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I am submitting herewith a dissertation written by Marina Valentinovna Shmakova entitled "On Black Holes and D-branes. " I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Physics.

George Siopsis, Major Professor

We have read this dissertation
and recommend its acceptance:

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Associate Vice Chancellor and
Dean of The Graduate School

ON BLACK HOLES AND D-BRANES

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ABSTRACT

N=2 supersymmetric extreme black holes associated with the different types of the moduli space are investigated. The explicit solutions for the black hole entropy as a function of the electric and magnetic charges of the black hole are found. Starting with the simple case of two moduli and two electric and magnetic charges (q_Λ and p^Λ with $\Lambda = 0, 1$) the investigation proceeds to the more complicated moduli spaces, such as the most general form of the Calabi-Yau moduli space with the arbitrary number of moduli fields. The global N=4 Supersymmetry transformations for the gauge-fixed κ symmetric Born-Infeld D3 brane action are calculated in the flat background using Killing gauge. One loop corrections to the effective Lagrangian for the D3 brane are found.

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1 Introduction

If the 80's can be named the era of the extended d=1 objects - strings, the 90's became the era of the d=n extended objects with dimensionality running from zero to 9. All kind of imaginable "branes" have appeared: membranes, p-branes, D-branes, 0-branes and so on. The number of possible theories grows with the number of extended dimensions only to lead to the realization that all of these possible theories can be regarded as parts of a gigantic puzzle called M-theory. This dissertation presents an attempt to look more closely at some particular pieces of this puzzle to find the connections between them and their place in the general picture.

Puzzle one: Black hole entropy.

The classical black hole solution in general relativity describes the black hole as a macroscopic object - a singularity of the spacetime metric. Therefore the similarity between the black hole law and the thermodynamic law is one of the most amazing properties of the black hole [1, 2], that hints at the possibility of describing the black hole from the microscopic point of view. The laws for the black hole with mass M and charge Q and the thermodynamic law for the system of volume V and energy U are

$$\left(\begin{array}{c} \text{Black hole law :} \\ dM = \frac{k}{8\pi G} dA + \Phi dQ \\ \text{for} \quad \delta A \geq 0 \end{array} \right) \Leftrightarrow \left(\begin{array}{c} \text{Thermodynamics law :} \\ dU = TdS - pdV \\ \text{for} \quad \delta S \geq 0 \end{array} \right), \tag{1.1}$$

where T is the temperature , S is the thermodynamic entropy, k is the surface gravity, A is the area of the event horizon and Φ is the electrostatic potential of the black hole. In his famous work Hawking showed [3] that black holes indeed radiate a thermal spectrum with temperature $T = \frac{\hbar k}{2\pi}$. The combination of the approaches of general relativity and quantum theory gives a definition of the black hole entropy : the thermodynamic entropy of the black hole (in Planck units) is equal to the one quarter of the area of the event horizon:

$$S_{BH} = \frac{A}{4G\hbar}. \tag{1.2}$$

The thermodynamic approach is only an approximation. In the classical theory the thermodynamic entropy of the statistical system is determined by the number of the degrees of freedom of this system, e^S. It is not clear what kind of states (degrees of freedom) are associated with the black hole entropy since the black hole entropy is a quantum mechanical effect and its statistical description will require application of some "quantum theory of gravity". Can supergravity and string theory be good candidates for the role of this "quantum theory of gravity"?

Significant progress has been made in understanding those degrees of freedom that give rise to the entropy of the certain black holes in string theory. This progress was possible due to the topological nature of the Hawking-Beckenstein entropy of extreme black holes.

Puzzle two: Supergravity and black hole geometry.

A classical Reissner-Nordstrom black hole with electric charge $Q \neq 0$ and zero angular momentum $J = 0$ is described by the metric:

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + \frac{dr^2}{(1 - \frac{2M}{r} + \frac{Q^2}{r^2})} + r^2(d\theta^2 + \sin^2\theta d\ ) \tag{1.3}$$

with the horizons $r_{H+} = M + \sqrt{M^2 - Q^2}$ and $r_{H-} = M - \sqrt{M^2 - Q^2}$. The special case $M^2 = Q^2$ and $r_{H+} = r_{H-}$ is called an extreme black hole. In this case there is no Hawking radiation and the extreme black hole is a stable object. The solution of the field equations for this case in the Reissner-Nordstrom throat region ($|r - Q| \ll Q$) is called Bertotti-Robinson solution. The Bertotti-Robinson metric in the spherically symmetric coordinates is

$$ds^2 = -\frac{r^2}{M_{BR}^2}dt^2 + \frac{M_{BR}^2}{r^2}(dr^2 + r^2 d\Omega). \tag{1.4}$$

The Bertotti-Robinson metric plays a special role in supergravity. Indeed, this kind of black hole solutions exists in $N = 2$ supersymmetric theories. In general, N=2 supergravity in d=4 dimensions can be considered as a result of the compactification of the d=10 Type II superstring on some d=6 Calabi-Yau manifold. N=2 supersymmetric black holes are described by the "special geometry" approach (see, for example, review [4] and references therein) where the interplay between the geometry of special Kähler manifolds and space-time geometry is used. This special geometry approach makes it possible to investigate different types of black hole solutions connected with different types of string theories. In particular, special and quaternionic geometry structures [4] were found in N=2 supergravity interacting with vector multiplets and hypermultiplets. In this case the theory contains the following multiplets:

gravitational multiplet contains the veilbein, the SU(2) doublet of gravitino $\psi_{A\mu}$ and the graviphoton $\mathrm{T}^-{}_{\mu\nu}$;

n_v **vector multiplets**, each contains a gauge boson with the field strength $\mathcal{F}^{i-}_{\mu\nu}$, a doublet of gauginos λ^{iA} and a complex scalar field z^i;

n_h **hypermultiplets**, each contains a doublet of hyperinos ζ_α and 4 real scalar fields q^u.

Scalar fields play a fundamental role in the theories with extended supersymmetries, N$\geq$ 2, where they appear as superpartners of the vector fields and transform in the same adjoint representation. In fact, complex scalar fields z^i $(i = 1, 2, ..., n_v)$ of N=2 vector multiplets can be regarded as coordinates of some scalar manifold $\mathcal{M}^v_{scalar}$, a special Kähler manifold with the metric $g^{i\bar{\jmath}}$ in the N=2 case. The transformations on this manifold are the diffeomorphisms:

$$z^i \longrightarrow t^i(z); \;\; \{t \; : \;\; \mathcal{M}^v_{scalar} \longrightarrow \mathcal{M}^v_{scalar}\}.$$

The diffeomorphism that leaves the metric invariant $t^* g = g$, an isometry, forms a global symmetry of the theory. As a global symmetry this transformation commutes with the supersymmetry and therefore it affects the vector superpartners of the scalar fields as well. This transformation for the vector fields turns out to be a duality rotation. The most general group of duality rotations is $Sp(2n, \mathbf{R})$ and the isometries of the scalar manifold $\mathcal{M}^v_{scalar}$ form a subgroup $\mathcal{I} \subset Sp(2n, \mathbf{R})$ of this transformation.

Introducing the formal operator $\{^*\}$ that transforms the vector field strength into its dual:

$$^*\mathcal{F}^{\Lambda}_{\mu_1\mu_2} \equiv \frac{1}{2}\epsilon_{\mu_1\mu_2\nu_1\nu_2}\mathcal{F}^{\Lambda\nu_1\nu_2}$$

with $\{^*\}^2 = -1$ we can write the kinetic term of the Lagrangian for the vector fields as:

$$L = \frac{1}{2}(\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}(z^i, \bar{z}^i)\mathcal{F}^{\Lambda}\mathcal{F}^{\Sigma} + \mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma}(z^i, \bar{z}^i)\mathcal{F}^{\Lambda *}\mathcal{F}^{\Sigma}), \tag{1.5}$$

which is an analog of the QCD Lagrangian $(\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{\theta}{16\pi}F^a_{\mu\nu}{}^*F^{a\mu\nu})$ where instead of simple summation over indexes a we have to deal with some kind of "metric" $\mathcal{N}_{\Lambda\Sigma}$. This complex matrix $\mathcal{N}_{\Lambda\Sigma}(z^i, \bar{z}^i)$ depends on the scalar fields z^i and the geometry of the scalar manifold ($\mathcal{N}$ is really a period matrix of the scalar manifold) and generalizes the coupling constant and θ-angle of ordinary QCD. At this point we can introduce a new tensor $\mathcal{G}_\Lambda$:

$$\mathcal{G}_\Lambda = \mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{\Sigma} - \mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}{}^*\mathcal{F}^{\Lambda}.$$

With this notation the Lagrangian becomes very simple $L = \frac{1}{2}\mathcal{F}^{\Lambda *}\mathcal{G}_\Lambda$. The form of the field equations and Bianchi identities for this Lagrangian is

$$\begin{aligned} \partial^{\mu *}\mathcal{F}^{\Lambda}_{\mu\nu} &= 0 \\ \partial^{\mu *}\mathcal{G}_{\Lambda\mu\nu} &= 0. \end{aligned}$$

We can introduce the $2n_v$ column vector: $V = \begin{pmatrix} ^*\mathcal{F}^{\Lambda} \\ ^*\mathcal{G}_{\Lambda} \end{pmatrix}$ which must satisfy the field equations $\partial V = 0$. The general transformation that keeps $\partial V = 0$ is:

$$\begin{pmatrix} {}^*\mathcal{F}^\Lambda \\ {}^*\mathcal{G}_\Lambda \end{pmatrix}' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} {}^*\mathcal{F}^\Lambda \\ {}^*\mathcal{G}_\Lambda \end{pmatrix} \tag{1.6}$$

for any matrix $\Lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2n_v, \mathrm{R})$. The requirement that the Lagrangian should also be invariant under this transformations restricts the form of the matrix so that they must belong to the symplectic group $\Lambda \in Sp(2n, \mathbf{R})$ with the condition on the coefficients:

$$\Lambda^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.7}$$

Vector fields transform as the column vector V (called the section of the symplectic vector bundle) of the electric plus magnetic field strength. The symplectic invariant combinations of $\mathcal{F}^\Lambda$ and $\mathcal{G}_\Lambda$ form the graviphoton T^- and the vector field strength $\mathcal{F}^{i-}_{\mu\nu}$:

$$\begin{aligned} \mathrm{T}^- &= M_\Lambda \mathcal{F}^\Lambda - L^\Lambda \mathcal{G}_\Lambda & (1.8) \\ \mathcal{F}^{-i} &= g^{i\bar{j}}(D_{\bar{j}} \bar{M}_\Lambda \mathcal{F}^\Lambda - D_{\bar{j}} \bar{L}^\Lambda \mathcal{G}_\Lambda), & (1.9) \end{aligned}$$

where M_Λ and L^Λ are functions of the scalar fields (moduli) z, see also Sec.(2.2) of Chapter (2).

The extended supersymmetry algebra for this system has a form:

$$\{Q_i, \bar{Q}_j\} = \delta_{ij} P_\mu \gamma^\mu + \varepsilon_{ij} Z \tag{1.10}$$

where Q_i $(i = 1, 2)$ are the supersymmetry charges, P_μ is the momentum and Z is called the central charge.

The central charge of the supersymmetry algebra is also connected with the symplectic invariants as follows:

$$Z = -\frac{1}{2} \int_{S_2} T^- = q_\Lambda L^\Lambda - p^\Lambda M_\Lambda \tag{1.11}$$

where q_Λ and p^Λ are the electric and magnetic charges of $\mathcal{F}$.

It turns out (see Sec.(2.2) of Chapter (2)) that the nontrivial solution of the field equations for this system[1] with unbroken N=2 supersymmetry has a metric

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}(dr^2 + r^2 d\Omega), \tag{1.12}$$

with $\Delta e^{-U(r)} = 0$ and constrains:

(1) the vector multiplet vector field strength to be zero: $\mathcal{F}^i_{\mu\nu} = 0$;

(2) the Ricci tensor to be the product of graviphoton field: $R^{BR}_{\alpha,\beta,\alpha',\beta'} = T_{\alpha,\beta}\bar{T}_{\alpha',\beta'}$;

(3) the scalar fields to be constant ($\partial_\mu z^i = 0$, $\partial_\mu q^u = 0$) with their values not arbitrary but determined by the consistency conditions (1) and (2) above.

The first consistency condition gives:

$$\frac{\partial}{\partial z} Z(z|_{horizon}) = 0 \tag{1.13}$$

and the central charge reach its extremum value at the horizon. The requirement of the unbroken N=2 supersymmetry leads to the condition that the energy of the space-time (or ADM mass) :

$$E = M_{ADM} = max|Z(q, p, z_{horizon})| \tag{1.14}$$

that looks like the analog of the classical condition $M^2 = Q^2$ for extreme black holes[2]. On the other hand the square of the minimal energy defines the area of the black hole horizon: $\pi E^2|_{extremum} = \frac{A}{4}$.

Near the horizon $r \to 0$ the expression $e^{-2U(r)}$ from (1.12) takes form $e^{-2U(r)} = \frac{A}{4\pi r^2} = \frac{M^2_{BR}}{r^2}$. In this case the metric becomes the Bertotti-Robinson one and ADM mass M_{ADM} coincides with the Bertotti-Robinson mass M_{BR}. The condition (1.14)

[1]The trivial solution is a flat vacuum with a flat metric, zero vector fields, and all scalar fields take arbitrary constant values: $ds^2 = dx^\mu dx^\nu \eta_{\mu\nu}$, $T_{\mu\nu} = \mathcal{F}^i_{\mu\nu} = 0$, $z^i = z^i_0$, $q^u = q^u_0$.

[2]In general, for the supersymmertic theories with a nonzero central charge there are the inequality on the masses and charges of the states $M \geq kZ$, where constant k depends on the parameters of the theory. The states that saturate this bound are called BPS states.

becomes:

$$M_{BR}^2 = |Z(q, p, z_{horizon})|^2 = \frac{A}{4\pi} = \frac{S}{\pi}. \tag{1.15}$$

This entropy-area formula of supersymmetric black holes has been found by S. Ferrara, R. Kallosh and A. Strominger [5]. It was proven by S. Ferrara and R. Kallosh [6] that the values of the moduli fields $z^i(r)$ demonstrate the attractor behavior as they approach the black hole horizon and their fixed values $z_{horizon}(q, p)$ are functions of the electric and magnetic charges only. To find these fixed values one must solve the nonlinear condition (1.13). This condition can be rewritten as two sets of equations:

$$\begin{aligned} p^\Lambda &= i(\overline{Z}L^\Lambda - Z\overline{L}^\Lambda), \\ q_\Lambda &= i(\overline{Z}M_\Lambda - Z\overline{M}_\Lambda). \end{aligned} \tag{1.16}$$

The central charge in (1.15) also becomes a function of the electric and magnetic charges only. Therefore the entropy of the supersymmetric black hole depends only on quantized electric and magnetic charges p and q :

$$S = \frac{A}{4} = \pi |Z_{fix}(p, q)|^2. \tag{1.17}$$

In our papers [7, 8, 9, 10] the equations (1.16) were solved for the first time. We have found explicit expressions for fixed moduli values and black hole entropy for different types of N=2 extreme black holes.

In our calculations we have connected two ideas (two puzzles) of the theory, black holes and supergravity, to find the macroscopic expression for the black hole entropy as a solution of supergravity equations. The microscopic approach to the black hole entropy makes us introduce a new puzzle of the theory - D branes.

Puzzle Three: D-branes.

D-branes (or Dirichlet branes) were first introduced by J. Polchinski [11] as a dual solution to the type II superstrings. Following J. Polchinski, we describe the D-branes as extended objects defined by the Dirichlet-Neumann boundary conditions, they break half of the supersymmetry of the type II superstrings and carry the complete set of electric and magnetic Ramond-Ramond charges. Another definition describes D-branes as hypersurfaces where open strings end. How can this kind of object be related to the black hole?

D-brane technology for the black hole microscopic entropy calculations was first proposed by A. Strominger and C. Vafa [12]. They calculated the degeneracy of the BPS states at weak coupling (bound states of several D-branes) with several Ramond-Ramond non-zero charges Q^i. It was proven that this system at weak coupling is dual to the d=5 extremal Reissner-Nordstrom black hole with the same charges Q^i at strong coupling. In this sense d=5 dimensional black holes can be considered as bound states of several D-branes and strings. For large Q^i the degeneracy turns out to be $(e^{\frac{A}{4G}})$ where A is the area of the horizon of the (dual) black hole with the same charges Q^i. This was the first time that the counting of microscopic states reproduced the Beckenstein-Hawking entropy formula. The amazing thing is that in these calculations the D-brane system was considered in the flat background and there was no "area" of any surface to start with.

This D-brane technology was quickly generalized for the calculations of the Beckenstein - Hawking entropy for other types of black hole. In particular these calculations were done for d=4 black holes [13, 14]. In the paper by J. Maldacena, A. Strominger and E. Witten [15] the extremal black holes in M-theory were studied. They considered d=11 dimensional M-theory compactified on the Calabi-Yau three-fold M and a circle $S^1, M \times S^1$. The resulting N=2, d=4 black hole with the charge configuration $(0, p^A, q_0, q_A)$ corresponds to the five-branes wrapping the five-cycle $P \times S^1$, where P is a four-cycle in M. They have compared their microscopic results for the black hole entropy with our macroscopic calculation for Calabi-Yau black holes [9] and found perfect agreement of the results:

$$S|_{micro} = S|_{macro} = 2\pi\sqrt{D\hat{q}},$$

where $D = D_{ABC}p^A p^B p^C$, see Chapter (4) and Equation (4.15).

D-branes make it possible to establish the connection between the macroscopic black hole entropy and the microscopic string states calculation. The significant role of the D-branes in the theory make it important to consider their description more closely.

Puzzle Four: D-branes and the Born-Infeld action.

The effective D-brane action was established from the one-loop corrections to the effective string action and was first derived for bosonic D-branes [16]:

$$S = \int_\Sigma d^{p+1}x e^{-\phi}\sqrt{-\det(G+F)}, \tag{1.18}$$

where Σ is a boundary of a type II string world-sheet surface, and $(p+1)$ of 10 coordinates x^a obey Neumann boundary conditions, while $(9-p)$ coordinates obey Dirichlet boundary conditions. The boundary, the D-brane, is restricted to the (p+1)-dimensional submanifold. ϕ is a dilaton, G is the induced metric on the D-brane and F is a field strength of the U(1) boundary gauge field (see Chapter (6) for details).

Recently significant progress has been achieved in the construction of the effective action for the supersymmetric D-brane [17]-[20]. This construction assumed the existence of a special local fermionic gauge symmetry on the worldvolume, the so-called κ- symmetry. The existence of a close relationship between κ- symmetry and supersymmetry leads to the existence of a global worldvolume supersymmetry (i.e. an equality of bosonic and fermionic degrees of freedom). The final form of the action depends on the choice of the κ gauge.

One of the interesting features of D-branes is their connection to supersymmetric Yang-Mills theory. We have performed calculations for the D3-brane action in the flat background [21] in the so-called Killing κ- gauge. Although this theory is N=4 supersymmetric, it is not conformally symmetric (as was the case for N=4 super-Yang-Mills) and has one-loop corrections and counterterms. We found it interesting to calculate these corrections and investigate the supersymmetry transformations in this case. We believe that our work will help in the understanding of more complicated cases, such as D-branes in a curved background, like AdS_5, where it was noticed that D3-branes have superconformal symmetry.

Summary of the following Chapters.

Closely following the papers [7]- [10], in this dissertation we will investigate N=2 extreme black holes associated with the different types of moduli spaces. Starting with the simple case [7] of two moduli and two electric and magnetic charges (q_Λ and p^Λ with $\Lambda = 0, 1$) in Chapter (2), we will then proceed to the more complicated moduli spaces with the arbitrary number of moduli fields. In Chapter (3) we will present the solution for moduli fields and entropy for so-called STU black holes [8]. In Chapter (4) we will investigate N=2 extreme black holes associated with the most general Calabi-Yau moduli space and prepotential $F = d_{ABC}\frac{X^A X^B X^C}{X^0}$, where X^A values of moduli of scalar fields and d_{ijk} is a characteristic of Calabi-Yau moduli space [9]. We will show that the black hole entropy-area formula depends on combinations of black hole charges p and q and parameters d_{ABC} and these combinations are the solutions of a simple system of algebraic equations. We will also find an explicit solution for the black hole entropy and fixed moduli and will investigate the uniqueness of this solution. In Chapter (5) we will generalize some of the black hole solutions to the five-dimensional theory [10]. In Chapter (6) we will calculate one-loop corrections to the effective Lagrangian for the D3 brane [21]. The gauge-fixing of the κ-symmetric Born-Infeld D3 brane action in a flat background is done using Killing gauge. The linearized supersymmetry of the gauge-fixed action coincides with that of the $\mathcal{N} = 4$ Yang-Mills theory. The helicity amplitude and unitarity technique is used in the one-loop amplitude calculations. The counterterms and the finite 1-loop corrections are of the form $(\partial F)^4$ and their supersymmetric generalization.

2 Freezing of moduli by N=2 dyons

2.1 Introduction

This chapter is based on the work done by the author of this manuscript in collaboration with Renata Kallosh and W. Kai Wong [7]. In our paper [7] we introduced the concept of the double-extreme black holes for the first time.

Supersymmetric black holes in most general version of ungauged N=2 supergravity interacting with arbitrary number n_v of vector multiplets and n_h of hypermultiplets have various properties which can be studied in a relatively easy way. This happens due to the existence of the well developed theory of local N=2 supersymmetry based on special and quaternionic geometry, see for example [22]-[25] and references therein.

There is a long standing problem to find explicitly the most general family of supersymmetric black holes. Following the paper [7] we will find all supersymmetric black holes solutions of N=2 theory for which moduli are frozen and do not change their values all the way from the horizon to infinity. As we will see these are highly non-trivial solutions since each of the $(n_v + 1)$ U(1) gauge group will be allowed to carry arbitrary electric and magnetic charges (p^Λ, q_Λ), $\Lambda = 0, \ldots n_v$, which will fix the complex n_v moduli $(z^i, \bar{z}^i)$, $i = 1, \ldots n_v$ of vector multiplets.

The most important properties of generic supersymmetric regular black holes with non-constant moduli in N=2 theory have been established recently: moduli of vector multiplets near the horizon become functions of charges only [5]. In fact it has been found that near the horizon supersymmetric black holes choose the size of the Bertotti-Robinson throat to be related to the extremal value of the central charge [6].

$$M^2_{\rm BR}(p,q) = \left(|Z|^2\right)_{\frac{\partial |Z(z,\bar{z},(p,q))|}{\partial z}=0} \ . \tag{2.1}$$

The term "non-extreme black hole" is usually applied to a charged black hole which in general has two horizons. When they coincide the black hole is called "extreme".

Typically extreme black holes have some unbroken supersymmetry. When they solve equations of motion of supergravities the mass of the extreme black hole depends on moduli as well as on quantized charges. When the mass is extremized in the moduli space at fixed charges, moduli become functions of charges. We define [7] *double-extreme black holes as extreme, supersymmetric black holes with the extremal value of the ADM mass equal to the Bertotti-Robinson mass.*

$$M_{\rm ADM}^2 = M_{\rm BR}^2(p,q) \ . \tag{2.2}$$

Double-extreme black holes have constant moduli both for vector multiplets as well as for hypermultiplets but unconstrained electric and magnetic charges (p^Λ, q_Λ) in each of $n_v + 1$ gauge group. The attractor diagram in Fig.1 of ref. [6] shows the double-extreme black holes on the horizontal line with the zero slope.

The relation between charges and moduli for supersymmetric black holes near the horizon was established in the following form [6]

$$\begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} = \mathrm{Re} \begin{pmatrix} 2i\bar{Z}L^\Lambda \\ 2i\bar{Z}M_\Lambda \end{pmatrix} , \tag{2.3}$$

where Z is the central charge depending on moduli and on conserved charges (p^Λ, q_Λ) and (L^Λ, M_Λ) are covariantly holomorphic sections depending on moduli. This is a highly non-trivial constraint between moduli and charges. Only few solutions to this constraint are known [6].

Following [7] we will show how to solve this constraint and find the double-extreme black holes moduli (or the near horizon values of moduli of extreme black holes with non-constant moduli) in case of two models. The first one has the prepotential $F = -iX^0X^1$ and there is also a symplectic transformation of this model to the one without the prepotential. The second one is the model known as ST[2,n] manifold and has no prepotential. Upon symplectic transformation this model is related to some classical Calabi-Yau moduli space described by the prepotential of the form $F = d_{ABC}\frac{X^AX^BX^C}{X^0}$. We will find the fixed values of $n_v = (n+1)$ complex moduli as the function of $2(n_v + 1) = 2(n + 2)$ electric and magnetic charges in this model. In the race for most general black hole solutions this seems at the moment to be the most general available case of relations between charges, space-time geometry and moduli. We will find out also that the expressions for the moduli in terms of charges are surprisingly elegant. Our example for the ST[2,2] case is related via symplectic transformation to the so-called STU model that will be described in Ch.3. Thus we will get the fixed values of these 3 complex moduli as the functions of 4 electric and 4 magnetic charges.

This chapter is organized as follows. In Section 2.2 we look for the solution of field equations for the double-extreme black holes in theories with arbitrary prepotentials or symplectic sections. The most important part of this which differs vastly from the standard routine of solving field equations for black holes is the procedure of solving equations for constant moduli. It is this place where the properties of special geometry and holomorphic properties are of crucial importance and supply the solutions. Section 2.3 presents double-extreme black holes and frozen moduli as function of charges for the theory with the prepotential $F = -iX^0X^1$ describing N=2 supergravity interacting with one vector multiplet corresponding to SO(4) version of N=4 supergravity. We also get the black holes and frozen moduli for the symplectic transformation of this model which is related to SU(4) version of N=4 supergravity and as N=2 theory has no prepotential. Finally Section 2.4 deals with ST[2,n] manifold which is an $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,n)}{SO(2)\times SO(n)}$ symmetric manifold. This N=2 theory has no prepotential. This theory is related via symplectic transformation to the one with the cubic holomorphic prepotential associated with particular Calabi-Yau moduli space. We will solve the moduli stabilization equations (2.3) and express vector multiplet moduli in terms of double-extreme dyon charges (p, q). In Discussion we summarize the new results.

2.2 N=2 Double-Extreme Black Holes

We will present here the minimal information on N=2 ungauged supergravity which will be necessary to explain the action which is our starting point for the derivation of N=2 black holes.

Ungauged d=4 , N=2 supergravity theory includes the following multiplets:

1. **gravitational multiplet** contains the veilbein, the SU(2) doublet of gravitino and the graviphoton;
2. $\mathbf{n_v}$ **vector multiplets**, each contains a gauge boson, a doublet of gauginos and a complex scalar field z;
3. $\mathbf{n_h}$ **hypermultiplets**, each contains a doublet of hyperinos and 4 real scalars q.

Complex scalar fields z^i $(i = 1, 2, ..., n_v)$ of N=2 vector multiplets can be regarded as coordinates of a special Kähler manifold of dimension n_v with additional constraint on curvature. Scalar fields q^u $(u = 1, ..., 4n_h)$ of n_h hypermultiplets can be considered as $4n_h$ coordinates of a quaternionic manifold, which is a $4n_h$ -dimensional real manifold with a metric $h_{uv}(q)$. Special Kähler manifold can be defined by constructing flat symplectic bundle of dimension $2n_v + 2$ over Kähler-Hodge manifold with symplectic section defined as

$$V = (L^\Lambda, M_\Lambda), \qquad \Lambda = 0, 1, ...n_v \ , \tag{2.4}$$

where (L, M) obey the symplectic constraint $i(\bar{L}^\Lambda M_\Lambda - L^\Lambda \bar{M}_\Lambda) = 1$ and $L^\Lambda(z, \bar{z})$ and $M_\Lambda(z, \bar{z})$ depend on scalar fields $z, \bar{z}$, which are the coordinates of the "moduli space". L^Λ and M_Λ are covariantly holomorphic (with respect to Kähler connection), e.g.

$$D_{\bar{k}} L^\Lambda = (\partial_{\bar{k}} - \frac{1}{2} K_{\bar{k}}) L^\Lambda = 0 \ , \tag{2.5}$$

where K is the Kähler potential. Symplectic invariant form of the Kähler potential can be found from this equation by introducing the holomorphic section (X^Λ, F_Λ):

$$L^\Lambda = e^{K/2} X^\Lambda \ , \qquad M_\Lambda = e^{K/2} F_\Lambda \ , \qquad (\partial_{\bar{k}} X^\Lambda = \partial_{\bar{k}} F_\Lambda = 0) \ . \tag{2.6}$$

The Kähler potential is $K = -\ln i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda)$. The Kähler metric is given by $g_{k\bar{k}} = \partial_k \partial_{\bar{k}} K$. Finally from special geometry one finds that there exists a complex symmetric $(n_v+1) \times (n_v+1)$ matrix $\mathcal{N}_{\Lambda\Sigma}$ such that

$$M_\Lambda = \mathcal{N}_{\Lambda\Sigma} L^\Sigma \ , \qquad \mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma} L^\Lambda \bar{L}^\Sigma = -\frac{1}{2} \ , \qquad D_{\bar{i}} \bar{M}_\Lambda = \mathcal{N}_{\Lambda\Sigma} D_{\bar{i}} \bar{L}^\Sigma \ . \tag{2.7}$$

The bosonic part of ungauged $N=2$ supergravity action is given by

$$\frac{1}{2}\int d^4x\sqrt{-g}\left\{-R + 2g_{i\bar{\jmath}}\,\nabla^\mu\, z^i\, \nabla_\mu\, \bar{z}^{\bar{\jmath}} + 2h_{uv}\nabla_\mu q^u \nabla^\mu q^v \right.$$
$$\left. +2(\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^\Lambda\mathcal{F}^\Sigma + \mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{\Lambda *}\mathcal{F}^\Sigma)\right\} . \tag{2.8}$$

The kinetic term of gauge fields is defined by the period matrix $\mathcal{N}_{\Lambda\Sigma}$ which depends only on scalar fields of the vector multiplets z^i. The vector field action can be also rewritten as $(\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^\Lambda\mathcal{F}^\Sigma + \mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{\Lambda *}\mathcal{F}^\Sigma) = \mathcal{F}^{\Lambda *}\mathcal{G}_\Lambda$, where $\mathcal{G}_\Lambda = \mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^\Sigma - \mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}{}^*\mathcal{F}^\Sigma$. The symplectic structure of equation of motion is manifest in terms of the Sp($2n_v$ +2) symplectic vector field strength $(\mathcal{F}^\Lambda, \mathcal{G}_\Lambda)$. These vector fields in the symplectic basis decompose in the susy basis into the vector field of the gravitational multiplet (graviphoton) and the vector fields of the vector multiplets. The graviphoton is given by the following symplectic invariant combination of the vector fields in the action

$$T = M_\Lambda \mathcal{F}^\Lambda - L^\Lambda \mathcal{G}_\Lambda \ .$$

The central charge formula (the charge of the graviphoton) for the general N=2 theories is given by [22] - [24]

$$Z(z,\bar{z},q,p) = e^{\frac{K(z,\bar{z})}{2}}(X^\Lambda(z) q_\Lambda - F_\Lambda(z)\, p^\Lambda) = (L^\Lambda q_\Lambda - M_\Lambda p^\Lambda) \ . \tag{2.9}$$

The n_v vector fields of the vector multiplets are given by the symplectic invariant combination

$$\mathcal{F}^{-i} = g^{i\bar{k}}(D_{\bar{k}} \bar{M}_\Lambda \mathcal{F}^\Lambda - D_{\bar{k}} \bar{L}^\Lambda \mathcal{G}_\Lambda) \ .$$

The equations of motion for $\mathcal{F}^*$, Bianchi identity for $\mathcal{F}^*$, equations of motion for z^i, equations of motion for q^u, and the Einstein-Maxwell equation are:

$$\begin{aligned}
\nabla_\mu(\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{\Sigma\mu\nu} + \mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma}{}^*\mathcal{F}^{\Sigma\mu\nu}) \equiv \nabla_\mu({}^*\mathcal{G}_\Lambda{}^{\mu\nu}) &= 0, &\quad (2.10)\\
\nabla_\mu({}^*\mathcal{F}^{\Lambda\mu\nu}) &= 0, &\quad (2.11)\\
\nabla_\mu(g_{i\bar{k}}\nabla^\mu z^i) - \partial_{\bar{k}} g_{i\bar{\jmath}}\nabla^\mu z^i \nabla_\mu \bar{z}^{\bar{\jmath}} - 2\partial_{\bar{k}}\mathrm{Im}\,(\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{+\Lambda}\mathcal{F}^{+\Sigma}) &= 0, &\quad (2.12)\\
\nabla_\mu(h_{uw}\nabla^\mu q^u) - \partial_w h_{uv}\nabla^\mu q^u \nabla_\mu q^v &= 0, &\quad (2.13)\\
R_{\mu\nu} + 2g_{i\bar{\jmath}}\nabla_{(\mu} z^i \nabla_{\nu)}\bar{z}^{\bar{\jmath}} + 2h_{uv}\nabla_{(\mu} q^u \nabla_{\nu)} q^v & \\
+4\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}(\mathcal{F}^\Lambda{}_{\mu\rho}\mathcal{F}^\Sigma{}_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}\mathcal{F}^\Lambda{}_{\sigma\rho}\mathcal{F}^{\Sigma\sigma\rho}) &= 0. &\quad (2.14)
\end{aligned}$$

Symplectic covariant charges are defined as follows

$$\begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} = \begin{pmatrix} \int \mathcal{F}^\Lambda \\ \int \mathcal{G}_\Lambda \end{pmatrix}. \qquad (2.15)$$

We will be looking for double-extreme black holes of N=2 theory with arbitrary charges (p, q) using the following assumptions. All scalars are constant:

$$\partial_\mu z^i = 0\ , \qquad \partial_\mu q^u = 0\ . \qquad (2.16)$$

Unbroken supersymmetry requires in this case [5, 6]

$$\mathcal{F}^{-i} = 0\ . \qquad (2.17)$$

We assume the standard form of the metric of supersymmetric black hole with spherical symmetry and asymptotic flatness.

$$ds^2 = e^{2U}dt^2 - e^{-2U}d\vec{x}^2, \qquad U = U(r)\ , \qquad U \to 0 \quad \text{as} \quad r \to \infty\ . \qquad (2.18)$$

Our ansatz is

$$\begin{aligned}
\mathcal{F}^\Lambda &= e^{2U}\frac{2Q^\Lambda}{r^2}\,dt\wedge dr - \frac{2P^\Lambda}{r^2}\,rd\theta\wedge r\sin\theta d\phi\ , &\quad (2.19)\\
e^{-U} &= 1 + \frac{M}{r}\ . &\quad (2.20)
\end{aligned}$$

We will find that the fields $\mathcal{F}^\Lambda$ solve Maxwell equation in curved spherical symmetric spacetime, with e^{-U} harmonic and the mass is given in terms of the matrix $\mathrm{Im}\,\mathcal{N}$ and the charges of $\mathcal{F}^\Lambda$.

$$M^2 = -2\,\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\,(Q^\Lambda Q^\Sigma + P^\Lambda P^\Sigma) = |Z|^2 \,. \tag{2.21}$$

To prove this consider the Maxwell Equation for $\mathcal{F}^\Lambda$. First consider (2.10), the equation of motion for $\mathcal{F}^\Lambda$. By (2.11) and the assumption that the moduli fields are constant, (2.10) becomes

$$\nabla_\mu(\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{\Sigma\mu\nu}) \;=\; 0\,. \tag{2.22}$$

We can also multiply (2.11) with $\mathrm{Im}\,\mathcal{N}$ to obtain the dual of the above equation

$$\nabla_\mu(\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}{}^*\mathcal{F}^{\Sigma\mu\nu}) \;=\; 0. \tag{2.23}$$

Let us note that a field strength $\mathcal{F}'$ in spherically symmetric spacetime satisfying the Maxwell equations

$$\nabla_\mu(\mathcal{F}'^{\mu\nu}) \;=\; 0\,, \tag{2.24}$$

$$\nabla_\mu({}^*\mathcal{F}'^{\mu\nu}) \;=\; 0\,, \tag{2.25}$$

have the following solution

$$\mathcal{F}' \;=\; \frac{1}{\sqrt{-g}}\frac{2Q'}{r^2}\,dt\wedge dr - \frac{2P'}{r^2}\,rd\theta\wedge r\sin\theta d\phi\,. \tag{2.26}$$

Therefore, the solution to (2.22, 2.23) is

$$\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^\Sigma \;=\; e^{2U}\frac{2Q'^\Lambda}{r^2}dt\wedge dr - \frac{2P'^\Lambda}{r^2}rd\theta\wedge r\sin\theta d\phi\,. \tag{2.27}$$

The matrix $\mathrm{Im}\,\mathcal{N}$ is negative definite, it can be inverted and

$$\mathcal{F}^\Lambda \;=\; e^{2U}\,\frac{\mathrm{Im}\,\mathcal{N}^{-1}_{\Lambda\Sigma}\,2Q'^\Sigma}{r^2}\,dt\wedge dr - \frac{\mathrm{Im}\,\mathcal{N}^{-1}_{\Lambda\Sigma}\,2P'^\Sigma}{r^2}\,rd\theta\wedge r\sin\theta d\phi \tag{2.28}$$

$$\;=\; e^{2U}\,\frac{2Q^\Lambda}{r^2}\,dt\wedge dr - \frac{2P^\Lambda}{r^2}\,rd\theta\wedge r\sin\theta d\phi\,, \tag{2.29}$$

which is (2.19), where Q^Λ and P^Λ are the electric and magnetic charges of $\mathcal{F}^\Lambda$ respectively.

Equations of motion for the hypermultiplet scalars for constant q^u is satisfied without any additional restriction. Thus black holes of Abelian theory do not seem to stabilize the coordinates of the quaternionic manifolds. However, equations of motion for vector multiplets moduli (2.12) are very restrictive when z^i are constants. We are going to show that it is satisfied with constant moduli fields taking into account that $\mathcal{F}^{-i} = 0$. When $\mathcal{F}^{-i} = 0$, one can find [22]-[24]

$$\begin{pmatrix} \mathcal{F}^{+\Lambda} \\ \mathcal{G}^{+}{}_{\Lambda} \end{pmatrix} = -ie^{K/2}T^{+} \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} . \tag{2.30}$$

To solve the equation of motion (2.12) for z we have to consider

$$\begin{aligned} \partial_{\bar{k}} \mathrm{Im}\,(\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{+\Lambda}\mathcal{F}^{+\Sigma}) &= \frac{i}{2}(\partial_{\bar{k}}\overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}\mathcal{F}^{-\Sigma} - \partial_{\bar{k}}\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{+\Lambda}\mathcal{F}^{+\Sigma}) \\ &= \frac{i}{2}\partial_{\bar{k}}\overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}\mathcal{F}^{-\Sigma} \end{aligned} \tag{2.31}$$

and take into account that $\mathcal{N}_{\Lambda\Sigma}$ is holomorphic ($F_\Lambda = \mathcal{N}_{\Lambda\Sigma}X^\Sigma$). Consider the complex conjugate of it, ignoring constant factors

$$\begin{aligned} \partial_k\mathcal{N}_{\Sigma\Lambda}\mathcal{F}^{+\Lambda}\mathcal{F}^{+\Sigma} &= -\partial_k\mathcal{N}_{\Sigma\Lambda}X^\Lambda X^\Sigma e^K T^{+2} \\ &= -\left[\partial_k(\mathcal{N}_{\Lambda\Sigma}X^\Lambda)X^\Sigma - \mathcal{N}_{\Lambda\Sigma}\partial_k X^\Lambda X^\Sigma\right] e^K T^{+2} \\ &= -(\partial_k F_\Sigma X^\Sigma - F_\Lambda\partial_k X^\Lambda)e^K T^{+2} \\ &= 0 \end{aligned} \tag{2.32}$$

The fact that $\partial_k F_\Sigma X^\Sigma - F_\Lambda \partial_k X^\Lambda = 0$ is a property of the special geometry of the moduli space [22] - [24].

Note that although the above derivation only requires z^i to be constant, it is not arbitrary. It is because in the case of $\mathcal{F}^{-i} = 0$, we have[6] that $D_i Z = \frac{1}{2}\int \mathcal{F}^{+\bar{\jmath}}g_{i\bar{\jmath}} = 0 \Rightarrow \partial_i|Z| = 0$, where Z is the central charge – a function of (p^Λ, q_Λ) and z, and so z is constrained to take the value for which when $|Z|$ is at extremum.

Now, we are going to solve the Einstein equation (2.14) to obtain (2.20, 2.21).

The non-vanishing components of $R_{\mu\nu}$ in the basis $(dt, dr, rd\theta, r\sin\theta d\phi)$ are

$$
\begin{aligned}
R_{tt} &= -e^{4U}\vec{\nabla}^2 U \ , && (2.33)\\
R_{rr} &= 2U'^2 - \vec{\nabla}^2 U \ , && (2.34)\\
R_{\theta\theta} &= -\vec{\nabla}^2 U \ , && (2.35)\\
R_{\phi\phi} &= -\vec{\nabla}^2 U \ , && (2.36)
\end{aligned}
$$

where U' is $\frac{\partial}{\partial r}U$ and $\vec{\nabla}^2$ is the spatial part of the Laplacian.

The energy momentum tensor for $\mathcal{F}^\Lambda$ is

$$
T_{\mu\nu} = -\frac{1}{2\pi}\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}(\mathcal{F}^\Lambda{}_{\mu\alpha}\mathcal{F}^{\Sigma\alpha}{}_\nu - \frac{1}{4}g_{\mu\nu}\mathcal{F}^\Lambda{}_{\alpha\beta}\mathcal{F}^{\Sigma\alpha\beta}) \ . \tag{2.37}
$$

To obtain $T_{\mu\nu}$, we first use (2.19) to calculate

$$
\begin{aligned}
\mathcal{F}^\Lambda{}_{t\alpha}\mathcal{F}^{\Sigma\alpha}{}_t &= -\frac{Q^\Lambda Q^\Sigma}{r^4}e^{6U} \ , && (2.38)\\
\mathcal{F}^\Lambda{}_{r\alpha}\mathcal{F}^{\Sigma\alpha}{}_r &= \frac{Q^\Lambda Q^\Sigma}{r^4}e^{2U} \ , && (2.39)\\
\mathcal{F}^\Lambda{}_{\theta\alpha}\mathcal{F}^{\Sigma\alpha}{}_\theta &= -\frac{P^\Lambda P^\Sigma}{r^4}e^{2U} \ , && (2.40)\\
\mathcal{F}^\Lambda{}_{\phi\alpha}\mathcal{F}^{\Sigma\alpha}{}_\phi &= -\frac{P^\Lambda P^\Sigma}{r^4}e^{2U} && (2.41)
\end{aligned}
$$

$$
\mathcal{F}^\Lambda{}_{\alpha\beta}\mathcal{F}^{\Sigma\alpha\beta} = 2\frac{e^{4U}}{r^4}(P^\Lambda P^\Sigma - Q^\Lambda Q^\Sigma) \ . \tag{2.42}
$$

From above and (2.37), the non-zero components of $T_{\mu\nu}$ are

$$
\begin{aligned}
T_{tt} &= \frac{1}{4\pi}\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\frac{e^{6U}}{r^4}(P^\Lambda P^\Sigma + Q^\Lambda Q^\Sigma) \ , && (2.43)\\
T_{rr} &= -\frac{1}{4\pi}\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\frac{e^{2U}}{r^4}(P^\Lambda P^\Sigma + Q^\Lambda Q^\Sigma) \ , && (2.44)\\
T_{\theta\theta} &= \frac{1}{4\pi}\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\frac{e^{2U}}{r^4}(P^\Lambda P^\Sigma + Q^\Lambda Q^\Sigma) \ , && (2.45)\\
T_{\phi\phi} &= \frac{1}{4\pi}\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\frac{e^{2U}}{r^4}(P^\Lambda P^\Sigma + Q^\Lambda Q^\Sigma) \ . && (2.46)
\end{aligned}
$$

Hence, the Einstein-Maxwell equation, (2.14),

$$
\begin{aligned}
R_{tt} - 8\pi T_{tt} &= 0 \,, &(2.47)\\
R_{rr} - 8\pi T_{rr} &= 0 \,, &(2.48)\\
R_{\theta\theta} - 8\pi T_{\theta\theta} &= 0 \,, &(2.49)\\
R_{\phi\phi} - 8\pi T_{\phi\phi} &= 0 \,, &(2.50)
\end{aligned}
$$

gives, respectively,

$$
\begin{aligned}
\vec{\nabla}^2 U + 2\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\, \frac{e^{2U}}{r^4}\, (P^\Lambda P^\Sigma + Q^\Lambda Q^\Sigma) &= 0 \,, &(2.51)\\
2U'^2 - \vec{\nabla}^2 U + 2\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\, \frac{e^{2U}}{r^4}\, (P^\Lambda P^\Sigma + Q^\Lambda Q^\Sigma) &= 0 \,, &(2.52)\\
\vec{\nabla}^2 U + 2\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\, \frac{e^{2U}}{r^4}\, (P^\Lambda P^\Sigma + Q^\Lambda Q^\Sigma) &= 0 \,, &(2.53)\\
\vec{\nabla}^2 U + 2\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\, \frac{e^{2U}}{r^4}\, (P^\Lambda P^\Sigma + Q^\Lambda Q^\Sigma) &= 0 \,. &(2.54)
\end{aligned}
$$

This amounts to

$$\vec{\nabla}^2 U = -2\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma} \frac{e^{2U}}{r^4} (P^\Lambda P^\Sigma + Q^\Lambda Q^\Sigma) \,, \tag{2.55}$$

and

$$\vec{\nabla}^2 e^{-U} = 0 \,, \tag{2.56}$$

since (2.56) is equivalent to $\vec{\nabla}^2 U - U'^2 = 0$.

Therefore, from (2.55, 2.56), e^{-U} is harmonic and has solution

$$
\begin{aligned}
e^{-U} &= 1 + \frac{M}{r} \,, &(2.57)\\
M^2 &= -2\,\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\, (Q^\Lambda Q^\Sigma + P^\Lambda P^\Sigma) \,, &(2.58)
\end{aligned}
$$

which was necessary to prove, as suggested in eqs. (2.20, 2.21).

When there are no vector multiplets, $\mathrm{Im}\,\mathcal{N} = -\frac{1}{2}$ by (2.8) and M^2 reduces to $P^2 + Q^2$, the well known Reissner-Nordstrom solution.

Now we will show that the mass of the black hole is actually equal to the extremal value of the central charge. For this purpose we have to rewrite our mass formula which follows from the solution of field equations and is given in terms of P and Q charges of $\mathcal{F}$'s to the one which employs the symplectic covariant charges (p,q), defined in eq. (2.15). In terms of the charges of $\mathcal{F}^\Lambda$,

$$\begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} = \begin{pmatrix} 2P^\Lambda \\ \mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma}\, 2P^\Sigma - \mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\, 2Q^\Sigma \end{pmatrix} \tag{2.59}$$

with the inverse

$$\begin{pmatrix} P^\Lambda \\ Q^\Lambda \end{pmatrix} = \frac{1}{2}\begin{pmatrix} p^\Lambda \\ (\mathrm{Im}\,\mathcal{N}^{-1}\mathrm{Re}\,\mathcal{N}\; p)^\Lambda - (\mathrm{Im}\,\mathcal{N}^{-1}\; q)^\Lambda \end{pmatrix}. \tag{2.60}$$

Thus using (2.58) we get

$$\begin{aligned} M^2 &= -2\,\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\,(Q^\Lambda Q^\Sigma + P^\Lambda P^\Sigma) \\ &= -\frac{1}{2}\left(p^\Lambda, q_\Lambda\right)\begin{pmatrix} (\mathrm{Im}\,\mathcal{N} + \mathrm{Re}\,\mathcal{N}\mathrm{Im}\,\mathcal{N}^{-1}\mathrm{Re}\,\mathcal{N})_{\Lambda\Sigma} & (-\mathrm{Re}\,\mathcal{N}\mathrm{Im}\,\mathcal{N}^{-1})_\Lambda{}^\Sigma \\ (-\mathrm{Im}\,\mathcal{N}^{-1}\mathrm{Re}\,\mathcal{N})^\Lambda_\Sigma & (\mathrm{Im}\,\mathcal{N}^{-1})^{\Lambda\Sigma} \end{pmatrix}\begin{pmatrix} p^\Sigma \\ q_\Sigma \end{pmatrix} \\ &= |Z|^2 + |Z_i|^2\,, \end{aligned} \tag{2.61}$$

where Z is the central charge and Z_i is given by $-\frac{1}{2}\int \mathcal{F}^{+\bar{\jmath}} g_{i\bar{\jmath}} = 0$ as $\mathcal{F}^{-i} = 0$ [22]-[24]. Hence, the mass of the double-extreme black hole is equal to the extremal value of the central charge.

$$M = |Z|_{Z_i=0}\;. \tag{2.62}$$

This is in accordance with what is found in [6], in particular, (48) of [6] give the mass of a pure magnetic black hole ($P \neq 0, Q = 0$) as $-\frac{1}{2}p^\Lambda \mathrm{Im}\,\mathcal{F}_{\Lambda\Sigma}p^\Sigma$, and $\mathrm{Im}\,\mathcal{F}_{\Lambda\Sigma}$ here can be replaced by $\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}$.

Thus the double-extreme black holes have the extremal ADM mass for a given set of (p,q) charges when there is a relation between the charges and vector multiplet moduli, given in eq. (2.3).

$$p^\Lambda = i(\overline{Z}L^\Lambda - Z\overline{L}^\Lambda)\,, \qquad q_\Lambda = i(\overline{Z}M_\Lambda - Z\overline{M}_\Lambda)\;. \tag{2.63}$$

This relation was derived in [6] from the constraint $D_iZ = Z_i = \mathcal{F}^{-i} = 0$ and as shown here is required for the solution of the equations of motion and the consistency of the total picture.

In the most general case when we have arbitrary sections (L, M) this is the best form of a constraint between the charges and moduli, which is available. Note that the central charge is a linear functions of (p^Λ, q_Λ) charges and depends on moduli as shown in eq. (2.9). In principle eqs. (2.63) can be solved for z^i in terms of (p^Λ, q_Λ). In what follows we will present few examples when this constraint can be solved so that the explicit form of the complex moduli z^i in terms of charges (p^Λ, q_Λ) can be found.

2.3 Axion-dilaton double-extreme black holes

Up to this point all the results we derived hold for a general prepotential F or for a general symplectic sections, when the prepotential is not available. In this section we consider a special case of N=2 supergravity interacting with one vector multiplet. The prepotential is given by $F = -iX^0X^1$, which is equivalent to $N = 4$ supergravity theory with two vector fields. We will express the moduli $z = \frac{X^1}{X^0}$ and the central charge $|Z|$ in terms of charges p^Λ and q_Λ where $\Lambda = 0, 1$and relate them to the corresponding moduli in N=4 theory.

With this prepotential, choosing the gauge $X^0 = 1$

$$L^\Lambda = e^{K/2}X^\Lambda = e^{K/2}\begin{pmatrix} 1 \\ z \end{pmatrix}, \tag{2.64}$$

$$M_\Lambda = e^{K/2}F_\Lambda = -ie^{K/2}\begin{pmatrix} z \\ 1 \end{pmatrix}, \tag{2.65}$$

where

$$e^K = \frac{1}{2(z+\overline{z})}. \tag{2.66}$$

As mentioned in the previous Section, z is constrained according to eqs. (2.63) which forces z to depend on the charges. Eliminating $\overline{Z}$ we can rewrite equations

(2.63) as

$$L^\Lambda q_\Sigma - p^\Lambda M_\Sigma = iZ(\overline{L}^\Lambda M_\Sigma - L^\Lambda \overline{M}_\Sigma) \;, \tag{2.67}$$

or in matrix form and using (2.64, 2.65),

$$e^{K/2} \begin{bmatrix} q_0 + ip^0 z & q_1 + ip^0 \\ q_0 z + ip^1 z & q_1 z + ip^1 \end{bmatrix} = -Ze^K \begin{bmatrix} z+\overline{z} & 2 \\ 2z\overline{z} & z+\overline{z} \end{bmatrix} \;. \tag{2.68}$$

From equations of the two diagonal components we can solve for z in terms of charges:

$$z = \frac{q_0 - ip^1}{q_1 - ip^0} \;. \tag{2.69}$$

This is consistent with equations from other components. This is the expression for z in terms of the charges. In order to keep e^K positive according to (2.66) one is further required to take Rez to be equal to the absolute value of the real part of eq. (2.69). i.e. to $|q_0 q_1 + p^0 p^1|$.

Now let us express the central charge in terms of p^Λ and q_Λ only. In terms of z, p^Λ and q_Λ, by (2.64, 2.65), the central charge is given by [22]-[24]

$$\begin{aligned} Z &= L^\Lambda q_\Lambda - M_\Lambda p^\Lambda && (2.70) \\ &= e^{K/2}\left[(q_0 + ip^1) + (q_1 + ip^0) z\right] \;. && (2.71) \end{aligned}$$

Substituting (2.69) and (2.66) we get

$$Z = \left(\frac{q_0 q_1 + p^0 p^1}{q_1^2 + p^{0\,2}}\right)^{1/2} (q_1 + ip^0) \;, \tag{2.72}$$

and the mass of the double-extreme black hole is a function of charges:

$$M^2 = |Z|^2 = |q_0 q_1 + p^0 p^1| \;. \tag{2.73}$$

Now consider the model related to this one by symplectic transformation [22]-[24]. It corresponds to N=2 reduction of the SU(4) formulation of pure N=4 supergravity.

$$\hat{X}^0 = X^0 , \qquad \hat{F}_0 = F_0 , \qquad \hat{X}^1 = -F_1 , \qquad \hat{F}_1 = X^1 . \tag{2.74}$$

and for charges we have

$$\hat{p}^0 = p^0 , \qquad \hat{q}_0 = q_0 , \qquad \hat{p}^1 = -q_1 , \qquad \hat{q}_1 = p^1 . \tag{2.75}$$

Our solution for moduli becomes

$$z = \frac{\hat{q}_1 + i\hat{q}_0}{\hat{p}^0 - i\hat{p}^1} , \tag{2.76}$$

and again it is required that Re$z = |\hat{p}^0\hat{q}_1 - \hat{q}_0\hat{p}^1|$. The double-extremal black hole mass in this version is given by:

$$M^2 = |Z|^2 = |\hat{p}^0\hat{q}_1 - \hat{q}_0\hat{p}^1| . \tag{2.77}$$

This coincides with the double-extreme axion-dilaton black holes in SU(4) version of N=4 supergravity [26], [27].

2.4 N=2 heterotic vacua

In heterotic string vacua space-time supersymmetry comes from the right-moving sector of the string theory. In particular, the graviton, an antisymmetric tensor, the dilaton and 2 abelian fields together with fermions constitute the vector-tensor multiplet. On shell an antisymmetric tensor combines with the dilaton into a complex scalar, which belongs to an N=2 vector multiplet. Other vector multiplets originate from the left-moving sector. The corresponding theory can be defined in terms of a prepotential [3]

$$F = \frac{1}{2} d_{ABC} t^A t^B t^C = S\, \eta_{ij}\, t^i t^j , \qquad X^0 = 1 , \tag{2.78}$$

where

$$t^1 = S, \qquad d_{ABC} = \begin{pmatrix} d_{1jk} = \eta_{jk} \\ 0 \text{ otherwise} \end{pmatrix}, \qquad A, B, C = 1, 2, \ldots, n+1 , \tag{2.79}$$

[3]We are using here notation of [25].

and

$$\eta_{ij} = \text{diag}(+,-,-\ldots,-), \quad i,j = 2,\ldots,n+1. \tag{2.80}$$

This prepotential corresponds to the product manifold $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,n)}{SO(2)\times SO(n)}$. The $\frac{SU(1,1)}{U(1)}$ coordinate is the axion-dilaton field S. The remaining n complex moduli t^i are special coordinates of the $\frac{SO(2,n)}{SO(2)\times SO(n)}$ manifold. In particular, when $n = 2$ we have

$$F = \frac{1}{2} d_{ABC} t^A t^B t^C = \frac{1}{2} S \left[(t^2)^2 - (t^3)^2 \right] . \tag{2.81}$$

If we introduce the notation

$$t^2 \equiv \frac{1}{2}(T+U) , \qquad t^3 \equiv \frac{1}{2}(T-U) , \tag{2.82}$$

the prepotential becomes

$$F = \frac{1}{2} STU . \tag{2.83}$$

This theory is defined by 3 complex moduli and 4 gauge groups and the corresponding manifold is $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,2)}{SO(2)\times SO(2)}$.

We would like to find the values of moduli of the double-extreme black holes for this model with $n = 2$ as well as for the general case of arbitrary n.

Our method is to use the version of this theory which is related to the one described above by a singular symplectic transformation [22]-[24]. This version does not have a prepotential and is defined in terms of a symplectic sections. To avoid complicated formulae we will not use hats to describe this version of the theory but the sections as well as charges should not be associated with the prepotential version above. Thus the starting point to describe the N=2 heterotic vacua is [22]-[24]

$$\begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} = \begin{pmatrix} X^\Lambda \\ S\, X_\Lambda \end{pmatrix} , \qquad X^\Lambda X_\Lambda \equiv X \cdot X = 0 , \qquad \Lambda = 0, 1, \ldots n+1. \tag{2.84}$$

The metric $\eta_{\Lambda\Sigma} = \text{diag}(+,+,-,\ldots,-)$ is used for changing the position of the indices: $X_\Lambda = \eta_{\Lambda\Sigma} X^\Sigma$. Note that X^Λ are not independent and satisfy the constraint $X \cdot X = 0$.

From sections one can derive K and $\mathcal{N}$ as follows:

$$K = -\ln\left[i(S-\overline{S})X\cdot\overline{X}\right] , \tag{2.85}$$

$$\mathcal{N}_{\Lambda\Sigma} = (S-\overline{S})\ \frac{X_\Lambda\overline{X}_\Sigma+\overline{X}_\Lambda X_\Sigma}{X\cdot\overline{X}} + \overline{S}\eta_{\Lambda\Sigma} . \tag{2.86}$$

The Kähler metric can be obtained from the second derivative of K:

$$g_{S\overline{S}} = \frac{\partial}{\partial S}\frac{\partial}{\partial\overline{S}}K = \frac{1}{(2\mathrm{Im}\, S)^2} , \tag{2.87}$$

$$g_{X^\Lambda\overline{X}^\Sigma} = \frac{\partial}{\partial X^\Lambda}\frac{\partial}{\partial\overline{X}^\Sigma}K = \frac{1}{X\cdot\overline{X}}(\frac{X_\Sigma\overline{X}_\Lambda}{X\cdot\overline{X}} - \eta_{\Lambda\Sigma}) . \tag{2.88}$$

Then the non-gravitational part of the Lagrangian is given by

$$\begin{aligned}\mathcal{L} &= \frac{1}{(2\mathrm{Im}\, S)^2}\partial S\partial\overline{S} + \frac{1}{X\cdot\overline{X}}(\frac{X_\Sigma\overline{X}_\Lambda}{X\cdot\overline{X}} - \eta_{\Lambda\Sigma})\partial X^\Lambda\partial\overline{X}^\Sigma \\ &+ \mathrm{Im}\, S\left[2(\frac{X_\Lambda\overline{X}_\Sigma+\overline{X}_\Lambda X_\Sigma}{X\cdot\overline{X}}) - \eta_{\Lambda\Sigma}\right]\mathcal{F}^\Lambda\mathcal{F}^\Sigma + \mathrm{Re}\, S\ \eta_{\Lambda\Sigma}\mathcal{F}^{\Lambda *}\mathcal{F}^\Sigma.\end{aligned} \tag{2.89}$$

Using $F_\Lambda = SX_\Lambda$ in the stabilization eqs. (2.63) and we can bring them to the following form

$$p^\Lambda = i\overline{Z}e^{K/2}X^\Lambda - iZe^{K/2}\overline{X}^\Lambda , \tag{2.90}$$

$$q_\Lambda = i\overline{Z}e^{K/2}SX_\Lambda - iZe^{K/2}\overline{S}\ \overline{X}_\Lambda . \tag{2.91}$$

We can contract these equations with X using the constraint $X\cdot X = 0$ and we get

$$X\cdot p = -iZe^{K/2}X\cdot\overline{X} , \tag{2.92}$$

$$X\cdot q = -iZe^{K/2}\overline{S}\ X\cdot\overline{X} . \tag{2.93}$$

These two equations imply

$$X\cdot q = \overline{S}X\cdot p . \tag{2.94}$$

The above equation is useful as all $X\cdot q$ and $\overline{X}\cdot q$ can be expressed in terms of $\overline{S}\ X\cdot p$ and $S\overline{X}\cdot p$ respectively. We contract equations (2.90) and (2.91) with p and q and get the following:

$$p^2 = i\overline{Z}e^{K/2}X \cdot p - iZe^{K/2}\overline{X} \cdot p \tag{2.95}$$

$$q \cdot p = i\overline{Z}e^{K/2}X \cdot q - iZe^{K/2}\overline{X} \cdot q \tag{2.96}$$

$$p \cdot q = i\overline{Z}e^{K/2}SX \cdot p - iZe^{K/2}\overline{S}\,\overline{X} \cdot p \tag{2.97}$$

$$q^2 = i\overline{Z}e^{K/2}SX \cdot q - iZe^{K/2}\overline{S}\,\overline{X} \cdot q \tag{2.98}$$

Using (2.94) to get rid of q on the right hand sides we get:

$$p^2 = i\overline{Z}e^{K/2}X \cdot p - iZe^{K/2}\overline{X} \cdot p \tag{2.99}$$

$$q \cdot p = i\overline{Z}e^{K/2}\overline{S}\,X \cdot p - iZe^{K/2}S\,\overline{X} \cdot p \tag{2.100}$$

$$p \cdot q = i\overline{Z}e^{K/2}S\,X \cdot p - iZe^{K/2}\overline{S}\,\overline{X} \cdot p \tag{2.101}$$

$$q^2 = i\overline{Z}e^{K/2}S\overline{S}\,X \cdot p - iZe^{K/2}S\overline{S}\,\overline{X} \cdot p \tag{2.102}$$

Now, comparing eq. (2.99) with eq. (2.102) we see that it can be satisfied only if

$$S\overline{S} = \frac{q^2}{p^2}\,. \tag{2.103}$$

The sum of eqs. (2.100) and (2.101) can be compared with eq. (2.99) from which we learn that the real part of the S-moduli is already defined in terms of charges.

$$S + \overline{S} = \frac{2p \cdot q}{p^2}. \tag{2.104}$$

From the above two equations we can obtain the fixed value of the axion-dilaton, which is the moduli on $\frac{SU(1,1)}{U(1)}$ manifold:

$$S = \frac{p \cdot q}{p^2} + i\frac{(p^2q^2 - (p \cdot q)^2)^{1/2}}{p^2}, \tag{2.105}$$

here the sign of $\operatorname{Im} S$ is chosen to be negative.

To calculate the central charge $|Z|$ one can multiply eq. (2.99) on eq. (2.102) and subtract the product of eqs. (2.101) and (2.100).

$$\begin{aligned} p^2q^2 - (p \cdot q)^2 &= |Z|^2 e^K (2S\overline{S} - \overline{S}^2 - S^2) X \cdot p\, \overline{X} \cdot p \\ &= |Z|^2 e^K (S - \overline{S})^2 |Z|^2 e^K (X \cdot \overline{X})^2 \\ &= |Z|^4, \end{aligned} \tag{2.106}$$

which leads to

$$|Z|^2 = (p^2q^2 - (p\cdot q)^2)^{1/2} \ , \tag{2.107}$$

where we used (2.92) and the expression for e^K.

The next step is to find the moduli on $\frac{SO(2,n)}{SO(2)\times SO(n)}$ manifold. For this purpose we multiply $\overline{S}$ on eq. (2.90) and subtract eq. (2.91):

$$\overline{S}p^\Lambda - q^\Lambda = i\overline{Z}e^{K/2}(\overline{S} - S)\ X^\Lambda, \tag{2.108}$$

which leads to a beautiful equation:

$$\frac{X^\Lambda}{X^\Sigma} = \frac{\overline{S}p^\Lambda - q^\Lambda}{\overline{S}p^\Sigma - q^\Sigma} \ . \tag{2.109}$$

Here $\overline{S}$ is given by

$$\bar{S} = \frac{p\cdot q}{p^2} + i\frac{(p^2q^2 - (p\cdot q)^2)^{1/2}}{p^2} \ . \tag{2.110}$$

Note that the ratio $\frac{X^\Lambda}{X^\Sigma}$ does not give us yet the moduli since $X \cdot X = 0$. This constraint can be solved, in particular using Calabi-Vesentini coordinates which can be identified with the special coordinates of rigid special geometry. However for local special geometry a suitable set of unconstrained moduli can be taken in the following form [22] - [25]

$$X^0 = - \ \tfrac{1}{2}\ \ (1 - \eta_{ij}\, t^i t^j) \ , \tag{2.111}$$

$$X^{i-1} = \ t^i \quad , \qquad\qquad i,j = 2,\ldots,n+1 \ , \tag{2.112}$$

$$X^{n+1} = \ \tfrac{1}{2}\ \ (1 + \eta_{ij}\, t^i t^j) \ . \tag{2.113}$$

This solution of the constraints has the property that $X^{n+1} - X^0 = 1$. The solution for moduli is

$$t^i = \frac{X^{i-1}}{X^{n+1} - X^0} = \frac{\overline{S}p^{i-1} - q^{i-1}}{\overline{S}(p^{n+1} - p^0) - (q^{n+1} - q^0)} \ . \tag{2.114}$$

Thus the double-extreme black hole of N=2 supergravity with $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,n)}{SO(2)\times SO(n)}$ symmetry has the following properties. The $n+1$ complex vector multiplet moduli are functions of charges:

$$t^1 = S = \frac{p\cdot q}{p^2} - i\,\frac{(p^2q^2-(p\cdot q)^2)^{1/2}}{p^2}\ , \tag{2.115}$$

$$t^i = \frac{\overline{S}p^{i-1} - q^{i-1}}{\overline{S}(p^{n+1}-p^0) - (q^{n+1}-q^0)}\ , \qquad i = 2,\ldots,n+1\ . \tag{2.116}$$

The mass (proportional to the area of the black hole horizon) is given by

$$M^2 = |Z|^2 = (p^2q^2-(p\cdot q)^2)^{1/2}\ , \tag{2.117}$$

and the scalars in the hypermultiplets can take arbitrary values not specified by vector field charges.

Note that the moduli could be also rewritten as functions of the charges of the version of the theory which has the prepotential. The explicit singular symplectic transformation which relate those 2 set of charges is available [22] - [25]. It seems however that the expressions for moduli are more complicated and will not be given here.

One can focus on a simple case of just 3 vector multiplets and $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,2)}{SO(2)\times SO(2)}$ model with the following charges: (p^0, q_0), (p^1, q_1), (p^2, q_2), (p^3, q_3) of the 4 gauge groups. The O(2,n) scalar products are

$$p^2 = (p^0)^2 + (p^1)^2 - (p^2)^2 - (p^3)^2\ , \tag{2.118}$$

$$q^2 = (q_0)^2 + (q_1)^2 - (q_2)^2 - (q_3)^2\ , \tag{2.119}$$

$$p\cdot q = (p^0q_0) + (p^1q_1) + (p^2q_2) + (p^3q_3)\ . \tag{2.120}$$

The frozen values of 3 complex moduli are

$$S = \frac{p\cdot q}{p^2} - i\frac{(p^2q^2-(p\cdot q)^2)^{1/2}}{p^2}\ ,$$

$$T = \frac{\overline{S}(p^1+p^2) - (q^1+q^2)}{\overline{S}(p^3-p^0) - (q^3-q^0)}\ ,$$

$$U = \frac{\overline{S}(p^1 - p^2) - (q^1 - q^2)}{\overline{S}(p^3 - p^0) - (q^3 - q^0)} . \tag{2.121}$$

The mass of the generic extreme black holes in this class depends on charges and moduli, it is minimal for the black holes with regular horizons and given in eq. (2.117) when moduli are taking the extremal values given in eqs. (2.121). This accomplishes our goal of finding the explicit form of the moduli as functions of charges for double-extreme black holes.

2.5 Discussion

Double-extreme black holes are characterized by the regular horizon and unbroken supersymmetry and their ADM mass is minimal for a given set of charges. The important new feature of these black holes is that the moduli are frozen in space at the values which minimize the ADM mass. We have found all N=2 dyonic double-extreme black holes.

The fact that the central charge of the gravitational dyons has an extremum in moduli space at the fixed values of charges follows from unbroken supersymmetry which is doubled near the black hole horizon [6]. However, not so many examples of the known black holes are available to verify this theorem. The new double-extreme black holes found in this chapter [7] for the most general coupling of N=2 supergravity to vector multiplets exhibit a relation between charges and moduli given in eq. (2.3) and found before in [6]. For the most general set of symplectic sections defining a given theory this relation, which we have called stabilization equation, is difficult to solve. However we have solved it for the so-called tree level heterotic N=2 vacua with arbitrary number of vector multiplets described by some of the classical Calabi-Yau moduli space, see eqs. (2.115,2.116). The new expressions for moduli in terms of electric andn magnetic charges have been found for arbitrary number of complex moduli, in particular for the complex values of S, T, U moduli in $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,2)}{SO(2)\times SO(2)}$

model with 4 electric and 4 magnetic charges, see eq. (2.121). Those are the main new technical results of this work.

3 STU blackholes

3.1 Introduction

The non-perturbative properties of the future fundamental theory manifests themselves in the duality properties of the area formulae of the supersymmetric black holes horizon. The universal entropy-area formula of supersymmetric black holes is given by the central charge extremized in the moduli space Z_{fix} and depends only on quantized charges. The universal formula obtained by S. Ferrara and R.Kallosh [6] is $S = \frac{A(p,q)}{4} = \pi |Z_{\text{fix}}|^{\alpha}$ with $\alpha = 2$ $(3/2)$ for $d = 4$ $(d = 5)$.

This universal formula has various implementations in different theories. A particularly rich class of area formulae may be expected to exist in N=2 supersymmetric theories which are characterized by different choice of the holomorphic prepotential and/or symplectic sections. A beautiful interplay between the geometry of special Kähler manifolds [22],[24],[28]-[31] and space-time geometry of supersymmetric black holes has been discovered recently [5], [6], [7].

In this chapter,following [8], we will find the 4d double-extreme black holes in a class of N=2 theories with the prepotential $F = d_{ABC}\frac{X^A X^B X^C}{X^0}$ [29]. These theories with real symmetric constant tensors d_{ABC} are related to geometry occurring in 5-dimensional supergravity [32] where the term $\int d_{ABC} F^A \wedge F^B \wedge A^C$ is present in the action. These theories are also related to the special geometry of Calabi-Yau moduli spaces where d_{ABC} are the intersection numbers of the Calabi-Yau manifold, $t^A = \frac{X^A}{X^0}$ are the moduli fields of the Kähler class [33]. The theories of this class are also referred to as "very special geometry" [34] and "real special geometry" [35].

We will focus mostly on STU-symmetric model $F = \frac{X^1 X^2 X^3}{X^0}$, and will find the moduli-independent $[SL(2,\mathbf{Z})]^3$ symmetric area formula. The moduli of this theory are coordinates of the $\left(\frac{SU(1,1)}{U(1)}\right)^3$ manifold. Duality symmetry of this theory is $[SL(2,\mathbf{Z})]^3$. The dual partners of these black holes (where one of the moduli, e.g. S is singled out and whose imaginary part plays the role of string coupling) are already known [7]. The moduli in this version of the theory are coordinates of $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,2)}{SO(2)\times SO(2)}$ manifold. S-duality, or $SL(2,\mathbf{Z})$ symmetry associated with the S moduli in string theory has a non-perturbative character, whereas T and U dualities, related to $SO(2,2)$ symmetry have perturbative character. Perturbative symmetries of string theory do not mix electric and magnetic charges. Stringy black holes treat one of the moduli on different footing than others. This is due to the fact that 11-dimensional supergravity has to be reduced to $d = 10$ first and this makes 11-th component of the metric or the dilaton, special. If however, we are looking for exact non-perturbative solution of 11-dimensional supergravity, we may expect some solutions where the radius of 11-th, 6-th and 5-th dimensions are all on equal footing. These are our STU black holes. They may be related to M, F, Y or whatever fundamental theory which is not the conventional theory of strings. To establish the relation between new (STU)-symmetric black holes and their dual $(S|TU)$ stringy partners is our main goal. In string triality picture [36] the role of S may be replaced by T or U but still there is one moduli different from the others and only one duality symmetry is non-perturbative whereas the other two are perturbative. We will find that all three S and T and U duality symmetries in "democratic" black hole solutions are non-perturbative. This is not too surprising: black holes are non-perturbative objects!

We will find that the area of the horizon in (STU)-symmetric theory equals the area of the horizon of the $(S|TU)$ dual theory

$$A^{(STU)}(p,q) = A^{(S|TU)}(\hat{p},\hat{q}) \ , \tag{3.1}$$

where the charges are related by particular $Sp(8,\mathbf{Z})$ duality transformation. This transformation has been found in [24] and relates the symplectic sections and charges in two theories.

The duality symmetry of this area formula is of an unusual form. The typical situation studied before was that the area as a function of charges was invariant under duality transformation.

$$A(p,q) = A(\hat{p},\hat{q}) \ . \tag{3.2}$$

For example, U-duality [4] invariant area formula is given by the quartic Cartan invariant of E_7 in d=4, [37], [6], [38], where the 2×28 unhatted (p,q) are charges before duality transformation and 2×28 hatted $(\hat{p},\hat{q})$ are charges after E_7 transformation. This duality transformation was a property of one specific theory: in this case, for example, N=8 supergravity in d=4. The equations of motion of this theory have hidden symmetry and it manifests itself in E_7 invariance of the area formula of the black holes of this theory with 1/8 of supersymmetry unbroken.

$$A^{N=8}(p,q) = A^{N=8}(\hat{p},\hat{q}) = 4\pi\sqrt{J(p,q)} = 4\pi\sqrt{J(\hat{p},\hat{q})} \ . \tag{3.3}$$

The new phenomenon which we observe here by studying the black holes in the framework of special geometry is the following. Black holes in two versions of the theory related by symplectic transformation have two different area formulae, when the area of the original version is expressed in terms of charges of original theory and the area of the transformed (dual) theory is expressed as a function of charges of dual theory. However, these two area formulae are related as in eq. (3.1). If one has the area in one theory and the transformation which defines the dual theory is known, the area can be found using (3.1). The reason for the area formulae to be different is that they carry different symmetries: $\left(\frac{SU(1,1)}{U(1)}\right)^3$ in one case and $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,2)}{SO(2)\times SO(2)}$ in the other case.

This chapter is organized as follows. In Sec. 2 we discuss the basic equations [6] defining the double-extreme black holes of N=2 theory [7] and the values of moduli as functions of charges. We refer to these equations as "stabilization equations".

[4]U-duality in the context of E_7-symmetry should not be confused with U-duality in the context of $SL(2,\mathbf{Z})$ symmetry related to U-moduli. Unfortunately, these two different dualities carry the same name in the current literature.

The main property of these equations relevant to present investigation is that they are symplectic covariant. Therefore once the solution for moduli in terms of charges is known in one version of the theory, the dual solution can be found by applying the symplectic transformation to the known solution. We explain this for the case of ST[2,n] manifold, $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,n)}{SO(2)\times SO(n)}$ symmetric theory which does not admit a prepotential and the dual version of it which admits the prepotential. We also explain that in both theories one has an option of solving for double-extreme black holes directly in each version, without using the information on the solution in the dual theory. Having obtained these two sets of double-extreme black holes one can check that the solutions are actually connected by symplectic transformation. Or one could use the solution available on one side and perform a relevant symplectic transformation to get the double-extreme black holes of the dual theory and the mass-area formula in terms of dual charges. One can verify that the transformed solution indeed solves the equations of the dual theory. In Sec. 3 we proceed with solving "stabilization equations" on the prepotential side and we consider the prepotential $F = \frac{X^1 X^2 X^3}{X^0}$. We derive the new mass-area formula for this theory. In Sec. 4 we show that alternative derivation of STU symmetric double-extreme black holes is possible: via symplectic transformation from the dual version of the theory without the prepotential, i.e. from the theory where S is not symmetric with T, U. In Sec. 5 these two sets of double-extreme black holes are studied from the perspective of string triality and the difference between the new and stringy black holes solutions is explained. In the Outlook we point out the implication of our new d=4 area formulae for Calabi-Yau moduli space and the corresponding d=5 area formulae. We also comment on string loop corrections and their possible effect on supersymmetric black holes and vice versa.

3.2 Symplectic covariance of "stabilization equations"

Stabilization equations for n_v complex moduli of supersymmetric black holes in N=2 theory near the horizon have the following form [6]

$$\begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} = \mathrm{Re} \begin{pmatrix} 2i\bar{Z}L^\Lambda \\ 2i\bar{Z}M_\Lambda \end{pmatrix}, \tag{3.4}$$

where the central charge [24]

$$Z(z,\bar{z},q,p) = e^{\frac{K(z,\bar{z})}{2}}(X^\Lambda(z)q_\Lambda - F_\Lambda(z)\,p^\Lambda) = (L^\Lambda q_\Lambda - M_\Lambda p^\Lambda) \tag{3.5}$$

depends on moduli and on $2n_v+2$ conserved charges (p^Λ, q_Λ). (L^Λ, M_Λ) are covariantly holomorphic sections depending on moduli. For double-extreme black holes [7] with frozen moduli these equations implicitly define the frozen moduli as functions of charges.

Symplectic transformation acts on charges as well as on sections

$$\begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \qquad \begin{pmatrix} \hat{X} \\ \hat{F} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} X \\ F \end{pmatrix}, \tag{3.6}$$

and provides the relation between the dual versions of the theory. Here

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n_v+2, \mathbf{Z}) \ . \tag{3.7}$$

Stabilization equations are covariant under symplectic transformations

$$\begin{pmatrix} \hat{p}^\Lambda \\ \hat{q}_\Lambda \end{pmatrix} = \mathrm{Re} \begin{pmatrix} 2i\bar{Z}\hat{L}^\Lambda \\ 2i\bar{Z}\hat{M}_\Lambda \end{pmatrix} , \tag{3.8}$$

since the central charge is invariant.

$$Z = (L^\Lambda q_\Lambda - M_\Lambda p^\Lambda) = (\hat{L}^\Lambda \hat{q}_\Lambda - \hat{M}_\Lambda \hat{p}^\Lambda) \ . \tag{3.9}$$

and the sections are covariant.

We are interested in dual relation between black holes of two theories. The first one can be defined in terms of a prepotential [5]

$$F = \frac{1}{2} d_{ABC} t^A t^B t^C = S\, \eta_{\alpha\beta}\, t^\alpha t^\beta \ , \qquad X^0 = 1 \ , \tag{3.10}$$

[5] We are using here notation of [25].

where

$$t^1 = S, \qquad d_{ABC} = \begin{pmatrix} d_{1\alpha\beta} = \eta_{\alpha\beta} \\ 0 \text{ otherwise} \end{pmatrix}, \qquad\qquad A, B, C = 1, 2, \ldots, n+1 \ , \quad (3.11)$$

and

$$\eta_{\alpha\beta} = \mathrm{diag}(+, -, - \ldots, -), \qquad \alpha, \beta = 2, \ldots, n+1. \quad (3.12)$$

This prepotential corresponds to the product manifold $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,n)}{SO(2)\times SO(n)}$. The $\frac{SU(1,1)}{U(1)}$ coordinate is the axion-dilaton field S. The remaining n complex moduli t^i are special coordinates of the $\frac{SO(2,n)}{SO(2)\times SO(n)}$ manifold. In particular, when $n = 2$ we have

$$F = \frac{1}{2} d_{ABC} t^A t^B t^C = \frac{1}{2} S \left[(t^2)^2 - (t^3)^2 \right] \ . \quad (3.13)$$

This theory is defined by 3 complex moduli and 4 gauge groups and the corresponding manifold is $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,2)}{SO(2)\times SO(2)}$.

If we introduce the notation

$$t^2 \equiv \frac{1}{\sqrt{2}}(T+U) \ , \qquad t^3 \equiv \frac{1}{\sqrt{2}}(T-U) \ , \quad (3.14)$$

the prepotential becomes

$$F = STU \ . \quad (3.15)$$

This theory has the symmetry of the manifold $\left(\frac{SU(1,1)}{U(1)}\right)^3$, which corresponds to the embedding of $\left(\frac{SU(1,1)}{U(1)}\right)^2$ into $\frac{SO(2,2)}{SO(2)\times SO(2)}$. The STU symmetric theory with the cubic holomorphic prepotential (3.15) is associated with particular Calabi-Yau moduli space. It is related to the dual version of the theory via symplectic transformation [24]. We will study this relation in the context of black holes in Sec. 4.

There are two possibilities to find double-extreme black holes in the theory with the prepotential. Either directly solve the stabilization equations or perform the dual rotation from the known solution. In the next section we will first find the black holes directly in STU-symmetric model.

3.3 Double-extreme black holes in STU-model

STU-model [39] is described by the prepotential:

$$F(X) = \frac{d_{ijk} X^i X^j X^k}{X^0} , \qquad i, j, k = 1, 2, 3, \tag{3.16}$$

For our case

$$F(X) = \frac{X^1 X^2 X^3}{X^0} . \tag{3.17}$$

Holomorphic section determined by that prepotential has a form: (X^Λ, F_Λ) with $F_\Lambda = \frac{\partial F}{\partial X^\Lambda}$ and $\Lambda = (0, i = 1, 2, 3)$. Special coordinates z^i are determined by

$$z^i = \frac{X^i}{X^0} , \qquad X^0 = 1 . \tag{3.18}$$

Corresponding Kähler potential is

$$K = -\log\left(- i d_{ijk} (z - \bar{z})^i (z - \bar{z})^j (z - \bar{z})^k \right) . \tag{3.19}$$

and we will use also

$$z^1 = S, \quad z^2 = T , \quad z^3 = U . \tag{3.20}$$

In terms of special coordinates the holomorphic sections are given by:

$$X^\Lambda = \begin{pmatrix} 1 \\ z^1 \\ z^2 \\ z^3 \end{pmatrix} , \qquad F_\Lambda = \begin{pmatrix} -z^1 z^2 z^3 \\ z^2 z^3 \\ z^1 z^3 \\ z^1 z^2 \end{pmatrix} . \tag{3.21}$$

The stabilization equations are:

$$p^\Lambda = i e^{K/2} (\bar{Z} X^\Lambda - Z \bar{X}^\Lambda) , \tag{3.22}$$

$$q_\Lambda = i e^{K/2} (\bar{Z} F_\Lambda - Z \bar{F}_\Lambda) . \tag{3.23}$$

We can eliminate $\bar{Z}$ from these equations so that:

$$X^\Lambda q_\Sigma - p^\Lambda F_\Sigma = i e^{K/2} Z (\bar{X}^\Lambda F_\Sigma - X^\Lambda \bar{F}_\Sigma) . \tag{3.24}$$

This is the matrix equation we used in [7] to solve for the solution of frozen moduli. In what follows we will solve for z^1 as a function of charges. The solution for z^2 and z^3 can be obtained in an analogous way as a result of symmetry between the three moduli.

Here are the components $((\Lambda, \Sigma) = (1,0),(0,1),(1,1),(2,3)$ and $(3,2)$ respectively) from the matrix equation (3.24) we need for the derivation of z^1:

$$\begin{aligned}
q_0 + p^1 z^2 z^3 &= ie^{K/2} Z(\bar{z}^1\bar{z}^2\bar{z}^3 - \bar{z}^1 z^2 z^3)\ , && (3.25)\\
q_1 - p^0 z^2 z^3 &= ie^{K/2} Z(z^2 z^3 - \bar{z}^2\bar{z}^3)\ , && (3.26)\\
q_1 z^1 - p^1 z^2 z^3 &= ie^{K/2} Z(\bar{z}^1 z^2 z^3 - z^1\bar{z}^2\bar{z}^3)\ , && (3.27)\\
q_3 z^2 - p^2 z^1 z^2 &= ie^{K/2} Z(\bar{z}^2 z^2 z^1 - z^2\bar{z}^2\bar{z}^1)\ , && (3.28)\\
q_2 z^3 - p^3 z^1 z^3 &= ie^{K/2} Z(\bar{z}^3 z^3 z^1 - z^3\bar{z}^3\bar{z}^1)\ . && (3.29)
\end{aligned}$$

Using (3.25) and (3.26) we can eliminate the factor $ie^{K/2}Z$ and obtain

$$z^2 z^3 = \frac{q_1\bar{z}^1 + q_0}{p^0\bar{z}^1 - p^1}\ . \qquad (3.30)$$

Using (3.27, 3.30) we can obtain a simple formula for $ie^{K/2}Z$

$$ie^{K/2}Z = \frac{(p^0 z^1 - p^1)}{(\bar{z}^1 - z^1)}\ . \qquad (3.31)$$

Substituting (3.31) into (3.28) and (3.29) respectively we can express z^2 and z^3 in terms of z^1 and the charges only:

$$\bar{z}^2 = \frac{(p^2 z^1 - q_3)}{(p^0 z^1 - p^1)}\ , \quad \text{and} \quad \bar{z}^3 = \frac{(p^3 z^1 - q_2)}{(p^0 z^1 - p^1)}\ . \qquad (3.32)$$

Finally, using (3.30) and the above equations to eliminate z^2 and z^3 we are getting a quadratic equation for z^1

$$(z^1)^2 + \frac{((p\cdot q) - 2p^1 q_1)}{(p^0 q_1 - p^2 p^3)} z^1 - \frac{(p^1 q_0 + q_1 q_2)}{(p^0 q_1 - p^2 p^3)} = 0\ , \qquad (3.33)$$

where

$$(p\cdot q) = (p^0 q_0) + (p^1 q_1) + (p^2 q_2) + (p^3 q_3) \equiv p^\Lambda q_\Lambda\ , \qquad (3.34)$$

and the solution for z^1 moduli is

$$z^1 = \frac{((p \cdot q) - 2p^1 q_1)}{2(p^3 p^2 - p^0 q_1)} \mp i \frac{\sqrt{W}}{2(p^3 p^2 - p^0 q_1)} , \tag{3.35}$$

where

$$\begin{aligned} W(p^\Lambda, q_\Lambda) &= -(p \cdot q)^2 - 4p^0 q_1 q_2 q_3 + 4q_0 p^1 p^2 p^3 \\ &+ 4((p^1 q_1)(p^2 q_2) + (p^1 q_1)(p^3 q_3) + (p^3 q_3)(p^2 q_2)). \end{aligned} \tag{3.36}$$

The function $W(p^\Lambda, q_\Lambda)$ is symmetric under transformations: $p^1 \leftrightarrow p^2 \leftrightarrow p^3$ and $q_1 \leftrightarrow q_2 \leftrightarrow q_3$. Finally solution for all three complex modulus are:

$$z^i = \frac{((p \cdot q) - 2p^i q_i) \mp i\sqrt{W}}{2(3d_{ijk} p^j p^k - p^0 q_i)} . \tag{3.37}$$

There is no summation over i in $p^i q_i$. For the solution to be consistent we have to require $W > 0$, otherwise the moduli are real and the Kähler potential is not defined

At this point the choice of signs in the imaginary part of the moduli is ambiguous. However, to preserve the obvious exchange symmetries, we want to choose common signs for all. In fact it turns out that only the "$-$" is consistent as we shall see.

With these expressions for the z^i the Kähler potential eq. (3.19) is easily computed. We find

$$e^{-K} = \pm \frac{W^{3/2}}{\omega_1 \omega_2 \omega_3} , \tag{3.38}$$

where

$$\omega_i = (3d_{ijk} p^j p^k - p^0 q_i) . \tag{3.39}$$

It is also useful to calculate the product of three ω_i's which appears to be positive:

$$\omega_1 \omega_2 \omega_3 = \frac{1}{4} \left((p^0)^2 W + [2p^1 p^2 p^3 - p^0 (p \cdot q)]^2 \right) > 0 \tag{3.40}$$

with $\Lambda = 0, 1, 2, 3$,

For the Kähler potential e^{-K} to be positive we have to pick up only one choice of sign for each imaginary part of the special coordinates in eq. (3.37), it has to be negative:

$$z^i = \frac{((p \cdot q) - 2p^i q_i) - i\sqrt{W}}{2(3d_{ijk} p^j p^k - p^0 q_i)} \qquad \Longrightarrow \qquad e^{-K} = \frac{W^{3/2}}{\omega_1 \omega_2 \omega_3} > 0 . \tag{3.41}$$

We can proceed now with the calculation of the central charge to find the black hole mass, which for double-extreme black holes is proportional to the area of the black hole horizon. We find that

$$e^K Z\bar{Z} = \frac{(p^0)^2 W + [2p^1p^2p^3 - p^0(p\cdot q)]^2}{4W} \,. \tag{3.42}$$

We deduce for the mass/area

$$Z\bar{Z} = M^2 = \frac{W^{3/2}}{\omega_1\omega_2\omega_3}\frac{(p^0)^2 W + [2p^1p^2p^3 - p^0(p\cdot q)]^2}{4W} \,, \tag{3.43}$$

which finally gives the beautiful result

$$Z\bar{Z} = M^2 = \frac{A}{4\pi} = \left(W(p^\Lambda, q_\Lambda)\right)^{1/2} \,. \tag{3.44}$$

This is a very nice and simple expression for the area which relies on the fact that the nominator of the second expression in (3.43) and the product of the ω_i cancel. Thus we have completely described the double-extreme black holes solutions with frozen moduli in the *STU* symmetric theory. The geometry is that of extreme Reissner-Nordström type with the mass/area formula, as function of quantized charges given in eq. (3.36).

$$ds^2 = \left((1 + \frac{[W(p,q)]^{1/4}}{r}\right)^{-2} dt^2 - \left((1 + \frac{[W(p,q)]^{1/4}}{r}\right)^{2} d\vec{x}^2 \,. \tag{3.45}$$

It is instructive to remind that our mass/area formula has also a nice symplectic invariant form, as explained in [6], [7].

$$M^2 = \tag{3.46}$$

$$= \frac{-1}{2}\left(p^\Lambda, q_\Lambda\right)\begin{pmatrix} (\mathrm{Im}\,\mathcal{N} + \mathrm{Re}\,\mathcal{N}\mathrm{Im}\,\mathcal{N}^{-1}\mathrm{Re}\,\mathcal{N})_{\Lambda\Sigma} & (-\mathrm{Re}\,\mathcal{N}\mathrm{Im}\,\mathcal{N}^{-1})_\Lambda{}^\Sigma \\ (-\mathrm{Im}\,\mathcal{N}^{-1}\mathrm{Re}\,\mathcal{N})^\Lambda_\Sigma & (\mathrm{Im}\,\mathcal{N}^{-1})^{\Lambda\Sigma} \end{pmatrix}_{\text{fix}}\begin{pmatrix} p^\Sigma \\ q_\Sigma \end{pmatrix}$$

$$= \frac{-1}{2}\left(p^\Lambda, q_\Lambda\right)\begin{pmatrix} (\mathrm{Im}\,F + \mathrm{Re}\,F\mathrm{Im}\,F^{-1}\mathrm{Re}\,F)_{\Lambda\Sigma} & (-\mathrm{Re}\,F\mathrm{Im}\,F^{-1})_\Lambda{}^\Sigma \\ (-\mathrm{Im}\,F^{-1}\mathrm{Re}\,F)^\Lambda_\Sigma & (\mathrm{Im}\,F^{-1})^{\Lambda\Sigma} \end{pmatrix}_{\text{fix}}\begin{pmatrix} p^\Sigma \\ q_\Sigma \end{pmatrix},$$

where the period matrix $\mathcal{N}$ as well as the second derivative of the prepotential F are functions of moduli which at the fixed point near the black hole horizon become functions of charges, as defined in eq. (3.41).

If we parametrize all 3 moduli in terms of axion-dilaton fields

$$z^i = a^i - ie^{-\eta_i} \tag{3.47}$$

where

$$a^i = \frac{((p \cdot q) - 2p^i q_i)}{2\omega_i} , \qquad e^{-\eta_i} = \frac{\sqrt{W}}{2\omega_i} , \tag{3.48}$$

Káhler potential is

$$e^{-K} = -8\, \mathrm{Im}S\ \mathrm{Im}T\ \mathrm{Im}U = 8e^{-\eta_1}e^{-\eta_2}e^{-\eta_3} . \tag{3.49}$$

This parametrization is possible under the condition that all three combination of charges are positive,

$$\omega_i > 0 . \tag{3.50}$$

3.4 Dual rotation of double-extreme black holes

The double-extreme black holes for this model without the prepotential for the general case of arbitrary n as well as for $n = 2$ have been found before [7]. The resume of this black hole for $n = 2$ is the following. Solution is defined in terms of 4 magnetic and 4 electric and charges $(\hat{p}^\Lambda, \hat{q}_\Lambda)$ and $\Lambda = 0, 1, 2, 3$. The frozen moduli are given by[6]

$$S = \frac{\hat{p} \cdot \hat{q} - i\, (\hat{p}^2 \hat{q}^2 - (\hat{p} \cdot \hat{q})^2)^{1/2}}{\hat{p}^2} , \tag{3.51}$$

$$T = \frac{\hat{X}^3 - \hat{X}^1}{\hat{X}^0 - \hat{X}^2} = \frac{\bar{S}(\hat{p}^3 - \hat{p}^1) - (\hat{q}^3 - \hat{q}^1)}{\bar{S}(\hat{p}^0 - \hat{p}^2) - (\hat{q}^0 - \hat{q}^2)} , \tag{3.52}$$

$$U = \frac{-\hat{X}^3 - \hat{X}^1}{\hat{X}^0 - \hat{X}^2} = \frac{\bar{S}(-\hat{p}^3 - \hat{p}^1) - (-\hat{q}^3 - \hat{q}^1)}{\bar{S}(\hat{p}^0 - \hat{p}^2) - (\hat{q}^0 - \hat{q}^2)} , \tag{3.53}$$

[6] We are choosing the negative sign for the imaginary part of S here for the sake of the dual rotation to the prepotential version, using the symplectic matrix (3.56).

and the mass/area formula is

$$Z\bar{Z} = M^2 = \frac{A}{4\pi} = \left(\hat{p}^2\hat{q}^2 - (\hat{p}\cdot\hat{q})^2\right)^{1/2} . \tag{3.54}$$

The symplectic transformation between the theory without the prepotential ("hatted" version) to the one with the prepotential ("unhatted" version) is[24]:

$$Sp(8,\mathbf{Z}) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \tag{3.55}$$

with

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \qquad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.56}$$

Starting with the prepotential $F = STU$ in terms of special coordinates we have the holomorphic section:

$$X^\Lambda = \begin{pmatrix} 1 \\ S \\ T \\ U \end{pmatrix}, \qquad F_\Lambda = \begin{pmatrix} -STU \\ TU \\ SU \\ ST \end{pmatrix}. \tag{3.57}$$

After symplectic transformation defined in (3.6), (3.56) we get for hatted sections:

$$\hat{X}^\Lambda = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 - TU \\ -(T+U) \\ -(1+TU) \\ T - U \end{pmatrix}, \qquad \hat{F}_\Lambda = \begin{pmatrix} S\hat{X}^0 \\ S\hat{X}^1 \\ -S\hat{X}^2 \\ -S\hat{X}^3 \end{pmatrix} = S\eta_{\Lambda\Sigma}\hat{X}^\Sigma , \tag{3.58}$$

where the metric is $\eta_{\Lambda\Sigma} = (++--)$. This theory does not admit the prepotential [24].

We can now relate the known results of the version without a prepotential (for which we use variables with a hat) to the ones obtained here. From eqs. (3.6), (3.56) we find the transformation between $\hat{p}, \hat{q}$ and p, q to be:

$$\begin{bmatrix} \hat{p}^\Lambda \\ \\ \hat{q}_\Lambda \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} p^0 - q_1 \\ -p^2 - p^3 \\ -p^0 - q_1 \\ p^2 - p^3 \\ p^1 + q_0 \\ -q_2 - q_3 \\ p^1 - q_0 \\ q_2 - q_3 \end{bmatrix} \tag{3.59}$$

This transformation gives us the relations:

$$\hat{p}^2 = 2(p^2 p^3 - p^0 q_1) = 2\omega_1 \ , \tag{3.60}$$

$$\hat{p} \cdot \hat{q} = p \cdot q - 2p^1 q_1 \ , \tag{3.61}$$

hence we find $S = z_1$ where S is the first moduli field of the version without the prepotential, and (with a little more work) we have

$$\hat{p}^2 \hat{q}^2 - (\hat{p} \cdot \hat{q})^2 = W \ . \tag{3.62}$$

For T and U we get, using the relation between charges (3.59)

$$T = \frac{\bar{S}(\hat{p}^3 - \hat{p}^1) - (\hat{q}^3 - \hat{q}^1)}{\bar{S}(\hat{p}^0 - \hat{p}^2) - (\hat{q}^0 - \hat{q}^2)} = \frac{\bar{S}p^2 - q_3}{\bar{S}p^0 - p^1} = z_2 \ , \tag{3.63}$$

$$U = \frac{\bar{S}(-\hat{p}^3 - \hat{p}^1) - (-\hat{q}^3 - \hat{q}^1)}{\bar{S}(\hat{p}^0 - \hat{p}^2) - (\hat{q}^0 - \hat{q}^2)} = \frac{\bar{S}p^3 - q_2}{\bar{S}p^0 - p^1} = z_3 \ . \tag{3.64}$$

3.5 String triality and STU black holes

Our results allow for a comparison with the string triality picture as described in [36]. There, a six-dimensional string, described by the low energy action

$$I_6 = \frac{1}{2\kappa^2} \int d^6x \sqrt{-G} e^{-\Phi} \Big[R_G + G^{MN} \partial_M \Phi \partial_N \Phi \\ - \frac{1}{12} G^{MQ} G^{NR} G^{PS} H_{MNP} H_{QRS} \Big] \tag{3.65}$$

with $M, N = 0, .., 5$ was considered. This string might be a truncated version of a heterotic or a Type II string. Upon toroidal compactification to $D = 4$ one obtains a $N = 2$ supergravity theory coupled to three vector multiplets. The four-dimensional metric is related to the six-dimensional one by

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} + A_\mu^m A_\nu^n G_{mn} & A_\mu^m G_{mn} \\ A_\nu^n G_{mn} & G_{mn} \end{pmatrix} , \tag{3.66}$$

where the space-time indices are $\mu, \nu = 0, 1, 2, 3$ and the internal indices are $m, n = 1, 2$. Two more vectors arise from the reduction of the B field.

One also finds six scalars, four of which are moduli of the 2-torus. We parametrize the internal metric and 2-form as

$$G_{mn} = e^{\eta_3 - \eta_2} \begin{pmatrix} e^{-2\eta_3} + a_3^2 & -a_3 \\ -a_3 & 1 \end{pmatrix}, \tag{3.67}$$

and

$$B_{mn} = a_2 \, \epsilon_{mn} \ . \tag{3.68}$$

η_1, the four-dimensional dilaton, is given by

$$e^{-\eta_1} = e^{-\Phi} \sqrt{\det G_{mn}} = e^{-(\Phi + \eta_2)} \ . \tag{3.69}$$

The sixth scalar is the axion a_1 which arises from dualization of the three-form field strength in four dimensions.

The scalars are typically combined into three complex scalars, which in notation suitable for our previous sections are:

$$\begin{aligned} z_1 = S &= a_1 - ie^{-\eta_1} , \\ z_2 = T &= a_2 - ie^{-\eta_2} , \\ z_3 = U &= a_3 - ie^{-\eta_3} . \end{aligned} \tag{3.70}$$

Here S is a dilaton-axion of the heterotic string, T and U are the Kähler form and the complex structure of the torus. These three scalars are obviously the ones considered so far in this paper. The four vectors are combined to a vector A^a_μ with $a = 1, 2, 3, 4$. Details can be found, e.g. , in [36]. In fact, the electric and magnetic charges can be put together to an $SP(8)$ vector as given in the earlier chapters.

The symmetry of this theory is $SL(2,Z) \times \frac{O(2,2,Z)}{O(2)\times O(2)}$. The $SL(2,Z)$ component is the famous S-duality, a conjectured non-perturbative symmetry of string theory. The second factor, which is just a product of two $SL(2,Z)$ plus their exchange, is related to perturbative T-duality symmetry. In the following, T-duality will denote the duality symmetry generated by the first $SL(2,Z)$, which acts on the Kähler form, whereas the second one is called U-duality and acts on the complex structure U. All three symmetries act on the scalars by

$$z_i \to \frac{a_i z_i + b_i}{c_i z_i + d_i} \tag{3.71}$$

with $a_i d_i - b_i c_i = 1$. The electric and magnetic charges transform as vectors under the three duality symmetries (where the $SP(8)$ vector has to be converted into an $SL(2)^3$ vector [36]).

This theory is precisely the one studied in [7]. In [36] it was found that the theory allows two (or five, according to taste) dual descriptions where the roles of S, T and U get interchanged. For example, S is the dilaton/axion field for the heterotic string, the Kähler form for the Type IIA string and the complex structure of the Type IIB string. However, all those theories were of the same type, in the sense that (at least in the truncated versions considered here) two symmetries were perturbative and one was non-perturbative.

This is easily seen by considering (for example) the four-dimensional heterotic Lagrangian (in the absence of axionic fields):

$$\begin{aligned} \mathcal{L} &= \sqrt{-g}\Big[R - \frac{1}{2}\sum(\partial\eta_i)^2 \\ &- \frac{1}{4}\left(e^{-\eta_1-\eta_2-\eta_3}F^1F^1 + e^{-\eta_1-\eta_2+\eta_3}F^2F^2 + e^{-\eta_1+\eta_2+\eta_3}F^3F^3 + e^{-\eta_1+\eta_2-\eta_3}F^4F^4\right)\Big]. \end{aligned} \tag{3.72}$$

Clearly, $\eta_2 \to -\eta_2$ and $\eta_3 \to -\eta_3$ (accompanied by an exchange of a few field strengths) are off-shell symmetries, where as $\eta_1 \to -\eta_1$ requires dualizations of field strengths.

How does the STU-model considered in this paper tie in with those three (or six) string theories? Obviously, it cannot correspond to either of those, since it treats S, T and U on equal footing. This is already clear from the prepotential, but also the action gives some insights. It can be obtained from

$$\mathcal{L}_V = \mathrm{Im}\mathcal{N}_{\Lambda\Sigma}\ \mathcal{F}^\Lambda \mathcal{F}^\Sigma + \mathrm{Re}\mathcal{N}_{\Lambda\Sigma}\ \mathcal{F}^\Lambda\ {}^*\mathcal{F}^\Sigma. \tag{3.73}$$

We find

$$F_{\Lambda\Sigma} = \begin{pmatrix} 2d_{ijk}z^i z^j z^k & -3d_{mij}z^i z^j \\ -3d_{lij}z^i z^j & 6d_{mli}z^i \end{pmatrix}, \tag{3.74}$$

from which one can deduce

$$\mathcal{N}_{\Lambda\Sigma} = \bar{F}_{\Lambda\Sigma} + 2i\frac{(\mathrm{Im}F_{\Lambda\Omega})(\mathrm{Im}F_{\Pi\Sigma})X^\Omega X^\Pi}{(\mathrm{Im}F_{\Omega\Pi})X^\Omega X^\Pi}, \tag{3.75}$$

where $X^\Lambda = (1, z^1, z^2, z^3)$. We do not try to express $\mathcal{N}$ in full generality, however we note that the lower three by three matrix $\mathcal{N}_{ij}$ has the extremely simple form

$$\mathcal{N}_{ij} = \begin{pmatrix} -ie^{+\eta_1-\eta_2-\eta_3} & a_3 & a_2 \\ a_3 & -ie^{-\eta_1+\eta_2-\eta_3} & a_1 \\ a_2 & a_1 & -ie^{-\eta_1-\eta_2+\eta_3} \end{pmatrix}. \tag{3.76}$$

The vector part of the Lagrangian in the absence of any axion-like fields is then given by

$$\begin{aligned} \mathcal{L}_V = -\Big(& e^{-\eta_1-\eta_2-\eta_3}\mathcal{F}^0\mathcal{F}^0 + e^{+\eta_1-\eta_2-\eta_3}\mathcal{F}^1\mathcal{F}^1 \\ &+ e^{-\eta_1+\eta_2-\eta_3}\mathcal{F}^2\mathcal{F}^2 + e^{-\eta_1-\eta_2+\eta_3}\mathcal{F}^3\mathcal{F}^3\Big). \end{aligned} \tag{3.77}$$

Note that this Lagrangian has perfect exchange ($1 \leftrightarrow 2 \leftrightarrow 3$) symmetry, but it is not invariant under any $\eta_i \to -\eta_i$ duality transformation (accompanied by the appropriate exchange of vector fields). Hence, the theory has neither S, T nor U duality (in the notation of [36]) realized off-shell!

We can also compare how S, T and U dualities (3.71) are realized as $SP(8)$ matrices. For this we go first in a basis $(S|TU)$ (heterotic string compactified on a two-torus). This can be done by the symplectic rotation

$$\mathcal{C}: \qquad (p^\Lambda, q_\Lambda) \to (\tilde{p}^\Lambda, \tilde{q}_\Lambda) = (p^0, -q_1, p^2, p^3 \,|\, q_0, p^1, q_2, q_3) \,. \tag{3.78}$$

In this new basis the $SL(2,Z)$ transformations are realized by a matrix (3.55) with [40]

$$S \to \frac{aS+b}{cS+d} \;: \quad \text{by} \quad a\,A = d\,D = \; a\,d\,\delta_{ij} \;; \quad b\,B = c\,C = b\,c \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{3.79}$$

and

$$T \to \frac{aT+b}{cT+d} \;: \quad \text{by} \quad B = C = 0 \,, \quad A = (D^T)^{-1} = \begin{pmatrix} d & 0 & c & 0 \\ 0 & a & 0 & -b \\ b & 0 & a & 0 \\ 0 & -c & 0 & d \end{pmatrix}, \tag{3.80}$$

$$U \to \frac{aU+b}{cU+d} : \quad \text{by} \quad B = C = 0 \,, \quad A = (D^T)^{-1} = \begin{pmatrix} d & 0 & 0 & c \\ 0 & a & -b & 0 \\ 0 & -c & d & 0 \\ b & 0 & 0 & a \end{pmatrix}. \tag{3.81}$$

As we can see, in this basis only the S-duality is non-perturbative (exchange of electric with magnetic charges) whereas the T-and U-duality acts diagonal, i.e. exchange electric with electric and magnetic with magnetic quantum numbers. Finally, to get the transformation for our original charges we have to invert the transformation $\mathcal{C}$. Combining all symplectic transformations we find for our charges in the STU basis the transformations

$$
\begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} \to \begin{pmatrix} d\,p^0 + c\,p^1 \\ b\,p^0 + a\,p^1 \\ d\,p^2 + c\,q_3 \\ d\,p^3 + c\,q_2 \\ a\,q_0 - b\,q_1 \\ -c\,q_0 + d\,q_1 \\ b\,p^3 + a\,q_2 \\ b\,p^2 + a\,q_3 \end{pmatrix}_{SL(2)_S} ; \quad \begin{pmatrix} d\,p^0 + c\,p^2 \\ d\,p^1 + c\,q_3 \\ b\,p^0 + a\,p^2 \\ d\,p^3 + c\,q_1 \\ a\,q_0 - b\,q_2 \\ b\,p^3 + a\,q_1 \\ -c\,q_0 + d\,q_2 \\ b\,p^1 + a\,q_3 \end{pmatrix}_{SL(2)_T} ; \quad \begin{pmatrix} d\,p^0 + c\,p^3 \\ d\,p^1 + c\,q_2 \\ d\,p^2 + c\,q_1 \\ b\,p^0 + a\,p^3 \\ a\,q_0 - b\,q_3 \\ b\,p^2 + a\,q_1 \\ b\,q_1 + a\,q_2 \\ -c\,q_0 + d\,q_3 \end{pmatrix}_{SL(2)_U} . \tag{3.82}
$$

As expected, one finds that W as given in eq. (3.36) or the area of the horizon (mass) is invariant under these transformations (or $(SL(2))^3$ symmetric).

Let us turn back to the Lagrangians (3.72) and (3.77). Both lagrangians are on-shell equivalent, (3.72) corresponds to the $(S|TU)$ basis whereas (3.77) is our (STU) basis and the transformation $\mathcal{C}$ maps both. This transformation is a dualization of $\mathcal{F}_1$ and a renaming $(\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \to 1/2(F_1, F_3, F_4, F_2)$. Note, that under dualization the pre-factor of $\mathcal{F}_1$ gets inverted. This dualization makes all duality symmetries in the (STU) basis non-perturbative. The statement that in this symmetric basis none of the dualities is perturbative is equivalent to the statement that there is no basis in which all of dualities are perturbative. The most we can achieve is to make only one non-perturbative and two perturbative.

We have considered the case where the S-transformation is non-perturbative. On equal footing we could take T or U. These three possibilities fix then the three underlying string theories (heterotic, type IIA or type IIB). When all three theories are symmetrised by going to the STU basis, immediately all dualities become non-perturbative.

These transformations can be nicely visualized in the form of a cube, as it was done in [36]. Figure 1 shows how the dualities transform the field strengths into each other and their duals in the (STU) model. Figure 1 and Figure 2 also illustrate the

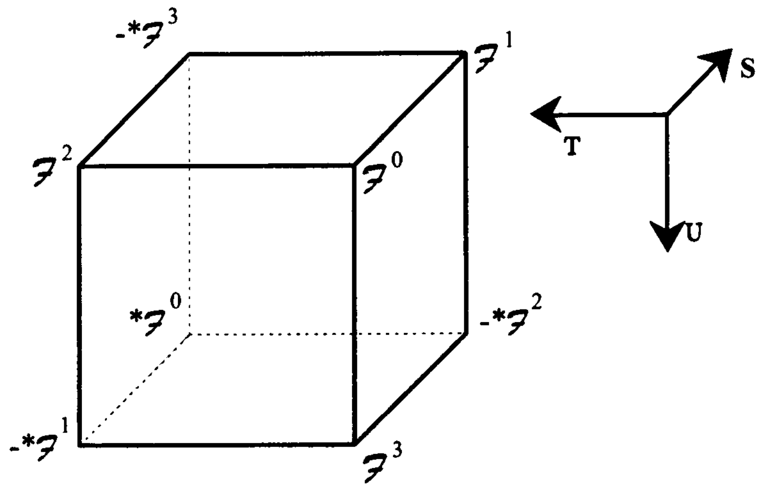


Figure 1: Duality transformations in the STU-model. The fundamental field strengths are not located on one side.

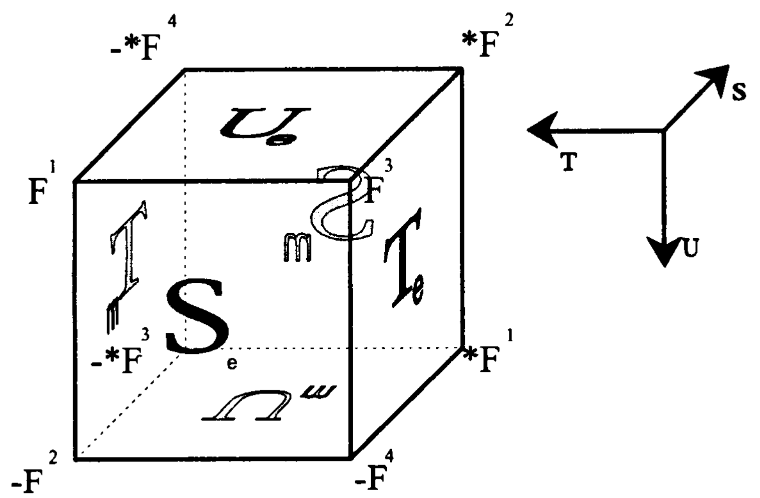


Figure 2: Duality transformations in the S,T and U strings. The fundamental field strengths are located on one side and two duality symmetries are perturbative. The field strengths have different indices from Figure one, because the ones here are fundamental S-string fields.

crucial difference with the $(S|TU)$-, $(T|US)$- and $(U|ST)$-models of [36] (Figure 2). In the STU theory, the fundamental field strengths (or electric charges) are located around $\mathcal{F}^0$, whereas in the $(S|TU)$, $(T|US)$ and $(U|ST)$ models the fundamental fields were located on one side of the cube, allowing 2 dualities to be perturbative.

The black holes with vanishing axions and finite scalars all have four charges. These charges must be located on four corners of the cube which are NOT connected by edges. Hence, the choices one has are q_0, p^1, p^2, p^3 (with product of 4 charges $q_0p^1p^2p^3$ positive) and p^0, q_1, q_2, q_3 (with product of 4 charges $p^0q_1q_2q_3$ negative), which is consistent with our results. In the S-, T-, or U- string picture one always needed two electric and two magnetic charges.

3.6 Outlook

In conclusion, we have found a new type of d=4 supersymmetric black holes in the context of N=2 special geometry related to Calabi-Yau threefold. The main difference with the existing supersymmetric black holes is a completely *democratic treatment of all moduli of the theory.* This is due to our use of the version of special geometry with the prepotential $F = d_{ABC}\frac{X^AX^BX^C}{X^0}$ [29] where d_{ABC} are real symmetric constant tensors. A particular model of this type with $F = STU$ gives no preference to any of the moduli and therefore none of them can play a role of coupling constants. This makes the new (STU) black holes different from stringy $(S|TU)$-, $(T|US)$- and $(U|ST)$ black holes [7], [36] where one of the moduli (S in heterotic case, T in type IIa case and U in type IIb case) does play the role of the coupling constant.

One may try to relate our new d=4 black holes to d=5 supersymmetric black holes described in [6]. The area formula found there depends on symmetric tensor d_{ABC} as follows.

$$Z_{\text{fix}} = \sqrt{(d^{AB}(q))^{-1} q_A q_B}\ , \qquad A \sim \left[\left(d^{AB}(q)\right)^{-1} q_A q_B\right]^{3/4}\ , \tag{3.83}$$

where $\left(d^{AB}(q)\right)^{-1} = \left(d^{AB}(t(z))|_{\partial_i Z=0}\right)^{-1}$ and $(d^{AB})^{-1}$ is the inverse of $d_{ABC}t^C$. Equation (3.83) applies in particular to eleven dimensional supergravity compactified on

Calabi-Yau threefold.

The new result for four-dimensional black holes is the value of the moduli t^A as the function of charges at the fixed point $\partial_i Z = 0$ for particular example of d_{ABC}. It can be used to find also the area of the five-dimensional black holes as the function of charges for this theory.

It remains to be seen if it is possible to address the issue of quantum corrections in string theory using extreme black holes of classical moduli spaces as the starting point. Following paper [8] we have established a duality relation between stringy $(S|TU)$-, $(T|US)$- and $(U|ST)$ black holes and "democratic" STU black holes. Stringy black holes were known to be related to each other by the so-called triality in such a way that only one of S, T, U-dualities was non-perturbative [36]. The "democratic" black holes give us some new insights into the spectrum of states of the fundamental theory: all dualities there are non-perturbative.

4 Calabi-Yau Blackholes

4.1 Introduction

Recently supersymmetric black holes become a "natural" theoretical laboratory for string theory. Calculation of black hole entropy by counting microscopic configurations [12, 13, 41, 42] and comparison of these results with one obtained by macroscopic calculations [6, 5, 43] is a very powerful tool in this "laboratory". In this chapter, following [9], we will focus on macroscopic calculations of the black hole entropy as the area of the black hole horizon [2],[3]. We will give a solution for N=2 extreme black holes associated with the general Calabi-Yau moduli space.

One of the most important properties of N=2 supersymmetric black holes in the general version of ungauged N=2 supergravity interacting with arbitrary number n_v of vector multiplets [22]- [24],[28],[44] has been established recently [5]. It was shown that the area of the extreme black hole horizon and moduli of vector multiplets near the horizon are functions of charges only.

$$A = A(z^i(p,q,d_{ijk}),p,q,d_{ijk}), \tag{4.1}$$

where A is an area of the black hole horizon, $z^i(p,q,d_{ijk})$ are moduli fields, p and q are electrical and magnetic charges of black hole and d_{ijk} is a characteristic of Calabi-Yau (classical intersection numbers). Relation between charges and moduli for supersymmetric black holes near the horizon has the following form [6]:

$$\begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} = \mathrm{Re}\begin{pmatrix} 2i\bar{Z}L^\Lambda \\ 2i\bar{Z}M_\Lambda \end{pmatrix}, \tag{4.2}$$

where Z is the central charge and (L^Λ, M_Λ) are covariantly holomorphic sections.

Classical Calabi-Yau moduli space is described by the prepotential of the form:

$$F = d_{ABC}\frac{X^A X^B X^C}{X^0} + \quad \text{corrections.} \tag{4.3}$$

Here d_{ABC} are the topological intersection numbers $d_{ABC} \equiv \int J_A \wedge J_B \wedge J_C$ and J_A Kähler cone generators, $J_A \in H^{1,1}(Y, \mathbf{Z})$. Special coordinates z^A of Kähler "moduli space" are connected with X^Λ by:

$$z^A = \frac{X^A}{X^0}; \qquad X^0 = 1; \quad A = 1, 2..., \quad \Lambda = 0, 1.... \tag{4.4}$$

According to [45, 46, 47] this form of the Calabi-Yau prepotential can be extended by addition of an extra topological term determined by second Chern class c_2 to the prepotential:

$$F_1(z) = \sum_A^h \frac{c_2 \cdot J_A}{24} z^A. \tag{4.5}$$

Symplectic invariant form of the Kähler potential is $K = -\ln i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda)$, where

$$L^\Lambda = e^{K/2} X^\Lambda, \quad M_\Lambda = e^{K/2} F_\Lambda. \tag{4.6}$$

In terms of special coordinates the Kähler potential is:

$$K(z, \bar{z}) = -\log(-i d_{ABC}(z - \bar{z})^A (z - \bar{z})^B (z - \bar{z})^C). \tag{4.7}$$

The central charge Z has a form [22]- [24],[28],[44]:

$$Z = e^{\frac{K(z,\bar{z})}{2}} (X^\Lambda q_\Lambda - F_\Lambda p^\Lambda). \tag{4.8}$$

In [47] it was proposed to introduce new variables:

$$Y^\Lambda \equiv \bar{Z} X^\Lambda. \tag{4.9}$$

Stabilization equations (4.2) in terms of these new variables become:

$$\begin{aligned} ip^\Lambda &= Y^\Lambda - \bar{Y}^\Lambda \\ iq_\Lambda &= F_\Lambda(Y) - \bar{F}_\Lambda(\bar{Y}). \end{aligned} \tag{4.10}$$

Mass of the double-extreme black hole and entropy then has a form:

$$\frac{S}{\pi} = M_{BPS}^2 = |Z|^2 = |Y^0|^2 exp(-K(z, \bar{z})). \tag{4.11}$$

In this chapter we will show the solution of equations (4.10) for most general form of d_{ABC} and arbitrary p and q charges. In Section 2 we will derive the expression for black hole entropy in terms of d_{ABC} and some combinations of p and q. In Section 3 solutions for moduli fields will be given and in Section 4 some examples of explicit solutions for entropy for particular Calabi-Yau intersection numbers will be considered.

4.2 Entropy of Calabi-Yau black holes.

Stabilization equations (4.10) and the entropy formula (4.11) lead to the general result for entropy of Calabi-Yau black hole with prepotential (4.3). For the most general case with $p^0 \neq 0, q_0 \neq 0$ and arbitrary charges (p^A, q_A) the entropy of the black hole depends only on these charges and numbers d_{ABC}:

$$\frac{S}{\pi} = \frac{1}{3p^0}\sqrt{\frac{4}{3}(\Delta_A \tilde{x}^A)^2 - 9\big(p^0(p\cdot q) - 2D\big)^2}\,. \tag{4.12}$$

Here $\Delta_A = 3D_A - p^0 q_A$, and $D_A = d_{ABC}p^B p^C$, $D = d_{ABC}p^A p^B p^C$, $p\cdot q = p^0 q_0 + p^A q_A$. In combination

$$(\Delta_A \tilde{x}^A)^2 = \Big((3D_A - p^0 q_A)\tilde{x}^A\Big)^2 \tag{4.13}$$

the variables $\tilde{x}^A$ are the *real* solutions of algebraic system:

$$d_{ABI}\tilde{x}^A \tilde{x}^B = \Delta_I. \tag{4.14}$$

which is determined by intersection numbers of the Calabi-Yau manifold and black hole charges. This system might have no analytical solution in general case. The number of equations in (4.14) is equal to the number of moduli fields n_v. Left-hand parts of each equations are quadratic forms with coefficients $d_{AB(I)}$ and right-hand parts are arbitrary and depend on values of electric and magnetic charges of the black hole. In other words, we have n_v second order $(n_v - 1)$-dimensional surfaces embedded in n_v-dimensional space and intersection of these surfaces is a solution of this system.

It is not clear yet if there is any connection between the geometry of Calabi-Yau manifolds and the geometry of this picture. We considered a few examples of Calabi-Yau manifolds with different sets of d_{ABC}. For each of them we found the expression for the black hole entropy. These examples will be considered in part 4 of this paper.

Solution of equations (4.10) becomes much simpler if $p^0 = 0$. In this case we have the system of linear (instead of quadratic) equations and there is a general solution for moduli fields in terms of black hole charges. The entropy of the black hole is then given by the formula:

$$\frac{S}{\pi} = \sqrt{\frac{D}{3}(q_B D^B + 12 q_0)} \ . \tag{4.15}$$

Here $D_{AB} = d_{ABC} p^C$, $D^{AB} = [D_{AB}]^{-1}$, and $D^A = D^{AB} q_B$.

It is straightforward to generalize these results to the prepotential with additional topological term $F_1 = \frac{c_2 J_A}{24} z^A$.

$$F(X) = d_{ABC} z^A z^B z^C + \sum_A^h \frac{c_2 \cdot J_A}{24} z^A \tag{4.16}$$

In [47] it was shown that the addition of this topological term leads to simple transformation of charges q_Σ:

$$\tilde{q}_\Sigma = q_\Sigma - W_{\Sigma\Lambda} p^\Sigma$$

$$W_{0A} = \frac{c_2 J_A}{24}. \tag{4.17}$$

Therefore in our expressions for entropy (4.12) and (4.15)we should use these new $\tilde{q}_\Sigma$ connected with old charges by the equations:

$$\tilde{q}_0 = q_0 - \frac{c_2 J_A}{24} p^A$$

$$\tilde{q}_A = q_A - \frac{c_2 J_A}{24} p^0. \tag{4.18}$$

4.3 Fixed Moduli solutions

Stabilization equations (4.10) define values of the moduli fields near the black hole horizon through the electric and magnetic charges of the black hole. From "p^Λ" equations (4.10) it follows immediately that $ImY^\Lambda = \frac{p^\Lambda}{2}$. Therefore the second part of (4.10), "q_Λ" -equations, is a system of equations on ReY^Λ. In new variables:

$$x^A = ReY^A - \frac{p^A ReY^0}{p^0}; \quad x^0 = ReY^0 \tag{4.19}$$

this system has a form:

$$2p^0 ReY^0 \Delta_C x^C = \left(p^0(p \cdot q) - 2D\right)\left(|Y^0|^2\right)^2 \tag{4.20}$$

$$d_{ABI} x^A x^B = \frac{\Delta_I}{3p^{0^2}} |Y^0|^2. \tag{4.21}$$

where $|Y^0|^2 = (ReY^0)^2 + \frac{p^{0^2}}{4}$. Solution of equations (4.20)and (4.21) can be expressed in terms of variables $\tilde{x}^A$ from (4.14), in that case :

$$x^A = \tilde{x}^A \sqrt{\frac{|Y^0|^2}{3p^{0^2}}} \tag{4.22}$$

and

$$|Y^0|^2 = \frac{p^{0^2}}{3} \frac{(\Delta_C \tilde{x}^C)^2}{\frac{4}{3}(\Delta_C \tilde{x}^C)^2 - 9(p^0(p \cdot q) - 2D)^2}. \tag{4.23}$$

Finally the moduli fields z^A are:

$$z^A = \frac{3}{2} \frac{\tilde{x}^A}{p^0(\Delta_C \tilde{x}^C)} (p^0(p \cdot q) - 2D) + \frac{p^A}{p^0} - i\frac{3}{2} \frac{\tilde{x}^A}{(\Delta_C \tilde{x}^C)} \frac{S}{\pi} \tag{4.24}$$

where the entropy S/π is given by (4.12) and $\tilde{x}^A$ are solutions of algebraic system:

$$d_{ABI} \tilde{x}^A \tilde{x}^B = \Delta_I. \tag{4.25}$$

In case $p = 0$ equations the system (4.10) becomes very simple:

$$6D_{AI} ReY^A = q_I ReY^0$$

$$3D_{AB}ReY^AReY^B - \frac{D}{4} = -q_0(ReY^0)^2 \tag{4.26}$$

and the solution for moduli is:

$$z^A = \frac{D^A}{6} - i\frac{p^A}{2}\sqrt{\frac{q_B D^B + 12q_0}{3D}} = \frac{D^A}{6} - i\frac{p^A}{2}\frac{DS}{\pi} \tag{4.27}$$

were S/π is given by (4.15).

4.4 Examples.

We will present below some examples of the entropy formulas for particular CY manifolds [45, 48, 46].

In case of two moduli $n_v = 2$ the algebraic system (4.14) has a general solution. Expression $(\Delta_C \tilde{x}^C)^2$ in entropy formula (4.12) has the form:

$$\begin{aligned}
(\Delta_C \tilde{x}^C)^2 &= \frac{1}{(\det \hat{d}_{123}) - 4\Delta_1\Delta_2}\Big(-2\Delta_A\Delta_B\Delta_C d_A^{BC} \det d_1 \det d_2 \\
&+ (\det \hat{d}_{123})(\Delta_1\Delta_A\Delta_B d_2^{AB} \det d_2 + \Delta_2\Delta_A\Delta_B d_1^{AB} \det d_1) \\
&+ 2(-{\Delta_1}^2 \det d_2 - {\Delta_2}^2 \det d_1 + \Delta_1\Delta_2(\det \hat{d}_{123}))^{3/2}\Big)
\end{aligned} \tag{4.28}$$

were $(\det \hat{d}_{123}) = d_{111}d_{222} - d_{112}d_{122}$ and

$$\begin{aligned}
\det d_1 = \det(d_{1AB}) \neq 0, &\qquad \det d_2 = \det(d_{2AB}) \neq 0, \\
d_1^{AB} = [d_{1AB}]^{-1}, &\qquad d_2^{AB} = [d_{2AB}]^{-1}.
\end{aligned} \tag{4.29}$$

Even if these conditions are not satisfied and there is no inverse matrix d_1^{AB} or d_1^{AB} a solution can still exist. One of particular choices of Calabi-Yau intersection numbers with $n_v = 2$ for which condition (4.29) does not work was given in [45]. In that case:

$$d_{111} = 8/6; \quad d_{112} = 4/6; \quad d_{122} = d_{222} = 0. \tag{4.30}$$

Solution of (4.14) exists and expression (4.13) has a form:

$$(\Delta_C \tilde{x}^C)^2 = 3/8\Delta_2(3\Delta_1 - 2\Delta_2)^2 \tag{4.31}$$

so that black hole entropy is:

$$\frac{S}{\pi} = \frac{1}{3p^0}\sqrt{\frac{\Delta_2}{2}(3\Delta_1 - 2\Delta_2)^2 - 9(p^0(p\cdot q) - 2D)^2}. \tag{4.32}$$

One of examples of model with $n_v = 3$ is STU model studied in [39] with prepotential $F(X) = d_{ABC}z^A z^B z^C$ and $d_{ABC} = d_{123} = 1/6$. Expressions for entropy and for moduli fields were already found in [8]. Here we show these expressions in terms of solutions of algebraic system (4.14) and Δ_A:

$$(\Delta_C \tilde{x}^C)^2 = \frac{9}{2}\frac{\Delta_1\Delta_2\Delta_3}{d_{123}}. \tag{4.33}$$

and entropy (4.12) is

$$\frac{S}{\pi} = \frac{1}{p^0}\sqrt{\frac{2}{3d_{123}}\Delta_1\Delta_2\Delta_3 - (p^0(p\cdot q) - 2D)^2} \tag{4.34}$$

this expression coincides with expression for entropy of STU black holes in [8]. Expression for moduli is:

$$z^A = \frac{3(p^0(p\cdot q) - 2D) + 6p^A\Delta_A}{2p^0\Delta_A} - i\frac{S}{2\pi\Delta_A} \quad \text{(no summation on A)} \tag{4.35}$$

which also coincides with expression for moduli from [8].

Some interesting examples of $n_v = 3$ Calabi-Yau manifolds were given in [48, 46]. Those examples of classical prepotentials correspond to Type II vacuum. Each of prepotentials are related with each other and with $F^0_{II} = STU + \frac{1}{3}U^3$ prepotential by linear transformations:

$$\begin{aligned} F^1_{II} &= \frac{4}{3}(z^1)^3 + (z^1)^2 z^2 + (z^1)^2 z^3 + z^1 z^2 z^3 \\ &\quad (z^1 = U,\ z^2 = T - U,\ z^3 = S - U); \end{aligned} \tag{4.36}$$

$$\begin{aligned} F^2_{II} &= \frac{4}{3}(z^1)^3 + 2(z^1)^2 z^2 + z^1 (z^2)^2 + (z^1)^2 z^3 + z^1 z^2 z^3, \\ &\quad (z^1 = U,\ z^2 = T - U,\ z^3 = S - T)\ ; \end{aligned} \tag{4.37}$$

$$F_{II}^3 = \frac{4}{3}(z^1)^3 + \frac{3}{2}(z^1)^2 z^2 + \frac{1}{2} z^1 (z^2)^2 + (z^1)^2 z^3 + z^1 z^2 z^3 \, , \qquad (4.38)$$
$$(z^1 = U, \, z^2 = T - U, \, z^3 = S - 1/2T - 1/2U) \, .$$

Solution of algebraic equation (4.14) exists for each of these prepotentials and the entropy for F_{II}^0 is:

$$\frac{S}{\pi} = \frac{1}{3p^0}\sqrt{2\Big(\Delta_3((\Delta_3)^2 + 12\Delta_1\Delta_2) + (W_0)^{\frac{3}{2}}\Big) - 9\Big(p^0(p\cdot q) - 2D\Big)^2} \qquad (4.39)$$

$$W_0 = \Big((\Delta_3)^2 - 4\Delta_1\Delta_2\Big), \qquad (4.40)$$

expressions $(\Delta_C \tilde{x}^C)^2$ for F_{II}^1, F_{II}^2 and F_{II}^3 are:

$$(\Delta_C\tilde{x}^C)^2_{F1} = \frac{3}{2}(\Delta_1 - \Delta_2 - \Delta_3)((\Delta_1 - \Delta_2 - \Delta_3)^2 + 12\Delta_2\Delta_3) + \frac{3}{2}(W_1)^{\frac{3}{2}},$$

$$W_1 = \Big((\Delta_1 - \Delta_2 - \Delta_3)^2 - 4\Delta_2\Delta_3\Big). \qquad (4.41)$$

$$(\Delta_C\tilde{x}^C)^2_{F2} = \frac{3}{2}(\Delta_1 - \Delta_2)((\Delta_1 - \Delta_2)^2 + 12\Delta_3(\Delta_2 - \Delta_3)) + \frac{3}{2}(W_2)^{\frac{3}{2}},$$

$$W_2 = \Big((\Delta_1 - \Delta_2)^2 - 4\Delta_3(\Delta_2 - \Delta_3)\Big). \qquad (4.42)$$

$$(\Delta_C\tilde{x}^C)^2_{F3} = \frac{3}{2}(\Delta_1 - \Delta_2 - \frac{1}{2}\Delta_3)((\Delta_1 - \Delta_2 - \frac{1}{2}\Delta_3)^2 + 12\Delta_3(\Delta_2 - \frac{1}{2}\Delta_3))$$
$$+ \frac{3}{2}(W_3)^{\frac{3}{2}},$$

$$W_3 = \Big((\Delta_1 - \Delta_2 - \frac{1}{2}\Delta_3)^2 - 4\Delta_3(\Delta_2 - \frac{1}{2}\Delta_3)\Big). \qquad (4.43)$$

Solutions (4.41), (4.42) and (4.43) are connected with each other and with $(\Delta_C\tilde{x}^C)^2$ solution for F_{II}^0 by linear transformations of Δ_A. For example, connection between (4.41) and (4.42) is : $\{\Delta_1, \Delta_2, \Delta_3\} \to \{\Delta_1, (\Delta_2 - \Delta_3), \Delta_3\}$ and between (4.41) and (4.43) is : $\{\Delta_1, \Delta_2, \Delta_3\} \to \{\Delta_1, (\Delta_2 - \frac{1}{2}\Delta_3), \Delta_3\}$.

The last example of Calabi-Yau manifolds that we are going to consider here was again given in [46]. This model is connected with $F = \frac{3}{8}U^3 + \frac{1}{2}UT^2 - \frac{1}{6}V_X^3$ by linear transformation:

$$z^1 = V_X, \quad z^2 = U - V_X, \quad z^3 = T - \frac{3}{2}U. \tag{4.44}$$

Classical prepotential in this case is:

$$\begin{aligned} F &= \frac{4}{3}(z^1)^3 + \frac{3}{2}(z^2)^3 + \frac{9}{2}(z^1)^2 z^2 + \frac{9}{2}z^1(z^2)^2 + \frac{3}{2}(z^1)^2 z^3 \\ &+ \frac{3}{2}(z^2)^2 z^3 + \frac{1}{2}z^1(z^3)^2 + \frac{1}{2}z^2(z^3)^2 + 3z^1 z^2 z^3. \end{aligned} \tag{4.45}$$

the solution for (4.12) is:

$$\frac{S}{\pi} = \frac{1}{3p^0}\sqrt{\frac{4}{3}\left(\sqrt{6(\Delta_2 - \Delta_1)^3} + \sqrt{\frac{2}{3}(\Delta_2 - 3\Delta_3)^3} - \sqrt{\frac{2}{3}(\Delta_2)^3}\,\right)^2 - 9\big(p^0(p\cdot q) - 2D\big)^2}$$

We have suppressed here the stabilized values of moduli fields although they are available for each case. The expressions for entropy of black holes are rather complicated, but they could be compared with entropy obtained by counting of microscopic degrees of freedom (stringy microstates, "D-branes").

4.5 Uniqueness of the Black Hole Entropy Solution

One of the important questions regarding the solution (4.12) and (4.24) is its uniqueness. This question is closely related to the question about the number and the form of the solutions for the algebraic system (4.14). This system in general can have 2^{n_v} solutions and only one of them give the true physical value of the black hole entropy. The "right" solution should correspond to the positive-definite Kähler metric of the moduli space G_{AB}, defined by

$$G_{AB} = \frac{\partial K}{\partial z^A \partial \bar{z}^B} \tag{4.46}$$

Substituting the Kähler potential (4.7) we will get:

$$G_{AB} = \frac{1}{S^2}\left(\Delta_A\Delta_B - \frac{2}{3}(\Delta_C\tilde{x}^C)d_{ABC}\tilde{x}^C\right) \tag{4.47}$$

or, alternatively, this can be given as

$$G_{AB} = \frac{\tilde{x}^C\tilde{x}^D}{S^2}\left(\Delta_B d_{CAD} - \frac{2}{3}\Delta_C d_{BAD}\right). \tag{4.48}$$

Let's look at the determinant of matrix G_{AB}. It contains the sum of two matrixes $\Delta_A\Delta_B$ and $-\frac{2}{3}(\Delta_C\tilde{x}^C)d_{ABC}\tilde{x}^C$ therefore

$$\det G_{AB} = \frac{1}{S^{2n}}\left(\det\Delta_A\Delta_B - \det(-\frac{2}{3}(\Delta_D\tilde{x}^D)d_{ABC}\tilde{x}^C) + \sum_{i=1}^{(n-1)}\det M_{(i)}\right) \tag{4.49}$$

where $M_{(i)}$ are the matrixes that contain i rows of $\Delta_A\Delta_B$ and $(n-i)$ rows of $-\frac{2}{3}(\Delta_C\tilde{x}^C)d_{ABC}\tilde{x}^C$ so that

$$\det M_{(i)} = (-\frac{2}{3}(\Delta_C\tilde{x}^C))^{(n-i)}\begin{vmatrix} \Delta_1\Delta_1 & \Delta_1\Delta_2 & \cdots & \Delta_1\Delta_n \\ \cdots & \cdots & \cdots & \cdots \\ \Delta_i\Delta_1 & \Delta_i\Delta_2 & \cdots & \Delta_i\Delta_n \\ d_{(i+1)1C}\tilde{x}^C & d_{(i+1)2C}\tilde{x}^C & \cdots & d_{(i+1)nC}\tilde{x}^C \\ \cdots & \cdots & \cdots & \cdots \\ d_{n1C}\tilde{x}^C & d_{n2C}\tilde{x}^C & \cdots & d_{nnC}\tilde{x}^C \end{vmatrix}.$$

It is easy to notice that $\det\Delta_A\Delta_B = 0$ as well as $\det M_{(i)} = 0$ for all $i > 1$ because these matrixes contain rows proportional to each other, therefore

$$\begin{aligned} &\det G_{AB} = \\ = \quad &(-\frac{2\Delta_D\tilde{x}^D}{3S^2})^n \det d_{ABC}\tilde{x}^C + (-\frac{2\Delta_D\tilde{x}^D}{3S^2})^{(n-1)}\frac{1}{S^2}\sum_{permut}\det M_{(1)AB}. \end{aligned} \tag{4.50}$$

There are n matrixes $M_{(1)}$ so that $M_{(1)}^{\{i\}}$ has "$\Delta_i\Delta_j$"in the i-th row:

$$\det M_{(1)AB}^{\{i\}} = \Delta_i \det\tilde{M}_{AB}^{\{i\}},$$

where

$$\tilde{M}_{AB}^{\{i\}} = \begin{pmatrix} d_{11C}\tilde{x}^C & d_{12C}\tilde{x}^C & \cdots & d_{1nC}\tilde{x}^C \\ \cdots & \cdots & \cdots & \cdots \\ d_{(i-1)1C}\tilde{x}^C & d_{(i-1)2C}\tilde{x}^C & \cdots & d_{(i-1)nC}\tilde{x}^C \\ \Delta_1 & \Delta_2 & \cdots & \Delta_n \\ d_{(i+1)1C}\tilde{x}^C & d_{(i+1)2C}\tilde{x}^C & \cdots & d_{(i+1)nC}\tilde{x}^C \\ \cdots & \cdots & \cdots & \cdots \\ d_{n1C}\tilde{x}^C & d_{n2C}\tilde{x}^C & \cdots & d_{nnC}\tilde{x}^C \end{pmatrix}.$$

Now it is time for a "trick" : system of quadratic equations

$$d_{ABI}\tilde{x}^A\tilde{x}^B = \Delta_I \tag{4.51}$$

may be rewritten as a "linear" one so that

$$\tilde{d}_{AB}\tilde{x}^B = \Delta_A, \tag{4.52}$$

where $\tilde{d}_{AB} = d_{ABC}\tilde{x}^C$. From this linear system "point of view" the solution is

$$\tilde{x}^I = \frac{\det \tilde{M}_{AB}^{\{I\}}}{\det \tilde{d}_{AB}} = \frac{\det \tilde{M}_{AB}^{\{I\}}}{\det d_{ABC}\tilde{x}^C},$$

therefore

$$\det \tilde{M}_{AB}^{\{I\}} = (\det d_{ABC}\tilde{x}^C)\tilde{x}^I. \tag{4.53}$$

Plugging the expression (4.53) into (4.50) we will get:

$$\det G_{AB} = \left(-\frac{2}{3}\frac{(\Delta_C\tilde{x}^C)}{S^2}\right)^n \frac{1}{2}\det d_{ABC}\tilde{x}^C. \tag{4.54}$$

This means that the determinant of the metric G_{AB} is proportional to the determinant of the matrix $\tilde{d}_{AB} = d_{ABC}\tilde{x}^C$. If the $\det d_{ABC}\tilde{x}^C$ become zero it means that the determinant of the metric vanishes too. Zero value of the $\tilde{d}_{AB}$ corresponds to the ambiguity in the solution for moduli equation (4.52) and therefore(4.51).

We will present below an example of the entropy and moduli solution for the simple case of the two moduli $n_v = 2$ CY manifold. In this case the algebraic system (4.14) has a general solution. In the notations

$$d_{111} = a; \quad d_{112} = b; \quad d_{122} = c; \quad d_{222} = d. \tag{4.55}$$

the system becomes:

$$ax^2 + 2bxy + cy^2 = \Delta_1, \tag{4.56}$$

$$bx^2 + 2cxy + dy^2 = \Delta_2. \tag{4.57}$$

It is easy to find that

$$\begin{aligned}(\Delta_C \tilde{x}^C)^2 \quad &= \quad \frac{\Delta_2(d\Delta_1^2 + b\Delta_2^2 - c\Delta_1\Delta_2)}{(bd - c^2)} \\ &- \quad \frac{((c^2 - bd)\Delta_1^2 + (b^2 - ac)\Delta_2^2 + (ad - bc)\Delta_1\Delta_2)x^2}{(bd - c^2)}.\end{aligned} \tag{4.58}$$

We do not have any ambiguity till now:

$$\begin{aligned}x_\pm^2 \quad &= \quad -\frac{(d\Delta_1 - c\Delta_2)(bc - ad) - 2(c\Delta_1 - b\Delta_2)(c^2 - bd)}{((bc - ad)^2 - 4(b^2 - ac)(c^2 - bd))} \\ &\pm \quad \frac{\sqrt{4(c^2 - bd)^2((c^2 - bd)\Delta_1^2 + (b^2 - ac)\Delta_2^2 + (ad - bc)\Delta_1\Delta_2)}}{((bc - ad)^2 - 4(b^2 - ac)(c^2 - bd))},\end{aligned} \tag{4.59}$$

and $y_\pm^2$ is given by:

$$y^2 = \frac{c\Delta_1 - b\Delta_2}{(c^2 - bd)} + \frac{(b^2 - ac)x^2}{(c^2 - bd)} \tag{4.60}$$

and

$$2xy = \frac{d\Delta_1 - c\Delta_2}{(bd - c^2)} + \frac{(bc - ad)x^2}{(bd - c^2)}. \tag{4.61}$$

Let's consider the metric for this case:

$$G_{AB} = \frac{\tilde{x}^C \tilde{x}^D}{S^2}\left(\Delta_B d_{CAD} - \frac{2}{3}\Delta_C d_{BAD}\right). \tag{4.62}$$

then

$$\det G_{AB} = \frac{4}{9}(\Delta_1 x + \Delta_2 y)^2 \Big((ac-b^2)x^2 + (bd-c^2)y^2 + (ad-bc)xy\Big) \tag{4.63}$$
$$- \frac{2}{3}(\Delta_1 x + \Delta_2 y)\Big(\Delta_1^2(cx+dx) - 2\Delta_1\Delta_2(bx+cy) + \Delta_2^2(ax+by)\Big).$$

Using (4.60),(4.61),(4.58) we will get:

$$\Big((ac-b^2)x^2 + (bd-c^2)y^2 + (ad-bc)xy\Big) =$$
$$\pm \ (\text{or} \mp)\sqrt{((c^2-bd)\Delta_1^2 + (b^2-ac)\Delta_2^2 + (ad-bc)\Delta_1\Delta_2)} \tag{4.64}$$

sign $\pm$ or $\mp$ depends on $\frac{|c^2-bd|}{c^2-bd} > (or \quad < \ )0$. Second term in (4.63) is

$$\Big(\Delta_1^2(cx+dx) - 2\Delta_1\Delta_2(bx+cy) + \Delta_2^2(ax+by)\Big) =$$
$$\Big(\Delta_1^2 c - 2\Delta_1\Delta_2 b + \Delta_2^2 a\Big)x + \Big(\Delta_1^2 d - 2\Delta_1\Delta_2 c + \Delta_2^2 b\Big)y \tag{4.65}$$

using (4.56)and (4.57) we can transform

$$\begin{aligned}
\Big(\Delta_1^2 c - 2\Delta_1\Delta_2 b + \Delta_2^2 a\Big)x &= \Delta_1(ac-b^2)x^3 - \Delta_1(bd-c^2)y^2x \\
&+ 2\Delta_2(ac-b^2)x^2y + \Delta_2(ad-bc)y^2x \\
\Big(\Delta_1^2 d - 2\Delta_1\Delta_2 c + \Delta_2^2 b\Big)y &= \Delta_1(ad-bc)x^2y + 2\Delta_1(bd-c^2)y^2x \\
&- \Delta_2(ac-b^2)x^2y + \Delta_2(bd-c^2)y^3,
\end{aligned}$$

Finally expression for metric is

$$\begin{aligned}
\det G_{AB} &= (\frac{4}{9} - \frac{2}{3})(\Delta_1 x + \Delta_2 y)^2 \Big((ac-b^2)x^2 + (bd-c^2)y^2 + (ad-bc)xy\Big) \\
&= \mp\frac{2}{9}(\Delta_1 x + \Delta_2 y)^2\sqrt{((c^2-bd)\Delta_1^2 + (b^2-ac)\Delta_2^2 + (ad-bc)\Delta_1\Delta_2)}.
\end{aligned}$$

In this simple example it is obvious that the physical case corresponds to the "+" solution and that the entropy solution is unique.

4.6 Outlook

We have found the entropy-area formula of N=2 extreme black hole for the most general case of Calabi-Yau moduli space. We have have shown that the entropy and the moduli fields in general case depend on the solution of algebraic system of quadratic equations(4.14). We have found the entropy of the black hole for some particular examples of Calabi-Yau moduli space. Although it is not obvious that the existence of the solutions can be regarded as general property of Calabi-Yau prepotentials we hope that there is a connection between them. It will be very interesting to find such a connection.

5 d5 Blackholes

5.1 Introduction

In the moduli space encompassing all known perturbative string theories, it has been discovered [49] that there exists a special point at which perturbative string theories are strongly coupled and are best described by a more fundamental eleven-dimensional theory. For low-energy excitations, this so-called M-theory is given by eleven-dimensional $N = 1$ supergravity [50] appended with gravitational anomaly cancellation terms. Not only providing a more fundamental description of non-perturbative string theories, M-theory is also indispensable for constructing a phenomenologically viable four-dimensional string compactifications. Witten [51] has shown that strongly coupled phase of $E_8 \times E_8$ heterotic string theory, which is most appropriately described by M-theory compactified on an orbifold $\mathbf{S}_1/\mathbf{Z}_2$, is crucial for unification of all gauge and gravitational couplings at the grand-unification scale [7].

The above consideration points to the importance of a better understanding of M-theory compactification vacua. With such motivation we will study M-theory compactified on Calabi-Yau three-folds CY_3 to five dimensions. The compactification appropriately encodes the strong coupling limit of type IIA compactification on the same Calabi-Yau three-fold CY_3. After the compactification, the low-energy dynamics is governed by a $d = 5, N = 2$ supergravity. Its field content and interactions are determined by the topological data of CY_3, which may be understood, for example, from eleven-dimensional supergravity compactified on CY_3[53, 54].

In understanding the web of moduli spaces of string compactification vacua, it has become clear that two aspects are essential. First, the classical moduli space

[7]It is to be noted that universal threshold corrections still leaves a possibility of coupling constant unification within weakly coupled heterotic string theory [52].

may undergo topology changing phase transitions accompanied by the appearance of massless states. Resolution of conifold singularities [55], symmetry enhancement by small instantons [56] and the appearance of tensionless strings [57] - [61] are well-known instances. At the same time, it has become clearer that BPS black holes with finite horizon serve as a spacetime probe of the compactified internal space [5]-[8] and [62] -[64]. For example, for $d = 4, N = 2$ supergravity theory, the so-called doubly extreme black holes attract Kähler moduli scalar fields to a critical point of the central charge, $\partial Z/\partial\phi^i = 0$. The extremized BPS black hole mass then becomes equal to the square root of the entropy, and depends only on topological data of the compactified space such as intersection numbers, the second Chern class and rational, elliptic instanton numbers.

In five dimensions, Witten [65] has shown that all phases of the M-theory compactification vacua are purely geometrical. Since the Kähler moduli do not admit complexificiation in five dimensions, the topology changing phase transitions in five dimensions are sharp. As such, the nature of phase transitions is quite distinct from that in four dimensions. It is therefore of interest to understand all distinct features of the phase transition explicitly by means of simple probes. As in thermodynamical systems, the nature of phase transitions and the corresponding critical behavior can be probed by various supersymmetry potentials of Landau-Ginzburg type. The BPS central charge Z, defined as the electric charge of the gravi-photon, which is the BPS mass of the electric 5d black holes, is one of such potentials. Similarly, the magnetic charge of the gravi-photon Z_m defines the tension of the magnetic 5d string and also can be used to probe the phase transition. Likewise, the black hole potential V and the gauged central charge potential P are other additional physical quantities that can be used for analyzing the topology-changing phase transitions. All these potentials are defined for fixed electric charges (wrapping numbers of M-branes on two-cycles of the Calabi-Yau three-fold) and may be regarded as independent thermodynamic potentials in the canonical ensemble[8]. The critical point in the vector moduli space is determined entirely by the chosen charge configuration for the potentials. The idea is that, by considering different charge configurations, the critical configuration of the Kähler moduli for each of the above physical quantities may be located at different phases of the Calabi-Yau three-fold, hence, probing the nature of phase transitions.

This chapter is organized as follows. In section 2 we review the compactification of M-theory on a Calabi-Yau threefold. The resulting low-energy approximation, $d = 5, N = 2$ supergravity coupled to abelian vector supermultiplets, is based on the structure of *very special geometry* [34]. In section 3, we derive the stabilization equations of five-dimensional vector moduli scalars for electrically charged particle states and magnetic string states. The central charge Z evaluated at these fixed scalar field values turns out to be a minimum as expected. In section 4, we introduce two five-dimensional potentials of interest :– the black hole potential V and the scalar potential P which arises in gauged supergravities. The central charge Z and these two potentials are explicitly discussed in a simple example. Section 5 constitutes the main results of our study. After reviewing briefly some basic notions of the five-dimensional extended Kähler cone, we probe the topology-changing phase transitions using the supersymmetry potentials Z, Z_m, V and P. Particular emphasis is put on flop transitions connecting two adjacent Kähler cones. We analyze how Z, V and P are affected by it. We find that the electric central charge and its first two derivatives are in fact continuous along the flop transition. Since, as we will also show, any extremum of Z has to be a minimum one finds uniqueness of the critical point of Z in the entire extended Kähler cone. Whereas the magnetic central charge (BPS tension) and its first derivative are continuous, also implying uniqueness of the stabilization point in the extended Kähler cone, V on the other hand shows a kink

[8]Yet another possible supersymmetry potential is the topological free energy [66] in which all BPS electric and magnetic charges are summed over. Following the paper [10], we will mainly consider canonical ensemble approach and do not consider the topological free energy.

at the flop transition. Before we conclude, our results are verified by an explicit example in section 6, where we study one particular extended Kähler cone in detail. The behavior of the potentials is graphically displayed. In addition we find new critical points for the potentials V and P. We observe that extremal transitions in five dimensions between Calabi-Yau spaces with different numbers of Kähler moduli often involve tensionless strings.

5.2 Very Special Geometry and Calabi-Yau Manifolds

We begin by recalling relevant aspects [53, 67] of M-theory compactification on a Calabi-Yau three-fold CY_3 with Hodge numbers $h_{(1,1)}$, $h_{(2,1)}$ and intersection numbers $C_{IJK}(I, J, K = 1, \cdots, h_{(1,1)})$. The bosonic fields of the supergravity multiplet in eleven dimensions consist of the metric g_{MN} and the 3-form antisymmetric tensor potential $\mathcal{A}_{\mathcal{MNP}}$. Denote the complex coordinates of CY_3 as $\mathcal{I}, \mathcal{J}, \overline{\mathcal{I}}, \overline{\mathcal{J}} = 1, 2, 3$. After compactification on the CY_3, the action contains one CY complex axion $\mathcal{A}_{\mathcal{IJK}} = \epsilon_{\mathcal{IJK}} b$, one universal axion a ($da \sim {}^*d\mathcal{A}$), the Calabi-Yau volume det $\mathcal{G}_{\mathcal{I}\overline{\mathcal{J}}}$, $h_{(1,1)} - 1$ real Kähler moduli $\mathcal{G}_{\mathcal{I}\overline{\mathcal{J}}}$ and $h_{(2,1)}$ pairs of complex scalars $(\mathcal{G}_{\mathcal{IJ}}, \mathcal{A}_{\mathcal{IJ}\overline{\mathcal{K}}})$. In addition, $h_{(1,1)}$ abelian vector potentials $\mathcal{A}_{\mu\mathcal{I}\overline{\mathcal{J}}}$ arise, one of which becomes the gravi-photon. They combine into $d = 5$ supermultiplets as follows. Taking into account one graviphoton in the supergravity multiplet, there are $h_{(1,1)} - 1$ vector multiplets consisting of uni-modular $(\mathcal{G}_{\mathcal{I}\overline{\mathcal{J}}}, \mathcal{A}_{\mu\mathcal{I}\overline{\mathcal{J}}})$, one universal hypermultiplet (det $\mathcal{G}_{\mathcal{I}\overline{\mathcal{J}}}, a, \mathcal{A}_{\mathcal{IJK}}$), and $h_{(2,1)}$ hypermultiplets containing $(\mathcal{G}_{\mathcal{IJ}}, \mathcal{A}_{\mathcal{IJ}\overline{\mathcal{K}}})$. Supersymmetry non-renormalization theorems [28, 68] ensure that the moduli space of the hypermultiplets containing the complex structure moduli (and the size of the entire Calabi-Yau space) is completely disconnected from the Kähler moduli space.

In this paper we will restrict ourselves to the moduli space of the vector multiplets, corresponding to the Kähler moduli t^I of the Calabi-Yau space. The Kähler moduli are derived from the Kähler form J on CY_3 by expanding into a fundamental 2-form basis J_I as

$$J = t^I J_I. \tag{5.1}$$

With this definition the moduli t^I correspond to the sizes of 2-cycles in CY_3 dual to the J_I.

The supersymmetric coupling of $d = 5, N = 2$ vector multiplets to N=2 supergravity has been constructed in [69][9]. The vector multiplet coupling is determined completely by specifying a prepotential

$$\mathcal{V} = C_{IJK} t^I t^J t^K, \qquad I = (1, \cdots, h_{(1,1)}), \tag{5.2}$$

which defines intrinsic 'very special geometry' [34]. The prepotential is related to the overall volume of Calabi-Yau three-fold. In five dimensions, the overall volume belongs to the so-called universal hypermultiplet. Therefore, the prepotential which encodes independent degrees of freedom of the vector multiplet moduli space is constrained to satisfy

$$\mathcal{V} = 1. \tag{5.3}$$

The variables t^I are related to the conventional very special coordinates [70] X^I by the rescaling

$$t^I = 6^{\frac{1}{3}} X^I. \tag{5.4}$$

The $h_{(1,1)} - 1$ independent, real scalar fields parametrizing the vector multiplet moduli space are denoted by ϕ^i, and the t^I are functions of them. It is often convenient to identify the ϕ^i with $h_{(1,1)} - 1$ of the t^I and to take the remaining Kähler moduli to be a function of the other ones defined by the constraint $\mathcal{V} = 1$.

In terms of the aforementioned normalization convention, the bosonic part of the action is given by

$$e^{-1}\mathcal{L} = -\frac{R}{2} - \frac{1}{4} G_{IJ}(\phi) F_{\mu\nu}{}^I F^{\mu\nu J} - \frac{1}{2} g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j + \frac{e^{-1}}{48} \epsilon^{\mu\nu\rho\sigma\lambda} C_{IJK} F^I_{\mu\nu} F^J_{\rho\sigma} A^K_\lambda \tag{5.5}$$

[9] We adopt different normalization of the action from the earlier works [67, 70] so that the Chern-Simons coupling C_{IJK} equals directly to the topological intersection number on Calabi-Yau three-folds.

The vector multiplet coupling is defined by

$$G_{IJ} = -\frac{1}{2 \cdot 6^{\frac{2}{3}}} \partial_I \partial_J (\ln \mathcal{V})|_{\mathcal{V}=1}, \tag{5.6}$$

while the moduli space metric g_{ij} is given by

$$g_{ij} = 6^{\frac{2}{3}} G_{IJ} t^I_{,i} t^J_{,j} = -3 C_{IJK} t^I t^J_{,i} t^K_{,j}. \tag{5.7}$$

The scalar fields ϕ^i, $(i = 1, \cdots, h_{(1,1)} - 1)$ parametrize the vector multiplet moduli space $\mathcal{M}_V$. For a given $\mathcal{M}_V$, consider the second covariant derivative of the very special coordinate with respect to the moduli scalar fields ϕ^i

$$(t^I)_{,i;j} = (2/3) g_{ij} t^I - \sqrt{2/3}\, T_{ijk} t^{I,k}. \tag{5.8}$$

The symmetric tensor T_{ijk}, which is a function of the moduli scalar fields, encodes local geometric structures of $\mathcal{M}_V$[10]. If T_{ijk} is covariantly constant, $T_{ijk;l} = 0$, the scalar manifold $\mathcal{M}_V$ is locally a symmetric space. $d = 5, N = 2$ supergravity with locally symmetric $\mathcal{M}_V$ has been studied extensively with the aid of Jordan algebras [69]. Generically, with $n \equiv h_{(1,1)} - 1$,

$$\mathcal{M} = SO(1,1) \times \frac{SO(1, n-1)}{SO(n-1)}. \tag{5.9}$$

In this case the prepotential $\mathcal{V} = C_{IJK} t^I t^J t^K$ should be factorizable into a linear and a quadratic polynomial parts. That is,

$$\mathcal{V} = C_{IJK} t^I t^J t^K = t^1 Q(t^\lambda) \tag{5.10}$$

where $\lambda = 2, .., n$ and Q is a quadratic form of signature $(+, -, -, \ldots, -)$. If $\mathcal{V}$ is not factorizable, $\mathcal{M}_V$ may or may not be a symmetric space. The cases with locally symmetric and homogeneous manifold $\mathcal{M}_V$, but with non-factorizable prepotential $\mathcal{V}$ have been classified in [69]. There are only four exceptional cases, so-called 'magic square' supergravities, for which $n = 3(1 + \mathrm{dim}\mathbf{A})$ where $\mathbf{A}$ is one of the division algebras and $n = 5, 8, 14, 26$.

[10]The same tensor appears in a number of places in the Lagrangian including 4-fermion terms.

In generic Calabi-Yau compactifications of M-theory, the prepotentials do not belong to the set based on the Jordan algebra classification of $d = 5$ supergravities. Possible exceptions are Calabi-Yau compactifications described by factorizable prepotentials of the form (5.10) or the 'magic square' cases $n = 5, 8, 14, 26$. In all other cases, the $d = 5$ compactifications do not share many of the properties of the known $d = 5$ supergravities based on the Jordan algebras. In following sections, we will study Calabi-Yau prepotentials with small dimensions of vector moduli spaces $n = 1, 2$, which are neither factorizable nor belong to the "magic square" cases. As we will see, these examples are simple enough to allow a detailed analysis, while rich enough to capture all the essentials of geometric phase transitions in five dimensions.

5.3 Stabilization Conditions

In $d = 5, N = 2$ supergravity theories, the gravi-photon field T is given by a linear combination of all vector fields F^I of the theory with moduli dependent coefficients: $T = t^I(\phi)G_{IJ}(\phi)F^I$. Associated to the gravi-photon is the electric central charge [67]

$$Z = t^I(\phi)q_I \tag{5.11}$$

where q_I is the quantized electric charge for each vector field $G_I \equiv G_{IJ}F^I$. In the context of M-theory the electrically charged point-particle states are identified with BPS winding states of the M-theory membrane around 2-cycles of the Calabi-Yau three-fold CY_3 [53, 67, 68]. Magnetically charged string states are identified with BPS winding states of the M-theory five-brane around 4-cycles. The magnetic central charge describing the string tension is given by [67]

$$Z_m = t_I(\phi)m^I \tag{5.12}$$

where the dual moduli t_I defined by

$$t_I = C_{IJK}t^Jt^K \tag{5.13}$$

encode the sizes of four-cycles of the CY_3. The mass spectrum of BPS states varies over the vector multiplet moduli space. As such, the mass of a BPS state can often be extremized as a function of moduli for given values of charges q_I [6]. It is therefore of interest to study critical behavior of the BPS mass-squared over the moduli space $\mathcal{M}_V$

$$M_{\rm BPS}^2 = Z^2(q_I, C_{IJK}, \phi^i) = t^I t^J q_I q_J. \tag{5.14}$$

The critical points of the BPS mass-squared are specified by the extremal value of the central charge $\partial_i Z = 0$ or by the vanishing of the central charge $Z = 0$ since

$$\frac{\partial M_{\rm BPS}^2}{\partial \phi^i} = 2Z\,\partial_i Z = 2(t^I)_{,i}\, q_I\, t^J q_J = 0. \tag{5.15}$$

In the $d = 4, N = 2$ theory the extremal configuration of the central charge $\partial_i Z = 0$ has been found to be a solution of a set of so-called stabilization equations, which define functional relations between microscopic charges and moduli scalar fields at the critical points [6]. Quite often, the stabilization condition equations are easier to solve than the problem of extremizing directly the central charge [7, 8, 62, 63, 64].

In what follows, we derive the corresponding stabilization condition equations for $d = 5, N = 2$ supergravity in two alternative ways. The first derivation extends the method utilized in the $d = 4, N = 2$ context [6]: the very special geometry that underlies the vector moduli space is used to invert the implicit functional relations $\partial_i Z = (t^I)_{,i} q_I = 0$ into equations between charges and moduli. The second method employs a Lagrange multiplier to the constraint equation $\mathcal{V}(t) = C_{IJK} t^I t^J t^K = 1$ and also effectively inverts the $(t^I)_{,i} q_I = 0$ condition.

5.3.1 Stabilization of the Electric Central Charge: Very Special Geometry

To invert $\partial_i Z = (t^I)_{,i} q_I = 0$ and find the expression for q_I's in terms of t^I's we will utilize some of the properties of the very special geometry presented in [70]. We first multiply $\partial_i Z$ by $g^{ij}t^J{}_{,j}$ and get

$$\partial_i Z = (t^I)_{,i} q_I = 0 \qquad \Longrightarrow \qquad g^{ij} t^I{}_{,i} t^J{}_{,j}\, q_I \equiv \Pi^{IJ} q_I = -\frac{1}{3}((C^{IJ} - t^I t^J) q_I = 0 \tag{5.16}$$

where the matrix C^{IJ} is the inverse matrix of $C_{IJ} = C_{IJK}t^K$. Multiplying this equation by C_{KJ}, we find that

$$q_K = C_{KJ} t^J Z = C_{IJK} t^J t^K Z \tag{5.17}$$

at $\partial_i Z = 0$. Thus the $d = 5$ stabilization condition equations have a particularly simple form

$$q_I = Z t_I \tag{5.18}$$

where t_I, as defined in (5.13), is the coordinate dual to the very special coordinate t^I.

5.3.2 Alternative Derivation: Lagrange Multiplier Method

We use the linear sigma model approach to the constraint of the prepotential. Consider a function:

$$\tilde{Z}(t^I) = Z + \lambda(\mathcal{V} - 1) = q_I t^I + \lambda(C_{IJK} t^I t^J t^K - 1), \tag{5.19}$$

where the t^I's are *a priori* unconstrained and λ denotes a Lagrange multiplier. Taking derivatives with respect to t^I we obtain a set of extremization equations:

$$q_I + 3\lambda C_{IJK} t^J t^K = 0. \tag{5.20}$$

Similarly, the extremization condition with respect to λ gives:

$$C_{IJK} t^I t^J t^K = 1. \tag{5.21}$$

Contracting (5.20) with t^I and using the constraint (5.21) we solve for λ as

$$3\lambda = -q_I t^I = -Z. \tag{5.22}$$

Substituting (5.21) into (5.20) we obtain

$$q_I - ZC_{IJK}t^J t^K = 0. \tag{5.23}$$

In terms of the dual variable t_I, the solution is

$$q_I = Zt_I, \tag{5.24}$$

yielding the same result as the very special geometry method.

For later use, we find it useful to put the $d = 5$ stabilization equation with a suitable rescaling. Let us call $\bar{t}^I$ the original very special coordinate rescaled by the square root of the central charge

$$\bar{t}^I \equiv \sqrt{Z}\, t^I. \tag{5.25}$$

Since the coordinates t^I of the very special geometry are real, the rescaled coordinates $\bar{t}^I$ have to be real for configurations with positive value of the central charge. In $d = 5$, one can also consider the case of imaginary $\bar{t}$ with negative Z. One can see however that this case is easily reproduced starting with the configuration with positive Z and real solutions for $\bar{t}$ by flipping the signs of all charges, as the critical points of Z and $-Z$ do coincide manifestly. It is then possible to rewrite the $d = 5$ stabilization equation into a rather suggestive form

$$q_I = C_{IJK}\bar{t}^J\bar{t}^K. \tag{5.26}$$

This equation implies that, for a given choice of Calabi-Yau manifold with specific topological intersection numbers and a given number of moduli fields, the value of the electric charge q_I defines the real (imaginary) values of rescaled moduli via eq.(5.26), which is quadratic in the rescaled moduli. The solution may or may not exist for the real (or purely imaginary) $\bar{t}^I$. In cases where the solution exists we have an explicit expression for the extremum of the BPS mass and for the black hole entropy. Once the solution to eq. (5.26) is found in the form $\bar{t}^I(q_L, C_{IJK})$ we find the critical value of the central charge in the form

$$\left(\bar{t}^I q_I\right)^2 = Z^3_{\rm cr}(q_L, C_{IJK}). \tag{5.27}$$

This in turn defines the entropy of the black hole [6, 70]

$$S(q_L, C_{IJK}) = \frac{\pi^2}{12}|Z_{\rm cr}|^{3/2}. \tag{5.28}$$

Recall that the near horizon geometry of the $d=5$ black hole is given by $adS_2 \times S^3$, hence, the black hole entropy is proportional to the volume of the S^3. The expression for the critical value of the central charge has been found previously [6, 70] in the form $(Z)^2_{\rm cr} = q_I q_J (C^{IJ})_{\rm cr}$. The new $d=5$ stabilization equation (5.26) enables to find the entropy of $d=5$ black holes explicitly as a function of the microscopic electric charges and of the CY topological intersection numbers.

Indeed, the structure of the $d=5$ stabilization equations are far simpler to solve than those in the $d=4$ case. In [9] it was found that the $d=4, N=2$ black hole entropy with classical Calabi-Yau prepotential can be understood completely once a real solution is found to the following set of algebraic equations: $\Delta_I = C_{IJK}\tilde{x}^J\tilde{x}^K$ where Δ_I is some particular combination of microscopic electric and magnetic charges and topological intersection numbers. When these equations have a solution with real $\tilde{x}^J$, an algorithm was presented for finding the fixed values of moduli, the central charge and the entropy of black holes. The algorithm was general enough but the explicit form of the result was rather complicated.

In $d=5$ the situation has been simplified considerably. The stabilization equations $q_I = C_{IJK}\bar{t}^J\bar{t}^K$ coincide with the algebraic equations of [9] $\Delta_I = C_{IJK}\tilde{x}^J\tilde{x}^K$ upon identification $\Delta_I \sim q_I$ and $\bar{t}^J \sim \tilde{x}^J$. Moreover, the algorithm for obtaining the critical value of the BPS mass and the entropy after $\bar{t}$ is found is very simple in $d=5$ and displayed in eqs. (5.27) and (5.28). If one would like to use the critical value of the $d=4$ central charge directly to find the critical value of its 5d counterpart the following restrictions have to be applied to the $d=4$ charges: $p^I = q_0 = p^0 - 1 = 0$.

5.3.3 Stabilization of the Magnetic String Tension

The magnetic central charge determining the tension of the magnetic string states [67] is a function of the moduli as well, hence, one can also have a critical point for fixed values of the microscopic magnetic charges m^I

$$\partial_i Z_m = (t_I)_{,i} m^I = 2C_{IJK} t^I (t^J)_{,i} m^K = 0. \tag{5.29}$$

To invert this relation we proceed similarly as before

$$C_{IJK} t^I (t^J)_{,i} (t^L)_{,j} g^{ij} m^K = C_{IJK} t^I m^K (C^{JL} - t^J t^L) = 0. \tag{5.30}$$

It follows that

$$m^L = t^L Z_m \qquad \Rightarrow \qquad t^L = \frac{m^L}{Z_m}. \tag{5.31}$$

The critical value of the dual coordinate is

$$t_I = \frac{C_{IJK} m^J m^K}{Z_m^2} \tag{5.32}$$

and the critical value of the BPS string tension is

$$(Z_m)^3_{\text{cr}} = (t_L m^L)^3_{\text{cr}} = C_{IJK} m^I m^J m^K. \tag{5.33}$$

For the electrically charged $d = 5$ BPS black holes, the critical value of the BPS mass was related to the horizon volume S^3, where the near horizon geometry was given by $adS_2 \times S^3$. This in turn was proportional to the black hole entropy. For the magnetically charged $d = 5$ BPS black string, we find similarly that the extremized value of the BPS tension is related to the volume of the S^2 as well as to the size of the infinite throat of adS_3 where the near horizon geometry is given by $adS_3 \times S^2$. The volume of S^2 is moduli independent and is proportional to $Z_m^2 = (C_{IJK} m^I m^J m^K)^{\frac{2}{3}}$. For $d = 4$ black holes the relevant no-axion black hole solution with the entropy proportional to $\sqrt{q_0 C_{IJK} m^I m^J m^K}$ was found in [62] and an empirical rule of state counting in M-theory was proposed in [42], [41] to explain the entropy. Upon decompactification to $d = 5$ this black hole solution was identified with the magnetic string in [71]. Here we have found that the stabilization of the magnetic string tension in $d = 5$ describes the same structure as the stabilization of the magnetic no-axion black hole in $d = 4$ with unit electric charge $q_0 = 1$. The critical value of the 5d magnetic tension can be obtained from the stabilization of the 4d black hole mass at $q_0 - 1 = q_I = p^0 = 0$. This result follows simply from comparing $d = 4$ results for the black hole with the $d = 5$ results for the magnetic string.

5.3.4 The Extremum of the BPS Mass is a Minimum

A natural question to address is under what conditions the extremum of the BPS mass is a *minimum*. An analogous issue has been already addressed in the context of $d = 4$ special geometry [72], where it was found that

$$D_i D_j Z^2 = \frac{1}{2} g_{ij} Z^2 \qquad \text{at} \qquad \partial_i Z = 0. \tag{5.34}$$

In $d = 5$ the corresponding calculation yields, using eq. (5.8):

$$D_i D_j Z = \partial_i \partial_j Z = \frac{2}{3} g_{ij} Z_{\text{cr}} \qquad \text{at} \quad \partial_i Z = 0 \ . \tag{5.35}$$

This shows that the extremum of the central charge is a minimum (maximum) if the central charge at the critical point is positive (negative), using the fact that the metric of the moduli space is positive definite. For the square of the BPS mass we get

$$D_i D_j (Z)^2 = \partial_i \partial_j (Z)^2 = \frac{4}{3} g_{ij} (Z^2)_{\text{cr}} \qquad \text{at} \quad \partial_i Z = 0 \ . \tag{5.36}$$

The same is true for the magnetic BPS mass. Thus, as long as the very special geometry is regular viz. g_{ij} and G_{IJ} are positive definite and non-degenerate, the BPS mass has a minimum at the critical point. Hence, for a given choice of charges and signs of moduli at infinity the corresponding minimum, if it exists, is unique.

5.4 The Black Hole and Gauged Central Charge Potentials

In the last section we have studied the critical points of the central charge Z in detail. For black holes, a quantity of more direct significance is the black hole potential V. In this section we introduce this potential along with a gauged central charge potential P, which arises in gauged supergravities, and discuss their respective critical points in detail. We will also compare the critical behavior of Z^2, V and P for an example defined by prepotential $\mathcal{V} = STU$.

5.4.1 The Black Hole Potential and Its Critical Points

In the context of $d = 4, N = 2$ supergravity, it was explained in [72] that the stabilization of moduli fields of BPS black holes near the black hole horizon depends on the properties of the so-called black hole potential V. For extremal (but not necessarily supersymmetric) black holes with regular horizon and regular moduli scalars, the following relations have been established. The entropy is defined (in eq. (26) of [72]) as the value of the black hole potential evaluated at the horizon:

$$\frac{S}{\pi} = V(p, q, \phi_h). \tag{5.37}$$

Furthermore, in [72] - [74], it was found that the ADM mass of a generic non-extreme black hole consists of three contributions: the black hole potential depending on the value of moduli at infinity, some function of the scalar charges and the $(2ST)^2$ term, where T is the black hole temperature:

$$(M_{\text{ADM}})^2 = V(p, q, \phi_\infty) - G_{ab}\Sigma^a\Sigma^b + (2ST)^2. \tag{5.38}$$

For the case of $d = 4, N = 2$ supergravity, utilizing special geometry relations, V was found to be equal to $|Z|^2 + |\partial Z|^2$. In the extreme case when $TS = 0$ there are two possibilities: (1) the extreme black hole is supersymmetric

$$(M_{\text{ADM}})^2 = (M_{\text{BPS}})^2 = |Z(p, q, \phi_\infty)|^2, \tag{5.39}$$

which requires for consistency that $|\partial Z(p, q, \phi_\infty)|^2 = G_{ab}\Sigma^a\Sigma^b$ and the entropy is given by $\frac{S}{\pi} = V(p, q, \phi_h) = |Z|^2$ at $\partial_i|Z| = 0$, or (2) the extreme black hole is not supersymmetric

$$(M_{\text{ADM}})^2 = |Z|^2 + |\partial Z|^2 - G_{ab}\Sigma^a\Sigma^b > |Z(p, q, \phi_\infty)|^2 = (M_{\text{BPS}})^2 \tag{5.40}$$

and the black hole horizon area is still given by the value of the potential at the horizon $\frac{S}{\pi} = V(p, q, \phi_h)$ if the moduli are regular (in such cases the critical point of the black hole potential is not defined by the enhancement of unbroken supersymmetry $(\partial_i |Z| = 0)$ as it happens for supersymmetric black holes). The potential has a critical point which describes the stabilization of moduli near the horizon of extreme non-supersymmetric black holes. These non-supersymmetric critical points have not been studied much previously. However, the extreme non-supersymmetric solutions with (or without) regular horizon have been known to exist [75, 76, 77, 78]. The most recent detailed study of such solutions in $N = 4$ and $N = 8$ supergravity theories was given in [79].

In what follows we will study the analogous black hole potential in $d = 5$. We will find as expected both types of critical points. The gauge kinetic term in the Lagrangian responsible for the black hole potential is proportional to $G_{IJ}F^I F^J$. Using eqs. (16,29) from [70] one finds that the $d = 5$ potential whose critical points describe the BPS mass and the stabilization of moduli for extreme black holes is given by

$$V = Z^2 + \frac{3}{2} g^{ij} \partial_i Z \partial_j Z = \left(t^I t^J + \frac{3}{2} g^{ij} (t^I)_{,i} (t^J)_{,j} \right) q_I q_J. \tag{5.41}$$

It can be shown that the potential is proportional to the gauge couplings

$$V = \frac{6^{1/3}}{4} G^{IJ}(\phi) q_I q_J \tag{5.42}$$

and the critical points of this potential are given by

$$\partial_i V = 4 \partial_i Z Z - \sqrt{6}\, T_{ikj} \partial^k Z \partial^j Z = 0. \tag{5.43}$$

There can be

1. **Supersymmetric critical points**. Obviously the critical point of the central charge is also a critical point of the potential

$$\partial_i V = 0 \qquad \text{at} \quad \partial_i Z = 0. \tag{5.44}$$

This applies to supersymmetric extreme black holes. The inverse is not true: it is possible that there is a

2. **Non-supersymmetric critical point** where

$$\partial_i V = 0 \qquad \text{at} \quad \partial_i Z \neq 0 \tag{5.45}$$

which give rise to stabilization of the moduli also for non-supersymmetric black holes. The calculation of the second derivative of the potential at the supersymmetric critical point is straightforward and is given by

$$(V)_{,i;j} = \partial_i \partial_i V = \frac{8}{3} g_{ij} V_{\rm cr} \qquad \text{at} \qquad \partial_i V = \partial_i Z = 0. \tag{5.46}$$

We conclude that for a positive definite metric of very special geometry the potential has a minimum at the supersymmetric critical point (5.44), which exists under the condition that the stabilization equation (5.26) for the moduli in terms of charges has a real solution.

The non-supersymmetric critical points (5.45) do not seem to allow a simple general treatment. We will study them for a few examples and show that indeed such critical points do exist.

5.4.2 The Gauged Central Charge Potential

The gauged central charge of $N = 2$ supergravity arises in theories in which an $U(1)$ subgroup of the $SU(2)$ automorphism group of the $N = 2$ supersymmetry algebra is gauged [69]. The gauging which breaks $SU(2)$ down to $U(1)$ is realized by making the gravitino and gaugino charged so that their covariant derivatives are

$$\begin{aligned} (D_\mu \lambda^i)^\alpha &= \nabla_\mu \lambda^{i\alpha} + g V_I A_\mu{}^I \delta^{\alpha\beta} \lambda^i{}_\beta & (5.47) \\ (D_\mu \psi_\nu)^\alpha &= \nabla_\mu \psi_\nu^\alpha + g V_I A_\mu{}^I \delta^{\alpha\beta} \psi_{\nu\beta} \,, \qquad \alpha, \beta = 1, 2. & (5.48) \end{aligned}$$

To maintain the $N = 2$ supersymmetry intact after the gauging, one has to add to the theory the gauged central charge potential proportional to $g^2 P$, where

$$P = Z^2 - \frac{3}{4} g^{ij} \partial_i Z \partial_j Z. \tag{5.49}$$

In the context of gauged supergravity the central charge is actually a moduli dependent combination of gravitino and gaugino charges defined by

$$Z = t^I V_I \tag{5.50}$$

where V_I is the charge defining the gravitino-gravitino-vector and gaugino-gaugino-vector interactions as may be deduced from their respective covariant derivatives. In addition to the potential, there are gravitino and gaugino mass terms proportional to

$$\begin{aligned} L^{\text{grav}}_{\text{mass}} &= -i\bar{\psi}^{\alpha}_{\mu}\Gamma^{\mu\nu}\psi^{\beta}_{\nu}\delta_{\alpha\beta}Z & (5.51)\\ L^{\text{gauge}}_{\text{mass}} &= -i\bar{\lambda}^{\alpha i}\lambda^{\beta j}\delta_{\alpha\beta}\left(g_{ij}Z + 4\sqrt{2}T_{ijk}Z^{,k}\right) & (5.52)\end{aligned}$$

and a mixing term proportional to the derivative of the central charge $\bar{\lambda}^{\alpha i}\Gamma^{\nu}\psi^{\beta}_{\nu}\delta_{\alpha\beta}Z_{,i}$. Thus in this theory the masses of gravitino and gaugino coincide with the BPS mass in the generic point of the moduli space up to terms proportional to the derivative of the central charge in the moduli space. This observation implies that at supersymmetric critical points where $Z_{,i} = 0$, one finds

$$M^{\text{grav}} = M^{\text{gauge}} = |(t^I)_{\text{cr}}V_I| = |Z_{\text{cr}}(V_I, C_{IJK})| \qquad \text{at} \qquad Z_{,i} = 0. \tag{5.53}$$

It is interesting to note that, as a function of gravitino charges and topological intersection numbers, the gravitino mass is given by the same topological formula which defines the entropy of the extreme supersymmetric black holes as function of black hole charges. The critical points of the P are given by

$$\partial_i P = Z\,\partial_i Z + \sqrt{3/2}\,T_{ijk}\partial^j Z\partial^k Z = 0. \tag{5.54}$$

As in previous cases it is straightforward to study supersymmetric critical point where the second derivative of the potential is proportional to the metric on the moduli space:

$$(P)_{,i;j} = \partial_i\partial_i P = \frac{2}{3}g_{ij}(Z^2)_{\text{cr}} \qquad \text{at} \qquad \partial_i P = \partial_i Z = 0. \tag{5.55}$$

Within the validity range of very special geometry, this potential has a minimum whenever the stabilization equation has a solution. In the case of potentials related to Jordan algebras all critical points of P have been classified in [69], where two types of critical points were described. It was found that the first derivative of the potential vanishes when the configuration of charges is such that either the derivative of the central charge is vanishing in our formalism or the potential is vanishing

$$\partial_i P \sim \partial_i Z\, P = 0 \qquad \text{at} \qquad \partial_i Z = 0 \quad \text{or} \qquad P = 0. \tag{5.56}$$

In the first case, the potential is non-zero at the minimum giving rise to a cosmological constant. This critical point is associated with supersymmetric black holes and extremization of their mass. The second case with the identically vanishing potential (cosmological constant) is specific to the theories associated with irreducible idempotents of the Jordan algebra.

As explained in section 2, M-theory compactified on a CY_3 manifold typically does not give rise to a factorizable prepotential nor corresponds to one of the magic square supergravity theories. Properties of the supersymmetric critical points of V and P can be studied using eqs. (5.44,5.55). However, critical points of any other nature are neither known nor excluded for the two potentials in case generic CY_3 compactifications. We will see however in section 6 by analyzing specific examples that generically there exist additional critical points.

To become familiar with the various kinds of potentials discussed so far we will examine a simple example defined by $\mathcal{V} = STU$. Although this prepotential does not directly correspond to a Calabi-Yau compactification it is useful enough to illustrate some essential ideas.

5.4.3 The $\mathcal{V} = STU = 1$ Example

This example corresponds to a factorizable case, hence, the full Jordan algebra apparatus applies and in fact this prepotential has been studied in detail in [69]. The solutions to the stabilization equations are given by

$$S = \left(\frac{q_T q_U}{q_S^2}\right)^{\frac{1}{3}}$$

$$
\begin{aligned}
T &= (\frac{q_U q_S}{q_T^2})^{\frac{1}{3}} \\
U &= (\frac{q_S q_T}{q_U^2})^{\frac{1}{3}}
\end{aligned}
\tag{5.57}
$$

and lead to the central charge

$$
Z = 3(q_S q_T q_U)^{\frac{1}{3}}. \tag{5.58}
$$

At this point we have not explicitly required positivity of the moduli fields. The result (5.57) shows that we find exactly one critical point of the central charge for a given set of charges. However, only if all of them are positive corresponding to no anti-brane configurations, we find the stabilization of the central charge for positive values of the moduli fields.

To understand the structure of the moduli space it is also instructive to analyze the moduli space metric g_{ij} and the gauge couplings G_{IJ}. Taking S as the dependent field, we find

$$
\begin{aligned}
g_{(TU)} &= \begin{pmatrix} \frac{1}{T^2} & \frac{1}{2TU} \\ \frac{1}{2TU} & \frac{1}{U^2} \end{pmatrix} \\
G &= \frac{1}{2 \times 6^{\frac{2}{3}}} \begin{pmatrix} \frac{1}{T^2} & 0 & 0 \\ 0 & \frac{1}{U^2} & 0 \\ 0 & 0 & T^2U^2 \end{pmatrix}.
\end{aligned}
\tag{5.59}
$$

Clearly, whenever either of the boundaries $T = 0$ or $U = 0$ is approached the corresponding matrix elements in g and G diverge. This implies that the boundaries are infinitely far away from a generic point in the moduli space. Also, it is to be noted that the vector field F_S becomes strongly coupled at the boundaries.

The potentials are easily calculated. We find

$$
\begin{aligned}
Z^2 &= (\frac{q_S}{TU} + q_T T + q_U U)^2 \\
V &= 3\left((\frac{q_S}{TU})^2 + (q_T T)^2 + (q_U U)^2\right) \\
P &= 3(TU q_T q_U + \frac{q_S q_U}{T} + \frac{q_S q_T}{U}).
\end{aligned}
\tag{5.60}
$$

As discussed earlier the stabilization equations provide only critical points of Z (and Z^2) which in turn are also critical points of V and P. However, already in this example we find new stabilization points of V. To see this consider for example the case of $q_T, q_U < 0, q_S > 0$. The result of (5.57) indicates the existence of an extremum only in the $U, T < 0$ domain. On the other hand, since in V the charges arise in quadratic form only, we do find an extremum also in the positive domain of U and T as well. Hence, critical points of the potential can also occur when $\partial_i Z \neq 0$. This behavior is visualized in Figure 3. We plot contour graphs for Z^2, V and P with charge configurations $(q_S, q_T, q_U) = (1,1,1), (1,-1,1), (1,-1,-1)$ in the neighborhood of $(T, U) = (1,1)$. Indeed, the central charge and P stabilize for positive charges only, whereas V has always a minimum. Hence, we see that the behavior of Z and V can be quite different from those of supersymmetric extreme black holes.

The gauged central charge potential P does admit also additional critical points for special charge configurations. If only one of the three charges is non-vanishing, Z and V do not have any critical points at all. However, P is *identically* zero in this case. This has been already observed in [69].

5.5 Boundaries of the Kähler and Extended Kähler Cones

In the previous sections we have concentrated on a particular version of $d = 5, N = 2$ supergravity with a given set of intersection numbers and have discussed critical points of the BPS mass, V and P respectively. Special geometry by itself did not require the moduli fields necessarily to be positive, as long as the gauge couplings and the moduli space metric are regular and positive definite. However, since these moduli correspond to sizes of 2-cycles in the Calabi-Yau three-fold they must be positive, thus defining the Kähler cone. If one of the moduli approaches zero one of the following three scenarios takes place [65]:

(1) a complex curve E (2-cycle) collapses as one approaches the boundary. This results in a topology change via a flop transition between two different but birationally

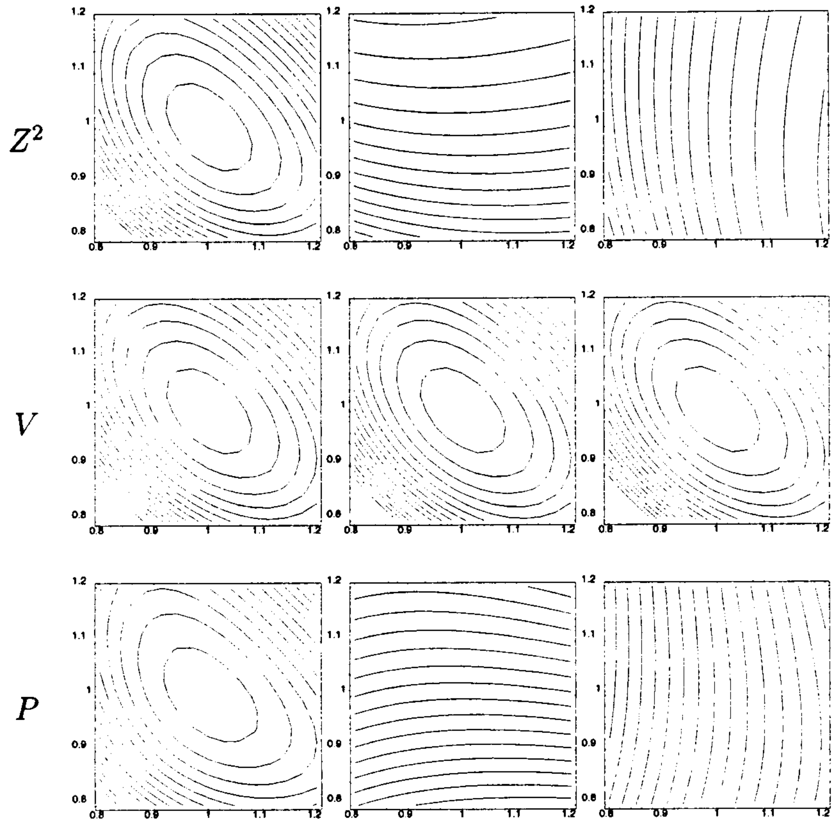


Figure 3: Contour plots of Z^2, V and P for charge configurations $(q_S, q_T, q_U) = (1,1,1), (1,-1,1), (1,-1,-1)$

equivalent CY_3 manifolds.

(2) a complex divisor D may collapse (a) to a curve, or (b) to a point. We first recapitulate some essential features of the flop transition. Detailed discussions can be found in [65, 67]. We start with a cubic surface defining an effective $d = 5$ supergravity before the phase transition

$$\mathcal{V} = C_{IJK} t^I t^J t^K \tag{5.61}$$

where all t^I are positive. Let the $t^2 = 0$ boundary of the Kähler cone constitute a flop transition. Generically, the gauge couplings G_{IJ} and the moduli space metric g_{ij} are regular and non-degenerate at the flop transition line.

A phase transition from a positive to a negative t^2 value has the following geometrical interpretation: a 2-cycle shrinks to a zero size and is subsequently blown up into a topologically distinct manner. The size of the new 2-cycle is given by $t'^2 = -t^2$. Mathematically, the effect is that the topological intersection numbers change by a new term $C_{222} = -\frac{1}{6}$ [67]. In [65] a physical interpretation of this topology change via one-loop renormalization was given. At the $t^2 = 0$ transition point, BPS states in a massive hypermultiplet with charge $q = (0, \pm 1, 0.., 0)$ and mass $m = t^2$ becomes massless. The fermions in this multiplet contribute to the parity-violating one-loop diagram involving three A^2 photon vertices. This one-loop diagram gives rise to an induced Chern-Simons interaction whose coupling is governed by the intersection number C_{222}. Across the transition the mass parameter m becomes formally negative. The induced Chern-Simons coupling is proportional to the sign change of these mass parameter. Once the numerical factors and multiplicity factor two are properly taken into account, it turns out that the parity-violating one-loop Feynman diagram induces a change of the topological intersection numbers of $\Delta C_{222} = -1/6$. By a supersymmetry non-renormalization theorem, there is no further perturbative nor non-perturbative renormalization to this result beyond one-loop. The prepotential of the new phase, when expressed in variables of the original CY space, is given by

$$\hat{\mathcal{V}} = C_{IJK} t^I t^J t^K - \frac{1}{6} t^2 t^2 t^2 \ . \tag{5.62}$$

In this form t^2 is negative, hence, while satisfactory for the original effective supergravity, this description is not satisfactory for the flopped CY manifold. We need to switch to different coordinates so that the new Kähler cone past the phase transition is well described. After a change of variables, hence, a change of basis for basic cycles in the new phase, of the type

$$t^2 = -t'^2 \qquad t^1 = t'^1 + t^2 \quad etc. \tag{5.63}$$

one finds that the new prepotential

$$\mathcal{V}' = C'_{IJK} t'^I t'^J t'^K \tag{5.64}$$

describes the new Kähler cone for the flopped but birationally equivalent CY_3 space where all intersection numbers C'_{IJK} and all moduli t'^I are positive.

The extended Kähler cone consists of the union of Kähler cones related by flop transitions. This enlarged region has boundaries [65] at which divisors shrink to a complex curve (2a) or a point (2b). In both cases the moduli space simply ends as opposed to four dimensional cases, where non-geometric conformal field theory phases exist beyond the extended Kähler cone.

In case (2a) the tension of a BPS string is proportional to the magnetic central charge Z_m and behaves as

$$Z_m \sim (t^k)^1 \tag{5.65}$$

where t^k denotes the shrinking 2-cycle. Associated with a shrinking 2-cycle is an A_1 singularity. Therefore, as approaching the type (2a) boundary of the extended Kähler cone, we expect a $SU(2)$ gauge symmetry enhancement.

The physics at the (2b) transition is more involved. Here, the BPS strings become tensionless with

$$Z_m \sim (t^k)^2. \tag{5.66}$$

This shows that excitations of this magnetic string become massless at the same rate as electric point-particle BPS states. Extremal transitions are possible here if the singularity left by the collapsing divisor can be removed by the appearance of new complex structure deformations. As in the $d = 4$ case, Kähler deformations can be traded for complex structure deformations.

We will not attempt to give a more detailed discussion of the general framework here. We are mainly interested in the behavior of various physical quantities across the boundary of flop phase transitions.

5.5.1 Physical Quantities at the Flop Transition

The natural question to address is how the electric central charge Z, the magnetic string tension Z_m and the other potentials are affected by these phase transitions, particularly by the flop phase transition. We will study the analyticity properties of various physical quantities at this transition line. Quantities of particular interest are

1. the electric central charge $Z = t^I q_I$ for particle states,
2. the gauge couplings G_{IJ},
3. the black hole potential $V = \frac{6^{\frac{1}{3}}}{4} G^{IJ} q_I q_J$, and
4. the magnetic central charge $Z_m = t_I m^I$ which determines the tension of BPS string states.

For this purpose we describe the system before and after the transition using the same set of special coordinates t^I. At the transition, one of the independent special coordinates (for concreteness t^2) changes its sign. Let the prepotential for positive t^2 be given by

$$\mathcal{V}(t^1, \ldots, t^{n-1}, t^n) \tag{5.67}$$

where t^n is the function of $t^1, ..., t^{n-1}$ defined by $\mathcal{V} = 1$. Then the change in topology induces a change in the prepotential for the region of negative t^2, corresponding to the induced change in the Chern-Simons coupling C_{222}. The prepotential in this Kähler cone is given by

$$\hat{\mathcal{V}} = \mathcal{V}(t^1, \ldots, t^{n-1}, \hat{t^n}) - \frac{1}{6} t^2 t^2 t^2. \tag{5.68}$$

The notation $\hat{t}^n$ is to emphasize the fact that the new term in the prepotential will change the functional form of the dependent variable as a function of the independent ones.

To understand the behavior of various potentials we first need to work out how t^n is affected by the transition, and how the charges in both CY spaces have to be identified. Let us first consider t^n. Close to the transition, the change enforced by the constraints $\mathcal{V} = 1$ and $\hat{\mathcal{V}} = 1$ in t^n is easily found from (5.67) and (5.68) to be

$$\Delta t^n = \hat{t}^n - t^n \sim -\frac{1}{6}(t^2)^3 \left(\frac{\partial \mathcal{V}}{\partial t^n}\right)^{-1} \qquad \text{as } t^2 \to 0. \tag{5.69}$$

The prepotential $\mathcal{V}$ is given by a function which may have a cubic, quadratic, linear and constant dependence on t^n. Therefore in generic cases the function $\left(\frac{\partial \mathcal{V}}{\partial t^n}\right)^{-1}$ is regular near the boundary. The case that the dependence on t^n enters only via $t^n t^2 t^I$ or $t^n t^2 t^2$ is excluded since this would lead to $G_{nI} \to 0$ and divergent gauge couplings at the phase transition. This would not constitute a flop transition. Thus, we have established the fact that the dependent variable is affected only very mildly, as it behaves as

$$\Delta t^n = \hat{t}^n - t^n \sim (t^2)^3 \to 0\,, \qquad \text{as } t^2 \to 0. \tag{5.70}$$

The question of the identification of the charges in both Calabi-Yau manifolds also finds an easy answer. As the charges $q_1, q_3, .., q_n$ correspond to membrane wrappings around two-cycles which do not shrink to zero at $t^2 = 0$, they clearly cannot change under the transition. Slightly more subtle is the issue of q_2. We will adopt the supergravity point of view [65] and note that the topology change gives rise to a change to the one-loop quantum effect of massive states. Besides the discrete shift of the Chern-Simons coupling the transition is perfectly smooth. We thus conclude that q_2 is unaffected as well by the flop if t^2 is taken to be negative after the transition. If one adopts the natural Calabi-Yau variables in this flopped side also, implying

$$t^2 \to \hat{t}^2 = -t^2 \tag{5.71}$$

then q_2 also changes its sign correspondingly[11]. However, we stress that in the variables of the original Calabi-Yau space the charges should be unchanged.

With these preliminaries it is straightforward to determine the analyticity properties of Z, as the central charge $\sum_1^n(t^I q_I)$ is affected by the topology change only via the induced change in t^n. Before and after the transition we have

$$Z = \sum_1^{n-1}(t^I q_I) + t^n(t^1, \dots t^{n-1})q_n \qquad \hat{Z} = \sum_1^{n-1}(t^I q_I) + \hat{t}^n(t^1, \dots t^{n-1})q_n \tag{5.72}$$

and therefore

$$\Delta Z = \hat{Z} - Z = (\hat{t}^n - t^n)q_n \sim (t^2)^3 q_n \to 0 \ , \qquad \text{as } t^2 \to 0. \tag{5.73}$$

Thus the electric central charge for a generic set of charges is continuous across the phase transition. This is also the case for the first and second derivatives over t^2. The third derivative of Z is discontinuous in general. This point is of immense importance, as it implies that there is *at most* one supersymmetric stabilization point throughout the entire extended Kähler cone! As Z and its first derivative are continuous at flop transitions, additional extrema would require at least one maximum or saddle point. However, special geometry is valid in any section of the extended cone, hence any extrema must be a minimum. This proves *uniqueness* of the supersymmetric stabilization throughout the entire extended Kähler cone.

This important observation bears the following implications. If a charge configuration is realized by a black hole in space-time the moduli fields at the horizon will take the values of the unique critical point. However, their asymptotic values can be taken arbitrarily, in particular, they can be chosen to the values of a different Kähler cone from the Kähler cone in which the near-horizon values are taken. It is clear that then the Calabi-Yau space at infinity is topologically different from the space close to the black hole. In this sense, the $d = 5$ BPS black hole induces an interpolation of CY spaces of different topology. Surrounding the black hole is a surface of codimension one at which the the topology of the internal CY space changes, and which has

[11] Generically there are more redefinitions necessary which also affect the charges.

additional massless states localized on it. This surface around the BPS black hole constitutes a $d = 4$ domain wall world with distinct massless physical spectrum.

We now turn to the gauge couplings G_{IJ}. They are defined by the second derivative of the prepotential $G_{IJ} = -\frac{1}{2\cdot 6^{\frac{2}{3}}}\partial_I\partial_J \ln \mathcal{V}|_{\mathcal{V}=1}$. To lowest order in t^2 they change at the transition line as

$$\Delta G_{IJ} = \frac{1}{2\cdot 6^{\frac{2}{3}}}\delta_I^2\delta_J^2 t^2 \to 0 \ , \qquad \text{as } \ t^2 \to 0. \tag{5.74}$$

The first derivative of G_{22} is discontinuous since

$$\partial_{t^2}\hat{G}_{IJ} - \partial_{t^2}G_{IJ} = \frac{1}{2\cdot 6^{\frac{2}{3}}}\delta_I^2\delta_J^2 \ , \qquad \text{as } t^2 \to 0. \tag{5.75}$$

The continuity and analyticity of the black hole potential $V = \frac{6^{\frac{1}{3}}}{4}q_I G^{IJ} q_J$ is determined by the properties of the gauge couplings. With (5.74) we find

$$\hat{V} - V = \frac{6^{\frac{1}{3}}}{4}q_I(\hat{G}^{IJ} - G^{IJ})q_J = -\frac{1}{8\cdot 6^{\frac{1}{3}}}t^2\,(q_I G^{I2})^2 \to 0 \ , \qquad t^2 \to 0. \tag{5.76}$$

Note that V always has an upward concave kink around the transition, since

$$\partial_{t^2}\hat{V} - \partial_{t^2}V = -\frac{1}{8\cdot 6^{\frac{1}{3}}}(q_I G^{I2})^2 \leq 0 \ , \qquad \text{as } t^2 \to 0. \tag{5.77}$$

This observation might turn out to be crucial for proving uniqueness of non-supersymmetric stabilization as well. Unfortunately, the nature of non-supersymmetric extrema is not quite understood yet. On the other hand, if the uniqueness of the critical point of V can be shown, then the immediate conclusion is that the black hole entropy is not affected by the topology change to a birationally equivalent CY_3 at infinity. This is also what one would expect. Clearly, the gauged central charge potential P has the same analyticity as the black-hole potential V, as it is a linear combination of Z and V.

Finally, it is very easy to study the analyticity properties of the BPS string tension

$$Z_m = t_I m^I. \tag{5.78}$$

As we also do not expect the string charges to be affected by the flop transition, any discontinuity in Z_m can only enter via the t_I which are given by

$$t_I = 4 \times 6^{1/3} G_{IJ} t^J. \tag{5.79}$$

Hence we find with (5.74) and (5.70)

$$\Delta Z_m \sim m^2 (t^2)^2. \tag{5.80}$$

In conclusion, the BPS tension and its first derivative are continuous across the flop transition, the second derivative generically is not. Nevertheless, the smoothness is sufficient to show the uniqueness also of the stabilization point of Z_m in the extended Kähler cone.

5.6 Examples of Calabi-Yau Compactifications

We now give an explicit example for the discussions above. We study an extended Kähler cone (displayed in Figure 4) consisting of two elliptically fibered CY_3 spaces with base F_1 but with different triangulations of the toric diagram. The first F_1 vacuum is denoted as model III and contains three Kähler moduli. Its second Chern class is characterized by

$$c_2(J_I) = \int c_2 \wedge J_I = (92, 36, 24). \tag{5.81}$$

In this space a flop can be performed to the other triangulation of the base F_1 (model II), leading to different intersection numbers and

$$c_2(J_I) = (92, 102, 36). \tag{5.82}$$

Here, a complex divisor can be shrunk to zero size and the resulting singularity can be resolved by blowing up 29 three-cycles leading to an elliptic fibration with base P_2, model I. This space contains only two Kähler moduli and has

$$c_2(J_I) = (102, 36). \tag{5.83}$$

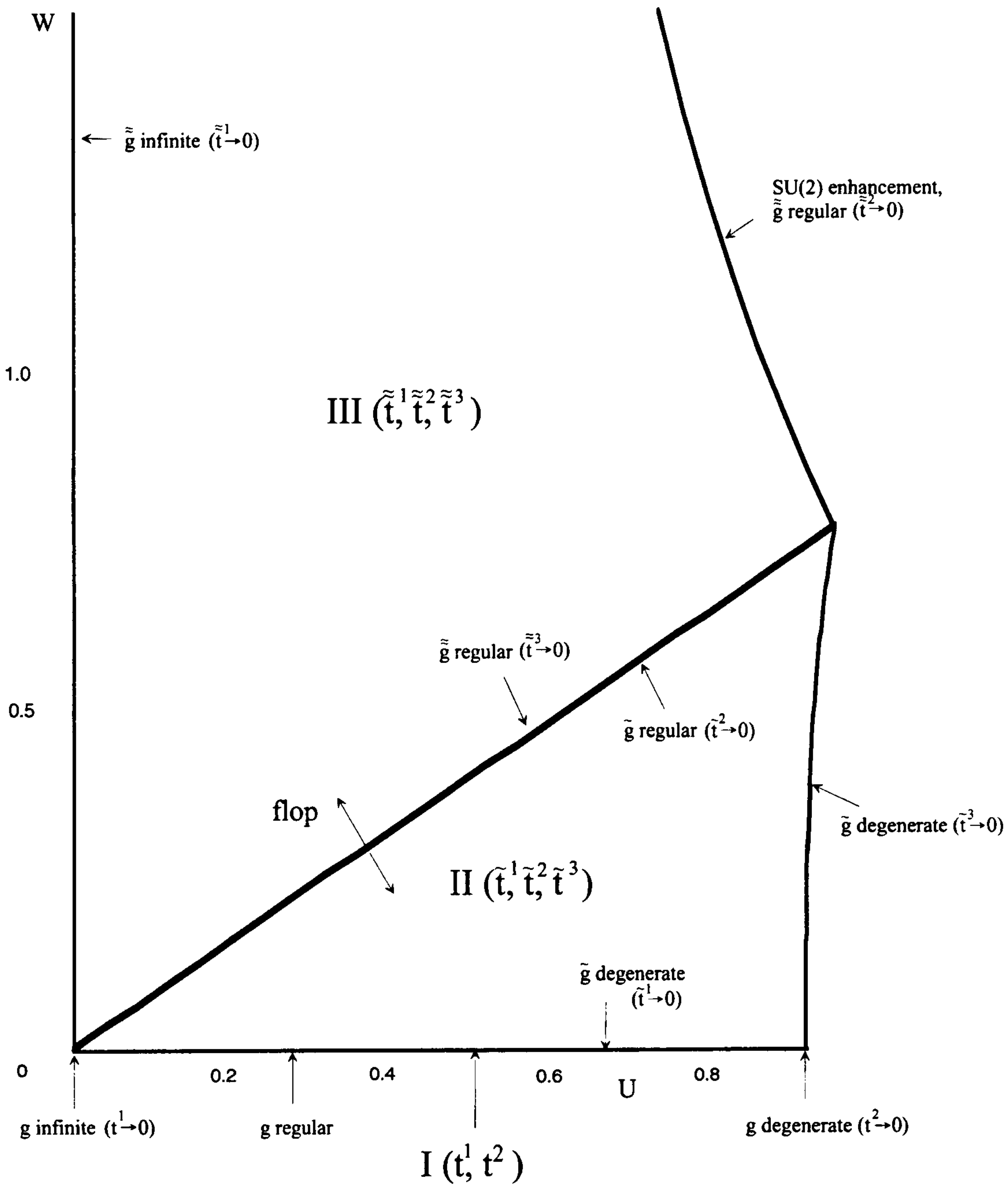


Figure 4: The extended moduli space

These CY_3 manifolds and their interrelation have been discussed extensively in [80, 81, 82].

We will explicitly study the stabilization of the central charge, Z, the black hole potential V and the gauged central charge potential P in all regions of the extended Kähler cone and analyze how various potentials behave under the flop transition. The main results of our study are as follows. We find new critical points for the gauged central charge P which were out of reach of the discussion in [69]. These extrema can be saddle points or maxima, as opposed to the critical points which coincide with the extrema of Z, as those are all minima. The supersymmetric stabilization equations are explicitly solved in all regions of the moduli space and we will graphically demonstrate the smoothness of Z and the kink in V along the flop transition. We also study the tension of the BPS strings at the boundaries of each Kähler cone and find that our results are in agreement with the general considerations of [65].

For convenience we start the discussion with the P_2 vacuum, then blow up a four-cycle and finally consider the flopped vacuum.

5.6.1 Model I: the Base P_2 Vacuum

This Calabi-Yau compactification corresponds to the lower boundary of Figure 4. It is a rather simple example as it admits only one independent moduli field. Yet, it incorporates rich and surprising structures.

Its prepotential is given by

$$\mathcal{V}_I = 9(t^1)^3 + 9(t^1)^2 t^2 + 3t^1(t^2)^2 = 1. \tag{5.84}$$

To achieve a consistent notation for all three connected models we perform a coordinate redefinition to variables U and T, related to the basic cycles t^1 and t^2 by

$$\begin{aligned} U &= 6^{\frac{1}{3}} t^1 \\ T &= 6^{\frac{1}{3}} (t^2 + \frac{3}{2} t^1). \end{aligned} \tag{5.85}$$

The constraint on the moduli fields translates to

$$\mathcal{V}_I = \frac{3}{8}U^3 + \frac{1}{2}UT^2 = 1. \tag{5.86}$$

For convenience we choose U as an independent variable. In this case, T is given by

$$T = \sqrt{\frac{2}{U}(1 - \frac{3}{8}U^3)}. \tag{5.87}$$

Note that it is the basic cycles t^1, t^2 of the Calabi-Yau manifold and not the T, U variables which should be restricted to be positive.

To understand the structure of the moduli space one has to compute the metric which in this case is a scalar. It is determined to be

$$g = \frac{6 - 9U^3}{8U^2 - 3U^5}. \tag{5.88}$$

Important are also the gauge couplings. We do not give the explicit form of G_{IJ} here but note that its determinant is given by

$$\det G_{IJ} = \frac{1}{24 \times 6^{1/3} U}(1 - \frac{3}{2}U^3). \tag{5.89}$$

In the region $0 \leq t^1, t^2$, g and $\det G$ are regular as expected. They diverge at $t^1 \sim U = 0$ and degenerate at

$$U_0 = (\frac{2}{3})^{1/3}. \tag{5.90}$$

This is also the point where t^2 turns negative. Hence, $0 \leq U \leq U_0$ defines the domain of validity. The regularity of T in this region and on the boundary $t^2 = 0$ is quite crucial as it implies that Z and ∂Z are regular here also.

These considerations allow us to predict some properties of the potentials. First of all, for a range of charge configurations (q_U, q_T) we do expect a critical point of the central charge in the regular domain which in turn is a minimum of the potentials V and P.

Since T diverges at $U \to 0$ all three potentials diverge to positive infinity at this boundary of the moduli space. On the other side of the domain of validity, Z and ∂Z

are regular but $g^{-1} \to \infty$, hence V generically diverges to positive infinity whereas P turns negative towards infinity. This implies a second new critical point for potentials of gauged supergravities[12]! In fact, this extrema must be a maximum of P (implying a minimum of the gauge potential, as we flipped the sign for convenience).

The second prediction we can make is that V *always* admits a minimum, as on both boundaries it diverges to positive infinity.

Let us verify our general considerations. First we solve the stabilization equations, leading to the critical points of the central charge. The charges are denoted by (q_U, q_T) where we take $q_U \geq 0$ without loss of generality, since $(q_U, q_W) \to -(q_U, q_W)$ leaves the potentials invariant. In the domain of validity we find the solution

$$\begin{aligned} U_{\text{crit}} &= \frac{2}{\sqrt{3Z^{\text{crit}}}}\sqrt{q_U - \sqrt{q_U^2 - \frac{9}{4}q_T^2}} \\ T_{\text{crit}} &= \sqrt{\frac{3}{Z^{\text{crit}}}}\sqrt{q_U + \sqrt{q_U^2 - \frac{9}{4}q_T^2}} \end{aligned} \tag{5.91}$$

with the central charge

$$Z^{\text{crit}} = \left[q_U \frac{2}{\sqrt{3}}\sqrt{q_U - \sqrt{q_U^2 - \frac{9}{4}q_T^2}} + q_T\sqrt{3q_U + 3\sqrt{q_U^2 - \frac{9}{4}q_T^2}}\right]^{2/3}. \tag{5.92}$$

The charge configurations allowing for a stabilization of the central charge are constrained by

$$q_U \geq \frac{3}{2}q_T \geq 0. \tag{5.93}$$

As anticipated, in the domain $g > 0$ ($t^1, t^2 > 0$) we find precisely one extremum of the central charge (5.93). This should coincide with the extremum of the potentials according to our considerations of the last section. We will not give the explicit form of V and P here as functions of U but rather consider the specific charge configurations $q_U = 4, q_T = 2$ and $q_U = 2, q_T = 2$. The results are plotted in Figure 5. This can be done without much loss of generality as the general structure of the potentials is

[12]It is well known that the stabilization issue of critical points in gauged supergravities is rather subtle. The stabilization of this new critical point remains to be analyzed.

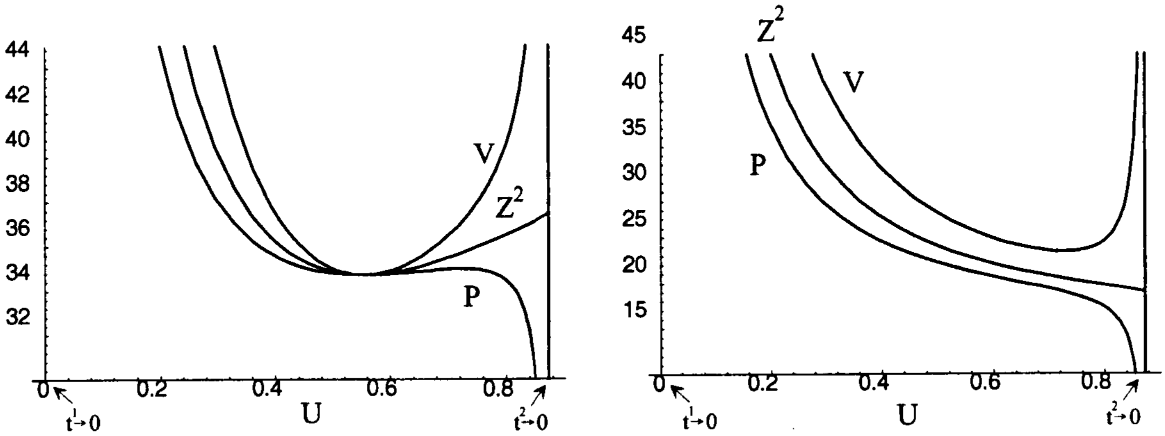


Figure 5: Z^2, V and P for $(q_U, q_W) = (4, 2)$ (left) and $(q_U, q_T) = (2, 2)$ (right)

rather insensitive to a specific choice of charges, as long as we avoid the boundary values $q_T = \frac{2}{3}q_U$ and $q_T = 0$.

For the first set of charges the picture is as expected. The minima of Z^2, P and V coincide exactly at the position predicted by (5.91). However, P indeed admits a new maximum which does not correspond to critical points of the central charge.

The situation is very different for the configuration $(q_U, q_T) = (2, 2)$. The central charge does not get stabilized anymore, also P does not admit critical points. However, V still has a minimum in agreement with our discussion above. This minimum corresponds to the stabilized values of the moduli in the non-supersymmetric extreme black hole solution with the same charges. In fact one finds that for *all* charge configurations stabilization of the scalar potential takes place!

There are also the two boundary points:

$$q_U = \frac{3}{2}q_T \quad \text{and} \quad q_T = 0. \tag{5.94}$$

Here, we find stabilization at the boundaries (i.e. on the singularities) and the potentials become finite there. An example is given in Figure 6, where the charge configuration $(q_U, q_T) = (3, 2)$ is shown to admit stabilization at the point $U = U_0$, i.e. at $t^2 = 0$. This is rather peculiar as it implies that stabilization takes place right on the edge of a Calabi-Yau space. Stabilization at $t^1 = 0$ occurs for $q_T = 0$.

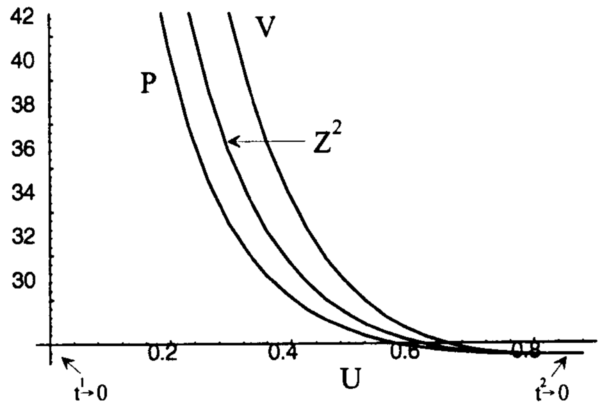


Figure 6: Z^2, V, P for $q_U = 3,\ q_T = 2$

To summarize the discussion of V we emphasize again that in this example for *all* charge configurations V has a minimum in the allowed region of moduli fields (or on its boundaries), as opposed to Z and P, where extrema are found only when (5.93) is satisfied.

It is known (see for example [80]) that in this Calabi-Yau manifold 29 complex structure moduli can be shrunk to zero size, and the resulting singularity can be resolved by blowing up a four-cycle. In this manner we obtain a new Calabi-Yau manifold with one more Kähler moduli field, which will be discussed now.

5.6.2 Model II: the Base F_1 Vacuum

This space (model II of Figure 4) is more complex than the previous one. Nevertheless, the general structure of critical points is similar to what we have learned above.

If a complex divisor is blown up in the P_2 space, resulting in a new two-cycle W, it turns out that the new prepotential is given by [80]

$$\mathcal{V}_{II} = \frac{3}{8}U^3 + \frac{1}{2}UT^2 - \frac{1}{6}W^3. \tag{5.95}$$

Clearly, in the limit $W \to 0$ one recovers (5.86).

To find the boundary of this Kähler cone a transformation to the basic cycles

$$\tilde{t}^1 = W$$

$$
\begin{aligned}
\tilde{t}^2 &= U - W \\
\tilde{t}^3 &= T - \frac{3}{2}U
\end{aligned}
\tag{5.96}
$$

has to be performed in terms of which the prepotential becomes

$$
\begin{aligned}
\mathcal{V}_{II} &= \frac{4}{3}(\tilde{t}^1)^3 + \frac{3}{2}(\tilde{t}^2)^3 + \frac{9}{2}(\tilde{t}^1)^2\tilde{t}^2 + \frac{9}{2}\tilde{t}^1(\tilde{t}^2)^2 + \frac{3}{2}(\tilde{t}^1)^2\tilde{t}^3 \\
&+ \frac{3}{2}(\tilde{t}^2)^2\tilde{t}^3 + \frac{1}{2}\tilde{t}^1(\tilde{t}^3)^2 + \frac{1}{2}\tilde{t}^2(\tilde{t}^3)^2 + 3\tilde{t}^1\tilde{t}^2\tilde{t}^3,
\end{aligned}
\tag{5.97}
$$

and the domain of this Kähler cone is then defined by $\tilde{t}^I \geq 0$.

For the discussion of the critical points the variables (U, T, W) are more convenient. If we choose U and W as independent variables, the constraint $\mathcal{V}_{II} = 1$ leads to

$$
T = \sqrt{\frac{24 - 9U^3 + 4W^3}{12U}}
\tag{5.98}
$$

and the determinant of the moduli space metric is given by

$$
\det \tilde{g}_{ij} = \frac{3W(6 - 9U^3 + W^3)}{2U^2(24 - 9U^3 + 4W^3)}.
\tag{5.99}
$$

The determinant of the gauge couplings is found to be

$$
\det \tilde{G}_{IJ} = \frac{W}{288U} - \frac{U^2W}{192} + \frac{W^4}{1728U}.
\tag{5.100}
$$

The boundary of the Kähler cone is reached by setting one (or more) of the fundamental cycles to zero. At these points one can have a flop transition or reaches the end of the extended Kähler cone. At $\tilde{t}^1 = W = 0$ (lower boundary in Fig. 4) we find $\det \tilde{g}_{ij} = 0$ and we recover model I as mentioned before. The moduli space metric also degenerates at the boundary $\tilde{t}^3 = T - \frac{3}{2}U = 0$ (right boundary in Fig. 4). We note that some gauge fields become strongly coupled at these boundaries.

$\tilde{t}^2 = U - W \to 0$ (upper left boundary) turns out to be a flop transition. Here, the metrics are completely regular and very special geometry is still valid if we pass this boundary. However, physically we enter the new Calabi-Yau space of model III with different intersection numbers.

We begin the discussion of critical points with a few general observations. If we choose a charge configuration which allows for a stabilization of the central charge then the arguments given for model I do apply again and we can deduce some of the properties of the potentials. Consider first P. At each of the two intersecting $\det \tilde{g} = 0$ lines (i.e. $\tilde{t}^1 = 0$ and $\tilde{t}^3 = 0$) P is expected to diverge to negative infinity. As the supersymmetric stabilization point must be a minimum, there ought to be three additional critical points: two saddle points and one maximum. Hence, we learn once more that the structure of critical points of gauged supergravities is rather rich. We also expect that V has new critical points for charge configurations which do not stabilize Z in the valid domain.

As for model I we verify our intuition by solving the stabilization equations and study the potentials with specific charge configurations. We find that the extremum of the central charge in the valid region is located at

$$\begin{aligned} U_{\text{crit}} &= \frac{2}{\sqrt{3Z^{\text{crit}}}}\sqrt{q_U - \sqrt{q_U^2 - \frac{9}{4}q_T^2}} \\ T_{\text{crit}} &= \sqrt{\frac{3}{Z^{\text{crit}}}}\sqrt{q_U + \sqrt{q_U^2 - \frac{9}{4}q_T^2}} \\ W_{\text{crit}} &= \sqrt{-\frac{6}{Z^{\text{crit}}}q_W} \end{aligned} \tag{5.101}$$

with the critical value of the central charge

$$Z^{\text{crit}} = \left(q_U\sqrt{\frac{4}{3}(q_U - \sqrt{q_U^2 - \frac{9}{4}q_T^2})} + q_T\sqrt{3(q_U + \sqrt{q_U^2 - \frac{9}{4}q_T^2})} + q_W\sqrt{-6q_W} \right)^{\frac{2}{3}} . \tag{5.102}$$

Here, $q_U > 0$ was chosen without loss of generality due to the $\mathbf{Z}_2$ $(q^I \to -q^I)$ symmetry of the potentials. Supersymmetric stabilization occurs only for the region

$$\begin{aligned} q_U &\geq \frac{3}{2}q_T \geq 0 \\ 0 &\geq q_W \geq -\frac{2}{9}\left(q_U - \sqrt{q_U^2 - \frac{9}{4}q_T^2}\right) . \end{aligned} \tag{5.103}$$

At the critical values (saturation of above inequalities) Z gets stabilized on the boundaries of the moduli space. To be explicit, we find for $q_U = \frac{3}{2} q_T$ stabilization at $\tilde{t}^3 = 0$ and for $q_W = 0$ at $\tilde{t}^1 = 0$. The latter observation is very much consistent with the expectation that absence of q_W drives the internal space in the vicinity of the black hole to the CY space with P_2 as a base.

If

$$q_W < -\frac{2}{9}\left(q_U - \sqrt{q_U^2 - \frac{9}{4}q_T^2}\right) \tag{5.104}$$

no stabilization occurs in the present Kähler cone. In fact, what we observe is that the critical point moves over the boundary $\tilde{t}^2 = U - W = 0$ into the adjacent Calabi-Yau space.

Let us now turn to the study of P via a specific example. As a typical case with supersymmetric stabilization we consider the charge configuration

$$q_U = 5, \quad q_T = 2, \quad q_W = -\frac{1}{6}, \tag{5.105}$$

as general properties of the potential are independent of a specific choice. Figure 7 shows beautifully the structure of P. This potential admits indeed four critical points as anticipated. The minimum agrees with the ones of V and Z^2, the two saddle points and the maximum are new. If q_W approaches 0, the pairs of critical points merge on the $W = 0$ line rendering the results of model I. Note that $\tilde{t}^2 = U - W$ is the only boundary where the moduli space metric is non-degenerate, hence stabilization points can smoothly "move" across this transition line.

If q_W becomes positive, the critical points of P and Z disappear, however, V still stabilizes, showing the same characteristics as in the last example. As long as q_W is below a certain value, the black hole potential stabilizes in the allowed domain of this Calabi-Yau. There is some critical value for which also the non-supersymmetric stabilization of V takes place in the adjoining Kähler cone in very much the same way as we already observed for Z.

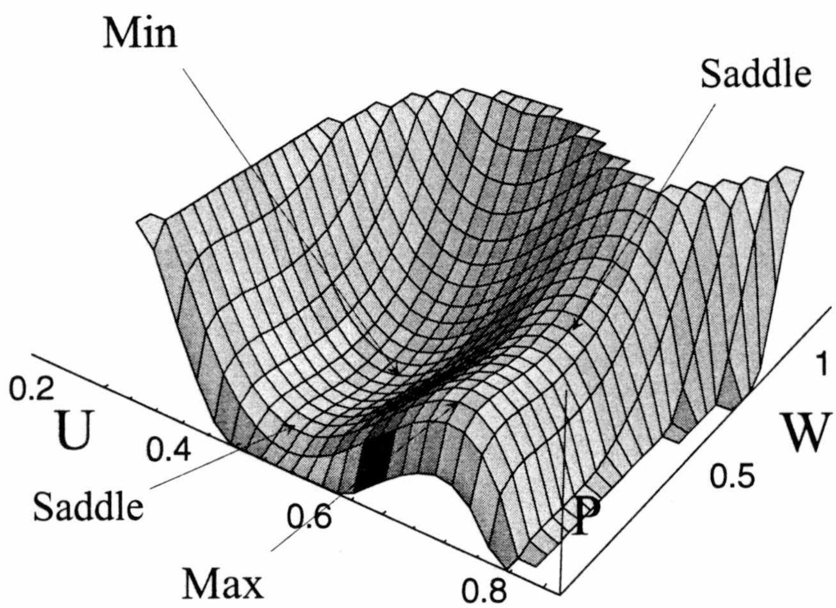


Figure 7: Potential P with 4 critical points

Let us also comment on the other two boundaries. In particular, we will consider the tension of solitonic BPS strings. The tension of these solitonic objects is given by

$$Z_m = \tilde{t}_I \tilde{m}^I \tag{5.106}$$

where the $\tilde{m}^I$ are the winding numbers of eleven-dimensional five-branes wrapping around the four-cycles $\tilde{t}_I$ defined in (5.13). At the boundary $\tilde{t}^1 \to 0$ (transition to model I) we find indeed tensionless strings consistent with the arguments given in [65]. This can be easily seen as the four-cycle moduli are given by

$$\begin{aligned}
\tilde{t}_1 &= \frac{4}{3}(\tilde{t}^1)^2 + 3\tilde{t}^1\tilde{t}^2 + \frac{3}{2}(\tilde{t}^2)^2 + \tilde{t}^1\tilde{t}^3 + \frac{1}{6}(\tilde{t}^3)^2 + \tilde{t}^2\tilde{t}^3 \\
\tilde{t}_2 &= \frac{3}{2}(\tilde{t}^1)^2 + 3\tilde{t}^1\tilde{t}^2 + \frac{3}{2}(\tilde{t}^2)^2 + \tilde{t}^1\tilde{t}^3 + \frac{1}{6}(\tilde{t}^3)^2 + \tilde{t}^2\tilde{t}^3 \\
\tilde{t}_3 &= \frac{1}{2}(\tilde{t}^1)^2 + \tilde{t}^1\tilde{t}^2 + \frac{1}{2}(\tilde{t}^2)^2 + \frac{1}{3}\tilde{t}^1\tilde{t}^3 + \frac{1}{3}\tilde{t}^2\tilde{t}^3.
\end{aligned} \tag{5.107}$$

If we consider the cycle $\tilde{t}_1 - \tilde{t}_2$ we find

$$\tilde{t}_1 - \tilde{t}_2 = -\frac{1}{6}(\tilde{t}^1)^2 \tag{5.108}$$

implying that the BPS string states with charges $\tilde{m} = \pm(1,-1,0)$ indeed become tensionless with $Z \sim (\tilde{t}^1)^2$! We argue that model I is reached via this mechanism.

The boundary $\tilde{t}^3 \to 0$ shows the same characteristics, the moduli space metric degenerates and the string states with charges[13]

$$\tilde{m} = \pm(0,1,-3) \tag{5.109}$$

become tensionless also with tension $\sim (\tilde{t}^3)^2$.

As expected, at the flop transition $\tilde{t}^2 = 0$ it is straightforward to verify that none of the string states becomes tensionless along the entire transition line, i.e. there is no linear combination of $\tilde{t}_I$ which vanishes here.

We will now study the flopped Calabi-Yau space in detail.

[13]As in five dimensions all moduli are real all states are bound states at threshold. Proving their existence is typically quite involved and we do not attempt to do so.

5.6.3 Model III: The Flopped F_1 Base

In model II $\tilde{t}^2 = U - W \to 0$ marks a topology change of the mild form. The Hodge numbers are unaffected, but the intersection numbers C_{IJK} change [67, 65]. These topological data changes lead to the new prepotential augmented by the term $-\frac{1}{6}(U-W)^3$

$$\mathcal{V}_{III} = \frac{5}{24}U^3 + \frac{1}{2}U^2W - \frac{1}{2}UW^2 + \frac{1}{2}T^2U = 1. \tag{5.110}$$

As discussed in section 5.1 the charges q_U, q_W, q_T are in fact unchanged by this flop transition.

U, W and T are again not the proper Calabi-Yau variables, the correct parametrization to determine the boundary of this space is

$$\begin{aligned} \tilde{\tilde{t}}^1 &= U \\ \tilde{\tilde{t}}^2 &= T - \frac{1}{2}U - W \\ \tilde{\tilde{t}}^3 &= W - U \end{aligned} \tag{5.111}$$

and $\tilde{\tilde{t}}^I > 0$ defines this Kähler cone. In these variables the prepotential becomes

$$\mathcal{V}_{III} = \frac{4}{3}(\tilde{\tilde{t}}^1)^3 + \frac{3}{2}(\tilde{\tilde{t}}^1)^2\tilde{\tilde{t}}^2 + \frac{1}{2}\tilde{\tilde{t}}^1(\tilde{\tilde{t}}^2)^2 + (\tilde{\tilde{t}}^1)^2\tilde{\tilde{t}}^3 + \tilde{\tilde{t}}^1\tilde{\tilde{t}}^2\tilde{\tilde{t}}^3 = 1. \tag{5.112}$$

The potentials are again analyzed in (U, T, W) variables. Due to the constraint, T is now given by

$$T = \frac{6 - 8U^3 - 9U^2W - 3UW^2}{6U(U+W)}. \tag{5.113}$$

We find for the moduli space metric and gauge couplings

$$\det \tilde{\tilde{g}}_{ij} = \frac{3 - 4U^3}{4U^2(U+W)^2} \tag{5.114}$$

and

$$\det \tilde{\tilde{G}}_{IJ} = \frac{3 - 4U^3}{864}. \tag{5.115}$$

Hence, $\tilde{\tilde{g}}$ diverges at $\tilde{\tilde{t}}^1 \to 0$, (left boundary of Fig. 4), but is regular at $\tilde{\tilde{t}}^2 \to 0$ (right boundary) and $\tilde{\tilde{t}}^3 \to 0$ (lower boundary). Regularity at the latter transition is no surprise, as we have just pass back into the Calabi-Yau space of above via the flop. However, $\tilde{\tilde{t}}^2 \to 0$ marks the end of the moduli space in *finite* distance. We will later interpret this line as the transition where a divisor shrinks to a complex curve. Noteworthy is that, as all metrics are regular here, there is no protection that ensures stabilization *before* the boundary, not even for V.

As the behavior of the potentials is generically not very different from model II we will not give many details. Nevertheless it is useful to consider the supersymmetric stabilization points

$$\begin{aligned} U_{\text{crit}} &= \frac{1}{2Z_{\text{crit}}^{\frac{1}{2}}}\sqrt{6q_U + 3q_W - 3\sqrt{4q_U^2 + 4q_U q_W + 9q_W^2 - 8q_T^2}} \\ T_{\text{crit}} &= \frac{3q_T}{U} = \frac{6q_T}{Z_{\text{crit}}^{\frac{1}{2}}\sqrt{6q_U + 3q_W - 3\sqrt{4q_U^2 + 4q_U q_W + 9q_W^2 - 8q_T^2}}} \\ W_{\text{crit}} &= \frac{U^2 - 6q_W}{2U} = \frac{6q_U - 21q_W - 3\sqrt{4q_U^2 + 4q_U q_W + 9q_W^2 - 8q_T^2}}{4Z_{\text{crit}}^{\frac{1}{2}}\sqrt{6q_U + 3q_W - 3\sqrt{4q_U^2 + 4q_U q_W + 9q_W^2 - 8q_T^2}}}. \end{aligned} \tag{5.116}$$

With some algebra we find that supersymmetric stabilization takes place if

$$q_U \geq \frac{3}{2}q_T \geq 0 \tag{5.117}$$

and

$$q_W \leq -\frac{2}{9}\left(q_U - \sqrt{q_U^2 - \frac{9}{4}q_T^2}\right). \tag{5.118}$$

If (5.117) is saturated the critical point lies directly on the $\tilde{\tilde{t}}^2 = T - \frac{1}{2}U - W = 0$ (right) boundary, whereas saturation of (5.118) implies stabilization on the flop transition line $\tilde{\tilde{t}}^3 = W - U = 0$. In fact, the stabilization points of model II and model III agree with each other for such a configuration as predicted. In both cases, the moduli metric and the gauge couplings are finite. At the flop transition we expect $Z, \partial Z$ and V to be continuous. However, V should have a kink, according to our discussion in section 5.1. Figure 8 and 9 graphically illustrate this behavior. The minima of V and Z^2 are marked by an ellipse.

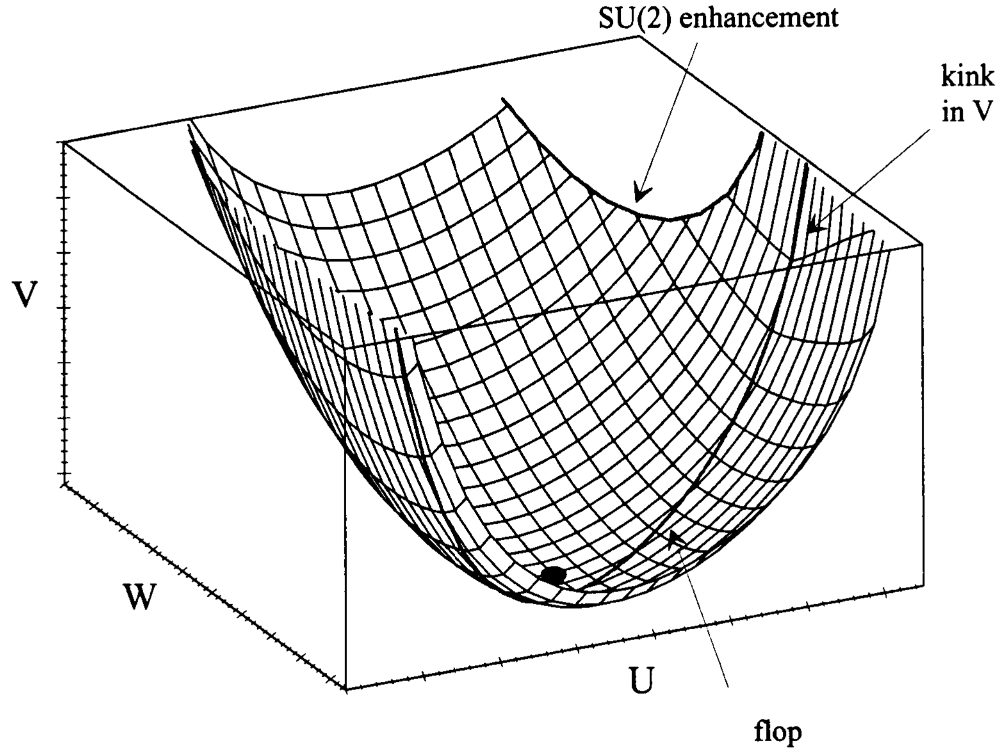


Figure 8: Continuity and kink of V

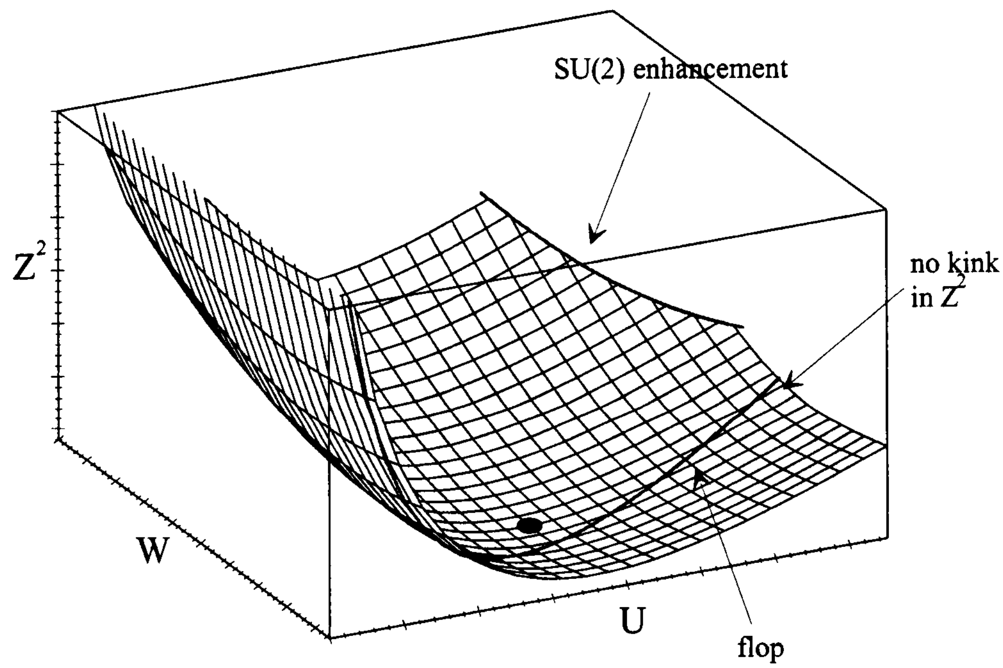


Figure 9: Continuity and smoothness of Z^2

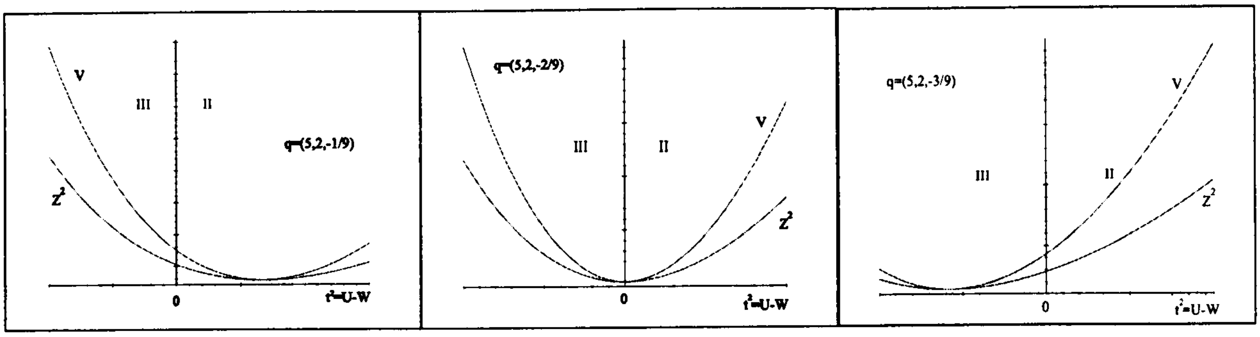


Figure 10: Stabilization for configurations (q_U, q_T, q_W) = (5,2,-1/9), (5,2,-2/9), (5,2,-3/9)

Figure 10 shows how the stabilization point smoothly moves over the flop boundary if we vary q_W.

The right hand side boundary $\tilde{\tilde{t}}^2 = T - \frac{1}{2}U - W = 0$ is different in the sense that there is no flop transition to another Calabi-Yau space possible, although the metric is also regular. Hence, the conclusion is that this line corresponds to an end of the moduli space in finite distance. Let us see again whether strings become tensionless at this point. Indeed, one finds that the magnetic central charge for string states with charges

$$\tilde{\tilde{m}} = \pm(1, -2, 1) \tag{5.119}$$

vanishes. However, in this case Z_m behaves as

$$Z_m \sim \tilde{\tilde{t}}^2 \tag{5.120}$$

which implies that the tension of the string is decreasing much slower than we found for the two non-flop boundaries of model II. This behavior again agrees with the one expected in [65] for the case where a divisor shrinks to a curve and symmetry enhancement to $SU(2)$ takes place. This symmetry enhancement is indeed expected in our model, as one can perform a redefinition of variables to write the prepotential as $\mathcal{V}_{III} = S'T'U' - \frac{1}{3}U'^3$. In these variables $\tilde{\tilde{t}}^2 \to 0$ corresponds to $T' = U'$ which is known to be a symmetry enhancement point.

At the other boundary $\tilde{\tilde{t}}^1 \to 0$ we again find the phenomenon of strings becoming tensionless faster than any particle becomes massless. We observe that at the end of the moduli space of our example special geometry remains valid if a divisor shrinks to a curve. If the divisor shrinks to a point, special geometry breaks down and the moduli space metric and gauge couplings degenerate or diverge.

5.7 Outlook

We have studied [10] phase transition of $d = 5$ M-theory compactifications on Calabi-Yau three-folds. We have probed the nature of the phase transitions utilizing several physical potentials including the electric BPS mass Z, the magnetic string tension Z_m, the black hole potential V and the gauged central charge potential P. They are direct supersymmetric and supergravity analogs of the Landau-Ginzburg potentials used in thermodynamic systems that undergo phase transitions.

A lot of attention recently was directed to the properties of the entropy of the extreme black holes. The entropy is given by the area of the horizon and is related to the size of the infinite throat of adS_2. It has a topological origin, being independent on the values of moduli far away from the horizon [5]. The value of the entropy is given by the central charge extremized in the moduli space [6]. We found that in five dimensions the corresponding stabilization equations take an extremely simple form.

For the extended objects $p > 0$ other than black holes, which have a vanishing area of the horizon, the analogous critical behavior was not studied before. Here we have found that the tension of the magnetic string which is not equal to the area of the horizon, but describes a size of an infinite throat of adS_3, is given by a critical value of the magnetic central charge. We found that the tension of the magnetic string in 5d can take a minimal value $T^3 = C_{IJK}m^I m^J m^K$ for a given choice of winding numbers m^I of an 11d fivebrane wrapped on four-cycles inside the CY space. We expect that new results on critical behavior of various extended objects will emerge based on these results.

We also have shown explicitly that V and P admit new extrema which are not related to supersymmetric stabilization.

One of the main goal in this work was to obtain a better understanding of the structure of the above supersymmetry potentials in the entire extended Kähler cone. In $d = 5$, because of absence of non-geometric phases, the extended Kähler cone is generically a geodesically incomplete, bounded domain. Since Z and ∂Z turned out to be continuous across flop transitions, and any critical point of Z must be a minimum, we were able to draw the important conclusion that supersymmetric stabilization can take place in at most one point in the entire extended Kähler cone. The same conclusion was drawn for the string tension Z_m. The analytic properties of V were shown to be slightly different. At the flop V is continuous but not differentiable. It remains to be shown that also the critical point of V (for fixed charges) is unique. This result would prove that the black hole entropy is unaffected even though a topology change takes place as one moves away from the black hole horizon to spatial infinity.

Finally we have studied in detail several explicit examples of Calabi-Yau compactifications with small dimensions of the Kähler moduli space. We have verified our general results of previous sections on these examples and found that at the boundaries of the extended Kähler cone, BPS magnetically charged strings become tensionless. In some cases, these tensionless strings can be related to extremal phase transitions to Calabi-Yau spaces with different Hodge numbers.

6 One-loop corrections to the D3 brane action

6.1 Introduction

The action for supersymmetric D3-branes in flat and curved type IIB supergravity backgrounds has been studied extensively during the last few years [17, 18, 19, 20]. In addition to extended supersymmetry these actions have a local κ-symmetry and half of the 32 fermionic fields can be "gauged away" by fixing the κ-gauge. Moreover, of the 32 supersymmetries of type IIB supergravity, 16 are realized linearly and 16 non-linearly. The interpretation of all these supersymmetries is closely related to the problem of the correspondence between supersymmetric $\mathcal{N} = 4$ $U(N)$-symmetric Yang-Mills theories and D3-brane Born-Infeld-type $U(1)$-symmetric actions [20]. The $\mathcal{N} = 4$ Yang-Mills theory in d=4 is not only renormalizable but even finite [83]. The D3 brane action has been considered so far only as an effective action as it is not renormalizable by power counting in d=4 . Still, there are 16+16 linear and non-linear global symmetries in the gauge-fixed action (or equivalently a local κ-symmetry of the classical action). Is it possible that these symmetries are strong enough to forbid the counterterms in the D3 brane action which have a higher number of derivatives than the classical Born-Infeld action? One has to take into account that the D3 brane action has terms of the form $((F_{\mu\nu})^n + \ldots)$ however, terms of the form $(\partial F)^n$ or $(\partial^2 F)^n$ are not present.

As a first step in resolving these issues, in particular to clarify the relation between the effective action of the open string theory, Yang-Mills theory and effective action of the D3-brane theory, we will treat the κ-symmetric action of the D3-brane on the same footing as supergravity in d=4 or supersymmetric Yang-Mills theory in $d > 4$. A priori these theories are not renormalizable. However a careful analysis of supersymmetric higher derivative counterterms was performed in the past, as well as some actual calculations [84].

In this paper we will present one-loop corrections to the D3-brane Abelian Born-Infeld action in a flat background, and we will show that there are ultraviolet divergences, in contrast to the finite d=4 $\mathcal{N} = 4$ Super-Yang-Mills [83], and that the form of the couterterm is $(s^2+t^2+u^2)F^4 \sim (\partial F)^4$. To fix the κ-symmetry we will employ the so-called Killing gauge used in [85] for gauge-fixing of the GS type IIB string in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. In this gauge we will be able to directly identify the linear supersymmetries of the gauge-fixed D3 brane action with the YM supersymmetries. We will compare this to the symmetries of the D3 brane action in $\theta_1 = 0$ gauge, proposed in [19]. The calculation of the tree amplitudes and one-loop corrections gives the same results in the other gauge, of course. However, in the Killing gauge the lowest order Lagrangian contains only quartic interaction terms while in $\theta_1 = 0$ gauge there are cubic as well as quartic interactions. Therefore it is easier to perform one-loop calculations in the Killing gauge.

It is interesting that the terms with the $(\partial F)^4$ structure were found in the effective action for the open superstrings in the paper by Andreev and Tseytlin [86] from the string S-matrix term $(s^2+t^2+u^2)F^4$. This kind of terms were also found in the more recent papers by Hashimoto and Klebanov [87] in which they computed the scattering amplitude for massless vectors on a D-brane in string theory. In the low energy limit their result also contain the term proportional to $(s^2+t^2+u^2)F^4$. The appearance of this structure in our calculations and in the string effective action could mean that the supersymmetry fixes this structure in the unique way. This would also mean that the local κ-symmetry after gauge fixing corresponds to the global $\mathcal{N} = 4$ supersymmetry at least in this approximation.

To perform one-loop calculations we will use the helicity amplitude method that has been developed extensively in the last 10 years (for reviews see [88, 89]). The main ideas of this method that we will employ here are the spinor helicity representation for polarization vectors [90], supersymmetry identities [91] and unitarity cuts [92]. This method has been used for loop calculations in Yang-Mills theory and recently also in gravity [93]. It turns out that it is extremely useful for the D3 brane calculations.

In Sec. 2 we will discuss the D3-brane action and find the SUSY transformations in the Killing gauge. In Sec. 3 the $\mathcal{N}=4$ supersymmetric Lagrangian of the Born-Infeld theory will be presented up to the quartic order in the fields. In Sec. 4 we will apply the helicity amplitude formalism to the D3-brane action. We will present tree and one-loop amplitudes for the quartic Lagrangian in the Killing gauge and give the expressions for the one-loop corrections and counterterms in this theory. In Sec. 5 we will discuss the form of the Lagrangian up to the quartic order in $\theta_1 = 0$ gauge and Sec. 6 is a conclusion.

6.2 The D3 brane action in a flat background

In this paper we will closely follow the notations of [20]. The supersymmetric D3-brane action is a sum of the Born-Infeld (BI) and Wess-Zumino (WZ) terms [17, 19, 20] that depend on superspace coordinates $X^M = (x^{\hat{a}}, \Theta^I)$ and an Abelian world-volume gauge field strength dA

$$S = S_{BI} + S_{WZ} \ , \tag{6.1}$$

where the Born-Infeld part is

$$S_{BI} = -T_3 \int_{M_4} d^4\sigma \sqrt{-\det\left(G_{ij} + 2\pi\alpha' \mathcal{F}_{ij}\right)} \tag{6.2}$$

where $T_3 \sim \frac{1}{(2\pi\alpha')^2}$ is the D3-brane tension [11] and $2\pi\alpha'$ is the inverse string tension. G_{ij} is the pullback to the d=4 world-volume metric of the d=10 Minkowski metric $\eta^{\hat{a}\hat{b}} = (- + ...+)$ with $i, j = 0...3$; $\hat{a}, \hat{b} = 0...9$:

$$G_{ij} = L_i^{\hat{a}} \eta^{\hat{a}\hat{b}} L_i^{\hat{b}}, \qquad L^{\hat{a}}(X(\sigma)) = d\sigma^i L_i^{\hat{a}} \tag{6.3}$$

and in the flat background the supervielbeins are

$$\begin{aligned} L^{\mathcal{I}} &= d\Theta^{\mathcal{I}}, \qquad \mathcal{I} = 1, 2 \\ L^{\hat{a}} &= dx^{\hat{a}} - \mathrm{i}\bar{\Theta}^{\mathcal{I}} \Gamma^{\hat{a}} d\Theta^{\mathcal{I}}. \end{aligned} \tag{6.4}$$

The field strength $F = \frac{1}{2}\mathcal{F}_{ij} d\sigma^i \wedge d\sigma^j$ is the supersymmetric extension of dA:

$$F = dA + 2\mathrm{i}\int_0^1 dt\ L_t^{\hat{a}} \wedge \bar{\Theta}\eta^{ab}\Gamma^{\hat{b}}\mathcal{K}L_t\ . \tag{6.5}$$

The WZ part of the action is

$$S_{WZ} = 2\mathrm{i}T_3 \int_{M_4}\int_0^1 dt\left(\frac{1}{6}\bar{\Theta}\widehat{L}_t\wedge\widehat{L}_t\wedge\widehat{L}_t\mathcal{E}L_t + 2\pi\alpha'\bar{\Theta}\widehat{L}_t\wedge \mathrm{F}_t\wedge\mathcal{J}L_t\right) + T_3\int_{M_4}\Omega_4^{(bos)}\ , \tag{6.6}$$

where $\widehat{L} = L^{\hat{a}}\Gamma^{\hat{a}}$ and $L_t^{\hat{a}}(x,\Theta) \equiv L^{\hat{a}}(x,t\Theta)$, $L_t(x,\Theta) \equiv L(x,t\Theta)$. The Θ independent part of the 4-form in the flat background is $\Omega_4^{(bos)} = d^4\sigma$. The $\Gamma^{\hat{a}}$ are the d=10 Dirac matrices and the matrices $\mathcal{E},\mathcal{J},\mathcal{K}$ in (6.6) and (6.5) act on the SO(2) indices of type IIB spinors $\Theta^{\mathcal{I}}$:

$$\mathcal{E} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{K} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.7}$$

The supersymmetry transformations with global parameter ε and κ-symmetry transformations with local parameter $\kappa(\sigma)$ for this action are [17, 20]

$$\delta_\varepsilon\Theta^{\mathcal{I}} = \varepsilon^{\mathcal{I}} \tag{6.8}$$

$$\delta_\varepsilon x^{\hat{a}} = -\mathrm{i}(\bar{\Theta}\Gamma^{\hat{a}}\varepsilon) \tag{6.9}$$

$$\delta_\varepsilon A = \mathrm{i}dx^{\hat{a}}\bar{\Theta}\Gamma^{\hat{a}}\mathcal{K}\varepsilon - \frac{1}{6}(d\bar{\Theta}\Gamma_{\hat{a}}\Theta)(\bar{\Theta}\Gamma^{\hat{a}}\mathcal{K}\varepsilon) - \frac{1}{6}(d\bar{\Theta}\Gamma_{\hat{a}}\mathcal{K}\Theta)(\bar{\Theta}\Gamma^{\hat{a}}\varepsilon) \tag{6.10}$$

$$\delta_\kappa\Theta^{\mathcal{I}} = \kappa^{\mathcal{I}} \tag{6.11}$$

$$\delta_\kappa x^{\hat{a}} = \mathrm{i}(\bar{\Theta}\Gamma^{\hat{a}}\kappa) \tag{6.12}$$

$$\delta_\kappa A = -\mathrm{i}dx^{\hat{a}}\bar{\Theta}\Gamma^{\hat{a}}\mathcal{K}\kappa + \frac{1}{2}(d\bar{\Theta}\Gamma_{\hat{a}}\Theta)(\bar{\Theta}\Gamma^{\hat{a}}\mathcal{K}\kappa) + \frac{1}{2}(d\bar{\Theta}\Gamma_{\hat{a}}\mathcal{K}\Theta)(\bar{\Theta}\Gamma^{\hat{a}}\kappa), \tag{6.13}$$

where the SUSY parameter $\varepsilon^{\mathcal{I}}$ is a constant type IIB Majorana-Weyl spinor. Half of the components of the κ-transformation parameter are projected out by the condition

$$\kappa = \frac{1}{2}(1+\Gamma)\kappa\,, \qquad \Gamma^2 = 1, \tag{6.14}$$

with

$$\Gamma = \begin{pmatrix} 0 & \zeta \\ \tilde{\zeta} & 0 \end{pmatrix} = \frac{\epsilon^{i_1\ldots i_4}\left(\frac{1}{4!}\Gamma_{i_1\ldots i_4}\mathcal{E} + \frac{2\pi\alpha'}{4}\Gamma_{i_1 i_2}\mathcal{F}_{i_3 i_4}\mathcal{J} + \frac{(2\pi\alpha')^2}{8}\mathcal{F}_{i_1 i_2}\mathcal{F}_{i_3 i_4}\mathcal{E}\right)}{\sqrt{-\det(G_{ij} + 2\pi\alpha'\mathcal{F}_{ij})}} \tag{6.15}$$

where $\Gamma_{i_1 \ldots i_n} \equiv \Lambda_{[i_1} \ldots \Lambda_{i_n]}$.

Following [85, 94] we will choose the "Killing gauge" for the κ-symmetry. In [85] $\Theta^{1,2}$ were choose to be the Killing spinors of the AdS_5 background, for the flat background we can simply introduce new variables $\vartheta^{\pm}$

$$\vartheta^{\pm} = U\Theta^{1,2} \quad \Rightarrow \quad \begin{pmatrix} \vartheta^+ \\ \vartheta^- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \Gamma^{(4)} \\ 1 & -\Gamma^{(4)} \end{pmatrix} \begin{pmatrix} \Theta^1 \\ \Theta^2 \end{pmatrix} \tag{6.16}$$

where $\Gamma^{(4)} = \Gamma^0\Gamma^1\Gamma^2\Gamma^3$ with

$$\begin{aligned} \Gamma^p &= \gamma^p \otimes 1, \quad p = 0, \ldots, 3 \\ \Gamma^t &= \gamma^5 \otimes \tilde{\gamma}^t, \quad t = 4, \ldots, 9 \end{aligned} \tag{6.17}$$

Here γ^p, and γ^5 are standard Dirac matrices with $\left\{\gamma^p, \gamma^{p'}\right\} = 2\eta^{pp'}$ and $\tilde{\gamma}^t$ are 8×8 six-dimensional Dirac matrices with $\left\{\tilde{\gamma}^t, \tilde{\gamma}^{t'}\right\} = 2\delta^{tt'}$.

The supersymmetry and κ-symmetry transformations (6.10) and (6.13) with the new parameters $\varepsilon^{\pm} = U\varepsilon^{\mathcal{I}}$, $\kappa^{\pm} = U\kappa^{\mathcal{I}}$ are

$$\begin{pmatrix} \delta_{\kappa+\varepsilon}\vartheta^+ \\ \delta_{\kappa+\varepsilon}\vartheta^- \end{pmatrix} = \begin{pmatrix} \varepsilon^+ \\ \varepsilon^- \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 - \frac{1}{2}C & \frac{1}{2}A \\ -\frac{1}{2}A & 1 + \frac{1}{2}C \end{pmatrix} \begin{pmatrix} \kappa^+ \\ \kappa^- \end{pmatrix}. \tag{6.18}$$

Here the matrices A and C are

$$A = \zeta\Gamma^{(4)} + \Gamma^{(4)}\tilde{\zeta}, \quad C = \zeta\Gamma^{(4)} - \Gamma^{(4)}\tilde{\zeta} \tag{6.19}$$

and explicit expressions for ζ and $\tilde{\zeta}$ are given in (6.15).

We gauge away half of the fermionic coordinates by imposing $\vartheta^- = 0$. This condition also requires that $\delta_{\kappa+\varepsilon}\vartheta^- = 0$. Gauge fixing κ-symmetry by the condition $\kappa^+ = 0$ we get:

$$\delta_{\kappa+\varepsilon}\vartheta^+ = \frac{1}{4}A\kappa^- + \varepsilon^+ \tag{6.20}$$

$$\delta_{\kappa+\varepsilon}\vartheta^- = \frac{1}{2}(1 + \frac{1}{2}C)\kappa^- + \varepsilon^- = 0. \tag{6.21}$$

Eq. (6.21) gives a relation between κ^- and ε^-,

$$\kappa^- = -2(1+\frac{1}{2}C)^{-1}\varepsilon^-. \tag{6.22}$$

Finally we get

$$\delta_{\kappa+\varepsilon}\vartheta^+ = \frac{1-\zeta\Gamma^{(4)}}{1+\zeta\Gamma^{(4)}}\varepsilon^- + \varepsilon^+. \tag{6.23}$$

Using this transformation on A^p and $x^{\hat{a}} = x^p,\ y^t$ $(p = 0,\cdots,3;\ t = 4,\cdots,9\ )$:

$$\delta_{\kappa+\varepsilon}x^p = i\bar{\vartheta}^+\Gamma^p(\tau\varepsilon^- - \varepsilon^+) \tag{6.24}$$

$$\delta_{\kappa+\varepsilon}y^t = -2i\bar{\vartheta}^+\Gamma^t\varepsilon^- \tag{6.25}$$

$$\delta_{\kappa+\varepsilon}A^p = 2i\bar{\vartheta}^+\Gamma^p\varepsilon^- - \frac{2}{3}\partial_p\bar{\vartheta}^+\Gamma_a\vartheta^+\bar{\vartheta}^+\Gamma^a\varepsilon^- - i\partial_p y^t\bar{\vartheta}^+\Gamma^t(\tau\varepsilon^- - \varepsilon^+) \tag{6.26}$$

where

$$\tau = \frac{1-\zeta\Gamma^{(4)}}{1+\zeta\Gamma^{(4)}}. \tag{6.27}$$

The additional general coordinate reparametrization is necessary to satisfy the static gauge ($x^p = \sigma^p$) condition $\delta x^p = 0$

$$\delta_\xi\vartheta = \xi^p\partial_p\vartheta \tag{6.28}$$

$$\delta_\xi x^{\hat{a}} = \xi^p\partial_p x^{\hat{a}} \tag{6.29}$$

then from $\delta_{\kappa+\varepsilon+\xi}x^p = 0$ and $\partial_i x^p = \delta_i^p$ it follows that $\xi^i = -i\bar{\vartheta}^+\Gamma^i(\tau\varepsilon^- - \varepsilon^+)$. Finally the SUSY-transformations for remaining nonzero fermions, scalars and vectors are:

$$\delta_{\kappa+\varepsilon+\xi}\vartheta^+ = (\tau\varepsilon^- + \varepsilon^+) + \xi^i\partial_i\vartheta^+ \tag{6.30}$$

$$\delta_{\kappa+\varepsilon+\xi}y^t = -2i\bar{\vartheta}^+\Gamma^t\varepsilon^- + \xi^i\partial_i y^t \tag{6.31}$$

$$\delta_{\kappa+\varepsilon+\xi}A^i = 2i\bar{\vartheta}^+\Gamma^i\varepsilon^- - \frac{2}{3}\partial_i\bar{\vartheta}^+\Gamma_a\vartheta^+\bar{\vartheta}^+\Gamma^a\varepsilon^- \tag{6.32}$$

$$- \ i\partial_i y^t\bar{\vartheta}^+\Gamma^t(\tau\varepsilon^- - \varepsilon^+) + (\xi^j\partial_j A^i + \partial_i\xi^j A^j) \tag{6.33}$$

It is easy to see that the transformations for y^t are linear and do not contain an ε^+ part.

From here on we will drop the "+" in the spinor notation i.e. $\vartheta^+ \Rightarrow \vartheta$. In the new variables the supervierbeins (6.4) are:

$$L = d\vartheta;\quad L^p = dx^p - i\bar{\vartheta}\Gamma^p d\vartheta;\quad L^t = dy^t. \tag{6.34}$$

6.3 The Quartic $\mathcal{N} = 4$ Lagrangian

The D3-brane BI action (6.2) simplifies considerably in the Killing gauge. In this case

$$\begin{aligned} G_{ij} + 2\pi\alpha'\mathcal{F}_{ij} &= \eta^{ij} - i\bar{\vartheta}\Gamma_i\partial_j\vartheta - i\bar{\vartheta}\Gamma_j\partial_i\vartheta - \bar{\vartheta}\Gamma_p\partial_i\vartheta\bar{\vartheta}\Gamma_p\partial_j\vartheta + \partial_i y^t\partial_i y^t \\ &+ 2\pi\alpha'(\mathrm{F}_{ij} - i\partial_i y^t\bar{\vartheta}\Gamma_p\partial_j\vartheta + i\partial_j y^t\bar{\vartheta}\Gamma_t\partial_i\vartheta) \end{aligned} \tag{6.35}$$

where $\mathrm{F}_{ij} = \partial_i A_j - \partial_j A_i$ and the WZ-term is

$$\begin{aligned} L_{WZ} &= -T_3(i\bar{\theta}\,\partial\!\!\!/\theta - \frac{i}{2}\partial_l\bar{\theta}\Gamma^{jkl}\Gamma^t\Gamma^{t'}\theta\partial_j y^t\partial_k y^{t'} + \frac{1}{2}((\bar{\theta}\,\partial\!\!\!/\theta)^2 - \bar{\theta}\Gamma^i\partial_j\theta\bar{\theta}\Gamma^j\partial_i\theta) \\ &- \frac{1}{4}\epsilon^{ijkl}\partial_k\bar{\theta}\Gamma^c\theta\partial_l\bar{\theta}\Gamma_c\Gamma^{(4)}\Gamma^t\Gamma^{t'}\theta\partial_i y^t\partial_j y^{t'} - \frac{1}{2}\epsilon^{ijkl}\partial_k\bar{\theta}\Gamma^t\Gamma^{(4)}\theta\partial_l\bar{\theta}\Gamma^{t'}\theta\partial_i y^t\partial_j y^{t'} \\ &+ \frac{i}{6}\epsilon^{ijkl}\epsilon_{abcl}\bar{\theta}\Gamma^a\partial_i\theta\bar{\theta}\Gamma^b\partial_j\theta\bar{\theta}\Gamma^c\partial_k\theta + \frac{1}{24}\epsilon^{ijkl}\epsilon_{abcd}\bar{\theta}\Gamma^a\partial_i\theta\bar{\theta}\Gamma^b\partial_j\theta\bar{\theta}\Gamma^c\partial_k\theta\bar{\theta}\Gamma^d\partial_l\theta \\ &- (2\pi\alpha')\frac{i}{2}\partial_l\bar{\theta}\Gamma^{ijkl}\mathrm{F}_{ij}\Gamma^t\theta\partial_k y^t) \end{aligned} \tag{6.36}$$

Using a series expansion for $\sqrt{\det(G_{ij} + 2\pi\alpha'\mathcal{F}_{ij})}$ and a standard redefinition of the fields we get the effective Lagrangian up to the fourth order in fields (and coupling constant $\alpha = 2\pi\alpha'$):

$$\begin{aligned} L^4_{BI+WZ} &= 2i\bar{\theta}\,\partial\!\!\!/\theta + \frac{1}{4}\mathrm{F}_{ij}\mathrm{F}_{ji} - \frac{1}{2}\partial_i y^t\partial_i y^t + \alpha^2((\bar{\theta}\,\partial\!\!\!/\theta)^2 - \bar{\theta}\Gamma^i\partial_j\theta\bar{\theta}\Gamma^j\partial_i\theta) \\ &+ \frac{\alpha^2}{8}(\mathrm{F}_{ij}\mathrm{F}^{jk}\mathrm{F}_{kl}\mathrm{F}^{li} - \frac{1}{4}(\mathrm{F}_{ij}\mathrm{F}_{ji})^2) + \frac{\alpha^2}{4}(\partial_i y^t\partial_j y^t\partial_i y^{t'}\partial_j y^{t'} - \frac{1}{2}(\partial_i y^t\partial_i y^t)^2) \\ &- \frac{\alpha^2}{2}(\partial_i y^t\partial_j y^t\mathrm{F}_{ik}\mathrm{F}^{kj} - \frac{1}{4}\partial_i y^t\partial_i y^t\mathrm{F}_{jk}\mathrm{F}^{kj}) \\ &- i\alpha^2((\bar{\theta}\Gamma^i\partial_j\theta)(\partial_i y^t\partial_j y^t) - \frac{1}{2}(\bar{\theta}\,\partial\!\!\!/\theta)(\partial_i y^t\partial_i y^t) + \frac{1}{2}\partial_k\bar{\theta}\Gamma^{ijk}\partial_i y^t\Gamma^t\partial_j y^{t'}\Gamma^{t'}\theta) \\ &+ i\alpha^2(\bar{\theta}\Gamma^i\partial_j\theta\mathrm{F}^{jk}\mathrm{F}_{ki} - \frac{1}{4}\bar{\theta}\,\partial\!\!\!/\theta\mathrm{F}^{ij}\mathrm{F}_{ji}) \\ &+ i\alpha^2(\bar{\theta}\partial_i y^t\Gamma^t\partial_j\theta\mathrm{F}_{ij} - \frac{1}{2}\bar{\theta}\partial_i y^t\Gamma^t\Gamma^{(4)}\partial_l\theta\mathrm{F}_{jk}\epsilon^{ijkl}) \end{aligned} \tag{6.37}$$

To establish the analogy between this theory and $\mathcal{N} = 4$ super-Yang-Mills theory it is helpful to use the chiral representation for $\tilde{\gamma}^t$ (see Appendix) where the SU(4) symmetry is manifest. Using this representation we will introduce new notations for the scalar fields $s^{IJ} = \frac{1}{2}(\tilde{\sigma}_t^{-1})^{IJ}y^t$ with SU(4) indices I, $J = 1,2,3,4$. The 16-

component spinor θ will be represented as four d=4 Majorana spinors,

$$\theta = \frac{\tilde{\psi}^{(I)}}{2} = \begin{pmatrix} \lambda_{\alpha I} \\ \bar{\lambda}^{\dot{\alpha} I} \end{pmatrix}, \tag{6.38}$$

where the extra factor 2 was introduced for convenience and the d=4 chiral projections are

$$\tilde{\psi}_I^+ = \frac{1}{2}(1+\gamma^5)\tilde{\psi}^{(I)}; \quad \tilde{\psi}^{-I} = \frac{1}{2}(1-\gamma^5)\tilde{\psi}^{(I)}.$$

The quartic $\mathcal{N}=4$ Lagrangian in terms of the new 4-dim YM variables $(g, \tilde{\psi}^I, s^{IJ})$ with $g^i = A^i$ is

$$\begin{aligned}
L^4_{BI+WZ} &= i\frac{1}{2}\bar{\tilde{\psi}}_I \not\partial\tilde{\psi}^I + \frac{1}{4}\mathrm{F}_{ij}\mathrm{F}_{ji} - \frac{1}{2}\partial_i s^{IJ}\partial_i s_{IJ} \\
&+ \frac{\alpha^2}{16}\Big((\bar{\tilde{\psi}}_I \not\partial\tilde{\psi}^I)^2 - \bar{\tilde{\psi}}_I\gamma^i\partial_j\tilde{\psi}^I\bar{\tilde{\psi}}_J\gamma^j\partial_i\tilde{\psi}^J\Big) \\
&+ \frac{\alpha^2}{8}\Big(\mathrm{F}_{ij}\mathrm{F}^{jk}\mathrm{F}_{kl}\mathrm{F}^{li} - \frac{1}{4}(\mathrm{F}_{ij}\mathrm{F}_{ji})^2\Big) + \frac{\alpha^2}{4}\Big(\partial_i s^{IJ}\partial_j s_{IJ}\partial_i s^{I'J'}\partial_j s_{I'J'} - \frac{1}{2}(\partial_i s^{IJ}\partial_i s_{IJ})^2\Big) \\
&- \frac{\alpha^2}{2}\Big(\partial_i s^{IJ}\partial_j s_{IJ}\mathrm{F}_{ik}\mathrm{F}^{kj} - \frac{1}{4}\partial_i s^{IJ}\partial_i s_{IJ}\mathrm{F}_{jk}\mathrm{F}^{kj}\Big) \\
&- \frac{i\alpha^2}{4}\Big((\bar{\tilde{\psi}}_I\gamma^i\partial_j\tilde{\psi}^I)(\partial_i s^{IJ}\partial_j s_{IJ}) - \frac{1}{2}(\bar{\tilde{\psi}}_I \not\partial\tilde{\psi}^I)(\partial_i s^{IJ}\partial_i s_{IJ}) \\
&- 2\partial_k\bar{\tilde{\psi}}_I\gamma^{ijk}\partial_i s^{IJ}\partial_j s_{JK}\tilde{\psi}^K\Big) \\
&+ \frac{i\alpha^2}{4}\Big(\bar{\tilde{\psi}}_I\gamma^i\partial_j\tilde{\psi}^I\mathrm{F}^{jk}\mathrm{F}_{ki} - \frac{1}{4}\bar{\tilde{\psi}}_I \not\partial\tilde{\psi}^I\mathrm{F}^{ij}\mathrm{F}_{ji}\Big) \\
&+ \frac{i\alpha^2}{2}\Big(\bar{\tilde{\psi}}_I^+\partial_i s^{IJ}\gamma^5\partial_j\tilde{\psi}_J^+\mathrm{F}_{ij} - \bar{\tilde{\psi}}^{I-}\partial_i s^{IJ}\gamma^5\partial_j\tilde{\psi}^{J-}\mathrm{F}_{ij} \\
&+ \frac{1}{2}\bar{\tilde{\psi}}_I^+\gamma^{ijkl}\partial_i s^{IJ}\gamma^5\partial_l\tilde{\psi}_J^+\mathrm{F}_{jk} - \frac{1}{2}\bar{\tilde{\psi}}_I^-\gamma^{ijkl}\partial_i s^{IJ}\gamma^5\partial_l\tilde{\psi}_J^-\mathrm{F}_{jk}\Big).
\end{aligned} \tag{6.39}$$

where $\gamma^{i_1\cdots i_k} \equiv \frac{1}{k!}\gamma^{[i_1}\cdots\gamma^{i_k]}$. The parameter τ from Eq.(6.27) is, up to linear order in the fields:

$$\tau_{IJ} \approx \frac{1}{2}\not\partial y^t\Gamma^t_{IJ} - \frac{1}{4}\gamma^{ij}\mathrm{F}_{ij}\delta_{IJ}. \tag{6.40}$$

The supersymmetry transformation now looks like:

$$\delta_{susy}\tilde{\psi}^{(I)} = \begin{pmatrix} -\frac{1}{2}\gamma^{ij}\mathrm{F}_{ij}\delta_{IJ} & 2\not\partial\gamma^5 s_{IJ} \\ -2\not\partial\gamma^5 s^{IJ} & -\frac{1}{2}\gamma^{ij}\mathrm{F}_{ij}\delta_{IJ} \end{pmatrix} \cdot \begin{pmatrix} \varepsilon_J^- \\ \varepsilon^{-J} \end{pmatrix}$$

$$\begin{aligned}
\delta_{susy} g^i &= i\bar{\tilde{\psi}}\gamma^i\varepsilon^- \\
\delta_{susy} s_{IJ} &= i\varepsilon^-{}_I\tilde{\psi}_J - i\varepsilon^-{}_J\tilde{\psi}_I + i\bar{\varepsilon}^-_K\bar{\tilde{\psi}}_L\epsilon^{IJKL}.
\end{aligned}$$

Finally, using a chiral representation for γ^i we get

$$\delta_{susy}\tilde{\psi}_I = -\frac{1}{2}\sigma^{ij}\mathrm{F}_{ij}\varepsilon^-{}_I - 2\sigma^i\partial_i s_{IJ}\bar{\varepsilon}^{-J} \tag{6.41}$$

$$\delta_{susy} g^i = -i\varepsilon^-_I\sigma^i\bar{\tilde{\psi}}_I + i\tilde{\psi}^I\sigma^i(\bar{\varepsilon}^-)^I \tag{6.42}$$

$$\delta_{susy} s_{IJ} = i\varepsilon^-{}_I\tilde{\psi}_J - i\varepsilon^-{}_J\tilde{\psi}_I + i\bar{\varepsilon}^-_K\bar{\tilde{\psi}}_L\epsilon^{IJKL} \tag{6.43}$$

which is exactly the $\mathcal{N} = 4$ transformation given in [95].

6.4 Helicity amplitudes.

To calculate one loop corrections to the D3-brane action we use the helicity amplitude technique (see for example the review paper [88] and references therein). In the spinor helicity formalism, positive- and negative-helicity massless spinors ψ are represented as

$$|p\pm\rangle = \psi_\pm(p) = \frac{1}{2}(1 \pm \gamma^5)\psi(p) \tag{6.44}$$

with antisymmetric spinor products

$$\langle p_i - |p_j+\rangle \equiv \langle p_i p_j\rangle \equiv \langle ij\rangle \tag{6.45}$$

$$\langle p_i + |p_j-\rangle \equiv [p_i p_j] \equiv [ij] \tag{6.46}$$

$$\langle ij\rangle[ji] = 2p_i \cdot p_j \equiv s_{ij}. \tag{6.47}$$

Negative- and positive-helicity polarization vectors also can be represented through spinors $|p\pm\rangle$,

$$\varepsilon^+_\mu(p;k) = \frac{\langle k - |\gamma_\mu|p-\rangle}{\sqrt{2}\langle kp\rangle}, \quad \varepsilon^-_\mu(p;k) = -\frac{\langle k + |\gamma_\mu|p+\rangle}{\sqrt{2}[kp]}, \tag{6.48}$$

where k_μ ($k^2 = 0$) is a "reference" momentum which corresponds to the particular choice of gauge for the external legs. Due to the gauge invariance of the amplitudes, the vector k drops out of the final expressions. The supersymmetry transformations

(6.41),(6.42) and (6.43) can also be rewritten in terms of bosonic states with definite helicity $g^{\pm}(p) = \varepsilon^{\pm}_{\mu}(p) g^{\mu}$ and spinors $\tilde{\psi}_{\pm}(p) = |p\pm\rangle$ as [91]

$$
\begin{aligned}
[Q_I(p), \tilde{\psi}^{\pm}_J(k)] &= \mp\Gamma^{\mp}(k,p) g^{\pm}(k)\delta_{IJ} \mp \Gamma^{\pm}(k,p) s^{\pm}_{IJ}, && (6.49)\\
[Q_I(p), g^{\pm}(k)] &= \mp i\Gamma^{\pm}(k,p)\tilde{\psi}^{\pm}_I, && (6.50)\\
[Q_I(p), s^{\pm}_{JK}(k)] &= \pm i\Gamma^{\mp}\delta^{IJ}\tilde{\psi}^{\pm}_K \mp i\Gamma^{\mp}\delta^{IK}\tilde{\psi}^{\pm}_J \pm i\Gamma^{\pm}\tilde{\psi}^{\pm}_L\epsilon^{IJKL}, && (6.51)
\end{aligned}
$$

where $\Gamma^{+}(k,p) = \bar{\eta}[pk], \quad \Gamma^{-}(k,p) = \eta\langle pk\rangle$, and η is a numerical Grassmann parameter.

Using the Lagrangian (6.39) it is easy to find the tree-level helicity amplitudes. For the vector bosons we have:

$$
\begin{aligned}
4!(\mathrm{F}_{ij}\mathrm{F}^{jk}\mathrm{F}_{kl}\mathrm{F}^{li} - \frac{1}{4}(\mathrm{F}_{ij}\mathrm{F}_{ji})^2) &= (t_8)_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} F_{\mu_1\nu_1}F_{\mu_2\nu_2}F_{\mu_3\nu_3}F_{\mu_4\nu_4}\\
&= 16(t_8)_{\mu_1\nu_1\cdots\mu_4\nu_4} k_1^{\mu_1}k_2^{\mu_2}k_3^{\mu_3}k_4^{\mu_4}\epsilon_1^{\nu_1}\epsilon_2^{\nu_2}\epsilon_3^{\nu_3}\epsilon_4^{\nu_4} \quad (6.52)
\end{aligned}
$$

where (t_8) is given in eq.(9.A.19) in [96]

$$
\begin{aligned}
(t_8)_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} &= \Big(-\frac{1}{2}(\delta_{\mu_1\mu_2}\delta_{\nu_1\nu_2} - \delta_{\mu_1\nu_2}\delta_{\nu_1\mu_2})(\delta_{\mu_3\mu_4}\delta_{\nu_3\nu_4} - \delta_{\mu_3\nu_4}\delta_{\nu_3\mu_4})\\
&+ \frac{1}{2}(\delta_{\nu_1\mu_2}\delta_{\nu_2\mu_3}\delta_{\nu_3\mu_4}\delta_{\nu_4\mu_1} + \text{antisym. of } [\mu_i\nu_i])\Big)\\
&+ \Big((1324) + (1342)\,\text{permutations}\Big) \quad (6.53)
\end{aligned}
$$

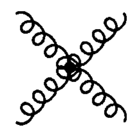
and k_i and ϵ_i are the momentum and polarization vectors for an external vector boson leg.

In the YM theory with non-Abelian gauge group $SU(N_c)$ the tree amplitudes are usually represented according to color decomposition [88, 97] as a sum of partial subamplitudes with fixed cyclic ordering of external legs multiplied by the color factors:

$$
\mathcal{A}_n^{tree}(1,2,\cdots n) = \sum_{\sigma\in\frac{S_n}{Z_n}} \mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}\cdots T^{a_{\sigma(n)}}) A_n^{tree}(\sigma(1),\sigma(2),\cdots\sigma(n)) \quad (6.54)
$$

where $\frac{S_n}{Z_n}$ is the set of all non-cyclic permutations and $T^{a_{\sigma(n)}}$ are matrices of the fundamental representation of the $SU(N_c)$ color group. In the D3-brane case the gauge group is an Abelian U(1) and color factors will be simply reduced to 1.

All four-particle tree amplitudes are proportional to α^2 (6.39) and to simplify the notations we will let $\alpha^2 = 1$. Due to the supersymmetric Ward identities (SWI) [91] the only nonzero four vector-boson tree amplitude in this case is $A_4^{tree}(g_1^- g_2^- g_3^+ g_4^+)$ (with two $-$ and two $+$ helicity bosons)



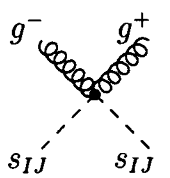
$$
\begin{aligned}
A_4^{tree}(1^-,2^-,3^+,4^+) &= -i\frac{4!}{8}\langle 3,\, 4|(\mathrm{F}_{ij}\mathrm{F}_{jk}\mathrm{F}_{kl}\mathrm{F}_{li} - \frac{1}{4}(\mathrm{F}_{ij}\mathrm{F}_{ji})^2)|1,\, 2\rangle \\
&= -i\frac{4}{2}(t_8)_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} k_1^{\mu_1} k_2^{\mu_2} k_3^{\mu_3} k_4^{\mu_4} \epsilon_1^{\nu_1} \epsilon_2^{\nu_2} \epsilon_3^{\nu_3} \epsilon_4^{\nu_4} \\
&= \frac{i}{2}\frac{st\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}
\end{aligned}
\tag{6.55}
$$

where $s = s_{12}$, $t = s_{14}$, $u = s_{13}$ are Mandelstam variables. This amplitude is related to the standard $\mathcal{N} = 4$ SYM four-gluon color-ordered sub-amplitude [88] by:

$$
A_4^{tree}(1^-,2^-,3^+,4^+) = \frac{1}{2} st A_4^{tree,YM}(1,2,3,4). \tag{6.56}
$$

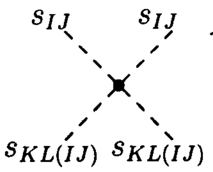
It is important to notice that the RHS of (6.56) is totally symmetric under external leg permutations [91].

The nonzero vector-scalar tree amplitudes are:



$$
\begin{aligned}
A_4^{tree}(\ g^-,\ \ & s_{IJ}^-, s_{IJ}^+, g^+) = \\
&= -i\frac{4}{2}(t_8)_{\mu_1 t \mu_2 t \mu_3 \nu_3 \mu_4 \nu_4} k_1^{\mu_1} k_2^{\mu_2} k_3^{\mu_3} k_4^{\mu_4} \epsilon_3^{\nu_3} \epsilon_4^{\nu_4} \\
&= \frac{i}{2}\langle 12\rangle\langle 13\rangle[43][42] \\
&= \frac{st\langle 12\rangle^2\langle 13\rangle^2}{2\langle 12\rangle^4} A_4^{tree,YM}(1,2,3,4).
\end{aligned}
\tag{6.57}
$$

There are two kinds of four-scalar amplitudes. One of them could contain two different kinds of scalar fields s_{IJ} and s_{KL} and the other contains the same scalar fields s_{IJ} :



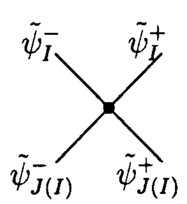
$$A_4^{tree}(\ s_{IJ}^-,\ \ s_{KL}^-, s_{IJ}^+, s_{KL}^+) = \frac{i}{2}st \qquad (6.58)$$

$$= \frac{st\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}{2\langle 12\rangle^4} A_4^{tree,YM}(1,2,3,4)$$

$$A_4^{tree}(\ s_{IJ}^-,\ \ s_{IJ}^-, s_{IJ}^+, s_{IJ}^+) = -\frac{i}{2}s^2 \qquad (6.59)$$

$$= \frac{st\langle 12\rangle^2\langle 34\rangle^2}{2\langle 12\rangle^4} A_4^{tree,YM}(1,2,3,4).$$

The four fermion tree amplitudes include two nonzero ones:



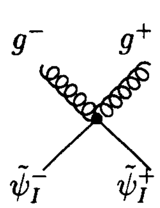
$$A_4^{tree}(\ \tilde{\psi}_I^-,\ \ \tilde{\psi}_J^-, \tilde{\psi}_J^+, \tilde{\psi}_I^+) = -\frac{i}{2}[13][42]\langle 12\rangle^2$$

$$= \frac{-\langle 12\rangle^2\langle 13\rangle\langle 24\rangle}{\langle 12\rangle^4} A_4^{tree}(1,2,3,4) \qquad (6.60)$$

$$A_4^{tree}(\ \tilde{\psi}_I^-,\ \ \tilde{\psi}_I^-, \tilde{\psi}_I^+, \tilde{\psi}_I^+) = -\frac{i}{2}[12][43]\langle 12\rangle^2$$

$$= \frac{-st\langle 12\rangle^3\langle 34\rangle}{2\langle 12\rangle^4} A_4^{tree,YM}(1,2,3,4). \qquad (6.61)$$

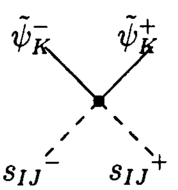
The next set of diagrams contains a mixture of two fermions and two bosons. There are five nonzero tree amplitudes. The amplitude with two vectors and two fermions is:



$$A_4^{tree}(\ g^-,\ \ \tilde{\psi}_I^-, \tilde{\psi}_I^+, g^+) = -(i)\frac{i}{2}[43][24]\langle 12\rangle^2 \qquad (6.62)$$

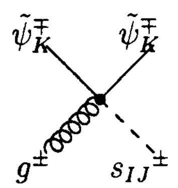
$$= (i)\frac{-st\langle 12\rangle^3\langle 13\rangle}{2\langle 12\rangle^4} A_4^{tree,YM}(1,2,3,4).$$

The amplitudes with two fermions and two scalars and "mixed" amplitudes with two fermions, one vector and one scalar are:



$$
\begin{aligned}
A_4^{tree}(\tilde{\psi}_I^-, s_{JK}^-, s_{JK}^+, \tilde{\psi}_I^+) &= -(i)\frac{i}{2}[13][24]\langle 12\rangle\langle 13\rangle \\
&= (i)\frac{st\langle 12\rangle\langle 13\rangle^2\langle 24\rangle}{2\langle 12\rangle^4} A_4^{tree,YM}(1,2,3,4)
\end{aligned} \tag{6.63}
$$

$$
\begin{aligned}
A_4^{tree}(\tilde{\psi}_I^-, s_{JK}^-, s_{IJ}^+, \tilde{\psi}_K^+) &= -(i)\frac{i}{2}[14][23]\langle 12\rangle\langle 13\rangle \\
&= (i)\frac{st\langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 23\rangle}{2\langle 12\rangle^4} A_4^{tree,YM}(1,2,3,4)
\end{aligned} \tag{6.64}
$$



$$
\begin{aligned}
A_4^{tree}(\tilde{\psi}_I^-, \tilde{\psi}_J^-, s_{IJ}^+, g^+) &= \\
&= i\frac{st\langle 12\rangle^2\langle 13\rangle\langle 23\rangle}{2\langle 12\rangle^4} A_4^{tree,YM}(1,2,3,4)
\end{aligned} \tag{6.65}
$$

$$
\begin{aligned}
A_4^{tree}(g^-, s_{IJ}^-, \tilde{\psi}_I^+, \tilde{\psi}_J^+) &= \\
&= -i\frac{st\langle 12\rangle^2\langle 13\rangle\langle 14\rangle}{2\langle 12\rangle^4} A_4^{tree,YM}(1,2,3,4)
\end{aligned} \tag{6.66}
$$

This set of tree amplitudes has exactly the same relative factors between them as for $\mathcal{N}=4$ SUSY Yang-Mills theory given in [93] (extra i's come from difference between metrics signature which are $g^{\mu\nu}=(+---)$ in [93] and $g^{\mu\nu}=(-+++)$ in this paper). The general expression connecting SYM amplitudes and the D3 brane amplitudes is:

$$
A_4^{tree}(l_1,l_2,l_3,l_4) = \frac{st}{2} A_4^{tree,YM}(l_1,l_2,l_3,l_4) \tag{6.67}
$$

where l_1,l_2,l_3,l_4 are the momenta of the particles from N=4 SUSY multiplet.

The main difference between the standard $\mathcal{N}=4$ SYM and the D3-brane effective theory (6.39) is that the first one is finite in d=4. The following calculation shows that the D3-brane action is not one-loop finite in the flat background.

A very useful relation for one-loop calculations in the standard $\mathcal{N}=4$ SYM theory is [93, 98]:

$$\sum_{S_1,S_2\in(N=4)} A_4^{tree,YM}(-l_1^{S_1},1,2,l_2^{S_2}) \times A_4^{tree,YM}(-l_2^{S_2},3,4,l_1^{S_1})$$
$$= \frac{-istA_4^{tree,YM}(1,2,3,4)}{(l_1-k_1)^2(l_2-k_3)^2} \qquad (6.68)$$

where the sum is over all particles in the $\mathcal{N}=4$ multiplet and $1,2,3,4$ stands for the momenta of the external particles. This statement is true for any dimension and its proof can be found in [98, 99]. For our case (with relation (6.67)) it becomes:

$$\sum_{S_1,S_2\in(N=4)} A_4^{tree(D3)}(-l_1^{S_1},1,2,l_2^{S_2}) \times A_4^{tree(D3)}(-l_2^{S_2},3,4,l_1^{S_1}) \qquad (6.69)$$
$$= \frac{-is^2}{4} stA_4^{tree,YM}(1,2,3,4)$$
$$= \frac{-is^2}{2} A_4^{tree(D3)}(1,2,3,4)$$

here $A_4^{tree(D3)}(1,2,3,4)$ is the four vector-boson D3-brane amplitude (6.56).

The $\mathcal{N}=4$ supersymmetry of the D3-brane action allows us to use a unitarity-based construction of the one loop amplitudes, as it has done in $\mathcal{N}=4$ Super-Yang-Mills theory (see review [89] and references therein). In this technique, one obtains the imaginary part of the one-loop amplitude from the product of the tree amplitudes, and then reconstructs the real part up to a possible polynomial function. The good ultraviolet behavior of $\mathcal{N}=4$ super-Yang-Mills theory makes it possible to reconstruct the whole amplitude without the additive polynomial ambiguity. The one-loop amplitudes that can be reconstructed that way ("cut-constructible") must satisfy a certain loop-momentum "power-counting" criterion given, for example, in [89].

One-loop amplitudes in the standard $\mathcal{N}=4$ SYM theory obtained by using the cut-reconstruction formalism can be expressed as:

$$A_4^{N=4,1-loop}(1,2,3,4) = ig^2 s_{12}s_{23}A_4^{tree}(1,2,3,4)(C_{1234}I_4^{1-loop}(s_{12},s_{23})$$
$$+ C_{3124}I_4^{1-loop}(s_{12},s_{13}) + C_{2314}I_4^{1-loop}(s_{23},s_{13})) \qquad (6.70)$$

where C_{1234} is a color factor for the non-Abelian gauge group and I_4^{1-loop} is a one-loop four-point (box) integral,

$$I_4^{1-loop}(s_{12}s_{23}) = \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2(p-k_1)^2(p-k_1-k_2)^2(p+k_4)^2}. \tag{6.71}$$

The expression (6.68) was used to simplify the direct product of the tree amplitudes.

The expressions (6.68)and (6.69) for $\mathcal{N}=4$ amplitudes are correct in all dimensions and can be used [93, 100] to reconstruct complete massless loop amplitudes. The D3-brane effective Lagrangian (6.39) in the Killing gauge does not contain three-particle vertices, hence only bubble diagrams will contribute to the one-loop corrections to the four-particle amplitudes. In this case the expression for one-loop corrections considerably simplifies and can be written as:

$$\begin{aligned}
&A_4^{D3,1-loop}(1,2,3,4) = \\
&= \frac{i\alpha^4}{4} s_{12}s_{23} A_4^{tree,YM}(1,2,3,4)(s_{12}^2 I_2^{1-loop}(s_{12}) + s_{13}^2 I_2^{1-loop}(s_{13}) + s_{14}^2 I_2^{1-loop}(s_{14})) \\
&= \frac{i\alpha^4}{2} A_4^{tree(D3)}(1,2,3,4)(s^2 I_2^{1-loop}(s) + t^2 I_2^{1-loop}(t) + u^2 I_2^{1-loop}(u)).
\end{aligned} \tag{6.72}$$

Here we have taken into account that $s_{12}s_{23}A_4^{tree,YM}(1,2,3,4)$ from (6.56) is symmetric under momenta permutations and color factors are gone because gauge field in (6.5) is an Abelian one. Instead of the box integrals (6.71) expression (6.72) contains only two-point one-loop integrals:

$$I_2^{1-loop}(s_{12}) = \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{1}{p^2(p-k_1-k_2)^2}. \tag{6.73}$$

This integral is ultra-violet divergent and in dimensional reduction it has the form:

$$\begin{aligned}
I_2^{1-loop}(s) &= i(4\pi)^{\epsilon-2} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\epsilon(1-2\epsilon)\Gamma(1-2\epsilon)} (\frac{-s}{\mu^2})^{-\epsilon} \\
&\sim \frac{i}{(4\pi)^2}\frac{1}{\epsilon}[1+\epsilon(2-\gamma-\log\frac{-s}{\mu^2}+\log 4\pi+\cdots)]
\end{aligned} \tag{6.74}$$

as $\epsilon \to 0$.

It is easy to find counterterms for effective D3-brane Lagrangian (6.39). Their structure at the quartic level will be

$$(s^2+t^2+u^2)(t_8)\mathrm{F}^4 \to (t_8)_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} \partial_\alpha \mathrm{F}_{\mu_1\nu_1} \partial_\alpha \mathrm{F}_{\mu_2\nu_2} \partial_\beta \mathrm{F}_{\mu_3\nu_3} \partial_\beta \mathrm{F}_{\mu_4\nu_4}, \tag{6.75}$$

or, in the notations of [93] and [84], the counterterms are:

$$T_4^{D3}(\mathrm{F}_{ij}\mathrm{F}^{jk}\mathrm{F}_{kl}\mathrm{F}^{li} - \frac{1}{4}(\mathrm{F}_{ij}\mathrm{F}_{ji})^2), \tag{6.76}$$

where

$$T_4^{D3} = -\frac{\alpha^4}{(4\pi)^2}\frac{1}{16\epsilon}(s^2 + t^2 + u^2) \tag{6.77}$$

Using the relations (6.55)-(6.66) between the amplitudes with different particle content it is easy to find the corrections and the counterterms for all other four-particle interactions of $\mathcal{N} = 4$ multiplets in this theory.

6.5 Different κ-gauges

The Killing gauge is obviously not the only choice for fixing the κ-gauge. In [19] it was proposed to use the gauge where $\theta_1 = 0$ and $\theta_2 = \lambda$. Then the supersymmetry transformations will be:

$$\begin{aligned} \delta\bar{\theta} &= \bar{\epsilon} + \bar{\kappa}(1 - \Gamma) + \xi^\mu \partial_\mu \bar{\theta} \\ \delta X^m &= \bar{\epsilon}\Gamma^m\theta - \bar{\kappa}(1 - \Gamma)\Gamma^m\theta + \xi^\mu \partial_\mu X^m. \end{aligned} \tag{6.78}$$

where $\Gamma = \begin{pmatrix} 0 & \zeta \\ \tilde{\zeta} & 0 \end{pmatrix}$ was defined in (6.15).

The requirement $\delta\bar{\theta}_1 = 0$ and the static gauge condition $\delta X^p = 0$ determine the connection between the κ and ϵ parameters, so that the final supersymmetry transformations are [19]:

$$\begin{aligned} \delta_{\kappa+\varepsilon+\xi}\bar{\lambda} &= \bar{\epsilon}_2 + \bar{\epsilon}_1\zeta + \xi^\mu\partial_\mu\bar{\lambda}, \\ \delta_{\kappa+\varepsilon+\xi} y^t &= (\bar{\epsilon}_2 - \bar{\epsilon}_1\zeta)\Gamma^t\lambda + \xi^\mu \partial_\mu y^t, \\ \delta_{\kappa+\varepsilon+\xi} A_\mu &= (\bar{\epsilon}_1\zeta - \bar{\epsilon}_2)(\Gamma_\mu + \Gamma_t \partial_\mu y^t)\lambda \\ &\quad + (\frac{1}{3}\bar{\epsilon}_2 - \bar{\epsilon}_1\zeta)\Gamma_m\lambda\bar{\lambda}\Gamma^m\partial_\mu\lambda + \xi^\rho\partial_\rho A_\mu + \partial_\mu\xi^\rho A_\rho. \end{aligned} \tag{6.79}$$

The index m is a d=10 index, which includes both $\mu(p)$ and t values (t is a six-dimensional index for scalar fields). The induced world-volume metric in this case is:

$$G_{\mu\nu} = \eta_{mn} L^m_\mu L^n_\nu, \qquad L^m_\mu = \partial_\mu X^m - i\bar{\lambda}\Gamma^m \partial_\mu \lambda \tag{6.80}$$

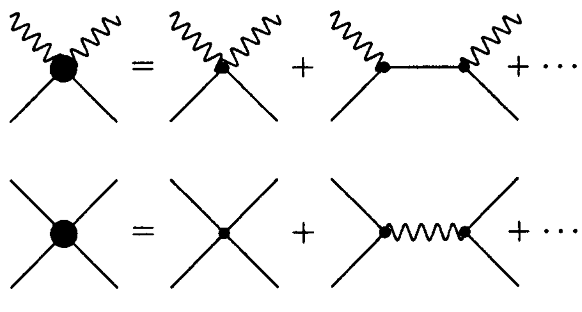
and

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} - i\bar{\lambda}(\Gamma_\mu + \Gamma_t \partial_\mu y^t)\partial_\nu \lambda + i\bar{\lambda}(\Gamma_\nu + \Gamma_t \partial_\nu y^t)\partial_\mu \lambda. \tag{6.81}$$

The choice of κ-gauge affects only the part of the Lagrangian that contains fermions leaving the purely bosonic terms unchanged. In the $\theta_1 = 0$ gauge the fermionic part of the Lagrangian up to the quartic order in the fields is:

$$\begin{aligned} L^{(4)}_{ferm} &= i\bar{\theta}\, \partial\!\!\!/\theta + i\alpha\bar{\theta}\Gamma^t \partial_\mu y^t \partial^\mu \theta + \frac{i\alpha}{2}(\bar{\theta}\Gamma^\mu \partial_\nu \theta - \mathrm{i}\bar{\theta}\Gamma^\nu \partial_\mu \theta)\mathrm{F}_{\mu\nu} \\ &+ \frac{\alpha^2}{2}(\bar{\theta}\Gamma^t \partial_\mu \theta \bar{\theta}\Gamma^t \partial_\mu \theta - (\bar{\theta}\, \partial\!\!\!/\theta)^2) - i\alpha^2(\partial_\mu y^t \partial_\nu y^t \bar{\theta}\Gamma^\nu \partial_\mu \theta - \frac{1}{2}\partial_\mu y^t \partial_\mu y^t \bar{\theta}\, \partial\!\!\!/\theta) \\ &+ \frac{i\alpha^2}{2}((\bar{\theta}\Gamma^\mu \partial_\nu \theta + \bar{\theta}\Gamma^\nu \partial_\mu \theta)\mathrm{F}_{\mu\lambda}\mathrm{F}_{\lambda\nu} - \frac{1}{2}\bar{\theta}\, \partial\!\!\!/\theta \mathrm{F}_{\mu\nu}\mathrm{F}_{\mu\nu}) \\ &+ \frac{i\alpha^2}{2}\mathrm{F}_{\mu\nu}(\bar{\theta}\Gamma^t \partial_\mu y^t \partial_\nu \theta - \bar{\theta}\Gamma^t \partial_\nu y^t \partial_\mu \theta). \end{aligned} \tag{6.82}$$

In this gauge the Lagrangian contains the three-particles vertices along with the quartic terms. Four-particle tree amplitudes with fermions will contain a set of tree diagrams where line $\sim\!\sim\!\sim$ denotes both gauge A_μ and scalar y^t bosons and —— stands for the fermions θ :



Calculations of these tree amplitudes show that they are the same for both cases of the κ-gauges. Both theories have the same one loop corrections and nontrivial counterterms in d=4.

6.6 Outlook

The Born-Infeld type actions have received a lot of attention in the last few years. This type of action plays a crucial role for the D3 brane theory. The rich structure of interaction terms in this theory makes it especially interesting to study them. The other interesting question that arises in this theory is a question about the form of supersymmetry transformations after the fixing of the $\kappa-$gauge.

We have presented the supersymmetry transformations, Feynman rules and one-loop corrections to the κ-gauge fixed D3 brane Born-Infeld theory up to the quartic order. We would like to stress here that the Feynman rules following from the gauge-fixed supersymmetric Born-Infeld action turned out to be rather simple at least for the 4-point one loop calculations, which has allowed us to perform the quantum calculation in this very unusual theory.

Our calculations show that D3 brane theory in the flat IIB background has a nontrivial counterterm in d=4. We have found that the structure of the counterterm as well as finite corrections includes terms like $(\partial \mathrm{F})^4$. We hope that our calculations may help to shed some light on the relations between the properties of the fundamental theory including the D3 branes and the quantum $\mathcal{N} = 4$ supersymmetric YM gauge theory.

We also consider these calculations as a preparation for the studies of the more complicated theories, e.g. the D3 brane action in $AdS_5 \times S^5$ background or, possibly, for a few interacting D3 branes with non-Abelian gauge field in the action. We have found that the use of the Killing gauge for κ -symmetry combined with the helicity amplitudes technique has simplified our calculations significantly. Hopefully these methods will be useful for calculations in some other class of interesting problems.

7 Conclusion

We have studied N=2 supersymmetric extreme black holes and found the expressions for the black hole entropy as a function of its electric and magnetic charges. The black hole entropy (given by $\frac{1}{4}$ of the area of the black hole horizon) is connected with the extremum of the central charge of N=2 extended supersymmetry algebra. Due to its topological nature the extremal value of the central charge depends only on the electric and magnetic charges of the black hole, although this dependence is hidden in the dependence on the scalar fields z^i (moduli fields). These scalar fields are described by the attractors and their values near the black hole horizon are independent of their values at infinity and again depend only on the topological characteristics of the black hole - its electric and magnetic charges. The fact that a black hole has no "scalar hair" manifests the topological nature of the black hole entropy.

The dependence of the moduli fields on the electric and magnetic charges is encoded in the system of the complicated nonlinear equations. We found the explicit solutions for this system for different types of compactification and Calabi-Yau manifolds and studied their behavior.

The new results presented in this dissertation are as follows:

(1) The new concept of a Double-extreme black hole was introduced.

(2) The stabilization equations were solved and the solution for the black hole entropy as a function of the charges was given.

(3) STU black holes were studied.

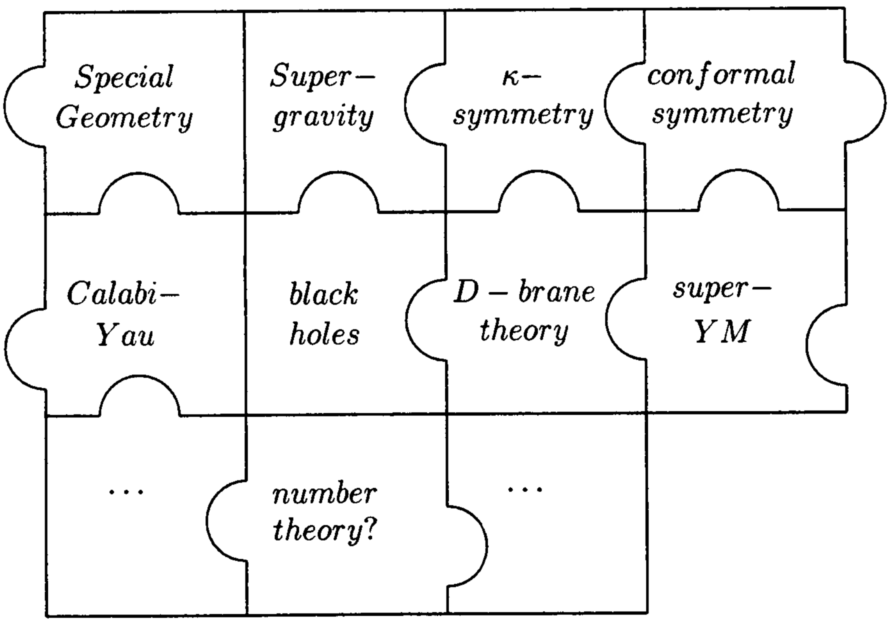
(4) The entropy of N=2 black holes associated with the general Calabi-Yau moduli space was found. Particular examples were studied.

(5) d=5 black holes were studied as well as the critical points of the effective potential and phase transitions for the different Calabi-Yau manifolds.

(6) We have also studied D3-branes and one-loop corrections to the Born-Infeld

action. The supersymmetry transformations in a particular $\kappa-$gauge (Killing gauge) are found. The connection between the helicity amplitude description and D-3 branes was established.

As our understanding of the theory grows, new facts and ideas emerge. Recent development in the theory of supersymmetric black holes and its possible connection to number theory proposed by G. Moore [102] raises new questions concerning the entropy of the black holes and the uniqueness of the solution for the stabilization equations. We have discussed this problem at the end of Ch.(4) and given some arguments concerning the uniqueness of the solution for the case when the moduli space belongs to the unique Calabi-Yau manifold. The uniqueness of the solution for the general picture when the moduli space contains different regions described by different Calabi-Yau geometry with possible phase transitions (as it was in case of d=5 black holes) is still a question and a subject for a future investigation. Puzzles of the theory continue to grow and new connections are established every day.



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APPENDIX

We use chiral representation for d=6 dimensional $\gamma^{(6)}$ matrices following M. Sohnius notations [95]. In this representation $\gamma^{(6)}$ is written as

$$\tilde{\gamma}^t = \begin{pmatrix} 0 & (\tilde{\sigma}_t)_{IJ} \\ (\tilde{\sigma}_t^{-1})^{IJ} & 0 \end{pmatrix} \tag{7.1}$$

where matrices $(\tilde{\sigma}_t)_{IJ}$ satisfies the conditions:

$$\begin{aligned} \mathrm{tr}\tilde{\sigma}_t\tilde{\sigma}_{t'}^{-1} &= 4\delta_{tt'} \\ (\tilde{\sigma}_t^{-1})^{IJ} &= -\frac{1}{2}\epsilon^{IJKL}(\tilde{\sigma}_t)_{KL} \end{aligned} \tag{7.2}$$

A general d=10 32 component complex spinor will be:

$$\theta = \begin{pmatrix} \lambda_{\alpha I} \\ \chi_\alpha^I \\ \bar{\omega}_I^{\dot{\alpha}} \\ \bar{\varphi}^{\dot{\alpha} I} \end{pmatrix}, \quad \bar{\theta} = \left(\omega^{\alpha I}, \varphi_I^\alpha, \bar{\lambda}_{\dot{\alpha}}^I, \bar{\chi}_{\dot{\alpha} I}\right) \tag{7.3}$$

The d=10 Majorana and chirality conditions [95] are:

$$\theta = C^{(10)}\bar{\theta}^T \quad , C^{(10)} = C^{(4)} \otimes C^{(6)} = \begin{pmatrix} -\epsilon_{\alpha\beta} & 0 \\ 0 & -\epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7.4}$$

$$\theta = \frac{1}{2}(1 - \Gamma^{(11)})\theta, \quad \Gamma^{(11)} = \gamma^5 \otimes \gamma^{(7)} = i\gamma^5 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.5}$$

They will give the constraints on the spinors components:

$$\bar{\varphi}^{\dot{\alpha} I} = \bar{\lambda}^{\dot{\alpha} I} = (\lambda_{\beta I})^\dagger \epsilon^{\dot{\beta}\dot{\alpha}}, \quad \bar{\omega}_I^{\dot{\alpha}} = \bar{\chi}_{\dot{\alpha} I} = (\chi_\beta^I)^\dagger \epsilon^{\dot{\beta}\dot{\alpha}} \tag{7.6}$$

$$\bar{\omega}_I^{\dot{\alpha}} = \chi_\alpha^I = 0. \tag{7.7}$$

The surviving 16 components are

$$\theta = \begin{pmatrix} \lambda_{\alpha I} \\ 0 \\ 0 \\ \bar{\lambda}^{\dot{\alpha} I} \end{pmatrix}; \quad \bar{\theta} = \left(0, \lambda_I^\alpha, \bar{\lambda}_{\dot{\alpha}}^I, 0\right). \tag{7.8}$$

The new scalar fields s_{IJ} with manifest SU(4) indexes $I, J = 1, \cdots 4$ are

$$s_{IJ} = -\frac{1}{2}(\tilde{\sigma}_t)_{IJ} y^t; \; s^{IJ} = \frac{1}{2}(\tilde{\sigma}_t^{-1})^{IJ} y^t$$

and $y^t y^t = s^{IJ} s_{IJ}$.

VITA

Marina Valentinovna Shmakova graduated from Moscow State University (Russia), Physics Department with the Master of Science degree in physics. For six years she worked as a researcher at the Lebedev Physics Institute, Academy of Science of Russia. In 1994 she joint the University of Tennessee to pursue the Doctor of Philosophy degree in Physics. From 1995 to 1999 she worked at the Theory group at Stanford Linear Accelerator Center as a graduate student. Her research was done in collaboration with the Theory group at the Physics department of Stanford University. The doctoral degree was received in 1999.