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To the Graduate Council:

I am submitting herewith a thesis written by Mohammed-Reza Bannazadeh-Mahani, entitled "Characterizations of the Eigenvalues and Eigenvectors of Matrices." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

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CHARACTERIZATIONS OF THE EIGENVALUES AND EIGENVECTORS OF MATRICES

A Thesis

Presented to

The Graduate Council of

The University of Tennessee

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

---

by

Mohammed-Reza Bannazadeh-Mahani

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ABSTRACT

The main purpose of this paper is to determine the eigenvalues and eigenvectors of especial types of matrices and to present basic theorems about them.

Along with the definitions of the generalized eigenvalues and the eigenvectors of matrices, a method to compute the generalized eigenvectors is given. e^{At} is computed by the use of eigenvalues of matrix A. A convex set and a convex function are defined. The localization of eigenvalues of matrices are studied. The doubly stochastic matrices, permutation matrices, compound matrices, irreducible matrices, and primitive matrices are defined. The eigenvalues and eigenvectors of companion matrices, Hessenberg matrices, and permutation matrices are investigated.

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INTRODUCTION

This study is an attempt to investigate the eigenvalues and eigenvectors of special types of matrices. In Chapter I eigenvalues and eigenvectors of general matrices are defined and a method of finding generalized eigenvectors of real matrices is described. The singular values and singular vectors are defined. The eigenvalues of matrices have been used to compute the exponential matrix e^{At}. Examples in Chapter I illustrate some of the theorems and propositions.

In Chapter II, a convex set and a convex function are defined. The doubly stochastic matrices, permutation matrices and compound matrices are defined, and the basic theorems about eigenvalues of compound matrices are stated. The eigenvalues and singular values of matrices are compared and the localization of eigenvalues of matrices are observed.

Chapter III deals with the eigenvalues and eigenvectors of special types of matrices. We will discuss a little about graph theory. Irreducible matrices and primitive matrices are defined. The basic theorems related to eigenvalues of imprimitive matrices are given. The Hessenberg matrices are defined and eigenvalues and eigenvectors of them are investigated. The properties of companion matrices are presented. A computation of generalized eigenvectors and eigenvalues is shown. A computation of eigenvalues and eigenvectors of special types of permutation matrices is shown by an example. We express theorems

about the eigenvalues and eigenvectors of permutation matrices. Finally, we present a method to compute the eigenvalues and eigenvectors of permutation matrices.

CHAPTER I

EIGENVALUES AND EIGENVECTORS OF REAL MATRICES

In section 1 the eigenvalues and eigenvectors are defined for general matrices. It will also show a method to find generalized eigenvectors of matrices. The definition of a singular value and a singular vector of a matrix is given. In section 2, the exponential matrix e^{At} is formulated.

Throughout this paper F will denote a field; and the field of complex numbers and the field of real numbers are denoted by C and R respectively. The capital letters A, B, D, and E shall denote matrices and $M_{m,n}(F)$ is the set of all $m \times n$ matrices with entries in F (when $m = n$, it is denoted by $M_n(F)$). Vector spaces are denoted by U, V, and W; and the capital letters X, Y, and Z shall designate vectors;and the small letters x, y, and z denote their components. Greek letters will designate eigenvalues. The n-tuple spaces over the real numbers and the complex numbers are denoted by R^n and C^n respectively.

I. A COMPUTATION OF EIGENVALUES AND EIGENVECTORS OF MATRICES

DEFINITION 1.1. Let V be a vector space over the field F and let L be a linear operator on V. An <u>eigenvalue</u> of L is a scalar λ in F such that there is a nonzero vector X in V with $LX = \lambda X$. If λ is an eigenvalue of L, then any nonzero vector X such that $LX = \lambda X$ is called an <u>eigenvector</u> of L corresponding to the eigenvalue λ.

If A is an $m \times n$ matrix over the field F and X is a nonzero vector in an n-tuple space over F such that

$$AX = \lambda X \tag{1.1}$$

then, X is an eigenvector of A corresponding to the eigenvalue λ. The function

$$f(\lambda) = \det (A - \lambda I) \tag{1.2}$$

is called the <u>eigenfunction</u> of A. The solutions of the eigenfunction of A are the eigenvalues of A.

DEFINITION 1.2. Let λ_0 be an eigenvalue of A and Z_k be a vector for which

$$(A - \lambda_0 I)^{k-1} Z_k \neq 0 \tag{1.3}$$

but for which

$$(A - \lambda_0 I)^k Z_k = 0 \tag{1.4}$$

Z will be called a generalized eigenvector or an eigenvector of rank k corresponding to the eigenvalue λ_0 . An eigenvector Z_1 is called a proper eigenvector, corresponding to the eigenvalue $_0$.

If $A \in M_n(f)$, then $(A - \lambda_0 I)^i Z_k$ are written as $n \times 1$ matrices. Those matrices denote vector Z_{k-i} such that

$$\begin{aligned} &(A - \lambda_0 I)\, Z_k = Z_{k-1} \\ &(A - \lambda_0 I)^2 Z_k = Z_{k-2} \\ &\cdots\cdots\cdots\cdots \\ &(A - \lambda_0 I)^{k-1} Z_k = Z_1 \ . \end{aligned} \tag{1.5}$$

REMARK 1.1. Let $A \in M_n(F)$ and let $\{\lambda_i\}$ be the set of its eigenvalues and Z_i be its eigenvectors corresponding to λ_i where λ_i are solutions of the eigenfunction

$$f(\lambda) = \det (A - \lambda I) = (\lambda - \lambda_1)^{r_1} (\lambda - \lambda_2)^{r_2} \ldots (\lambda - \lambda_n)^{r_n} . \tag{1.6}$$

We will call λ_i the r_i-fold eigenvalue of A , when λ_i is multiplicity of r_i . If λ_i is the r_i-fold eigenvalue and rank of $(A - \lambda_i I)$ is equal to n-1, there is a single proper eigenvector $Z_1(\lambda_i)$ that satisfy (1.1) and $r_i - 1$ generalized eigenvectors $Z_2(\lambda_i) \ldots Z_k(\lambda_i)$ which satisfy the $r_i - 1$ equations in (1.5).

THEOREM 1.1. Let $A \in M_n(R)$ and $\{\lambda_i\} \subset R$ be the set of

its eigenvalues. If λ_i is an r-fold eigenvalue of A , then the set of generalized eigenvectors belonging to A is given by

$$Z_j(\lambda_i) = \frac{1}{(j-1)!} Z^{(j-1)}(\lambda_i) \quad j = 2, 3 \ldots k \tag{1.7}$$

where $Z^{(j)}(\lambda_i)$ is jth derivative of $Z(\lambda_i)$ respect to λ_i .

PROOF. The eigenvalues λ_i of a matrix A are the solutions of the function $f(\lambda) = \det(A - \lambda I)$. If λ is not an eigenvalue of A , $f(\lambda) \neq 0$ and the equation

$$(A - \lambda I)\, Z(\lambda) = (f(\lambda) , 0 \ldots 0)^t \tag{1.8}$$

has a unique column vector $Z(\lambda)$ as a solution whose components are polynomials in λ . We shall omit the transpose symbol t and denote a column by a row in brackets. If the eigenvalues are all simple, a single eigenvector belongs to each eigenvalue and is given by $Z(\lambda_i)$ provided $Z(\lambda_i) \neq 0$;for when $\lambda = \lambda_i$, Equation (1.8) reduces to

$$(A - \lambda_i I)\, Z_i = 0$$

If λ_i is a r-fold eigenvalue and the rank of $(A - \lambda_i I)$ is equal to n-1, then there is a proper eigenvector Z_1 , and r-1 generalized eigenvectors $Z_2, Z_3, \ldots Z_k$. These satisfy equation (1.5) and r-1 equations obtained from it by differentiating r-1 times with respect to λ . Putting $\lambda = \lambda_i$ in (1.8) we have

$$(A - \lambda I)\, Z'(\lambda) - Z(\lambda) = (f'(\lambda), 00\ldots0) ,$$

$$(A - \lambda I)\, Z''(\lambda) - 2Z'(\lambda) = (f''(\lambda\ 00\ldots0) ,$$

. . . . . .

. . . . . .

$$(A - \lambda I)\, Z^{(r-1)}(\lambda) - (r-1)\, Z^{(r-2)}(\lambda) = (f^{r-1}(\lambda), 00\ldots0) .$$

Since λ_i is an r-fold eigenvalue and a solution of $f(\lambda)$, all vectors on the right vanish when $\lambda = \lambda_i$ and yield the system

$$(A - \lambda_i I)\, Z'(\lambda_i) - Z(\lambda_i) = (00\ldots00) ,$$

$$(A - \lambda_i I)\, Z''(\lambda_i) - 2Z'(\lambda_i) = (00\ldots00) ,$$

. . . . . .

$$(A - \lambda_i I)\, Z^{(r-1)}(\lambda_i) - (r-1)\, Z^{(r-2)}(\lambda_i) = (00\ldots00) ,$$

then we can write the system as follows:

$$(A - \lambda_i I)\, Z'(\lambda_i) = Z(\lambda_i) ,$$

$$(A - \lambda_i I)\, 1/2Z''_i(\lambda_i) = Z'(\lambda_i) ,$$

. . . . . .

. . . . . .

$$(A - \lambda_i I)\frac{1}{(r-1)!}\, Z^{r-1}(\lambda_i) = \frac{1}{(r-2)!}\, Z^{(r-2)}(\lambda_i) .$$

A comparison with equation $(A - \lambda_i I)\, Z_i = Z_{i-1}$ now reveals that a set of generalized eigenvectors belonging to λ_i is given by

$$Z_j(\lambda_i) = \frac{1}{(j-1)!}\, Z^{(j-1)}(\lambda_i), \quad j = 2, 3, \ldots r .$$

PROPOSITION 1.1. Let $A \in M_n(F)$, then the generalized

eigenvectors of different ranks are linearly independent.

PROOF. Let $\{Z_1, Z_2, \ldots Z_r\}$ be a set of generalized eigenvectors of different ranks.

If $C_1 Z_1 + C_2 Z_2 + C_3 Z_3 + C_4 Z_4 \ldots C_r Z_r = 0$

and premultiplying it r-1 times by the matrix $(A - \lambda_i I)$, then

$$(A - \lambda_i I)(C_1 Z_1 + C_2 Z_2 + \ldots C_r Z_r) = 0 ,$$

$$C_1(A - \lambda_i I) Z_1 + C_2(A - \lambda_i I) Z_2 \ldots\ldots C_r(A - \lambda_i I) Z_r = 0 .$$

By $(A - \lambda_i I) Z_i = Z_{i-1}$, we have

$$C_2 Z_1 + C_3 Z_2 \ldots C_r Z_{r-1} = 0 .$$

If we continue this action r-1 times, we will have

$$C_2 Z_1 + C_3 Z_2 \ldots\ldots C_r Z_{r-1} = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$C_{r-1} Z_1 + C_r Z_2 = 0$$
$$C_r Z_1 = 0 .$$

Since $Z_1 \neq 0$ then $C_r = 0$; and on passing upward through those equations we find that all the constants are zero.

EXAMPLE 1.1. In this example we want to show a computation of eigenvalues and generalized eigenvectors.

Let

$$A = \begin{pmatrix} 3 & 2 & -1 \\ -3 & -2 & 2 \\ -1 & -1 & 2 \end{pmatrix}$$

then,

$$f(\lambda) = (A - \lambda I) \equiv - (\lambda - 1)^3$$

and 1 is a 3-fold eigenvalue. By Equation (1.8) we can write

$$\begin{pmatrix} 3-\lambda & 2 & -1 \\ -3 & -2-\lambda & 2 \\ -1 & -1 & 2-\lambda \end{pmatrix} \quad Z(\lambda) = \begin{pmatrix} f(\lambda) \\ 0 \\ 0 \end{pmatrix} ,$$

where

$$Z(\lambda) = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} .$$

Then we have

$$\begin{pmatrix} 3-\lambda & 2 & -1 \\ -3 & -2-\lambda & 2 \\ -1 & -1 & 2-\lambda \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} f(\lambda) \\ 0 \\ 0 \end{pmatrix} .$$

We can find $z_1 = \lambda^2-2$, $z_2 = 4 - 3\lambda$, $z_3 = 1 - \lambda$, therefore $Z(\lambda) = (\lambda^2-2, 4-3\lambda , 1-\lambda)$. Since rank of (A-I) is 2 , this yields the only one proper eigenvector $Z_1 = Z(1) = (-1,1,0)$. The two generalized eigenvectors are

$$Z_2 = Z'_{(1)} = (2 - 3 - 1) ,$$

and

$$Z_3 = Z''_{(1)} = (2, 0, 0) .$$

DEFINITION 1.3. A Jordan block is a square matrix whose elements are zero except for those on the principal diagonal, which are all equal, and those on the first superdiagonal, which are all equal to unity; thus

$$J = \begin{pmatrix} \lambda & 1 & 0 & & 0 \\ 0 & \lambda & 1 & & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & & \lambda \end{pmatrix} .$$

PROPOSITION 1.2. For a matrix $A \in M_n(R)$ assume λ_j is an r-fold eigenvalue of A, with a proper eigenvector Z_1 and r-1 generalized eigenvectors $Z_2, Z_3 \ldots Z_r$. Then for any nonzero vector Z in the space spanned by $\{Z_1, Z_2 \ldots Z_r\}$, $(A - \lambda_j I)^{r-1} Z = CZ_1$.

PROOF. A set of generalized eigenvectors of λ_i in Proposition 1.1 are linearly independent. Therefore Z can be expressed in the form

$$Z = C_1 Z_1 + C_2 Z_2 \ldots C_r Z_r .$$

Since

$$(A - \lambda_j I)\ Z_i = Z_{i-1}, \quad i = 2 \ldots r, \text{ and } (A - \lambda_j I)^i Z_i = 0 ,$$

it follows that

$$\begin{aligned} (A - \lambda_j I)\, Z &= (A - \lambda_j I)\ (C_1 Z_1 + C_2 Z_2 \ldots C_r Z_r) \\ &= C_2 Z_1 + C_2 Z_3 \ldots C_r Z_{r-1} \\ &\quad \cdot \quad \cdot \quad \cdot \\ (A - \lambda_j I)^{r-2} Z &= C_{r-1} Z_1 + C_r Z_2 \\ (A - \lambda_j I)^{r-1} Z &= C_r Z_1 . \end{aligned}$$

DEFINITION 1.4. A Jordan matrix is a matrix whose diagonal is formed by Jordan blocks, such that

$$J = \begin{pmatrix} J_1 & & & & 0 & 0 \\ & \ddots & & & 0 & 0 \\ & & J_2 & & & \\ & & & J_j & & \\ & & & & \ddots & \\ 0 & 0 & & & & \\ 0 & 0 & \cdots & \cdots & J_k & \end{pmatrix},$$

where J_i are Jordan blocks.

THEOREM 1.2. If $A \in M_n(F)$, a nonsingular matrix exists such that

$$Q^{-1}AQ = \begin{pmatrix} J_1 & \cdots & 0 & \cdots & 0 \\ 0 & \ddots & 0 & & 0 \\ & & J_2 & & \\ & & & J_j & \\ 0 & \cdots & & & J_k \end{pmatrix}, \qquad (1.9)$$

where the J_i are Jordan blocks. The same eigenvalue may occur in different blocks, but the number of distinct blocks corresponding to a given eigenvalue is equal to the number of independent eigenvectors corresponding to that eigenvalue.

From Proposition 1.2, we can arrange n linearly independent generalized eigenvectors (including proper eigenvectors) $Z_1, Z_2, \ldots, Z_n$ in group according to the distinct eigenvalues to form the matrix

$$E = (Z_1, Z_2, \ldots, Z_n) \qquad (1.10)$$

Since the vectors Z_i are linearly independent, we can form a set $\{Z^1, Z^2, \ldots, Z^n\}$ for which

$$Z_i Z^j = \delta_{ij} = \begin{cases} 1 , & \text{if } i = j \\ 0 , & \text{if } i \neq j \end{cases} .$$

These equations show that the matrix having Z^i as rows the inverse of E :

$$E^{-1} = \begin{pmatrix} Z^1 \\ Z^2 \\ \vdots \\ Z^n \end{pmatrix} . \qquad (1.11)$$

THEOREM 1.3. Let $A \in M_n(R)$, then

$$E^{-1} A E = \begin{pmatrix} J_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ & J_j & \\ \vdots & \ddots & \vdots \\ 0 & & J_t \end{pmatrix}$$

where E , and E^{-1} are matrices in (1.10) and (1.11)

PROOF. Let $A \in M_n(R)$ and $E = (Z_1, Z_2 \ldots Z_n)$

where E^{-1} is the matrix in (1.11) , then

$$E^{-1} A E = \begin{pmatrix} Z^1 \\ Z^2 \\ \vdots \\ Z^n \end{pmatrix} A (Z_1, Z_2 \ldots Z_n) = \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{pmatrix} (AZ_1, AZ_2 \ldots AZ_n) .$$

Since $AZ_1 = \lambda Z_1$ and

$$(A - \lambda_i I)\, Z_i = Z_{i-1} \quad \text{or} \quad AZ_i = \lambda_i Z_i + Z_{i-1} ,$$

then

$$E^{-1} A E = \begin{pmatrix} Z^1 \\ \vdots \\ Z^n \end{pmatrix} (\lambda_1 Z_1, \lambda Z_2 + Z_1, \lambda_1 Z_3 + Z_2 \cdots \lambda_1 Z_1 + Z_{r-1} \cdots \lambda_t Z_n) ,$$

where λ_i is an r-fold eigenvalue.

Then

$$E^{-1} A E = \begin{pmatrix} Z^1\lambda Z_1 & Z^1(\lambda Z_2 - Z_1) \cdots Z^1(\lambda_t Z_n) \\ Z^2\lambda_1 Z_1 & Z^2\lambda_1 Z_2 \\ \cdot & \cdot \\ Z^n\lambda_1 Z_1 & Z^n(\lambda_1 Z_n Z_1) \cdots Z^n\lambda_t Z_n \end{pmatrix} .$$

Therefore

$$E^{-1} A E = \begin{pmatrix} \lambda_1 & 1 & 0 \cdots 0 & 0 \\ 0 & \lambda_1 & 1 \cdots 0 & 0 \\ 0 & 0 & 0 \cdots 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_t \end{pmatrix} = \begin{pmatrix} J_1 & \cdots & 0 & 0 \\ 0 & J_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & J_t \end{pmatrix} ,$$

where J_i are Jordon blocks.

REMARK 1.2. If $f(\lambda)$ is chosen as the jth component, instead of the first component, all the equations yield the same eigenvector $Z(\lambda)$. The computation is shown in the following example, and also

we will show the computation of Jordan matrix for A .

EXAMPLE 1.2.

Let

$$A = \begin{pmatrix} 2 & 2 & -6 \\ 4 & 2 & -3 \\ 12 & 3 & -8 \end{pmatrix} .$$

Then,

$$f(\lambda) = \det (A - \lambda I) = (1 - \lambda)^3 ,$$

Therefore 1 is a 3-fold eigenvalue.

Since

$$\begin{pmatrix} 2-\lambda & 2 & -6 \\ 4 & 2-\lambda & -3 \\ 12 & 3 & 8-\lambda \end{pmatrix} \begin{pmatrix} z_{11} \\ z_{12} \\ z_{13} \end{pmatrix} = \begin{pmatrix} f(\lambda) \\ 0 \\ 0 \end{pmatrix} ,$$

we have

$$z_{11} = (\lambda-1)(\lambda+7), z_{12} = 4(\lambda-1), z_{13} = (\lambda-1)12 ,$$

then

$$Z_1 = (\lambda-1)(\lambda+7, 4, 12) .$$

Let $Y_1 = Z_1(\lambda)/\lambda-1 = (\lambda + 7, 4, 12)$.

If we choose $f(\lambda)$ as a second component in computation, then

$$\begin{pmatrix} 9-\lambda & 2 & -6 \\ 4 & 2-\lambda & 3 \\ 12 & 3 & -8-\lambda \end{pmatrix} \begin{pmatrix} z_{21} \\ z_{22} \\ z_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ f(\lambda) \\ 0 \end{pmatrix} .$$

Let $Z_2 = (z_{21}\ z_{22}\ z_{23})$

then, $z_{21} = 2(\lambda-1)$, $z_{22} = (\lambda-1)(\lambda)$, $z_{23} = (\lambda-1)3$.

Hence

$$Z_2 = (\lambda-1)\ (2,\ \lambda,\ 3) .$$

Let

$$Y_2 = Z_2/(\lambda-1) = (2,\ \lambda,\ 3) .$$

If we choose $f(\lambda)$ as the third component, then

$$\begin{pmatrix} 2-\lambda & 2 & -6 \\ 4 & 2-\lambda & 3 \\ 12 & 3 & -8-\lambda \end{pmatrix} \begin{pmatrix} z_{31} \\ z_{32} \\ z_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ F(\lambda) \end{pmatrix}$$

Let $Z_3 = (z_{31},\ z_{32},\ z_{33})$,

then $z_{31} = (\lambda-1)6$, $z_{32} = (\lambda-1)3$, $z_{33} = (\lambda-1)(10-\lambda)$.

Hence $Z_3 = (\lambda-1)(6,\ 3,\ 10-\lambda)$, $Y_3 = z_3/(\lambda-1) = (6,\ 3,\ 10-\lambda)$.

When $\lambda = 1$, the proper eigenvectors are the same.

$Y_1(1) = 4(2,1,\ 3)$, $Y_2(1) = (2,\ 1,\ 3)$ $Y_3(1) = 3(2,\ 1,\ 3)$.

But since the rank of $(A-I)$ is equal to 1 , there must be a second proper eigenvector Z_3 and a generalized eigenvector $Z_2 = Z_1'$

corresponding to $Z_1 = (2, 1, 3)$ which is given by

$$1/4\ Y_1' = (1/4, 0, 0) = Z_{12} ,$$

or

$$Y_2' = (0, 1, 0) = Z_{22}, \text{ and } 1/3\ Y_3' = (0, 0, -1/3) = Z_{23} .$$

Since a generalized eigenvector satisfies (1.5), we have with $\lambda_1 = 1$

$(A-\lambda_1 I)\ Z_{21} = Z_1$, $(A-\lambda_1 I)\ Z_{22} = Z_1$, $(A-\lambda_1 I)\ (Z_{21} - Z_{22}) = 0$,

that is Z_{21}, Z_{22} are proper eigenvectors. Thus we have three new proper eigenvectors.

$$Z_{21} - Z_{22} = (1/4, -1, 0) = 1/4(1, 4, 0) ,$$

$$Z_{21} - Z_{23} = (1/4, 0, 1/3) = 1/12(3, 0, 4),$$

and

$$Z_{22} - Z_{23} = (0, 1, 1/3) = 1/3\ (0, 3, 1) .$$

Thus any of the vectors $(1, 4, 0)$, $(3, 0, 4)$, $(0, 3, 1)$ may be chosen for Z_3 and all three lie in a plane with Z_1 . The generalized eigenvector corresponding to $Z_1 = (2, 1, 3)$ is simply chosen as $Z_2 = (0, 1, 0)$.

If Z_3 is referred to Z_1 , then the Jordan matrix of A is

$$E^{-1} A E = \begin{pmatrix} Z^1 \\ Z^2 \\ Z^3 \end{pmatrix} A\ (Z_1, Z_2, Z_3)$$

$$= \begin{pmatrix} Z^1 \\ Z^2 \\ Z^3 \end{pmatrix} (Z_1, Z_2 + Z_1, Z_3) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

If Z_3 is referred to Z_2, then the Jordan matrix of A is

$$E^{-1} A E = \begin{pmatrix} z^1 \\ z^2 \\ z^3 \end{pmatrix} A (Z_1, Z_2, Z_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} z^1 \\ z^2 \\ z^3 \end{pmatrix} (Z_1, Z_2, Z_3 + Z_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} .$$

REMARK 1.3. Let an arbitrary set of linearly independent vectors X_i be chosen and assigned to arbitrary eigenvalues λ_i singly, or in groups. In each group specify which vectors are to be proper and which are generalized eigenvectors, and refer each generalized eigenvector to its proper companion. For each specific and order and grouping we can construct a matrix A which has the eigenvalues and eigenvectors thus specified. This is rather surprising since the generalized eigenvectors must conform to equations

$$(A - \lambda I) X_i = X_{i-1} .$$

To achieve this result, let J be the Jordan matrix which corresponds to the order and grouping of the eigenvalues with due regard to the reference of generalized to proper eigenvectors. Then the required matrix A may be obtained at once from the equation.

$$E^{-1} A E = J , \quad \text{or} \quad A = E J E^{-1}, \tag{1.12}$$

where E has the columns formed by generalized eigenvectors in (1.10) in the order corresponding to J. We note again that each generalized eigenvector must be referred to its proper companion. To be specific,

if the proper Z_1 has the generalized Z_2 and Z_3 referred to it, we write

$$Z_2 = Z_1' \ , \ Z_3 = Z_1'' \ .$$

EXAMPLE 1.3. Find a matrix such that the eigenvalues and the corresponding eigenvectors are

$$\lambda_1 = 1, \ X_1 = (2, 0, 1), \lambda_2 = 2, \ X_2 = (3, 2, 1),$$

and

$$\lambda_3 = 3, \ \ X_3 = (2, 2, 1) \ .$$

SOLUTION. In this case,

$$E = \begin{pmatrix} 2 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} , \qquad E^{-1} = \begin{pmatrix} 0 & -1/2 & 1 \\ -1 & 1/2 & 2 \\ 1 & 0 & -2 \end{pmatrix} ,$$

and

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} .$$

Since

$$A = E \, J \, E^{-1} \ ,$$

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1/2 & 1 \\ -1 & 1/2 & 2 \\ 1 & 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 1 & 8 \\ 2 & 2 & 2 \\ 1 & 1/2 & -1 \end{pmatrix} .$$

DEFINITION 1.4. The inner product of two $n \times 1$ column vectors X, Y with real or complex elements is a scalar quality, denoted by

$$(X, Y) = X^*Y ,$$

where X^* is the conjugate transpose of X .

DEFINITION 1.5. The vectors Y_i such that

$$Y^* A = \lambda_i Y^* \qquad A^*Y_i = \overline{\lambda}_i Y_i \tag{1.13}$$

are called left eigenvectors of $A \in M(C)$. To distinguish these from X_i such that $AX_i = \lambda X_i$ we call the X_i right-eigenvectors of A .

DEFINITION 1.6. Two vectors X and Y are orthogonal if

$$(X, Y) = 0 .$$

THEOREM 1.4. A right-eigenvector X_i and left-eigenvector corresponding to distinct eigenvalues λ_i, λ_j are orthogonal.

THEOREM 1.5. If $A \in M_n(C)$ has n linear independent right-eigenvectors X_i , then to each X_i there corresponds a left-eigenvector Y_i such that

$$(Y_j, X_i) = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

PROPOSITION 1.3. Using the condition and notation of Theorem

1.5. Let P, Q, denote the matrices whose jth columns are X_j, Y_j respectively. Then

(i) $Q^*P = P^* Q = I$,

(ii) $Q^*AP = P^* AP = D$,

where D is a diagonal matrix.

DEFINITION 1.7. Two finite sets of vectors $\{X_1, X_2 \ldots X_n\}$ and $\{Y_1, Y_2 \ldots Y_n\}$ are biorthogonal if $(Y_i, X_j) = 0$ $i, j = 1, 2, \ldots n$ $i \neq j$

THEOREM 1.6. If sets $\{X_i\}$ and $\{Y_i\}$ of vectors are biorthogonal and $(X_i, Y_i) \neq 0$ if a vector Z can be expressed as a linear combination of the X_i, then the coefficients in the expansion are given by

$$\alpha_i = \frac{(Y_i, Z)}{(Y_i, Z_i)} \tag{1.14}$$

where

$$Z = \sum_{i=1}^{n} \alpha_i X_i .$$

DEFINITION 1.8. If $A \in M_{m,n}(C)$, μ is a nonzero number, and X,Y are vectors such that

$$AX = \mu Y, \quad A^*Y = \mu X \tag{1.15}$$

then μ is called a <u>singular value</u> of A and X, Y are called a pair of singular vectors corresponding to A .

REMARK 1.5. The equations

$$AX = \mu Y \ , \quad A^*Y = \mu X$$

imply

$$\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \begin{pmatrix} Y \\ X \end{pmatrix} = \mu \begin{pmatrix} Y \\ X \end{pmatrix} \tag{1.16}$$

so that μ is an eigenvalue of a hermitian matrix, and therefore real. Also, if X, Y are singular vectors corresponding to μ, then X, $-Y$ are singular vectors corresponding to $-\mu$. Hence, singular values occur in pairs $\pm \mu$ we need to consider only singular values that are greater than or equal to zero.

From (1.15)

$$A^*AX = \mu A^*Y = \mu^2 X \ . \tag{1.17}$$

Hence X is an eigenvector of A^*A corresponding to eigenvalue μ^2. Similarly Y is an eigenvector of AA^* corresponding to the same eigenvalue μ^2.

THEOREM 1.7. If X_i, $i = 1 \ldots k$, is an orthonormal set of eigenvectors of A^*A corresponding to the nonzero eigenvalues

$$\mu^2_1 \ , \ \mu^2_1 \ \ldots \ \mu^2_k$$

defining $\quad Y_i = \mu_i^{-1} A X_i, \quad i = 1, \ldots k \ . \tag{1.18}$

Then:

(i) Y_i is an orthonormal set of eigenvectors of AA^* corresponding to the nonzero eigenvalues.

(ii) X_i, Y_i are a pair of vectors corresponding to the singular value μ_1 .

II. A COMPUTATION OF THE EXPONENTIAL MATRIX e^{At}

DEFINITION 1.9. Let $A \in M_n(F)$, and $e^{\lambda t}$ be the exponential function, and e^{At} be the exponential matrix. Then

$$e^{\lambda t} * e^{At} = \int_0^t e^{\lambda t}\, e^{A(t-\tau)}\, d\tau \tag{1.19}$$

is called convolution product.

PROPOSITION 1.4. For any matrix $A \in M_n(C)$ and any number λ, we have

$$e^{At} = e^{\lambda t} I + (A - \lambda I)\, e^{\lambda t} * e^{At}. \tag{1.20}$$

PROOF. Let

$$G(t) = e^{\lambda t} I + (A - \lambda I)\, e^{\lambda t} * e^{At}.$$

To prove that $G(t) = e^{At}$, it is sufficient to show that $G'(t) = AG(t)$ and $G(0) = I$. But, by using the following formula: if

$$G(u) = \int_{a(u)}^{b(u)} H(x, u)\, dx, \quad \text{then} \quad \frac{dG}{du} = \int_a^b \frac{dH}{du}\, dx + H(b, u)\, \frac{db}{du} - H(a, u)\, \frac{da}{du},$$

we can get,

$$(e^{\lambda t} * e^{At})' = \left(\int_0^t e^{\lambda t}\, e^{A(t-\tau)}\right)' = e^{\lambda t} I + Ae^{\lambda t} * e^{At}.$$

Now we have

$$G(t) = e^{\lambda t} I + (A - \lambda I)\, e^{\lambda t} * e^{At},$$

$$G'(t) = \lambda e^{\lambda t} I + (A - \lambda I)(e^{\lambda t} I + Ae^{\lambda t} * e^{At}) = AG(t),$$

and

$$G(0) = I + \int_0^0 e^{A\tau}\, d\tau = I .$$

Therefore

$$G(0) = I .$$

Hence $G(t) = e^{At}$.

PROPOSITION 1.5. For any matrix $A \in M_n(C)$ and any numbers $\lambda_1, \lambda_2 \ldots \lambda_n$ (not necessarily distinct), we have

$$e^{At} = \sum_{k=1}^{n-1} P_k \cdot r_{k+1}(t) + P_n \cdot r_n(t) * e^{At}, \qquad (1.21)$$

where

$$P_0 = I, \; P_k = \prod_{i=1}^{k} (A - \lambda_i I), \quad k = 1, 2, \ldots n$$

and

$$r_1(t) = e^{\lambda_1 t}, \; r_k(t) = e^{\lambda_1 t} * e^{\lambda_2 t} * \ldots * e^{A_k t} \quad k = 2,3 \ldots n.$$

PROOF. We can prove this proposition by induction. For n=1, this is exactly Proposition 1.4 with

$$\lambda = \lambda_1 \; , \; e^{At} = e^{\lambda t} I + (A - \lambda I)\, e^{\lambda t} * e^{At} .$$

If $n=\ell$,

$$e^{At} = \sum_{k=0}^{\ell-1} P_k \cdot r_{k+1}(t) + P_\ell \cdot r_\ell(t) * e^{At} . \qquad (1.22)$$

We want to prove that Formula (1.21) is true for $n=\ell+1$.

If $\lambda = \lambda_{\ell+1}$, (1.20) becomes

$$e^{At} = e^{\lambda_{\ell+1} t} I + (A - \lambda_{\ell+1} I) e^{\lambda_{\ell+1} t} * e^{At} .$$

Then from (1.22)

$$e^{At} = \sum_{k=0}^{\ell-1} P_k \cdot r_{k+1}(t) + P_\ell \cdot r_\ell(t) * [e^{\lambda_{\ell+1} t} I + (A - \lambda_{\ell+1}) e^{\lambda_{\ell+1} t} * e^{At}],$$

$$e^{At} = \sum_{k=0}^{\ell} P r_{k+1} + P_{\ell+1} \cdot r_{\ell+1} * e^{At} .$$

Hence the formula is true for $n = \ell + 1$.

Therefore,

$$e^{At} = \sum_{k=0}^{n-1} P_k \cdot r_{k+1} + P_n \cdot r_n * e^{At}, \text{ for any } n > 0.$$

THEOREM 1.8. If $A \in M_n(C)$ and $\lambda_1 \lambda_2 \ldots \lambda_n$ are the eigenvalues (including multiplicities) of the matrix A, then

$$e^{At} = \sum_{k=0}^{n-1} P_k \cdot r_{k+1}, \qquad (1.23)$$

where P_k and $r_k(t)$ are defined as in Proposition 1.5 .

PROOF. By Proposition 1.5 , we have

$$e^{At} = \sum_{k=1}^{n-1} P_k \cdot r_{k+1}(t) + P_n \cdot r_n * e^{At} .$$

By the Cayley-Hamilton theorem,

$$P_n = \prod_{i=1}^{n} (A - \lambda_i I) = 0 .$$

Then $$e^{At} = \sum_{k=0}^{n-1} P_k r_{k+1}(t) .$$

Let us define some new notations for any $A \in M_n(C)$ and any sequence $\lambda_1, \lambda_2 \ldots \lambda_m$ of distinct numbers, let

$$P_k(x) = \prod_{\substack{i=1 \\ i \neq k}}^{m} (X - \lambda_i), \quad k = 1, 2, \ldots n , \tag{1.24}$$

$$f(t) = \sum_{k=1}^{m} e^{\lambda_k t} \frac{P_k(A)}{P_k(\lambda_k)} , \tag{1.25}$$

$$g(t) = \sum_{k=1}^{m} \frac{e^{\lambda_k t}}{P_k(\lambda_k)} , \tag{1.26}$$

$$P = \prod_{k=1}^{m} (A - \lambda_k I) , \tag{1.27}$$

$g^{*k}(t)$ will denote the kth convolution product of $g(t)$ with itself.

PROPOSITION 1.6. With the above notation, and any sequence $\lambda_1, \lambda_2 \ldots \lambda_m$ of distinct numbers,

$$e^{At} = f(t) + P \cdot g(t) * e^{At} . \tag{1.28}$$

PROOF. Let $H(t) = f(t) + P \cdot g(t) * e^{At}$, it is sufficient to show that $H'(t) = AH(t)$ and $H(0) = I$. But

$$f'(t) + P \cdot g(t) = \sum_{k=1}^{m} e^{\lambda_k t} \frac{P_k(A)}{P_k(\lambda_k)} + \prod_{k=1}^{m} (A-\lambda_k I)\left(\sum_{k=1}^{m} \frac{e^{\lambda_k t}}{P_k(\lambda_k)}\right) ,$$

then

$$f'(t) + P \cdot g = \sum_{k=1}^{m} \lambda_k e^{\lambda_k t} \frac{P_k(A)}{P_k(\lambda_k)} + \sum_{k=1}^{m} e^{\lambda_k t} A \frac{P_k(A)}{P_k(\lambda_k)}$$

$$- \sum_{k=1}^{m} \lambda_k e^{\lambda_k t} \frac{P_k(A)}{P_k(\lambda_k)}$$

$$= A \sum_{k=1}^{m} e^{\lambda_k t} \frac{P_k A}{P_k(\lambda_k)} = A\, f(t) .$$

Therefore ,

$$\begin{aligned} H'(t) &= f'(t) + P\cdot(g(t) * e^{At})' \\ &= f'(t) + P \cdot g(t)\, I + P \cdot A \cdot g(t) * e^{At} \\ &= A\, f(t) + A \cdot P\, g(t) * e^{At} = A \cdot H(t). \end{aligned}$$

Also $H(0) = f(0) + P \cdot g(0)\, e^{A\cdot 0}$. It can be shown that $g(0) = 0$. Furthermore, the Lagrange interpolation polynomial $\sum_{k=1}^{m} \frac{P_k(A)}{P_k(\lambda_k)}\, q_k(\lambda_k I)$ for the function of (A) is exact for polynomials in A of degree $\leq m$, so $A^0 = \sum_{k=1}^{m} \frac{P_k(A)}{P_k(\lambda_k)}\, I = I$.

Therefore $H(0) = f(0) = \sum_{k=1}^{m} \frac{P_k(A)}{P_k(\lambda_k)} = I$.

Thus $H(t) = e^{At}$ as required.

REMARK 1.6. If $A \in M_n(C)$ with n distinct eigenvalues $\lambda_1, \lambda_2 \ldots \lambda_n$, then $P = \prod_{k}^{m} (A - \lambda_k I) = 0$, by the Cayley-Hamilton theorem. Hence

$$e^{At} = f(t) .$$

PROPOSITION 1.7. Still using the above notation, and any sequence $\lambda_1, \lambda_2 \ldots \lambda_m$ of distinct numbers, then for any integer $q \geq 2$ we have

$$e^{At} = f(t) + \sum_{j=1}^{q-1} p^j\, g^{*j}(t) * f(t) + P^q \cdot g^{*q}(t) * e^{At} \quad (1.29)$$

PROOF. We will use induction on q to prove this theorem. By Proposition 1.6, we have

$$e^{At} = f(t) + P \cdot g(t) * e^{At}$$

If we use Formula (1.28) twice, then we will have

$$e^{At} = f(t) + P \cdot g(t) * (f(t) + P \cdot g(t) * e^{At})$$

$$= f(t) + P \cdot g(t) * f(t) + P^2 \cdot g(t) * g(t) * e^{At} .$$

Hence the Formula (1.29) is true for $q = 2$. Suppose that the Formula (1.29) is true for $q = n$,

$$e^{At} = f(t) + \sum_{j=1}^{n-1} P^j \cdot g^{*j}(t) * f(t) + P^n . g^{*n} * e^{At}. \quad (1.30)$$

For $q = n + 1$, by using Formula (1.28) in (1.30) we will have

$$e^{At} = f(t) + \sum_{j=1}^{n-1} P^j . g^{*j}(t) * f(t) + P^n \; g^{*n} . (f(t)+P \cdot g(t) * e^{At})$$

$$= f(t) + \sum_{j=1}^{n} P^j \cdot g^{*j}(t) * f(t) + P^{n+1} . g^{*n+1} * e^{At} .$$

Hence this proposition is true for $q \geq 2$.

THEOREM 1.9. Let $A \in M_n(C)$ and $\lambda_1, \lambda_2 \ldots \lambda_m$ be the distinct eigenvalues of A , and let q be the maximum multiplicity of any of them, then

$$e^{At} = f(t) + \left(\sum_{j=1}^{q-1} P^j \; g^{*j}(t)\right) * f(t). \quad (1.31)$$

PROOF. We will use the same notation as we used in Propositions 1.6, 1.7 . By using Proposition 1.7 we have,

$$e^{At} = f(t) + \sum_{j=1}^{q-1} p^j . g^{*j}(t) * f(t) + p^q . g^{*q}(t) * e^{At}. \quad (1.32)$$

From Cayley-Hamilton theorem $p^q = 0$. (In case $q = 1$, the right hand sum of (1.32) is zero by convention.)

Then

$$e^{At} = f(t) + \sum_{j=1}^{q-1} p^j \cdot g^{*j}(t) * f(t) .$$

Hence,

$$e^{At} = f(t) + \left(\sum_{j=1}^{q-1} p^j \cdot g^{*j}(t)\right) * f(t) .$$

REMARK 1.7. The various convolution products in Formulas (1.21 and (1.31) can all be calculated by successive computation of linear combinations of terms of the form $e^{at} * t^m e^{bt}$. For completeness, we mention the following easily verified formula:

$$e^{at} * t^m e^{bt} = \begin{cases} 1/m+1 \; t^m e^{at}, \text{ if } b = a , \\ e^{bt} \sum_{k=0}^{m} (-1)^k \frac{t^{m-k}}{(b-a)^{k-1}} \frac{m!}{(m-1)!} \\ +(-1)^{m+1} e^{at} \frac{m!}{(b-a)^{m+1}} , \quad \text{if } b \neq a . \end{cases} \quad (1.33)$$

EXAMPLE 1.4. Compute e^{At} when

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

SOLUTION. The eigenvalues of A are $\lambda_1=0$ $\lambda_2=1$ $\lambda_3=2$.

By Formula (1.23) ,

$$e^{At} = \sum_{k=0}^{n-1} P_k \cdot r_{k+1}(t) ,$$

where

$$P_0=I,\ P_k = \prod_{i=1}^{k}(A-\lambda_i I),\ r_1(t) = e^{\lambda_1 t}; r_k(t)=e^{\lambda_1 t}*e^{\lambda_2 t}*\ldots*e^{\lambda_k t}.$$

Therefore

$$P_0 = I,\ P_1 = (A - \lambda_1 I) = A ,$$

$$P_2 = (A - \lambda_1 I)(A - \lambda_2 I) = A(A - I),$$

and

$$r_1(t) = e^{0t}=1,\ r_2(t) = e^{\lambda_1 t}*e^{\lambda_2 t} = \int_0^t e^{\lambda_1 t}e^{\lambda_2 (t-\tau)}\ d\tau$$

$$= e^t - 1 ,$$

or

$$r_2(t) = e^t - 1$$

$$r_3(t) = e^{-\lambda_1 t}*e^{\lambda_2 t}*e^{\lambda_3 t} = e^{0t}*e^t*e^{2t}$$

$$= (e^t-1)*e^{2t} = -\int_0^t (e^t-1)e^{2(t-\tau)}\ d\tau$$

$$= 1/2e^{3t} + 1/2e^{2t} - 1/2e^t + 1/2.$$

Hence,

$$e^{At} = P_0 r_1(t) + P_1 r_2(t) + P_2 r_3(t)$$

$$= I + A(e^t - 1) + 1/2A(A - I)(1 - e^t + e^{2t} - e^{3t}).$$

CHAPTER II

LOCALIZATION OF EIGENVALUES OF MATRICES

In this chapter a convex set and a convex function are defined. The basic theorems related to eigenvalues of matrices are stated, then the eigenvalues and singularvalues of matrices are compared. Finally, the localization of eigenvalues of matrices has been investigated.

Also, we shall be mainly interested in subsets of R^n and $M_{m,n}(R)$. Some of the statements that follow are true for both R^n and $M_{m,n}(R)$, in which case we shall designate either of the sets by U .

DEFINITION 2.1. If X and Y are elements in U and then the set

$$\{Z \in U \mid Z = \alpha X + (1 - \alpha) Y \quad 0 \leq \alpha \leq 1\} \tag{2.1}$$

is called the line segment joining X and Y . If with any two elements of a subset S of U , the line segment joining them is a subset of S , we say that the set S is convex.

DEFINITION 2.2. A function f defined on a convex subset S of U is convex, if

$$f(\alpha x + (1 - \alpha)Y) \leq \alpha f(X) + (1 - \alpha) f(Y). \tag{2.2}$$

For any X and Y in S and all $\alpha, (0 \leq \alpha \leq 1)$. If the inequality

in (2.2) is strict for all distinct X and Y in S and for all α, $(0 < \alpha < 1)$, then f is strictly convex. A function g is concave (strictly concave), if $-g$ is convex (strictly convex). We will also have occasion to consider a convex function f on C^n with values in R. This is defined by the inequality in which X and Y are now taken to be in some convex subset S of C^n which contains $\alpha X + (1 - \alpha)Y$ whenever $X, Y \in S$ and $0 \leq \alpha \leq 1$.

DEFINITION 2.3. A matrix $A \in M_n(R)$ is called doubly stochastic if all its entries are nonnegative and its row sums and all its column sums are equal to one. Let Ω_n denote a doubly stochastic set; then Ω_n is convex.

DEFINITION 2.4. Let $B = (b_{ij}) \in M_n(C)$ be a unitary matrix. Then the matrix $E \in M_n(R)$ whose (i, j) entry is $|b_{i,j}|^2$ is doubly stochastic. Such a doubly stochastic matrix is called orthostochastic.

PROPOSITION 2.1. If $X = (x_1, \ldots x_n), Y = (y_1, \ldots y_n)$ and $x_1 \geq x_2 \geq \ldots \geq x_n$, $y_1 \geq y_2 \geq \ldots \geq y_n$, $x_1 + x_2 \ldots x_k \leq y_1 + y_2 + \ldots y_k$, $(k = 1, \ldots n-1)$, $x_1 + x_2 + \ldots + x_n = y_1 + y_2 \ldots y_n$. then there exists $Q \in \Omega_n$ such that $x = Qy$. In fact Q may be chosen to be orthostochastic.

DEFINITION 2.5. Let f be a function on C^n with values in R if $\Lambda = (\lambda_1 \ldots \lambda_n) \in C^n$ and $B \in \Omega_n$, then we may define

a function g on Ω_n to R by

$$g(B) = f(B_{(1)}, \Lambda), \ldots (B_{(n)}, \Lambda) ,$$

where $B_{(i)}$, is the ith row of B.

PROPOSITION 2.2. If the function f in Definition 2.5 is convex (concave), then g is a convex (concave) function on Ω_n with values in R.

DEFINITION 2.6. Let σ be one to one function on the set $\{1, 2 \ldots n\}$, then σ is called a <u>permutation</u>. If $A \in M_n(R)$ for which $a_{i\sigma(i)} = 1$, $(i = 1 \ldots n)$, and $a_{ij} = 0$ otherwise, A is called a <u>permutation</u> matrix.

DEFINITION 2.7. Let $A \in M_n(F)$, and let r-rowed minor determinants be represented by $a \begin{matrix} i_1 \cdots i_r \\ j_1 \cdots j_r \end{matrix}$. The matrix of order $m = \binom{n}{r} = \frac{n!}{r!(n-r)!}$ whose elements are the numbers $a \begin{matrix} i_1 \cdots i_r \\ j_1 \cdots j_r \end{matrix}$ in the lexacographic order is called the r-th <u>compound matrix</u> or the <u>adjugate</u> of A, it is denoted by $C_r(A)$.

PROPOSITION 2.3. If $A \in M_{n,m}(F)$ and $B \in M_{n,k}(F)$ and $1 \leq r \leq \min(m, n, k)$, then,

$$C_r(AB) = C_r(A)\, C_r(B) . \qquad (2.2)$$

PROPOSITION 2.4. If $A \in M_n(F)$, $1 \leq r \leq n$ and A is

nonsingular, then

$$(C_r(A))^{-1} = C_r(A^{-1}), \tag{2.3}$$

$$C_r(A^*) = (C_r(A))^*, \tag{2.4}$$

$$C_r(A^t) = (C_r(A))^t, \tag{2.5}$$

$$C_r(\overline{A}) = (\overline{C_r(A)}), \tag{2.6}$$

Where A^* is the conjugate transpose, A^t is the transpose, and $\overline{A}$ is the conjugate of A.

For any $K \in F$

$$C_r(KA) = K^r C_r(A), \tag{2.7}$$

$$C_r(I_n) = I_{\binom{n}{r}},$$

where I_n is the n-square identity matrix.

PROPOSITION 2.5. If $A \in M_n(C)$ and $1 \leq r \leq n$, then the characteristic roots of $C_r(A)$ are the $\binom{n}{r}$ products $\lambda_1(A)\ldots\lambda_r(A)$, where $\lambda_i(A)$ is an eigenvalue of A.

REMARK 2.1. Let f be a real valued convex function on the convex subset S of C^n (see Definition 2.2) where S is defined as the totality of points $B\Lambda$, for $B \in \Omega_n$ and $\Lambda = (\lambda_1\ldots\lambda_n)$ the n-tuple of eigenvalue a normal matrix $A \in M_n(C)$. Then by Proposition 2.2

$$g(B) = f((B_1, \Lambda), \ldots (B_n, \Lambda))$$

is convex on Ω_n and, moreover $g(B) \leq g(P)$ where P is an appropriate permutation matrix in Ω_n.

PROPOSITION 2.6. If A is a permutation matrix in $M_n(C)$ and $X_1,\ldots,X_n$ are orthonormal vectors in C^n, then

$$\sum_{j=1}^{k} \lambda_{r-j+1} \leq \sum_{j=1}^{k} (AX_j, X_j) \leq \sum_{j=1}^{k} \lambda_j, \tag{2.8}$$

where $\lambda_1 \geq \ldots \geq \lambda_n$ are the eigenvalues of A.

THEOREM 2.1. If $A \in M_n(C)$ has eigenvalues

$$\lambda_1, \ldots, \lambda_n, \quad |\lambda_1| \geq \ldots \geq |\lambda_n|,$$

and singular values

$$\mu_1 \geq \cdots \geq \mu_n,$$

then

$$\prod_{j=1}^{k} |\lambda_i| \leq \prod_{i=1}^{k} \mu_i, \quad k = 1 \ \ldots \ n \tag{2.9}$$

with equality for $k = n$.

PROOF. Let X_1 be a unit eigenvector of A corresponding to the eigenvalue λ_1. Then

$$|\lambda_1|^2 = (\lambda_1 X_1, \lambda_1 X_1) = (AX_1, AX_1) = (A^*AX_1, X_1).$$

But A^*A is a nonnegative hermitian and its largest eigenvalue is μ_1^2. Hence by Proposition 2.6 with $k = 1$, $(A^*A X_1, X_1) \leq \mu_1^2$.

Then

$$|\lambda_1| \leq \mu_1 .$$

Now apply the preceding to the matrix $C_k(A)$. By Proposition 2.5 the largest eigenvalue of $C_k(A)$ is $\prod_{i=1}^{k} \lambda_i$. By Proposition 2.4,

$$(C_k(A))^* C_k(A) = C_k(A^*) C_k(A) = C_k(A^*A) .$$

Therefore the largest singular value of $C_k(A)$ is $\prod_{i=1}^{k} \mu_i$. If we apply the result for k = 1 to $C_k(A)$ we can obtain

$$\prod_{i=1}^{k} |\lambda_i| \leq \prod_{i=1}^{k} \mu_i .$$

The equality in (2.9) occurs for k = n because of

$$|\det (A)| = (\det (A^*A))^{1/2} .$$

THEOREM 2.2. If $A \in M_n(C)$ has eigenvalues $\lambda_1, \ldots, \lambda_n$,

$$|\lambda_1| \geq |\lambda_2| \ldots \geq |\lambda_n| ,$$

and singular values

$$\mu_1 \geq \ldots \geq \mu_n , \text{ and } r > 0 , \text{ then}$$

$$\sum_{j=1}^{k} |\lambda_j|^r \leq \sum_{i=1}^{k} \mu_1^r . \qquad (2.11)$$

PROOF. We may assume A is nonsingular so that $|\lambda_n| > 0$. Then by taking the logarithm on both sides of (2.9) it follows that

$$\sum_{j=1}^{k} \ln |\lambda_j| \leq \sum_{j=1}^{k} \ln \mu_i , \quad k = 1, \ldots, n \qquad (2.12)$$

with equality for $k = n$. Let

$$u = (\ln |\lambda_1|, \ln |\lambda_2| \ldots \ln |\lambda_n|) ,$$

and

$$\upsilon = (\ln \mu_1, \ln \mu_2 \ldots \ln \mu_n) .$$

By Proposition 2.1, $u = B\nu$ for $B \in \Omega$. Next define

$$f(t) = f(t_1 \ldots . t_n) = \sum_{j=1}^{k} e^{rt_i} , \qquad (2.13)$$

the function f is convex on the positive intervals, hence

$$g(E) = f((E_{(1)}, \nu), \ldots, (E_{(n)}, \nu)) = f(E\nu), \quad (2.14)$$

where $E \in \Omega_n$ is convex on Ω_n . From Remark 2.1, $g(E) \leq g(P)$ where P is some permutation matrix. Thus $E = B$

$$f(\upsilon) = f(B\nu) = g(B) \leq g(P) = f((P_1,\nu) \ldots (P_{(n)},\nu)) \quad (2.15)$$

$$= f(\nu_{\sigma(1)} \cdots \nu_{\sigma(n)})$$

where σ is a permutation. Evaluating both sides of Inequality (2.15) we can obtain

$$\sum_{i=1}^{k} e^{rh|\lambda_i|} \leq \sum_{i=1}^{k} e^{r(\ln \mu_{\sigma(i)})}$$

or

$$\sum_{j=1}^{k} |\lambda_j|^r \leq \sum_{j=1}^{k} \mu^r{}_{\sigma(i)} \leq \sum_{j=1}^{k} \mu^r{}_j .$$

If $A \in M_n(C)$, let $B = (b_{ij}) = \frac{A+A^*}{2}$ and $E = (e_{ij} = \frac{A-A^*}{2i}$, both of which are hermitian. Let $\lambda_1, \ldots \lambda_n$,

$$|\lambda_1| \geq \ldots \geq |\lambda_n|, \quad \mu_1 \geq \cdots \geq \mu_n \qquad \nu_1 \geq \quad \geq \nu_n$$

be the eigenvalue of A, B, E respectively and let $g=\max |a_{ij}|$, $g'=\max |b_{ij}|$, $g''=\max |e_{ij}|$.

THEOREM 2.3. (Bendixon's Theorem [20]). If $A \in M_n(R)$, then

$$|I\ m\ (\lambda_i)| \leq g''\sqrt{n(n-1)/2} \ , \tag{2.16}$$

where $I\ m\ (\lambda_i)$ is an imaginary part of λ_i .

THEOREM 2.4. If $A \in M_n(R)$, then $\mu_n \leq Re\ (\lambda_t) \leq \mu_1$ where $Re(\lambda_t)$ is the real part of λ_t .

THEOREM 2.5. If $A \in M_n(C)$, then

$$|\lambda_t| \leq ng \ ,$$

and

$$|Re\ \lambda_t| \leq ng' \ ,$$
$$|I\ m\ (\lambda_t) \leq ng'' \ .$$

If $A + A^t$, is real, then this last inequality can be replaced by

$$|I\ m\ (\lambda_t)| \leq g''\ \sqrt{\frac{n(n-1)}{2}} \quad .$$

PROPOSITION 2.7. If $A = (a_{ij}) \in M_n(C)$ has eigenvalues λ_i, $i = 1, 2, \ldots n$, then

$$\sum_{t=1}^{n} |\lambda^2{}_t| \leq \sum_{i,j=1}^{n} |a_{ij}|^2 \ . \tag{2.17}$$

with equality if and only if A is normal.

PROPOSITION 2.8. If $A \in M_n(C)$ has eigenvalues λ_i, $i = 1,2...n$, and singular values $\mu_1 \geq \ldots \geq \mu_n$,

then

$$\mu_n \leq |\lambda_i| \leq \mu_1 . \tag{2.18}$$

PROOF. Let X be an eigenvector of unit length corresponding to λ_i. Then $AX = \lambda_i X$. Therefore

$$(A^*AX, X) = (AX, AX) = \lambda_i \overline{\lambda}_i (X, X) = |\lambda_i|^2 .$$

Now A^*A is hermitian and its greatest and least eigenvalues are μ^2_1 and μ^2_n, respectively. Hence by Proposition 2.6

$$\sum_{j=1}^{k} \mu_{n-j+1} \leq \sum_{j=1}^{k} (A^*AX_j, X_j) \leq \sum_{j=1}^{k} \mu_j ,$$

then

$$\mu^2_n \leq |\lambda_i|^2 \leq \mu^2_1 \quad \text{for} \quad k = 1$$

or

$$\mu_n \leq |\lambda_i| \leq \mu_1 .$$

PROPOSITION 2.9. Let N_i and T_j denote the sum of absolute values of the entries of A in the ith row and the sum of the absolute values of the entries in the jth column respectively.

Let $N = \max N_i$, $T = \max T_i$. If $A \in M(C)$, then

$$|\lambda_i| \leq \frac{N + T}{2} \tag{2.19}$$

$$|\lambda_i| \leq \max \frac{N_i + T_i}{2} \tag{2.20}$$

$$|\lambda_i| \leq (NT)^{1/2} \qquad (2.21)$$

PROPOSITION 2.10. Let $P_i = N_i - |a_{ii}|$ and $Q_j = T_j - |a_{jj}|$. If $A \in M_n(C)$ and

$$|a_{ii}| > P_i , \quad i = 1 \ldots n ,$$

then det $(A) \neq 0$.

THEOREM 2.4. The eigenvalues of an n-square complex matrix A, $A \in M_n(C)$ lie in the closed region of the Z-plane consisting of the union of the discs.

$$|Z-a_{ii}| \leq P_i, \; i = 1 \ldots n . \qquad (2.22)$$

PROOF. Let λ_j be an eigenvalue of $A \in M_n(C)$, then, det $(\lambda_j I_n - A) = 0$, and by Proposition 2.10 ,

$$|\lambda_j - a_{jj}| \leq P_i$$

for at least one i .

THEOREM 2.5. Let $A \in M(C)$ and λ_k be an eigenvalue of A . then

$$|\lambda_k| \leq \min (N, T) . \qquad (2.23)$$

PROOF. By Theorem 2.4 there exists an i such that,

$$|\lambda_k - a_{ii}| \leq P_i .$$

Therefore

$$|\lambda_k| \leq P_i + |a_{ii}| = N_i \leq N .$$

Similary $|\lambda_k - a_{jj}| \leq Q_j$ for some j ,

$$|\lambda_k| \leq Q_j + |a_{jj}| = T_i \leq T .$$

Hence

$$|\lambda_k| \leq \min (N, T).$$

PROPOSITION 2.11. If $A \in M_n(C)$, and

$$|a_{ii}||a_{jj}| > Q_iQ_j, \ |a_{ii}||a_{jj}| > P_iP_j, \ i,j = 1,\ldots,n; \ i \neq j, \quad (2.24)$$

then $\det (A) \neq 0$.

THEOREM 2.6. If $A \in M_n(C)$, then each of the eigenvalues of A lies in at least one of the $n(n-1)/2$ Ovals of cassini

$$|Z-a_{ii}||Z-a_{jj}| \leq P_iP_j, \ (i,j = 1, 2 \ldots n, \ i \neq j),$$

and in at least one of the ovals

$$|z-a_{ii}||z-a_{jj}| < Q_iQ_j .$$

PROOF. Let λ_k be an eigenvalue of A. Then $\det (\lambda_k I - A) = 0$ and by Proposition 2.11 ,

$$|\lambda_k-a_{ii}||\lambda_k-a_{jj}| > P_iP_j$$

cannot hold for all i, j . Hence there exists (i,j) such that,

$$|\lambda_k-a_{ii}||\lambda_k-a_{jj}| \leq P_iP_j .$$

Similarly,

$$|\lambda_k-a_{ii}||\lambda_k-a_{jj}| \leq Q_iQ_j .$$

THEOREM 2.7. Let $C_1, C_2 \ldots C_n$ be positive numbers, and K the maximum of n numbers

$$K_r = \frac{C_1(a_{r_1}) + \ldots C_n(a_{rn})}{C_r}, \quad r = 1, \ldots n, \qquad (2.25)$$

then

$$|\lambda_k| \leq K .$$

PROOF. For $C_1 = C_2 = \ldots = C_n = 1$,

$$K_r = \sum_{i=1}^{n} a_{ri} = T_r ,$$

then by Theorem 2.5, $|\lambda_k| \leq K$. In the general case, let E be a diagonal matrix whose diagonal is $C_1, C_2 \ldots C_n$, and let $E^{-1}AE$ be a matrix. Then A and $E^{-1}AE$ have the same eigenvalues. If λ_k be an eigenvalue of A, then

$$N_r = a_{r_1} c_1/c_r + a_{r_2} c_1/c_r + \ldots + a_{r_r} c_r/c_r + \ldots + a_{r_n} c_n/c_r$$

and

$$T_r = a_{1r} c_r/c_1 + a_{2r} c_r/c_2 + \ldots + a_{r_r} + \ldots + a_{n_r} c_n/c_r .$$

When A is nonnegative

$$K_r = \frac{c_1|a_{1n}| + \ldots + c_n|a_{rn}|}{c_r} = N_r$$

then by Formula (2.23)

$$|\lambda_k| < \min (N, T) .$$

Hence

$$|\lambda_k| < K .$$

CHAPTER III

EIGENVALUES AND EIGENVECTORS OF SPECIAL TYPES OF MATRICES

The eigenvalues and eigenvectors of imprimitive matrices, Hessenberg matrices, and permutation matrices are investigated.

I. THE EIGENFUNCTION OF AN IMPRIMITIVE MATRIX

DEFINITION 3.1. A graph is a system consisting of two sets T and Φ together with a mapping g of Φ into $T \times T$, the elements of T and Φ are called the vertices and edges of the directed graph, respectively, and g is called its direct incidence mapping. if $g(\phi) = (\upsilon_1, \upsilon_2)$, υ_1 is called the initial vertex of edge and υ_2 is called terminal vertex.

DEFINITION 3.2. A path of length n is a sequence of distinct edges $\phi_1, \phi_2, \ldots, \phi_n$ such that for an appropriate sequence of $n + 1$ vertices $\upsilon_1, \upsilon_2, \ldots, \upsilon_{n+1}$ we have $g(\phi_i) = (\upsilon_i, \upsilon_{i+1})$ for $i = 1, 2, \ldots, n$.

DEFINITION 3.3. A cycle in a directed graph G is a path from a vertex i to the same vertex i. A circuit is a cycle in which no vertex is repeated.

DEFINITION 3.4. A directed graph G is cyclically k-partite ($k \geq 2$) if and only if the vertex set T can be partitioned into

k sets, $T_1, T_2, \ldots, T_k$, such that (i, j) is on the edge of G only if $i \in T_p$, $j \in T_q$ with $p - q = 1$ or $1 - k$.

PROPOSITION 3.1. If G has a cycle and is cyclically k-partite then k is a divisor of the length of the cycle.

DEFINITION 3.5. A directed graph G is strongly connected if and only if, for every ordered pair (i,j) with $i \neq j$, there is a path from i to j.

DEFINITION 3.6. A matrix whose rows and columns correspond to the vertices of a graph and whose elements indicate the number of edges joining the corresponding pairs of vertices is called a vertex matrix.

DEFINITION 3.7. A graph G is called a complete graph if every two distinct vertices are adjacent.

DEFINITION 3.8. If a graph G is strongly connected, then its vertex matrix is said to be irreducible. An irreducible nonnegative matrix A is primitive if and only if there exists a positive integer m such that every entry of A^n is positive for $n \geq m$. A matrix which is not primitive is imprimitive.

THEOREM 3.1. If A is a matrix and if G is cyclically k-partite, then, corresponding to the partitioning $T_1, T_2, \ldots, T_k$, there exists a permutation matrix P such that

$$P^{-1}AP = \begin{pmatrix} 0_1 & B_1 & & 0 \\ & 0_2 & B_2 & \\ & & \ddots & \\ & & & B_{k-1} \\ B_k & & & 0_k \end{pmatrix} \qquad (3.1)$$

where the matrices $0_1, 0_2, \ldots, 0_k$ are square zero matrices, and, in fact every element of $P^{-1}AP$ which is not in one of $B_1, B_2, \ldots, B_k$ is zero.

DEFINITION 3.9. The matrices $B_1, B_2, \ldots, B_k$, in the above theorem, will be called the cyclic components of A relative to P or to the partitioning $T_1 T_2 \ldots, T_k$. A matrix A for which G is cyclically k-partite has been called weakly cyclic of index k.

PROPOSITION 3.2. If a matrix A for which G is cyclically k-partite, then we have

$$P^{-1}A^kP = \begin{pmatrix} A_1 & & & 0 & 0 \\ & A_2 & & 0 & 0 \\ & & \ddots & & \vdots \\ 0 & 0 & \cdots & & 0 \\ 0 & 0 & \cdots & & A_k \end{pmatrix}, \qquad (3.2)$$

where

$$A_1 = B_1 B_2 B_3 \ldots B_k,$$
$$A_2 = B_2 B_3 B_4 \ldots B_k B_1,$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$A_k = B_k B_1 B_2 \ldots B_{k-1}.$$

DEFINITION 3.10. The matrices $A_1, A_2, \ldots, A_k$ in Proposition 3.2 will be called the diagonal components of A^k relative to P or to partitioning $T_1, T_2, \ldots, T_k$.

PROPOSITION 3.3. An irreducible nonnegative matrix is primitive if and only if the eigenvalue of maximum modulus is positive.

PROPOSITION 3.4. Let A be an irreducible matrix and let d be the greatest common divisor of the lengths of the cycles of G. If $d > 1$, then G is cyclically d-partite.

DEFINITION 3.11. If A is irreducible and nonnegative, then d in Proposition 3.4 is called the index of imprimitivity of A and $d = 1$ if and only if G is primitive. If $d > 1$ such a matrix has been called cyclic of index d.

DEFINITION 3.12. A graph $G'(T',\phi',g')$ is a subgraph of $G = (T,\Phi, g)$ if $T' \subset T$, $\Phi' \subset \Phi$ and $g'_{(\upsilon)} = g_{(\upsilon)}$ for every edge $\upsilon \in \Phi'$.

PROPOSITION 3.5. If A is an n by n matrix, a subgraph G' consisting of m edges of G is the union on vertex disjoint circuits if and only if the set S of nonzero elements of A which correspond to the edges of G' have the following properties;

(1) No two elements of S are in the same row or column of A.

(2) The m by m minor of A which contains the elements of S contains m diagonal elements of A. These diagonal elements

may be zero.

PROPOSITION 3.6. If A is an n by n matrix and if, in the eigenfunction $c_0 \lambda^n + c_1 \lambda^{n-1} + \ldots + c_n$, the coefficient c_m is not zero, then G has a number of vertex disjoint circuits the sum of the lengths of which is m.

PROOF. Considering the expansion of the determinant $|A - \lambda I|$, a term in λ^{n-m} arises by selecting a λ from $n - m$ of the diagonal elements of which has properties (i) and (ii) of Proposition 3.5. The coefficient c_m is a sum of numbers each of which is the product of the elements of such a set S. Thus Proposition 3.6 follows from Proposition 3.5.

PROPOSITION 3.7. If A is a matrix for which the graph G is cyclically k-partite and if the coefficient c_m of λ^{n-m} in the eigenfunction of A is not zero, then k divides m.

PROOF. From Proposition 3.6, m is a sum of integers each of which is the length of a circuit. Since every circuit is a multiple of k, the result follows.

PROPOSITION 3.8. Let $E \in M_{n,m}(F)$ and $B \in M_{m,n}(F)$ be matrices $(n \leq m)$ and if the eigenfunction of EB is $g(\lambda)$, then the eigenfunction of BE is $g(\lambda)\ (\lambda^{m-n})$.

PROPOSITION 3.9. If $E \in M_{n,m}(F)$ and $B \in M_{m,n}(F)$ then BE and EB have the same elementary divisors for each nonzero eigenvalue.

THEOREM 3.2. If A is a matrix such that G is cyclically k-partite and if $A_1, A_2, \ldots, A_{k-1}, A_k$ constitute a set of diagonal components of A^k, then there exist a monic polynomial $f(\lambda)$ and a nonnegative integer p such that

(i) $f(0) \neq 0$,

(ii) the characteristic polynomial of A is $f(\lambda^k)\ \lambda^p$,

(iii) the characteristic polynomial of A^k is $[f(\lambda)]^k\ \lambda^p$,

(iv) there exist nonnegative integers $p_1, p_2, \ldots, p_k$ with sum p such that the eigenfunction of A_i is $f(\lambda)\ \lambda^{p_i}$ for $i = 1, 2, \ldots, k$,

(v) for any nonzero root of $f(\lambda)$ the elementary divisors are the same for each A_i, $i = 1, 2, \ldots, k$.

PROOF. If zero is an eigenvalue of A with multiplicity n, then $f(\lambda) = 1$, $p = n$ and the theorem is trivial.

(i) If the multiplicity p of the eigenvalue zero is less than the eigenvector $g(\lambda)$ of A can be written as

$$\begin{aligned} g(\lambda) &= C_0\ \lambda^n + C_1\ \lambda^{n-1} \ldots C_{n-p}\ \lambda^n \\ &= \lambda^p\ (C_0\ \lambda^{n-p} + C_1\ \lambda^{n-p-1} \ldots C_{n-p}) . \end{aligned}$$

Therefore there exist $q(\lambda)$ such that

$$g(\lambda) = \lambda^p\ q(\lambda) \quad \text{and} \quad q(0) \neq 0$$

(ii) By contraposition of Proposition 3.7 there exists a polynomial

$$q(\lambda) = f(\lambda^k),\ f(0) = q(0) \neq 0 \text{ and } g(\lambda) = \lambda^p\ (f(\lambda^k).$$

(iii) We can prove it by induction on $K \geq 2$.

(iv) and (v) By Propositions 3.8 and 3.9

$$A = B_1(B_2,B_3 \ldots B_k) \text{ and } A_2 = (B_2,B_3 \ldots B_k)B_1$$

have the same nonzero eigenvalue with the same multiplicities and in fact the same elementary divisors for each nonzero eigenvalue. The pairs of matrices A_2 and A_3, A_3 and $A_4 \ldots A_k$ and A_1 are similarly related. Thus there exist a monic polynomial $h(\lambda)$ and nonnegative integers $p_1, p_2 \ldots p_k$ such that A_i has an eigenfunction $h(\lambda)\ \lambda^{p_i}$, $h(0) \neq 0$, it follows that

$$f(\lambda)\ \lambda^p = [h(\lambda)]^k\ \lambda^{\Sigma p_i},\ f(\lambda) = 0,\ h(\lambda) \neq 0$$

and hence

$$f(\lambda) = h(\lambda) \quad \text{and} \quad P = \sum_{i=1}^{k} p_i \ .$$

THEOREM 3.3. Let A be a matrix for which G is cyclically k-partite. Let $B_1, B_2, \ldots, B_k$ be the cyclic components of A and let $A_1, A_2, \ldots, A_k$ be diagonal components of A corresponding to a permutation matrix P. Let λ be an eigenvalue of A_1 and X_1 be an eigenvector of A_1 defined $X_1 A_1 = X_1\lambda$. Let $\lambda_1,\lambda_2, \ldots,\lambda_k$ be the distinct kth roots of $\lambda(\lambda^k_1=\lambda^k_2=\ldots\lambda^k_k= \lambda)$, so that $\lambda_1,\lambda_2, \ldots\lambda_k$ are distinct eigenvalues of A.

Then an eigenvector of $P^{-1}AP$ corresponding to λ_j is

$$Y_j = (X_1\ \lambda^k_j,\ X_1\ B_1\ \lambda^{k-1}_j,\ X_1\ B_1\ B_2\ \lambda^{k-2}_j \ldots X_1\ B_1\ B_2 \ldots B_{k-1}\ \lambda_j).$$

Furthermore, the construction is reversible, X_j uniquely determines

X_1 apart from a constant multiple.

PROOF. Since $X_1 A_1 = X_1 \lambda = X_1 \lambda_1^k$,

$$Y_j P^{-1}AP = (X_1 B_1 B_2 \ldots B_{k-1} B_k \lambda_j, X_1 B_1 \lambda_j^k, X_1 B_1 B_2 \lambda_j^{k-1}, \ldots, X_1 B_1 B_2 \ldots B_k \lambda_j^2)$$

$$= (X_1 A_1, X_1 B_1 \lambda_j^{k-1} \ldots X_1 B_1 B_2 \ldots B_{k-1} \lambda_j)$$

$$= Y_j \lambda_j$$

Conversely, let Y_j be an eigenvector of $P^{-1}AP$ corresponding to the root λ_j. Put $Y_j = (w_1, w_2 \ldots w_k)$ where the number of components of w_i is the number of rows in B_i. The equation $Y_j P^{-1}AP = \lambda_j Y_j$ implies the equations

$$w_1 B_1 = \lambda_j w_2 ,$$

$$w_2 B_2 = \lambda_j w_3 ,$$

$$w_3 B_3 = \lambda_j w_4$$

$$\ldots\ldots\ldots$$

$$w_k B_k = \lambda_j w_1 .$$

Hence $w_1 A_1 = w_1 B_1 B_2 B_3 \ldots B_k = a\, w_2 B_2 \ldots B_k = \ldots \lambda\, w_1$. That is w_1 is an eigenvector of A_1 corresponding to λ and furthermore,

$$Y_j = (w_1, \frac{w_1 B_1}{\lambda_j}, \frac{w_2 B_1 B_2}{\lambda_j^2}, \ldots \frac{w_1 B_1 B_2 \ldots B_{k-1}}{\lambda_j^{k-1}}) .$$

THEOREM 3.4. Let A be a matrix for which G is cyclically k-partite. Let $B_1, B_2, \ldots, B_k$ be the cyclic components of A, and permutation matrix P. Let λ b an eigenvalue of A_1 and X_i be an eigenvector of A_i which corresponds to the root λ. Then for i = 1,2,...k-1, $X_{i+1} = X_i B_i$, and

$$B_1 = X_k B_k .$$

PROOF. Since X_1 is an eigenvector of A_1,

$$X_1 A_1 = X_1 \lambda ,$$

$$X_1 B_1 B_2 \ldots B_n = X_1 \lambda ,$$

$$(X_1 B_1) B_2 \ldots B_2 B_1 = X_1 B_1 \lambda ,$$

$$(X_1 B_1) A_2 = (X_1 B_1) \lambda .$$

Thus $X_2 = X_1 B_1$. The other results follow similarly.

II. ON THE EIGENVALUES AND EIGENVECTORS OF A CLASS OF HESSENBERG MATRICES

DEFINITION 3.13. An n-square matrix $H_n \in M_n(C)$ is called Hessenberg matrix if $h_{ij} = 0$, $j \geq i + 2$ where h_{ij} be an element of H_n, then ,

$$H_n = \begin{pmatrix} h_{11} & a_1 & 0 & \ldots & 0 \\ h_{21} & h_{22} & a_2 & \ldots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ \vdots & \vdots & & & a_n \\ h_{n1} & h_{n2} & \ldots & & h_{nn} \end{pmatrix} . \qquad (3.3)$$

THEOREM 3.5. (Franklin [11]). If H_n be a Hessenberg matrix and $f_n(\lambda) = \det (H_n - \lambda I)$ be the eigenfunction of H_n , then

$$f_n(\lambda) = (h_{nn} - \lambda) \det (H_{n-1} - \lambda I_{n-1}) + \sum_{k=1}^{n-1} (-1)^{n-k} h_{nk} a_k \ldots a_{n-1} \det (H_{k-1} - \lambda I_{k-1}) . \qquad (3.4)$$

THEOREM 3.6. (Franklin [11]). If H_n be a Hessenberg matrix and $X = (x_1, x_2, \ldots, x_n)$ be an eigenvector of H_n, then,

$$X_n = \det (H_{n-1} - I_{n-1}), \quad X_k = (-1)^{n-k} a_k + \ldots + a_{n-1} \det (H_{k-1} - \lambda I_{k-1}) . \qquad (3.5)$$

In this section we consider a class of Hessenberg matrices as the following form

$$H_n = \begin{pmatrix} h & 1 & 0 & \cdots & 0 & 0 \\ h^2 & h & 1 & \cdots & 0 & 0 \\ \ddots & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & & 1 \\ h^{n-1} & h^{n-2} & \cdots & \cdots & h & 1 \\ h^n & h^{n-1} & \cdots & \cdots & h^2 & h \end{pmatrix} . \tag{3.6}$$

THEOREM 3.7. Let H_n be the Hessenberg matrix

$$H_n = \begin{pmatrix} h & 1 & \cdots & 0 & 0 \\ h^2 & h & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ h^n & h^{n-1} & & h^2 & h \end{pmatrix} .$$

then H_n has $m = [n/2]$ eigenvalues equal to zero and $n - m$ eigenvalues are

$$4\, h \cos^2 \frac{k\pi}{n+2} , \tag{3.7}$$

where $[n/2]$ denotes the largest integer not exceeding $n/2$.

PROOF. The eigenvalues of H_n are given by the roots of the equation

$$\det (H_n - \lambda I) = 0 . \tag{3.8}$$

Let A_n be an n-square matrix:

$$A_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -h & 1 & & \vdots \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & -h & 1 \\ 0 & \cdots & -h & 1 \end{pmatrix} \qquad (3.9)$$

then premultiplying $(H_n - \lambda I)$ by A_n we see that (3.8) is equivalent to

$$\det A_n(H_n - \lambda I) = 0 , \qquad (3.10)$$

because of $\det A_n = 1$.

Then we have

$$A_n(H_n - \lambda I) = \begin{pmatrix} h-\lambda & 1 & 0 & \cdots & 0 \\ h\lambda & -\lambda & 1 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & h\lambda & -\lambda & 1 \\ 0 & \cdots & \cdots & h\lambda & -\lambda \end{pmatrix}$$

Let T_n be the n-square determinant

$$T_n = \begin{vmatrix} -\lambda & 1 & \cdots & 0 \\ h\lambda & -\lambda & & 0 \\ \vdots & \ddots & \ddots & \vdots \\ & h\lambda & -\lambda & 1 \\ 0 & \cdots & h\lambda & -\lambda \end{vmatrix} \quad ; \qquad (3.11)$$

then expanding the left hand side of (3.10) along the first columns, we obtain

$$(h - \lambda)\, T_{n-1} - h\lambda\, T_{n-2} = 0 . \qquad (3.12)$$

The determinant T_n satisfies the recurrences relation

$$T_n + \lambda T_{n-1} + h \lambda T_{n-2} = 0 \qquad (3.13)$$

with $T_0 = 1$, $T_1 = -\lambda$. If we put $\alpha = (\lambda^2 - 4h\lambda)^{1/2}$, $\beta = (-\lambda + \alpha)/2$, $\gamma = (-\lambda - \alpha)/2$, then the solution of (3.13) can be written in the following form:

$$T_n = 1/\alpha \, (\beta^{n+1} - \gamma^{n+1}) . \qquad (3.14)$$

Substituting (3.14) into the left-hand side of (3.12), we have

$$(h - \lambda) T_{n-1} - h \lambda T_{n-2} = -\frac{1}{\lambda\alpha} \{\beta^{n+2} - \gamma^{n+2}\}$$

$$= \frac{-1}{\lambda\alpha} \prod_{k=0}^{n+1} (\beta - \omega^k \gamma) , \qquad (3.15)$$

$1, \omega, \ldots \omega^{n+1}$ being the roots of $X^{n+2} - 1 = 0$. Using the relations

$$\omega^k + \omega^{n+2-k} = 2 \cos \frac{2k\pi}{n+2} \qquad (3.16)$$

Let $P = [(n+1)/2] = n - m$, in (3.15),

$$(h-\lambda)T_{n-1} - h\lambda T_{n-2} = \begin{cases} \frac{1}{\lambda} \prod_{k=1}^{p} (\lambda^2 - 4h\lambda \cos^2 \frac{k\pi}{n+2}) \cdot (-\lambda) = \lambda^m \prod_{k=1}^{p} (\lambda - 4 \cos^2 \frac{k\pi}{n+2}), \\ n \text{ is even} ; \\ -\frac{1}{\lambda} \prod_{k=1}^{p} (\lambda^2 - 4h\lambda \cos^2 \frac{k\pi}{n+2}) = -\lambda^m \prod_{k=1}^{p} (\lambda - 4 \cos^2 \frac{k\pi}{n+2}) , \\ n \text{ is odd} . \end{cases} \qquad (3.17)$$

This proves the theorem.

PROPOSITION 3.13. Let H_n be a Hessenberg matrix in form of (3.6) and λ be an eigenvalue of H_n. If $X = (x_1, x_2, \ldots, x_n)$ is an eigenvector of H_n, then,

$$x_n = \det (H_{n-1} - \lambda I),\ x_j = h^{j-1}\ 2^{j-2}\ \cos^{j-2}\ \phi\ \frac{\sin(j+1)\phi}{\sin \phi}$$

PROOF. By Theorem 3.6 ,

$$x_n = \det(H_{n-1} - \lambda I) ,$$

and

$$x_j = (-1)^{n-j} \det(H_{j-1} - \lambda I) . \qquad (3.18)$$

From Equation (3.13) we can write

$$\det(H_n - \lambda I) + \lambda \det(H_{n-1} - \lambda I) + h\ \lambda\ \det(H_{n-2} - \lambda I) = 0 \quad (3.19)$$

if $\det(H_0 - \lambda I) = 1$, $\det(H_1 - \lambda I) = \alpha - \lambda$.

The solution of (3.19) is

$$\det(H_n - \lambda I) = (-h)^n\ 2^{n-1}\ \cos^{n-1}\ \phi\ \frac{\sin(n+2)\phi}{\sin \phi} \quad (3.20)$$

where

$$\lambda = 4h\ \cos^2\ \phi \quad \text{and} \quad \phi = \frac{k\pi}{n+2} .$$

Hence, Formula (3.18) can be written as

$$x_j = h^{j-1}\ 2^{j-2} \cos^{j-2} \phi\ \frac{\sin(j+1)\phi}{\sin \phi} .$$

III. THE COMPANION MATRIX

DEFINITION 3.14. The companion matrix of a polynomial

$$f(x) = x^n + a_1\ x^{n-1} + a_2\ x^{n-2} \ \ldots\ a_n$$

is defined as

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{pmatrix} \qquad (3.21)$$

In which the first super diagonal consists entirely of one and all other elements above the last row are zero. The Companion matrix of $x + a_1$ is $(-a_1)$.

THEOREM 3.8. The companion matrix A of the polynomial f(x) has the eigenfunction as

$$(-1)^n f(\lambda) .$$

PROOF. If

$$f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n ,$$

then

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ & & & & \\ 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{pmatrix} .$$

By definition of eigenfunction we have

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & \cdots & 0 & 0 \\ 0 & -\lambda & \cdots & 0 & 0 \\ & & & & \\ 0 & 0 & \cdots & -\lambda & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a-\lambda \end{vmatrix} .$$

If we multiply columns 2, 3, ..., n of this determinant by $\lambda, \lambda^2, \ldots, \lambda^{n-2}$ and add them to the first column, all elements of this become zero except the last which is now $-f(\lambda)$, such that

$$\det (A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -f(\lambda) & -a_{n-1} & \cdots & \cdots & -a_2 & -a_1-\lambda \end{vmatrix}.$$

Since the cofactor of this element is $(-1)^{n+1}$, the eigenfunction of A is

$$\det (A - \lambda I) = (-1)^n f(\lambda) .$$

THEOREM 3.9. If the polynomial f(x) of degree n has $m \leq n$ distince zero λ_i then its companion matrix of f(x) has exactly m independent eigenvectors,

$$X_i = (1, \lambda^2_i, \lambda^3_i, \ldots, \lambda^{n-1}_i)$$

corresponding to eigenvalues λ_i.

PROOF. By the definition of eigenvectors and eigenvalues of matrices we have

$$AX(\lambda) = \lambda X(\lambda) ,$$

then

$$\begin{pmatrix} -\lambda & 1 & 0 & & 0 & 0 \\ 0 & -\lambda & 1 & & 0 & 0 \\ 0 & 0 & 0 & \ldots & -\lambda & 1 \\ -a_n & -a_{n-1} & \cdots & & -a_2 & -a_1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} ,$$

where

$$X(\lambda) = (x_1, x_2, \ldots, x_n) .$$

Therefore

$$\begin{aligned} -x_1 \lambda + x_2 &= 0 , \\ -x_2 \lambda + x_3 &= 0 , \\ &\cdots\cdots\cdots \\ -x_{n-1}\lambda + x_n &= 0 , \\ -a_n x_1 - a_{n-1} x_2 \ldots (a_1-\lambda) x_n &= 0 . \end{aligned}$$

Let x_1 be arbitrary, if $x_1 = 1$, then

$$x_2 = \lambda, \quad x_3 = \lambda^2, \ldots, x_i = \lambda^{i-1}, \ldots, x_n = \lambda^{n-1} .$$

Hence

$$X(\lambda) = (1, \lambda , \lambda^2 \ldots \lambda^{n-1}).$$

PROPOSITION 3.14. Let A be a companion matrix of the polynomial

$$f(x) = x^n + a_n x^{n-1} + a_{n-1} \lambda^{n-2} + \ldots + a_1 ,$$

where A has an eigenvalue λ_0 of multiplicity k . Then A has one proper eigenvector Z_1 , such that

$$A Z_1 = Z_1 \lambda_0$$

and k - 1 linear independent generalized eigenvectors are given by

$$Z_{j+1}(\lambda) = \frac{Z_1^{(j)}(\lambda)}{j !} \qquad j = 1, 2, \ldots, k-1 ,$$

where $Z_1^{(j)}(\lambda)$ is the jth derivative of $Z_1(\lambda)$ respect to λ .

PROPOSITION 3.15. If A is a companion matrix of the polynomial $f(\lambda)$, then a nonsingular matrix B exists such that

$$B^{-1}AB = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & J_k \end{pmatrix} ,$$

where the J_i are Jordan blocks, and $B = (Z_1, Z_2 \ldots Z_n)$ where Z_i are linearly independent generalized eigenvectors (including proper eigenvectors) .

EXAMPLE 3.1. Let $f(x) = x^5 - 7x^4 + 14x^3 - 2x^2 - 15x + 9$ be a polynomial. Compute the eigenvalues and proper and generalized eigenvectors of the companion matrix A of the polynomial $f(x)$.

SOLUTION. We can write

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -9 & +15 & 2 & -14 & 7 \end{pmatrix} .$$

By Theorem 3.8 the eigenfunction of A is

$$(-1)^n f(\lambda) = -\lambda^5+7\lambda^4-14\lambda^3+\lambda^2+15+9 .$$

Then

$$(\lambda+1)(\lambda-1)^2(\lambda-3)^2 = 0 ,$$

therefore the eigenvalues are

$$\lambda_1 = -1, \lambda_2 = 1 , \lambda_3 = 3 .$$

By Theorem 3.9 proper eigenvectors X_i corresponding to eigenvalues λ_i are given by

$$X_i(\lambda_i) = (1, \lambda_i, \lambda_i^2, \lambda_i^3, \lambda_i^4) .$$

Then

$$X_1(-1) = (1, -1, 1-1, 1) ,$$

$$X_2(1) = (1, 1, 1, 1, 1) ,$$

$$X_3(3) = (1, 3, 9, 27, 81) .$$

By Proposition 3.14, we have linearly independent generalized eigenvectors

$$Z_1(-1) = X_1(-1) = (1, -1, 1, -1, 1) ,$$

$$Z_2(1) = X_2(1) = (1, 1. 1, 1, 1) ,$$

$$Z_3(1) = X'_2(1) = (0, 1, 2, 3, 4),$$

$$Z_4(3) = X_3(3) = (1, 3, 9, 27, 81) ,$$

$$Z_5(3) = X'_3(3) = (0, 1, 6, 27, 108) .$$

Let $B = (Z_1, Z_2, Z_3, Z_4, Z_5)$ and

$$B^{-1} = \begin{pmatrix} Z^1 \\ Z^2 \\ Z^3 \\ Z^4 \\ Z^5 \end{pmatrix} ,$$

where $Z^i Z_j$ is a Kronecker delta δ_{ij} .

Then

$$B^{-1}AB = \begin{pmatrix} Z^1 \\ Z^2 \\ Z^3 \\ Z^4 \\ Z^5 \end{pmatrix} A \quad (Z_1, Z_2, Z_3, Z_4)$$

$$= \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} .$$

IV. EIGENVALUES AND EIGENVECTORS OF PERMUTATION MATRICES

DEFINITION 3.15. Let S_n be the set of all permutations on $\{1, 2, \ldots, n\}$. A cycle in S_n in a permutation σ which has the property that there exists a subset of $\{1, \ldots, n\}$ as $\{i_1, i_2, \ldots, i_k\}$, such that

$$\sigma(i_1) = i_2 \, , \, \sigma(i_2) = i_3 \ldots \sigma(i_{k-1}) = i_k \, , \, \sigma(i_k) = i_1$$

and $\sigma(j) = j$ for $j \neq i_t$ $(t = 1 \ldots k)$

DEFINITION 3.16. Any permutation σ is a product of cycles acting on disjoint subsets of $\{1, \ldots, n\}$ (disjoint cycles) and this factorization is unique to within order. If the length of these cycles are $p_1 \ldots p_m$, then σ is said to have the cycle structure

$$[p_1 \cdots p_m] \, .$$

EXAMPLE 3.2. Let $P \in M_n(R)$ be permutation matrix of the form

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \qquad (3.22)$$

compute the eigenvalues and eigenvectors of P.

SOLUTION. Since

$$\det (P - \lambda I) = \begin{vmatrix} -\lambda & 0 & & 0 & 1 \\ 1 & -\lambda & & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & & 1 & -\lambda \end{vmatrix} = \lambda^n - 1 = 0 \ ,$$

the eigenvalues λ_k are the nth roots of 1 ,

$$\lambda_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$$

$$= e^{2ik\pi/n} \ , \qquad \text{where } k = 0, 1 \ldots n-1 \ .$$

Let

$$\omega_j = (\delta_{1j}, \delta_{2j} \ldots \delta_{nj}) \text{ where } \delta_{ij} \text{ is a Kronecker delta,}$$

then

$$P\omega_1 = \omega_2 \ , \ P\omega_2 = \omega_3, \ \ldots, \ P\omega_{n-1} = \omega_n, \ P\omega_n = \omega_1 \ .$$

Let

$$X_t = \sum_{k=1}^{n} \varepsilon^{kt} \omega_k \ , \quad \text{where } \varepsilon^k = e^{2i k\pi/n}, \ t = (1,\ldots,n), (3.22)$$

then

$$PX_t = \sum_{k=1}^{n} \varepsilon^{kt} P\omega_k$$

$$= \varepsilon^{t}\omega_2 + \varepsilon^{2t}\omega_3 + \ldots \varepsilon^{(n-1)t}\omega_n + \varepsilon^{nt}\omega_1$$

$$= \varepsilon^{-t}\,(\varepsilon^{(n+1)t}\omega_1 + \varepsilon^{2t}\omega_2 + \ldots + \varepsilon^{nt}\omega_n)$$

$$= \varepsilon^{-t} \sum_{k=1}^{n} \varepsilon^{kt}\omega_k$$

$$= \varepsilon^{-t} X_t = \varepsilon^{n-t} X_t \ .$$

Therefore

$$P\, X_t = \varepsilon^{n-t} X_t \ .$$

Moreover for $r \neq t$

$$(X_r, X_t) = (\sum \varepsilon^{kr}\omega_r, \sum \varepsilon^{kt}\omega_t)$$

$$= \sum_{k=1}^{n} \varepsilon^{(r-t)k} = \frac{\varepsilon^{(r-t)} - \varepsilon^{(r-t)(n+1)}}{1 - \varepsilon^{(r-t}} = 0. \quad (3.24)$$

And also,

$$||X_t||^2 = (X_t, X_t)$$

$$= \sum_{k=1}^{n} \varepsilon^{(t-t)k} = n \ . \quad (3.25)$$

Therefore we can say that P has an orthonormal set of eigenvectors $Y_t = X_{n-t}/\sqrt{n}$ such that

$$P\, Y_t = \varepsilon^{t} Y_t \qquad t = 1, \ldots, n \ .$$

DEFINITION 3.17. If $A \in M_n(f)$ and $B \in M_n(f)$, then the direct sum of A and B, denoted by

$$A \oplus B = E ,$$

is a matrix in $M_{nm}(F)$ whose entries are given by

$$e_{ij} = a_{ij} , \quad (i, j = 1, \ldots, n)$$

$$e_{n+1,n+j} = b_{ij}, \quad (i, j = 1, \ldots, n) ,$$

$$e_{ij} = 0, \quad \text{otherwise} .$$

In general, we will write the direct sum of m matrices as the following:

$$\bigoplus_{i=1}^{m} A_i .$$

PROPOSITION 3.16. Let A be an n-square permutation matrix and let the permutation have cycle structure $[P_1 \ldots P_m]$. Then there exists an n-square permutation matrix P such that

$$P^{-1}AP = P^{*}AP = \bigoplus_{i=1}^{m} A_i$$

in which A_i is a p_i-square matrix of the form (3.22).

THEOREM 3.10. Let A be an n-square permutation matrix and the permutation σ has cycle structure $[P_1 \ldots P_m]$, then the eigenvalues of A are just the totality of roots of the m polynomial

$$\lambda^{p_i} - 1 , \quad (i = 1, \ldots, m) .$$

PROOF. By Proposition 3.16 there exists an n-square permutation matrix such that

$$P^{-1}AP = P^{*}AP = \bigoplus_{i=1}^{m} A_i \quad .$$

Since $\det\ (A - \lambda I) = \det\ P^{-1}(A - \lambda I)P$

$$= \det\ (P^{-1}AP - \lambda I)$$

then the eigenvalues of A and $P^{-1}AP$ are the same. The eigenvalues of $\bigoplus_{i=1}^{m} A_i$ is the set of all eigenvalues of $A_i, i=1, \ldots, m$. By Example 3.3 the eigenvalues of A_i are the roots of polynomial $\lambda^{p_i} - 1$. Hence the theorem is proved.

EXAMPLE 3.4. Compute the eigenvalue of a permutation matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad .$$

SOLUTION. The permutation σ for which $a_i\ \sigma\ (\tau) = 1$ is given by

$$\sigma_{(1)} = 2,\ \sigma_{(2)} = 3,\ \sigma_{(3)} = 1,\ \sigma_{(4)} = 5,\ \sigma_{(5)} = 4 \ .$$

Then A has a cycle structure of $[2, 3]$, by Proposition 3.16, there exists a permutation matrix P for which

$$P^{-1}AP = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad .$$

Then by Example 3.3 , the eigenvalues of A are

$$\lambda_k = e^{i2\pi k/3},\ (k = 0,1,2),\ \lambda_{k+3} = e^{i2k\pi/2},\ k = 0,\ 1\ .$$

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