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I am submitting herewith a dissertation written by Chih-Kang Eric Chen entitled "Locating Undesirable Facilities on a Network." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Management Science.

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LOCATING UNDESIRABLE FACILITIES ON A NETWORK

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ABSTRACT

This dissertation gives a systematic study of the problem of locating undesirable facilities on a network, where the new facilities can be located anywhere on the network. The undesirable facilities are further divided into two classes, namely, the "strictly undesirable facilities" and the "partially undesirable facilities." Models are developed for locating these two classes of facilities.

A new model which utilizes both the network shortest path distance measurement and Euclidean distance measurement is proposed for the partially undesirable facility location problems on a network. Two examples of this new model are examined and solved by an application of an interactive computer graphics technique.

The strictly undesirable facility location problem on a network is modeled with both median-type objective function and center-type objective function. Theoretical and computational results are given for both formulations.

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CHAPTER 1

INTRODUCTION

This dissertation deals with the problem of locating certain undesirable or controversial facilities on a network. Examples of facilities with this nature include dump sites, airports, factories, prisons, nuclear power plants, etc. A distinct characteristic of this class of facilities is that due to considerations of odor, noise, safety, etc., it is undesirable for them to be close to the general public. This is a departure from the fundamental assumption held in most of the ordinary facility location problems which states that facilities will provide for "better service" if they are "closer" to their potential users. Although facility location problems have been the subject of a great deal of research in the mathematical programming literature, only a handful of papers consider locating facilities having an undesirable nature. Since this is a relatively new area of interest, the question of how to locate undesirable facilities is a topic which needs to be explored.

1.1 Brief Review of Location Models

This section gives an overview of the mathematical programming facility location models. The mathematical

programming approach to the facility location problems [Francis & White (1974)] takes the form of

$$\min \text{ (max) } f(A,X)$$

$$X \in S$$

A: Location of the set of existing facilities which is the input of the model.

X: Location of the set of new facilities which is the output of the model.

S: Solution space.

In location theory, "facility" is used as a very general term. Facilities may be (1) macro entities like population centers, schools, or plants, or (2) micro entities like students, machines, or workers.

Within the location literature, models can be categorized by the following features:

1. Private sector vs. public sector (the economic context). In the private sector, one is concerned with the trade-off between establishment costs and operating costs of the new facilities. Typical examples are the plant location problems where the firm seeks the optimal configuration of plants in order to minimize the total costs, including building, production, and transportation costs [Alcouffe et al. (1976), McGinnis (1977)]. For locating public facilities such as schools, fire stations, libraries, it is often difficult to measure the involved social costs

and social benefits. The service distance or service time of the facilities is, therefore, usually employed as a surrogate measure of public utility [Church & ReVelle (1974), Toregas & ReVelle (1972)].

2. Solution space (mathematical context). According to the problem setting, the relevant solution space may be different. The following are the most generally encountered spaces:

(a) Continuous space—The solution space is on the plane or sphere. Problems of location on the plane arise naturally in such cases as locating a market place for several villages or locating a radio station for several towns where the market place and radio station can be located anywhere on the plane [Elzinga & Hearn (1972), Ostresh (1978)]. When the area under consideration is large, it is probably more accurate to use a sphere as the solution space to take account of the curvature of the Earth's surface [Katz (1978)].

(b) Discrete space—This is a subset of continuous space. In this case, the location of new facilities is limited to a discrete set. For example, only certain cities are selected as the potential location of new plants [Alcouffe et al. (1976)].

(c) Network—This includes road networks, pipeline networks, inland waterway networks. The new facilities

can be located anywhere on the network, such as locating new pump stations along a pipeline network or locating bus stops of a transit system [Garfinkel et al. (1974 & 1977)].

3. Distance measurement. The l_p distance (1.1) is usually employed for location problems on the plane. Given any two points (x_1,y_1), (x_2,y_2) on the plane, the l_p distance for $p \geqslant 1$ between these two points is defined as

$$(|x_1 - x_2|^p + |y_1 - y_2|^p)^{1/p} . \qquad (1.1)$$

Among the l_p distances, two most popular distance measurements are:

(a) The rectilinear distance (when $p = 1$, it is also known as the Manhattan distance) which is relevant for problems like locating fire stations in a large city.

(b) The Euclidean distance (when $p = 2$) which is relevant for problems like locating radio stations [Cabot et al. (1970), Elzinga & Hearn (1972), Drezner & Wesolowsky (1978)].

For location problems on networks, another distance measurement is usually used which is:

(c) Network shortest path distance—For any two points on a network, there may be several paths connecting them. The shortest one is called the shortest path

and its distance is used as the distance between these two points.

4. Single new facility vs. multiple new facilities. The multiple facility problem is usually more complicated than the single facility case. For multifacility problems, extra considerations are required. The consideration may be whether an existing facility "interacts" with all new facilities or with its closest new facility only, or it may be whether there are "interactions" among the new facilities [Francis & White (1974)].

5. Objective functional form. In the private sector location problems, minimizing the total costs is the usual objective. In the public sector, the following three types of objective function are usually employed to measure the quality of location.

(a) Center-type objective function (measuring equity)—The objective is to maximize "closeness" in the extreme, i.e., minimize the maximum service distance or service time from any existing facility to the new facilities [Elzinga & Hearn (1972), Garfinkel et al. (1977)].

(b) Median-type objective function (measuring efficiency)—The objective is to maximize "closeness" on the average, i.e., minimize the total service distance or service time from existing facilities to

the new facilities [Garfinkel et al. (1974), Ostresh (1978)].

(c) Maximal covering problem (measuring effectiveness)—The objective is to maximize the number of existing facilities within proximate range of new facilities [Church & Meadows (1979)].

6. Deterministic vs. probabilistic. Probabilistic models take account of the uncertainty characteristics of the location or the changes of service requirements of the existing facilities [Mirchandani & Odoni (1979), Rosenthal et al. (1978)].

7. Unconstrained vs. constrained. Various types of constraints have been incorporated in location models. They may be capacity constraints of the new facilities [Dearing & Newruck (1979)], investment or budget constraints [Rojeski & ReVelle (1970)], maximum service distance constraints [Church & Meadows (1977)], or constraints associated with the presence of forbidden regions [Cooper & Katz (1979)].

8. Ordinary vs. undesirable facility. Ordinary facility location models try to achieve proximity of existing facilities to new facilities. As indicated by the name itself, the undesirable facility is objectionable to be close to the general public or to its potential users [Church & Garfinkel (1978), Cornuejols et al. (1977)].

Through a reasonable combination of the different features, there are a great variety of models in the facility

location literature. Among the above features, probabilistic, constrained, and undesirable problems are relatively new fields of research. Although the consideration of uncertainties, constraints, or undesirabilities adds complication to the solution scheme, it enables the mathematical programming models to better resemble real world situations. This dissertation is concerned with locating undesirable facilities on a network, where the whole network is considered as the solution space and new facilities can be located anywhere on the network. This important aspect makes this dissertation differ from most of the other undesirable facility location models, where the solution spaces are finite discrete spaces. The undesirable facilities are further classified into strictly undesirable and partially undesirable classes. A number of possible models for locating these two types of facilities will be developed. While the strictly undesirable facility models only take into account the undesirable nature of the facilities, the partially undesirable facility models consider both the undesirabilities and accessibilities. To develop the models, the median-type and center-type objective functions will be employed. In terms of distance measurements, both Euclidean and network shortest path distance measurements will be used.

1.2 Structure of Later Chapters

General models for two classes of undesirable facility location problems on a network, namely, the "partially undesirable" and "strictly undesirable" location problems are discussed in Chapter 2. Two examples for the partially undesirable facility location problems are investigated in Chapter 3. To obtain the solutions for the examples, an interactive computer graphics solution technique is employed. Chapters 4 and 5 deal with two variations of the strictly undesirable facility location models. While models with a median-type objective function are discussed in Chapter 4, models with a center-type objective function are in Chapter 5. Both chapters present theoretical results as well as algorithms for solving the problems. The "generalized diameter problem" and its extensions are introduced in Chapter 6. Interest in this problem was originally spurred by the search for a finite optimal solution set for the strictly undesirable facility location problem, although the concept is of interest in its own right. A number of topics for further research are suggested in Chapter 7.

CHAPTER 2

DEVELOPMENT OF UNDESIRABLE FACILITY LOCATION MODELS

Two classes of undesirable facilities will be examined in this dissertation. Facilities like dump sites, airports, nuclear power plants will be denoted "partially undesirable." Although they are undesirable neighbors, they have to be accessible for the potential users. Therefore, in addition to undesirability, accessibility has to be considered. Another class is "strictly undesirable," where the object is to locate the new facilities as far away from the existing facilities as possible. For example, suppose one would like to put measuring devices to collect such information as water speed, water level, and water temperature along an inland waterway network. There are already q such devices (existing facilities), and a budget for p more devices (new facilities) is available. In order to obtain the maximum amount of information, one would like to locate the p new devices as far away from the q existing devices as possible. Prior to developing these two classes of facility location models on a network, a discussion of the notation, concepts, and results of the network facility location problem will be given.

2.1 Network Facility Location Problems

The two types of problem formulations that are most often adopted in facility location problems on a network are the p-median and the p-center problems. The problem formulations are formally introduced by Hakimi (1964 and 1965), who presents the formulations by taking examples from communication networks and highway networks. The p-median problem is related to the location of p telephone switching stations where the objective is to minimize the total length of wire needed to connect the switching stations and the telephone terminals. The p-center problem is related to the problem of policing a highway network where the objective is to locate p policemen in order to minimize the maximum distance between the policemen and points along the highway.

As ReVelle, Marks, and Liebman (1970) point out, the p-median and p-center models are especially suitable for locating public facilities. Instead of working under cost/benefit considerations, as is usually done by the private sector, the median-type and center-type formulation could serve as surrogate measures of social utility. Basically, public facility location problems involve the decision of where to locate p similar facilities, such as post offices, libraries, or fire stations so as to best provide service to the general public. The network interpretation arises naturally in correspondence with the road network where

the new facilities can be located anywhere on the network and the demand for the facilities is generated from the nodes. For general service facilities (post offices, libraries), the minimization of the total distance or time for the potential users to go to the facilities will reflect the efficiency of the various facility location configurations. For emergency facilities (fire stations, ambulance depots), the minimization of the maximum response distance or time for the potential user to reach the facilities will reflect the equity of the various location configurations.

Before giving the formal mathematical definitions of the p-median and p-center problems, notations for a graph (network) should be introduced. More detailed and extensive discussions of graphs can be found in books on graph theory, such as Christofides (1975).

$G = (N,A)$ is a connected, undirected simple network.

N : the finite node set. The existing facilities are assumed to be located on the node set. The existing facilities may be population centers, warehouses, machines, etc.

A : the arc set that connects pairs of nodes which indicate the relationship between the existing facilities.

w_i : nonnegative weight attached to the existing facility i to indicate its relative importance.

There will be no existing facility on node i if $w_i = 0$. The weight may be population, quantity demanded, etc.

c_{ij} : positive length of arc (i,j) which represents some relevant measure of the relationship between existing facilities, such as road distance or travel time. The triangle inequality for the arc lengths will not be assumed. That is, it is possible for

$$c_{ij} + c_{jk} < c_{ik} .$$

c_{ix} : the length of arc segment (i,x) where x is an interior point of arc (i,j).

s_{ij} : the shortest path length between nodes i, j.

s_{ix} : the shortest path length between node i and an interior point x of some arc (m,n), where $s_{ix} = \min\{s_{im} + c_{mx}, s_{in} + c_{nx}\}$.

s_{xy} : the shortest path length between two interior points x, y on some arcs.

s_{iX} : $\min_{k=1,\ldots,p} s_{i,x_k}$ where $X = (x_1, x_2, \ldots, x_p)$ is a set of p points on the graph G. These points can be either nodes or interior points.

Figure 2.1 is an example of a seven node network. The weights w_i are assumed to be 1 in this example.

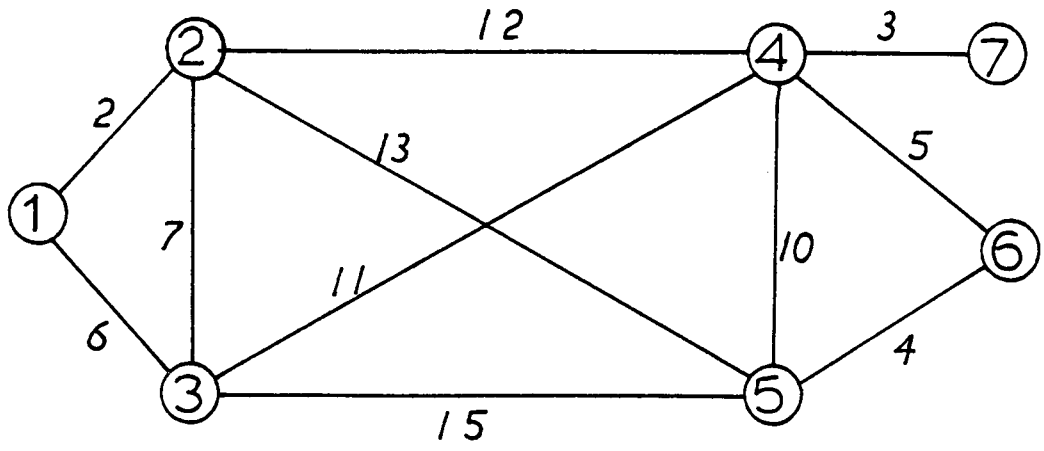


Figure 2.1. An example of a seven node network.

Note that the triangle inequality is not satisfied by this graph. For instance, $c_{46} + c_{65} < c_{45}$. This sort of situation is realistic in real world terms, especially if these arc lengths actually correspond to travel time.

P-Median Problem

The p-median problem is to minimize the sum of the weighted shortest path distances between the nodes and the set of new location points, i.e.

$$\min \sum_{i \in N} w_i s_{iX} . \qquad (2.1)$$

Using the seven node network in Figure 2.1 as an example, the optimal solution for the p = 2 case is to locate the new facilities at nodes 1 and 4 and the other nodes are assigned as in Figure 2.2. The objective function value is 25.

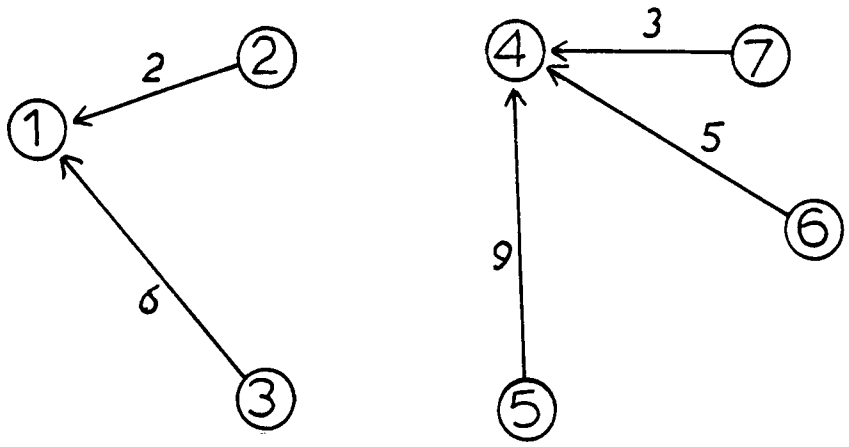


Figure 2.2. The solution for an example of the p-median problem.

Hakimi's principle [Hakimi (1965)] states that an optimal solution for problem (2.1) can be found in the node set N. This important result permits the network structure to be ignored since there is no need to consider locating new facilities on arcs. As a result, the p-median problem is reduced to the problem of selecting p locations among the possible $|N|$ locations. ($|\cdot|$ represents the cardinality of a set.) Hence, it can be formulated and solved as a 0-1 integer programming problem. The formulation was first introduced by ReVelle and Swain (1970).

$$\text{Min} \sum_{i,j\in N} w_i s_{ij} x_{ij} \tag{2.2}$$

s. t.

$$\sum_{j\in N} x_{ij} = 1 \qquad \text{all } i\in N \tag{2.3}$$

$$\sum_{j \in N} x_{jj} = p \tag{2.4}$$

$$x_{jj} \geq x_{ij} \qquad \text{all } i,j \in N,\ i \neq j \tag{2.5}$$

$$x_{ij} \in \{0,1\} \tag{2.6}$$

where $x_{ij} = \begin{cases} 1, & \text{if node } i \text{ assigns to node } j, \\ 0, & \text{otherwise.} \end{cases}$

A new facility is located at node j if $x_{jj} = 1$. The constraint (2.3) guarantees that all of the nodes have been assigned to one new facility. Constraint (2.5) avoids the situation of false assignment to a nonexistent facility.

A number of exact algorithms have been developed to solve the integer programming problem (2.2)-(2.6). In order to obtain an integer optimal solution, a branch and bound algorithm is employed either with the direct LP relaxation [El-Shaieb (1973), and Järvinen et al. (1972)], or with the Lagrangian relaxation [Narula et al. (1977)]. Other exact algorithms include the decomposition approach used by Garfinkel et al (1974) and Swain (1974), the generalized assignment problem formulation by Ross and Soland (1977), and the dual-based procedure developed by Erlenkotter (1978).

Heuristic methods are also available for the p-median problem. Among them, the most generally used are the greedy assignment method by Kuehn and Hamburger (1963), the

partition method by Marazana (1965), the vertex substitution method by Teitz and Bart (1968), and the dynamic programming approximation method by Baker (1974). In Chapter 3, both the greedy assignment method and the vertex substitution method will be employed to solve an example of the partially undesirable facility location problem.

P-Center Problem

The p-center problem minimizes the maximum weighted shortest path distance to any node, i.e.

$$\min \max_{i \in N} w_i s_{iX} . \qquad (2.7)$$

Since Hakimi's principle does not hold in this case, location cannot be limited to the node set. Again, using the seven node network as an example, the optimal solution for the p = 2 case is to locate the new facilities at points s, t on arcs (1,3) and (4,6) respectively, as Figure 2.3 shows. The objective function value is 6. It can be shown that the optimal solution of problem (2.7) is in the node set N plus the finite set of "edge bottleneck points" [Garfinkel et al, (1977)]. The existence of a finite feasible set also allows the p-center problem to be formulated and solved by using integer programming [Garfinkel et al. (1977)].

For network facility location problems, since the infinite number of points on the network is very difficult

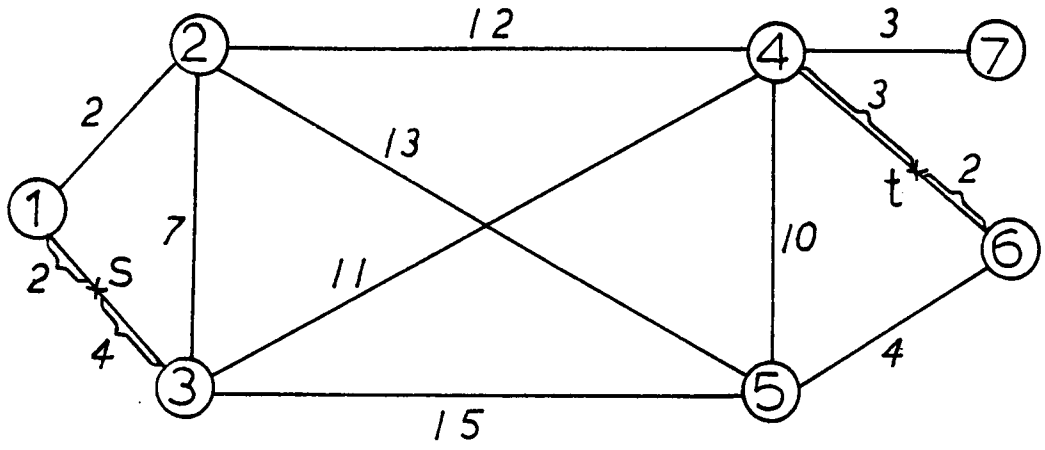


Figure 2.3. The solution for an example of the p-center problem.

to keep track of, it is important that a finite set is identified which contains the optimal solution. Such a finite set will be referred as a "finite optimal solution set" (FOSS). After this set is identified, the problem can be formulated and solved by integer programming. For this reason, success in obtaining a FOSS for the p-median and p-center problems is the key to achieving optimal solutions. Thus, the question of the existence of a FOSS for the undesirable facility location problems on a network will be one of the main efforts in the later chapters.

2.2 Partially Undesirable Facility Location Problems

For partially undesirable facilities, the traditional least-cost approach which completely ignores the possible opposition to the undesirable feature of the facilities

cannot give satisfactory results. This inefficiency is pointed out by Mumphrey et al. (1971) who propose a total expected cost approach to take account of the undesirability effects. The approach is demonstrated by locating a new expressway in a metropolitan area, where the possible locations are assumed to be a known finite set. After the potential groups who might oppose the expressway to be located in each of the locations are identified, the probability of each group raising opposition and the cost of this opposition is estimated. The optimal solution is the location that has the minimum total expected cost, which includes the expected physical costs (design, site acquisition, construction costs) and the expected opposition costs (legal costs, delays, redesign costs). In order to reduce the probability and cost of opposition, the authors also consider the possibility of establishing "auxiliary facilities" on each location to compensate for the inconveniences caused by the expressway. They distinguish long-run from short-run models, depending on whether or not possible future opposition is taken into account. It is clear that the success of both the short-run and long-run models depends heavily on the ability to correctly estimate the probability of opposition and their relevant costs. This is a very subjective and difficult task.

Austin (1974) works on a location problem with a generalized environment and a "place utility" approach. The place utility is assumed to be related to a set of environmental attributes such as air quality, noise level, crime rate, accessibility to goods and services, etc. Depending on the type of facility, he argues that the marginal place utility function u_{ij} of a residential location i to the new facility located at j is a function of the Euclidean distance e_{ij} and takes four basic forms as shown in Figure 2.4. The functional form of (a) is suitable for site-noxious facilities, (b) and (c) are for site-preferred facilities, and (d) is for site-neutral facilities. The site-noxious facilities correspond to the "partially undesirable" facilities, whose functional form may be composed of two components, shown in Figure 2.5. The first component is the undesirability effect due to some environmental attributes. It has a negative marginal utility which eventually approaches zero since the facility ceases to be undesirable past a certain distance. The second component is the positive effect in other attributes which remain reasonably constant with distance. By the use of utility functions, the model can take account the undesirable features of the new facilities. Like Mumphrey et al., Austin assumes that the possible locations for new facilities are a known finite set. After estimating all the marginal utilities, the

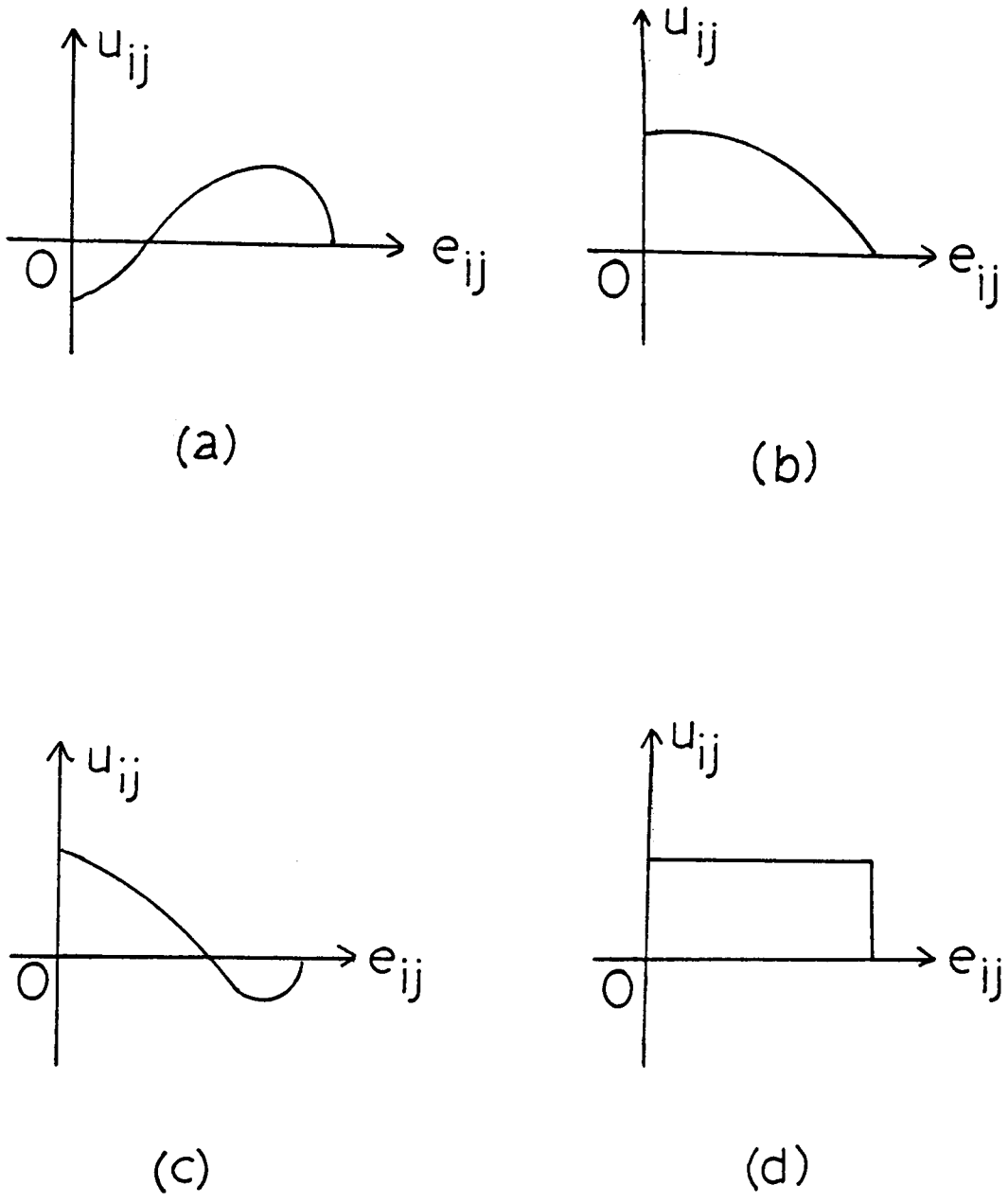


Figure 2.4. Marginal place utility functions.

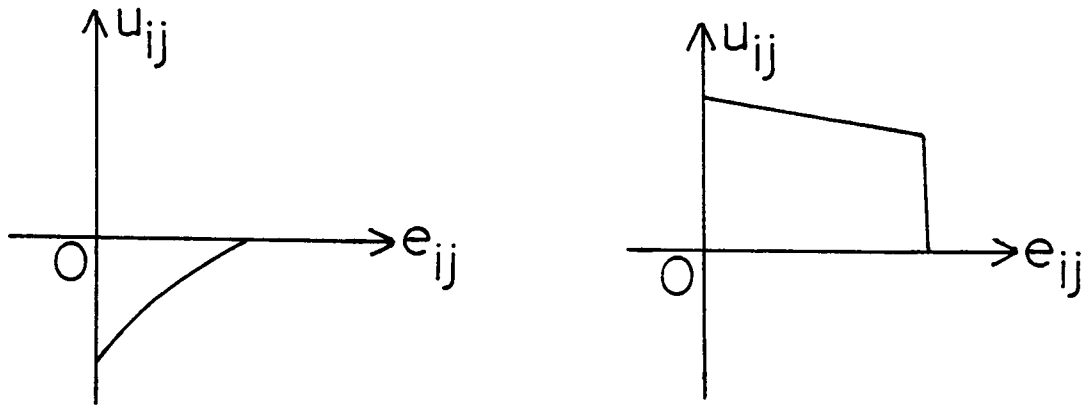


Figure 2.5. Components of a marginal place utility function.

optimal solution is then obtained with the maximum total utility. However, public utility is very difficult to obtain.

A new mathematical programming model will be introduced for the partially undesirable facility location problem. The model differs in two ways from the two approaches mentioned above. The first is the assumption of solution space. Both Mumphrey et al. and Austin assume the possible location of new facilities to be a known finite discrete space. In contrast, this research will study the location problem with a network solution space where the new facilities can be located anywhere on the network. The second characteristic of the new model is the distance measurement employed. Partially undesirable facilities like airports, factories, dumps, and nuclear power plants, provide services

and are necessities. Due to noise, odor, and safety considerations, however, proximity to these facilities is found to be objectionable by their potential users or the general public. In other words, partially undesirable facilities have both desirable and undesirable features. Because the facility is desirable, it has to be accessible, and one gets to it along the road network. Thus, the measure of accessibility or desirability is a measure of shortest path distance on the road network. A facility is undesirable or objectionable in the sense that either people can see it, or hear it, or smell it. But neither the sight, nor the sound, nor the odor travels along the road network. The undesirability is measured with a different metric and it is assumed to be the Euclidean metric. In short, the model will employ both Euclidean distance and network shortest path distance measurements to take account of the objectionable and desirable features of this class of facilities.

As a simple example, consider the network of Figure 2.6 from the viewpoint of the population at node 3. Location at y of a partially undesirable facility (e.g., a dump) would be preferred by residents at node 3 to a location at x. Although y is farther away from node 3 by Euclidean distance, it is closer to node 3 by road distance than x. The consideration of both Euclidean distance and network distance will

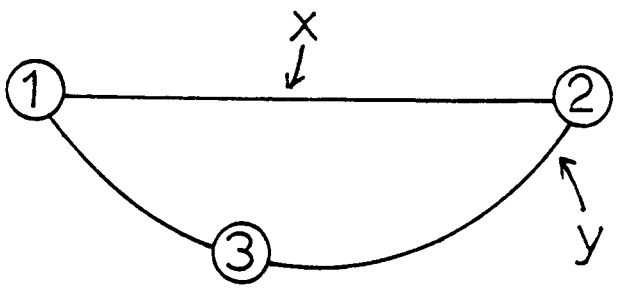


Figure 2.6. An example of a three node network.

naturally increase the complexity of the model. For instance, it is no longer valid to simply add the two functions in Figure 2.5 to produce Figure 2.4(a). In practical location problems, however, it is realistic to assume that both Euclidean distance and network distance are relevant yet quite dissimilar. This consideration motivates the investigation of a more complicated but realistic model.

Define the following:

e_{i,x_j} = Euclidean distance from node i to facility j.

$e_{i,X} = \min_{k=1,\ldots,p} e_{i,x_k}$ where $X = (x_1, x_2, \ldots, x_p)$.

$g_i(x_j) = g_i(s_{i,x_j})$ = accessibility of facility j to node i.

$f_i(x_j) = f_i(e_{i,x_j})$ = undesirability of facility j to node i.

$G_i(X) = G_i(g_i(x_1), \ldots, g_i(x_p))$ = total accessibility to node i of the facility configuration X.

$F_i(X) = F_i(f_i(x_1), \ldots, f_i(x_p))$ = total undesirability to node i of the facility configuration X.

$G(X) = G(G_1(X),\ldots,G_{|N|}(X))$ = total undesirability to all nodes of configuration X.

$F(X) = F(F_1(X),\ldots,F_{|N|}(X))$ = total undesirability to all nodes of configuration X.

The F and G functions serve as fundamental building blocks for the partially undesirable facility location models. No specific functional forms are assigned and these functions have the total flexibility to fit real world problems. Thus, different forms of functions F and G can be employed to model specific situations of various applications. For instance, the accessibility function G can take the p-median or p-center functional form, and, ignoring F one obtains the most common facility location problems on a network. It can also cover the strictly undesirable facility location problem on a network which will be examined in the next section. The function F may take a nonlinear form since the effect of undesirability is likely to be very powerful in its immediate vicinity, but may be negligible past a certain distance. The function F may also be influenced by such factors as terrain, environmental regulations, seismic activity, and population density.

After the proper forms of the F and G functions have been determined, the next step in building the model is to decide on the objective functional form. Two methods can be adopted for this purpose. One way is to employ a

multiobjective approach, where both F(X) and G(X) are included in the objective function. In this case, a proper relationship (i.e., proper weighting) between F(X) and G(X) has to be decided. Since different metrics are used in these two functions, the determination of the relationship is not an easy task. In addition, the objectives F(X) and G(X) are generally in conflict and there will not exist a set of locations which both minimizes F(X) and maximizes G(X). Since the two objectives will conflict rather directly, one would expect the set of efficient (noninferior or non-dominated) solutions to be quite large. Another approach is the single objective function approach, where one of F(X) or G(X) is treated as the objective function and the other as a constraint. Under this approach, the effect of F(X) and G(X) can be considered separately, and thus a direct assessment of the relationship between the two functions is avoided. The single objective function approach is used in this dissertation. Chapter 3 will present two examples of this approach. One example employs G(X) as the objective function, while F(X) is used as the constraint. The other employs F(X) as the objective function and G(X) as the constraint.

2.3 Strictly Undesirable Facility Location Problems

Cornuejols et al. (1977) consider the problem of optimal location of bank accounts with an objective of maximizing

the total check clearing time. In this case, one would like to locate the bank accounts as far away from the check receiving clients as possible. This is a strictly undesirable facility location problem with a median-type objective function. Since the possible location of bank accounts is limited to a finite number of cities, this problem can be formulated as a variation of the 0-1 integer programming problem introduced in (2.2) through (2.6) with a maximization objective function. All of the solution techniques mentioned earlier for problems (2.2) to (2.6) can be used to solve this problem. The authors employ the "greedy assignment heuristic" plus a "vertex substitution heuristic" to obtain a lower bound on the optimal solution and come out with favorable results. They also employ the Lagrangian relaxation techniques to obtain an upper bound on the optimal solution.

Nemhauser and Sheu (1978) consider the problem of locating k noxious facilities which are to be placed as far away from other facilities as possible. They use a max-min objective function and this is a strictly undesirable facility location problem with a center-type objective function. The possible location of the noxious facilities is limited to the node set. The authors then apply a general bottleneck optimization technique developed by Edmonds and Fulkerson (1970) to obtain the optimal solution.

Although the above two papers are presented with a network structure, limiting the new facilities to the node set essentially enables one to ignore the network structure and makes the problems little different from ordinary facility location problems. It is obvious that the limitation to the node set as the solution space would not be suitable for problems such as locating p new measuring devices where the objective is to locate far from the nodes (the existing q devices). This dissertation studies strictly undesirable facility location problems on a network with both median-type and center-type formulations without the limitation of location to the node set. It will be shown that certain results of ordinary location problems on a network no longer hold for the strictly undesirable cases.

In this sense, the only paper which deals with strictly undesirable facility location problems on a network is by Church and Garfinkel (1978). They consider the problem of locating a point on a network so as to maximize the sum of its weighted shortest path distances to the nodes. For this problem, Hakimi's principle no longer holds and location cannot be restricted to the node set. As they have shown, however, there exists an optimal solution for this problem which is in the set of "arc bottleneck points" plus the set of "dangling nodes." (A dangling node is a node which is incident to only one arc.) They also provide an algorithm

which iteratively computes the best point for every arc to find the optimal solution. Since the problem requires that the new facilities be as far away from the nodes as possible, it is not surprising that optimal location on a node is very unusual. In fact, it happens only in the case where the node is a dangling node.

The location of p new facilities for the above problem will be investigated in Chapter 4. The location of both single facility and multiple facilities for the strictly undesirable facility problem on a network with a center-type objective function will be studied in Chapter 5.

CHAPTER 3

PARTIALLY UNDESIRABLE FACILITY LOCATION PROBLEMS ON A NETWORK

This chapter deals with partially undesirable facility location problems on a connected, undirected, simple network $G = (N,A)$, where the whole network is considered as the solution space. As pointed out in Section 2.2, partially undesirable facilities like airports, dumps, and nuclear power plants are undesirable to be close to their potential users or the general public, but they are also necessities and cannot be totally inaccessible. The general model developed in that section introduces the accessibility function $G(X)$, which is a function of the network shortest path distances, as well as the undesirability function $F(X)$, a function of Euclidean distances. The functions $F(X)$ and $G(X)$ may assume various forms depending on the particular circumstances.

In this chapter, two examples of possible realizations of the model are presented and solution techniques for them are proposed. Both examples use the single objective function approach, where either $F(X)$ or $G(X)$ is considered as the objective function. In the first example, $F(X)$ is used as the objective function which takes a max-min functional form, while $G(X)$ is treated as a constraint such that the

facility has to be located on a network to insure accessibility. The second example uses G(X) as the objective function which takes the p-median functional form, while F(X) is treated as the constraints to specify a minimum Euclidean distance which is required for the new facilities to be away from the general public.

Since both network shortest path distance and Euclidean distance are considered in the model, it is important that these two distances between any two points on the network can be obtained. In order to compute the Euclidean distance, the actual physical locations or the coordinates on the plane of the points is required. In other words, the actual configuration or the actual map of the road network is needed so that proper Euclidean distances can be determined. This is a fundamental distinction from ordinary facility location problems on the network, where only the structure, not the actual configuration, of the network is relevant.

The requirement of working with the actual map of the road network motivates application of the interactive computer graphics technique as the solution method. Instead of working with maps on paper, the road map is inputed, stored, and displayed on a graphics terminal screen ready for users to work on. This approach also has the advantage of utilizing the computer to obtain the required Euclidean distances and network shortest path distances.

Interactive computer graphics has been employed in previous mathematical programming models. Schneider (1972) uses interactive computer graphics to aid the planning of transportation terminal systems in urban regions. Brady and Rosenthal (1980) employ this technique to solve min-max location problems on the plane. In these applications, the capability of user/computer interactions in the solution process is greatly emphasized. In the case of "partially undesirable facility" location problems, however, the necessity of working with the actual road map by itself is a strong motivation for the use of the computer graphics technique.

The computer graphics devices in The University of Tennessee, Knoxville, which are employed in this study include the Tektronix 4015 graphics terminal, the graphics input device "TABLET," and the hard copy machine. The graphic software package PLOT-10, which is a package of FORTRAN subroutines to perform basic graphics functions, such as "move" and "draw," is incorporated in the establishment of the interactive computer programs.

3.1 Max-Min Euclidean Distance with Accessibility Constraints

This section considers the simple problem of locating a single, new, partially undesirable facility on a connected, undirected, simple network G = (N,A). The function F(X) is

used as the objective function and takes the max-min functional form. This problem is formulated as:

$$\max_{x \in G} \min_{i \in N} w_i e_{ix} \tag{3.1}$$

where w_i are positive weights. The location on the network is sought to maximize its minimum weighted Euclidean distance to any node. This is a simple and reasonable formulation for locating partially undesirable facilities. Suppose a new dump site that serves several communities (nodes) has to be determined. It is reasonable to consider its location along the road network with an objective of maximizing its minimum Euclidean distance to any of those communities. The constraint $x \varepsilon G$ ensures the accessibility of the new facility through the road network. Without this constraint, the optimal solution for this problem would be simply to locate the new facility as far as possible from the nodes.

Although (3.1) has not been considered in the mathematical programming literature, its counterpart, min-max location problems on the plane, has been thoroughly studied. [For example, Elzinga and Hearn (1972), Brady and Rosenthal (1980)]. The formulation is given below:

$$\min \max w_i e_{a_i x} \tag{3.2}$$

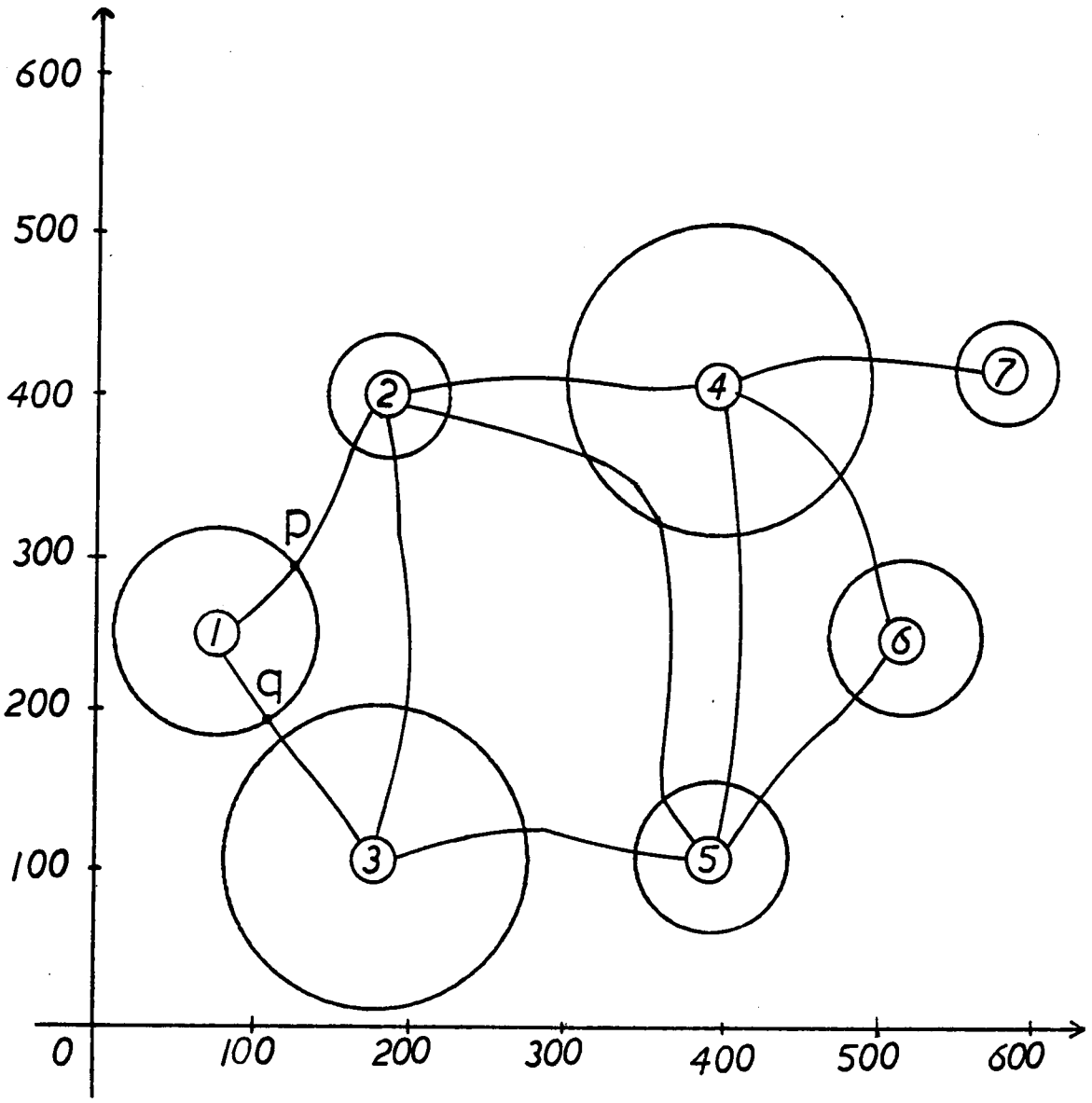
where a_i are the location of the existing facilities on the plane and x can be any point on the plane.

A geometric algorithm similar to those developed for (3.2) will be provided which finds the optimal solution for (3.1). Define the "set covered with radius r" as follows:

$$C(r) = \bigcup_{i \in N} C_i(r) \text{ and } C_i(r) = \{y \mid y \in G, \text{ and } w_i e_{iy} \leqslant r\} \tag{3.3}$$

where r is nonnegative. The elements of $C_i(r)$ are those points on the network whose Euclidean distances to node i are less than or equal to r/w_i. Similarly, the elements of C(r) are those points on the network whose weighted Euclidean distances to at least one of the nodes are less than or equal to r. As an example, the set $C_i(100)$ for each node i is indicated on the road map of Figure 3.1. It can be seen that $C_1(100)$ includes the arc segments $\overline{1p}$ and $\overline{1q}$.

For r = 0, C(r) = N. As r increases, C(r) includes more and more points on the network until finally C(r) = G. From (3.1), it follows that if $C(r) \subset G$, any point $x \in C(r)$ has objective function value less than or equal to r, and any point $y \in G - C(r)$ has objective function value greater than r. It follows that the optimal solution for (3.1) is achieved with the minimum value r* such that C(r*) = G. This leads to the following algorithm.



$w_1 = 1.5$ $w_5 = 2$
$w_2 = 2.5$ $w_6 = 2$
$w_3 = 1$ $w_7 = 3$
$w_4 = 1$

Figure 3.1. An example of a road map.

Algorithm 3.1

Step 1. Let lower bound q = 0, and upper bound u be a large number such that C(u) = G. Go to Step 2.

Step 2. Let $r = \frac{1}{2}(u + q)$. Go to Step 3.

Step 3. If u - q is within a prescribed tolerance, go to Step 6. Otherwise, go to Step 4.

Step 4. Draw a circle with radius r/w_i around each node i. Go to Step 5.

Step 5. Decide whether C(r) = G or not. If C(r) = G, let u = r; otherwise, let q = r. Go to Step 2.

Step 6. The optimal location of the new facility is the last covered point $x^* \varepsilon G$ with an objective function value $r^* = r$.

From Step 2 and Step 5, it can be seen that this is a binary search algorithm. At each iteration, the current search region (u - q) is divided into two equal subregions, one of which contains the optimal solution r*. Since there is no analytical method to represent the physical locations of the points on the network, a human decision is required in Step 5 to decide which is the subregion containing the optimal solution r*. The iteration continues until the current search region is smaller than a prescribed precision requirement. Binary search is known to be very efficient and converges within $o(\log_2 n)$ iterations. For instance, for a problem with starting upper bound 5,000 miles and

prescribed tolerance level one mile, Algorithm 3.1 achieves the optimal solution in 13 iterations.

At each iteration, Step 4 involves the drawing of $|N|$ circles around the nodes with various radii r/w_i. To do this manually could be time consuming. In addition, the quality of the drawing would be dependent on the skill of the researcher. This motivates the application of computer graphics to this problem where the road map is inputed and stored in a computer disk file. This road map information can be called out and displayed on the graphics terminal screen whenever it is desired. The drawing of circles around the nodes with different values can also be easily done by the computer.

Input of the Road Map

The pictorial display on the graphics terminal screen is composed of dots. The screen of the Tektronix 4015 graphics terminal has 4096 dots in either axis. In other words, there are $(4096)^2$ addressable points on the screen. The graphics input device "TABLET" is a large plate which is attached to the terminal. This plate has the same number of addressable points corresponding to the points on the screen. The road map can be put on the plate and a magnetic pen can be used to trace points on the map. The relative coordinates in x-axis and y-axis of these points can be

recorded and stored for later use. Even for a relatively small network, using the complete road map requires a tremendous amount of computer space to store the coordinates of all the points on the network. The computer time spent in retrieving and processing these detailed coordinates information could also be significant. Thus, a digitized network representation of the actual road map will be employed.

The map will be represented by a limited number of predetermined points on the arcs. In other words, the actual road map is replaced by its linearized approximation. Figure 3.2 is an example of the digitized road network for the actual road map in Figure 3.1. The precision of this digitized approximation depends on the number of grid points on the arcs; there is a trade-off between precision and processing costs. Table 3.1 lists part of the coordinate information stored in the disk data file for the map displayed in Figure 3.2. It indicates the number of grid points on each arc and the x-axis and y-axis coordinates for these points. The task of inputing the digitized network can also be accomplished without using the "TABLET." The coordinates of the grid points can be measured and inputed directly into the disk file such as the one shown in Table 3.1.

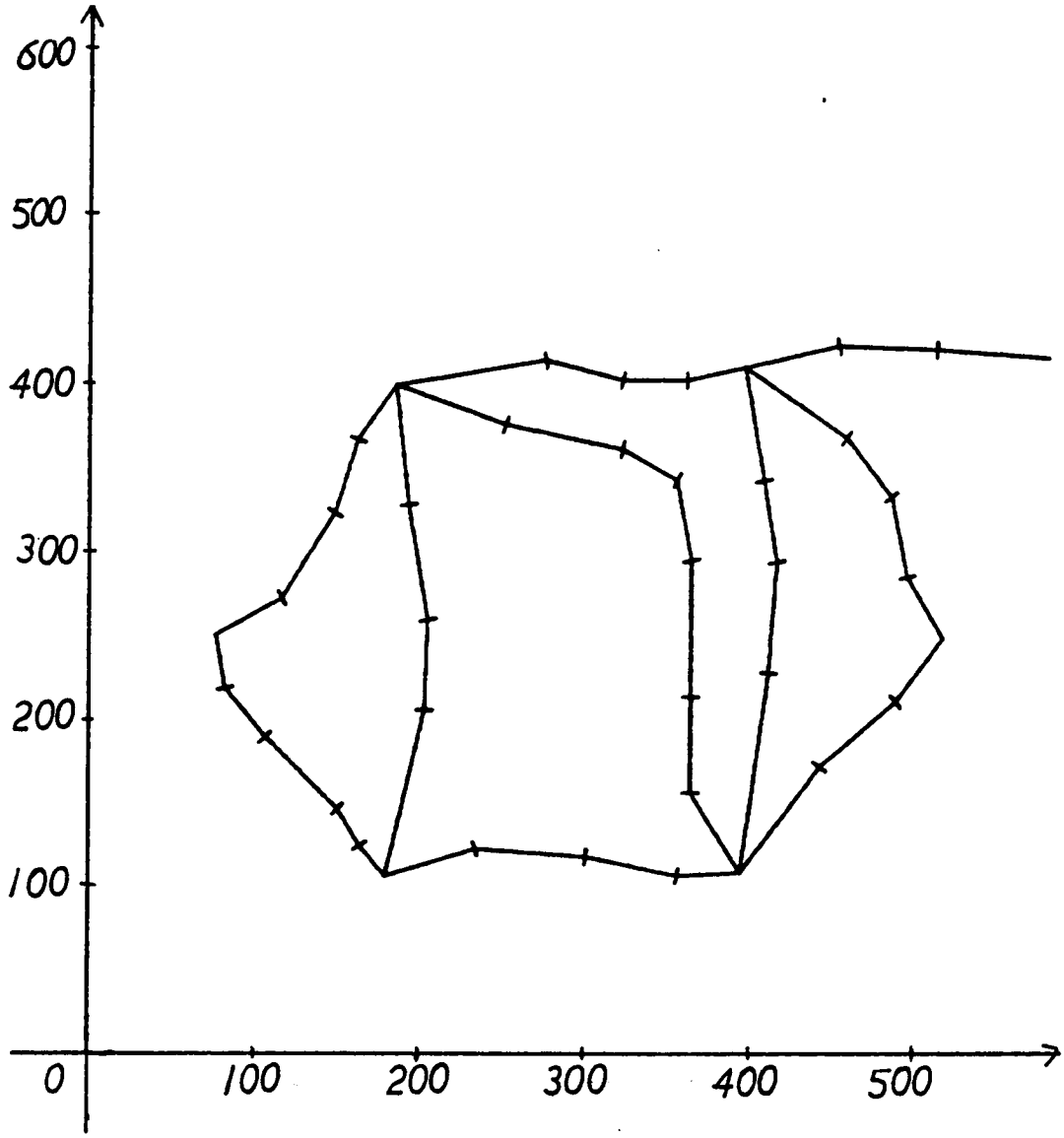


Figure 3.2. An example of a digitized road map.

Table 3.1. Data file for Figure 3.2

1[a]	2[a]	3[b]
78[c]	258[d]	
119	280	
152	332	
166	377	
190	410	
1	3	4
78	258	
84	226	
109	197	
154	153	
168	130	
184	111	
2	3	3

[a]End point of the arc.
[b]Number of grid points on the arc.
[c]X-axis coordinate.
[d]Y-axis coordinate.

An interactive computer graphics program to solve problem (3.1) was developed and run on the DEC-10 computer. Upon execution of the program, it gives the user the following options: (1) whether or not to input the digitized road network through "TABLET"; (2) to change the weights assigned to the nodes; and (3) to provide initial upper and lower bound values. The program then proceeds to perform Algorithm 3.1 and automatically draws the circles around the nodes at each iteration. The user interacts with the

computer in Step 5, where he has to check and decide whether C(r) = G and to issue commands for the program to carry on the iterations of the algorithm. Figure 3.3 shows the results of the final iteration for this example. As the figure indicates, the optimal solution for (3.1) for the network in Figure 3.2 is at point t with an objective function value of 210.5.

The main advantage of the computer graphics approach is its efficiency. For a complicated network, it is true that the digitization and inputing of the network could be fairly time consuming. Once the network is digitized and inputed, however, the program is able to efficiently execute Algorithm 3.1. The graph of the network and circles are drawn within seconds, after which the program is ready for the user to determine whether C(r) = G. Since the digitized network instead of the actual network is used, the quality of the solution cannot be guaranteed. However, this shortcoming can be overcome. Essentially, the program can be used to obtain an approximation of r* as the first stage of finding the optimal solution for (3.1). The approximate value can be incorporated with the actual road map to obtain the optimal solution r*. The computer graphics approach is especially valuable for the situation where solutions for different set of weights for nodes are required.

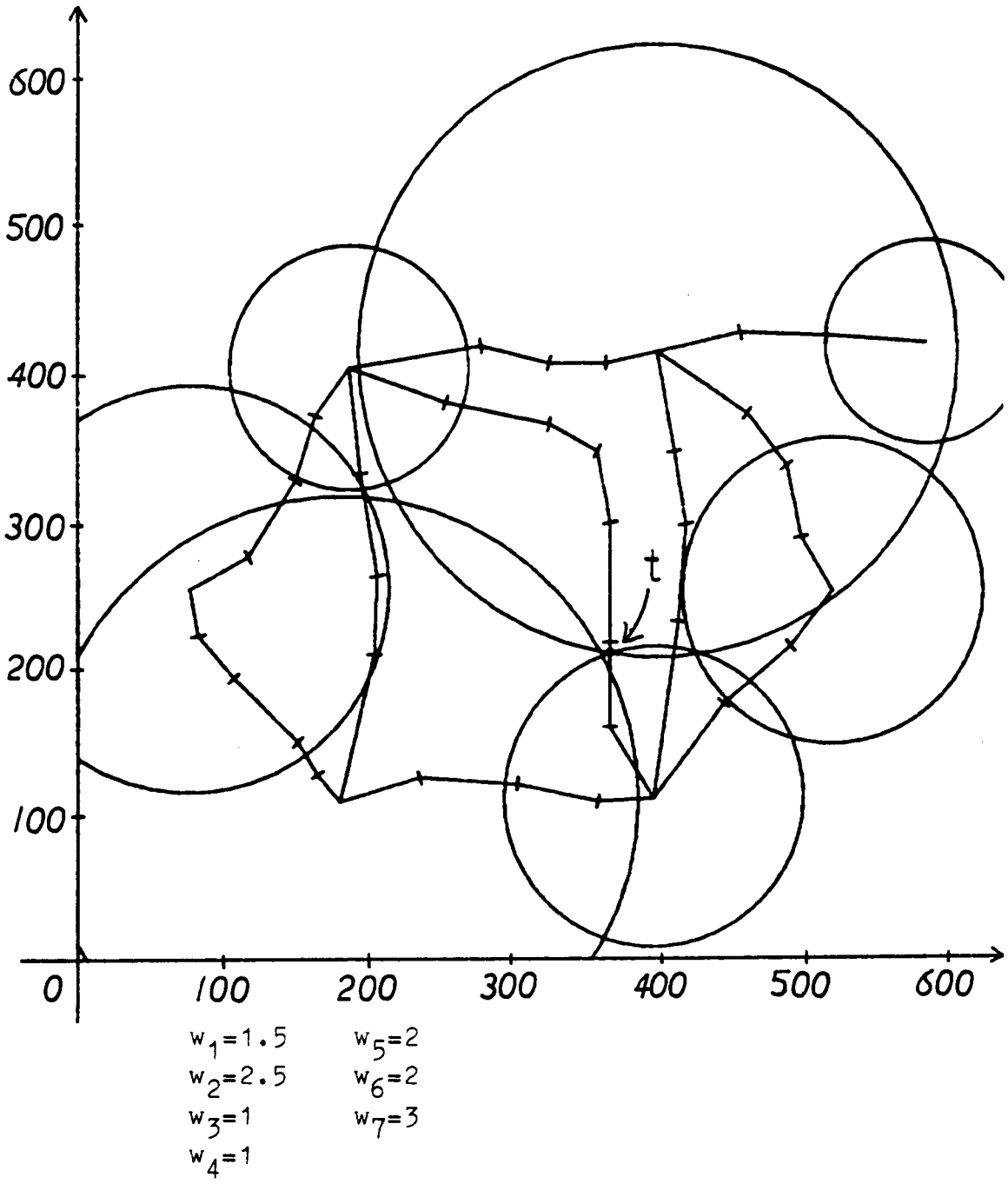


Figure 3.3. The solution for an example of (3.1).

3.2 P-Median Problem with Forbidden Circles around Nodes

This section deals with another realization of the general model for the partially undesirable facility location problem on the network. The problem is to locate p new facilities where G(X) is used as the objective function and F(X) is used as the constraints. This problem is formulated as follows:

$$\min \sum_{i \in N} w_i s_{iX} \tag{3.4}$$

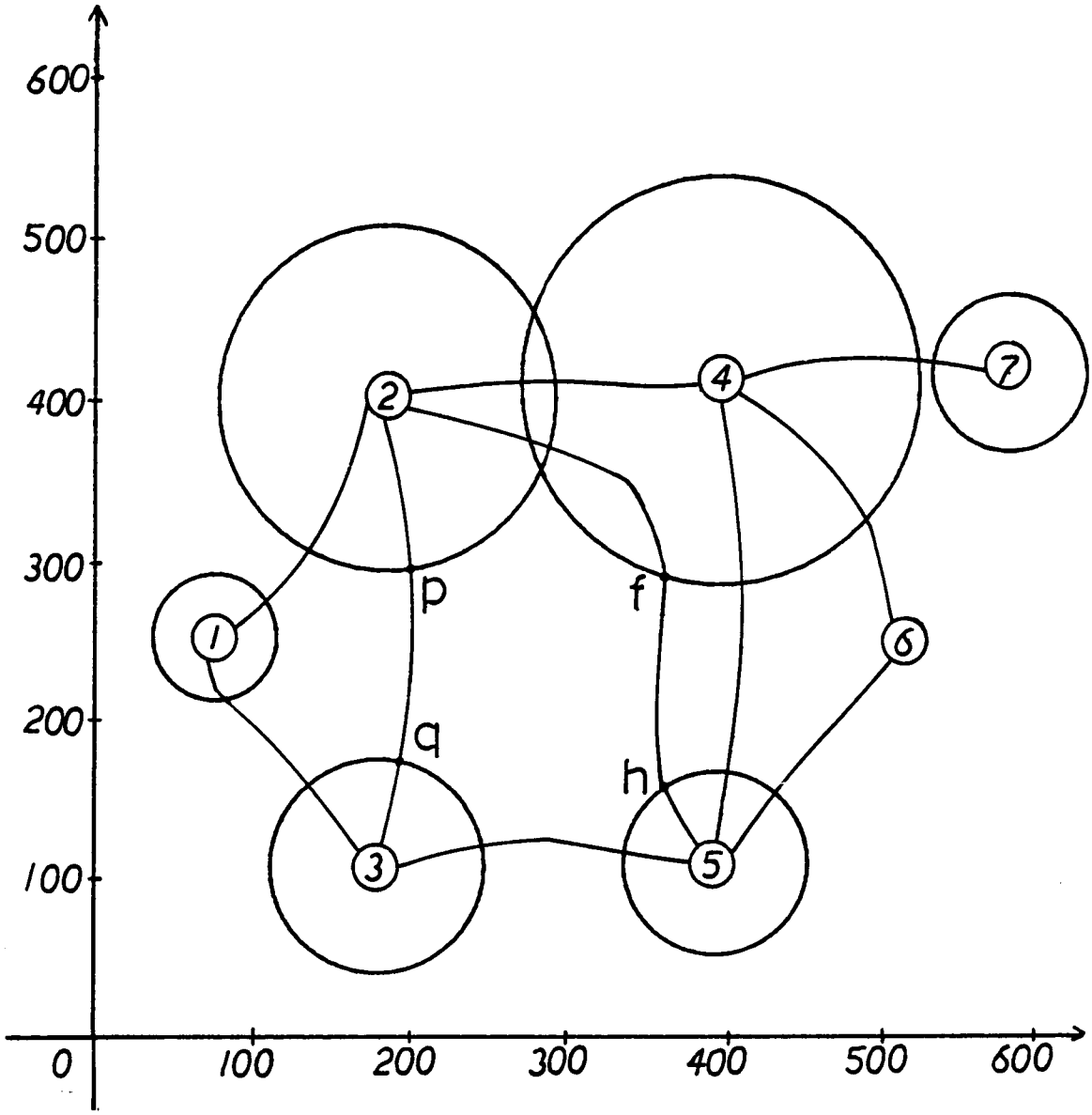
$$\text{s.t. } e_{ix_j} \geqslant b_i \qquad i \in N,\ j=1,\ldots,p \tag{3.5}$$

The objective function (3.4) is the same as that of the p-median problem. The constraints (3.5) state that no new facility can be placed "closer" to node i than b_i in terms of Euclidean distance. This formulation is suitable for locating partially undesirable facilities, such as airports, or bus terminals, where the accessibility of their potential customers is maximized, subject to the prescribed Euclidean distances. A similar problem on the plane has been considered by Orloff (1977). The b_i's are the specified minimum tolerable distances for various communities and may have different values according to such factors as political power, terrain, and people's perception. If the p new

facilities are not similar, then different b_{ij} value could be assigned for each of the i, j (node-facility) pairs. Figure 3.4 is an example of this problem with certain specified values of b_i.

Since new facilities are not allowed to be located inside the circles around the nodes, this problem can be called "the p-median problem with forbidden circles." From (3.5), it follows that the feasible region consists of those arc segments outside of the forbidden circles. It will be shown in Theorem 3.1 that an optimal solution can be found in the set of intersection points of the circles and the network.

Let $n_i(i,j)$ denote the nearest feasible intersection point on arc(i,j) with respect to its endpoint i. As an example, in Figure 3.4, $p = n_2(2,3)$, $q = n_3(2,3)$, $f = n_2(2,5)$, $h = n_5(2,5)$, and $6 = n_6(4,6) = n_6(5,6)$. There is no nearest feasible intersection point on arcs (2,4) and (4,7) since the entire arcs are inside the forbidden circles. Note that by definition, the nearest feasible intersection point is unique if it exists (since it is the nearest one). Note also that by the continuity of the arc, there will exist a nearest feasible intersection point on an arc if a portion of the arc is feasible (outside forbidden circles). Let I denote the set of all the nearest feasible intersection points. Since there are at most two such points on any arc, $|I| \leq 2|A|$.



$b_1=40$ $b_5=60$
$b_2=110$ $b_6=0$
$b_3=70$ $b_7=50$
$b_4=130$

Figure 3.4. An example of a p-median problem with forbidden circles.

Theorem 3.1. An optimal solution $(x_1^*, x_2^*, \ldots, x_p^*)$ can be found for (3.4) and (3.5) where $x_i^* \in I$ for $i = 1, \ldots, p$.

Proof: Let $X = (x_1, x_2, \ldots, x_p)$ be an optimal solution to (3.4) and (3.5). Assume $x_k \notin I$ and x_k is an interior point of arc (q,m). It follows that both $n_q(q,m)$ and $n_m(q,m)$ exist. Let $k_1 = n_q(q,m)$ and $k_2 = n_m(q,m)$. It follows that x_k is on the arc segment $\overline{k_1 k_2}$. Let N_k be the set of nodes which has x_k as their nearest new facility. In other words, the nodes in set N_k will obtain services from facility x_k. Define

$$N_k(k_1) = \{i \mid i \in N_k,\ s_{ik_1} \leqslant s_{ik_2}\}$$

and

$$N_k(k_2) = \{i \mid i \in N_k,\ s_{ik_1} > s_{ik_2}\} . \tag{3.6}$$

Without loss of generality, assume that

$$d(x_k) = \sum_{i \in N_k(k_1)} w_i - \sum_{i \in N_k(k_2)} w_i \geqslant 0 . \tag{3.7}$$

It follows that

$$\left[\sum_{i \in N_k} w_i s_{ix_k} \right] - \left[\sum_{i \in N_k} w_i s_{ik_1} \right]$$

$$= \left[\sum_{i \in N_k(k_1)} w_i s_{ik_1} + \sum_{i \in N_k(k_1)} w_i s_{k_1 x_k} \right.$$

$$\left. + \sum_{i \in N_k(k_2)} w_i s_{ik_2} + \sum_{i \in N_k(k_2)} w_i s_{k_2 x_k} \right]$$

$$- \left[\sum_{i \in N_k(k_1)} w_i s_{ik_1} + \sum_{i \in N_k(k_2)} w_i s_{ik_2} + \sum_{i \in N_k(k_2)} w_i s_{k_1 k_2} \right]$$

$$= \left[\sum_{i \in N_k(k_1)} w_i s_{k_1 x_k} + \sum_{i \in N_k(k_2)} w_i s_{k_2 x_k} \right] - \left[\sum_{i \in N_k(k_2)} w_i s_{k_1 k_2} \right]$$

$$= \left[\sum_{i \in N_k(k_1)} w_i s_{k_1 x_k} + \sum_{i \in N_k(k_2)} w_i s_{k_2 x_k} \right] - \left[\sum_{i \in N_k(k_2)} w_i s_{k_1 x_k} + \sum_{i \in N_k(k_2)} w_i s_{k_2 x_k} \right]$$

$$= \sum_{i \in N_k(k_1)} w_i s_{k_1 x_k} - \sum_{i \in N_k(k_2)} w_i s_{k_1 x_k}$$

$$= \left[\sum_{i \in N_k(k_1)} w_i - \sum_{i \in N_k(k_2)} w_i \right] s_{k_1 x_k}$$

$$= d(x_k) s_{k_1 x_k} . \qquad (3.8)$$

From (3.7) and (3.8),

$$\left[\sum_{i \in N_k} w_i s_{ix_k} \right] - \left[\sum_{i \in N_k} w_i s_{ik_1} \right] \geqslant 0 . \qquad (3.9)$$

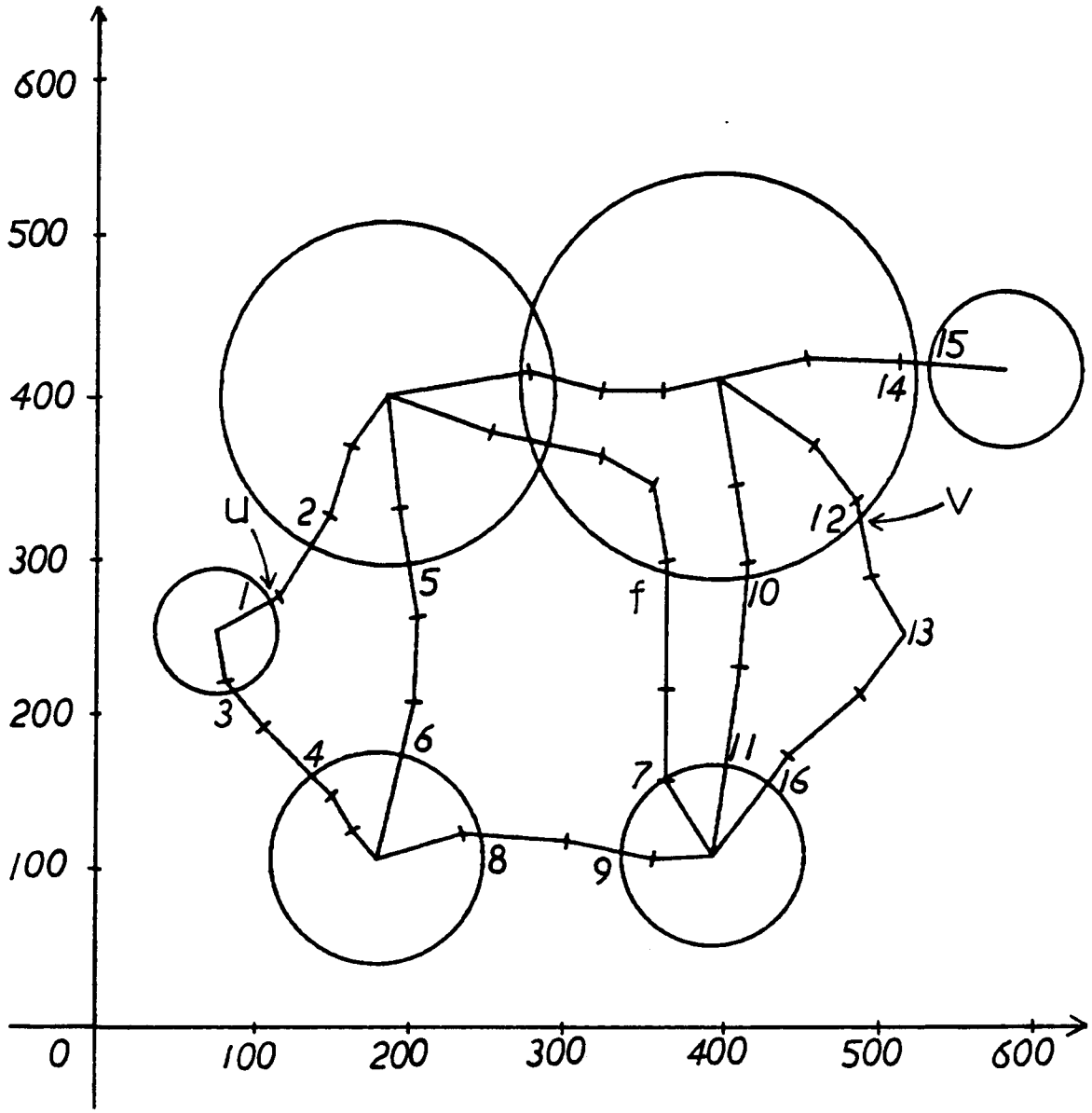
From (3.9), it follows that the location x_k can be replaced by the nearest feasible intersection point k_1 with a smaller objective function value. The proof of the theorem is complete.

From Theorem 3.1, it follows that once the finite set I is identified, (3.4) and (3.5) become the p-median problem. It can then be formulated as an integer program introduced in (2.3) through (2.7). All of the exact and heuristic techniques introduced there can be used to solve it. Another important implication of the theorem is that the optimal configuration of new facilities will be on the circles of minimum allowed Euclidean distances from the nodes. Hence, the b_i values are rather critical and it is relevant to solve and to compare for different set of b_i values.

With an actual road map and a set of b_i values, it is not difficult to identify the set I. It involves drawing circles around nodes on the road map and determining the nearest feasible intersection points on each arc. The network shortest path distances from points in I to the nodes can then be measured. Again, the possible application of interactive computer graphics technique to solve (3.4) and (3.5) is examined. The graphics approach is especially desirable when comparison of solutions for different sets of b_i values is required. In this case, the repeated work of drawing circles and computing shortest path distances can be easily accomplished by the computer.

An interactive computer graphics program was developed and run at The University of Tennessee, Knoxville, to solve the p-median problem with forbidden circles around nodes. Upon executing the program, the computer gives the user the options of whether or not to input the digitized road network through "TABLET," and whether or not to provide new values of b_i for the nodes. It then proceeds to identify the set I as the potential locations of the new facilities. The program traces through each arc and checks whether it has intersection points with the two circles corresponding to its two end points. If an intersection exists, the point needs to be further examined as to whether it belongs to I. After all the arcs have been checked, the network itself, the circles, and the current set of I are displayed on the graphics terminal screen. As Figure 3.5 indicates, 15 elements in I have been identified. At this stage, the user is requested to identify points in I that are left out in the above procedure, such as point f in Figure 3.5.

The main purpose for using the user-computer interaction is to reduce computer processing time. For instance, to be able to mathematically identify the point f in Figure 3.5 requires the examination of all the forbidden circles for possible intersection points in tracing all the arcs. This involves a substantial amount of computation. Human eyes, however, can easily identify whether the point f is feasible or not.



b_1=40 b_5=60
b_2=110 b_6=0
b_3=70 b_7=50
b_4=130

Figure 3.5. Elements of the set I.

After the finite set I has been identified, the remaining problem is to choose p points from I which will minimize the objective function (3.4). This is a p-median problem. The greedy assignment plus the vertex substitution heuristic method are employed and coded to solve this problem. This approach has been used by Cornuejols et al. (1977) who report favorable results. The heuristic solution for the $p = 2$ and $w_i = 1$ case is to locate the two new facilities at points u and v, as shown in Figure 3.5, with an objective function value of 1227.1. After a heuristic solution is obtained, the program will ask the user whether he would like to solve the problem with a different set of b_i values. If the answer is yes, the program will start again from the beginning, and the whole process can be repeated as many times as the user desires. Also, the user can make hard copies of the results from different b_i values through the hard copy machine for further comparison.

The interactive computer graphics program can efficiently achieve the task of identifying the set I and computing the approximate shortest path distances (since digitized network is used) between the elements of set I to nodes and between the elements themselves, as well as the task of finding a heuristic solution for (3.4) and (3.5). Since it is not difficult to identify the set I on the actual road map with a set of specific b_i values, it may not be desirable to adopt

the graphics approach if problems (3.4) and (3.5) are required to be solved with only one set of b_i values. In the planning stage of the location problem, however, a comparison of the solutions of (3.4) and (3.5) for a number of different b_i values is usually required. Therefore, it is advantageous to digitize and store the network in a disk data file and utilize the computational capability of the computer.

CHAPTER 4

STRICTLY UNDESIRABLE FACILITY LOCATION WITH A MEDIAN-TYPE OBJECTIVE FUNCTION

This chapter considers the problem of locating strictly undesirable facilities on a network. The entire network is employed as the solution space and the objective is to maximize the sum of the weighted network shortest path distances between the new and existing facilities.

4.1 Single New Facility

Church and Garfinkel (1978) consider the problem of locating a single facility on a network such that the sum of its weighted shortest path distances to all nodes is maximized, i.e.,

$$\max \sum_{i \in N} w_i s_{ix} . \tag{4.1}$$

This is a median-type objective function except that it is a maximization problem. They illustrate this model by considering the problem of locating a house or business (new facility) which is to be far away from the crime pockets (existing facilities) in a city. The weight w_i reflects the relative degree of danger from each of the crime pockets. As pointed out in Section 2.3, this is the

only work on undesirable facility location problems on a network which does not constrain the possible new facility locations to the nodes. Note that this problem can be viewed as a special case of the model developed for "partially undesirable" facility location problems discussed in Section 2.2. In that section, the accessibility function G(X), which is a function of network shortest path distances, is introduced to measure the accessibility of the new facilities. The undesirability function F(X), which is a function of Euclidean distances, is used to measure the undesirability of the new facilities. In (4.1), the function G(X) takes a negative accessibility measure and no F(X) is considered.

The formulation (4.1) is also suitable for the problem of locating measuring devices on an inland waterway network mentioned in Section 2.3. In this case, the weight w_i may be assigned to be the same for each facility or may be determined by other considerations. For instance, if the devices lose their precision in time and need to be collected and adjusted periodically, then the weight w_i can reflect the time since installation. A smaller weight means that maintenance for this device is soon due and that placing a new device nearby as a replacement is not too objectionable. The weights could also reflect the relative importance of the locations of existing devices. A smaller weight may mean that information about the waterway near this spot is

more relevant than that of other parts of the network and that a new facility can be installed fairly close to insure the collection of such information.

There is an optimal solution to (4.1) in the finite set $D \cup B_A$, where D is the set of dangling nodes and B_A is the set of "arc bottleneck points" [Church & Garfinkel (1978)]. In other words, $D \cup B_A$ is a FOSS (finite optimal solution set) for (4.1). The search for FOSS's for related problems plays a large role in the remainder of the chapter.

The set B_A is defined by

$$B_A = \bigcup_{k \in N} B_A(k)$$

and

$$B_A(k) = \{y \mid y \text{ is interior to some arc } (i,j) \text{ and } s_{ki} + c_{iy} = s_{kj} + c_{jy}\} . \qquad (4.2)$$

In other words, an arc bottleneck point is a point on an arc which has equal shortest path distances to some node through either end point of the arc. From definition (4.2), arc (i,j) contains an element of $B_A(k)$ if and only if

$$|s_{ik} - s_{jk}| < c_{ij} . \qquad (4.3)$$

(4.3) is a useful condition in identifying elements of B_A. For instance, arc (1,2) in Figure 4.1 has three arc bottleneck points a, b, c that belong to $B_A(4)$, $B_A(5)$, and $B_A(3)$,

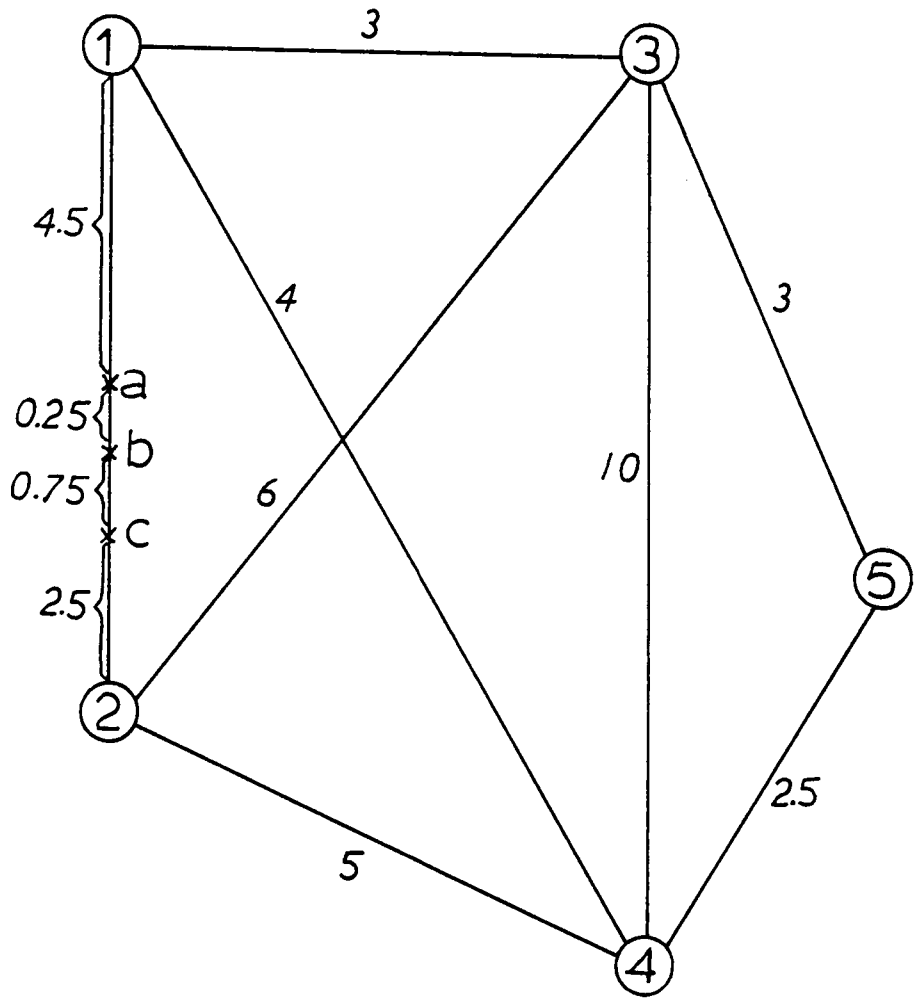


Figure 4.1. An example of arc bottleneck points.

respectively. There exists at most one arc bottleneck point on a given arc with respect to any arc-node pair so that the maximum total number of elements in B_A is $|N| \cdot |A|$.

The following is an example of locating one new measuring device on a waterway network with formulation (4.1). Using the network in Figure 4.1 and assuming $w_i = 1$ for all i, Figure 4.2 indicates the best points to locate the new device on each arc. The points are found by using the algorithm provided by Church and Garfinkel (1978). The results are summarized in Table 4.1. The optimal solution is to locate the new device at point f on arc (3,4) with an objective function value of 35.75.

4.2 Multiple New Facilities

In this section, the extension of problem (4.1) to locate p new strictly undesirable facilities on a network is studied. Specifically, using the measuring devices location problem as an example, the problem is to locate p new devices, where there already are q existing devices. The objective is to locate these p new devices away from the existing q devices as well as from each other in order to obtain the maximum amount of information from the whole network. This problem could be formulated with a median-type objective function as follows:

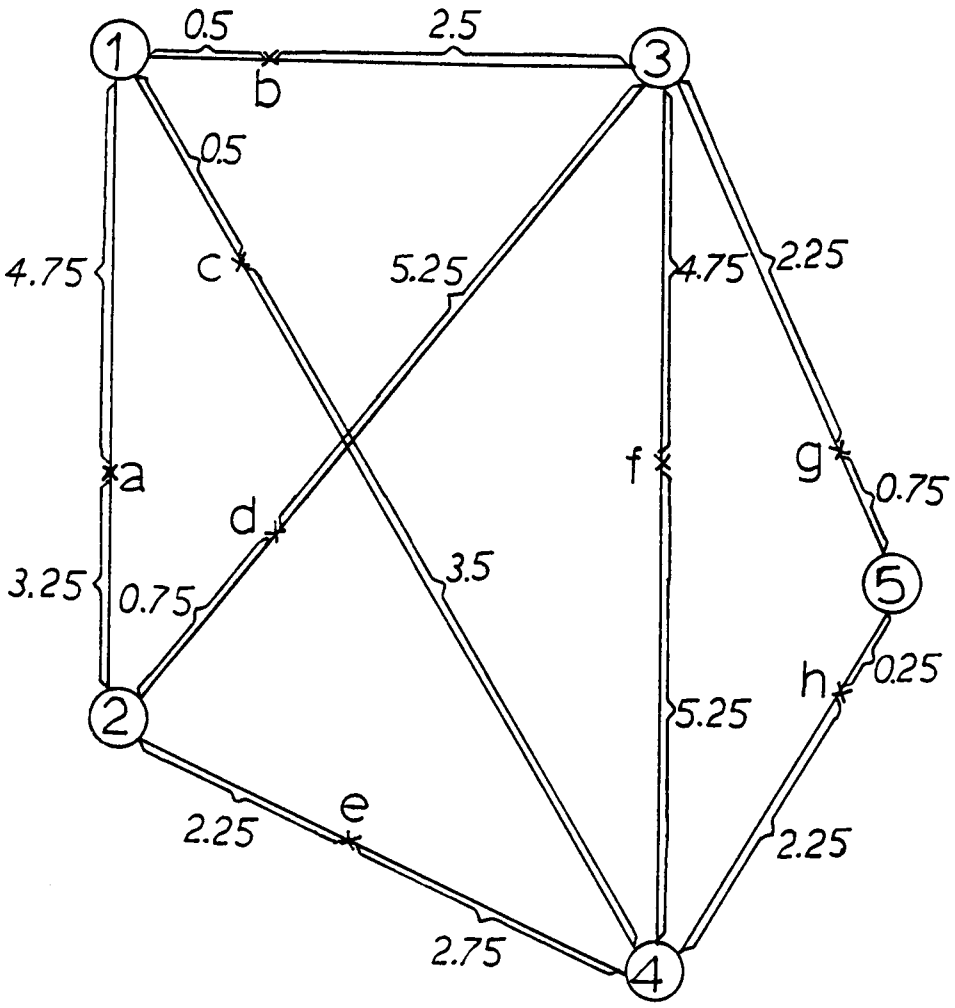


Figure 4.2. The best points on arcs.

Table 4.1. An example of the best points on arcs

Arcs	The Best Points on Arcs	Objective Function Values
(1,2)	$a \varepsilon B_A(5)$	34.75
(1,3)	$b \varepsilon B_A(2)$	21.50
(1,4)	$c \varepsilon B_A(2)$	22.00
(2,3)	$d \varepsilon B_A(5)$	28.25
(2,4)	$e \varepsilon B_A(3)$	28.75
(3,4)	$f \varepsilon B_A(5)$	35.75*
(3,5)	$g \varepsilon B_A(2)$	19.75
(4,5)	$h \varepsilon B_A(1)$	19.25

$$\max \left\{ \sum_{j=1}^{p} \sum_{i \varepsilon N} w_i s_{ix_j} + \sum_{1 \leq k < q \leq p} v s_{x_k x_q} \right\} . \qquad (4.4)$$

The w_i and v are nonnegative weights.

There are two components in (4.4). The first component measures the interactions between the existing and new facilities and the second measures the interactions between the new facilities. For ordinary facility location problems, it is reasonable to assume that service will be provided by, or obtained from, the nearest facility. This means that no new facility other than the nearest one has effect on the existing facility. Therefore, in the p-median problem, each

existing facility is assumed to interact only with its nearest new facility. For undesirable facilities, however, although the second nearest facility may have less influence than the nearest facility, the existing facilities are not only affected by their nearest new facilities, but by all the new facilities. For this reason, the existing facilities in (4.4) are assumed to interact with all new facilities. Problem (4.4) also assumes that all new facilities are similar and common weights w_i and v are applied to them. If the new facilities under consideration are not similar, different weights, w_{ij} and v_{kq}, should be assigned for each existing-new facility pair i, j, and for each new facility pair k, q, respectively.

If there is no interaction between new facilities, i.e., v = 0, problem (4.4) reduces to

$$\max \sum_{j=1}^{p} \sum_{i \in N} w_i s_{ix_j} . \qquad (4.5)$$

The optimal solution for (4.5) is to locate all p new facilities on the same optimal point as the p = 1 case. For instance, in the network in Figure 4.2, consider the p = 2 case. There is no reason to locate the two facilities anywhere but on the point f. However, identical location of all new facilities is not satisfactory for some cases, such as the measuring devices location problem. Theorem 4.1

provides an upper bound on v for new facilities to be located on the same point.

Theorem 4.1. If $v \geq \sum_{i \in N} w_i$ in (4.4), it is not optimal to locate the new facilities on the same point.

Proof: Let $f(x_1,x_1,\ldots,x_1)$ denote the objective function value of (4.4) for the location configuration $X = (x_1,x_1,\ldots,x_1)$ where all p new facilities are located on the same point x_1. Consider the location configuration $(x_1,x_2,\ldots,x_p)$ where $x_i \neq x_j$ for $1 \leq i < j \leq p$, and let $f(x_1,x_2,\ldots,x_p)$ denote its objective function value. Let

$$d = f(x_1,x_2,\ldots,x_p) - f(x_1,x_1,\ldots,x_1) .$$

It follows that

$$d = \left(\sum_{j=1}^{p} \sum_{i \in N} w_i s_{ix_j} + \sum_{1 \leq k < q \leq p} v s_{x_k x_q} \right) - \left(\sum_{j=1}^{p} \sum_{i \in N} w_i s_{ix_1} \right)$$

or

$$d = \sum_{k=1}^{p-1} \sum_{q=k+1}^{p} v s_{x_k x_q} - \sum_{j=2}^{p} \sum_{i \in N} w_i (s_{ix_1} - s_{ix_j}) . \quad (4.6)$$

From the triangle inequality, $s_{ix_1} \leq s_{ix_j} + s_{x_1 x_j}$. Thus, (4.6) becomes

$$d \geqslant \sum_{k=1}^{p-1} \sum_{q=k+1}^{p} vs_{x_k x_q} - \sum_{j=2}^{p} \sum_{i \in N} w_i s_{x_1 x_j} . \qquad (4.7)$$

Since $x_k \neq x_q$, $s_{x_k x_q} > 0$ and $\sum_{k=2}^{p-1} \sum_{q=k+1}^{p} vs_{x_k x_q} > 0$.

From (4.7), it follows that

$$d > \sum_{q=2}^{p} vs_{x_1 x_q} - \sum_{j=2}^{p} \sum_{i \in N} w_i s_{x_1 x_j}$$

or

$$d > \sum_{j=2}^{p} (v - \sum_{i \in N} w_i) s_{x_1 x_j} . \qquad (4.8)$$

From (4.8) and the assumption of the theorem,

$$d > 0 .$$

Since $f(x_1, x_2, \ldots, x_p) > f(x_1, x_1, \ldots, x_1)$, having a "spread out" configuration $(x_1, x_2, \ldots, x_p)$ is better than locating all new facilities on the same point x_1, and the proof is complete.

In order to be able to solve (4.4), it is essential that a FOSS be identified. The first component in (4.4) has FOSS given by DUB_A. The question remains whether there is a FOSS for the second (new facility interactions) component and then for (4.4) as a whole. The second component of (4.4) can be restated as follows:

$$\max \sum_{1 \leq k < q \leq p} s_{x_k x_q} . \tag{4.9}$$

This problem is to locate p points on a network such that the sum of the shortest path distances between them is maximized. Since it is an interesting problem by itself, a complete treatment of (4.9) will be presented in Chapter 6. It will be shown in that chapter that for the p = 2 case, problem (4.9) has FOSS NUB_A and that no apparent FOSS can be identified for the $p \geq 3$ case.

For p = 2, problem (4.4) becomes

$$\max \left\{ \sum_{i \in N} w_i s_{ix_1} + \sum_{i \in N} w_i s_{ix_2} + v s_{x_1 x_2} \right\} . \tag{4.10}$$

This problem is to locate two new facilities on the network and the objective is to maximize the sum of their weighted shortest path distances to all the existing facilities and between the new facilities themselves. The first and second terms are similar to (4.1) and have FOSS DUB_A. The third term is the "generalized diameter problem" which has FOSS NUB_A. A natural conjecture is that NUB_A also serves as a FOSS for (4.10). (Note that the dangling node set D is a subset of N.)

The following is a counterexample to that conjecture. This example employs the network in Figure 4.1 (page 55) and assumes $w_i = 1$ and v = 2. Examining this network for

the optimal solution, it is found that arcs (1,2) and (3,4) are the potential candidates for optimal location of the two new facilities. From Table 4.1, it is seen that arcs (1,2) and (3,4) contain the first and second best points on arcs. In addition, the generalized diameter of the same network is 13 and its end points lie on arcs (1,2) and (3,4). (See the figure on page 105.) Therefore, locating the two new facilities on any other arc will not be optimal. The optimal solution to (4.10) for this example is found to be on arcs (1,2) and (3,4). As shown in Figure 4.3, optimal solutions are (x_1^*, x_2^*) where x_1^* belongs to the arc segment $\overline{bb'}$ and x_2^* belongs to the arc segment $\overline{ff'}$. For any x_1^* on $\overline{bb'}$, its corresponding point x_2^* is the point on $\overline{ff'}$ with $s_{x_1^* x_2^*} = 13$. For instance, (b,f') and (b',f) are two optimal solutions with objective function value 96.

Using condition (4.3) to check the arc bottleneck points, it is found that arc (1,2) has arc bottleneck points a, b, c, and arc (3,4) has arc bottleneck points d, e, f, g, and h. These arc bottleneck points are also indicated in Figure 4.3. Table 4.2 lists all the shortest path distances of these arc bottleneck points to all the nodes and Table 4.3 gives the shortest path distances among the arc bottleneck points themselves. These two tables can be used as a basis to verify the optimality of the above solution. Table 4.2 also indicates that for any optimal solution (x_1^*, x_2^*), at

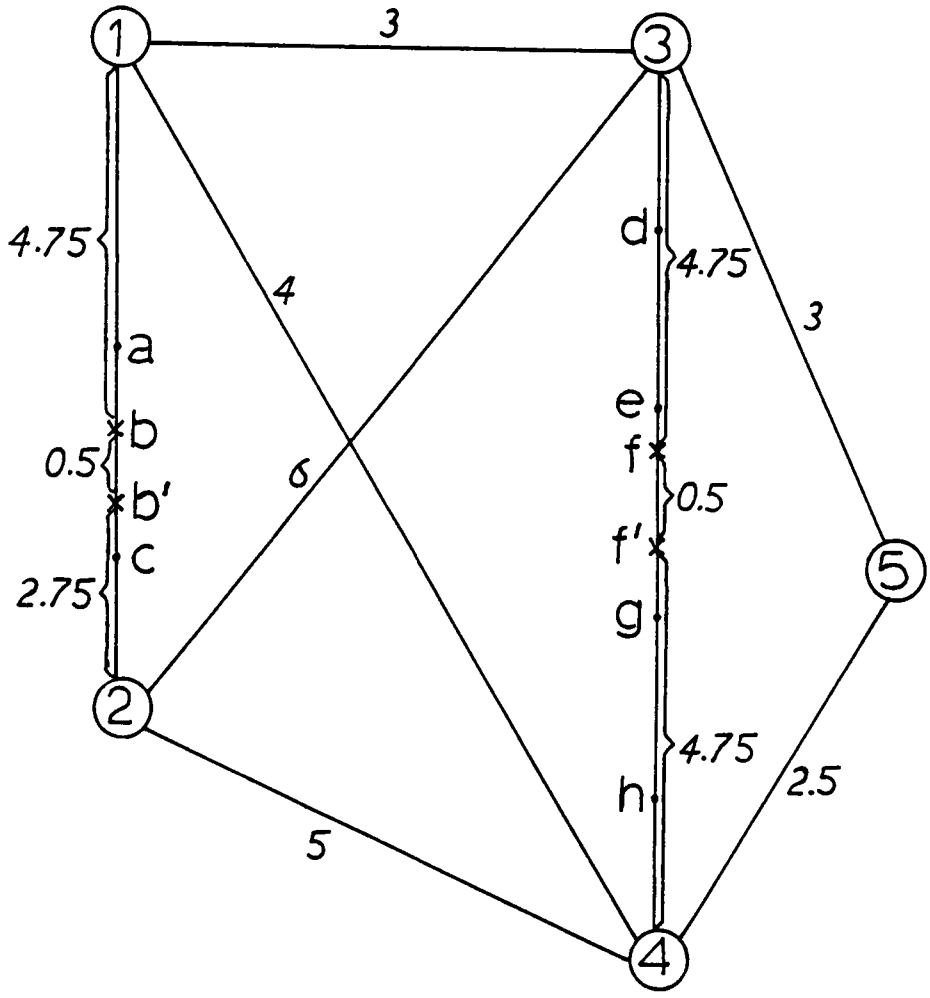


Figure 4.3. The solution for an example of (4.10).

Table 4.2. Shortest path distances to nodes

	Nodes					
	1	2	3	4	5	Sum
Points on arc (1,2)						
$a \varepsilon B_A(4)$	4.50	3.50	7.50	8.50	10.50	34.50
$b \varepsilon B_A(5)$	4.75	3.25	7.75	8.25	10.75	34.75
$b' \notin B_A$	5.25	2.75	8.25	7.75	10.25	34.25
$c \varepsilon B_A(3)$	5.50	2.50	8.50	7.50	10.00	34.00
Points on arc (3,4)						
$d \varepsilon B_A(4)$	5.25	8.25	2.25	7.75	5.25	28.75
$e \varepsilon B_A(2)$	7.50	10.50	4.50	5.50	7.50	35.50
$f \varepsilon B_A(5)$	7.75	10.25	4.75	5.25	7.75	35.75
$f' \notin B_A$	8.25	9.75	5.25	4.75	7.25	35.25
$g \varepsilon B_A(1)$	8.50	9.50	5.50	4.50	7.00	35.00
$h \varepsilon B_A(3)$	6.25	7.25	7.75	2.25	4.75	28.25

Table 4.3. Shortest path distances between points on arcs (1,2) and (3,4)

	a	b	b´	c
d	9.75	10.00	10.50	10.75
e	12.00	12.25	12.75	13.00
f	12.25	12.50	13.00	12.75
f´	12.75	13.00	12.50	12.25
g	13.00	12.75	12.25	12.00
h	10.75	10.50	10.00	9.75

least one of the new facility locations x_1^* does not belong to the set B_A. In other words, an optimal solution for (4.10) may not be found in the set $N \cup B_A$.

Although the optimal solutions for the above counter-example are not in the set $N \cup B_A$, they have an important characteristic. The optimal solution points x_1^* and x_2^* are arc bottleneck points with respect to each other. In other words, the shortest path distances from x_1^* to x_2^* through both endpoints of the arc on which x_1^* is located are equal. This characteristic leads to the following identification of a FOSS for (4.10).

Let B_B denote the set of arc bottleneck points of the point $q \in B_A$. That is

$$B_B = \bigcup_{q \in B_A} B_B(q) \text{ and } B_B(q) = \{y \mid y \text{ is interior to some arc } (i,j) \text{ and } s_{qi} + c_{iy} = s_{qj} + c_{jy}\} . \quad (4.11)$$

Condition (4.3) can be used to check whether arc (i,j) contains an arc bottleneck point with respect to arc bottleneck point q. As an example, points b´ and f´ in Figure 4.3 are in $B_B(f)$ and $B_B(b)$, respectively. There exists at most one arc bottleneck point on a given arc with respect to any $q \in B_A$-arc pair. Since the maximum number of elements in B_A is $|N| \cdot |A|$, the maximum number of elements in B_B is $|N| \cdot |A|^2$.

Theorem 4.2. $N \cup B_A \cup B_B$ is a FOSS for (4.10).

Proof: Let $f(x_1,x_2)$ denote the objective function value of (4.10) for the location configuration (x_1,x_2). Assume (x_1^*,x_2^*) is an optimal solution to (4.10) where $x_1^* \neq x_2^*$. (If $x_1^* = x_2^*$, (4.10) reduces to (4.1) which has FOSS $D \cup B_A$). It will be shown that x_1^* and x_2^* can be found in the set $N \cup B_A \cup B_B$. Consider the following three cases.

Case 1. (Either x_1^* or x_2^* is in the set N.) Without loss of generality, assume $x_1^* = t \in N$. Let $w_t' = w_t + v$, and $w_i' = w_i$ for all $i \in N - \{t\}$.

$$f(x_1^*,x_2^*) = f(t,x_2^*)$$

$$= \sum_{i \in N} w_i s_{it} + \sum_{i \in N} w_i s_{ix_2^*} + v s_{tx_2^*} ,$$

or

$$f(x_1^*,x_2^*) = \sum_{i \in N} w_i' s_{ix_2^*} + c_1 , \qquad (4.12)$$

where $c_1 = \sum_{i \in N} w_i s_{it}$. (4.12) is the objective function value of (4.1) for the optimal location x_2^* plus a positive constant c_1. From the fact that (4.1) has FOSS $D \cup B_A$ [Church & Garfinkel (1978)], it follows that x_2^* can be found in the set $D \cup B_A$.

Case 2. (Either x_1^* or x_2^* is in the set B_A.) Without loss of generality, assume $x_1^* \in B_A$ and x_1^* is an interior point of arc (m,n). Denote the network under consideration as G = (N,A). Consider a new network G′ = (N′,A′) with one more node x_1^*, i.e., let $N' = N \cup \{x_1^*\}$ and $A' = [A \cup \{(m,x_1^*), (n,x_1^*)\}] - \{(m,n)\}$. Network G′ contains exactly the same points as network G except that interior point x_1^* in G is considered as a node in G′. The shortest paths between points of network G′ are equal to those of network G. The set of arc bottleneck points B_A' for network G′ equals $B_A \cup B_B(x_1^*)$. It is also true that D′ = D. In other words, $D' \cup B_A' = D \cup B_A \cup B_B(x_1^*)$. Let $w_{x_1^*} = v$, then

$$f(x_1^*,x_2^*) = \sum_{i \in N} w_i s_{ix_1^*} + \sum_{i \in N} w_i s_{ix_2^*} + v s_{x_1^* x_2^*} ,$$

or

$$f(x_1^*,x_2^*) = \sum_{i \in N'} w_i s_{ix_2^*} + c_2 , \qquad (4.13)$$

where $c_2 = \sum_{i \in N} w_i s_{ix_1^*}$. (4.13) is the objective function value of (4.1) for the optimal point x_2^* plus a positive

constant c_2. It follows that x_2^* can be found in the set $D' \cup B_A'$, i.e., in the set $D \cup B_A \cup B_B(x_1^*)$.

Case 3. (Neither x_1^* nor x_2^* is in the set $N \cup B_A$.) Assume x_1^* and x_2^* are interior points of arcs (p,q), (m,n), respectively. This case can be separated into two subcases.

Case 3(a). [x_1^* (or x_2^*) is an arc bottleneck point with respect to x_2^* (or x_1^*). In other words, the shortest path distances from x_1^* (or x_2^*) through nodes p, q (or m, n) to x_2^* (or x_1^*) are equal.] It will be shown that in this case, x_1^* and x_2^* must be arc bottleneck points with respect to each other. Let s_{ijx} denote $s_{ij} + c_{jx}$. Without loss of generality, assume x_1^* is an arc bottleneck point with respect to x_2^*, that is,

$$s_{x_1^* x_2^*} = c_{px_1^*} + s_{px_2^*} = c_{qx_1^*} + s_{qx_2^*} \ . \tag{4.14}$$

From the assumption that x_1^* and x_2^* are not in B_A, it follows that

$$\begin{aligned} s_{pmx_2^*} &\neq s_{pnx_2^*} \ , \\ s_{qmx_2^*} &\neq s_{qnx_2^*} \ , \\ s_{mpx_1^*} &\neq s_{mqx_1^*} \ , \\ s_{npx_1^*} &\neq s_{nqx_1^*} \ . \end{aligned} \tag{4.15}$$

Without loss of generality, assume $s_{pmx_2^*} < s_{pnx_2^*}$. That is, assume

$$s_{px_2^*} = s_{pmx_2^*} \cdot \tag{4.16}$$

It will be shown that

$$s_{qx_2^*} = s_{qnx_2^*} , \tag{4.17}$$

$$s_{mx_1^*} = s_{mpx_1^*} , \tag{4.18}$$

$$s_{nx_1^*} = s_{nqx_1^*} \cdot \tag{4.19}$$

Assume (4.17) does not hold, that is

$$s_{qx_2^*} = s_{qmx_2^*} \cdot \tag{4.20}$$

From (4.20) and (4.14), it follows that

$$s_{x_1^*x_2^*} = c_{qx_1^*} + s_{qm} + c_{mx_2^*} \cdot \tag{4.21}$$

From (4.16) and (4.14),

$$s_{x_1^*x_2^*} = c_{px_1^*} + s_{pm} + c_{mx_2^*} \cdot \tag{4.22}$$

From (4.21) and (4.22), it follows that

$$c_{qx_1^*} + s_{qm} = c_{px_1^*} + s_{pm} \cdot \tag{4.23}$$

From the definition of arc bottleneck points, (4.23) implies that $x_1^* \epsilon B_A(m)$ which contradicts the assumption that x_1^* and x_2^* are not in $N \cup B_A$. It follows that (4.17) must be true. Similar arguments can be applied to show that (4.18) and (4.19) hold.

From (4.16) and (4.17), rewrite (4.14) as

$$s_{x_1^* x_2^*} = c_{px_1^*} + s_{pm} + c_{mx_2^*} ,$$

or

$$s_{x_1^* x_2^*} = c_{qx_1^*} + s_{qn} + c_{nx_2^*} . \tag{4.24}$$

From (4.18) and (4.19), (4.24) becomes

$$s_{x_1^* x_2^*} = s_{mx_1^*} + c_{mx_2^*} = s_{nx_1^*} + c_{nx_2^*} , \tag{4.25}$$

and (4.25) implies that x_2^* is an arc bottleneck point with respect to x_1^*. Since x_1^* and x_2^* are arc bottleneck points with respect to each other, moving either x_1^* or x_2^* from its present position will reduce the value of $s_{x_1^* x_2^*}$. However, the value of $s_{x_1^* x_2^*}$ in (4.24) will not change by moving x_1^* and x_2^* simultaneously with an infinitesimal positive amount δ toward p, n (or q, m), respectively. Essentially, the value of $s_{x_1^* x_2^*}$ will remain unchanged with an increasing δ value until one of the four end points is reached or until (4.15) no longer holds (i.e., an arc bottleneck is reached).

This result will be employed in the following to show that at least one of x_1^* or x_2^* can be found in the set $N \cup B_A$ for (4.10).

For any interior point z on arc (i,j), define

$$N_i(z) = \{k \epsilon N | s_{ki} + c_{iz} \leqslant s_{kj} + c_{jz}\}$$

and

$$N_j(z) = \{k \epsilon N | s_{ki} + c_{iz} > s_{kj} + c_{jz}\} . \qquad (4.26)$$

From the definition of arc bottleneck points if $z \notin B_A$, then

$$N_i(z) \cap N_j(z) = \phi . \qquad (4.27)$$

Also define

$$W_i(z) = \sum_{k \epsilon N_i(z)} w_k$$

and

$$W_j(z) = \sum_{k \epsilon N_j(z)} w_k . \qquad (4.28)$$

From the assumption that x_1^* and x_2^* are not in $N \cup B_A$, it follows that

$$N_p(x_1^*) \cap N_q(x_1^*) = \phi$$

and

$$N_m(x_2^*) \cap N_n(x_2^*) = \phi . \qquad (4.29)$$

Let

$$d(x_1^*,x_2^*) = [W_p(x_1^*) + W_n(x_2^*)] - [W_q(x_1^*) + W_m(x_2^*)] \ . \tag{4.30}$$

Without loss of generality, assume

$$d(x_1^*,x_2^*) \geqslant 0 \ . \tag{4.31}$$

Consider two points x_1', x_2', where $c_{px_1'} = c_{px_1^*} + \delta$, $c_{nx_2'} = c_{nx_2^*} + \delta$, and $s_{x_1'x_2'} = s_{x_1^*x_2^*}$. That is, move x_1^* and x_2^* simultaneously toward nodes q and m by an equal infinitesimal positive amount δ which ensures that $s_{x_1^*x_2^*}$ remains unchanged. Then

$$f(x_1',x_2') - f(x_1^*,x_2^*) = \left\{ \sum_{i\in N} w_i s_{ix_1'} + \sum_{i\in N} w_i s_{ix_2'} + vs_{x_1'x_2'} \right\} - \left\{ \sum_{i\in N} w_i s_{ix_1^*} + \sum_{i\in N} w_i s_{ix_2^*} + vs_{x_1^*x_2^*} \right\} ,$$

or

$$f(x_1',x_2') - f(x_1^*,x_2^*) = \delta d(x_1^*,x_2^*) \geqslant 0 \ . \tag{4.32}$$

Equation (4.32) holds as δ increases until x_1' (or x_2') reaches node q (or m) or an arc bottleneck point [where either (4.15) or (4.31) no longer holds]. This shows that a solution (x_1',x_2') can be obtained with either x_1' or x_2' belonging to the set $N \cup B_A$ and the solution achieves an objective function value greater than or equal to $f(x_1^*,x_2^*)$. By employing the results

of Case 1 (when $x_i' \epsilon N$) and Case 2 (when $x_i' \epsilon B_A$), it follows that an optimal solution can be found in the set $N \cup B_A \cup B_B$.

Case 3(b). [x_1^* (or x_2^*) is not an arc bottleneck point with respect to x_2^* (or x_1^*).] Without loss of generality, assume x_1^* is not an arc bottleneck point with respect to x_2^*, i.e.,

$$c_{px_1^*} + s_{px_2^*} \neq c_{qx_1^*} + s_{qx_2^*} .$$

Again, without loss of generality, assume

$$c_{px_1^*} + s_{px_2^*} > c_{qx_1^*} + s_{qx_2^*} ,$$

which is equivalent to assuming

$$s_{x_1^* x_2^*} = c_{qx_1^*} + s_{qx_2^*} . \tag{4.33}$$

Let

$$d(x_1^*) = W_p(x_1^*) - [W_q(x_1^*) + v] . \tag{4.34}$$

Without loss of generality, assume

$$d(x_1^*) \geqslant 0 . \tag{4.35}$$

Consider interior point x_1' on arc (p,q) where $c_{px_1'} = c_{px_1^*} + \delta$. That is, move point x_1^* toward node q by an infinitesimal positive amount δ. Then

$$f(x_1',x_2^*) - f(x_1^*,x_2^*) = \left\{ \sum_{i\epsilon N} w_i s_{ix_1'} + \sum_{i\epsilon N} w_i s_{ix_2^*} + v s_{x_1'x_2^*} \right\}$$

$$- \left\{ \sum_{i\epsilon N} w_i s_{ix_1^*} + \sum_{i\epsilon N} w_i s_{ix_2^*} + v s_{x_1^*x_2^*} \right\}$$

$$= \sum_{i\epsilon N} w_i (s_{ix_1'} - s_{ix_1^*}) + v(s_{x_1'x_2^*} - s_{x_1^*x_2^*}) ,$$

or

$$f(x_1',x_2^*) - f(x_1^*,x_2^*) = \delta d(x_1^*) \geqslant 0 . \qquad (4.36)$$

Problem (4.36) holds as δ increases as long as (4.35) holds. That is, (4.36) holds as δ increases until x_1' reaches the set $N \cup B_A$ or until x_1' becomes an arc bottleneck point with respect to x_2^*. This shows that a point x_1' which is in the set $N \cup B_A$ or is an arc bottleneck point with respect to x_2^* can be found, where $f(x_1',x_2^*)$ is greater than or equal to $f(x_1^*,x_2^*)$.

If $x_1' \epsilon N \cup B_A$, applying results from Case 1 or Case 2 shows that an optimal solution can be found in the set $N \cup B_A \cup B_B$. If x_1' is an arc bottleneck point with respect to x_2^*, applying results from Case 3(a) shows that an optimal solution can be found in the set $N \cup B_A \cup B_B$. The proof of this theorem is now complete.

From the proof of Theorem 4.2, it is seen that at least one of x_1^* or x_2^* can be found in the set $N \cup B_A$. An algorithm for (4.10) which utilizes this property is given below.

Algorithm 4.1

Step 1. Calculate the shortest path distance matrix, $S = (s_{ij})$, and let $q = 0$.

Step 2. Generate a point from the set $N \cup B_A$ which has not already been generated. If none exists, go to Step 5. If the generated point belongs N, go to Step 3. Otherwise, go to Step 4.

Step 3. The generated point is a node t. Let $x_1^* = t$, and solve for x_2^* using the objective function (4.12). If $q \geq f(x_1^*, x_2^*)$, go to Step 2. If $q < f(x_1^*, x_2^*)$, let $q = f(x_1^*, x_2^*)$ and record the current solution (x_1^*, x_2^*) as the "incumbent" solution and go to Step 2.

Step 4. The generated point is an arc bottleneck point s. Let $x_1^* = s$ and solve for x_2^* using the objective function (4.13). If $q \geq f(x_1^*, x_2^*)$, go to Step 2. If $q < f(x_1^*, x_2^*)$, let $q = f(x_1^*, x_2^*)$ and record the current solution (x_1^*, x_2^*) as the "incumbent" solution and go to Step 2.

Step 5. The optimal solution is the last recorded incumbent solution with objective function value 1.

Step 3 and Step 4 in Algorithm 4.1 involve the optimal solution of problem (4.1). The algorithm developed by Church and Garfinkel (1978) can be used in these two steps. Since the maximum number of elements in $N \cup B_A$ is $|N| \cdot (|A| + 1)$, Algorithm 4.1 may require the solution of (4.1) $|N| \cdot (|A| + 1)$ times in the worst case.

For the $p \geqslant 3$ case, since no apparent FOSS can be identified for (4.6), as will be shown in Chapter 6, the task of identifying a FOSS for the general problem (4.4) is not accomplished. Therefore, no method has been found to produce the exact optimal solution to (4.4) for the $p \geqslant 3$ case. One simple and intuitively appealing heuristic technique is to treat the location of p new undesirable facilities as p single new undesirable facility location problems. In other words, this heuristic approach considers the location of p new facilities in (4.4) one at a time as in (4.1) until all p facilities are located. Once a new facility is located, it is treated as an existing facility with weight v before proceeding to locate the next new facility.

A simple example of (4.4) with $p = 3$ is solved by applying the above heuristic method. Use the network in Figure 4.1 (see page 55) and assume $w_i = 1$ for all i and $v = 2$. The three new facilities are located one by one, and formulation (4.1) is used as the objective function in locating each of them. The first new facility is located at point f, as shown in Figure 4.3 (see page 64), with objective function value of 35.75. This newly located facility is then treated as an existing facility with weight $w_f = 2$. The location of the second new facility is considered with the objective of maximizing the sum of

its weighted shortest path distances to the six nodes shown in Figure 4.4. After checking Table 4.2 (see page 65) and Table 4.3 (see page 66), it follows that the second new facility should be located at point t with an objective function value of 60.25. Again, this facility at point t is treated as an existing facility with $w_t = 2$ before considering the location of the third new facility. Figure 4.5 indicates the best points to locate the third new facility on each arc. Table 4.4 lists these best location points and their objective function values. From this table, the third new facility should be located at point h on arc (4,f) with objective function value 62. The heuristic solution for this example is to have new facility configuration (f,t,h) with objective function value 35.75 + 60.25 + 62.0 = 168. In short, this is a greedy heuristic type approach which iteratively solves (4.1) p times and after each iteration, the number of existing facilities is increased by one. The main advantage of this heuristic method is its simplicity due to the fact that (4.1) is a well solved problem. However, its quality cannot be examined since no exact optimal solution is available for comparison.

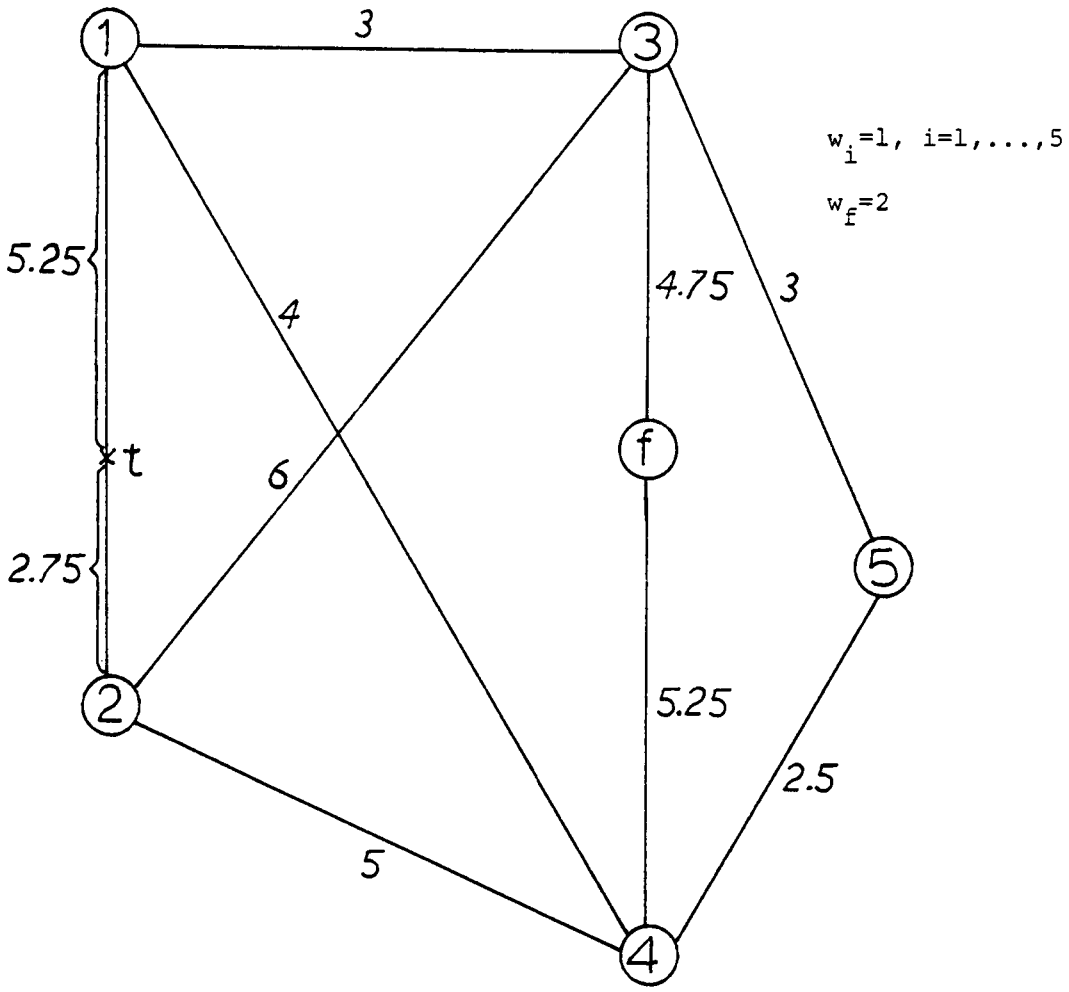


Figure 4.4. The location of the second facility.

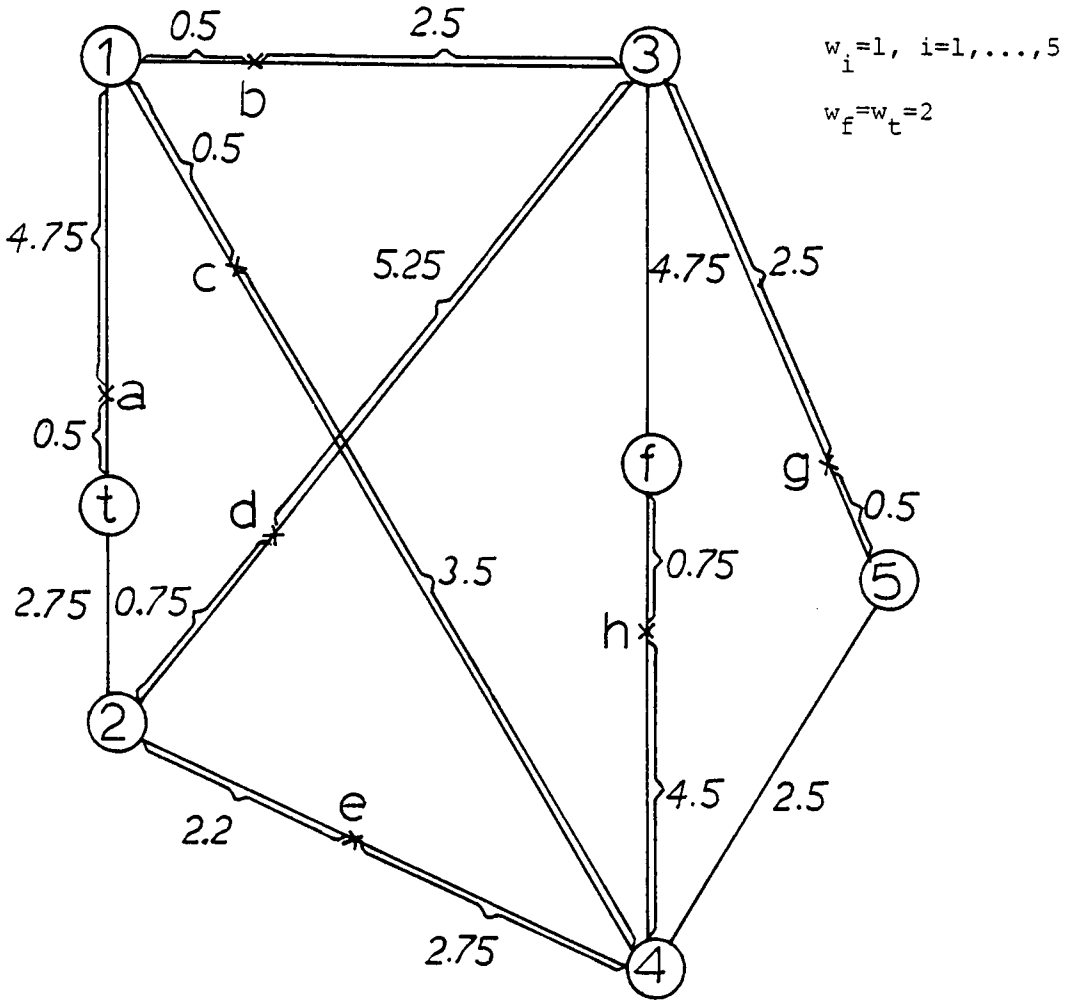


Figure 4.5. The location of the third facility.

Table 4.4. Best points on arcs for the third new facility

Arcs	The Best Points on Arcs	Objective Function Values
(1,t)	$a \varepsilon B_A(5)$	60.75
(1,3)	$b \varepsilon B_A(2)$	47.50
(1,4)	$c \varepsilon B_A(2)$	50.00
(2,t)	t	60.25
(2,3)	$d \varepsilon B_A(5)$	55.25
(2.4)	$e \varepsilon B_A(3)$	54.25
(3,f)	f	61.75
(3,5)	$g \varepsilon B_A(t)$	55.50
(4,f)	$h \varepsilon B_A(1)$	62.00*
(4,5)	5	55.00

CHAPTER 5

STRICTLY UNDESIRABLE FACILITY LOCATION WITH A CENTER-TYPE OBJECTIVE FUNCTION

This chapter deals with the problem of locating strictly undesirable facilities on a network with a center-type objective function. Again, the single new facility problem is studied separately before the problem of locating multiple new facilities is addressed.

5.1 Single New Facility

Let G = (N,A) be a connected, undirected simple network. Consider the following:

$$\max f(x) = \min_{i \in N} w_i s_{ix} \tag{5.1}$$

where the w_i are nonnegative weights. This problem is to locate a new facility on the network such that the minimum weighted shortest path distance from the new facility to any of the existing facilities is maximized. This is a location problem with a center-type objective function. For the problem of locating a house or business in a city to be far away from crime pockets, this formulation will obtain the location which is as far from the closest crime pocket as possible. If the w_i are equal for all $i \in N$, the

optimal solution for (5.1) is the midpoint of the longest arc. Hence, it will be assumed that this is not the case. Denote $s_{kqx} = s_{kq} + c_{qx}$, and consider the following definition of "center bottleneck point" which is an important concept for later discussions.

Definition 5.1. Point y is a center bottleneck point iff

1. y is interior to arc (i,j),
2. there exist two distinct nodes m, n, such that

$$w_m s_{miy} = w_n s_{njy},\ s_{miy} < s_{mjy},\ \text{and}\ s_{njy} < s_{niy}\ .$$

The point y is a bottleneck point in the sense that it cannot be moved in either direction without decreasing one of the two weighted shortest path distances from nodes m and n. The set of center bottleneck points with respect to nodes m and n will be denoted by $B_c(m,n)$, and the complete set of center bottleneck points will be denoted by B_c. [If the w_i are equal, then Definition 5.1 reduces to the case considered by Garfinkel et al. (1977)]. As an example, the network in Figure 5.1 has three center bottleneck points indicated as e, f, g, where $e \varepsilon B_c(1,2)$, $f \varepsilon B_c(1,3)$, and $g \varepsilon B_c(2,3)$. The weighted shortest path distances from these center bottleneck points to the corresponding nodes are 2, 3, and 6, respectively.

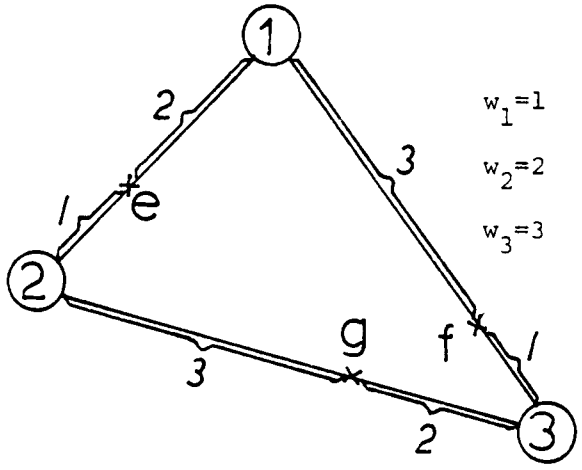


Figure 5.1. An example of center bottleneck points.

Each of the three center bottleneck points, e, f, and g, is the best point on its corresponding arc to locate a new facility for problem (5.1), and the optimal solution for (5.1) is on the point g with an objective function value of 6. A natural conjecture is that B_C may serve as a FOSS for (5.1). The following counterexample, however, demonstrates that this is not the case. Consider the network in Figure 5.2.

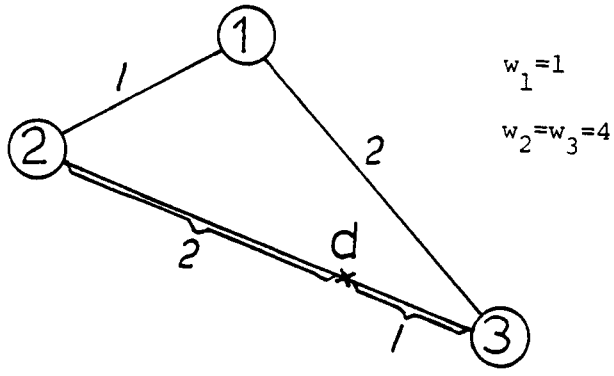


Figure 5.2. The solution for an example of (5.1).

The optimal solution for (5.1) is point d with objective function value 3. However, point d is not a center bottleneck point. It is in $B_A(1)$.

Theorem 5.1. $N \cup B_A \cup B_C$ is a FOSS for (5.1).

Proof: Suppose x^* is an optimal solution for (5.1) which is an interior point on arc (i,j) and $x^* \notin B_A \cup B_C$. Let

$$I = \{k \mid k \in N,\ f(x^*) = w_k s_{kx^*}\} \tag{5.2}$$

be the index set of all nodes which achieve the maximum weighted shortest path distance with point x^*. Clearly, $I \neq \phi$. Since $x^* \notin B_A$, it follows that

$$s_{qix^*} \neq s_{qjx^*} \qquad \forall\, q \in I\ .$$

Arbitrarily pick a node $m \in I$ and without loss of generality, assume

$$s_{mix^*} < s_{mjx^*}\ . \tag{5.3}$$

Since $x^* \notin B_C$, it follows that

$$s_{nix^*} < s_{njx^*} \qquad \forall\, n \in I,\ n \neq m\ . \tag{5.4}$$

If $s_{nix^*} > s_{njx^*}$, for any $n \in I$, $n \neq m$, then from (5.2) and

(5.3), $w_m s_{mix^*} = w_n s_{njx^*}$. This means that $x^* \varepsilon B_C$ which contradicts the assumption that $x^* \notin B_C$.

The above argument demonstrates that all shortest paths from node $q \varepsilon I$ to the point x^* go through the same node i. If x^* is moved along arc (i,j) toward j by an infinitesimal positive amount δ to the point y^*, i.e., $c_{iy^*} = c_{ix^*} + \delta$, then

$$f(y^*) - f(x^*) = \delta(\min_{q \varepsilon I} w_q) > 0 \ .$$

This contradicts the assumption that x^* is the optimal solution. Thus, $x^* \varepsilon N \cup B_A \cup B_C$ and the proof is complete.

An algorithm will be provided to solve problem (5.1). The algorithm iteratively checks each arc to find the best point for locating the new facility on that arc. Before giving the algorithm, properties and conditions to obtain elements of B_C are discussed.

For any [(i,j),m,n] combination where $(i,j) \varepsilon A$, m, $n \varepsilon N$, $m \neq n$, the shortest path from m to n which includes arc (i,j) can only take one of the following two forms shown in Figure 5.3: (1) the shortest path is of the form [p(m,i), j,p(j,n)]; or (2) the shortest path is of the form [p(n,i), j,p(j,m)], where p(m,n) denotes the set of arcs on the shortest path from nodes m to n.

Let p_1 be the length of path [p(m,i),j,p(j,n)] and p_2 be the length of path [p(n,i),j,p(j,m)]. Let x_1 be the

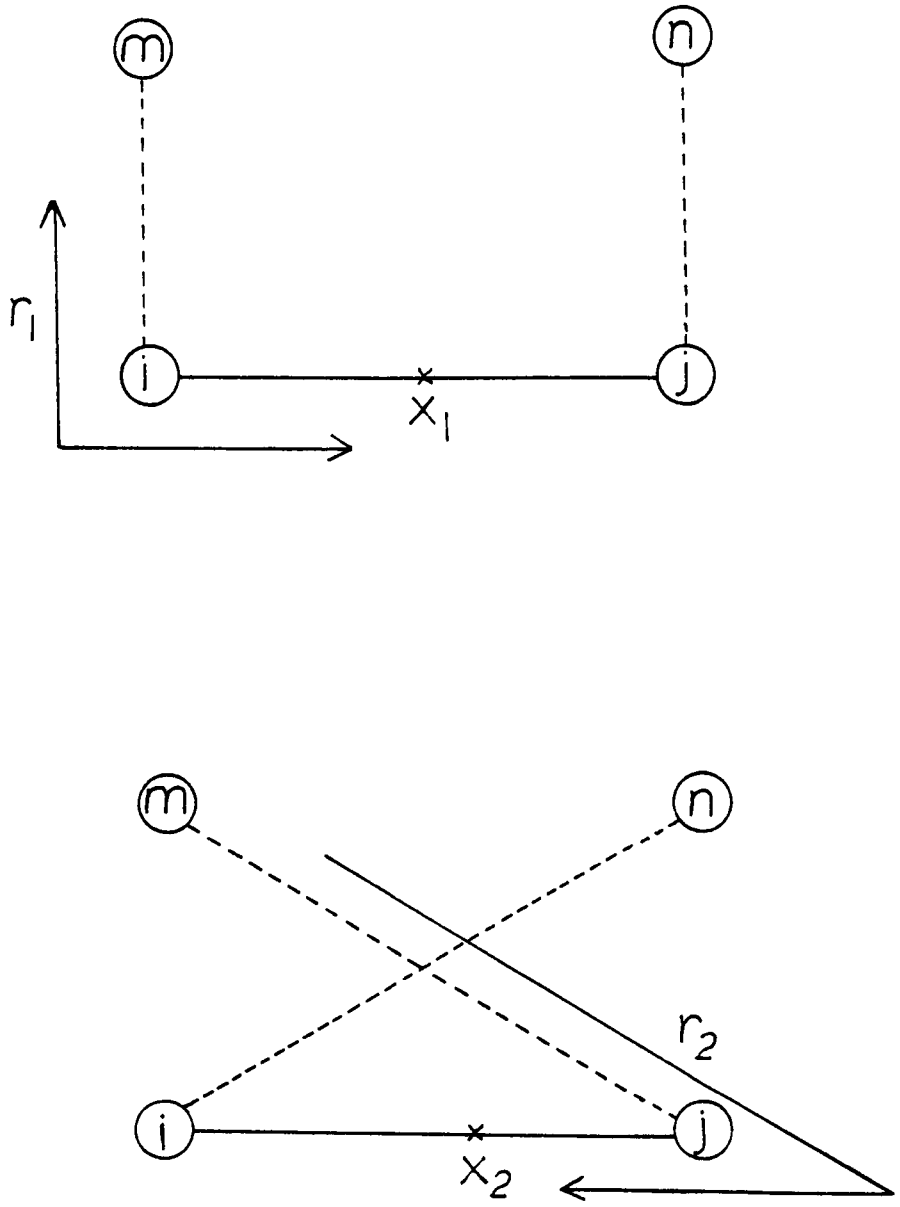


Figure 5.3. Paths from node m to node n which include arc (i,j).

point on path [p(m,i),j,p(j,n)] which is $r_1 = (\frac{w_n}{w_m + w_n})p_1$ units away from m, and let x_2 be the point on path [p(n,i), j,p(j,m)] which is $r_2 = (\frac{w_n}{w_m + w_n})p_2$ units away from m. Then x_1 and x_2 are the equally weighted distance points from nodes m and n. The weighted shortest path distances from x_1 to nodes m and n equal $(\frac{w_m w_n}{w_m + w_n})p_1$ and from x_2 to nodes m and n equal $(\frac{w_m w_n}{w_m + w_n})p_2$. These two points are the candidate points for the center bottleneck points with respect to the combination [(i,j),m,n]. The following theorem helps identify the center bottleneck points if they exist.

Theorem 5.2. Assume x_1 and x_2 are distinct points on arc (i,j),

(1) if $p_1 \geq p_2$, then $x_1 \notin B_C$

(2) if $p_2 \geq p_1$, then $x_2 \notin B_C$.

Proof: From the definitions of x_1 and x_2,

$$w_m s_{mix_1} = w_n s_{njx_1}$$

and

$$w_m s_{mjx_2} = w_n s_{nix_2} .$$

(1) If $p_1 \geq p_2$, then

$$w_m s_{mix_1} = w_n s_{njx_1} \geq w_m s_{mjx_2} = w_n s_{nix_2} . \qquad (5.5)$$

Assume $x_1 \varepsilon B_C$. By Definition 5.1,

$$s_{mix_1} < s_{mjx_1} , \tag{5.6}$$

and

$$s_{nix_1} > s_{njx_1} . \tag{5.7}$$

(a) If $c_{ix_1} \leqslant c_{ix_2}$, then

$$s_{nix_2} \geqslant s_{nix_1} . \tag{5.8}$$

From (5.7) and (5.8),

$$s_{nix_2} > s_{njx_1}$$

and

$$w_n s_{nix_2} > w_n s_{njx_1}$$

which contradicts (5.5). Thus, $x_1 \notin B_C$.

(b) If $c_{ix_2} \leqslant c_{ix_1}$,

$$s_{mjx_1} \leqslant s_{mjx_2} . \tag{5.9}$$

From (5.6) and (5.9),

$$s_{mix_1} < s_{mjx_2}$$

and

$$w_m s_{mix_1} < w_m s_{mjx_2}$$

which contradicts (5.5). Thus, $x_1 \notin B_C$.

(2) In the case of $p_2 \geqslant p_1$, the same arguments show that $x_2 \notin B_C$.

Corollary 5.1. There exists at most one center bottleneck point for any [(i,j),m,n] combination.

Proof: For any two real numbers, p_1 and p_2, either $p_1 > p_2$, or $p_1 = p_2$, or $p_1 < p_2$. From Theorem 5.2,

(1) if $p_1 > p_2$, then $x_1 \notin B_C$

(2) if $p_1 < p_2$, then $x_2 \notin B_C$

(3) if $p_1 = p_2$, then $x_1, x_2 \notin B_C$.

The following algorithm solves (5.1):

Algorithm 5.1

Step 1. Compute the shortest path distance matrix $S = (s_{ij})$ and let the lower bound $q = 0$.

Step 2. Generate an arc (i,j) which has not yet been generated. If none exists, go to Step 6; otherwise, go to Step 3. Let the upper bound u be a very large number, say 10^9.

Step 3. Test for B_A.

(1) Generate a node m which has not yet been generated. If none exists, go to Step 4; otherwise, go to (2).

(2) If $|s_{mi} - s_{mj}| < c_{ij}$, go to (3); otherwise, there is no arc bottleneck point on arc (i,j) for node m. Go to (1).

(3) If $w_m(\frac{s_{mi} + s_{mj} + c_{ij}}{2}) < u$, let $u = w_m(\frac{s_{mi} + s_{mj} + c_{ij}}{2})$ and record the current arc bottleneck point, which is in $B_A(m)$ on arc (i,j), as the "candidate" solution and go to (1).

Step 4. Test for B_C. The test consists of the following three stages:

(1) Generate a pair of nodes m and n, $m \neq n$, which has not yet been generated. If none exists, go to Step 5. Otherwise, go to (2).

(2) If $s_{mi} + c_{ij} + s_{nj} = s_{mj} + c_{ij} + s_{ni}$, then $x_1, x_2 \notin B_C$. Go to (1). If $s_{mi} + c_{ij} + s_{nj} \neq s_{mj} + c_{ij} + s_{ni}$, go to (3).

(3) If $s_{mi} + c_{ij} + s_{nj} > s_{mj} + c_{ij} + s_{ni}$, then $x_1 \notin B_C$. Go to (4). If $s_{mi} + c_{ij} + s_{nj} < s_{mj} + c_{ij} + s_{ni}$, check the following conditions for x_1 to be in B_C:

(a) If $s_{mi} < (\frac{w_n}{w_m + w_n})(s_{mi} + c_{ij} + s_{nj}) < s_{mi} + c_{ij}$, then x_1 is interior to arc (i,j). Go to (b). Otherwise, x_1 is not an interior point to arc (i,j). Go to (1).

(b) If either $s_{miy} \geq s_{mjy}$ or $s_{njy} \geq s_{niy}$, then the definition 5.1 is not satisfied and

$x_1 \notin B_C$. In this case, go to (1). Otherwise, go to (c).

(c) $x_1 \in B_C$. If $w_m(\frac{w_n}{w_m + w_n})(s_{mi} + c_{ij} + s_{nj}) < u$, let $u = w_m(\frac{w_n}{w_m + w_n})(s_{mi} + c_{ij} + s_{nj})$ and record the current center bottleneck point x_1 as the "candidate" solution. Go to (1).

(4) Using similar steps as (3a) to (3c), check for x_2 to be in B_C.

Step 5. If $q \geq u$, go to Step 2. If $q < u$, let $q = u$ and record the current "candidate" solution as the "incumbent" solution. Go to Step 2.

Step 6. The optimal solution is the last recorded "incumbent" solution with objective function value q.

Step 1 requires the solution of a well-solved problem and can be achieved in $o(|N|^3)$ by a number of algorithms [Yen (1975), Chapter 3]. Since Step 3 checks arc-node pairs, it can be done in $o(|A| \cdot |N|)$. Step 4 checks an arc with a pair of nodes which can be achieved in $o(|A| \cdot |N|^2)$. Therefore, the overall computational complexity of Algorithm 5.1 is $o(|A| \cdot |N|^2)$.

5.2 Multiple New Facilities

Consider the extension of (5.1) to locate p new facilities on a network with an objective of maximizing the minimum weighted shortest path distances between these new

facilities and between the new and existing facilities. That is,

$$\max \{\min_{i\varepsilon N} w_i s_{iX}; \min_{1\leqslant k<q\leqslant p} v s_{x_k x_q}\} . \qquad (5.10)$$

The new facilities are assumed to be similar and common weights w_i and v are employed. If v is positive, new facilities will be located in different positions. If there is no interaction between new facilities, i.e., v = 0, (5.10) becomes

$$\max \min_{i\varepsilon N} w_i s_{iX} . \qquad (5.11)$$

The optimal solution for (5.11) is to locate all p new facilities on the same optimal point x* as for the p = 1 case.

In order to solve (5.10), the question of whether there exists a FOSS for (5.10) needs to be examined. The following counterexamples indicate that there does not exist a FOSS for (5.10). Consider the simple network with two nodes and a single arc in Figure 5.4. When p = 2 and v = 1, the

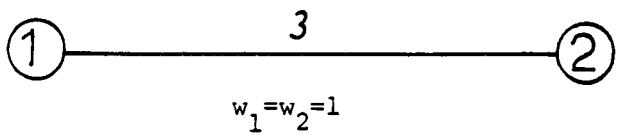


Figure 5.4. An example of a two node network.

optimal solution for (5.10) is to locate the two new facilities on points m and n, shown in Figure 5.5(a), with an objective function value of 1. Points m and n do not belong to any apparent finite set, such as B_A, B_C, or B_B. When p = 2 and v = 2, the optimal solution for (5.10) is to locate the two new facilities on points p and q, shown in Figure 5.5(b), with an objective function value of 1.2. Again, points p and q do not belong to any apparent finite set.

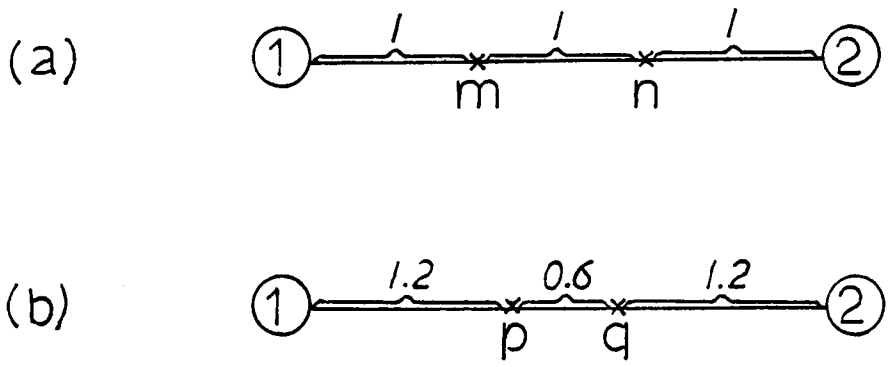


Figure 5.5. The solutions for an example of (5.10).

The above two simple examples demonstrate that for different values of v, the optimal solution for (5.10) will be on different points on arc (1,2). In other words, no FOSS can be identified for (5.10). It follows that no method has been found to produce the exact optimal solution for (5.10) for the $p \geqslant 2$ case.

The greedy heuristic method introduced at the end of Chapter 4 can be employed to solve (5.10). The heuristic approach solves the problem of locating p new facilities as p separate single facility location problems using (5.1). New facilities are located one at a time and are treated as existing facilities with weight v before locating the next new facility. The quality of this solution cannot be examined since no known exact optimal solution can be obtained for comparison.

CHAPTER 6

THE GENERALIZED DIAMETER PROBLEM AND ITS EXTENSIONS

This chapter considers the problem of locating p points on a connected, undirected simple graph, G = (N,A), with an objective of maximizing the sum of all the shortest path distances between them. As given in Chapter 4, this problem can be formulated as follows:

$$\text{Max} \sum_{1 \leq k < q \leq p} s_{x_k x_q} \tag{6.1}$$

where x_i, i=1,2,...,p, are points on the network and s_{xy} represents the shortest path distance between any two points x, y on the network. The motivation and major purpose in studying this problem is to identify a FOSS for (6.1) so as to gain more insight into problem (4.4).

6.1 Generalized Diameter Problems

The diameters of graphs are widely discussed in the graph theory literature and are defined as the distance between that pair of nodes which is farthest apart in terms of shortest path distances, or

$$d(G) = \max_{i,j \in N} s_{ij} . \tag{6.2}$$

The concept of diameter is generalized to include all of the points on arcs of the graph as follows:

$$\hat{d}(G) = \max_{x,y \in G} s_{xy} \ . \tag{6.3}$$

In other words, the "generalized diameter" of a graph is the shortest path distance between a pair of points on the graph which are farthest away from each other. Clearly, (6.3) is the p = 2 case of (6.1). Figure 6.1 gives an example of (6.2) and (6.3), where d(G) = 8 between nodes 1 and 2 and $\hat{d}(G)$ = 13 between points a and a´.

Let $\hat{p}(G)$ denote the path whose length is $\hat{d}(G)$. It will be shown that $N \cup B_A$ is a FOSS for (6.3). In other words, there exists $\hat{p}(G)$ whose endpoints are in the set $N \cup B_A$, where B_A is the set of "arc bottleneck points" defined in (4.2). The following theorem gives an upper bound on the shortest path distance between any pair of points on the network.

Theorem 6.1. For any two points x and y on arcs (m,n) and (i,j), respectively,

$$s_{xy} \leq u_{xy} = \min \left\{ \frac{c_{mn}+c_{ij}+s_{mi}+s_{nj}}{2} \ , \ \frac{c_{mn}+c_{ij}+s_{mj}+s_{ni}}{2} \right\} \ . \tag{6.4}$$

Proof: As shown in Figure 6.2, there are four possible routes to go from point x to point y. From the definition

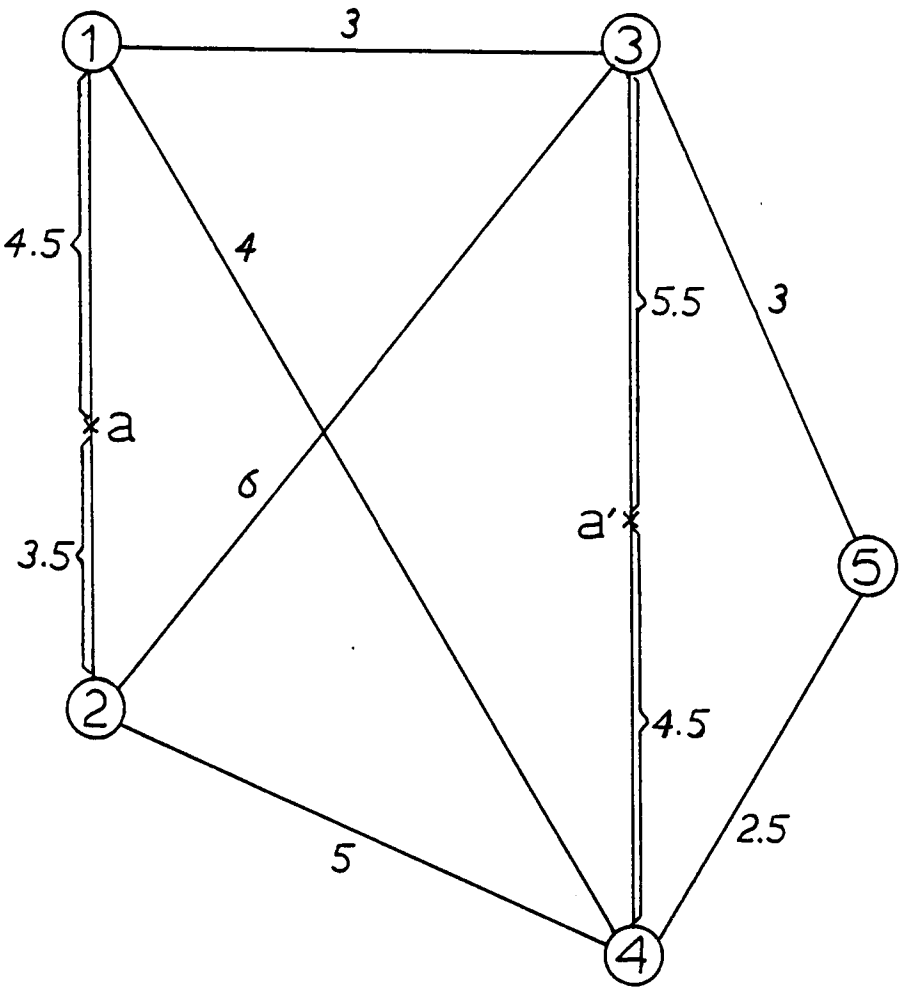


Figure 6.1. An example of (6.2) and (6.3).

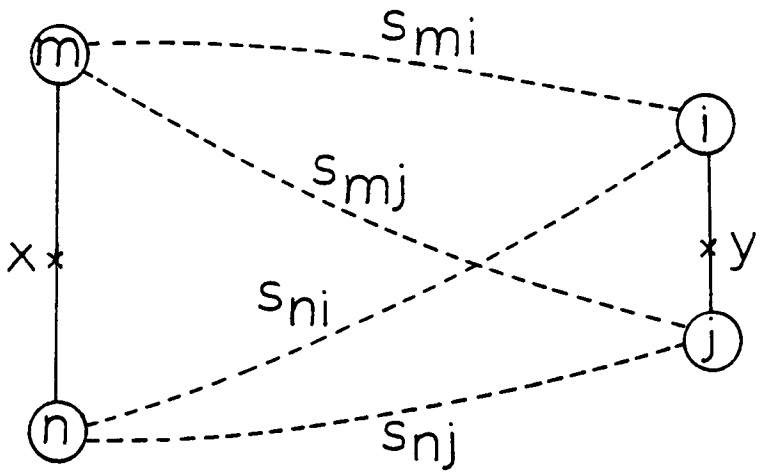


Figure 6.2. Possible routes from point x to point y.

of shortest path distance, it follows that

$$s_{xy} \leq c_{xm} + s_{mi} + c_{iy} , \tag{6.5}$$

$$s_{xy} \leq c_{xn} + s_{nj} + c_{jy} , \tag{6.6}$$

$$s_{xy} \leq c_{xm} + s_{mj} + c_{jy} , \tag{6.7}$$

$$s_{xy} \leq c_{xn} + s_{ni} + c_{iy} . \tag{6.8}$$

From (6.5) and (6.6), it follows that

$$s_{xy} \leq \frac{c_{mn} + c_{ij} + s_{mi} + s_{nj}}{2} . \tag{6.9}$$

From (6.7) and (6.8), it follows that

$$s_{xy} \leq \frac{c_{mn} + c_{ij} + s_{mj} + s_{ni}}{2} . \tag{6.10}$$

Problems (6.9) and (6.10) yield (6.4). The proof of Theorem 6.1 is complete.

The next corollary is derived directly from Theorem 6.1.

Corollary 6.1.

(1) When arcs (m,n) and (i,j) share a node, say m = i, then $s_{xy} \leqslant u_{xy} = \frac{c_{mn} + c_{mj} + s_{nj}}{2}$.

(2) When arcs (m,n) and (i,j) are identical, then $s_{xy} \leqslant u_{xy} = \frac{c_{mn} + s_{mn}}{2}$.

Theorem 6.2. NUB_A is a FOSS for (6.3).

Proof: Let x*, y* be an optimal solution for (6.3) which are interior points on arcs (m,n) and (i,j), respectively. From Theorem 6.1, it follows that the objective function value s_{x*y*} is smaller than or equal to the upper bound u_{xy}. It will be shown that there are $\bar{x}$, $\bar{y} \epsilon NUB_A$ on arcs (m,n) and (i,j), respectively such that $s_{\bar{x}\bar{y}} = u_{xy}$. In other words, an optimal solution $\bar{x}$, $\bar{y}$ can be found in the set NUB_A, where $s_{\bar{x}\bar{y}} = s_{x*y*} = u_{xy}$.

Without loss of generality, assume that u_{xy} in (6.4) is

$$u_{xy} = \frac{c_{mn} + c_{ij} + s_{mi} + s_{nj}}{2},$$

which is equivalent to assuming that

$$s_{mi} + s_{nj} \leqslant s_{mj} + s_{ni}. \tag{6.11}$$

Let $k^*(i,j)$ be the farthest point in terms of shortest path distances from node k on arc (i,j). It has been shown by Church and Garfinkel (1978) that $k^*(i,j)$ is either node i, node j, or $B_A(k)$ on arc (i,j) where

$$s_{kk^*(i,j)} = \frac{c_{ij} + s_{ki} + s_{kj}}{2} . \tag{6.12}$$

Let $\bar{x} = j^*(m,n)$ as defined in (6.12) so that

$$s_{j\bar{x}} = s_{jn\bar{x}} = \frac{c_{mn} + s_{mj} + s_{nj}}{2} . \tag{6.13}$$

If $\bar{x}$ is interior to arc (m,n), then $\bar{x}\varepsilon B_A(j)$. Similarly, let $\bar{y} = m^*(i,j)$ so that

$$s_{mj\bar{y}} = s_{mi\bar{y}} = \frac{c_{ij} + s_{mi} + s_{mj}}{2} . \tag{6.14}$$

Again, if $\bar{y}$ is interior to arc (i,j), then $\bar{y}\varepsilon B_A(m)$.

From (6.13), it follows that

$$s_{mj\bar{x}} + c_{j\bar{y}} = s_{jn\bar{x}} + c_{j\bar{y}} ,$$

or

$$s_{mj} + c_{m\bar{x}} + c_{j\bar{y}} = s_{jn} + c_{n\bar{x}} + c_{j\bar{y}} . \tag{6.15}$$

From (6.14), it follows that

$$s_{mj\bar{y}} + c_{m\bar{x}} = s_{mi\bar{y}} + c_{m\bar{x}} ,$$

or

$$s_{mj} + c_{j\bar{y}} + c_{m\bar{x}} = s_{mi} + c_{i\bar{y}} + c_{m\bar{x}} . \tag{6.16}$$

From (6.15) and (6.16), it follows that

$$s_{mj} + c_{m\bar{x}} + c_{j\bar{y}} = s_{jn} + c_{n\bar{x}} + c_{j\bar{y}} = s_{mi} + c_{i\bar{y}} + c_{m\bar{x}} . \tag{6.17}$$

From (6.11)

$$s_{mi} + s_{nj} + c_{m\bar{x}} + c_{n\bar{x}} + c_{i\bar{y}} + c_{j\bar{y}} \leqslant s_{mj} + s_{ni} + c_{m\bar{x}} + c_{n\bar{x}} + c_{i\bar{y}} + c_{j\bar{y}}$$

or

$$(s_{nj} + c_{n\bar{x}} + c_{j\bar{y}}) + (s_{mi} + c_{m\bar{x}} + c_{i\bar{y}}) \leqslant (s_{mj} + c_{m\bar{x}} + c_{j\bar{y}}) + (s_{ni} + c_{n\bar{x}} + c_{i\bar{y}}) . \tag{6.18}$$

From (6.17) and (6.18)

$$s_{mj} + c_{m\bar{x}} + c_{j\bar{y}} \leqslant s_{ni} + c_{n\bar{x}} + c_{i\bar{y}} . \tag{6.19}$$

From the definition of shortest path distances and from (6.17) and (6.19), it follows that

$$s_{\bar{x}\bar{y}} = s_{mj} + c_{m\bar{x}} + c_{j\bar{y}} = s_{jn} + c_{n\bar{x}} + c_{j\bar{y}} = s_{mi} + c_{i\bar{y}} + c_{m\bar{x}} . \tag{6.20}$$

From (6.20),

$$2s_{\bar{x}\bar{y}} = (s_{nj} + c_{n\bar{x}} + c_{j\bar{y}}) + (s_{mi} + c_{i\bar{y}} + c_{m\bar{x}})$$

$$= s_{nj} + s_{mi} + c_{mn} + c_{ij}$$

$$= 2u_{xy} ,$$

so that $s_{\bar{x}\bar{y}} = u_{xy} = \hat{d}(G)$, and the theorem is proved. [Note: If (m,n) and (i,j) share one or two nodes, the proof goes through by substitution of the appropriate subscripts.]

Since the proof of Theorem 6.2 provides the "farthest" pair of points on a given pair of arcs, clearly an $o(|A|^2)$ algorithm is possible once the shortest path distance matrix, $S = (s_{ij})$, has been calculated.

Algorithm 6.1

Step 1. Calculate S, and let $q = d(G)$.

Step 2. Generate an arc pair which has not already been generated. If none exists, go to Step 4. Otherwise, go to Step 3.

Step 3. Compute u_{xy} from (6.4) for the current arc pair. If $q \geqslant u_{xy}$, go to Step 2. If $q < u_{xy}$, let $q = u_{xy}$ and record the current arc pair as the "incumbent" pair and go to Step 2.

Step 4. $\hat{d}(G) = q$. Calculate $\bar{x}$ and $\bar{y}$ on the incumbent arc pair (m,n) and (i,j) as described in (6.13) and (6.14).

Note that if the graph is connected and has more than three nodes, there is no need to consider locating x and y on one arc, since

$$\frac{s_{mn} + c_{mn}}{2} \leqslant \frac{c_{mn} + c_{mj} + s_{nj}}{2} .$$

Also note that in Step 4 there may actually be an infinite number of optimal solutions. Pairs of points from the two arc segments [m*(i,j),n*(i,j)] and [i*(m,n),j*(m,n)] form the set of optimal solutions. Assume

$$u_{xy} = \frac{c_{mn} + c_{ij} + s_{mi} + s_{nj}}{2} .$$

For point $x_1 \varepsilon [i^*(m,n), j^*(m,n)]$, the corresponding point $y_1 \varepsilon [m^*(i,j), n^*(i,j)]$ can be found by solving

$$c_{mx_1} + s_{mi} + c_{iy_1} = q = u_{xy}$$

or

$$c_{nx_1} + s_{nj} + c_{jy_1} = q = u_{xy} .$$

Using the network in Figure 6.1 (see page 98) as an example, Figure 6.3 shows that the optimal solutions for (6.3) are x_1, y_1, where x_1 belongs to the arc segment $\overline{ab}$ and y_1 belongs to the arc segment $\overline{a'b'}$ with $\hat{d}(G) = s_{x_1y_1} = 13$. Note that $a = 4^*(1,2)$, $b = 3^*(1,2)$, $a' = 1^*(3,4)$, and $b' = 2^*(3,4)$.

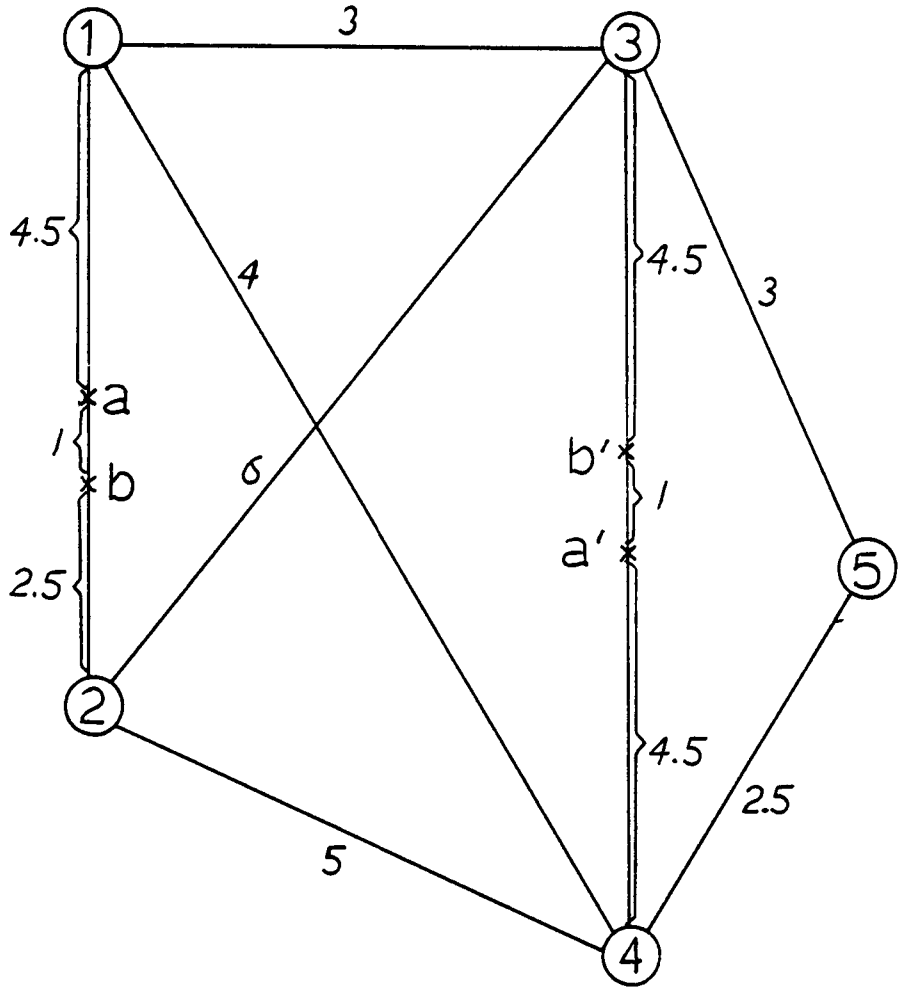


Figure 6.3. The solution for an example of (6.3).

Computational Results

A FORTRAN program for Algorithm 6.1 was developed and run on the DEC-10 computer. Test problems on connected graphs were randomly generated with arc density 0.35. The algorithm developed by Hu (1967) was used in calculating the shortest path distance matrix S in Step 1. Table 6.1 gives the results of several test runs. The run times do not include the time spent generating the test graphs or the time to calculate S. Table 6.2 is the relationship between run time and $|A|^2$. From Table 6.2, the ratio converges to approximately $3.3 \cdot 10^{-5}$.

6.2 Extensions of Generalized Diameter Problems

Consider the p = 3 case of (6.1) which is the simplest extension of the "generalized diameter problem." The formulation of the problem is

$$\max \{s_{x_1x_2} + s_{x_2x_3} + s_{x_1x_3}\} . \tag{6.21}$$

This problem is to locate 3 points on the network and the objective is to maximize the sum of the shortest path distances between them. Although it will be shown in Section 6.3 that (6.21) can be solved with linear programming, the first attempt was to isolate a FOSS analogous to NUB_A for the generalized diameter problem. Counterexamples to some natural conjectures are given below.

Table 6.1. Run time in seconds

Run	Number of Nodes				
	10	20	40	80	120
1	0.08 (19)*	0.28 (69)	2.66 (266)	40.67 (1097)	201.15 (2465)
2	0.08 (19)	0.28 (69)	2.67 (266)	40.75 (1098)	
3	0.08 (19)	0.29 (70)	2.66 (266)	40.62 (1098)	
4	0.08 (19)	0.28 (69)	2.68 (267)		
5	0.08 (19)	0.28 (68)	2.68 (267)		
Average	0.08	0.28	2.67	40.68	201.15

*( ) : Number of arcs generated.

Table 6.2. Relationship between run time and $|A|^2$

$\|A\|$	$\|A\|^2$	Run Time (sec.)	Ratio of Time/$\|A\|^2$
19	365	0.08	$21.92 \cdot 10^{-5}$
69	4,761	0.28	$5.88 \cdot 10^{-5}$
266	70,756	2.67	$3.77 \cdot 10^{-5}$
1098	1,205,604	40.68	$3.37 \cdot 10^{-5}$
2465	6,076,225	201.15	$3.31 \cdot 10^{-5}$

The first obvious conjecture is that the set $N \cup B_A$ is a FOSS for (6.21). Figure 6.4 is a counterexample for this conjecture where the unique optimal solution is on the midpoints of three arcs with an objective function value of 14. There is, however, no arc bottleneck point at all for this network.

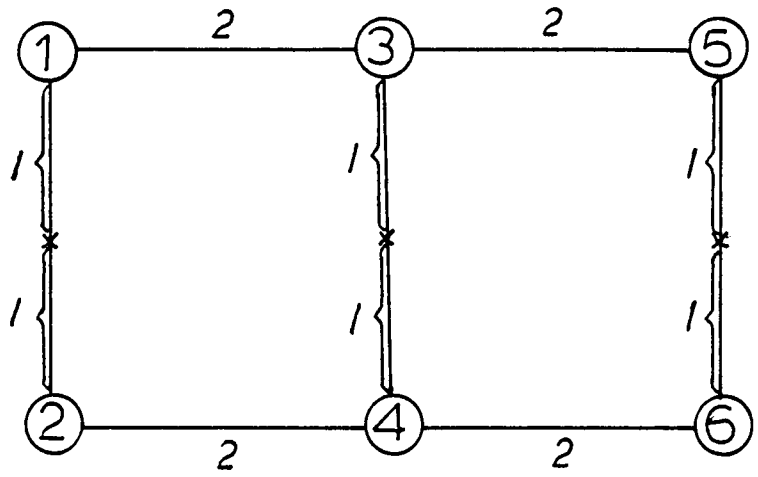


Figure 6.4. A counterexample which shows that $N \cup B_A$ is not a FOSS for (6.21).

The next conjecture is that the set $N \cup B_A \cup M \cup B_M \cup B_B$ is a FOSS for (6.21), where M denotes the set of midpoints of arcs of a network, and B_M denotes the set of arc bottleneck points of the point $m \in M$. That is,

$$B_M = \bigcup_{m \in M} B_M(m)$$

and

$$B_M(m) = \{y \mid y \text{ is interior to some arc } (i,j) \text{ and } s_{mi} + c_{iy} = s_{mj} + c_{jy}\} . \quad (6.22)$$

Definition (6.22) is similar to the definition of B_A in (4.2) and the definition of B_B in (4.11). Condition (4.3) can be used to check whether arc (i,j) contains an arc bottleneck point with respect to a midpoint m. Again, there exists at most one arc bottleneck point on a given arc with respect to any $m \in M$, and the maximum number of elements in B_M is $|A| \cdot (|A| - 1)$.

The following is a counterexample to the conjecture that the set $N \cup B_A \cup M \cup B_M \cup B_B$ is a FOSS for (6.21). Consider the six-node network in Figure 6.5, where the unique optimal solution for (6.21) is (x,y,z) with an objective function value of 69.5. It will be shown that point x does not belong to the set $N \cup B_A \cup M \cup B_M \cup B_B$. Therefore, the optimal solution for (6.21) cannot be found in the set $N \cup B_A \cup M \cup B_M \cup B_B$.

Figure 6.6 indicates all the elements in B_A where arcs (1,2), (2,4), and (4,6) do not have any arc bottleneck point with respect to nodes. Figure 6.7 indicates the elements in M. Using Figure 6.6 and Figure 6.7, all elements in the set $B_A \cup M \cup B_M \cup B_B$ on arc (1,2) are identified and indicated in Figure 6.8. There are only three such points, namely $t \in B_M(m_6)$, $u = m_1 \in M$, and $v \in B_M(m_8)$. It is seen that the

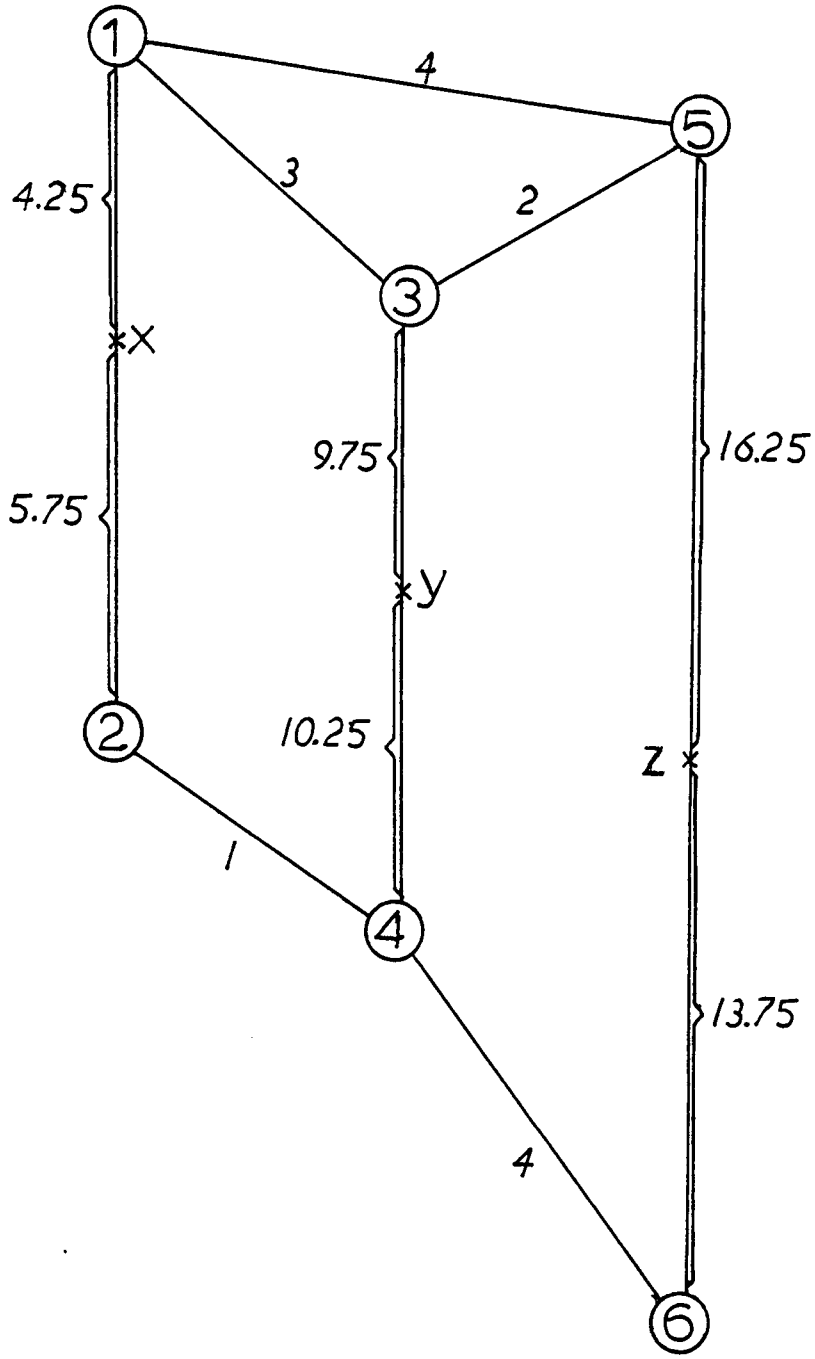


Figure 6.5. The solution for an example of (6.21).

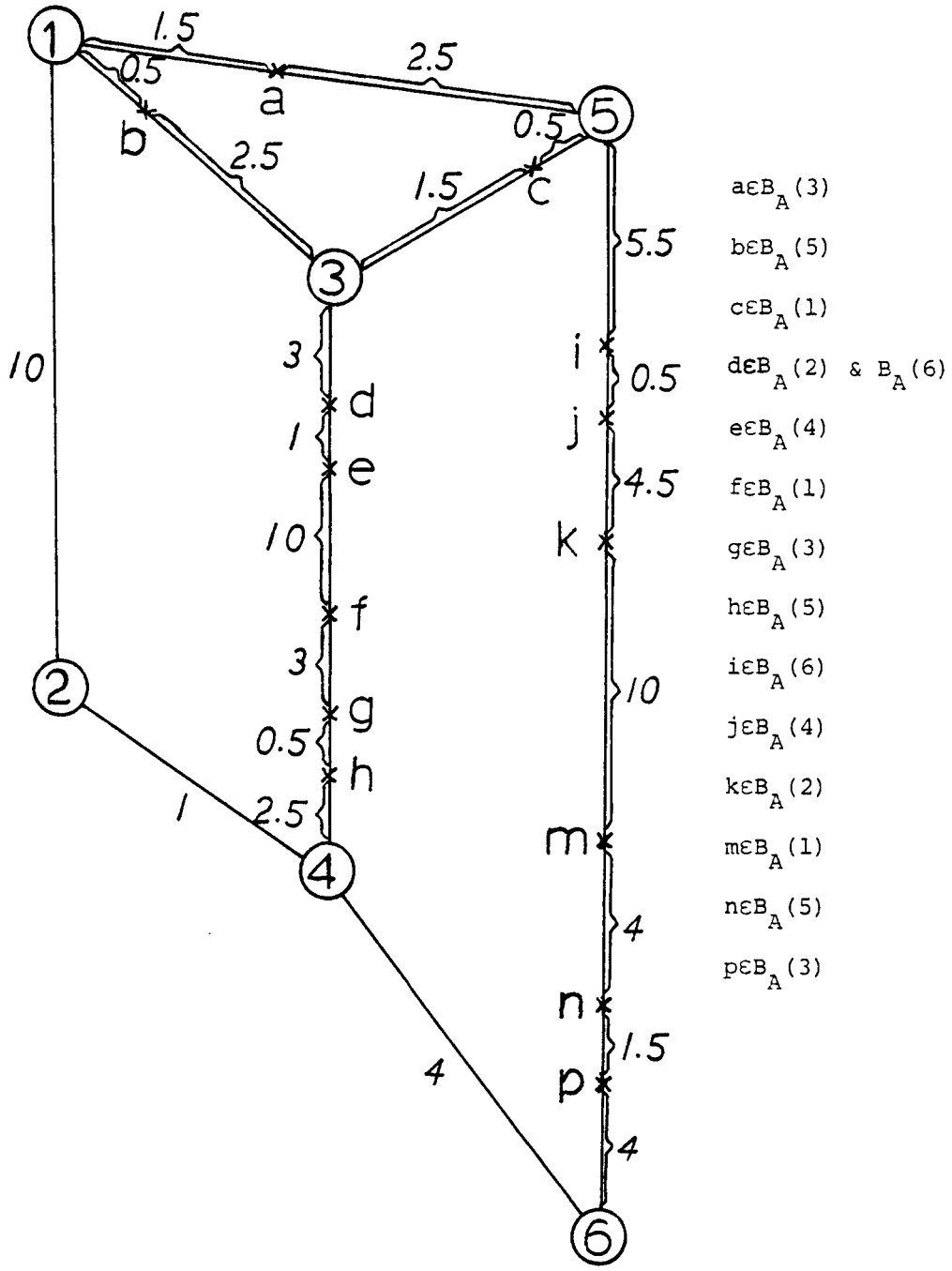


Figure 6.6. All of the bottleneck points of a network.

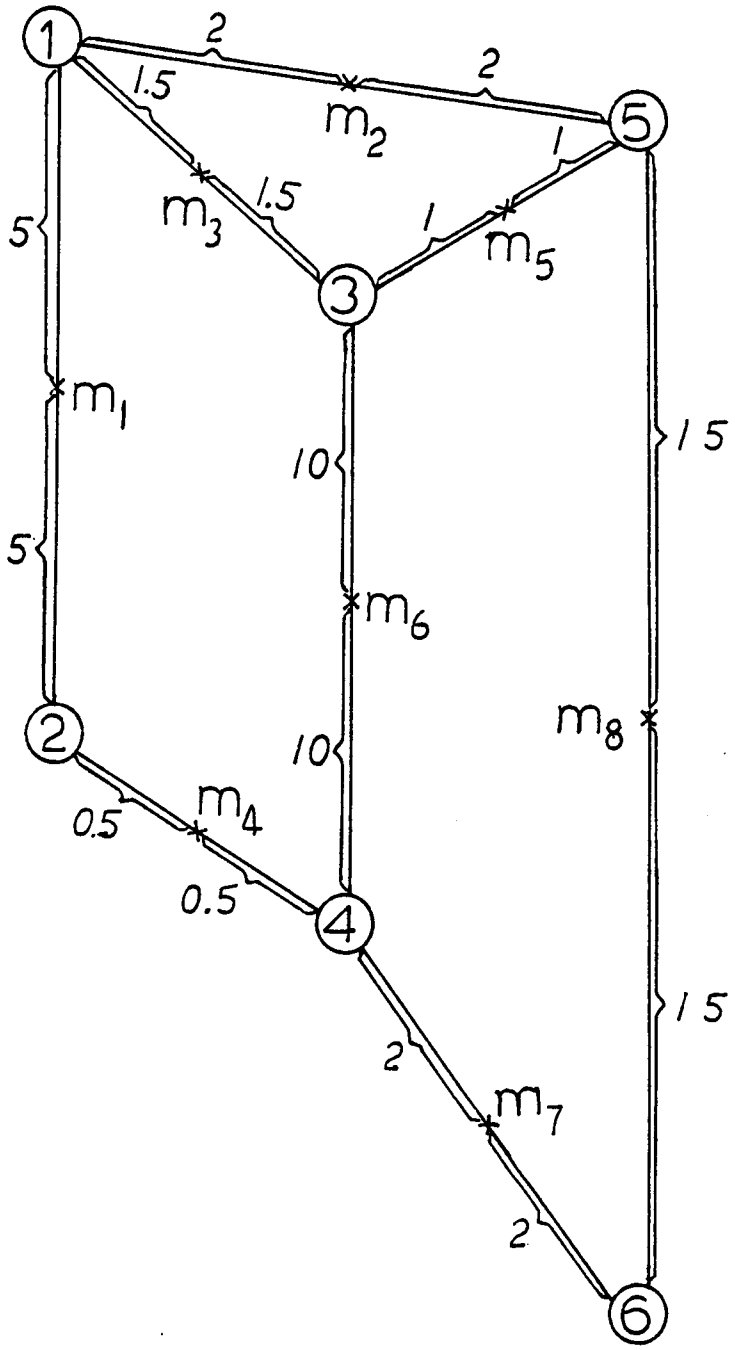


Figure 6.7. All of the midpoints of arcs of a network.

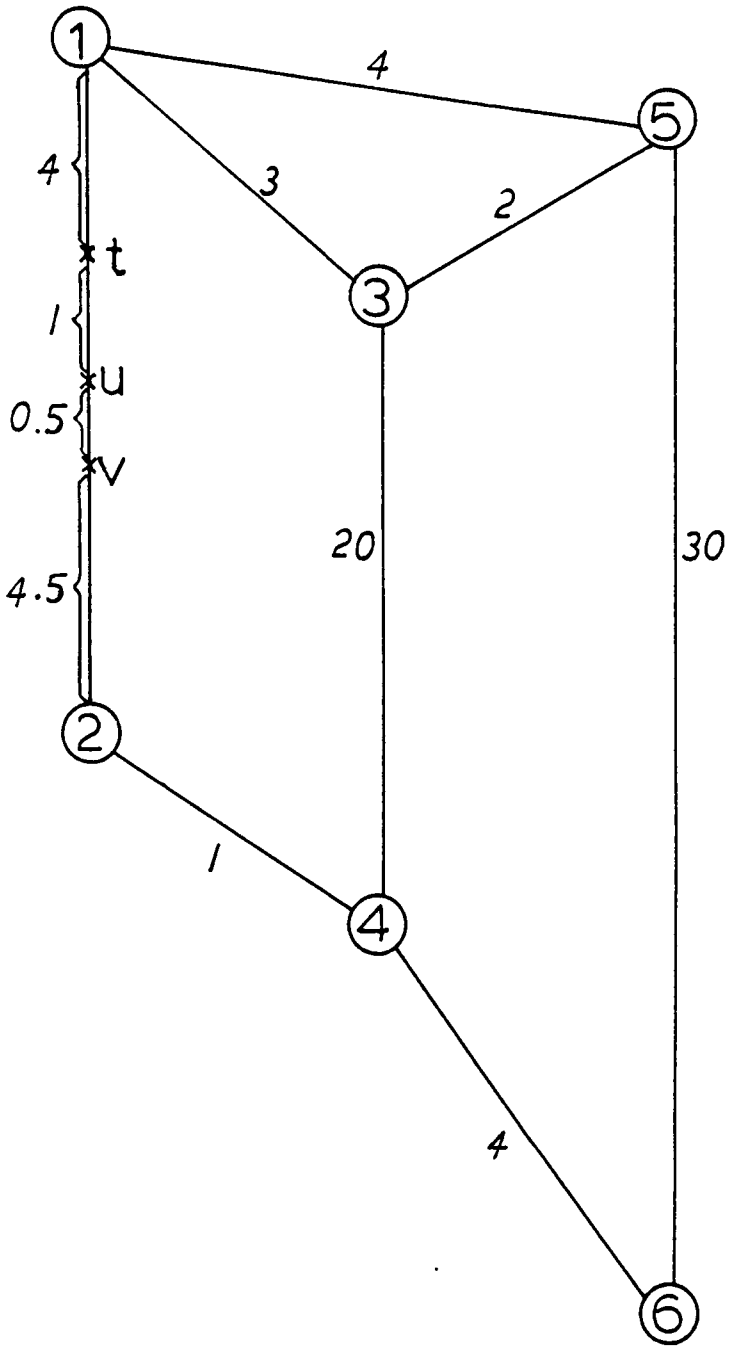


Figure 6.8. Elements in the set $B_A \cup M \cup B_M \cup B_B$ on arc (1,2).

point x does not coincide with t, u, or v. Hence, $N \cup B_A \cup M \cup B_M \cup B_B$ is not a FOSS for (6.21). The set $N \cup B_A \cup M \cup B_M \cup B_B$ exhausts all of the obvious conjectures for a FOSS for (6.21).

6.3 Linear Programming Formulation for (6.21)

Problem (6.21) is to locate three points on a network such that the sum of the shortest path distances between them is maximized. If the three points are restricted to be located on a triplet of arcs instead of the whole network, the following LP formulation obtains the optimal locations on these three arcs. Assuming that arcs (1,2), (3,4) and (5,6), shown in Figure 6.9, are the arc triplet under consideration, the LP formulation is:

$$\max \; s_{xy} + s_{yz} + s_{xz}$$

$$\text{s.t.} \quad c_{1x} + s_{13} + c_{3y} \geqslant s_{xy}$$

$$c_{1x} + s_{14} + c_{4y} \geqslant s_{xy}$$

$$c_{2x} + s_{23} + c_{3y} \geqslant s_{xy}$$

$$c_{2x} + s_{24} + c_{4y} \geqslant s_{xy}$$

$$c_{1x} + s_{15} + c_{5z} \geqslant s_{xz}$$

$$c_{1x} + s_{16} + c_{6z} \geqslant s_{xz}$$

$$c_{2x} + s_{25} + c_{5z} \geqslant s_{xz}$$

$$c_{2x} + s_{26} + c_{6z} \geqslant s_{xz}$$

$$c_{3y} + s_{35} + c_{5z} \geqslant s_{yz}$$

$$c_{3y} + s_{36} + c_{6z} \geqslant s_{yz}$$

$$c_{4y} + s_{45} + c_{5z} \geqslant s_{yz}$$

$$c_{4y} + s_{46} + c_{6z} \geqslant s_{yz}$$

$$c_{1x} + c_{2x} = c_{12}$$

$$c_{3y} + c_{4y} = c_{34}$$

$$c_{5z} + c_{6z} = c_{56}$$

$$c_{1x}, c_{2x}, c_{3y}, c_{4y}, c_{5z}, c_{6z}, s_{xy}, s_{xz}, s_{yz} \geqslant 0 .$$

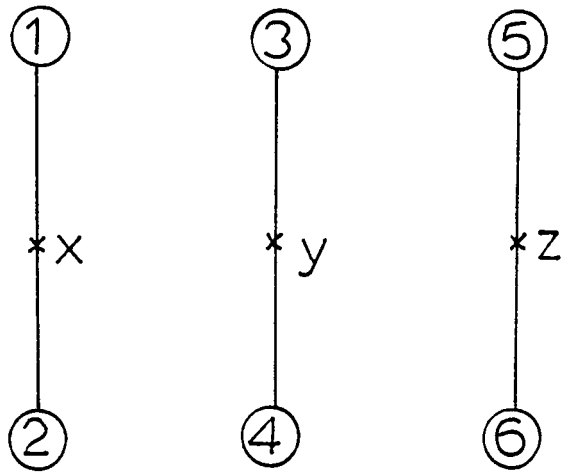


Figure 6.9. An example of three arcs.

This is a linear programming problem with 9 variables and 15 constraints. The decision variables are the c's and s_{xy}, s_{xz}, s_{yz}. Once their value is decided, the locations of x, y, and z can be easily obtained.

There are only four possible ways to go from point x to point y, namely, through nodes 1 and 3, nodes 1 and 4, nodes 2 and 3 or nodes 2 and 4. The first four constraints ensure that the shortest path distance s_{xy} among these four possible paths is accounted in the objective function. The next eight constraints serve the same purpose for s_{xz} and s_{yz}, respectively.

Problem (6.21) can be solved by iteratively examining all the triplets of arcs with the above linear programming formulation. This is a polynomial algorithm since the number of triplet of arcs is $\binom{|A|}{3}$ and the simple LP can be solved in a polynomial time as shown by Khachian (1979).

By the triangle inequality property of the shortest path distances, it is true that $s_{xy} + s_{xz} + s_{yz} \geqslant s_{xy} + s_{xy}$ for any point z on the network. Thus, it is guaranteed that the optimal solution for (6.21), which is (6.1) with $p = 3$, will be on three distinct points. However, for the $p \geqslant 4$ case, it may be optimal for (6.1) to have locations on the same point, and a similar linear programming cannot be used to solve it. As an example, consider the $p = 4$ case of (6.1) on the network shown in Figure 6.5 (see page 110). As indicated in Figure 6.10, the unique optimal solution is (x,y,z,w), where z and w are located on the same point, with an objective function value of 122. This is true since it can be verified that $s_{xw} + s_{yw} > s_{xw'} + s_{yw'} + s_{zw'}$

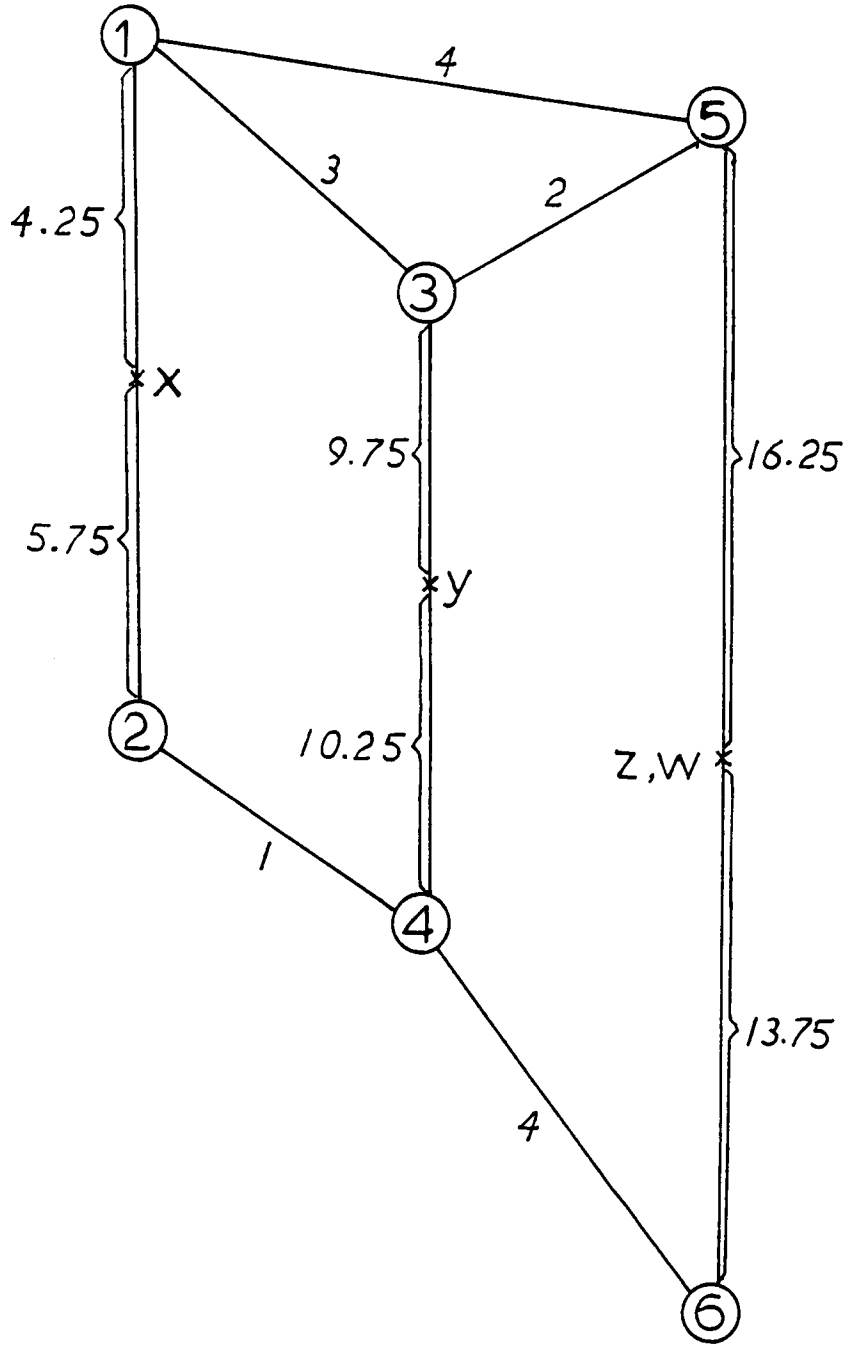


Figure 6.10. The solution for an example of (6.1).

for any other point w′ on the network. Therefore, the optimal solution for this example cannot be obtained by a linear programming formulation which examines all quadruplets of arcs.

CHAPTER 7

SUMMARY AND TOPICS FOR FURTHER RESEARCH

For network facility location problems, it is important that a FOSS can be identified. Otherwise, the infinite number of points on the network has to be considered as possible locations, and the task of finding optimal solution becomes formidable. After the FOSS is found, the problem can be formulated and solved by integer programming. In this dissertation, great effort has been spent in finding FOSS's for various models for the strictly undesirable facility location problem on a network. The results are summarized in Table 7.1.

Table 7.1. Summary of FOSS's for various models

Models	p = 1	p = 2	p ⩾ 3
Generalized Diameter Problem and Extensions--(6.1)		$N \cup B_A$	FOSS is not in $N \cup B_A \cup B_B \cup M \cup B_M$
Median-type Objective Function--(4.4)	$D \cup B_A$ [Church & Garfinkel (1978)]	$N \cup B_A \cup B_B$	
Center-type Objective Function--(5.10)	$N \cup B_A \cup B_C$	FOSS does not exist.	

From Table 7.1, it can be seen that FOSS's exist for the generalized diameter problem, for the median-type objective functional formulation with the p = 1 and p = 2 cases, and for the center-type objective functional formulation with p = 1 case. For all these cases, proofs and algorithms are provided in the related chapters. Counter-examples are provided for cases where no FOSS exists.

Locating undesirable facilities on a network is a new field of research and potential areas for further research are abundant. Based on the results of this dissertation, a few such topics are identified in the following sections.

7.1 Study of Objective Functional Forms

The models discussed in this dissertation employ the two most popular functional forms as the objective functions, namely, the median- and center-type objectives. An interesting study would be to search for other types of objective functional forms in modeling the location of undesirable facilities. This would be especially valuable for the model developed in Section 2.2 for partially undesirable facility location problem where no specific functional form is provided for either the accessibility function G(X) or undesirability function F(X). An excellent reference for this area of research is by Church and Bell (1978).

7.2 Enhancement of the Computer Graphics Capabilities

The capability and efficiency of the interactive computer graphics applications are heavily dependent upon the available graphics hardware. For instance, using the Tektronix 4015 graphics terminal, the interactive computer graphics program developed for algorithm 3.1 erases and redraws the whole display for each iteration. A graphics terminal that can perform partial erasure would reduce the amount of required drawing by only erasing the unneeded portion of the display (e.g., circles drawn in previous iteration) and keeping the background (e.g., the digitized road network) for the next iteration. The amount of drawing can also be reduced by using a color graphics terminal where different colors may be utilized to represent the results of different iterations which in turn will avoid the generation of the whole display for every iteration. Rosenthal (1979) describes the transfer of an elementary implementation (Tektronix 4010 graphics terminal/DEC 10 host computer) of an interactive computer graphical location analysis system to an advanced implementation with a Chromatics CG 1999 color graphic microcomputer. As a result of this conversion, the speed, accuracy, user convenience and program applicability are substantially enhanced. The potential of using more advanced graphics hardware to solve similar realizations

of the partially undesirable facility location problem presented in Chapter 3 is an area which could be further explored.

7.3 A Better Approximation of the Road Distances

Due to storage and processing considerations, the digitized road map, instead of the actual road map, is used in Chapter 3 in the application of interactive computer graphics techniques. The network shortest path distances (or the road travel distances) between grid points are approximated by their corresponding Euclidean distances. In other words, the linearized distances between grid points is used as an approximation for the actual road distances. This approximation introduces errors and this is the major shortcoming in using the graphics technique to solve the partially undesirable facility location problem on a network. If a better approximation of the actual road distances other than the Euclidean distances can be obtained and an error bound can be specified for the approximation, the graphics approach will become much more appealing. A pioneering study on this subject is found in Love and Morris (1979) where statistical tests are performed to compare the accuracy of using various functional forms in estimating urban and rural road distances. By choosing a proper functional form and an appropriate number of grid points, the accuracy of the graphics solutions can be greatly improved.

7.4 Strictly Undesirable Facility Location Problems with Constraints

For a general problem of locating p strictly undesirable new facilities, it has been shown that there is no FOSS for either median- or center-type formulation [(4.4) and (5.10)]. Since the entire network has to be considered as the solution space, the task of finding the optimal solution becomes formidable. In this case, unless a new formulation is considered, a heuristic solution is usually the only resort.

A possible new approach is presented as follows. Both (4.4) and (5.10) contain two components, one of which measures the interactions between existing and new facilities and the other between the new facilities themselves. The new facilities interaction component prevents locating the new facilities on the same point. The following formulation can achieve the same purpose:

$$\max \sum_{j=1}^{p} \sum_{i \in N} w_i s_{ix_j} \tag{7.1}$$

s.t.

$$s_{x_k x_q} > v \qquad 1 \leqslant k < q \leqslant p \tag{7.2}$$

Comparing this formulation with (4.4), it is seen that (7.1) is a simpler objective function which contains only the existing and new facilities interaction component. The

constraint (7.2) prevents the new facilities from being located on the same point. Although the effect is similar to the second component of (4.4), it is carried out in a stronger form. [Under (4.4), it is still possible to have new facilities located on the same point.]

A similar approach can be applied to (5.10) to obtain

$$\max \min_{i \in N} w_i s_{iX} \tag{7.3}$$

s.t.

$$s_{x_k x_q} > v \qquad 1 \leqslant k < q \leqslant p \tag{7.4}$$

Problems (7.1) to (7.4) are reasonable formulations for the strictly undesirable facility location problem. The study of properties as well as solution techniques for (7.1) through (7.4) will provide more insight into the strictly undesirable facility location problem on a network.

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