To the Graduate Council:

I am submitting herewith a thesis written by Wen Hsiang Lu entitled "An Investigation of a Resource-Consumer System by Using Continuous System Modeling Program and Stability Analysis." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

Thomas G. Hallam, Major Professor

We have read this thesis
and recommend its acceptance:

Accepted for the Council:

The Graduate School

AN INVESTIGATION OF A RESOURCE-CONSUMER SYSTEM BY USING CONTINUOUS SYSTEM MODELING PROGRAM AND STABILITY ANALYSIS

A Thesis

Presented for the

Master of Science

Degree

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ABSTRACT

This thesis presents two resource-consumer systems, one based upon the classical Gallopin model and another based upon the classical Gallopin model combined with the Smith's hypothesis. The models are studied via numerical and analytical techniques. The Continuous System Modeling Program is used to study the effect of parameters on the response of these two systems. The analytical approach used in the stability analysis consists of linearization combined with the Hurwitz stability criterion.

Necessary and sufficient conditions for local stability for both the extended Gallopin and Smith systems are found. In general, more restrictions are needed to guarantee stability in the extended Smith system than are necessary for the extended Gallopin model. From the computer output, a similar result is obtained. More damping and restrictions on parameters are observed.

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CHAPTER I

INTRODUCTION

An ecosystem has properties derived from its component populations, from the interactions between these populations, and from the physical environment. The complexity of an ecosystem is a result not only of the number of component elements (populations and environmental variables), but also the characteristics of their interactions. In order to study such a system, it is often important to reduce their constituent elements to essentials, and to center the research on the interactions between the constituents. The objective of this study is to develop a simple population model which could be used for theoretical studies of ecosystems, and to investigate the role of higher order rates of change in model formulations. This population model includes two main variables: population size and resource utilization. It describes the growth of a hypothetical population in a resource-limited environment.

In general, it is reasonable to postulate that a population growing in a limited environment will tend to grow as far as the environment allows it. This implies that for a given set of environmental conditions, there is one and only one maximum sustainable size for a given population. This size may or may not be reached at a given time, but it will be reached or oscillated around, given enough time. It is reasonable to consider a population growing in a limited environment as a control system in which the controlled variable is its own size. In particular, consider a population producing or maintaining biomass

in a given set of environmental conditions. Such a population can be described as a black box characterized by a set of state variables. The system receives an input (resource), loses part of the input and produces an output (biomass). This is illustrated in Figure 1 with the population being embedded in the environment. The environment can act upon the size of the population in many conceivable ways including:

1. Directly affecting the resource input by altering the population size.
2. Affecting the state variables, which in turn will affect the resource input and the loss.

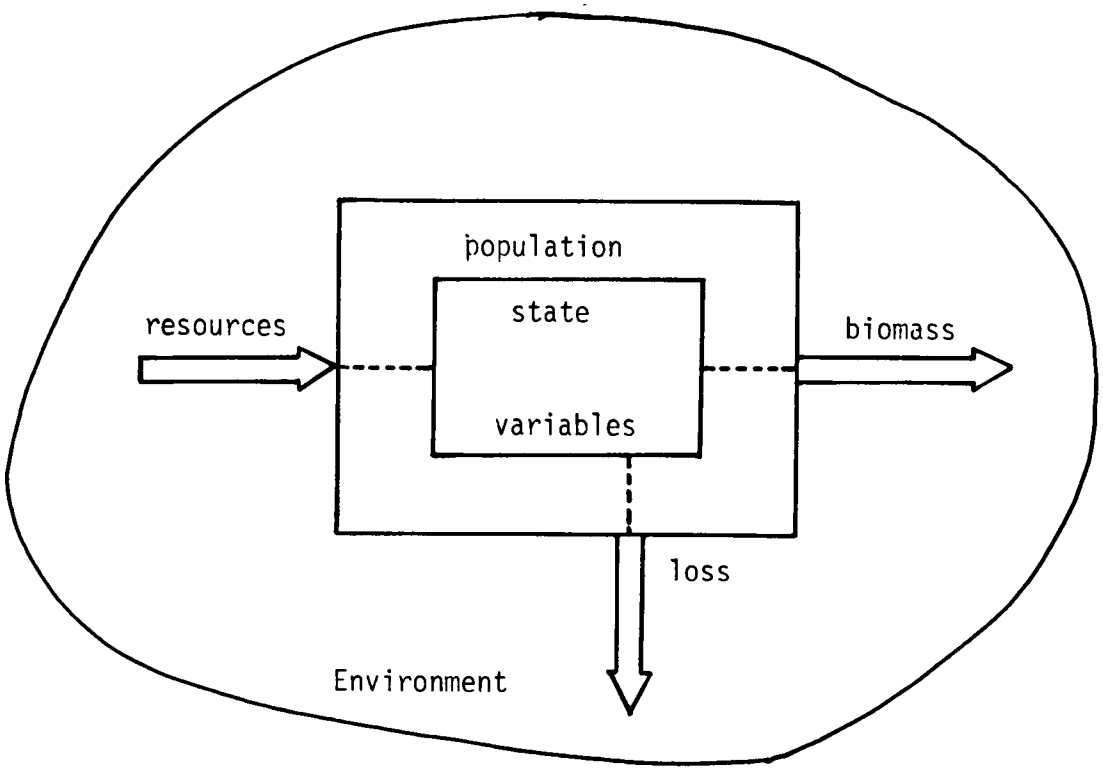

Figure 1. A general population-resource system

Within the context given above, a limiting factor (or set of factors) will be defined as one that determines the maximum size the population can reach in any given situation. Different sets of limiting factors can act at different points in time. Therefore, the size of the population will depend on the effects of all the relevant factors, including the limiting factors in the environment, and the factors inherent in the population itself.

The objective of this thesis is to derive mathematical models of resource-consumer systems based upon the Gallopin model and Smith's hypothesis. Based on the derived mathematical models, Continuous System Modeling Program, an IBM/360 program for the simulation of continuous system, is applied. Furthermore, the Hurwitz stability criterion is used for stability analysis.

CHAPTER II

GENERAL PROPERTIES OF A RESOURCE-CONSUMER SYSTEM

Before the model of ecosystem studied here is delineated, the following assumptions are noted:

Assumption 1. The environment, other than the resource, will be considered constant in order to simplify the situation and allow the study of the basic relations.

Assumption 2. There is a single limiting factor, the resource, acting on the consumer population.

Assumption 3. It will be assumed that the resource enters the system at a constant rate.

Assumption 4. The biomass of the population and the resource are homogeneously distributed in the physical space.

Assumption 5. The excreted resource is not available for reingestion.

Assumption 6. The relevant average physiological state of the organisms is reflected in the biomass and its derivatives.

As in the system shown in Figure 1, the consumer population will ingest part of the resource, and the ingested resource will be transformed into biomass except for the part that is lost by defecation or egestion. In order to study the resource-population model carefully, variables of the ecosystem will be defined and discussed.

1. Rate of Resource Input (f)

The rate of resource input to the system is constant, and it is

assumed to not be affected by the population. Therefore,

$$f = \frac{dF}{dt} = \text{constant} \tag{1}$$

where F is the amount of food entering the system.

2. Amount of Resource Available (A)

The amount of resource available for ingestion at any time is the difference between the resource that has entered the system and the resource that has been ingested by the population. Therefore,

$$A = F - C \tag{2}$$

where C is the amount of food ingested by the population.

The lifetime (θ) of the resource is defined as the time that the resource is available to the population. After θ has elapsed, the resource may be destroyed, may decay, or may be eaten by other populations.

3. Ingestion Rate (c)

The definition of the ingestion rate is more difficult to specify. The ingestion rate is related to: (1) the amount of food available (A), (2) the biomass of the population (M), (3) the time derivatives of the biomass ($\frac{dM}{dt}$, $\frac{d^2M}{dt^2}$, etc.) and (4) the characteristics and distribution of the resource and of the physiological state of the individuals. The relationship between the ingestion rate and the

characteristics and distribution of the resource is quite complicated and is discussed in Ivlev's paper (1961). In order to simplify our problem, assumptions 4 and 6 are being added.

As in Smith's paper (1963), it is proposed that the ingestion rate is related to some of the time derivatives of the biomass. This could account for the fact that growing organisms consume more food per unit mass than non-growing ones. In addition, it is suspected that the second order term (d^2M/dt^2) may have effects on the ingestion rate. That is, the faster growing organisms may consume more food per unit mass than slower growing ones. Therefore, the first and second time derivatives are considered in the present study.

It is evident that the ingestion rate cannot be infinite, even if there is an infinite amount of food available, and the amount of food ingested in a given period of time cannot exceed F. With all the above considerations, one has

$$C = g_1(M, dM/dt, d^2M/dt^2, \ldots, A) \quad . \tag{3}$$

Consider a simple relationship between the ingestion rate and biomass. From Gallopin's paper (1971), the rate of change in the ingestion rate per unit change in the food available is proportional to the hunger of the population. That is, if the population is satiated, an increase in available food will not increase the ingestion rate; but if the population is hungry, an increase in the food available will produce a significant increase in the ingestion rate. Therefore,

$$\frac{\partial c}{\partial A} = h(M)(aM - c) \tag{4}$$

where:

a: a constant; the ingestion rate per unit mass which corresponds to maintenance of complete satiation

Dimension: (1/time)

aM: the total ingestion rate at maintenance of satiation

aM - c: instantaneous hunger of the whole population

h(M): a proportionality coefficient depending on the biomass of the population.

Integrating with respect to A,

$$c = aM + K(M)e^{-h(M)A} \quad . \tag{5}$$

When $A = 0$, $K(M) = -aM$. Therefore,

$$c = aM(1 - e^{-h(M)A}) \tag{6}$$

This is the general expression for the ingestion rate of the population. The coefficient $h(M)$ may have different forms. The simplest assumption is that:

$$h(M) = \alpha/M \tag{7}$$

with α a dimensionless constant so that an increase in A has a greater effect on the ingestion rate when M is small than when M

is large. Then, we have

$$c = aM(1 - e^{-\alpha A/M}) \quad . \tag{8}$$

Following Smith's idea, the time derivatives are considered in the model. Assuming that a growing population (positive dM/dt) has a higher ingestion rate than a non-growing population of the same biomass leads to

$$\frac{\partial c}{\partial A} = \frac{\alpha}{M}[(aM + b\frac{dM}{dt}) - c] \quad . \tag{9}$$

Now integrating with respect to A ,

$$c = (aM + b\frac{dM}{dt})(1 - e^{-\alpha A/M}) \tag{10}$$

4. Loss Rate (λ)

The loss rate is defined as

$$\lambda = e + \gamma + x + d$$

where

e = egestion rate

γ = rate of respiration of consumer

x = rate of excretion of consumer

d = rate of death of the population

The simplest assumption is that the loss rate (λ) is proportional to

the biomass of the population. Therefore

$$\lambda = kM \tag{12}$$

By considering the terms individually involved, it is possible to formulate a more general model.

(i) Egestion Rate (e)

This is the rate at which the non-assimilated resource is being discarded; hence, the egestion rate can be represented as

$$e = g_2(c) \quad . \tag{13}$$

Since the egestion rate is proportionally related to the ingestion rate, it is reasonable to assume a first order approximation:

$$e = uc \tag{14}$$

where u is a constant for the simplest case, or a variable in the more complex ones. That is, a higher ingestion rate will have a higher egestion rate.

(ii) Respiration Rate (γ)

For a constant environment, the respiration rate can be assumed to be proportional to the biomass of the population and its derivatives. This is based on Smith's assumption that a growing population has a higher ingestion rate, a higher respiration rate, and a higher excretion

rate than a non-growing population of the same biomass. In addition, a faster growing population may have a higher respiration rate and excretion rate than a slower growing one. Therefore, in general,

$$\gamma = g_3(M, \frac{dM}{dt}, \frac{d^2M}{dt^2}, \ldots) \tag{15}$$

In particular, if one assumes γ is linear in its variables,

$$\gamma = k_1 M + \gamma_1 \frac{dM}{dt} + \gamma_2 \frac{d^2M}{dt^2} \tag{15'}$$

where k_1, γ_1, and γ_2 are constants.

(iii) Excretion Rate (x)

Based on Smith's assumption, the excretion rate is written as:

$$x = g_4(M, \frac{dM}{dt}, \frac{d^2M}{dt^2}) = k_2 M + w_1 \frac{dM}{dt} + w_2 \frac{d^2M}{dt^2} \tag{16}$$

where k_2, w_1, and w_2 are constants.

(iv) Death Rate (d)

In general, the death rate is assumed to be:

$$d = g_5(M, \frac{dM}{dt}, \ldots, P_1) \tag{17}$$

where P_1 = predation rate.

In the absence of predation, the natural death rate is assumed to be a function of the biomass and its derivatives. Therefore the natural death rate may be:

$$d = g_6(M, \frac{dM}{dt}, \ldots) = k_3 M - q_1 \frac{dM}{dt} - q_2 \frac{d^2M}{dt^2} \tag{18}$$

where q_1 and q_2 are positive constants.

Equation (18) implies that the death rate increases when the biomass of the population is decreasing $(dM/dt < 0)$ and the rate of the population biomass is decreasing $(d^2M/dt^2 < 0)$, and it increases when the population biomass is growing $(dM/dt > 0)$ and the rate of population biomass is increasing.

In conclusion, the total loss rate is

$$\begin{aligned}
\lambda &= e + \gamma + x + d \\
&= uc + k_1 M + v_1 \frac{dM}{dt} + v_2 \frac{d^2M}{dt^2} + k_2 M + w_1 \frac{dM}{dt} + w_2 \frac{d^2M}{dt^2} \\
&\qquad + k_3 M - q_1 \frac{dM}{dt} - q_2 \frac{d^2M}{dt^2} \\
&= uc + (k_1 + k_2 + k_3)M + (v_1 + w_1 - q_1) \frac{dM}{dt} \\
&\qquad + (v_2 + w_2 - q_2) \frac{d^2M}{dt^2}
\end{aligned} \tag{19}$$

where $k = k_1 + k_2 + k_3$ = maintenance cost per unit mass of the population with dimensions (1/time), $L = v_1 + w_1 - q_1$ is dimensionless and $P = v_2 + w_2 - q_2$ with dimension (time/mass).

To put all the above variables of the system together, the relationship between the ingestion rate, the growth rate and the loss rate is assumed to be:

$$\frac{dM}{dt} = c - \lambda \tag{20}$$

Substitution of equation (19) into equation (20) and rearrangement leads to

$$c(1 - u) = P \frac{d^2M}{dt^2} + (L + 1) \frac{dM}{dt} + kM \tag{21}$$

Let

$$\gamma = \frac{P}{1 - u} \ , \ \sigma = \frac{1 + L}{1 - u} \ , \ \beta = \frac{k}{1 - u} \quad ;$$

equation (21) becomes

$$c = \gamma \frac{d^2M}{dt^2} + \sigma \frac{dM}{dt} + \beta M \tag{22}$$

(I) Basic Equations of the Model

According to Gallopin's model, the general fundamental equations are:

$$\begin{aligned} A &= F(t) - C \\ c &= aM(1 - e^{-h(M)A}) \\ c &= \frac{dM}{dt} + \lambda \end{aligned} \tag{23}$$

It is assumed that $h(M) = \alpha/M$ and the lifetime of the resource (θ) is infinite. Substituting equations (8) and (22) into equation (23) and then differentiating A with respect to t, the resulting model is a system of two non-linear simultaneous differential equations:

$$\frac{dA}{dt} = f - c(t) = f - aM(1 - e^{-\alpha A/M}) \tag{24}$$

(A)

$$\gamma \frac{d^2M}{dt^2} + \sigma \frac{dM}{dt} + M(\beta - a + ae^{-\alpha A/M}) = 0 \tag{25}$$

The system can be represented by the block diagram as in Figure 2.

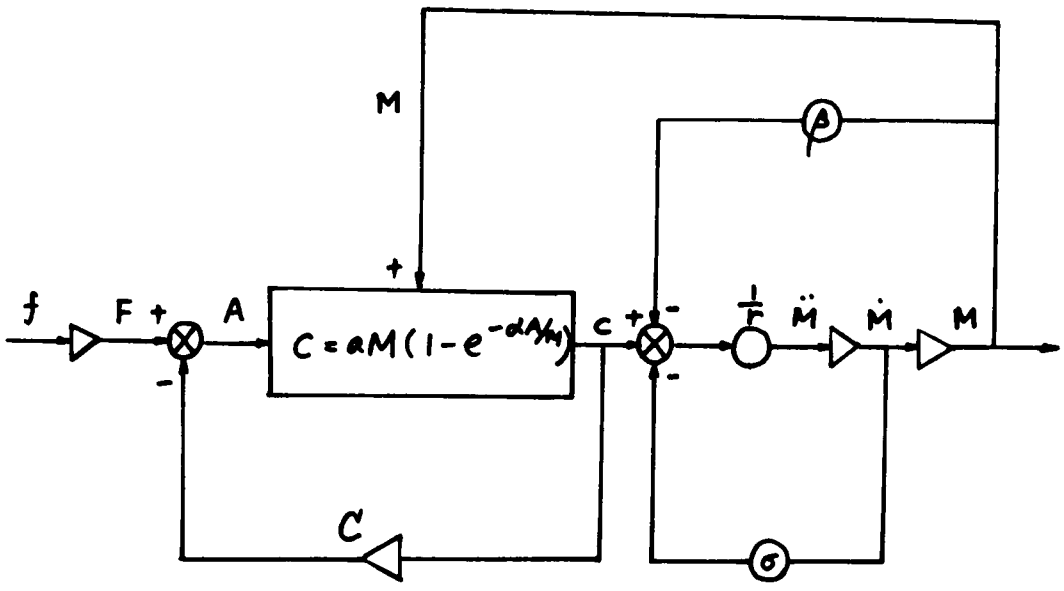

Figure 2. Block diagram of the resource-population system.

If Smith's model (equation 10) and equation (22) are substituted into equation (23), then the system of equations becomes:

$$\frac{dA}{dt} = f - c(t) = f - (aM + b\frac{dM}{dt})(1 - e^{-\alpha A/M}) \tag{26}$$

(B)

$$\gamma \frac{d^2M}{dt^2} + (\sigma - b + be^{-\alpha A/M})\frac{dM}{dt} + M(\beta - a + ae^{-\alpha A/M}) = 0 \tag{27}$$

The block diagram of this system is in Figure 3.

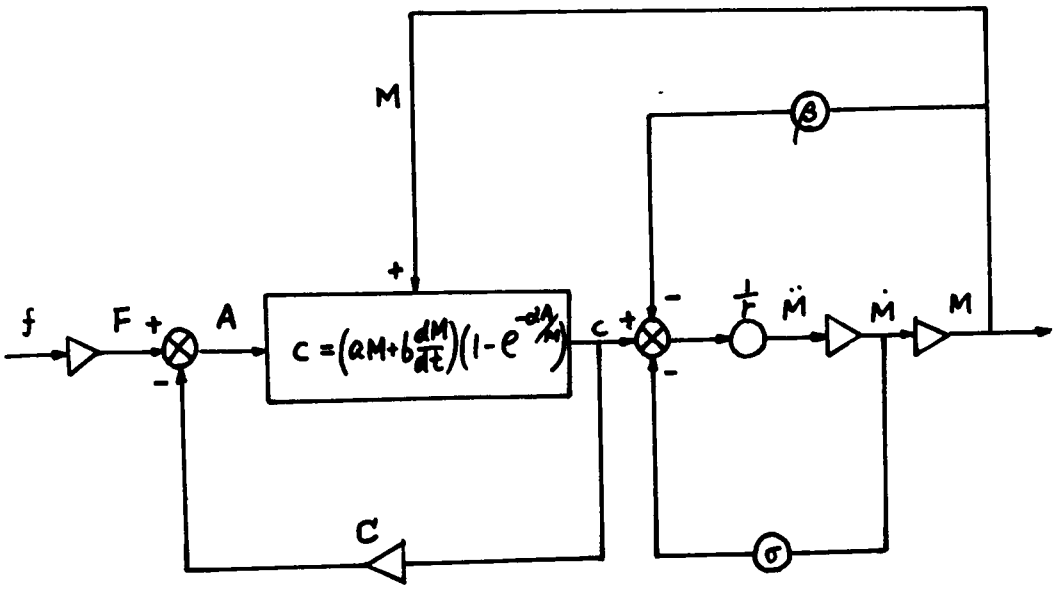

Figure 3. Block diagram of the resource-population system using Smith's model

(II) Sensitivity of the Steady State Values of the Variables to Changes in the Parameters

The steady state of a system is the state in which all the rates of change of the variables with respect to time are zero. Therefore, if

the system under consideration is at a steady state, then

$$\frac{d^2M}{dt^2} = \frac{d^2A}{dt^2} = 0 \quad \text{and} \quad \frac{dM}{dt} = \frac{dA}{dt} = 0 \quad \text{and} \quad M = M_0 = \text{constant};$$

$A = A_0 = \text{constant}$ and the system reduces to:

$$0 = f - aM_0(1 - e^{-\alpha A_0/M_0}) \tag{28}$$

$$0 = M_0(\beta - a + ae^{-\alpha A_0/M_0}) \tag{29}$$

Note that $M_0 = 0$ does not satisfy the equation (28) unless $f = 0$. For f different from zero, the equilibrium points of the system are:

$$M_0 = \frac{f}{\beta} = \frac{f(1 - u)}{k} \tag{30}$$

$$A_0 = -\frac{M_0}{\alpha} \ln(1 - \frac{\beta}{a}) = -\frac{f(1 - u)}{k\alpha} \ln(1 - \frac{k}{a(1 - u)}) \tag{31}$$

For the Smith model, the equilibrium points of system (B) are the same as system (A).

The sign of the derivatives of M_0, A_0 with respect to the parameters indicates whether a change in the value of the parameter will increase, decrease, or not affect the steady state values of the variables. Furthermore, the numerical value of the derivative gives an indication of whether the value of the variable at the steady state is strongly or weakly affected by the variation in the parameter. The

following development will be made for the case in which $k < a(1 - u)$, the only biologically interesting situation, in which the maintenance cost per unit mass of the population (k) is smaller than the assimilation rate per unit mass $(a(1 - u))$ and therefore the population is able to grow if provided with food.

Taking the derivatives of M_0 with respect to the parameters, the following may be obtained:

$$\frac{\partial M_0}{\partial a} = 0 \quad , \quad \frac{\partial M_0}{\partial \alpha} = 0$$

$$\frac{\partial M_0}{\partial \beta} = -\frac{f}{\beta^2} < 0 \qquad \frac{\partial M_0}{\partial (1 - u)} = \frac{f}{k} > 0$$

$$\frac{\partial M_0}{\partial k} = -\frac{(1 - u)f}{k^2} < 0$$

$$\frac{\partial M_0}{\partial f} = \frac{(1 - u)}{k} > 0$$

Similarly for A_0, a table for the analysis is formed.

From Table 1, the value of M_0 can be increased by increasing β or f, but it will decrease with an increase in the maintenance cost k, increasing instead the value of A_0, which is increased also with an increase in the resource input rate, as is to be expected; an increase in a, α, β will decrease the value of A_0. The ingestion rate at the steady state will be increased only by an increase in f.

Table 1. The qualitative effect of the parameters of the model on the steady state values of M, A, and C

	M_0	A_0	C_0
a	0	-	0
α	0	-	0
(1 - u)	+	-	0
k	-	+	0
f	+	+	+

CHAPTER III

COMPUTER SIMULATION

In order to show explicitly the behavior of the model under different conditions, Continuous System Modeling Program (CSMP) is applied in this study. The Fourth-order Runge-Kutta method of numerical integration over a fixed interval is used. Two systems: system (A) - equations (24) and (25), and system (B) - equations (26) and (27), are examined. The results are plotted from Figure 4 to Figure 17 for system (A) and system (B) is shown in Figures 18 to 33.

Figure 4 shows the behavior of the resource and the biomass simultaneously for a constant resource input rate when $a = 0.6$, $\alpha = 0.8$, $k = 0.5$, $\sigma = 2.0$, $\beta = 0.5$, $\gamma = 1.0$ and $f = 3.0$. Initially, the resource accumulates, because the population is too small to be able to ingest the resource at the same rate it enters. Then, the population overshoots its steady state value until it consumes the accumulated resource. It declines until it reaches the stable population size with the given resource input. The values of a, β, α, k, σ, γ, and f are the same as in the Gallopin's paper. The value of σ is 2.0 which is chosen to fit the results obtained by Gallopin. The value of σ cannot be negative. Otherwise, the system will be unstable. Figures 5 and 6 show the effect of a variation in the value of k over the population size and the resource available, respectively. An increase in k results in a decrease in the steady state value of the biomass, but it increases the steady state value of the resource available. Furthermore, the increase in k decreases the rate of

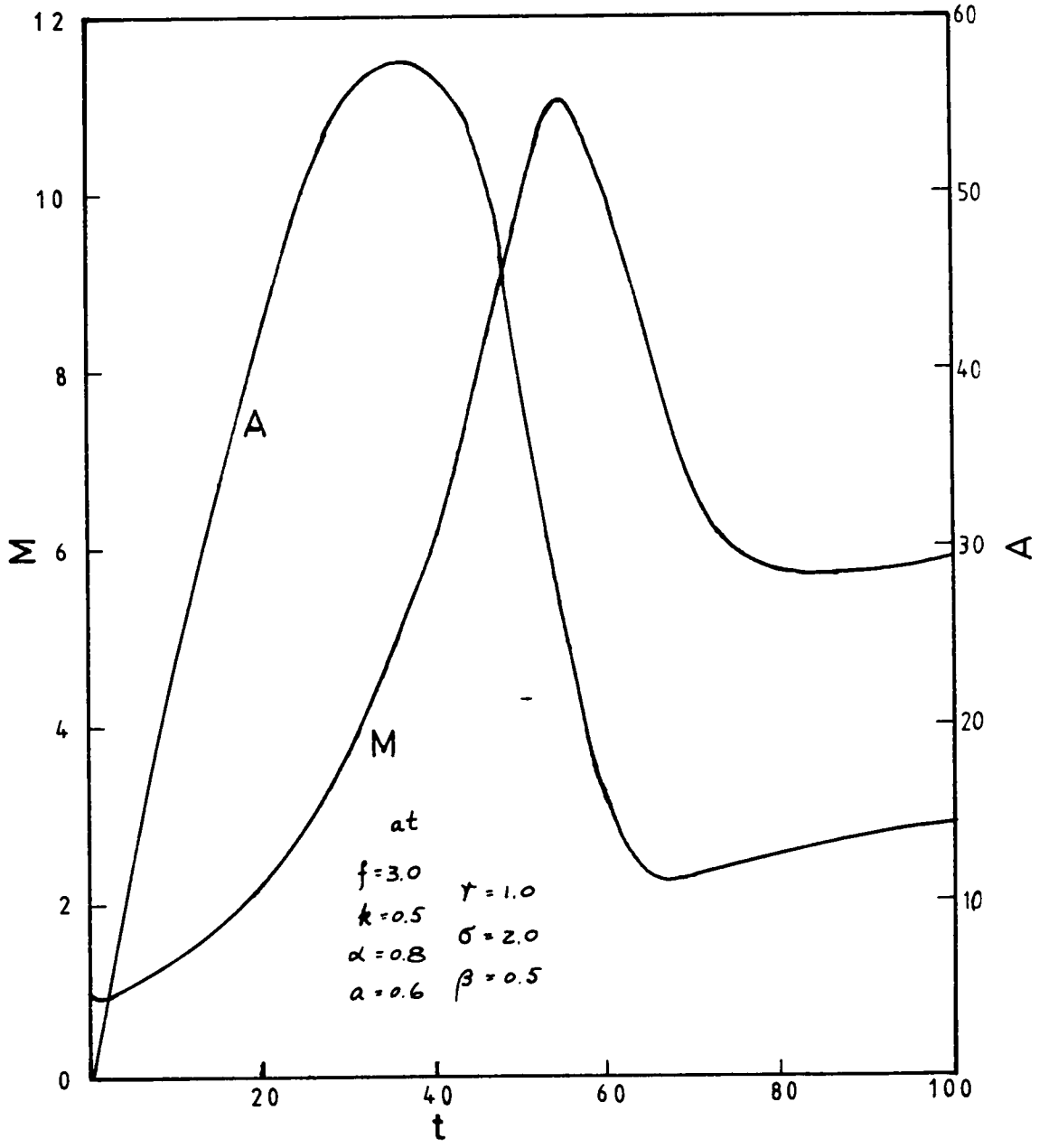

Figure 4. Behavior of the resource and the biomass for a constant resource input rate for System A

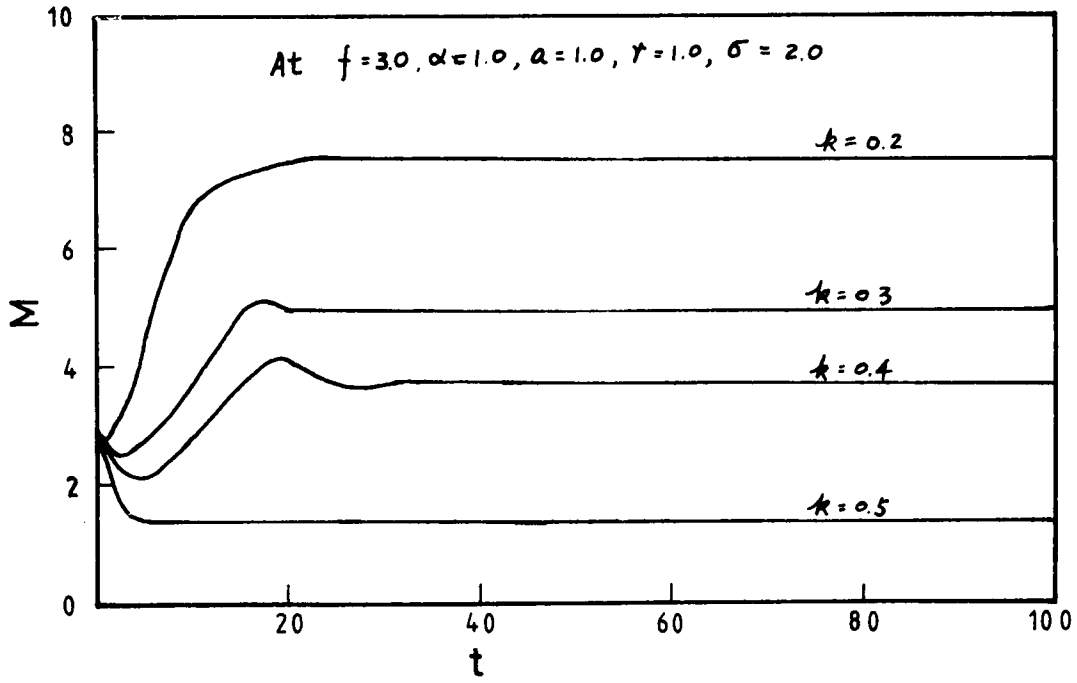

Figure 5. Effects of k on the behavior of M for System A

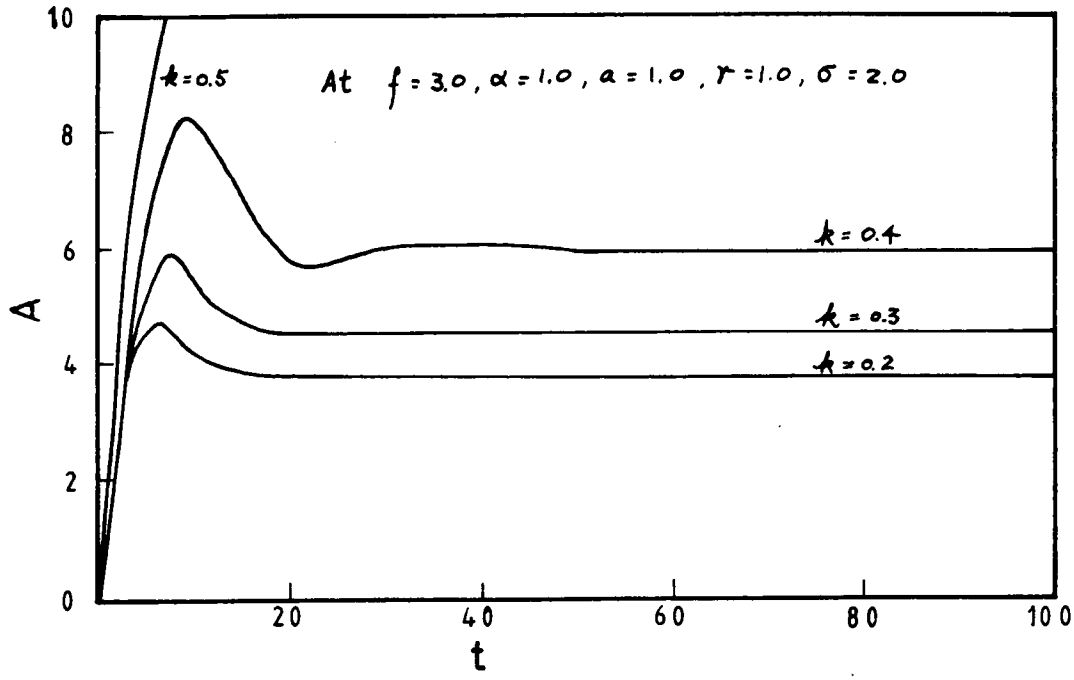

Figure 6. Effects of k on the behavior of A for System A

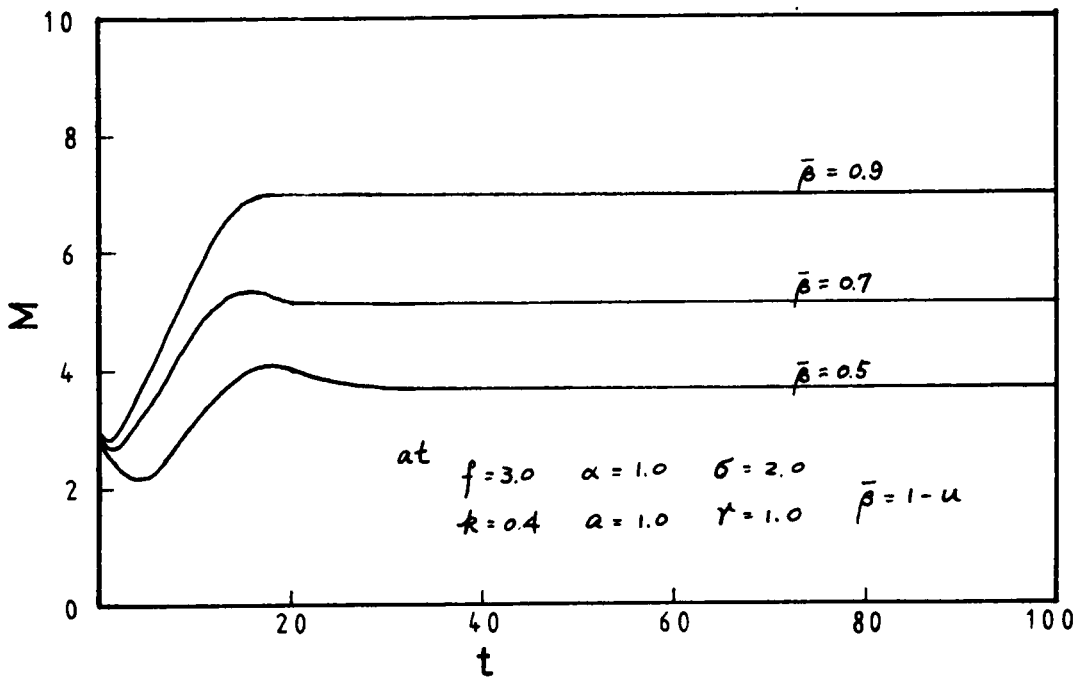

Figure 7. Effects of $\bar{\beta}$ on the behavior of M for System A

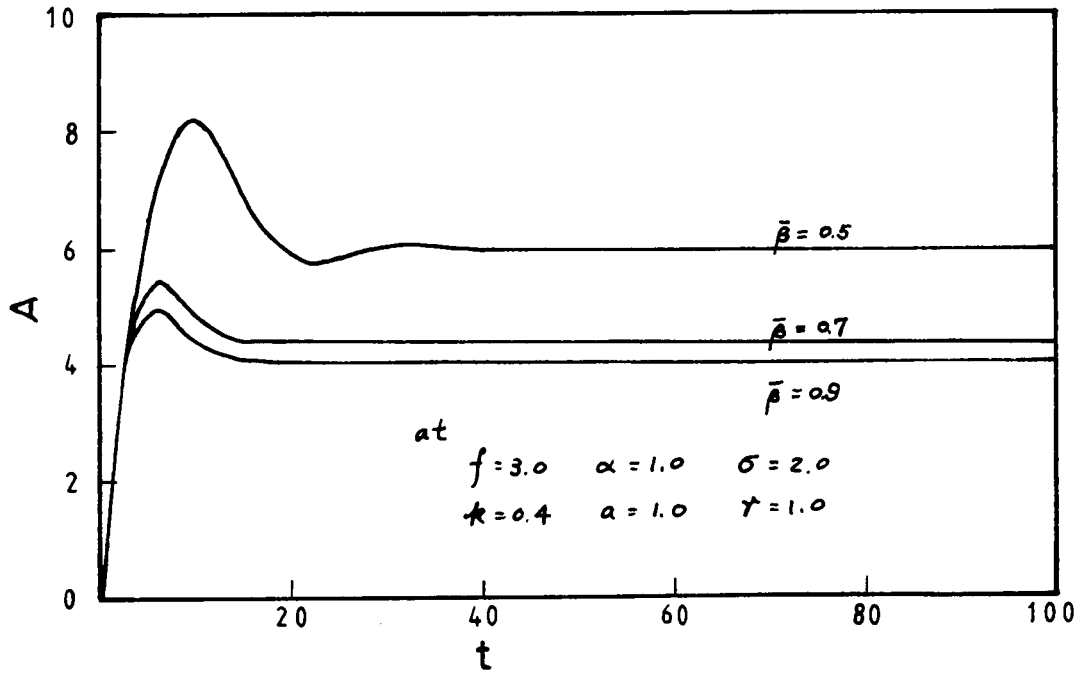

Figure 8. Effects of $\bar{\beta}$ on the behavior of A for System A

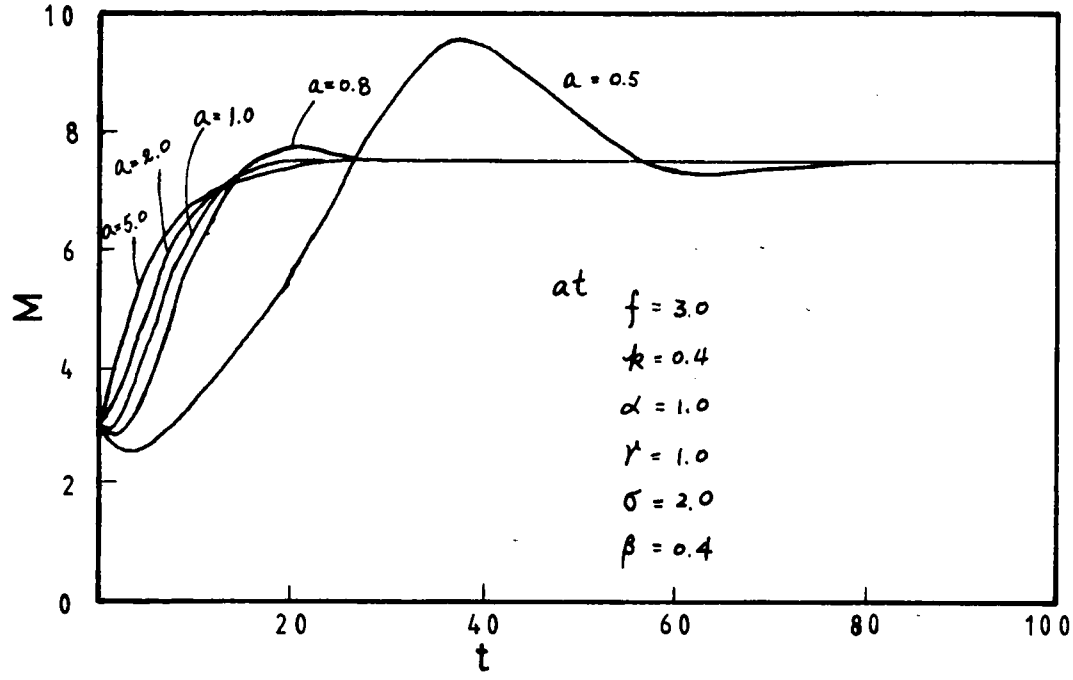

Figure 9. Effects of a on the behavior of M for System A

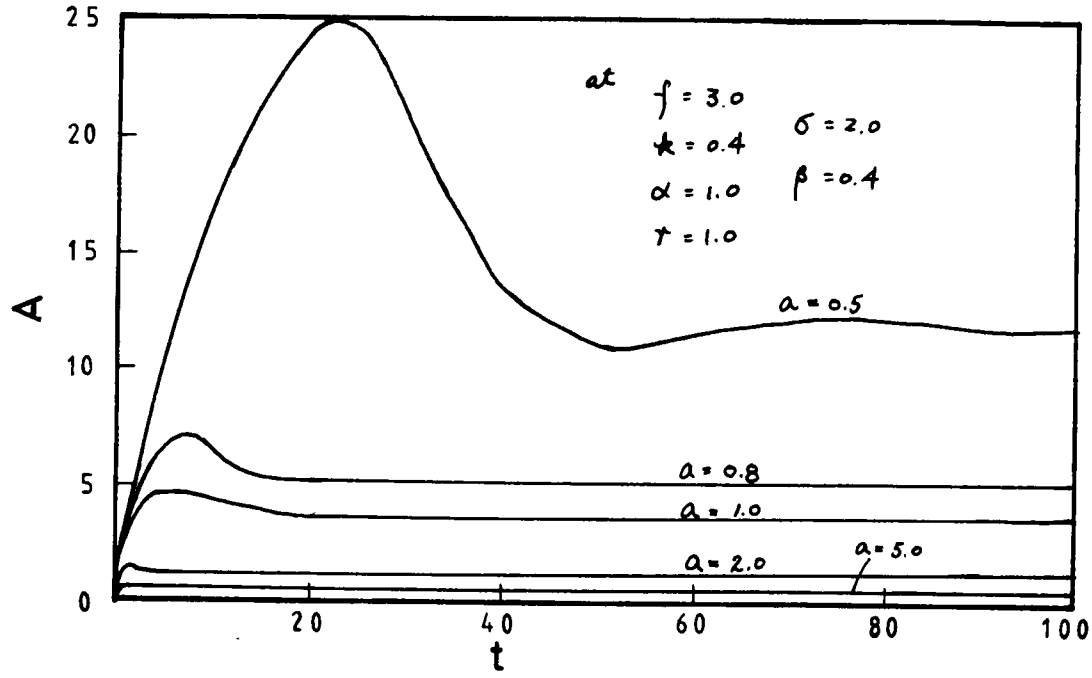

Figure 10. Effects of a on the behavior of A for System A

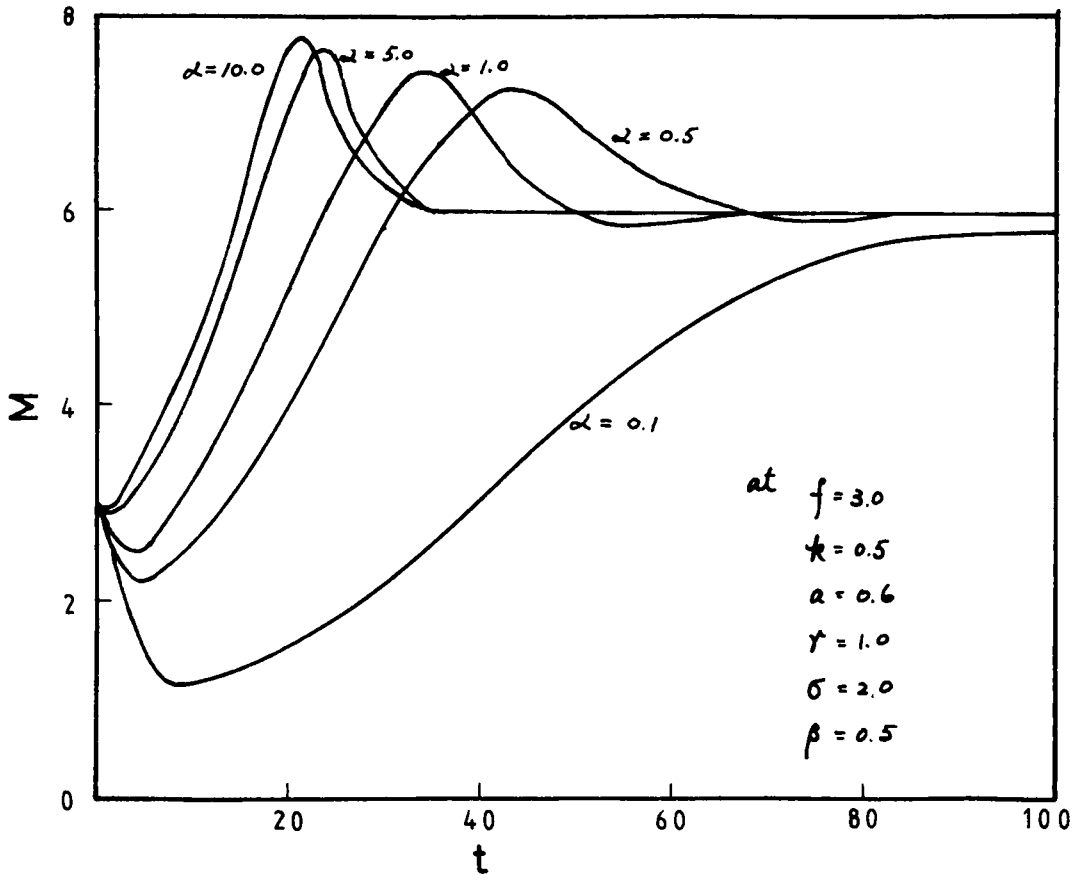

Figure 11. Effects of α on the behavior of M for System A

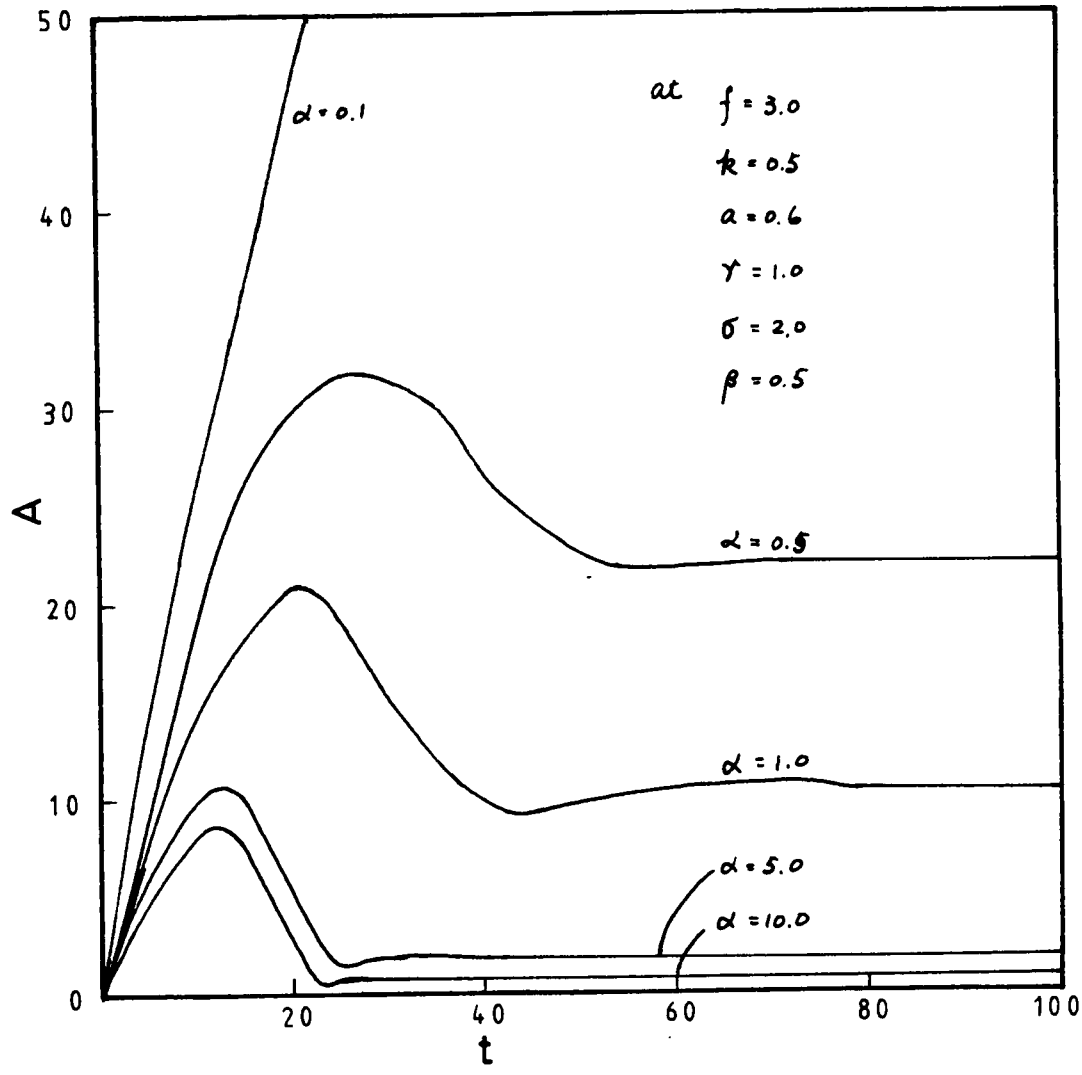

Figure 12. Effects of α on the behavior of A for System A

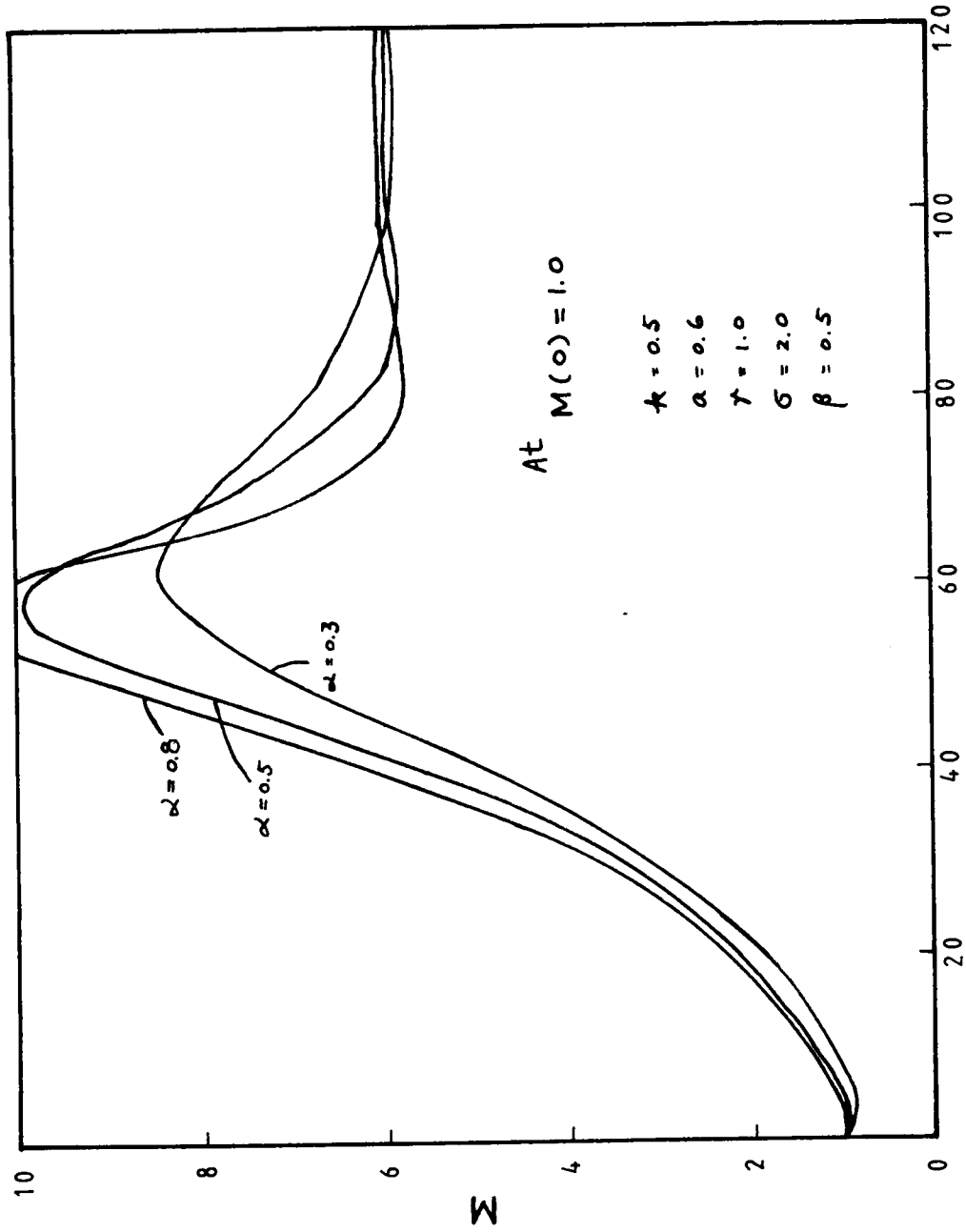

Figure 13. Effects of α on the behavior of M for M(0) = 1.0 different from Figure 11

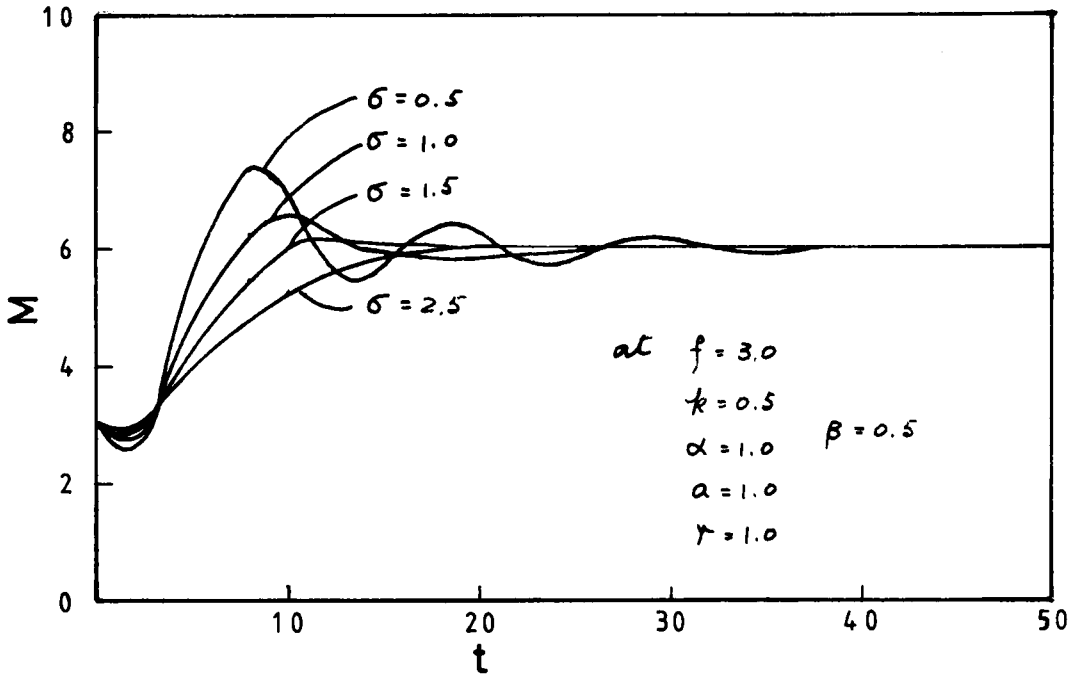

Figure 14. Effects of σ on the behavior of M for System A

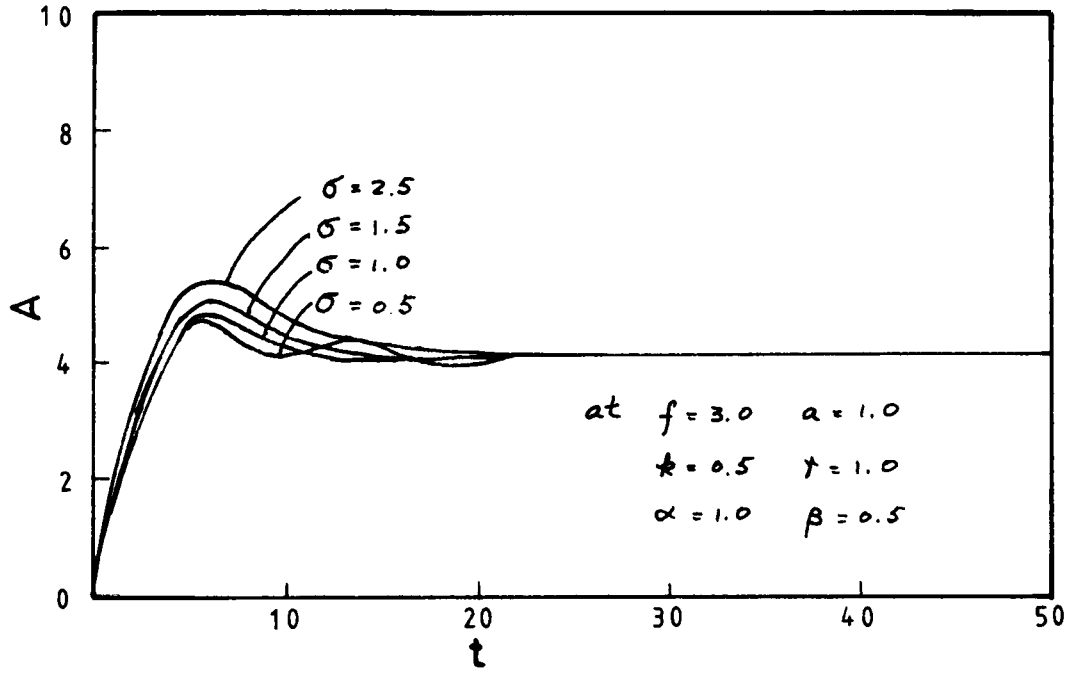

Figure 15. Effects of σ on the behavior of A for System A

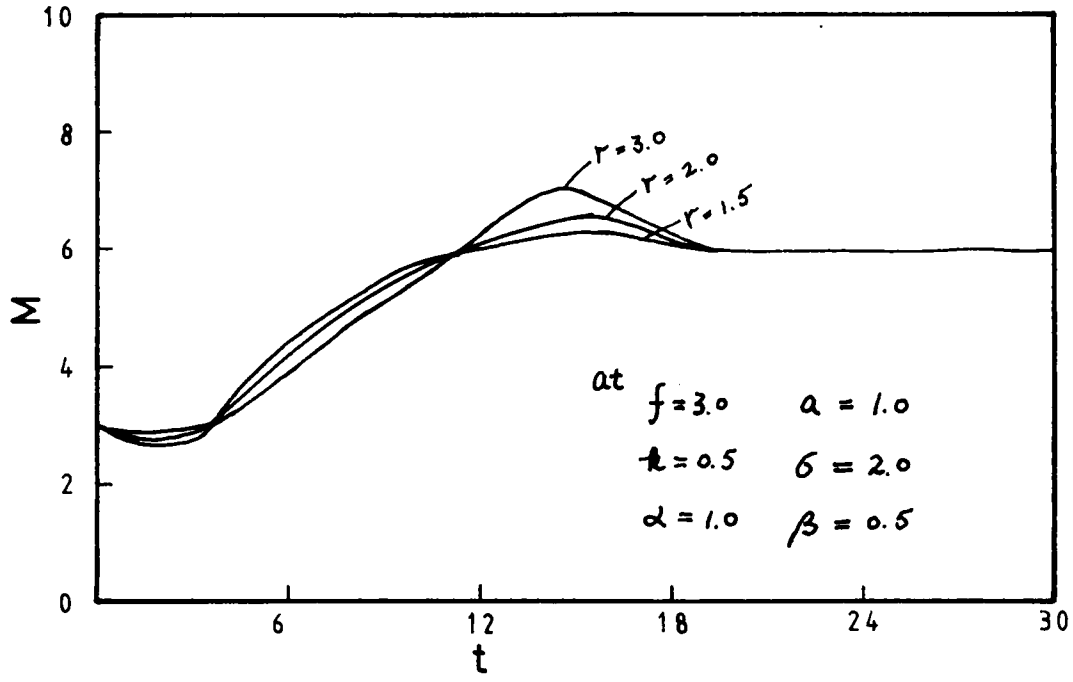

Figure 16. Effects of γ on the behavior of M for System A

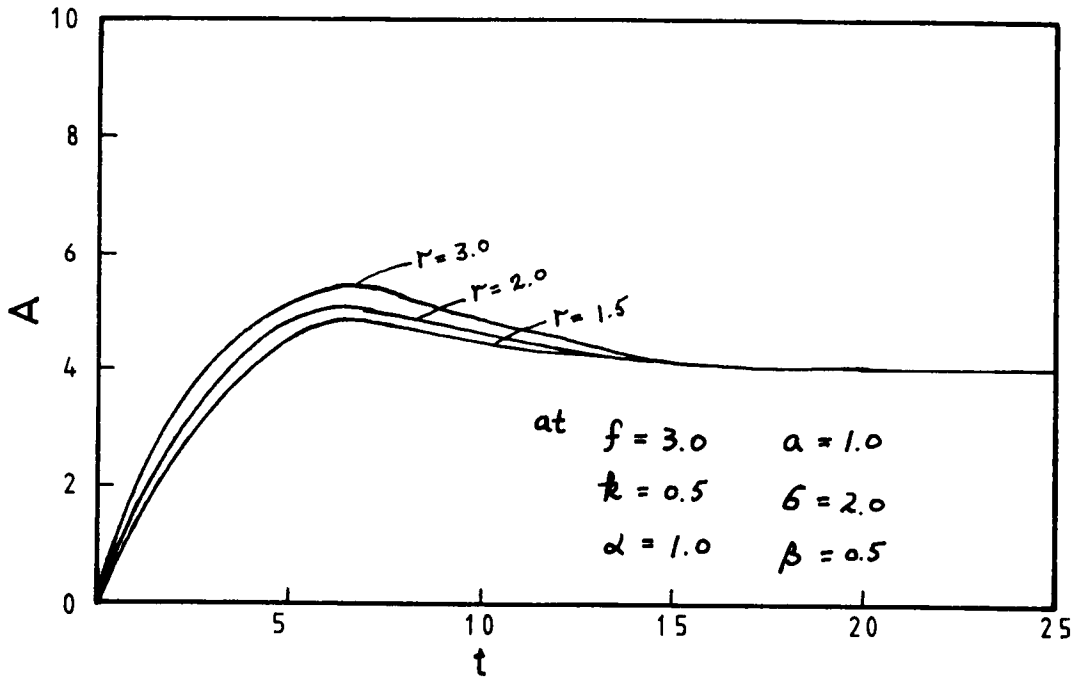

Figure 17. Effects of γ on the behavior of A for System A

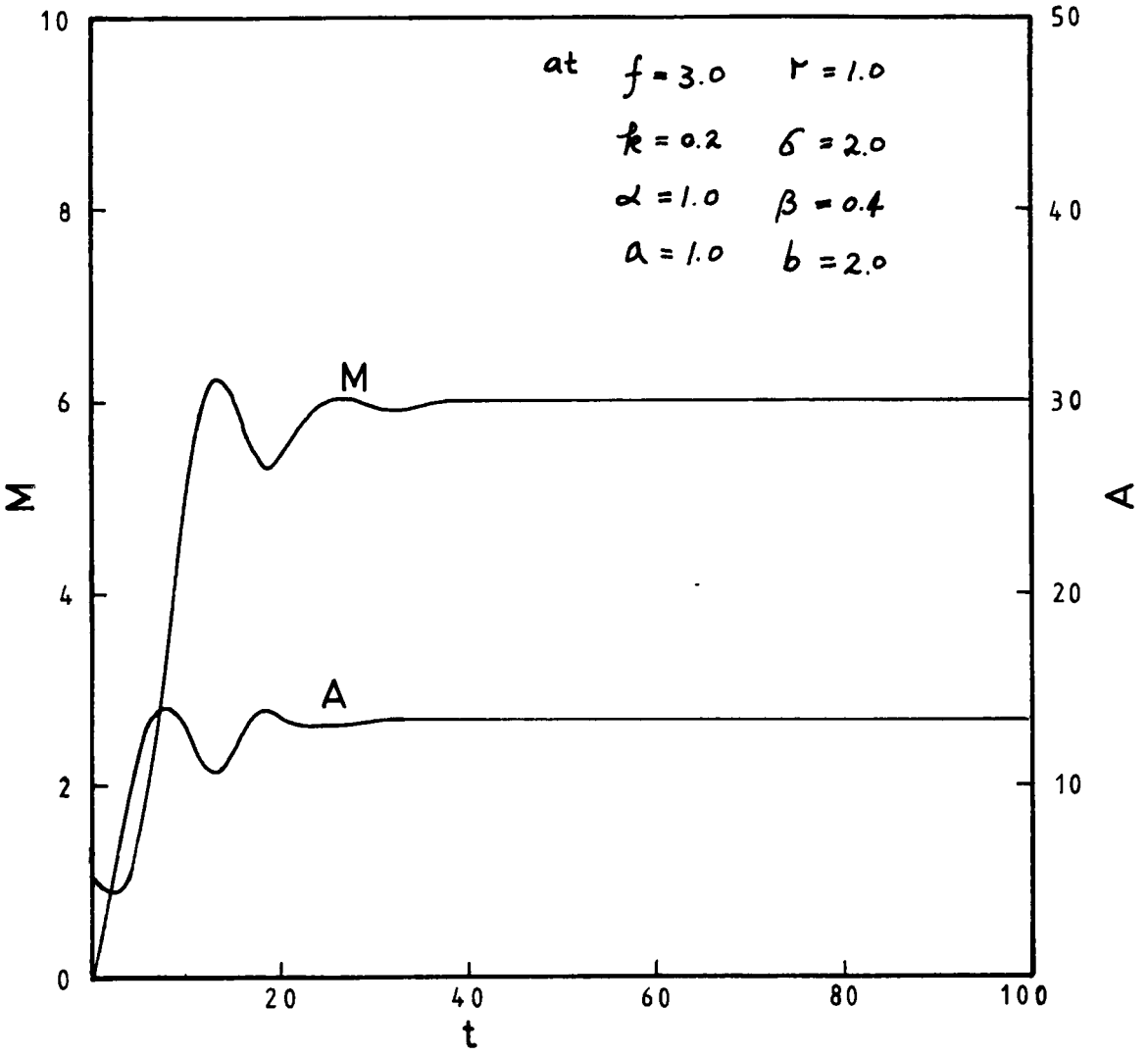

Figure 18. Behavior of the resource and the biomass for a constant resource input rate for System B

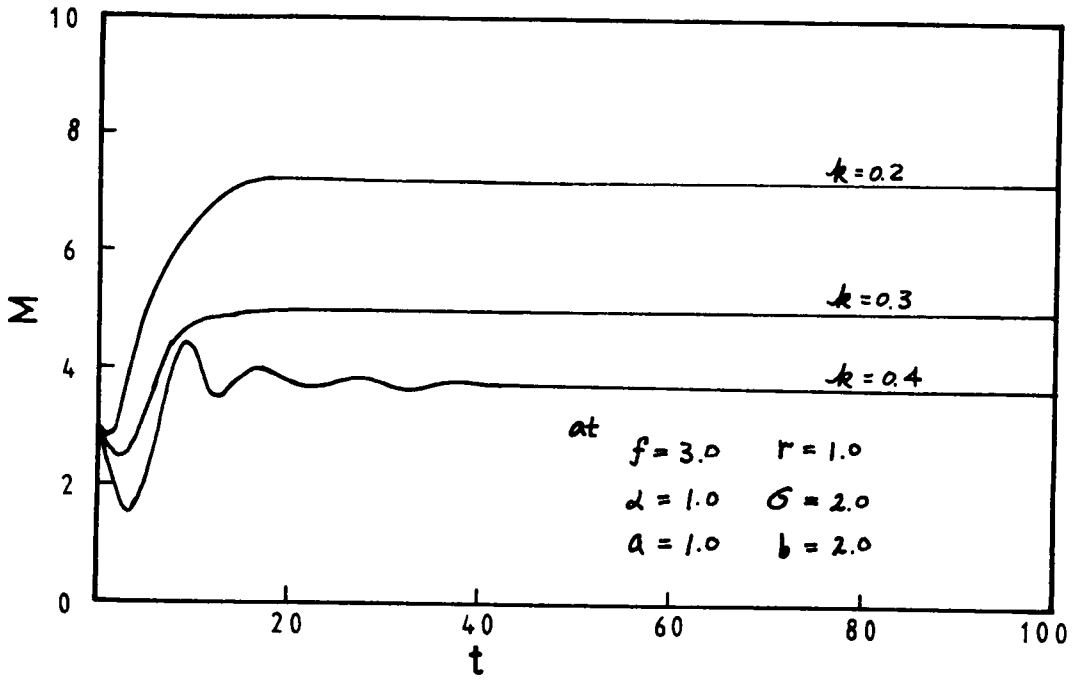

Figure 19. Effects of k on the behavior of M for System B

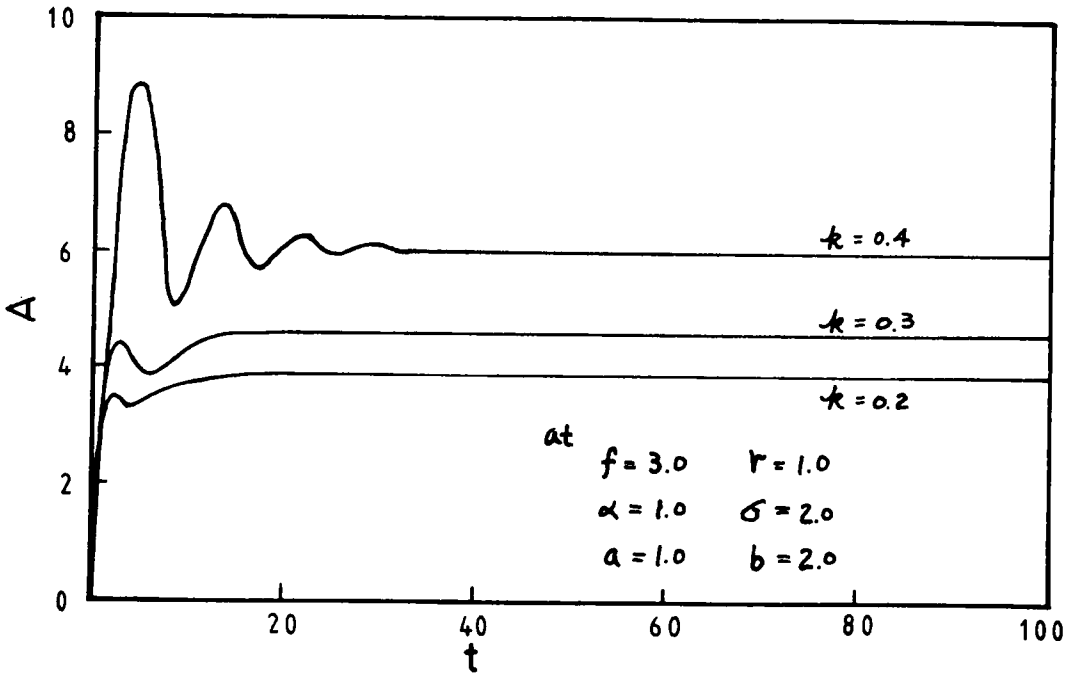

Figure 20. Effects of k on the behavior of A for System B

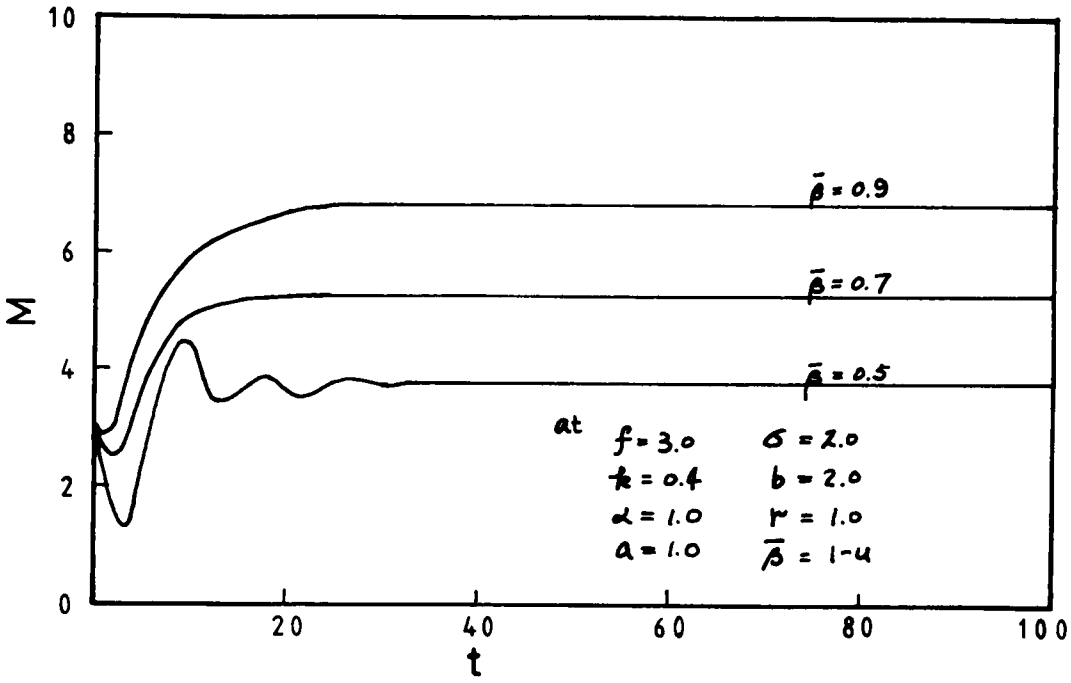

Figure 21. Effects of $\bar{\beta}$ on the behavior of M for System B

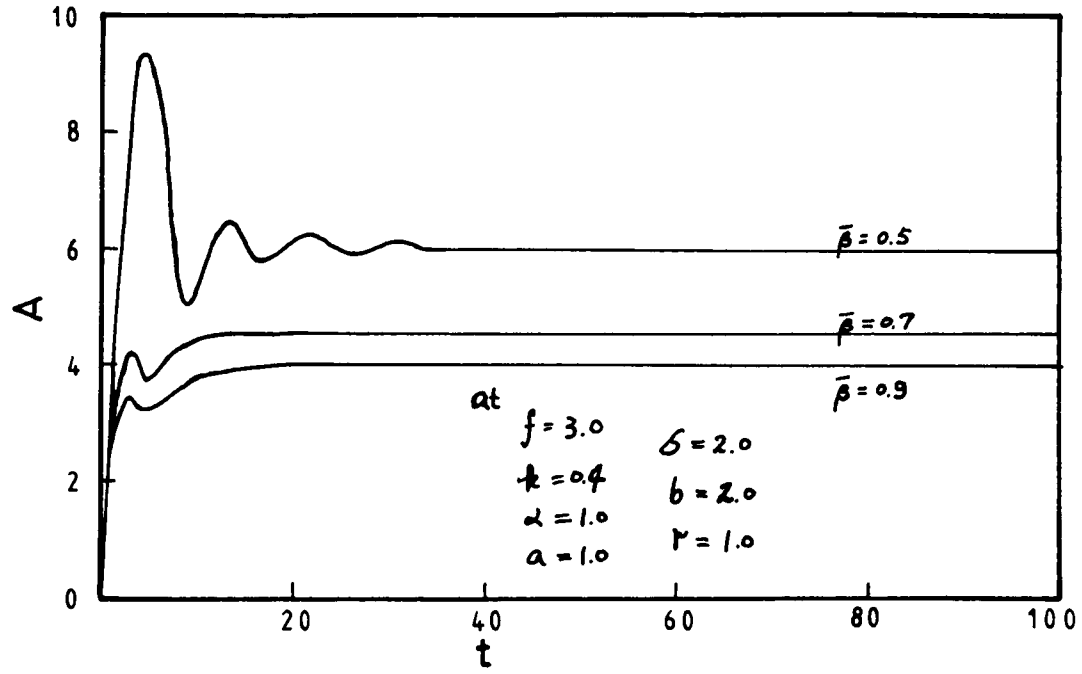

Figure 22. Effects of $\bar{\beta}$ on the behavior of A for System B

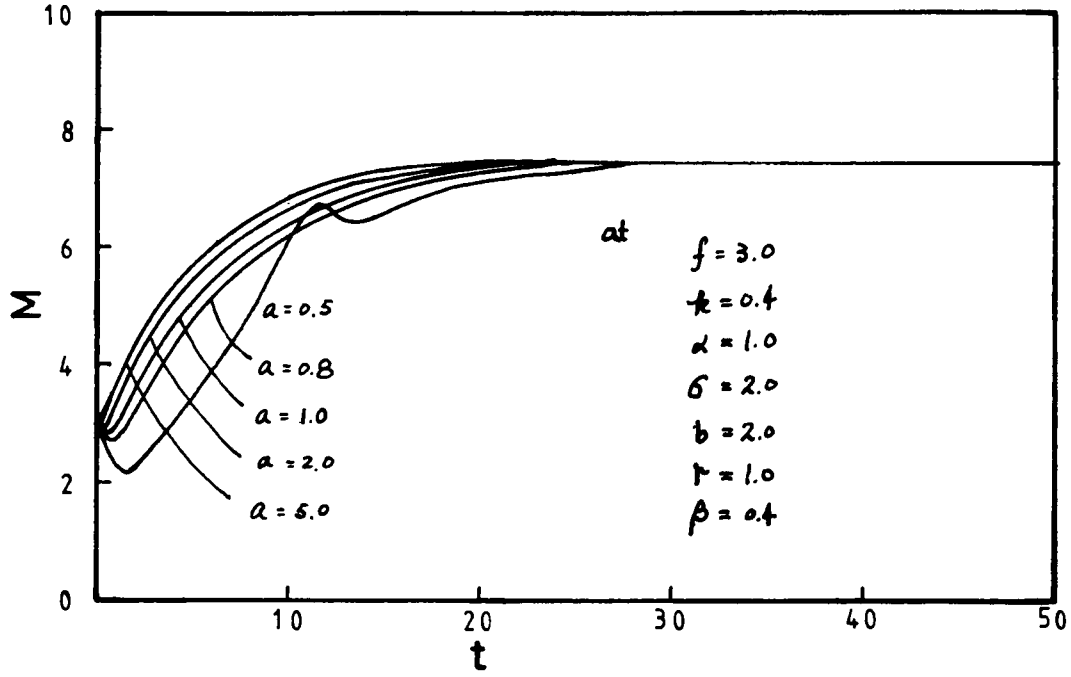

Figure 23. Effects of a on the behavior of M for System B

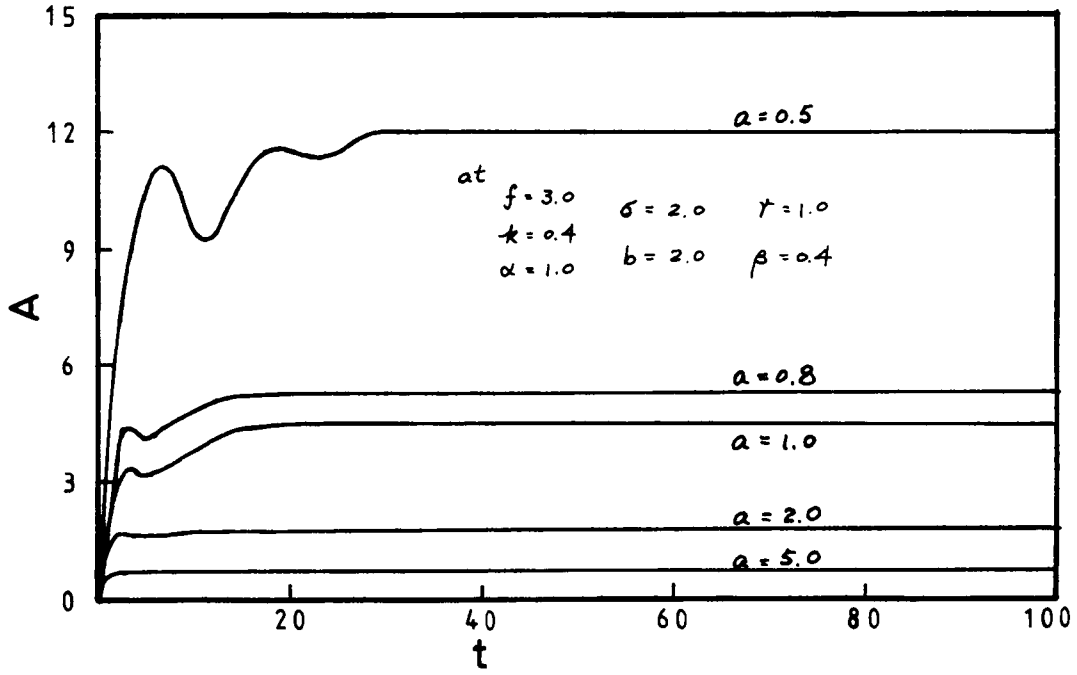

Figure 24. Effects of a on the behavior of A for System B

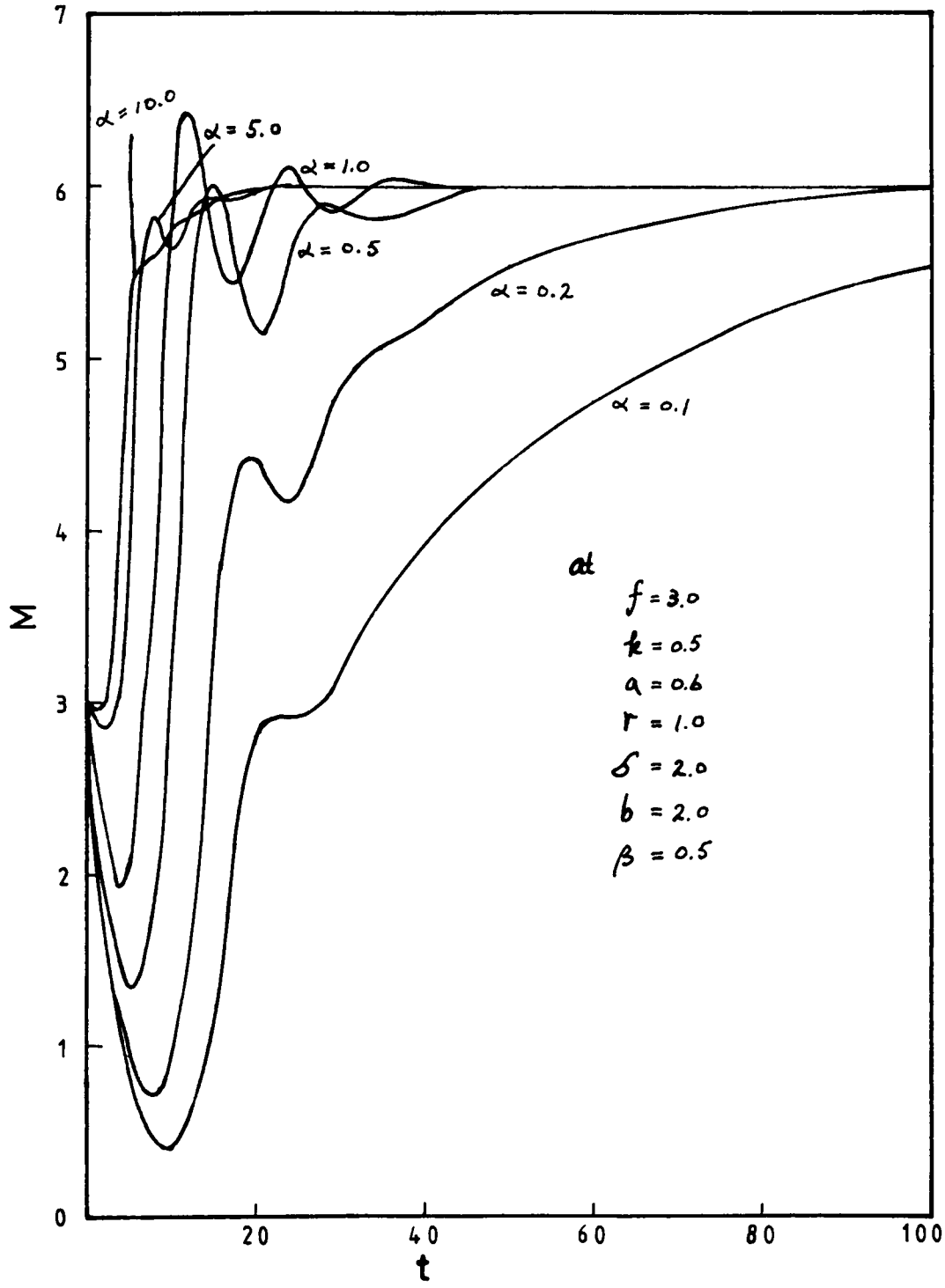

Figure 25. Effects of α on the behavior of M for System B

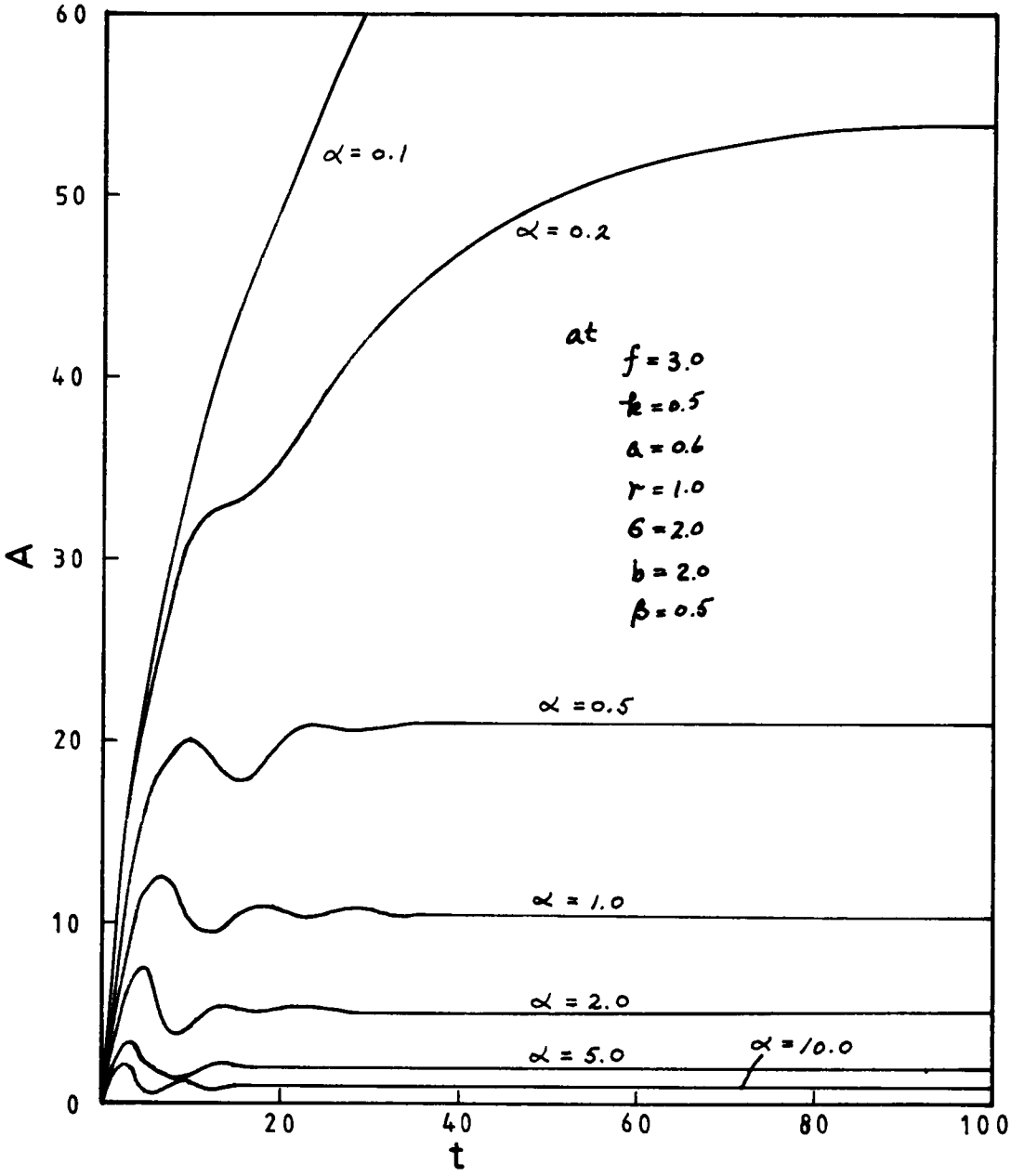

Figure 26. Effect of α on the behavior of A for System B

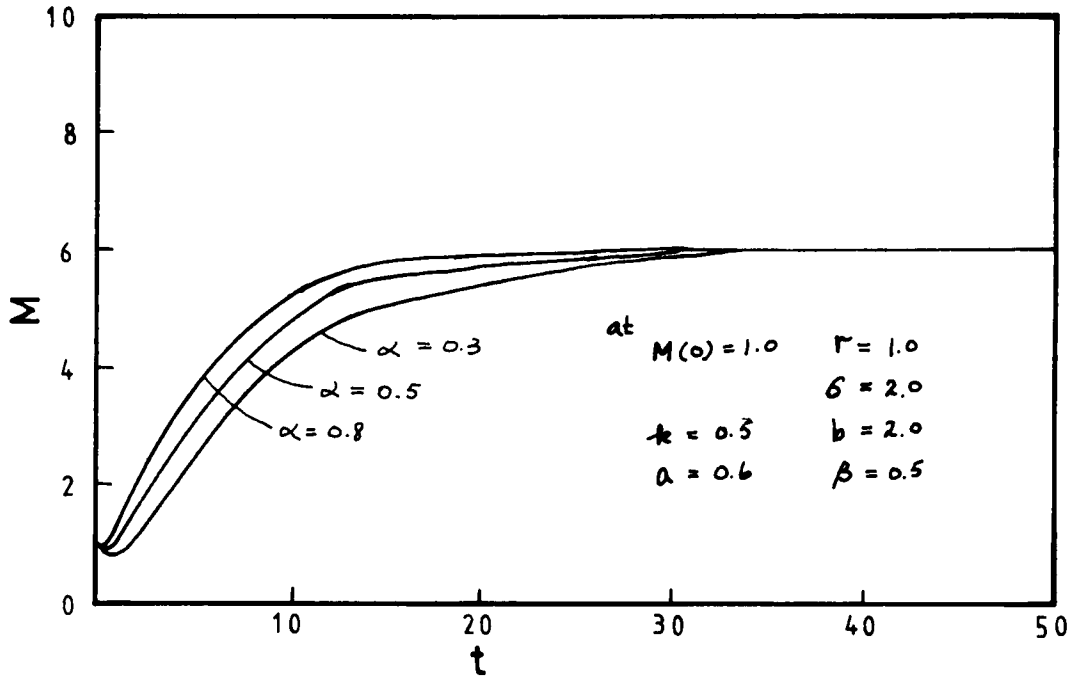

Figure 27. Effects of α on the behavior of M for M(0) = 1.0 different from Figure 25

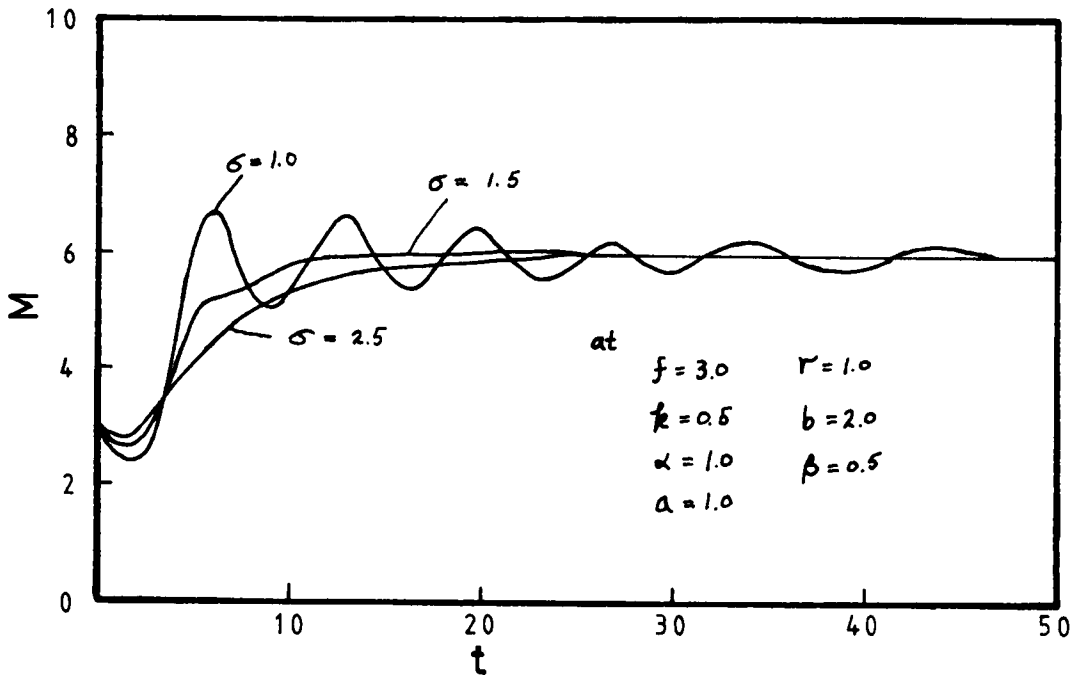

Figure 28. Effects of σ on the behavior of M for System B

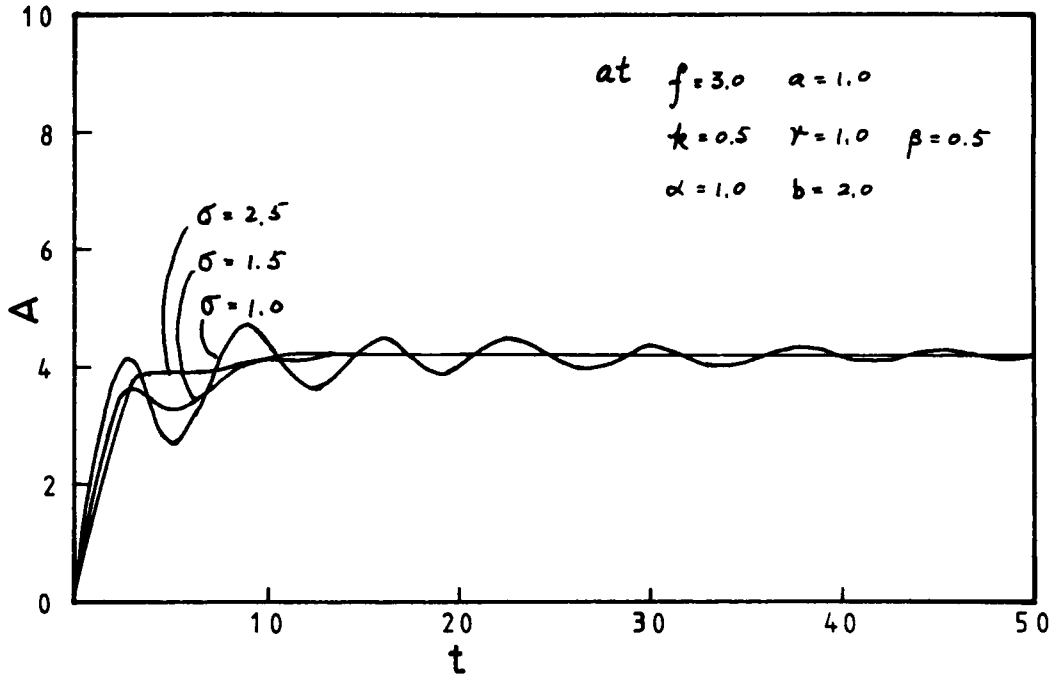

Figure 29. Effects of σ on the behavior of A for System B

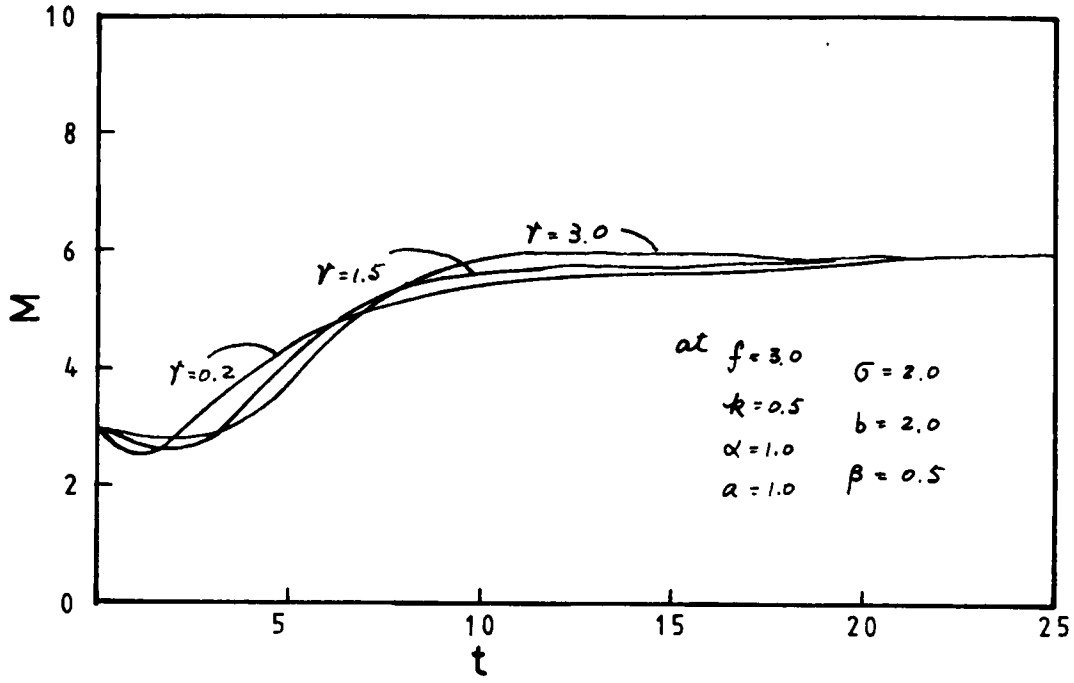

Figure 30. Effects of γ on the behavior of M for System B

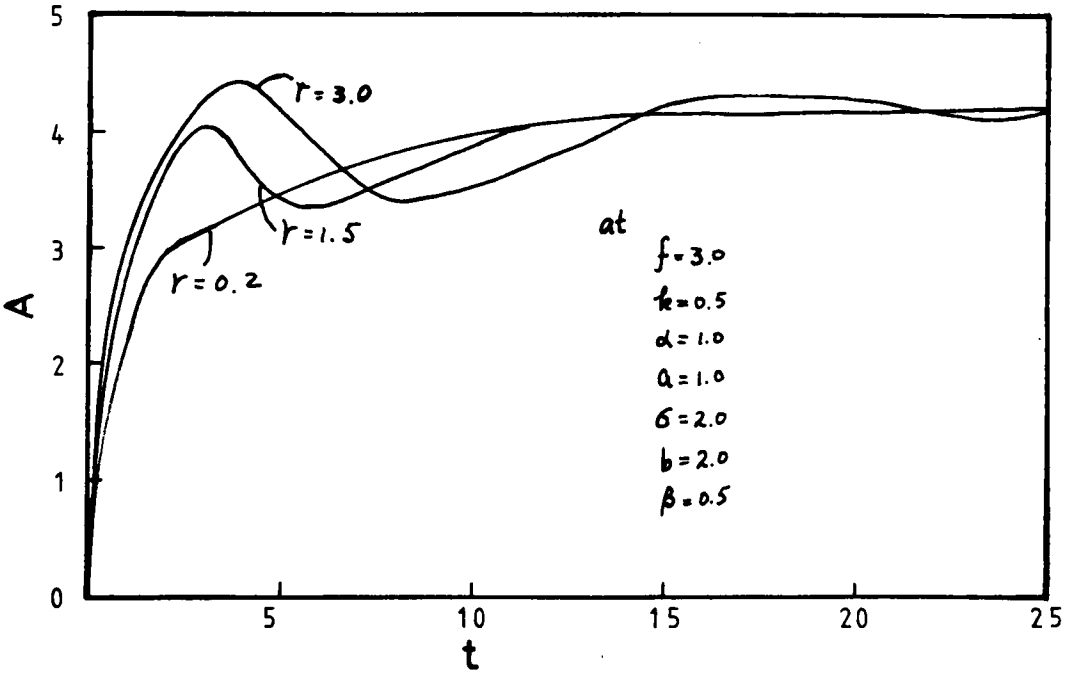

Figure 31. Effects of γ on the behavior of A for System B

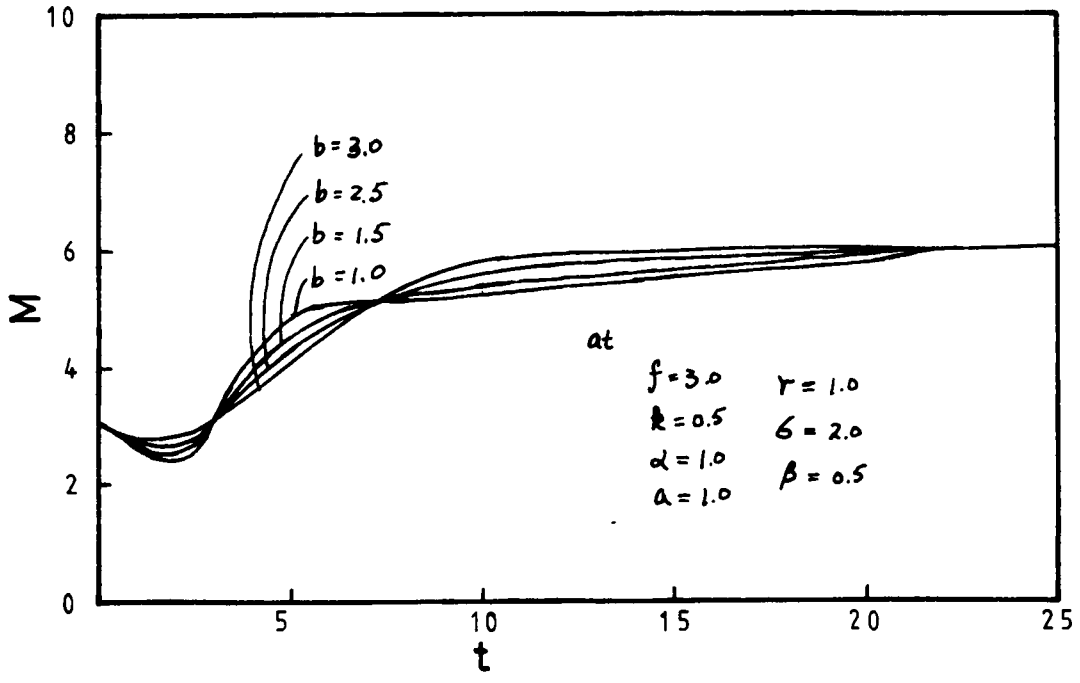

Figure 32. Effects of b on the behavior of M for System B

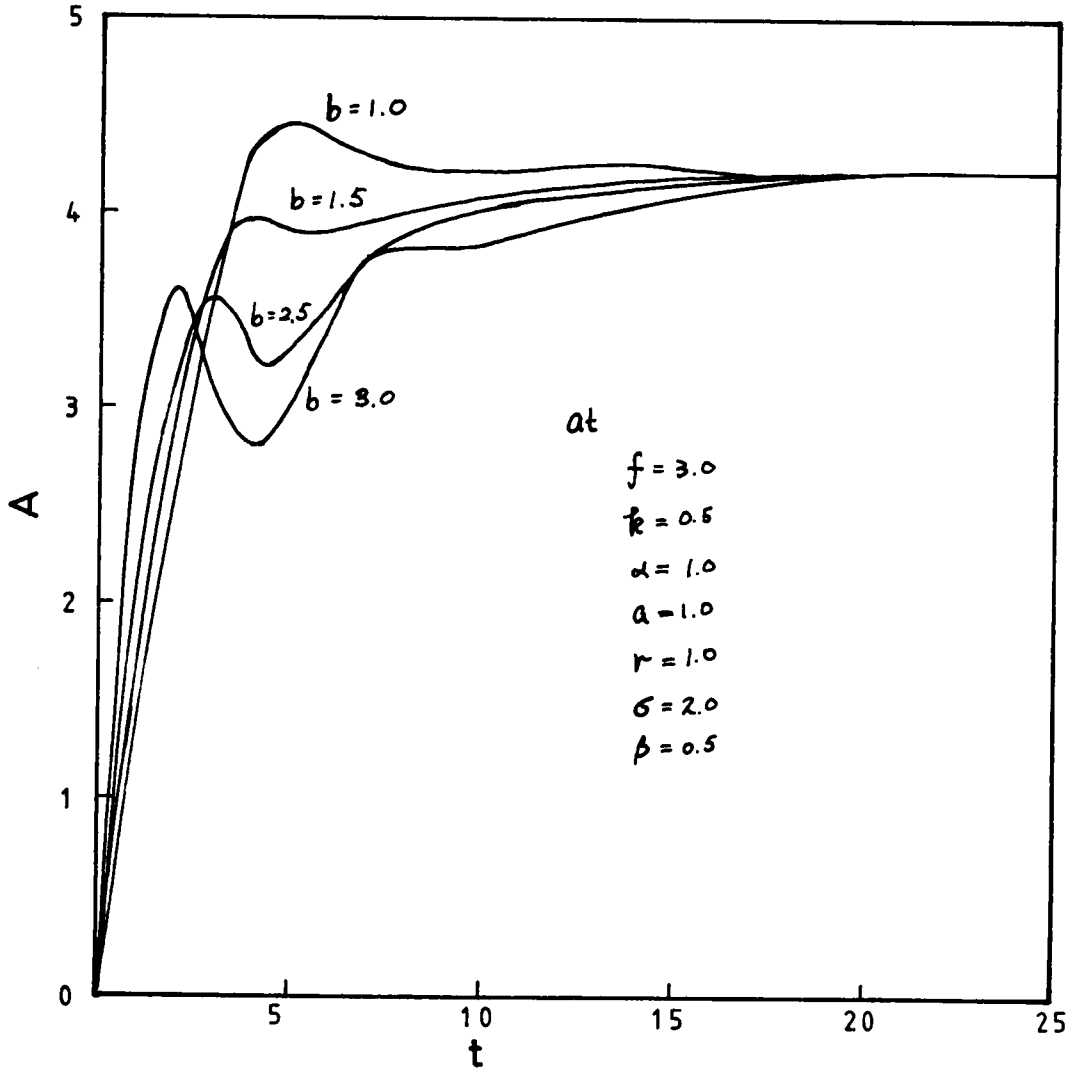

Figure 33. Effects of b on the behavior of A for System B

growth of the population. This is observed from the slope change in Figure 5. Figures 7 and 8 show the effect of varying $\bar{\beta}$ ($\bar{\beta} = 1 - u$) on the population size and the resource available, respectively. Here an increase in $\bar{\beta}$ increases the steady state value of the biomass whereas $\bar{\beta}$ has the opposite effect on the available resource A. Figures 9 and 10 show the effect of an increase in a on M and A, respectively. The change of a does not affect the steady state value of M, but it decreases the steady state value of A. An increase in a increases the growth rate of the population, but decreases the size of the overshoot. The effect of α upon M and A are given in Figures 11 and 12, respectively. The value of α does not affect the steady state value of M, but it affects the steady state value of A. An increase in α increases the growth rate of the population and the size of the overshoot. Figure 13 shows the same effect of the increase in α on the biomass for different initial conditions. Note that the curve for $\alpha = 0.5$ in Figure 13 shows a higher overshoot than the corresponding curve of Figure 11. The only difference between the two cases is the initial value of the population. The steady state value of M remains the same. Figures 14 and 15 show the effect of σ upon M and A respectively. The value of σ does not affect the steady state value of M and A. An increase in σ decreases the growth rate of population and the size of the overshoot; however, the size of the overshoot for A is increased. Note that the value of σ has a damping effect. For $\sigma < 0.2$, the response will be underflow. Figures 16 and 17 show that the steady state value of M and A are not affected by changes in γ. An increase in γ increases the height of overshoot for both M and A.

For system (B), where the equations of the model are derived from Smith's model, the simulation results are presented in Figures 18 thru 33. For a constant resource input rate, the behavior of the resource and the biomass is shown in Figure 18 with $f = 3.0$, $k = 0.2$, $\alpha = 1.0$, $a = 1.0$, $\gamma = 1.0$, $\sigma = 2.0$, $\beta = 0.4$, and $b = 2.0$. The result is the same as system (A) in Figure 4 except it has a more damping effect. Figures 19 and 20 show the effect of varying k on M and A respectively. The result is similar to system (A), except for higher overshoot and more damping results for lower k. Figures 21 and 22 exhibit the effect of changing $\overline{\beta}$ on M and A separately. An increase in $\overline{\beta}$ increases the efficiency of assimilation and the value of steady state of M, but decreases the steady state value of A. More damping is observed for lower $\overline{\beta}$. The effects of a are shown in Figures 23 and 24. Similar results to those obtained for system (A) are found for system (B), except no overshoot is observed for system (B). Figures 25 and 26 give the effect of α upon M and A respectively. Lower values of α tend to make M converge to the steady state slower. The growth rate is larger than in system (A). A decrease of α increases the steady state value of A, but decreases the damping effect. Figure 27 shows the effect of α on the biomass for different initial values. The lower initial value of M tends to decrease the damping effect. This can be obtained by comparing Figures 25 and 27 for $\alpha = 0.5$. Figures 28 and 29 show the effect of σ upon M and A. The steady state value of M and A are not affected. Comparing with system (A) shown in Figures 14 and 15, more damping effect results for a smaller value of σ. Note that for the case $\sigma = 0.5$, no result is

obtained (underflow). The effects of γ are given in Figures 30 and 31. Comparing both system (B) and system (A), system (B) has less overshoot for the biomass M, but more damping for the resource A. The steady state values of both M and A are unaffected. Figures 32 and 33 show the effect of b upon M and A. The steady state values of M and A do not change. An increase of b decreases the growth rate of the population, and decreases the overshoot of A.

CHAPTER IV

STABILITY ANALYSIS

Consider the two systems (A) and (B) discussed in Chapter 2:

$$\text{System A}\begin{cases} \dfrac{dA}{dt} = f - c(t) = f - aM(1 - e^{-\alpha A/M}) & (32) \\ \gamma \dfrac{d^2M}{dt^2} + \sigma \dfrac{dM}{dt} + M(\beta - a + ae^{-\alpha A/M}) = 0 & (33) \end{cases}$$

$$\text{System B}\begin{cases} \dfrac{dA}{dt} = f - c(t) = f - (aM + b\dfrac{dM}{dt})(1 - e^{-\alpha A/M}) & (34) \\ \gamma \dfrac{d^2M}{dt^2} + (\sigma - b + be^{-\alpha A/M})\dfrac{dM}{dt} + M(\beta - a + ae^{-\alpha A/M}) = 0 & (35) \end{cases}$$

The methods employed to study the stability of these systems are studied in this chapter. A linearization method discussed by Andronov (1966), Butenin (1965), and Saaty and Bram (1964) is coupled with the Hurwitz stability criterion in our study.

(I) System (A)

The nonlinear differential equations for system A are given as:

$$\frac{dA}{dt} = f - aM(1 - e^{-\alpha A/M})$$

$$\gamma \frac{d^2M}{dt^2} + \sigma \frac{dM}{dt} + M(\beta - a + ae^{-\alpha A/M}) = 0$$

Just as in Chapter 2, f is a constant. Also, let the equilibrium point of the system be (M_0, A_0) as given in equations (30) and (31). The procedure of investigating the stability of this equilibrium is based on the study of the character of motion of the state point in a neighborhood of this equilibrium state. In order to do that, the variables x, y, and z are introduced. Let

$$\begin{cases} x = M - M_0 \\ y = \dot{M} \\ z = A - A_0 \end{cases} \tag{36}$$

Equation (25) can be rewritten into two first order differential equations:

$$\dot{z}_1 = z_1 = \dot{M} \tag{37}$$

$$\begin{aligned} \dot{z}_2 &= -\frac{\sigma}{\gamma} z_2 - \frac{1}{\gamma} z_1(\beta - a + ae^{-\alpha A/M}) \\ &= -\frac{\sigma}{\gamma} \dot{M} - \frac{1}{\mu} M(\beta - a + ae^{-\alpha A/M}) \end{aligned} \tag{38}$$

Let

$$\begin{aligned} \dot{z}_1 &= H_1(M, \dot{M}, A) \\ \dot{z}_2 &= H_2(M, \dot{M}, A) \\ \dot{A} &= \frac{dA}{dt} = G(M, \dot{M}, A) \end{aligned} \tag{39}$$

By expanding the functions $H_1(M, \dot{M}, A)$, $H_2(M, \dot{M}, A)$ and $G(M, \dot{M}, A)$ in a neighborhood of the point $(M_0, 0, A_0)$ in a Taylor's series in x, y, and z , one obtains:

$$H_1(M, \dot{M}, A) = H_1(M_0, 0, A_0) + \frac{\partial M_1}{\partial M} x + \frac{\partial H_1}{\partial \dot{M}} y + \frac{\partial H_1}{\partial A} z + H_a(x, y, z)$$

$$H_2(M, \dot{M}, A) = H_2(M_0, 0, A_0) + \frac{\partial H_2}{\partial M} x + \frac{\partial H_2}{\partial \dot{M}} y + \frac{\partial H_2}{\partial A} z + H_b(x, y, z) \quad (40)$$

$$G(M, \dot{M}, A) = G(M_0, 0, A_0) + \frac{\partial G}{\partial M} x + \frac{\partial G}{\partial \dot{M}} y + \frac{\partial G}{\partial A} z + G(x, y, z)$$

The equilibrium points are found by solving $H_1(M, \dot{M}, A) = 0$, $H_2(M, \dot{M}, A) = 0$ and $G(M, \dot{M}, A) = 0$; therefore, $H_1(M_0, 0, A_0) = 0 = H_2(M_0, 0, A_0) = G(M_0, 0, A_0)$. Then the equations become

$$\begin{aligned}
\frac{dx}{dt} &= \left.\frac{\partial H_1}{\partial M}\right|_{M_0,0,A_0} x + \left.\frac{\partial H_1}{\partial \dot{M}}\right|_{M_0,0,A_0} y \\
&\quad + \left.\frac{\partial H_1}{\partial A}\right|_{M_0,0,A_0} z + H_a(x, y, z) \\
\frac{dy}{dt} &= \left.\frac{\partial H_2}{\partial M}\right|_{M_0,0,A_0} x + \left.\frac{\partial H_2}{\partial \dot{M}}\right|_{M_0,0,A_0} y \\
&\quad + \left.\frac{\partial H_2}{\partial A}\right|_{M_0,0,A_0} z + H_b(x, y, z) \quad (41) \\
\frac{dz}{dt} &= \left.\frac{\partial G}{\partial M}\right|_{M_0,0,A_0} x + \left.\frac{\partial G}{\partial \dot{M}}\right|_{M_0,0,A_0} y \\
&\quad + \left.\frac{\partial G}{\partial A}\right|_{M_0,0,A_0} z + G(x, y, z)
\end{aligned}$$

where $H_a(x, y, z)$, $H_b(x, y, z)$ and $G(x, y, z)$ are analytic functions in which the variables x, y, and z occur with at least the second power. Now, neglecting $H_a(x, y, z)$, $H_b(x, y, z)$, and $G(x, y, z)$ and letting

$$\left.\frac{\partial H_1}{\partial M}\right|_{M_0,0,A_0} = a \quad ; \quad \left.\frac{\partial H_1}{\partial \dot{M}}\right|_{M_0,0,A_0} = a_2 \quad ; \quad \left.\frac{\partial H_1}{\partial A}\right|_{M_0,0,A_0} = a_3$$

$$\left.\frac{\partial H_2}{\partial M}\right|_{M_0,0,A_0} = b_1 \quad ; \quad \left.\frac{\partial H_2}{\partial \dot{M}}\right|_{M_0,0,A_0} = b_2 \quad ; \quad \left.\frac{\partial H_2}{\partial A}\right|_{M_0,0,A_0} = b_3$$

$$\left.\frac{\partial G}{\partial M}\right|_{M_0,0,A_0} = c_1 \quad ; \quad \left.\frac{\partial G}{\partial \dot{M}}\right|_{M_0,0,A_0} = c_2 \quad ; \quad \left.\frac{\partial G}{\partial A}\right|_{M_0,0,A_0} = c_3$$

The system reduces to

$$\left.\begin{aligned} \frac{dx}{dt} &= a_1x + a_2y + a_3z \\ \frac{dy}{dt} &= b_1x + b_2y + b_3z \\ \frac{dz}{dt} &= c_1x + c_2y + c_3z \end{aligned}\right\} \tag{42}$$

The system of equations (42) is called the linear system equations of the first approximation. The characteristic equation of the system has the form:

$$\begin{vmatrix} a_1 - s & a_2 & a_3 \\ b_1 & b_2 - s & b_3 \\ c_1 & c_2 & c_3 - s \end{vmatrix} = 0$$

or

$$s^3 - (a_1 + b_2 + c_3)s^2 + (a_1c_3 + b_2c_3 + a_1b_2 - a_3c_1 - b_3c_2 - a_2b_1)s \tag{43}$$
$$- (a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2) = 0$$

This equation has three roots: s_1, s_2, and s_3. To analyze the system further, the coefficients a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, and c_3 are found from equations (37) and (38):

$$\left.\frac{\partial H_1}{\partial M}\right|_{M_0,0,A_0} = 0 \quad ; \quad \left.\frac{\partial H_1}{\partial \dot{M}}\right|_{M_0,0,A_0} = 1 \quad ; \quad \left.\frac{\partial H_1}{\partial A}\right|_{M_0,0,A_0} = 0$$

$$\left.\frac{\partial H_2}{\partial M}\right|_{M_0,0,A_0} = -\frac{1}{\gamma}(\beta - a + ae^{-\alpha A_0/M_0}) - \frac{\alpha}{\gamma} M_0(\alpha A_0/M_0^2)\, e^{-\alpha A_0/M_0}$$

$$\left.\frac{\partial H_2}{\partial \dot{M}}\right|_{M_0,0,A_0} = -\frac{\sigma}{\gamma} \tag{44}$$

$$\left.\frac{\partial H_2}{\partial A}\right|_{M_0,0,A_0} = \frac{a\alpha}{\gamma} e^{-\alpha A_0/M_0}$$

$$\left.\frac{\partial G}{\partial M}\right|_{M_0,0,A_0} = -a(1 - e^{-\alpha A_0/M_0}) + a\alpha \frac{A_0}{M_0} e^{-\alpha A_0/M_0}$$

$$\left.\frac{\partial G}{\partial \dot{M}}\right|_{M_0,0,A_0} = 0 \quad ; \quad \left.\frac{\partial G}{\partial A}\right|_{M_0,0,A_0} = -a\alpha e^{-\alpha A_0/M_0}$$

But from the equilibrium point (M_0, A_0), the expression of A_0 is

$$A_0 = -\frac{M_0}{\alpha} \ln\left(1 - \frac{\beta}{\alpha}\right) \quad \text{where} \quad \beta = \frac{k}{1-u} = \frac{k}{\bar{\beta}}$$

Therefore,

$$A_0/M_0 = -\frac{1}{\alpha} \ln\left(1 - \frac{k}{\alpha\bar{\beta}}\right) \tag{45}$$

and

$$e^{-\alpha A_0/M_0} = 1 - \frac{k}{\alpha\bar{\beta}} \tag{46}$$

Substitute these equations into system of equations (44), and one has

$$\left.\begin{aligned}
&\left.\frac{\partial H_1}{\partial \dot{M}}\right|_{M_0,0,A_0} = a_1 = 0; \quad \left.\frac{\partial H_2}{\partial \dot{M}}\right|_{M_0,0,A_0} = a_2 = 1 ; \\
&\left.\frac{\partial H_1}{\partial A}\right|_{M_0,0,A_0} = a_3 = 0 \\
&\left.\frac{\partial H_2}{\partial \dot{M}}\right|_{M_0,0,A_0} = b_1 = \frac{a}{\gamma}\left(1 - \frac{k}{\alpha\bar{\beta}}\right) \ln\left(1 - \frac{k}{\alpha\bar{\beta}}\right) \\
&\left.\frac{\partial H_2}{\partial \dot{M}}\right|_{M_0,0,A_0} = b_2 = -\frac{\sigma}{\gamma}
\end{aligned}\right\} \tag{47}$$

$$\left.\frac{\partial H_2}{\partial \dot{M}}\right|_{M_0,0,A_0} = b_3 = \frac{a\alpha}{\gamma}\left(1 - \frac{k}{\alpha\bar{\beta}}\right)$$

$$\left.\frac{\partial G}{\partial M}\right|_{M_0,0,A_0} = c_1 = -\left[a\left(1 - \frac{k}{\alpha\bar{\beta}}\right)\ln\left(1 - \frac{k}{\alpha\bar{\beta}}\right) + \frac{k}{\bar{\beta}}\right] \qquad (47)\ \text{(con't)}$$

$$\left.\frac{\partial G}{\partial \dot{M}}\right|_{M_0,0,A_0} = c_2 = 0$$

$$\left.\frac{\partial G}{\partial A}\right|_{M_0,0,A_0} = c_3 = -a\alpha\left(1 - \frac{k}{\alpha\bar{\beta}}\right) \quad .$$

Applying the Hurwitz stability criterion, for equation (43) to be stable, it is necessary and sufficient that

$$-(a_1 + b_2 + c_3) = -\left(-\frac{\sigma}{\gamma} - a\alpha\left(1 - \frac{k}{\alpha\bar{\beta}}\right)\right) = \frac{\sigma}{\gamma} + a\alpha\left(1 - \frac{k}{\alpha\bar{\beta}}\right) > 0 \qquad (48)$$

$$\frac{a\sigma}{\gamma^2}\left(1 - \frac{k}{\alpha\bar{\beta}}\right)\left[\sigma\alpha - \ln\left(1 - \frac{k}{\alpha\bar{\beta}}\right)\right] + \frac{a^2\alpha}{\gamma}\left(1 - \frac{k}{\alpha\bar{\beta}}\right)\left[\sigma\alpha - \ln\left(1 - \frac{k}{\alpha\bar{\beta}}\right)\right] \qquad (49)$$

$$- \frac{a\alpha k}{\gamma\bar{\beta}}\left(1 - \frac{k}{\alpha\bar{\beta}}\right) > 0$$

and

$$\frac{a\alpha k}{\gamma\bar{\beta}}\left(1 - \frac{k}{\alpha\bar{\beta}}\right) > 0 \qquad (50)$$

Since a, α, k, γ, and $\bar{\beta}$ are assumed to be positive, therefore, from equation (50), we have

$$a\bar{\beta} > k \tag{51}$$

If $(1 - \frac{k}{\alpha\bar{\beta}}) > 0$, then rearranging equation (49), one has

$$[\frac{\sigma}{\gamma} + a\alpha][\sigma\alpha - \ln(1 - \frac{k}{\alpha\bar{\beta}})] - \frac{\sigma k}{\bar{\beta}}[1 + \sigma\alpha - \ln(1 - \frac{k}{\alpha\bar{\beta}})] > 0 \tag{52}$$

Since $\frac{\alpha k}{\bar{\beta}}$ is always positive, in order for equation (52) to hold, at least

$$\sigma\alpha - \ln(1 - \frac{k}{\alpha\bar{\beta}}) > 0 \ ;$$

therefore

$$\begin{aligned} &\sigma\alpha > \ln(1 - \frac{k}{\alpha\bar{\beta}}) \\ &e^{\sigma\alpha} > 1 - \frac{k}{\alpha\bar{\beta}} > 0 \end{aligned} \tag{53}$$

σ and α can be any value and satisfy this condition (53) .

Now let $\sigma\alpha - \ln(1 - \frac{k}{\alpha\bar{\beta}}) = d$; hence, α is a parameter that depends on σ, α, k, a, and $\bar{\beta}$. Then equation (52) becomes

$$[\frac{\sigma}{\gamma} + a\alpha] \cdot \alpha - \frac{\alpha k}{\bar{\beta}}[1 + d] > 0$$

$$d[\frac{\sigma}{\gamma} + a\alpha] > \frac{\alpha k}{\bar{\beta}}[1 + d]$$

Therefore, in order to have a stable equilibrium point, all the parameters should satisfy equations (51), (53), and (54) . This shows the existence of a stable steady state for the system. Since there are some restrictions on the values of the parameters, the system (A) may not have a global equilibrium point. Therefore, the addition of the second order term in equation (25) of system (A) tends to increase the instability of the system. Local stability around the singular point exists and global stability is not guaranteed for the system (A).

Example:

Given that $f = 3.0$, $k = 0.5$, $\alpha = 0.8$, $a = 0.6$, $\beta = 0.5$, then at the equilibrium point

$$M_0 = \frac{f(1 - u)}{k} = \frac{f}{\beta} = \frac{3.0}{0.5} = 6.0$$

$$A_0 = - \frac{M_0}{\alpha} \ln(1 - \frac{\beta}{a})$$

$$= - \frac{6}{0.8} \ln(1 - \frac{0.5}{0.6})$$

$$= - 7.5 \cdot (-1.791759)$$

$$= +13.4382 .$$

Therefore, equation (51) is satisfied because

$$a\overline{\beta} = 0.6 \cdot 1 > k = 0.5$$

Now look at equation (53):

$$\sigma\alpha > \ln(1 - \frac{k}{\alpha\beta}) = -1.791759$$

$$\sigma > \frac{-1.791759}{0.8} = 2.2397$$

Therefore, as long as $\sigma > -2.2397$, the system will be stable.

Examining equation (54) with $d = 3.391759$ shows:

$$(\frac{\sigma}{\gamma} + 0.48) \cdot 3.391759 > 0.4[1 + 3.391759]$$

$$(\frac{\sigma}{\gamma} + 0.48) > \frac{1.756704}{3.391759} = 0.518$$

$$\frac{\sigma}{\gamma} > 0.038$$

Therefore, γ can be any value satisfying this equation.

(II) System B

The representation of system (B) is

$$\frac{dA}{dt} = f - (aM + b\frac{dM}{dt})(1 - e^{-\alpha A/M}) \qquad (26)$$

$$\gamma\frac{d^2M}{dt^2} + (\sigma - b + be^{-\alpha A/M})\frac{dM}{dt} + M(\beta - a + ae^{-\alpha A/M}) = 0 \qquad (27)$$

Now using a linearization method the same as in System (A), let

$$x = M = M_0$$

$$y = \dot{M}$$

$$z = A - A_0$$

Then equation (27) becomes

$$\dot{z}_1 = \dot{M} \tag{55}$$

$$\dot{z}_2 = -\frac{\sigma - b + be^{-\alpha A/M}}{\gamma}\dot{M} - \frac{1}{\gamma}M(\beta - a + ae^{-\alpha A/M}) \tag{56}$$

Let $H_1(M_0, 0, A_0) = H_2(M_0, 0, A_0) = G(M_0, 0, A_0) = 0$ and expand the function H_1, H_2, G in the neighborhood of the point $(M_0, 0, A_0)$ in Taylor's series:

$$\frac{dx}{dt} = \frac{\partial H_1}{\partial M}\Big|_{M_0,0,A_0} x + \frac{\partial H_1}{\partial \dot{M}}\Big|_{M_0,0,A_0} y + \frac{\partial H_1}{\partial A}\Big|_{M_0,0,A_0} z + H_a(x,y,z)$$

$$\frac{dy}{dt} = \frac{\partial H_2}{\partial M}\Big|_{M_0,0,A_0} x + \frac{\partial H_2}{\partial \dot{M}}\Big|_{M_0,0,A_0} y + \frac{\partial H_2}{\partial A}\Big|_{M_0,0,A_0} z + H_b(x,y,z)$$

$$\frac{dz}{dt} = \frac{\partial G}{\partial M}\Big|_{M_0,0,A_0} x + \frac{\partial G}{\partial \dot{M}}\Big|_{M_0,0,A_0} y + \frac{\partial G}{\partial A}\Big|_{M_0,0,A_0} z + G(x,y,z)$$

where

$$\frac{\partial H_1}{\partial \dot{M}}\Big|_{M_0,0,A_0} = 1 = a_2 \quad ; \quad \frac{\partial H_1}{\partial M} = 0 \quad , \quad \frac{\partial H_1}{\partial A} = 0$$

$$\left.\frac{\partial H_2}{\partial M}\right|_{M_0,0,A_0} = + \frac{a}{\gamma}\left(1 - \frac{k}{\alpha\bar{\beta}}\right) \ln\left(1 - \frac{k}{\alpha\bar{\beta}}\right) = b_1$$

$$\left.\frac{\partial H_2}{\partial \dot{M}}\right|_{M_0,0,A_0} = - \frac{\sigma - b + b\left(1 - \frac{k}{\alpha\bar{\beta}}\right)}{\gamma} = b_2$$

$$\left.\frac{\partial H_2}{\partial A}\right|_{M_0,0,A_0} = \frac{a\alpha}{\gamma}\left(1 - \frac{k}{\alpha\bar{\beta}}\right) = b_3$$

$$\left.\frac{\partial G}{\partial M}\right|_{M_0,0,A_0} = - \frac{k}{\bar{\beta}} - a\left(1 - \frac{k}{a\bar{\beta}}\right) \ln\left(1 - \frac{k}{a\bar{\beta}}\right) = c_1$$

$$\left.\frac{\partial G}{\partial \dot{M}}\right|_{M_0,0,A_0} = - \frac{kb}{a\bar{\beta}} = c_2$$

$$\left.\frac{\partial G}{\partial A}\right|_{M_0,0,A_0} = -\alpha\left(1 - \frac{k}{a\bar{\beta}}\right)a = c_3$$

Therefore, the system reduces to

$$\begin{aligned} \frac{dx}{dt} &= a_1 x + a_2 y + a_3 z \\ \frac{dy}{dt} &= b_1 x + b_2 y + b_3 z \\ \frac{dz}{dt} &= c_1 x + c_2 y + c_3 z \end{aligned} \tag{57}$$

The characteristic equation is

$$s^3 - (b_2 + c_3)s^2 + (b_2c_3 - b_1 - b_3c_2)s - (c_1b_3 - b_1c_3) = 0 \quad (58)$$

From the Hurwitz stability criterion, one has

$$-(b_2 + c_3) = -\left(-\frac{\sigma - b + b\left(1 - \frac{k}{\alpha\beta}\right)}{\gamma} + \left(-\alpha\left(1 - \frac{k}{\alpha\beta}\right)a\right)\right) > 0 \quad (59)$$

$$= \frac{\sigma - b + b\left(1 - \frac{k}{\alpha\beta}\right)}{\gamma} + \alpha a\left(1 - \frac{k}{\alpha\beta}\right) > 0$$

$$-(b_2 + c_3)(b_2c_3 - b_1 - b_3c_2) - (b_1c_3 - b_3c_1) > 0$$

$$\left(\frac{\sigma - b + b\left(1 - \frac{k}{\alpha\beta}\right)}{\gamma} + a\alpha\left(1 - \frac{k}{\alpha\beta}\right)\right)\left(\frac{\sigma - b + b\left(1 - \frac{k}{\alpha\beta}\right)}{\gamma} \cdot\right.$$

$$\left. \cdot\, \alpha a\left(1 - \frac{k}{\alpha\beta}\right) - \frac{a}{\gamma}\left(1 - \frac{k}{\alpha\beta}\right) \ln\left(1 - \frac{k}{\alpha\beta}\right) + \frac{a\alpha}{\gamma}\left(1 - \frac{k}{\alpha\beta}\right) \cdot \left(\frac{kb}{\alpha\beta}\right)\right)$$

$$- \left(1 - \frac{k}{\alpha\beta}\right)\frac{\alpha}{\gamma} \cdot a \cdot \frac{k}{\beta} > 0 \quad (60)$$

$$\frac{a}{\gamma}\left(1 - \frac{k}{\alpha\beta}\right)\left\{\left[\sigma\alpha - \ln\left(1 - \frac{k}{\alpha\beta}\right)\right]\left[\frac{\sigma - \frac{bk}{\alpha\beta}}{\gamma} + a\gamma\left(1 - \frac{k}{\alpha\beta}\right)\right]\right.$$

$$\left. - \frac{k^{\alpha}}{\alpha\beta}\left(1 - \frac{k}{\alpha\beta}\right)\right\} > 0 \quad (61)$$

$$(b_1c_3 - b_3c_1) = \left(1 - \frac{k}{\alpha\beta}\right)\frac{\alpha}{\gamma}\, a\, \frac{k}{\beta} > 0 \quad (62)$$

Assuming α, k, β, b, a are positive, then, from equation (62), we have

$$1 - \frac{k}{\alpha\bar{\beta}} > 0 \qquad \text{or} \quad a\bar{\beta} > k \tag{63}$$

Then equation (60) becomes

$$[\sigma\alpha - \ln(1 - \frac{k}{\alpha\bar{\beta}})][\frac{\sigma - \frac{bk}{\alpha\bar{\beta}}}{\gamma} + a\gamma(1 - \frac{k}{\alpha\bar{\beta}})] - \frac{k\alpha}{\bar{\beta}}(1 - \frac{k}{\alpha\bar{\beta}}) > 0$$

or

$$[\sigma\alpha - \ln(1 - \frac{k}{\alpha\bar{\beta}})][\frac{\sigma}{\gamma} - \frac{bk}{a\bar{\beta}\gamma} + a\gamma(1 - \frac{k}{\alpha\bar{\beta}})] > \frac{k\alpha}{\bar{\beta}}(1 - \frac{k}{\alpha\bar{\beta}})$$

and equation (59) becomes

$$\frac{\sigma}{\gamma} - \frac{bk}{a\bar{\beta}\gamma} + a\alpha - \frac{k\alpha}{\bar{\beta}} > 0 \tag{64}$$

Therefore, all parameters should satisfy equations (59), (61), and (63). This shows the existence of a stable steady state for the system. Since there are restrictions on the values of the parameters, the system (B) may not be structurally stable. Comparing with system (A), more restrictions apply to system (B). For example, for system (A) condition (48) is $\frac{\sigma}{\gamma} + a\alpha(1 - \frac{k}{\alpha\bar{\beta}}) > 0$ whereas for system (B) $\frac{\sigma}{\gamma} + a\alpha(1 - \frac{k}{\alpha\bar{\beta}})$ needs to be greater than $\frac{bk}{a\bar{\beta}\gamma}$. Similarly, equation (61) has more restrictions on parameters than equation (53). Therefore, the addition of the first order term in system (B) tends to increase the instability of the system and adds more restrictions on the system. Local stability around the singular point exists.

CHAPTER V

CONCLUSION

Systems (A) and (B), which are derived from Gallopin's model and with Smith's hypothesis, have very similar behaviors. Comparing with the Gallopin's model, the second order term of equations (25) and (27) tends to increase the instability of the sytem and the restrictions to the parameters used in the systems. The requirement of a constant resource input rate to systems (i.e., f = constant) is a necessary and sufficient condition for the existence of a steady state in the study. No other case is investigated here.

From the simulation results, one can find that the parameters a and α affect only the transient behavior of the population (its speed of approaching the steady state and the size and position of the overshoot), while the parameter k and $\overline{\beta}$ affect the steady state biomass of the population. All parameters affect both the transient and steady state value of the resource. The parameters r, σ, and b affect only the transient behavior of the population and resource. The values of r, σ, and b all depends on other parameters chosen. Therefore, the selection of these parameters are very important because they will affect the stability of the system. Thus, the second order term in both systems (A and B) and the $b\frac{dM}{dt}$ term in system B are quite influential.

It is interesting to compare the changes in the values of the parameters on the steady state value of the biomass. Comparing the figures, one can find that a unit increase in $\overline{\beta}$ will increase M_0

more than a unit increase in k will reduce it, if $\bar{\beta}$ is smaller than k, and vice versa. Furthermore, from calculation one can find that a unit increase in f will increase M_0 more than a unit increase in k will reduce it, if f is smaller than k, and vice versa. Therefore, from these figures, one finds that for a population to maximize its steady state biomass, and having no control over the input rate of resource to the system, the optimal strategy is to maximize the ingestion efficiency, $\bar{\beta}$ and minimize its maintenance cost, k. Moreover, it is more efficient for the population to increase $\bar{\beta}$ than to decrease k up to the point in which $\bar{\beta}$ is equal to k, and thereafter it is more efficient to decrease k than to increase $\bar{\beta}$.

Due to restrictions applied to both systems, one can only find local stable points. No global stability are found in this study for both systems. The parameters γ, σ, and b are three influential parameters whose degree of influence depends upon several restrictions. For example, when $f = 3$, $k = .05$, $\alpha = 1.0$, $a = 1.0$, $\gamma = 1.0$, $\beta = 0.5$ and $\sigma = 0.1$, the response on Figure 14 will become underflow. The system becomes nonstable and no stable points can be found. Furthermore, System B has more applicable restrictions than System A; that is, System B has smaller stability range for σ and γ. This is shown from Figure 28 where σ must be greater than 0.8. Therefore one can say that System B is likely to be more unstable than System A.

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