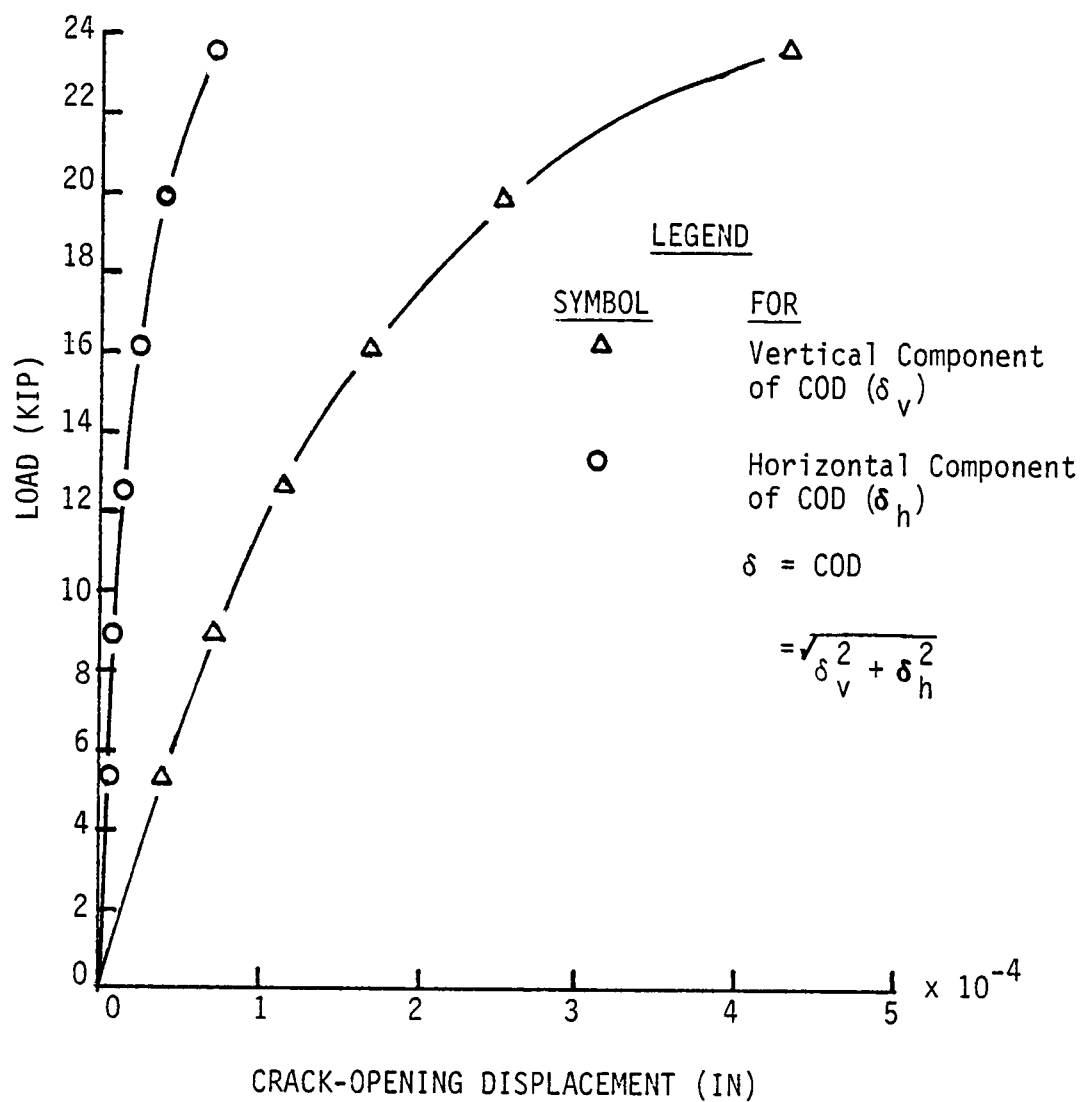


Figure 47. Variation in the components of crack-opening displacement for unidirectional B-A1 composite for $a/w = 0.2$ and $\theta = 30^\circ$.

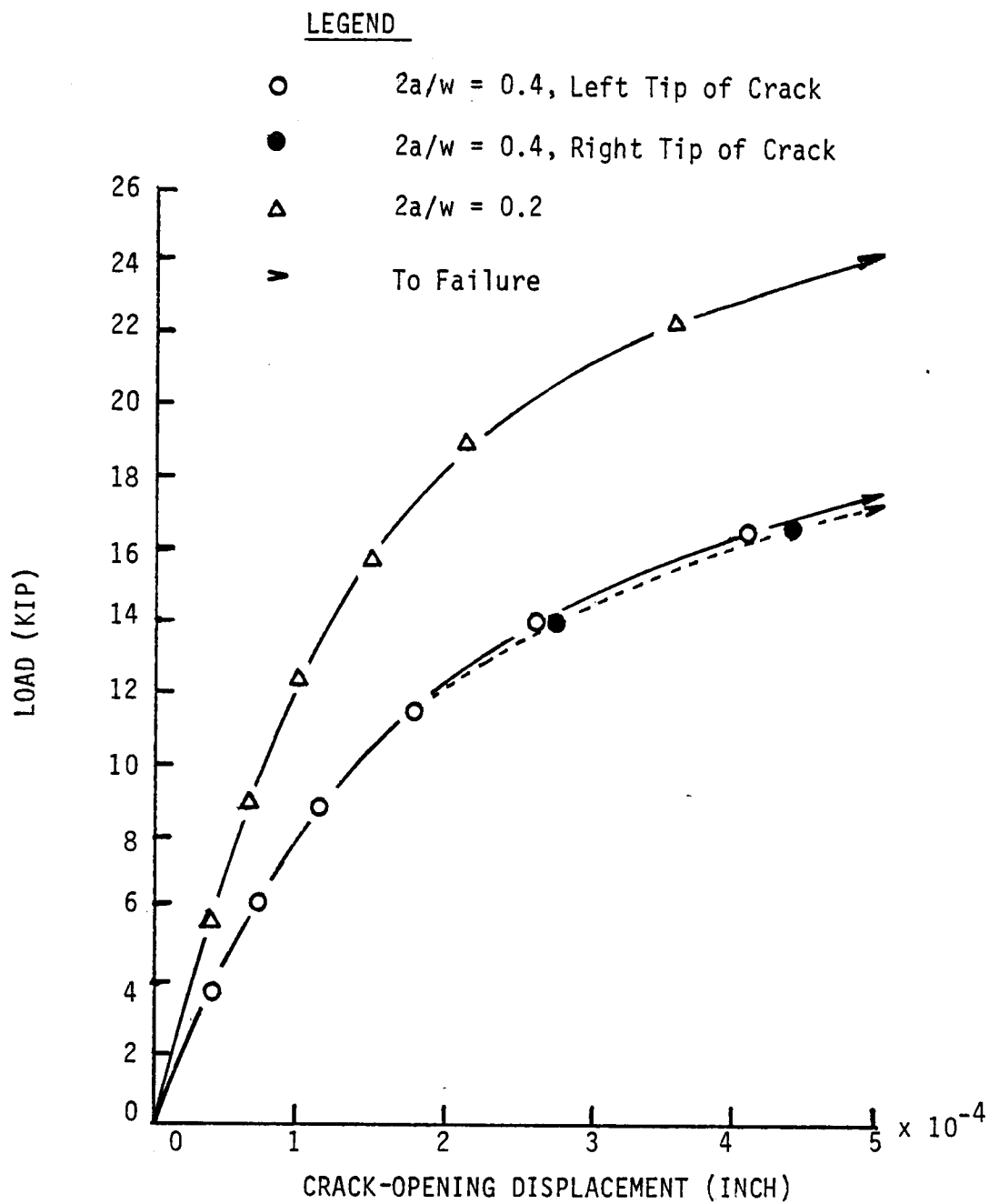


Figure 48. Variation of crack-opening displacement with load for unidirectional B-Al composite with $\theta = 45^\circ$.

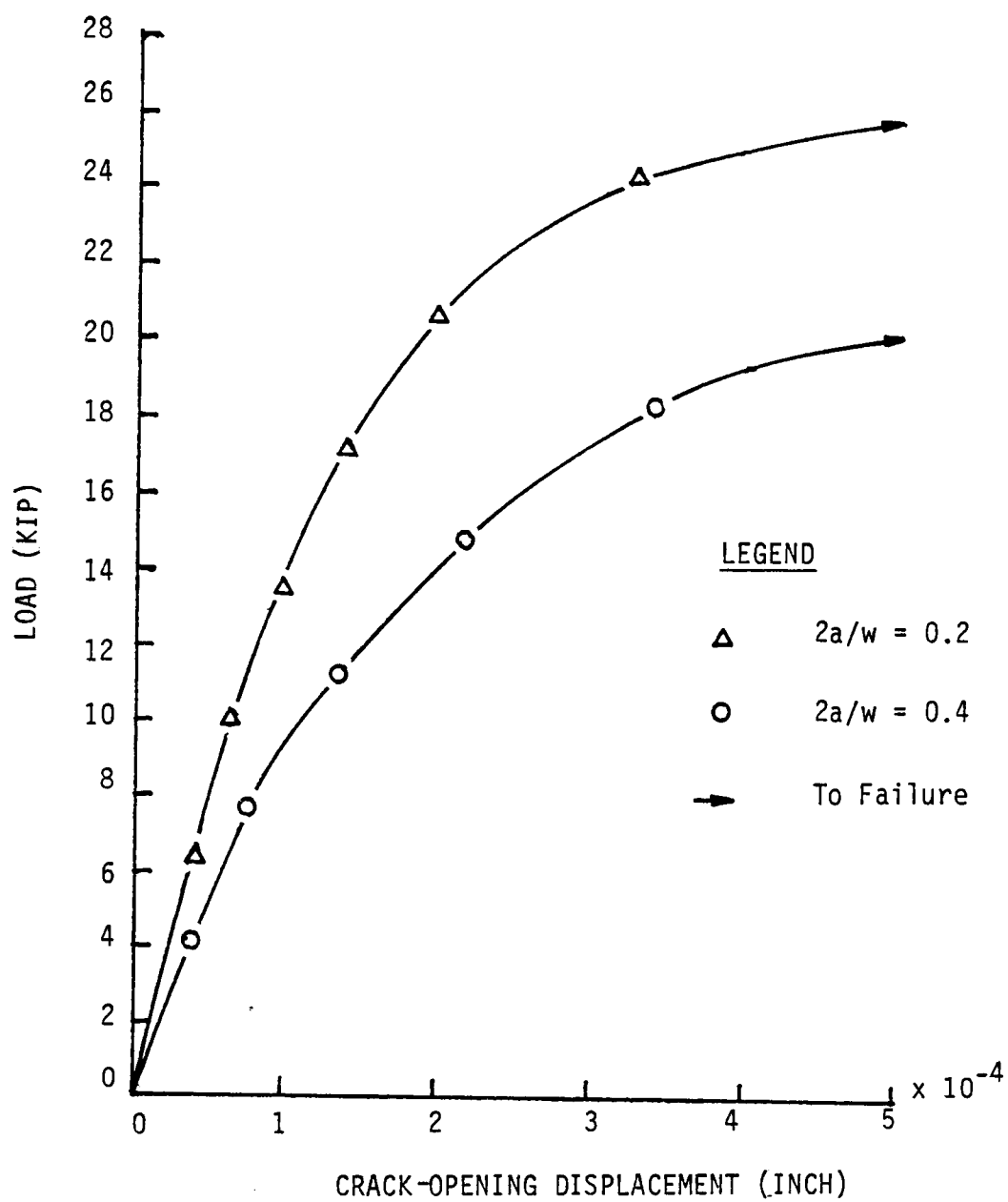


Figure 49. Variation of crack-opening displacement with load for unidirectional B-Al composite with $\theta = 60^\circ$.

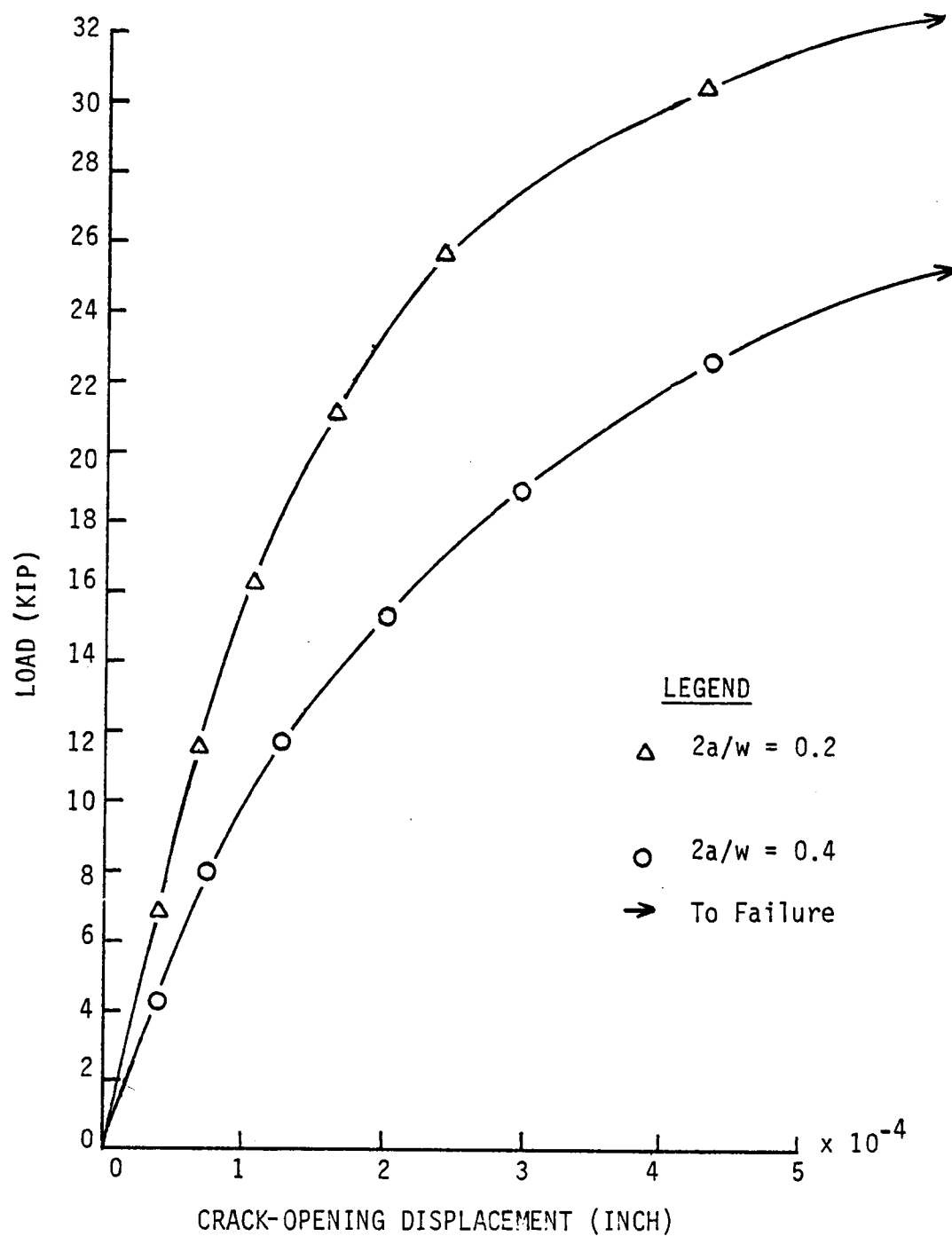


Figure 50. Variation of crack-opening displacement with load for unidirectional B-Al composite with $\theta = 75^\circ$.

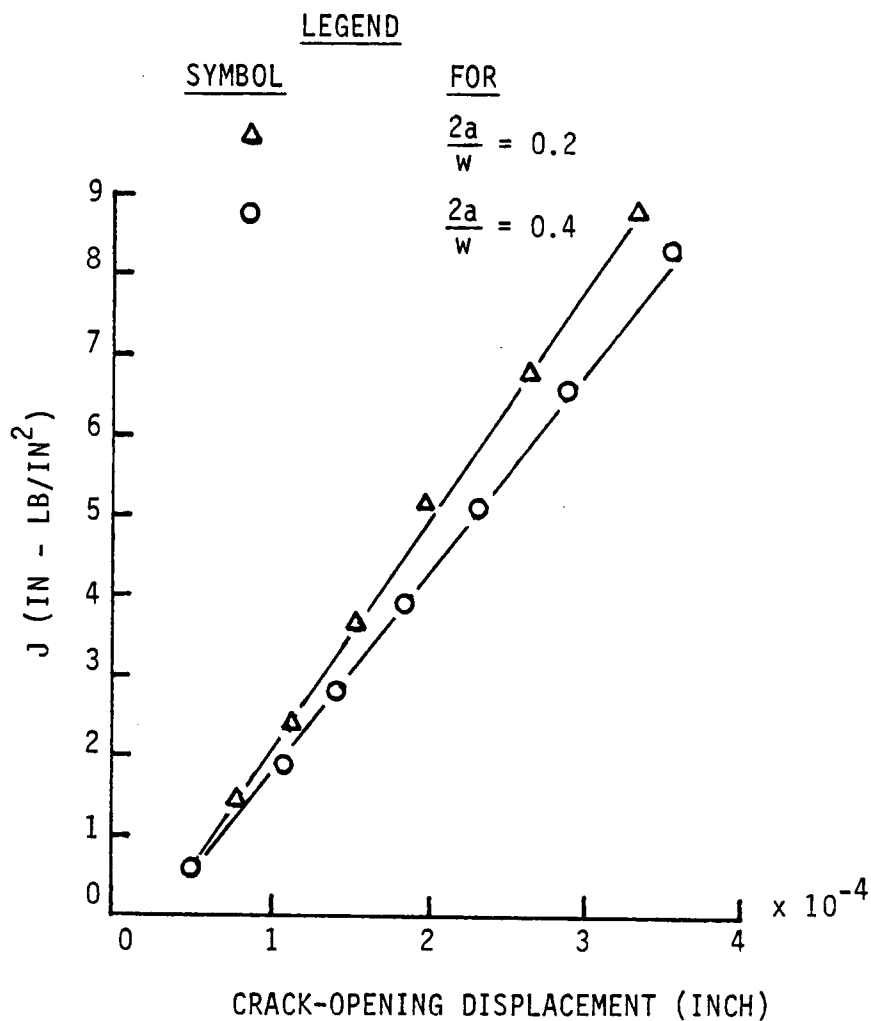


Figure 51. Variation of J-integral with crack-opening displacement for unidirectional B-Al composite with $\theta = 0^\circ$.

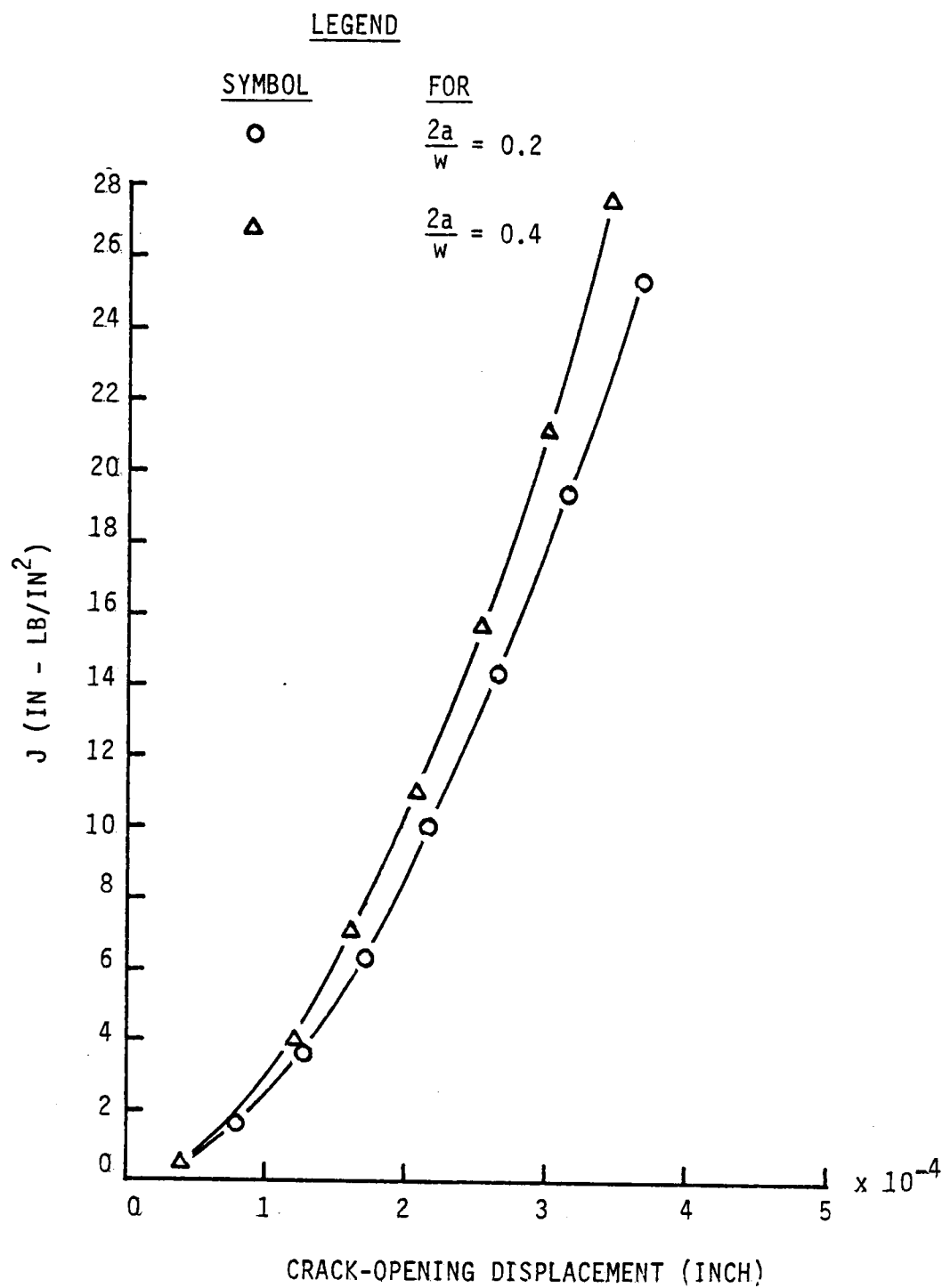


Figure 52. Variation of J-integral with crack-opening displacement for unidirectional B-Al composite with $\theta = 90^\circ$.

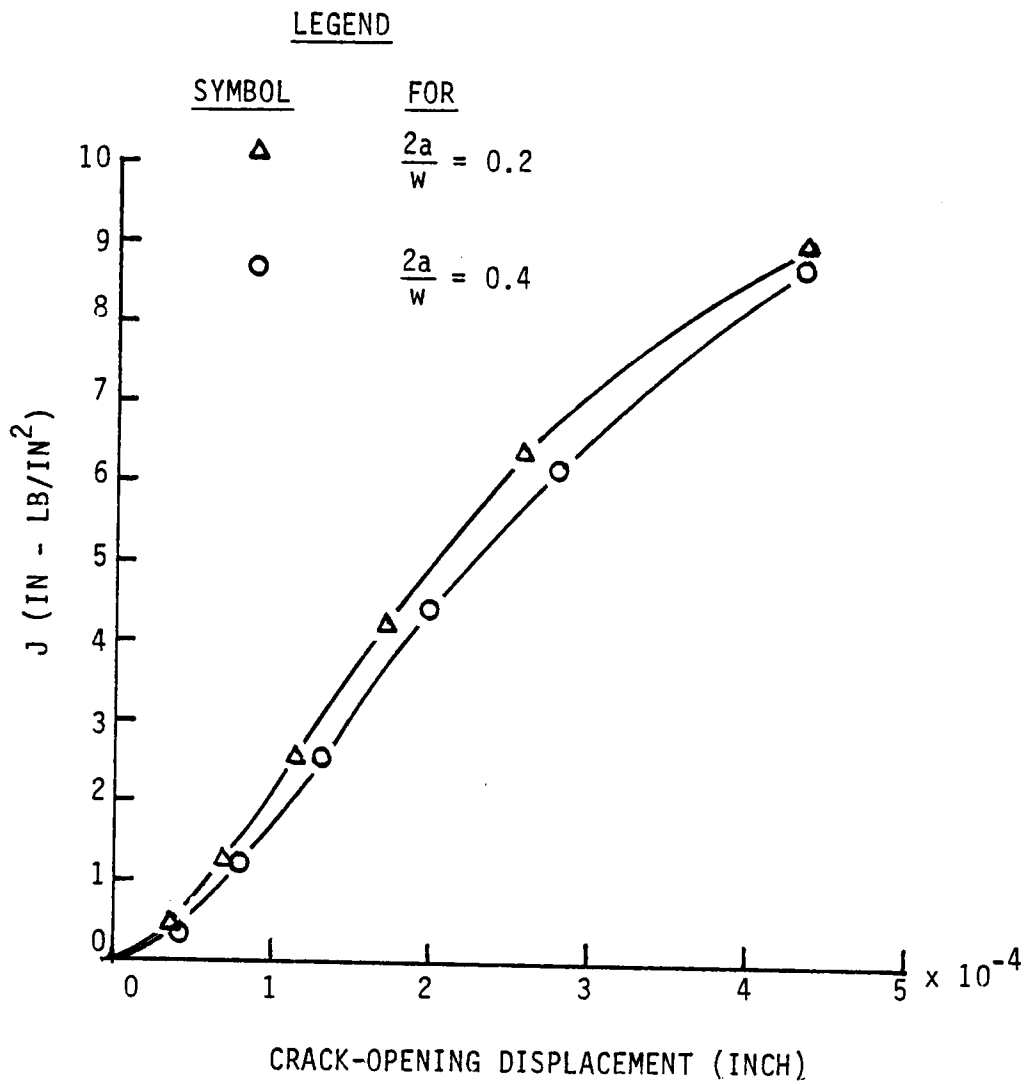


Figure 53. Variation of J-integral with crack-opening displacement for unidirectional B-Al composite with $\theta = 30^\circ$.

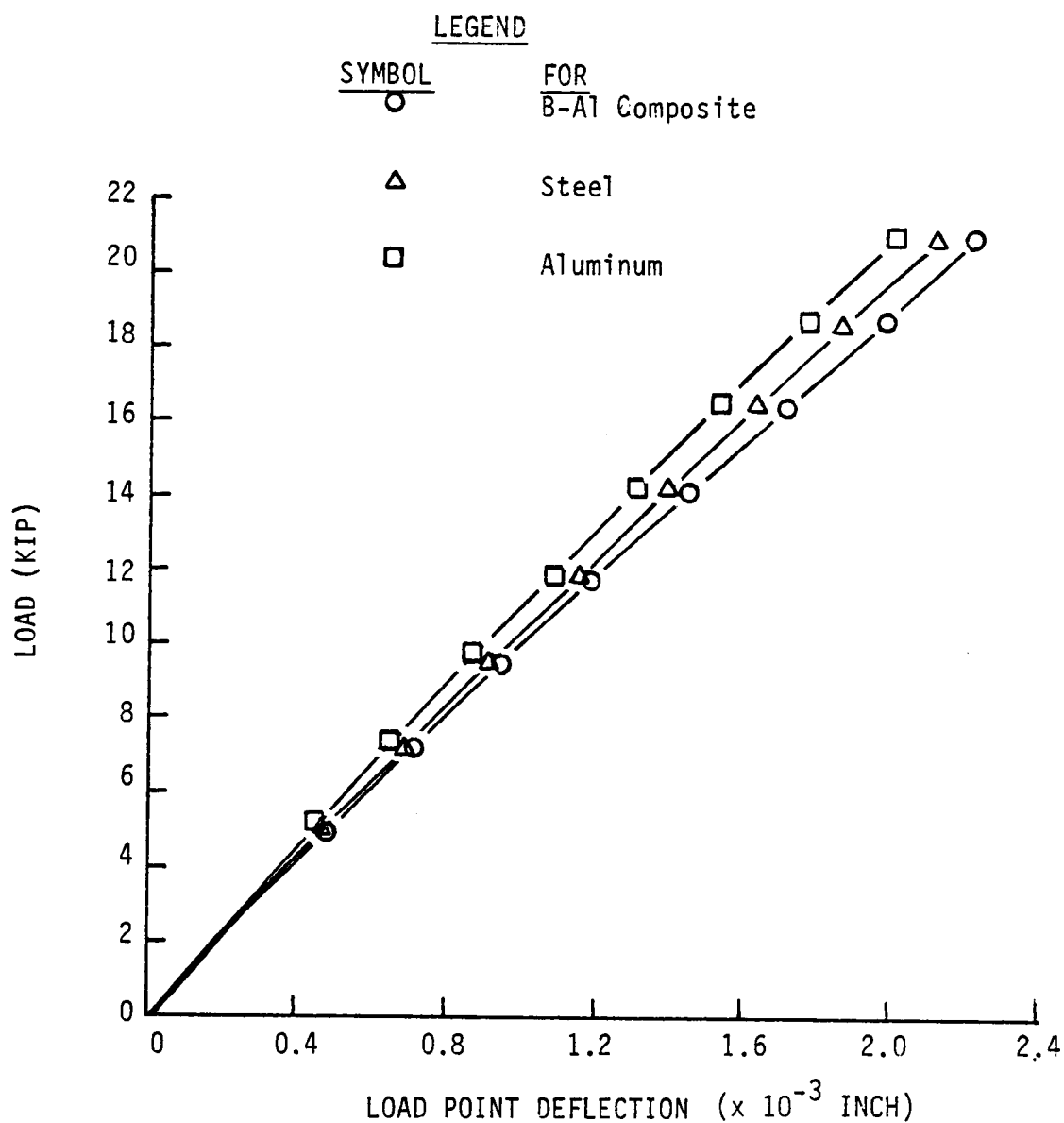


Figure 54. Effect of pin material on the load versus load point deflection curves for $0^{\circ}/90^{\circ}$ B-Al composite with $a/w = 0.4$.

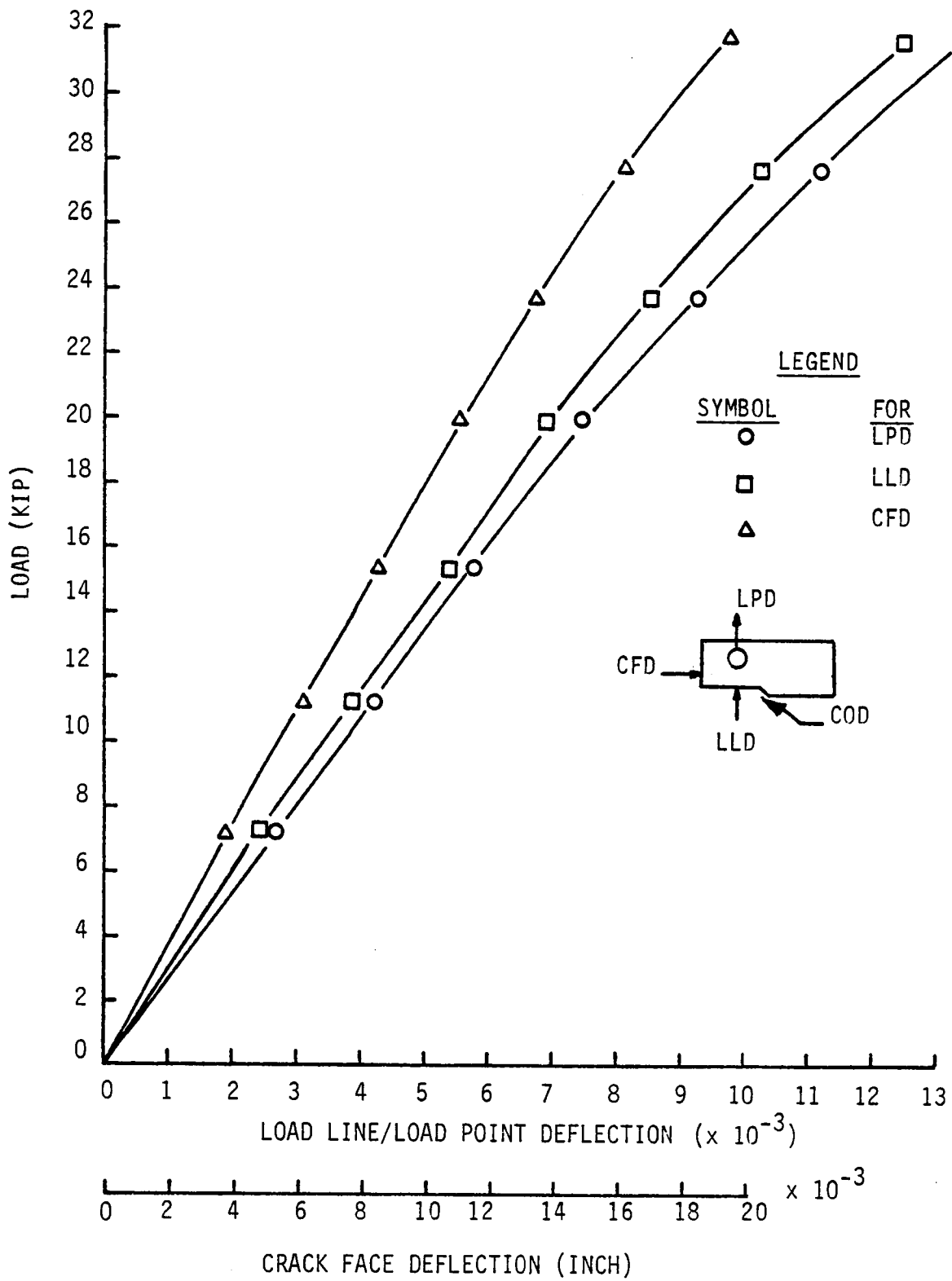


Figure 55. Variation of deflection with load for $0^\circ/90^\circ$ B-Al composite with $a/w = 0.2$.

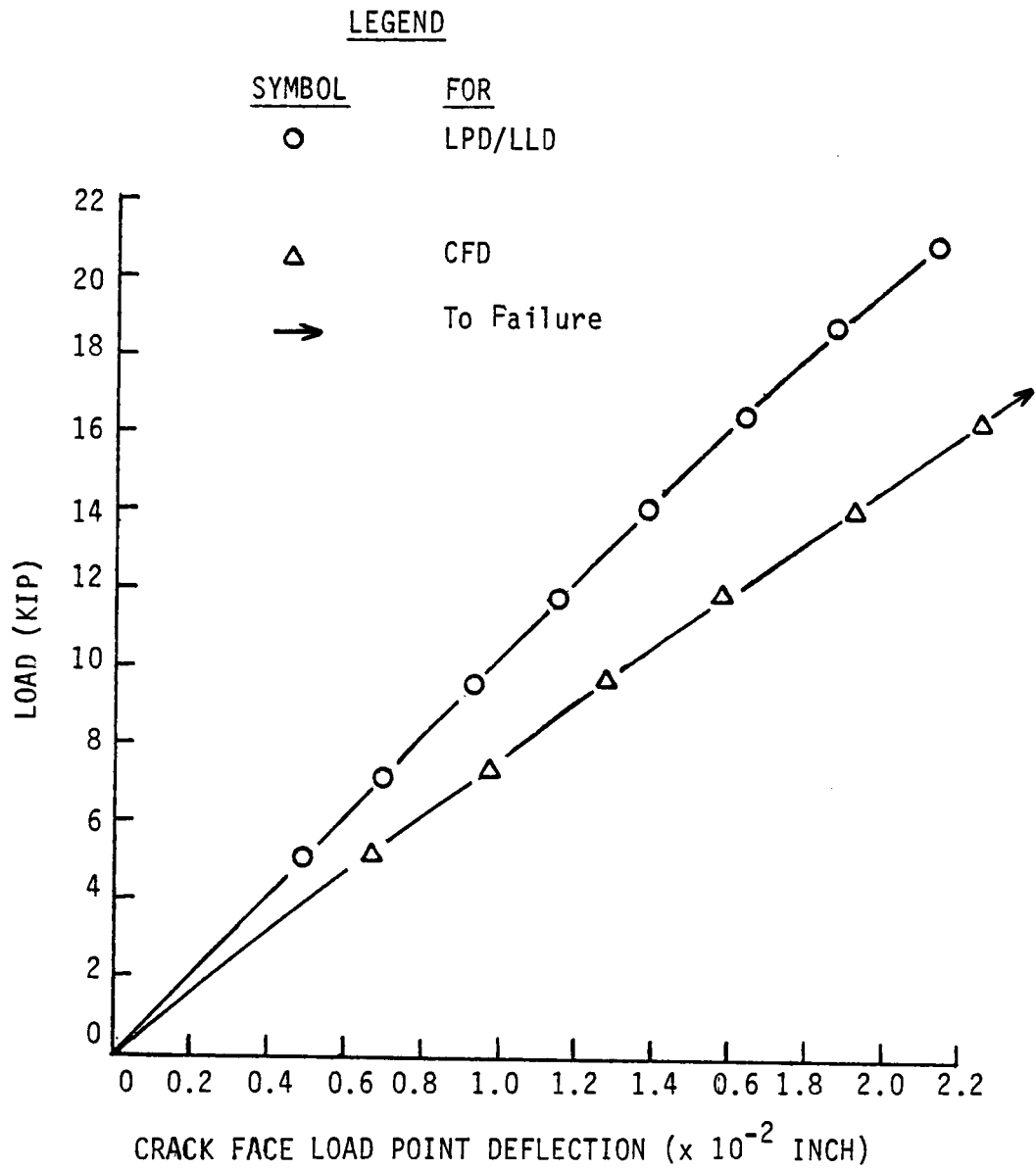


Figure 56. Variation of deflection with load for $0^\circ/90^\circ$ B-A1 composite with $a/w = 0.4$.

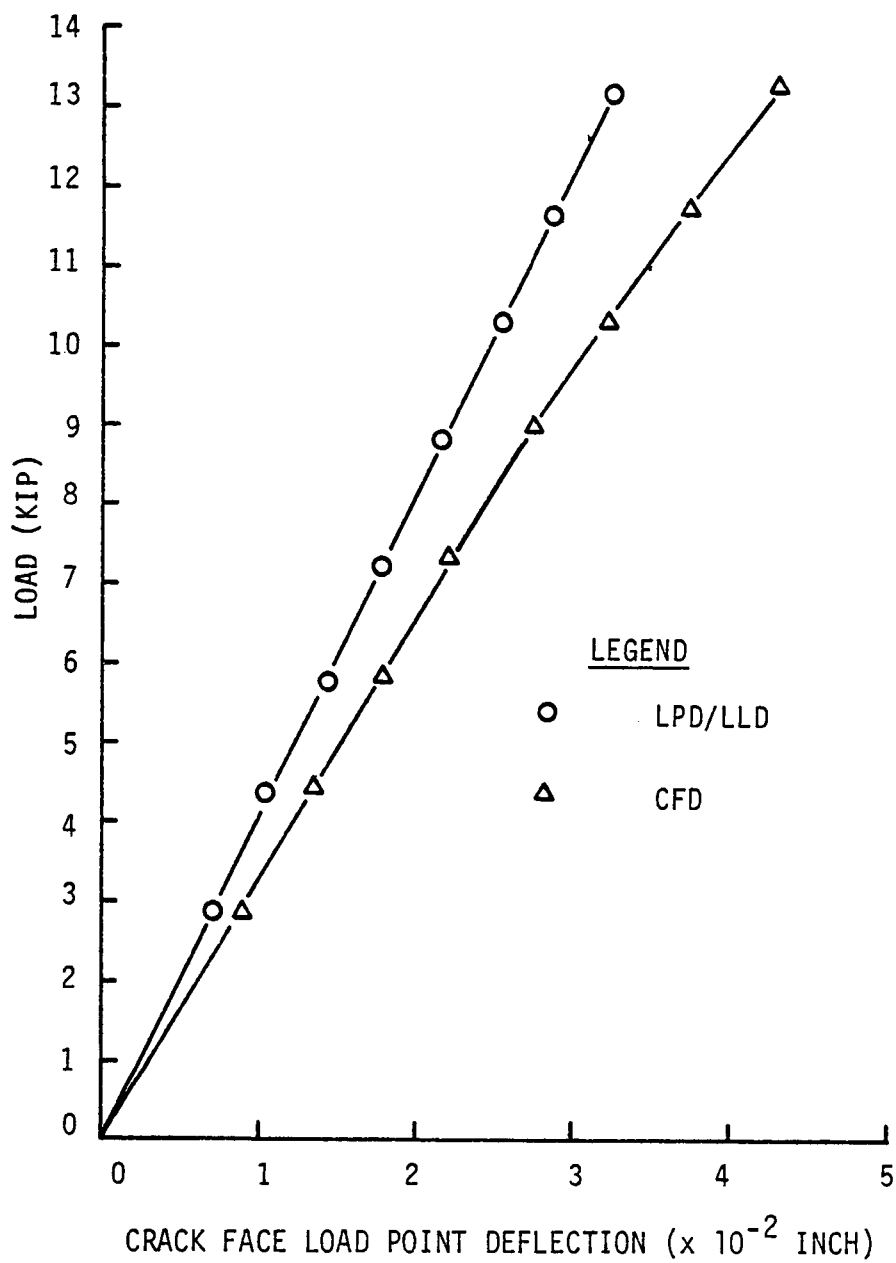


Figure 57. Variation of deflection with load for $0^\circ/90^\circ$ B-Al composite with $a/w = 0.6$.

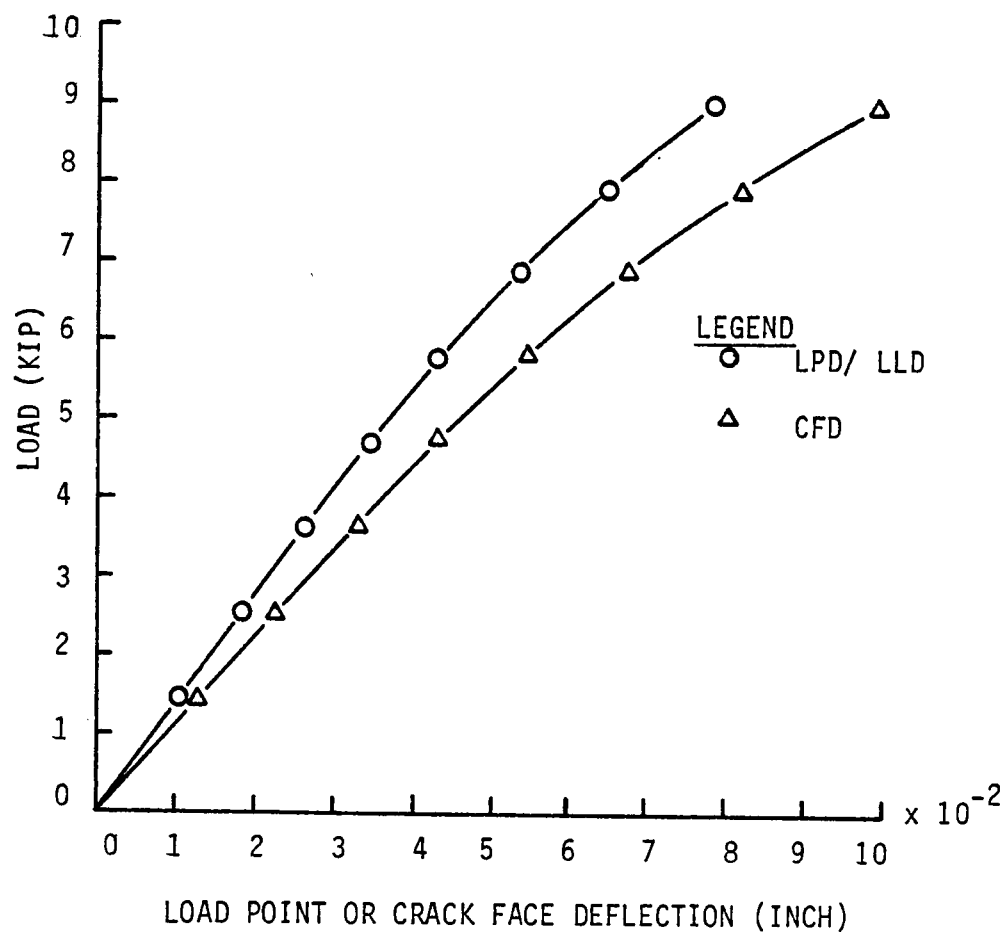


Figure 58. Variation of deflection with load for 0°/90° B-AI composite with $a/w = 0.8$.

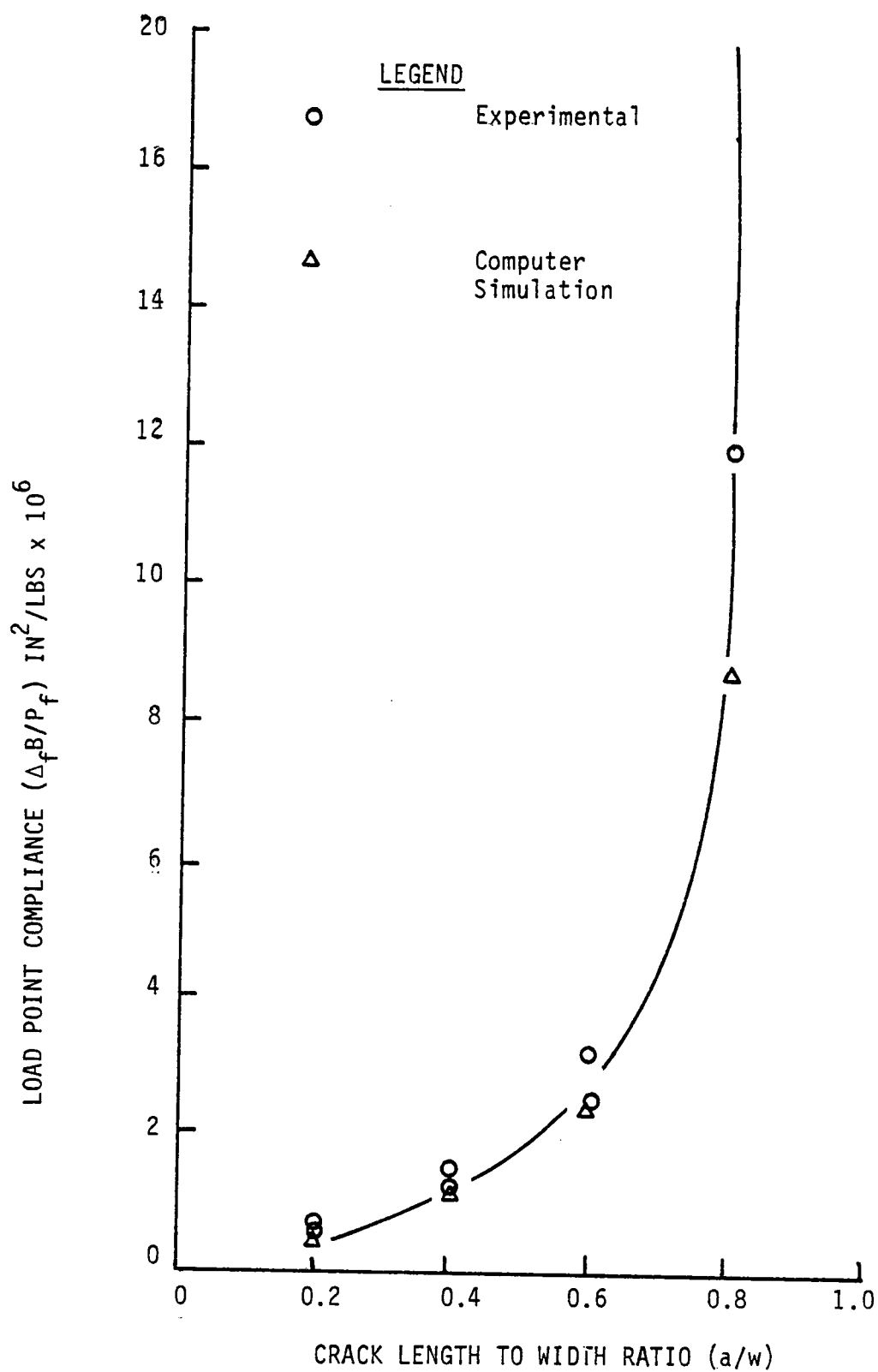


Figure 59. Variation of load point compliance with crack length for $0^\circ/90^\circ$ B-Al composite.

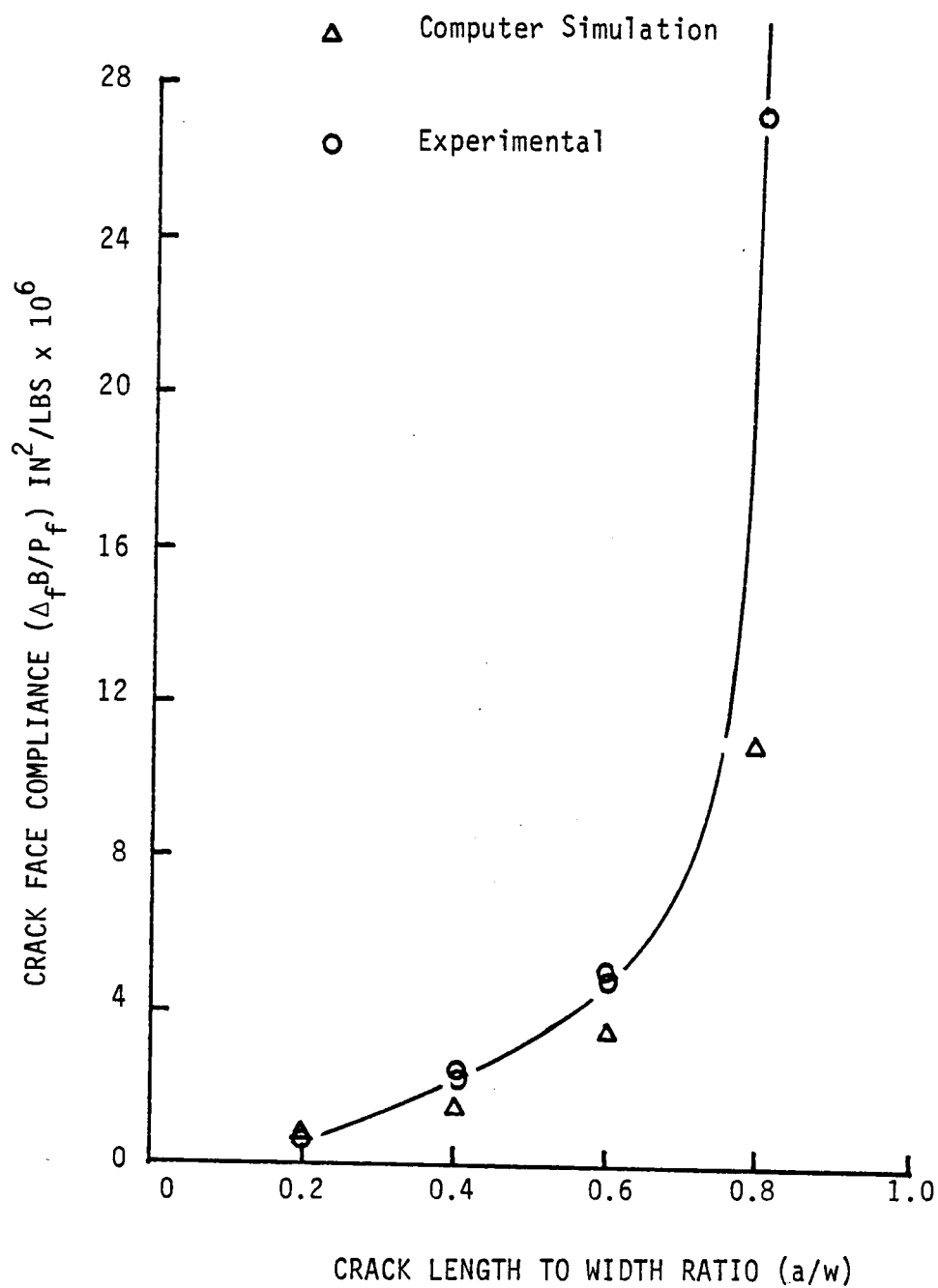
LEGEND

Figure 60. Variation of crack face compliance with crack length for 0°/90° B-Al composite.

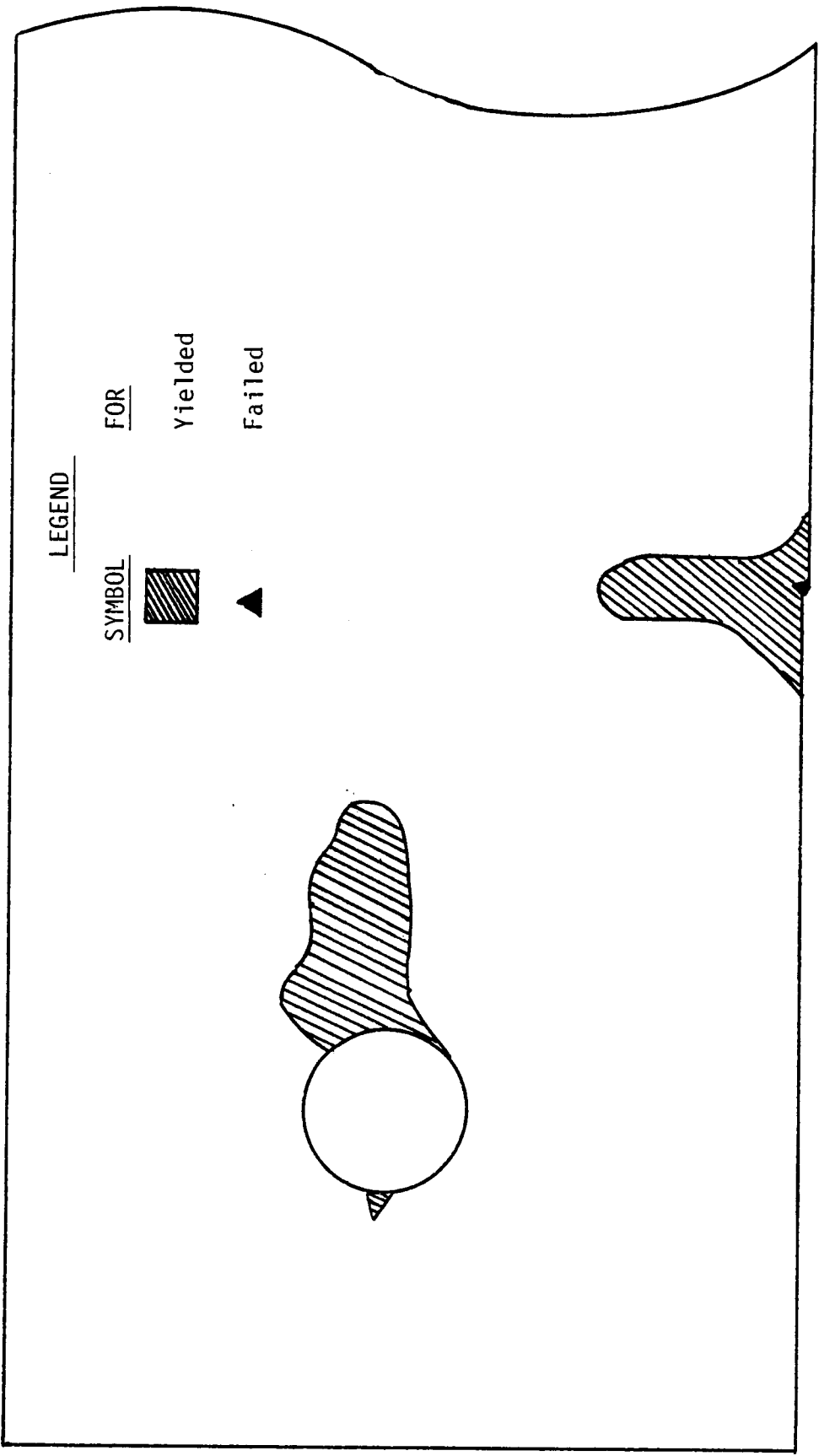


Figure 61. Typical yielded regions in a compact tension specimen of 0°/90° B-Al composite at failure load.

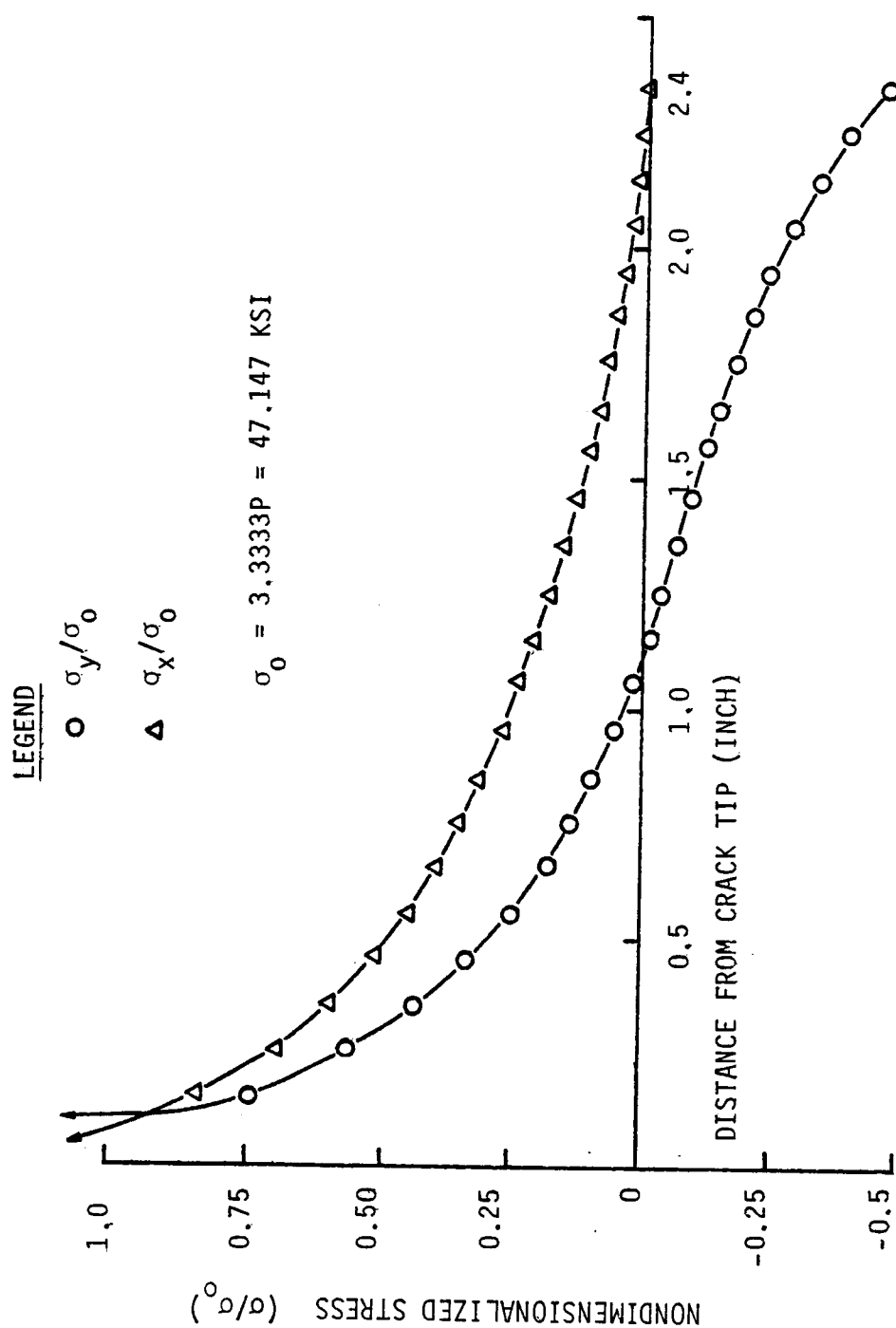


Figure 62. Variation of nondimensionalized normal stresses ahead of the crack tip ($y = 0.01575 \text{ in.}$) in a compact tension specimen of $0^\circ/90^\circ$ B-Al composite.

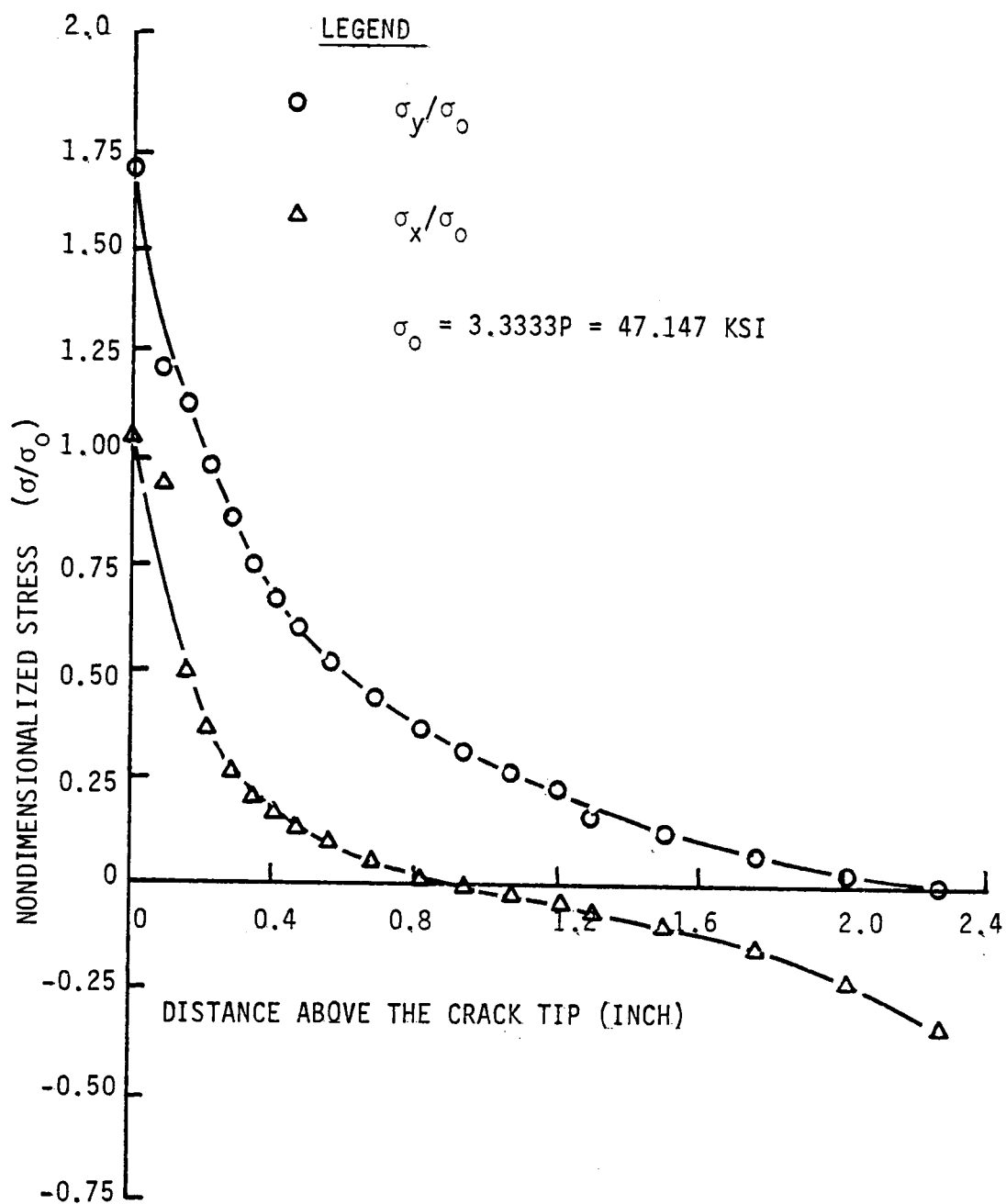
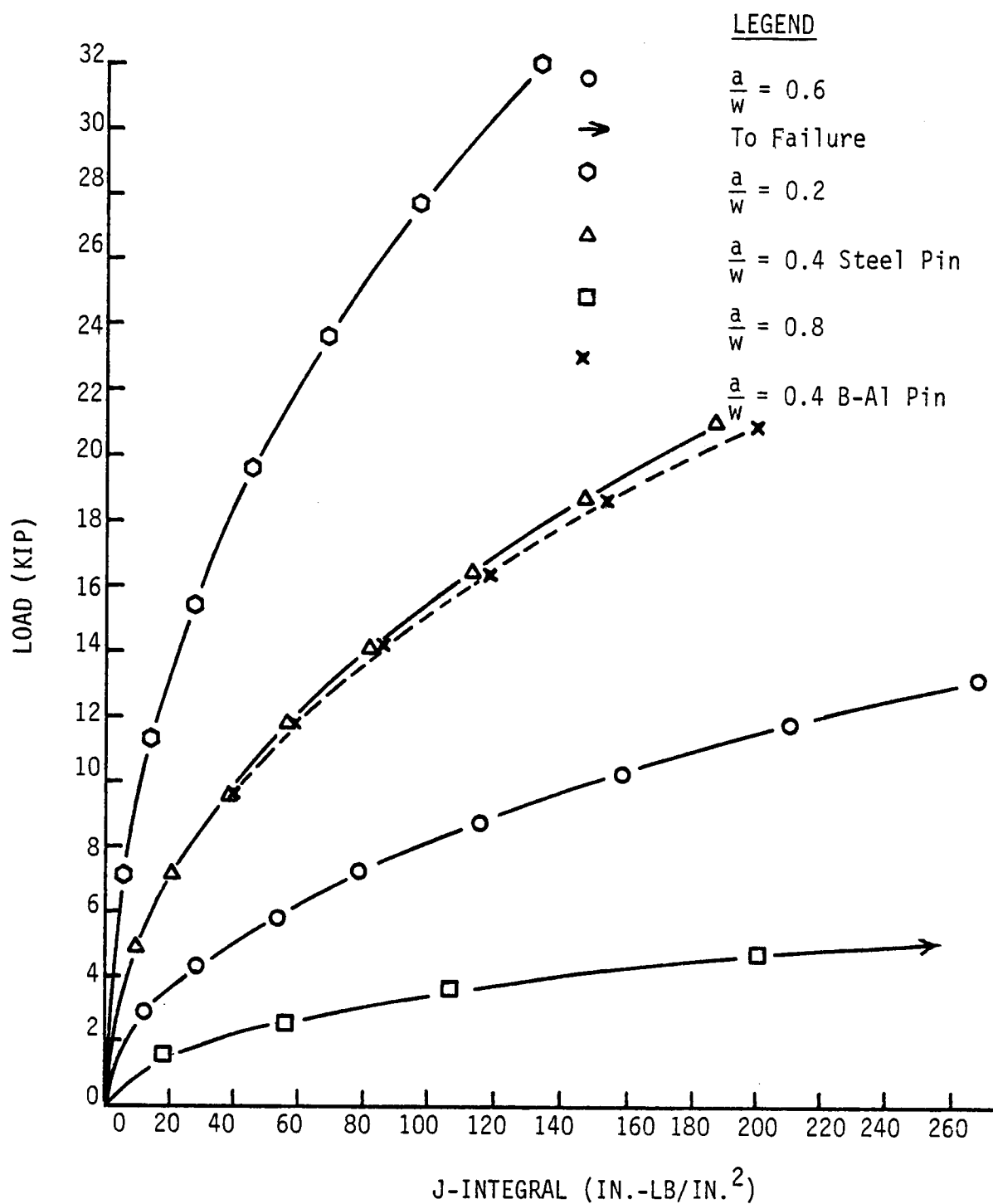


Figure 63. Variation of nondimensionalized normal stresses across the crack tip ($x = 2.60$ in.) in a compact tension specimen of $0^\circ/90^\circ$ B-Al composite.



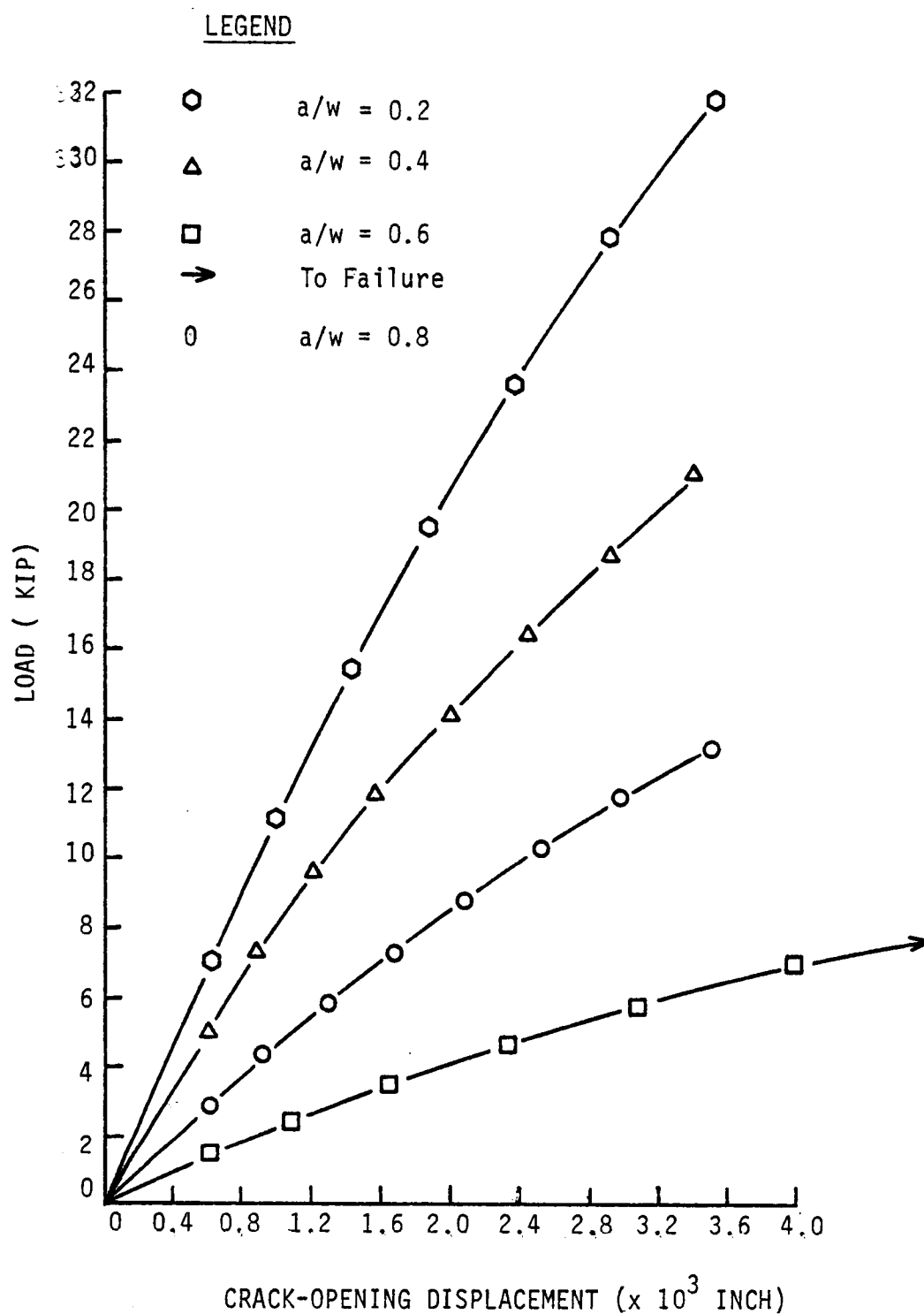
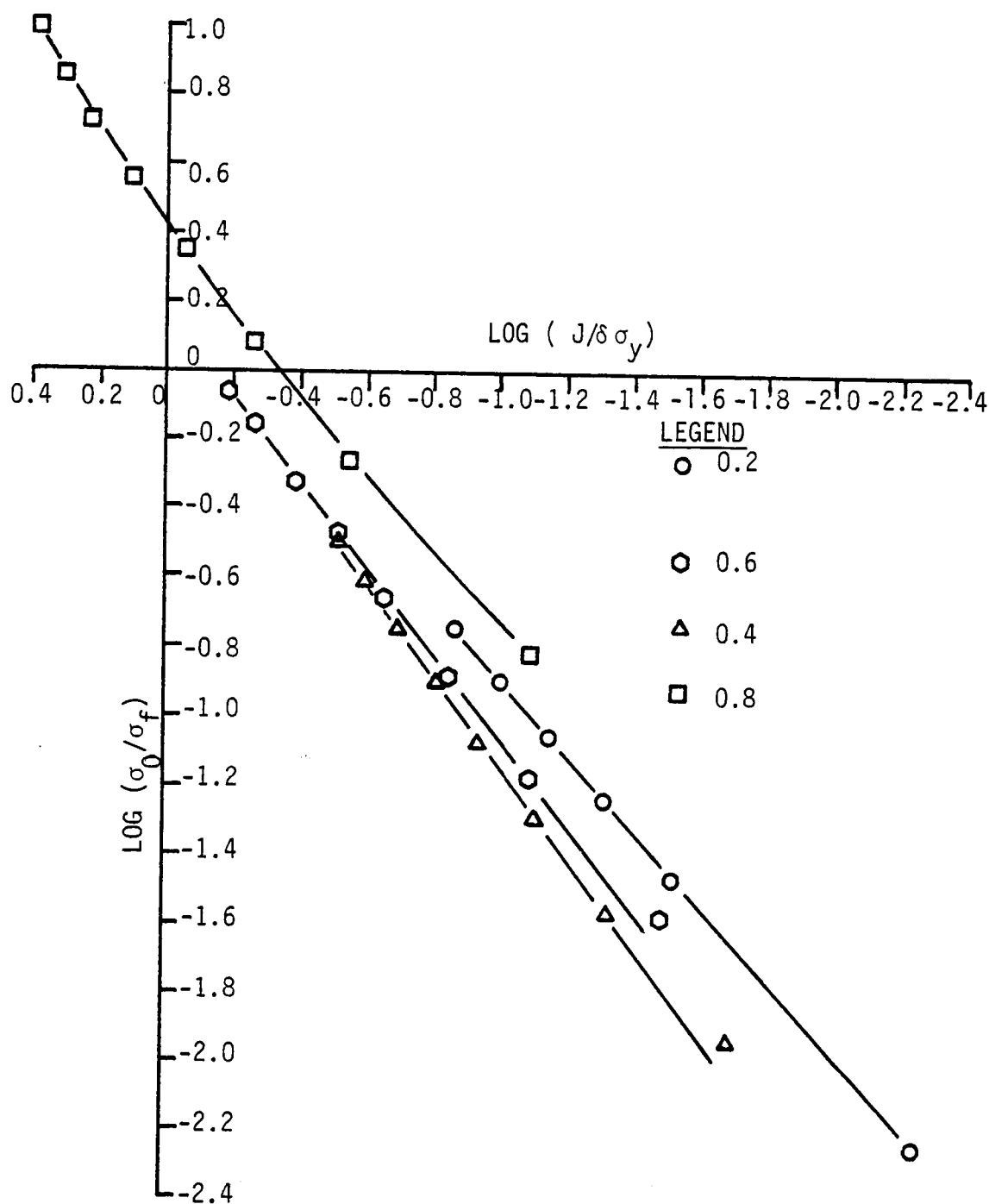


Figure 65. Variation of crack-opening displacement with load for $0^\circ/90^\circ$ B-A1 composite for various crack lengths.



APPENDIX B

TABLES

Table 1. Transformation Tables of Strength Parameters of Tsai-Wu Theory [27]

Strength Tensor		Constant	$\cos \theta$	$\sin \theta$	$\cos 2\theta$	$\sin 2\theta$	$\cos 4\theta$	$\sin 4\theta$
Transformation of F_i		F_1'	T_1	0	0	T_2	F_6	
		F_2'	T_1	0	0	$-T_2$	F_6	
		F_6'	0	0	0	$2F_6$	$-2T_2$	
		F_4'	0	$2F_4$	$2F_5$	0	0	
		F_5'	0	$2F_4$	$2F_5$	0	0	
		F_3'	0	0	0	0	0	
Transformation of F_{ij}		F_{11}'			U_2	$2U_6$	U_3	U_7
		F_{22}'	U_1		U_2	$-2U_6$	U_3	U_7
		F_{12}'	U_4		0	0	$-U_3$	$-U_7$
		F_{66}'	$4U_5$		0	0	$-4U_3$	$-4U_7$
		F_{16}'	0		$2U_6$	$-U_2$	$2U_7$	$-2U_3$
		F_{26}'	0		$2U_6$	$-U_2$	$-2U_7$	$2U_3$
		F_{13}'	V_1		V_2	$F_{36}/2$	0	0
		F_{23}'	V_1		$-V_2$	$-F_{36}/2$	0	0
		F_{36}'	0		F_{36}	$-V_2$	0	0

Table 1. (continued)

Strength Tensor		Constant	$\cos \theta$	$\sin \theta$	$\cos 2\theta$	$\sin 2\theta$	$\cos 4\theta$	$\sin 4\theta$
Transformation of F_{ij}	F_{44}'	W_1			W_2	$-F_{45}/2$	0	0
	F_{55}'	W_1			$-W_2$	$F_{45}/2$	0	0
	F_{45}'	0			$F_{45}/2$	W_2	0	0
	F_{33}'	F_{33}			0	0	0	0

X', Y' axes are rotated through an angle θ with respect to X, Y .

Transformations are shown for a rectangular coordinate system rotated through an angle θ .

$$\begin{aligned}
 T_1 &= (F_1 + F_2)/2 = \text{Invariant} \\
 T_2 &= (F_1 - F_2) \\
 U_1 &= (3F_{11} + 3F_{22} + 2F_{12} + F_{65})/8 = \text{Invariant} \\
 U_2 &= (F_{11} - F_{22})/2 \\
 U_3 &= (F_{11} + F_{22} - 2F_{12} - F_{66})/8 \\
 U_4 &= (F_{11} + F_{22} + 6F_{12} - F_{66})/8 = \text{Invariant} \\
 U_5 &= (F_{11} + F_{22} - 2F_{12} + F_{66})/8 = \text{Invariant} \\
 U_6 &= (F_{16} + F_{26})/4 \\
 U_7 &= (F_{16} - F_{26})/4 \\
 V_1 &= (F_{13} + F_{23})/2 = \text{Invariant} \\
 V_2 &= (F_{13} - F_{23})/2 \\
 W_1 &= (F_{44} + F_{55})/2 = \text{Invariant} \\
 W_2 &= (F_{44} - F_{55})/2,
 \end{aligned}$$

Table 2. Coefficient I_n of Strain-Hardening Stress Intensity [54]

n (Strain-Hardening Parameter)		$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{9}$	$\frac{1}{13}$
Tensile Crack Plane Stress	I_n	3.86	3.41	3.03	2.87
Tensile Crack Plane Strain	I_n	5.51	5.01	4.60	4.40
Shear Crack Plane Strain	I_n	0.694	0.458	0.269	0.181

Table 3. Three-Point Test Data for BS4360 Grade 50C Steel [101]

Specimen Description		δ		K	J	$\frac{J}{\delta \sigma_y}$
a/w	w m	10^{-3} m	JE/K ²	Mpa.m ^{1/2}	kJ/m ²	
0.50	0.050	0.063	1.53	69.5	32.6	1.429
		0.112	2.20	78.0	58.7	1.448
		0.161	3.00	80.3	85.0	1.458
		0.259	4.80	83.7	148.0	1.579
		0.358	6.47	85.0	205.0	1.582
		0.456	8.20	86.3	269.0	1.630
		0.554	9.87	86.7	326.0	1.626
0.17	0.050	0.058	0.85	64.2	15.4	0.733
		0.102	1.33	73.9	31.9	0.864
		0.145	1.80	78.2	48.5	0.924
		0.233	3.20	81.2	92.8	1.100
		0.320	4.43	83.9	137.2	1.118
		0.408	5.63	85.6	181.0	1.225
		0.496	6.80	86.9	225.9	1.258
0.50	0.025	0.026	0.93	55.6	12.6	1.339
		0.046	1.50	61.2	24.7	1.483
		0.068	2.20	63.1	38.5	1.564
		0.109	3.60	64.2	65.2	1.652
		0.151	4.77	65.5	90.0	1.646
		0.192	5.90	67.9	120.0	1.727
		0.234	6.90	70.3	150.0	1.771
0.20	0.025	0.050	1.16	53.2	14.5	0.801
		0.083	1.90	59.7	27.3	0.909
		0.116	2.76	61.8	46.4	1.105
		0.181	4.46	64.2	80.8	1.233
		0.246	6.00	66.4	116.4	1.307
		0.310	7.60	68.1	154.8	1.379
		0.375	9.13	69.3	193.2	1.423

$$\sigma_y = 362 \text{ MPa.}$$

Table 4. Properties of Unidirectional GFRE
along Principal Material Directions

Direction	Modulus MSI	Tensile Strength KSI	Compressive Strength KSI
Longitudinal	5.0	59.7	45.0
Transverse	1.7	3.3	12.0
Inplane Shear	0.4	9.0	9.0

Fiber Reinforcement = 50%.

Inplane Poisson Ratio = 0.16.

Table 5. Properties of 0°/90° Cross-Ply GFRE along Principal Material Directions

Direction	Modulus MSI	Tensile Strength KSI	Compressive Strength KSI
Longitudinal or Transverse	3.0	43.0	40.0
Inplane Shear	0.4	9.0	9.0

Major Poisson Ratio = 0.16.

Table 6. Properties of Unidirectional B-Al Composite
with 50% Reinforcement

Angle with Crack Axis, Degrees	Primary Modulus MSI	Secondary Modulus MSI	Yield Stress KSI	Failure Stress KSI	Failure Strain
90	35.0	30.0	35.5	200.0	0.0065
0	17.0	0.0	30.0	30.0	0.002
30	19.5	0.0	23.6	23.6	0.0052
45	23.2	0.0	22.5	22.5	0.011
60	28.7	0.0	24.6	24.6	0.0127
75	33.8	0.0	30.3	30.3	0.012

Inplane properties when crack and load axes coincide with
principal materials' axes:

Shear Modulus = 8.0 MSI.

Shear Strength = 12.0 KSI.

Major Poisson Ratio = 0.22.

Table 7. Comparison Between the Results of Uniform and Gradually Graded Meshes

	Case Ia	Case Ib	Case Ic	Case II	Case III	Case IV
Total Number of Nodes	178	234	246	2,931	860	860
Total Number of Elements	168	230	288	2,991	853	853
Number of Displacement Elements	168	230	288	444	243	853
Number of Hybrid Elements	0	0	0	2,547	610	0
Largest Element ($\times a^2$)	0.25	0.25	0.25	0.25	0.25	0.25
Smallest Element ($a^2 \times 10^4$)	39.1	9.77	0.305	6.25	15.6	15.6
Semi-bandwidth	48	56	80	156	80	80
Maximum Stress/ Applied Stress	1.94	2.25	7.72	6.46	3.99	3.68
Computer Time (Seconds)	37	56	78	3,763	415	350

Cases Ia, Ib, Ic: Gradually Graded Mesh.

Cases II to IV: Uniform Mesh.

Specimen: Center-Cracked Tension.

Length of Crack = 0.2 in.

Table 8. Comparison of Uniform and Gradually Graded Meshes with Experimental Results

Result from	Value 10^{-3} in.	Comments
Experiment [68]	9.46	Average of three experimental results
Tada [87]	9.05	Linear elastic solution
Uniform Mesh Finite Element	9.07	Quarter plate solution with Quarter plate solution with 860 nodes
Uniform Mesh Finite Element	9.11	Quarter plate solution with 2,931 nodes
Gradually Graded Finite Element	9.22	Full plate solution with 708 nodes
Gradually Graded Finite Element	8.45	Full plate solution with 512 nodes

All deflections are measured at load point.

Specimen: Center-Cracked Tension.

Crack Length to Width Ratio = 0.2.

Materials' Properties: $E = 14.0$ MSI, $G = 6.36$ MSI, $\nu = 0.1$.

Stress = 40 KSI.

Gage Length = 3 in.

Table 9. Effect of Fiber Orientation on Strength of Notched and Unnotched Specimens of GFRE [19]

Fiber Orientation, Degrees	Mean Failure Stress, KSI				
	Crack Length, Inches				
	0	0.125	0.25	0.50	0.625
90	59.70 ₂	53.95 ₃	40.97 ₃	23.97 ₃	20.9
80	12.70 ₃	9.2	7.03 ₃	4.5 ₂	3.35 ₂
75	12.30	6.1	4.85 ₄	2.77 ₃	2.05 ₂
30	3.50 ₃	---	2.0 ₂	1.10 ₂	1.0
20	3.30 ₂	---	1.73 ₄	1.25 ₂	---
0	3.30 ₃	---	2.0 ₃	1.2 ₂	1.0 ₂

Subscripts denote number of tests from which mean was taken.

Table 10. Effect of Fiber Orientation on Strength of
Cross-Ply 0°/90° GFRE Specimens [19]

Orientation, Degrees	<u>Mean Failure Stress, KSI</u>				
	<u>Crack Length, Inches</u>				
	0	0.11	0.205	0.31	0.405
90	43.0				
80	33.02	28.47	27.23	23.05	19.57
70	23.35	20.75	18.02	15.71	12.67
60	19.53	17.88	14.40	12.21	10.68
45	17.00				

Table 11. Strength Coefficients of Tsai-Wu Theory for Unidirectional and 0°/90° Cross-Ply GFRE Specimens

Coefficient	Unidirectional	Cross-Ply
F_1	$5.4 \times 10^{-6} \text{ in.}^2/\text{lb}$	$1.74 \times 10^{-6} \text{ in.}^2/\text{lb}$
F_2	$-220 \times 10^{-6} \text{ in.}^2/\text{lb}$	$1.74 \times 10^{-6} \text{ in.}^2/\text{lb}$
F_{11}	$3.722 \times 10^{-10} \text{ in.}^4/\text{lb}^2$	$5.814 \times 10^{-10} \text{ in.}^4/\text{lb}^2$
F_{22}	$252.525 \times 10^{-10} \text{ in.}^4/\text{lb}^2$	$5.814 \times 10^{-10} \text{ in.}^4/\text{lb}^2$
F_{66}	$625.0 \times 10^{-10} \text{ in.}^4/\text{lb}^2$	$123.46 \times 10^{-10} \text{ in.}^4/\text{lb}^2$
F_{12}	$-15.329 \times 10^{-10} \text{ in.}^4/\text{lb}^2$	$-2.907 \times 10^{-10} \text{ in.}^4/\text{lb}^2$

$F_3 = F_{16} = F_{26} = 0$ for both composites.

Table 12. Comparison Between Predicted and Experimental Strengths of Unidirectional Composite for Different Fiber Orientations

Fiber Orientation	Crack Length, Inches				
	0.0	0.125	0.25	0.50	0.625
90° Experimental (KSI)	59.7	53.95	40.97	23.97	20.90
80° Experimental (KSI)	12.97	9.2	7.03	4.5	3.35
Predicted (KSI)	22.4	11.48	8.72	5.1	4.45
Error (%)	43.3	19.9	19.4	11.8	29.7
75° Experimental (KSI)	12.3	6.1	4.85	2.77	2.05
Predicted (KSI)	15.2	11.11	8.49	4.94	4.31
Error (%)	19.1	45.1	42.5	43.9	52.4
30° Experimental (KSI)	3.5	---	2.0	1.1	1.0
Predicted (KSI)	3.95	3.16	2.40	1.41	1.23
Error (%)	11.4	---	16.7	29.1	18.7
20° Experimental (KSI)	3.3	---	1.73	1.25	---
Predicted (KSI)	3.6	2.98	2.26	1.33	1.16
Error (%)	8.3	---	23.4	6.0	---
0° Experimental (KSI)	3.3	---	2.0	1.20	1.0
Predicted (KSI)	3.3	2.98	2.26	1.33	1.16
Error (%)	0	---	11.5	9.8	13.8

Table 13. Comparison Between Predicted and Experimental Strengths of 0°/90° Cross-Ply GFRE Composite for Different Fiber Orientations

Fiber Orientation	Crack Length, Inches				
	0.0	0.11	0.205	0.31	0.405
90° Experimental (KSI)	43.0	41.95	33.47	31.26	25.54
80° Experimental (KSI)	33.02	25.47	27.23	23.05	19.57
Predicted (KSI)	33.37	33.21	25.70	24.01	19.61
Error (%)	1.0	14.3	-6.0	4.0	-0.2
70° Experimental (KSI)	23.35	20.75	18.02	15.71	12.67
Predicted (KSI)	23.55	22.78	18.18	16.98	13.87
Error (%)	0.8	8.9	0.7	7.5	8.6
60° Experimental (KSI)	19.53	17.88	14.40	12.21	10.68
Predicted (KSI)	18.79	19.65	15.20	14.20	11.60
Error (%)	-3.9	6.1	5.3	14.1	7.9
50° Experimental (KSI)	17.58	16.23	13.49	10.96	9.22
Predicted (KSI)	16.89	17.15	13.68	12.78	10.44
Error (%)	-4.1	6.5	1.4	16.6	13.2
45° Experimental (KSI)	17.00	---	---	---	---
Predicted (KSI)	16.67	16.59	13.23	12.36	10.10
Error (%)	-2.0	---	---	---	---

Table 14. Comparison Between Predicted and Experimental Critical Stress Intensity Factors of Unidirectional GFRE for Different Fiber Orientations

Orientation, Degrees	Critical Stress Intensity Factor, $K_{\theta c}$ in KSI in.					
	Crack Length, Inches				Mean $K_{\theta c}$	Predicted $K_{\theta c}$
	0.125	0.25	0.50	0.625	KSI $\sqrt{in.}$	KSI $\sqrt{in.}$
90	23.91	25.67	21.24	20.71	22.88	22.88
80	4.08	4.41	3.99	3.32	3.95	4.87
75	2.70	3.04	2.46	2.03	2.56	4.87
30	---	1.25	0.97	0.99	1.07	1.34
20	---	1.08	1.11	---	1.10	1.26
0	---	1.25	1.06	0.99	1.10	1.26

$$\text{Predicted } K_{\theta c} = K_{0c} \times \sigma_{\theta f} / \sigma_{0c}$$

Table 15. Comparison Between Predicted and Experimental Critical Stress Intensity Factors of Cross-Ply GFRE Composite for Different Fiber Orientations

Orientation, Degrees	Critical Stress Intensity Factor, $K_{\theta c}$ in $\text{KSI}\sqrt{\text{in.}}$					Mean $K_{\theta c}$ $\text{KSI}\sqrt{\text{in.}}$	Predicted $K_{\theta c}$ $\text{KSI}\sqrt{\text{in.}}$
	Crack Length, Inches						
	0.11	0.205	0.31	0.405			
80	11.83	14.54	16.08	15.61	14.74	14.74	
70	8.63	10.23	10.96	10.06	9.97	10.42	
60	7.43	8.17	8.52	8.52	8.16	8.72	
50	6.75	7.66	7.65	7.35	7.35	7.85	
45	---	---	---	---	---	7.59	
90	17.44	18.99	21.82	20.37	19.66	19.20	

$$\text{Predicted } K_{\theta c} = K_{80c} = \sigma_{\theta f} / \sigma_{80c}$$

Table 16. Critical Region Size (r_c) as Calculated from Experimental Data of Unidirectional and Cross-Ply ($0^\circ/90^\circ$) GFRE

	Angle, Degrees	r_c , Inches
Unidirectional	0	0.023
	10	0.015
	15	0.015
	60	0.015
	70	0.018
	90	0.018
Cross-Ply ($0^\circ/90^\circ$)	10	0.031
	20	0.028
	30	0.028
	40	0.028
	45	0.035

Table 17. Variation of Critical Stress Intensity Factors of Unidirectional B-Al Composite for Different Fiber Orientations

Fiber Orientation, Degrees	$2a/w = 0.2$	$2a/w = 0.4$	Average	Error, %
90	31.09	29.82	30.46	± 2.1
75	19.76	10.86	20.31	± 2.7
60	15.60	17.37	16.49	± 5.3
45	14.27	15.21	14.74	± 3.2
30	13.20	13.02	13.11	± 0.6
0	12.22	11.88	12.05	± 1.4

Table 18. Spread of Plastic Zones at Failure Loads for Unidirectional B-Al Composite for Different Fiber Orientations

	Fiber Orientation, Degrees	$2a/w = 0.2$	$2a/w = 0.4$
Yielding at Grip	90	Yes	Yes
	75	Yes	No
	60	Yes	No
	45	Yes	No
	30	Yes	No
	0	No	No
Yielding at Center of Crack	90	Yes	Yes
	75	Yes	Yes
	60	Yes	Yes
	45	Yes	No
	30	Yes	No
	0	No	No

Table 19. Comparison Between Estimated and Actual Failure Loads for Unidirectional B-Al Composite with Various Crack Lengths

Crack Length, Inches	Fiber Orientation, Degrees	Estimated Value (KSI)	Actual Value (KSI)	Error, %
0.2	90	62.0	55.5	-11.7
	75	31.7	35.2	10.1
	60	25.0	27.8	10.2
	45	22.9	25.5	10.0
	30	23.0	23.5	2.3
	0	23.2	21.8	-6.4
0.4	90	41.0	37.6	-9.0
	75	24.0	26.3	8.7
	60	19.4	21.9	11.4
	45	16.4	19.2	14.2
	30	15.9	16.4	3.0
	0	15.7	15.0	-4.7

Estimation of failure load is based on:

Critical COD = 4.0×10^{-4} in.

Table 20. Load and Deflection Values of a Fully (180°) Contacted Compact Tension Specimen ($a/w = 0.4$) of 0°/90° B-A1 Composite

Load KIP	Load Point Deflection 10 ⁻² in. LPD	Load Line Deflection 10 ⁻² in. LLD	Crack- Opening Displacement 10 ⁻³ in. COD	Crack Face Displacement 10 ⁻² in. CFD
5.00	0.4782	0.4662	0.6033	0.6630
7.29	0.6968	0.6795	0.8793	0.9661
9.57	0.9204	0.8975	1.2255	1.2756
11.86	1.1482	1.1194	1.5795	1.5909
14.14	1.3839	1.3486	2.0094	1.9164
16.43	1.6263	1.5843	2.4550	2.2514
18.72	1.8755	1.8266	2.9324	2.5950
21.00	2.1332	2.0770	3.3996	2.9490

Table 21. Load and Deflection Values of a Partially (120°)
Contacted Compact Tension Specimen ($a/w = 0.4$)
of 0°/90° B-AI Composite

Load KIP	Load Point Deflection 10 ⁻² in. LPD	Load Line Deflection 10 ⁻² in. LLD	Crack- Opening Displacement 10 ⁻³ in. COD	Crack Face Displacement 10 ⁻² in. CFD
5.00	0.4750	0.4644	0.6032	0.6543
7.29	0.6922	0.6768	0.8791	0.9535
9.57	0.9144	0.8942	1.2255	1.2592
11.86	1.1411	1.1156	1.5797	1.5711
14.14	1.3760	1.3446	2.0104	1.8937
16.43	1.6169	1.5793	2.4558	2.2240
18.72	1.8634	1.8194	1.9323	2.5616
21.00	2.1186	2.0680	3.3994	2.9098

Table 22, Compliance Values of 0°/90° Cross-Ply B-Al Composite
for Different Crack Lengths

	Load KIP	Load Point Compliance 10 ⁻⁶ in.2/lb	Load Line Compliance 10 ⁻⁶ in.2/lb	Crack Face Compliance 10 ⁻⁶ in.2/lb
a/w = 0.2	7.14	0.4	0.6	0.6
	11.26	0.4	0.6	0.6
	15.39	0.4	0.6	0.6
	19.51	0.4	0.6	0.6
	23.63	0.4	0.6	0.6
	27.75	0.4	0.6	0.6
	31.88	0.4	0.6	0.6
a/w = 0.4	5.00	1.0	0.9	1.3
	7.29	1.0	0.9	1.3
	9.57	1.0	0.9	1.3
	11.86	1.0	0.9	1.3
	14.14	1.0	1.0	1.3
	16.43	1.0	1.0	1.3
	18.72	1.0	1.0	1.3
	21.00	1.0	1.0	1.3
a/w = 0.6	2.93	2.3	3.1	3.1
	4.39	2.3	3.1	3.1
	5.87	2.4	3.1	3.1
	7.83	2.4	3.0	3.0
	8.80	2.4	3.1	3.1
	10.27	2.4	3.2	3.2
	11.74	2.4	3.2	3.2
	13.20	2.4	3.3	3.3
a/w = 0.8	1.47	7.1	8.9	8.9
	2.54	7.1	8.9	8.9
	3.62	7.2	9.1	9.1
	4.67	7.3	9.2	9.2
	5.77	7.5	9.4	9.4
	6.85	7.7	9.8	9.8
	7.42	8.2	10.3	10.3
	9.00	8.7	10.9	10.9

Compliance = $C' = \Delta B/P$.

APPENDIX C

ADDENDA I AND II

ADDENDUM I

The stress-strain relation for orthotropic materials in plane stress condition is given by

$$\{\sigma\}_{LT} = [D]\{\epsilon\}_{LT} , \quad (\text{AI-1})$$

$$\{\sigma\}_{LT}^T = \{ \sigma_L \quad \sigma_T \quad \tau_{LT} \} , \quad (\text{AI-2})$$

$$\{\epsilon\}_{LT}^T = \{ \epsilon_L \quad \epsilon_T \quad \gamma_{LT} \} , \quad (\text{AI-3})$$

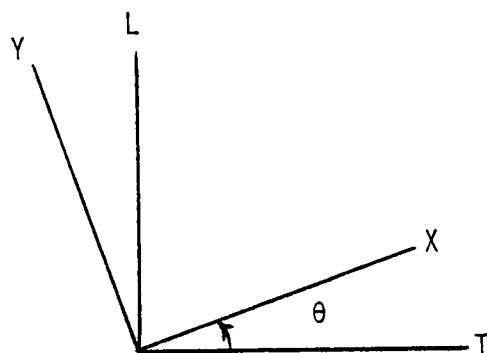
$$[D] = \begin{bmatrix} R & R_{TL} & 0 \\ R_{TL} & \frac{RE_{TT}}{E_{LL}} & 0 \\ 0 & 0 & G_{LT} \end{bmatrix} , \quad (\text{AI-4})$$

$$R = \frac{E_{LL}}{1 - \nu_{LT} \nu_{TL}} . \quad (\text{AI-5})$$

From the reciprocity relation,

$$\frac{\nu_{TL}}{E_{TT}} = \frac{\nu_{LT}}{E_{LL}} . \quad (\text{AI-6})$$

If the transverse material axis (T) is rotated through an angle (θ), the stress-strain relation can be developed as follows (rotation of axis, right-hand system):



From any strength of material test, it can be easily shown that

$$\{\epsilon\}_{LT} = [T] \{\epsilon\}_{xy} , \quad (\text{AI-7})$$

$$[T] = \begin{bmatrix} m^2 & n^2 & -mn \\ n^2 & m^2 & mn \\ 2mn & -2mn & m^2 - n^2 \end{bmatrix} , \quad (\text{AI-8})$$

where,

$$m = \sin\theta \quad \text{and} \quad n = \cos\theta .$$

From equality of work,

$$\{\sigma\}_{LT}^T \{\epsilon\}_{LT} = \{\sigma\}_{xy}^T \{\epsilon\}_{xy} , \quad (\text{AI-9})$$

$$\begin{aligned}
 \text{LHS} &= \{\epsilon\}_{LT}^T [D]^T \{\epsilon\}_{LT} = \{\epsilon\}_{LT}^T [D] \{\epsilon\}_{LT} \\
 &= \{\epsilon\}_{xy}^T [T]^T [D] [T] \{\epsilon\}_{xy} \quad , \quad (\text{AI-10})
 \end{aligned}$$

$$\text{RHS} = \{\epsilon\}_{xy}^T [D]_0 \{\epsilon\}_{xy} \quad (\text{AI-11})$$

Equating Eqs. (AI-10) and (AI-11),

$$[D]_\theta = [T]^T [D] [T] \quad (\text{AI-12})$$

ADDENDUM II

The modified Tsai-Wu theory for plane stress conditions is given by

$$f \equiv F_{11}\sigma_x^2 + F_{22}\sigma_y^2 - 2F_{12}\sigma_x\sigma_y - 2F_{16}\sigma_x\tau_{xy} - 2F_{26}\sigma_y\tau_{xy} \\ + F_{66}\tau_{xy}^2 - F_1\sigma_x - F_2\sigma_y - F_6\tau_{xy} - 1 = 0 . \quad (\text{AII-1})$$

For an equivalent stress along the applied direction, the load is similarly given by

$$f \equiv F_{22}\sigma_e^2 - F_2\sigma_e - 1 = 0 . \quad (\text{AII-2})$$

Equating Eqs. (AII-1) and (AII-2),

$$F_{22}\sigma_e^2 - F_2\sigma_e = F_{11}\sigma_x^2 + F_{22}\sigma_y^2 - 2F_{12}\sigma_x\sigma_y \\ - 2F_{16}\sigma_x\tau_{xy} - 2F_{26}\sigma_y\tau_{xy} + F_{66}\tau_{xy}^2 . \quad (\text{AII-3})$$

$$- F_1\sigma_x - F_2\sigma_y - F_6\tau_{xy} . \quad (\text{AII-3})$$

If $F_1 = F_2 = F_6 = 0$, then Eq. (AII-3) reduces to, as a special case,

$$\bar{\sigma}_e = \frac{1}{F_{11}} (F_{11}\sigma_x^2 + F_{22}\sigma_y^2 + F_{66}\tau_{xy}^2 + 2F_{12}\sigma_x\sigma_y \\ + 2F_{16}\sigma_x\tau_{xy} + 2F_{26}\sigma_y\tau_{xy}) . \quad (\text{AII-4})$$

VITA

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