To the Graduate Council:

I am submitting herewith a dissertation written by Charlotte A. Knotts-Zides entitled "Extremal Properties of Eigenvalues." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree Doctor of Philosophy, with a major in Mathematics.

Don Hinton

Don B. Hinton, Major Professor

We have read this dissertation
and recommend its acceptance:

Suzanne Lenhart

Philip Schaefer

John D. Landes

Accepted for the Council:

C.W. Minkel

Associate Vice Chancellor and
Dean of The Graduate School

EXTREMAL PROPERTIES OF EIGENVALUES

A Dissertation
Presented for the
Doctor of Philosophy
Degree
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Charlotte A. Knotts-Zides

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DEDICATION

This dissertation is dedicated

to my parents, Ray and Sandra Knotts,

who have always believed in me (and taught me early to believe in myself) and who have supported me with abundant prayers and words of encouragement,

and,

to my husband, Steve Zides,

whose sense of humor made me laugh even when I didn't feel like laughing and whose love and support nourished me throughout this process.

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ABSTRACT

In this work, we examine eigenvalues of ordinary and generalized Sturm-Liouville problems. We begin by considering the general boundary value problem $-y''(t)+q(t)y(t) = \lambda y(t)$, $(\cos\alpha)y(a) - (\sin\alpha)y'(a) = 0$, $(\cos\beta)y(b) - (\sin\beta)y'(b) = 0$, where $-\infty < a \leq t \leq b < \infty$, $0 \leq \alpha < \pi$, $0 < \beta \leq \pi$, and $q(t)$ is a real piecewise continuous function. This self-adjoint problem has associated eigenvalues $\lambda_0(q) < \lambda_1(q) < \ldots < \lambda_n(q) < \ldots$ If we let $S := \{\lambda_0(q) : \int_a^b \mid q(t) \mid dt = M\}$ where $M > 0$, our goal is to determine $\lambda_0^{\#} := \sup S$ and to demonstrate there exists a function $q^{\#}(t)$ that satisfies $\lambda_0(q^{\#}) = \lambda_0^{\#}$. For certain values of α and β, e.g. $\alpha = 0$ and $\beta = \pi$, this problem has been investigated by several authors. We present a new approach to this problem for general α and β and provide rigorous solutions to cases where only formal solutions have been presented. For $0 \leq \alpha \leq \frac{\pi}{2}$ and $\frac{\pi}{2} \leq \beta \leq \pi$, we show that $\lambda_0^{\#}$ is the root of a transcendental equation and that $q^{\#}(t)$ exists. For $\frac{\pi}{2} < \alpha < \pi$ and $\frac{\pi}{2} \leq \beta \leq \pi$, our technique yields a candidate for $\lambda_0^{\#}$, a value λ_0^{**} which is a root of a transcendental equation, is an upper bound for S, and is the eigenvalue of the above problem where q contains a delta function. This form of q leads us to consider a more general problem.

In the latter half of this work, we consider the generalized differential system $dy = (dP)\,z$, $dz = (dQ - \lambda dW)y$, and establish certain basic results such as the existence and uniqueness of the solution to the initial value problem. For a sequence of initial value problems, we show that the solutions y_n and z_n converge, the former uniformly and the latter pointwise. Furthermore, the eigenvalues of the boundary value problem are the roots of an entire function; under separated boundary conditions, the eigenvalues are real and simple and a sequence of eigenvalues (corresponding to a sequence of boundary value problems) converge; i.e., the kth

eigenvalue of the nth problem converges to the kth eigenvalue of the limit problem as n tends to infinity.

This last result enables us to prove that there is a sequence $\{\lambda_0^{(n)}\}$ in S that converges to λ_0^{**} and hence λ_0^{**} is the least upper bound of S, i.e., $\lambda_0^{\#} = \lambda_0^{**}$. Moreover, the eigenfunctions (corresponding to the above-mentioned sequence of eigenvalues) also converge, uniformly in y_n and pointwise in z_n.

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CHAPTER I

INTRODUCTION

In this paper, we will study eigenvalues of ordinary and generalized regular Sturm-Liouville problems with general self-adjoint boundary conditions. Previous work on the maximal and minimal values of eigenvalues of certain problems dates back to Krein [9] in 1955, but many papers have been written in the last two decades in response to a problem posed by A.G. Ramm [11] in the 1982 Notices of the American Mathematical Society. In short, the problem was to find the maximum of all first eigenvalues of the problem $-y'' + qy = \lambda y$, $y(0) = y(1) = 0$, under the additional constraint that $\int_0^1 |q(t)|dt = 1$. The first of these papers, written by Talenti [13] in 1984, uses the technique of rearrangements to establish a least upper bound for the eigenvalues of the regular Sturm-Liouville problem with simple boundary conditions. Essén [4] extends the results of Talenti in his paper three years later. Ashbaugh and Harrell [1] present further results for a general class of problems (in the setting of elliptic partial differential equations) with an interpretation for problems in mathematical physics. Using techniques similar to the methods of Talenti, Egnell [3] presents some formal and some rigorous solutions to the extremal properties of the first eigenvalue of a fairly general class of Sturm-Liouville problems. Our paper uses a different technique, but our results agree with Egnell's and, further, we provide rigorous solutions to cases where his solutions are only formal.

Our work differs from that of previous authors in that we utilize an inequality (derived by Brown, Hinton, and Schwabik [2] via a Prüfer transformation argument) to develop an upper bound on the eigenvalues of the differential equation $-y'' + qy = \lambda y$ with general self-adjoint boundary conditions; analysis of this inequality in section 2.2 reveals the conditions under which the maximal eigenvalue can be achieved. Section 2.3 applies these conditions to the regular problem with Dirichlet boundary conditions. In section 2.4, we look at the regular problem with general boundary conditions of the form $(\cos\alpha)y(a) - (\sin\alpha)y'(a) = 0$ and $(\cos\beta)y(b) - (\sin\beta)y'(b) = 0$ where $0 \leq \alpha < \pi$ and $0 < \beta \leq \pi$. When $0 \leq \alpha \leq \frac{\pi}{2}$ and $\frac{\pi}{2} \leq \beta \leq \pi$, the maximal eigenvalue can be found using the conditions established in section 2.2. However, if $\frac{\pi}{2} < \alpha < \pi$ and $\frac{\pi}{2} \leq \beta \leq \pi$, the conditions of section 2.2 suggest but do not establish the supremum of the eigenvalues. For particular values of α and β, we consider examples of this case in section 2.5 and, using the examples as motivation, we prove the existence of an improved upper bound, denoted λ_0^{**}, of the eigenvalues in section 2.6. The proof that this value is, in fact, the supremum of all the eigenvalues in the second case must wait until Chapter 4.

Chapter 3 establishes certain basic results of the generalized Sturm-Liouville problem with the goal of proving convergence of a sequence of eigenvalues corresponding to a sequence of generalized problems under certain conditions. We apply the results of Hinton [6] (summarized in section 3.1) to prove the existence and uniqueness of the solution to the initial value problem of the generalized differential system consisting of

$$dy = (dP)\,z,$$
$$dz = (dQ - \lambda dW)y.$$

For a sequence of such problems, we show in section 3.4 that the solutions y_n and z_n converge, the former uniformly and the latter pointwise. In section 3.3, we prove that the eigenvalues of the boundary value problem are the roots of an entire function $d(\lambda)$ and, in section 3.5, we verify that a sequence $d_n(\lambda)$ of these entire functions (corresponding to a sequence of boundary value problems) converges uniformly. A little more analysis shows that the eigenvalues (which are the roots of $d_n(\lambda)$) also converge.

Finally, in chapter 4, we recall that λ_0^{**} is an upper bound of the eigenvalues in case B of the general boundary conditions from section 2.6. From section 3.5, we can show that there exists a sequence of eigenvalues which converge to λ_0^{**} from below. Hence, λ_0^{**} is the supremum we seek. Moreover, the eigenfunctions (corresponding to these eigenvalues) also converge, uniformly in y_n and pointwise in z_n. We mention here the related work of Kong and Zettl [8] in which the authors show that the eigenvalues of a Sturm-Liouville problem depend continuously and smoothly on all the parameters of the problem including the coefficients. In particular, they show that if $q_n \to q$ in $\mathcal{L}^1$-norm, then $\lambda(q_n) \to \lambda(q)$ and $y_n(t, q_n) \to y_n(t, q)$ uniformly. In this paper, we have obtained a similar result with a weaker hypothesis on q and q_n.

CHAPTER II

THE CLASSICAL STURM-LIOUVILLE PROBLEM

2.1 The Regular Problem

We begin by considering the general problem

$$-y''(t) + q(t)y(t) = \lambda y(t) \tag{2.1}$$

on the interval $[a, b]$ where $-\infty < a < b < \infty$ with self-adjoint boundary conditions of the form

$$(\cos\alpha)y(a) - (\sin\alpha)y'(a) = 0 \tag{2.2}$$
$$(\cos\beta)y(b) - (\sin\beta)y'(b) = 0$$

where $0 \leq \alpha < \pi$ and $0 < \beta \leq \pi$; here, $q(t)$ is a real piecewise continuous function [14]. This self-adjoint problem has associated eigenvalues

$$\lambda_0(q) < \lambda_1(q) < \ldots < \lambda_n(q) < \ldots$$

Given $M > 0$, our goal is to find, for each n, the supremum of the set of all eigenvalues subject to the constraint $\int_a^b |q(t)|dt = M$, which we denote

$$\lambda_n^{\#} := \sup\{\lambda_n(q) : \int_a^b |q(t)|dt = M\}, \tag{2.3}$$

and to show, where possible, there exists $q^{\#}(t)$ such that the eigenvalue $\lambda_n(q^{\#})$ corresponding to $q^{\#}$ equals $\lambda_n^{\#}$. Here, we solve only the case where $n = 0$.

We begin with some definitions.

A function $f(x)$ defined on the real interval (a, b) is said to be *absolutely continuous* if for any $\epsilon > 0$, there is a $\delta > 0$ such that for any sequence of mutually disjoint intervals (a_i, b_i) in (a, b), with $\sum(b_i - a_i) < \delta$, then $\sum|f(b_i) - f(a_i)| < \epsilon$.

If $g(y)$ is a Lebesgue-integrable function on a bounded interval $[a,b]$, the function $f(x) = \int_a^x g(y)dy$ is absolutely continuous, and every absolutely continuous function f with $f(a) = 0$ has this representation.

Let $AC[a,b]$ represent the set of all absolutely continuous functions on $[a,b]$.

Define

$$f_+ := \begin{cases} f, & f > 0 \\ 0, & f \leq 0 \end{cases}$$

and

$$f_- := \begin{cases} 0, & f \geq 0 \\ -f, & f < 0. \end{cases}$$

For simplicity, we let $[a,b] = [0,1]$. Furthermore, we can choose $q(t) \geq 0$ without loss of generality by examining Rayleigh quotients. Here, we consider the case where $\alpha = 0$ and $\beta = \pi$ and define the function $Q(y,q)$ by

$$Q(y,q) = \frac{\int_0^1 \left[|y'(t)|^2 + q(t)|y(t)|^2\right] dt}{\int_0^1 |y(t)|^2 dt}.$$

Then, the eigenvalue $\lambda_0(q)$ has the property that $\lambda_0(q) = \min_{y \in \mathcal{D}} Q(y,q)$ where $\int_0^1 |q(t)|dt = M$ and

$$\mathcal{D} := \{y \in AC[0,1] : \int_0^1 |y'(t)|^2 dt < \infty \text{ and } y(0) = y(1) = 0\}.$$

We can define $\tilde{q}(t)$ by $\tilde{q}(t) = |q(t)|$ in order that $\tilde{q}(t)$ satisfy $\int_0^1 \tilde{q}(t)\, dt = M$. Note that

$$\int_0^1 q_+(t)dt \leq \int_0^1 |q(t)|dt = \int_0^1 \tilde{q}(t)dt.$$

Now,

$$Q(y,q) \leq Q(y,q_+) \leq Q(y,\tilde{q})$$

and, taking the minimum of each term in the inequality over $y \in \mathcal{D}$, we have

$$\lambda_0(q) \leq \lambda_0(q_+) \leq \lambda_0(\tilde{q}).$$

Since our goal is to find the supremum over all eigenvalues $\lambda_0(q)$ which satisfy the constraint $\int_0^1 |q(t)|dt = M$, we have just shown that it suffices to choose $q(t) \geq 0$.

2.2 An Upper Bound on the Eigenvalues of the Regular Problem

We begin by considering an established upper bound for λ_n (see [2]) from which we can derive necessary conditions to obtain a least upper bound.

Theorem 2.1 *Let q be a real piecewise continuous function and let $\lambda_0 < \lambda_1 < ...$ and $\phi_0, \phi_1, ...$ denote the eigenvalues and real orthonormal eigenfunctions, respectively, of the problem* (2.1) *and* (2.2). *Let $\lambda < \lambda_n$. Then the eigenvalue λ_n satisfies the inequality*

$$\lambda_n \leq \lambda + \frac{1}{4}\left([\theta(1)-\theta(0)] + \sqrt{[\theta(1)-\theta(0)]^2 + 4\int_0^1 \{\lambda - q\}_- dt}\right)^2 \qquad (2.4)$$

where $\theta(t) = \cot^{-1}\left(\frac{1}{\sqrt{\lambda_n - \lambda}} \cdot \frac{\phi_n'}{\phi_n}\right)$.

Proof: In $-\phi_n'' + q\phi_n = \lambda_n\phi_n$, we introduce the Prüfer transformation $\phi_n = r\sin\theta$ and $\phi_n' = (\sqrt{\lambda_n - \lambda})r\cos\theta$, where $r(t) > 0$. By differentiating ϕ_n and equating the result to ϕ_n' , we have

$$r\theta'\cos\theta + r'\sin\theta = (\sqrt{\lambda_n - \lambda})r\cos\theta. \qquad (2.5)$$

By differentiating ϕ_n' and substituting into $-\phi_n'' + q\phi_n = \lambda_n \phi_n$, we have

$$-(-\sqrt{\lambda_n - \lambda}\, r\theta' \sin\theta + \sqrt{\lambda_n - \lambda}\, r' \cos\theta) + qr\sin\theta = \lambda_n r \sin\theta$$

which simplifies to

$$r\theta' \sin\theta - r'\cos\theta = \frac{(\lambda_n - q) r \sin\theta}{\sqrt{\lambda_n - \lambda}}. \tag{2.6}$$

Now, multiplying equation (2.5) by $\sin\theta$ and equation (2.6) by $\cos\theta$ and subtracting the two equations, we obtain an expression for r', namely,

$$\begin{aligned} & r\theta' \cos\theta \sin\theta + r' \sin^2\theta - r\theta' \sin\theta\cos\theta + r'\cos^2\theta \\ & \qquad = \sqrt{\lambda_n - \lambda}\, r \cos\theta\sin\theta - \frac{(\lambda_n - q) r \sin\theta\cos\theta}{\sqrt{\lambda_n - \lambda}} \end{aligned}$$

or

$$r' = r\left(\sqrt{\lambda_n - \lambda} + \frac{q - \lambda_n}{\sqrt{\lambda_n - \lambda}}\right)\cos\theta\sin\theta. \tag{2.7}$$

Similarly, multiplying equation (2.5) by $\cos\theta$ and equation (2.6) by $\sin\theta$ and adding the two equations, we have

$$\begin{aligned} & r\theta' \cos^2\theta + r'\sin\theta\cos\theta + r\theta'\sin^2\theta - r'\sin\theta\cos\theta \\ & \qquad = \sqrt{\lambda_n - \lambda}\, r\cos^2\theta + \frac{(\lambda_n - q) r \sin^2\theta}{\sqrt{\lambda_n - \lambda}} \end{aligned}$$

or, simplified,

$$r\theta' = \sqrt{\lambda_n - \lambda}\, r\cos^2\theta + \frac{(\lambda_n - q) r\sin^2\theta}{\sqrt{\lambda_n - \lambda}}.$$

Now, dividing this equation by r and working with the last term above, we have

$$\begin{aligned} \theta' &= \sqrt{\lambda_n - \lambda}\cos^2\theta + \frac{(\lambda - q + \lambda_n - \lambda)\sin^2\theta}{\sqrt{\lambda_n - \lambda}} \\ &= \sqrt{\lambda_n - \lambda}\cos^2\theta + \frac{(\lambda - q)\sin^2\theta}{\sqrt{\lambda_n - \lambda}} + \sqrt{\lambda_n - \lambda}\sin^2\theta \end{aligned}$$

so that finally

$$\theta' = \sqrt{\lambda_n - \lambda} + \frac{(\lambda - q)\sin^2\theta}{\sqrt{\lambda_n - \lambda}}. \tag{2.8}$$

Since $(\lambda - q) \geq -\{\lambda - q\}_-$ and $\sin^2\theta \leq 1$, we obtain from above that

$$\theta' \geq \sqrt{\lambda_n - \lambda} - \frac{\{\lambda - q\}_- \sin^2\theta}{\sqrt{\lambda_n - \lambda}} \geq \sqrt{\lambda_n - \lambda} - \frac{\{\lambda - q\}_-}{\sqrt{\lambda_n - \lambda}}.$$

Thus, integrating over $[0,1]$ yields

$$\begin{aligned} \theta(1) - \theta(0) &\geq \sqrt{\lambda_n - \lambda} - \int_0^1 \frac{\{\lambda - q\}_- \sin^2\theta}{\sqrt{\lambda_n - \lambda}}\,dt && (2.9) \\ &\geq \sqrt{\lambda_n - \lambda} - \int_0^1 \frac{\{\lambda - q\}_-}{\sqrt{\lambda_n - \lambda}}\,dt && (2.10) \end{aligned}$$

so that

$$[\theta(1) - \theta(0)]\sqrt{\lambda_n - \lambda} \geq (\lambda_n - \lambda) - \int_0^1 \{\lambda - q\}_- dt. \tag{2.11}$$

Let $A = \lambda_n - \lambda$, $B = \theta(1) - \theta(0)$, and $C = \int_0^1 \{\lambda - q\}_- dt$. Then the above equation becomes $B\sqrt{A} \geq A - C$ or $A - B\sqrt{A} - C \leq 0$ which is quadratic in $\sqrt{A}$. Hence, we have

$$\sqrt{A} \leq \frac{B + \sqrt{B^2 + 4C}}{2}$$

or, equivalently,

$$\sqrt{\lambda_n - \lambda} \leq \frac{1}{2}\left([\theta(1) - \theta(0)] + \sqrt{[\theta(1) - \theta(0)]^2 + 4\int_0^1 \{\lambda - q\}_- dt} \right).$$

Squaring both sides, we obtain

$$\lambda_n \leq \lambda + \frac{1}{4}\left([\theta(1)-\theta(0)] + \sqrt{[\theta(1)-\theta(0)]^2 + 4\int_0^1 \{\lambda - q\}_- dt}\right)^2$$

which is inequality (2.4). ■

Note that the right hand side of the inequality (2.4) functions as an upper bound for λ_n. We seek the least upper bound of all λ_n under the constraint $\int_0^1 q(t)dt = M$ where $q(t) \geq 0$ and, hence, we seek equality in (2.4). The proof of Theorem 2.1 yields the conditions which are necessary for equality to hold in (2.4). Namely, we need to have equality in (2.9) and in (2.10), which reduces to

$$\int_0^1 \{\lambda - q\}_- \sin^2\theta \, dt = \int_0^1 \{\lambda - q\}_- dt \tag{2.12}$$

or, equivalently,

$$\int_0^1 \{\lambda - q\}_- \cos^2\theta \, dt = 0$$

and implies that $\{\lambda - q\}_- \cos^2\theta = 0$ a.e. Also, we need to have equality in (2.11):

$$[\theta(1)-\theta(0)]\sqrt{\lambda_n - \lambda} = (\lambda_n - \lambda) - \int_0^1 \{\lambda - q\}_- dt. \tag{2.13}$$

However, by integrating over $[0, 1]$ in (2.8), we obtain

$$\begin{aligned} \theta(1) - \theta(0) \quad = \quad & \sqrt{\lambda_n - \lambda} + \frac{1}{\sqrt{\lambda_n - \lambda}}\int_0^1 \{\lambda - q\}_+ \sin^2\theta \, dt \\ & - \frac{1}{\sqrt{\lambda_n - \lambda}}\int_0^1 \{\lambda - q\}_- \sin^2\theta \, dt. \end{aligned}$$

Using (2.12), we have

$$\theta(1)-\theta(0) = \sqrt{\lambda_n-\lambda}+\frac{1}{\sqrt{\lambda_n-\lambda}}\int_0^1\{\lambda-q\}_+\sin^2\theta\,dt$$
$$-\frac{1}{\sqrt{\lambda_n-\lambda}}\int_0^1\{\lambda-q\}_-dt$$

or

$$[\theta(1)-\theta(0)]\sqrt{\lambda_n-\lambda}=(\lambda_n-\lambda)+\int_0^1\{\lambda-q\}_+\sin^2\theta dt-\int_0^1\{\lambda-q\}_-dt.$$

Now, comparing this equation with equation (2.13), we find that we must require $\int_0^1\{\lambda-q\}_+\sin^2\theta\,dt=0$ which implies that $\{\lambda-q\}_+\sin^2\theta=0$ a.e. If we suppose that, for some t_0, $\sin\theta(t_0)=0$, i.e., $\theta(t_0)=n\pi$, then $\theta'(t_0)>0$ from equation (2.8). Thus, for small $\epsilon>0$, we know that $\sin\theta(t_0+\epsilon)\neq 0$ and, hence, $\sin\theta(t)\neq 0$ a.e. We conclude that $\{\lambda-q\}_+=0$ a.e. so that $\lambda-q(t)\leq 0$ a.e. or $q(t)\geq\lambda$ a.e. and furthermore $\lambda-q=-\{\lambda-q\}_-$. Thus, the two necessary conditions for equality to hold in (2.4) become

$$q(t)\geq\lambda \text{ a.e.} \tag{2.14}$$

$$\{\lambda-q\}\cos^2\theta=0 \text{ a.e.} \tag{2.15}$$

2.3 Dirichlet Boundary Conditions

We first consider the case of the specific problem with Dirichlet boundary conditions obtained when $\alpha=0$ and $\beta=\pi$, i.e.,

$$-y''(t)+q(t)y(t)=\lambda y(t),$$

$$y(0)=0=y(1),$$

and prove that the necessary conditions for equality to hold in (2.4) suffice to show that $\lambda_0^{\#} = \frac{1}{4}\left(\pi + \sqrt{\pi^2 + 4M}\right)^2$, that a function q exists that satisfies the constraint $\int_0^1 q(t)dt = M$ and that its corresponding eigenvalue $\lambda_0(q)$ is the value $\lambda_0^{\#}$.

From the Prüfer transformation $\phi_n = r\sin\theta$ and the boundary conditions $\phi_n(0) = 0 = \phi_n(1)$, the eigenfunction ϕ_n has exactly n zeros in $(0,1)$ [10, p. 145] so we may take $\theta(0) = 0$ and $\theta(1) = (n+1)\pi$. In particular, for $n = 0$, we have $\theta(0) = 0$ and $\theta(1) = \pi$. Choosing $\lambda = 0$ (since $\lambda_0(q) > 0$ in this case, which can be shown by analyzing the Rayleigh quotient), $n = 0$, $q \geq 0$ and $\int_0^1 q(t)dt = M$, inequality (2.4) becomes

$$\lambda_0 \leq \frac{1}{4}\left(\pi + \sqrt{\pi^2 + 4M}\right)^2 \tag{2.16}$$

and the two requirements for equality to hold in (2.16) are

$$q(t) \geq 0 \text{ a.e.}$$
$$q(t)\cos^2\theta = 0 \text{ a.e.}$$

Thus, either $\cos^2\theta(t) = 0$ a.e. or $q(t) = 0$ a.e. In other words, on the set $S_1 = \{t : \theta(t) \neq \frac{\pi}{2}\}$, $q(t) = 0$ a.e. and, from equation (2.8), $\theta'(t) = \sqrt{\lambda_0}$ a.e. On the set $S_2 = \{t : \theta(t) = \frac{\pi}{2}\}$, equation (2.8) gives us that, at any limit point t_0 of S_2,

$$0 = \theta'(t_0) = \sqrt{\lambda_0} + \frac{(-q(t_0))\sin^2(\frac{\pi}{2})}{\sqrt{\lambda_0}} = \sqrt{\lambda_0} - \frac{q(t_0)}{\sqrt{\lambda_0}}$$

so that $q(t_0) = \lambda_0$. Thus, $q(t)$ and $\theta(t)$ must necessarily have the graphs given in Figure 2.1 in order for equality to hold in (2.16).

The function $q(t)$ defined by Figure 2.1 gives equality in (2.16) if and only if

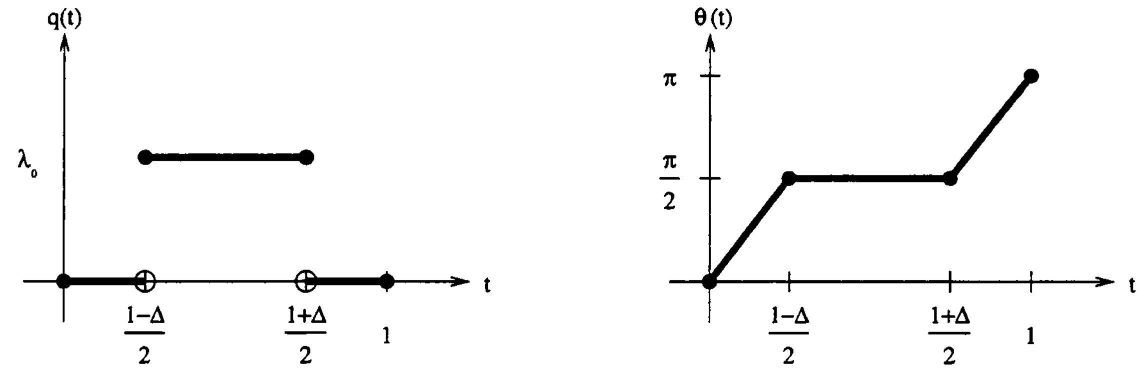

Figure 2.1: Graph of functions $q(t)$ and $\theta(t)$ in the case of Dirichlet boundary conditions

it satisfies the following three requirements (given $M > 0$):

$$\lambda_0 \Delta = M, \tag{2.17}$$

$$\lambda_0 = \frac{1}{4}\left(\pi + \sqrt{\pi^2 + 4M}\right)^2, \tag{2.18}$$

$$\sqrt{\lambda_0}\left(\frac{1-\Delta}{2}\right) = \frac{\pi}{2}. \tag{2.19}$$

Note that equation (2.17) requires $q(t)$ to have the appropriate area, namely $\int_0^1 q(t)dt = M$, equation (2.18) is the definition of equality in (2.16), and equation (2.19) defines the slope of $\theta(t)$ between the points $(0,0)$ and $(\frac{1-\Delta}{2}, \frac{\pi}{2})$ as well as between the points $(\frac{1+\Delta}{2}, \frac{\pi}{2})$ and $(1,\pi)$ (see equation (2.8)). Despite being three equations in two unknowns, these requirements are not overdetermined since (2.17) and (2.18) imply (2.19), which can be easily shown. Thus, we have the following theorem:

Theorem 2.2 *If* $M > 0$, $\lambda_0^* = \frac{1}{4}\left(\pi + \sqrt{\pi^2 + 4M}\right)^2$, $\Delta = \frac{M}{\lambda_0^*}$,

$$q^*(t) = \begin{cases} 0, & 0 \leq t < \frac{1-\Delta}{2} \\ \lambda_0^*, & \frac{1-\Delta}{2} \leq t \leq \frac{1+\Delta}{2} \\ 0, & \frac{1+\Delta}{2} < t \leq 1, \end{cases}$$

$$\theta(t) = \begin{cases} \sqrt{\lambda_0^*}\, t, & 0 \leq t < \frac{1-\Delta}{2} \\ \frac{\pi}{2}, & \frac{1-\Delta}{2} \leq t \leq \frac{1+\Delta}{2} \\ \sqrt{\lambda_0^*}\, t + (\pi - \sqrt{\lambda_0^*}), & \frac{1+\Delta}{2} < t \leq 1, \end{cases}$$

and $r(t)$ *solves the initial value problem*

$$r' = r\left(\sqrt{\lambda_0^*} + \frac{q^* - \lambda_0^*}{\sqrt{\lambda_0^*}}\right)\cos\theta\sin\theta$$

$$r(0) = 1$$

then $\phi_0 = r\sin\theta$ *satisfies*

$$-\phi_0'' + q^*\phi_0 = \lambda_0^*\phi_0$$

$$\phi_0(0) = 0 = \phi_0(1)$$

except at points of discontinuity of $q^*(t)$.

Remark: Note that the initial value problem which r solves is composed of equation (2.7) and an arbitrarily chosen initial condition, since equation (2.7) is homogeneous and linear in r and hence any constant multiple of a solution is also a solution. •

Proof: First, we verify the boundary conditions:

$$\phi_0(0) = r(0)\sin\theta(0) = r(0)\sin 0 = 0,$$

$$\phi_0(1) = r(1)\sin\theta(1) = r(1)\sin\pi = 0.$$

We have $\phi_0 = r\sin\theta$ and, since $\phi_0' = \sqrt{\lambda_0^*}\,r\cos\theta$, we know that

$$\phi_0'' = \sqrt{\lambda_0^*}\,r'\cos\theta - \sqrt{\lambda_0^*}\,r\theta'\sin\theta.$$

On the set S_2, we have $\theta(t) = \frac{\pi}{2}$, $\theta'(t) = 0$, and $q^*(t) = \lambda_0^*$ a.e.; thus,

$$\begin{aligned}
-\phi_0'' + q^*\phi_0 - \lambda_0^*\phi_0 &= -\sqrt{\lambda_0^*}\,r'\cos\theta + \sqrt{\lambda_0^*}\,r\theta'\sin\theta + q^*r\sin\theta - \lambda_0^* r\sin\theta \\
&= -\sqrt{\lambda_0^*}r'\cdot 0 + \sqrt{\lambda_0^*}r\cdot 0 + \lambda_0^* r\cdot 1 - \lambda_0^* r\cdot 1 \\
&= 0.
\end{aligned}$$

On the set S_1, we have $\theta' = \sqrt{\lambda_0^*}$ and $q^* \equiv 0$ a.e.; thus,

$$\begin{aligned}
r' &= r\left(\sqrt{\lambda_0^*} + \frac{q^* - \lambda_0^*}{\sqrt{\lambda_0^*}}\right)\cos\theta\sin\theta \\
&= r\left(\sqrt{\lambda_0^*} + \frac{0 - \lambda_0^*}{\sqrt{\lambda_0^*}}\right)\cos\theta\sin\theta \\
&= 0
\end{aligned}$$

and, finally,

$$\begin{aligned}
-\phi_0'' + q^*\phi_0 - \lambda_0^*\phi_0 &= -\sqrt{\lambda_0^*}\,r'\cos\theta + \sqrt{\lambda_0^*}\,r\theta'\sin\theta + q^*r\sin\theta - \lambda_0^* r\sin\theta \\
&= -\sqrt{\lambda_0^*}\cdot 0\cdot\cos\theta + \sqrt{\lambda_0^*}\,r\sqrt{\lambda_0^*}\sin\theta + 0\cdot r\sin\theta - \lambda_0^* r\sin\theta \\
&= 0.
\end{aligned}$$

This completes the proof. ■

Theorem 2.1 established that $\lambda_0^* = \frac{1}{4}(\pi + \sqrt{\pi^2 + 4M})^2$ is an upper bound for all eigenvalues $\lambda_0(q)$ of the problem $-y'' + qy = \lambda y$, $y(0) = 0 = y(1)$. Theorem

2.2 shows furthermore that $\lambda_0^* = \lambda_0(q^*)$ is an eigenvalue of the above problem with $\int_0^1 q^*(t)\,dt = M$ so that

$$\lambda_0^* \in \{\lambda_0(q) : \int_0^1 q(t)dt = M\}$$

and, hence, we have the following corollary:

Corollary 2.2.1 *Given the hypotheses of Theorem 2.2, the value of λ_0^* is the maximum of the set $S = \{\lambda_0(q) : \int_0^1 q(t)dt = M\}$, i.e.,*

$$\lambda_0^{\#} = \frac{1}{4}\left(\pi + \sqrt{\pi^2 + 4M}\right)^2,$$

and the eigenvalue $\lambda_0(q^*)$, which corresponds to q^* defined in Theorem 2.2, is the value $\lambda_0^{\#}$.

2.4 General Boundary Conditions

Now, we want to consider the Sturm-Liouville problem (2.1) on $[0,1]$ with general boundary conditions (2.2) where $0 \le \alpha < \pi$ and $0 < \beta \le \pi$.

First, we need to determine the dependence of $\theta(0)$ and $\theta(1)$ on the values of α, β, and λ_0. If $\alpha = 0$, then $0 = y(0) = r(0)\sin\theta(0)$ so that we can choose $\theta(0) = 0$. If $0 < \alpha < \pi$, then, from (2.2) and the Prüfer transformation, we know

$$\cot\alpha = \frac{y'(0)}{y(0)} = \frac{\sqrt{\lambda_0}\,r(0)\cos\theta(0)}{r(0)\sin\theta(0)} = \sqrt{\lambda_0}\cot\theta(0)$$

so that $\theta(0) = \cot^{-1}\left(\frac{\cot\alpha}{\sqrt{\lambda_0}}\right)$. Similarly, if $\beta = \pi$, we can choose $\theta(1) = \pi$. If $0 < \beta < \pi$, then, as above, $\theta(1) = \cot^{-1}\left(\frac{\cot\beta}{\sqrt{\lambda_0}}\right)$. Again, we look at $n = 0$ and let

$\int_0^1 q(t)dt = M$ where $q(t) \geq 0$ and, without loss of generality, we consider two cases for α and β:

$$\text{A. } 0 \leq \alpha \leq \frac{\pi}{2}, \ \frac{\pi}{2} \leq \beta \leq \pi$$

$$\text{B. } \frac{\pi}{2} < \alpha < \pi, \ \frac{\pi}{2} \leq \beta \leq \pi$$

Case A: Let $0 \leq \alpha \leq \frac{\pi}{2}$ and $\frac{\pi}{2} \leq \beta \leq \pi$. Here, $0 \leq \theta(0) \leq \frac{\pi}{2}$ and $\frac{\pi}{2} \leq \theta(1) \leq \pi$. We can again let $\lambda = 0$ in Theorem 2.1 since we know that $\lambda_0(q) > 0$ by analyzing the Rayleigh quotient; that is, if $\mathcal{D} := \{y \in AC[0,1] : \int_0^1 (y')^2\, dt < \infty\}$ with the condition $y(0) = 0$ added if $\alpha = 0$ and the condition $y(1) = 0$ added if $\beta = \pi$ and $Q(y) = \int_0^1 (-y'' + qy)y\, dt$, then

$$\lambda_0(q) \quad = \quad \min_{y \in \mathcal{D}} \frac{Q(y)}{\int_0^1 y^2 dt}$$

with equality for $y = y_0$, the eigenfunction corresponding to $\lambda_0(q)$. Then,

$$\lambda_0(q) \quad = \quad \frac{-(\cot\beta)(y_0(1))^2 + (\cot\alpha)(y_0(0))^2 + \int_0^1 (y_0')^2 dt + \int_0^1 q y_0^2 dt}{\int_0^1 y_0^2 dt}.$$

If $\alpha = 0$, then $y_0(0) = 0$; otherwise, if $0 < \alpha \leq \frac{\pi}{2}$, then $\cot\alpha \geq 0$. If $\beta = \pi$, then $y_0(1) = 0$; otherwise, if $\frac{\pi}{2} \leq \beta < \pi$, then $-\cot\beta \geq 0$. Hence, $-(\cot\beta)(y_0(1))^2 + (\cot\alpha)(y_0(0))^2 \geq 0$. Now, y_0 cannot be identically zero on $[0,1]$ (and $q \geq 0$) so that we have $\int_0^1 q y_0^2\, dt \geq 0$, $\int_0^1 y_0^2\, dt > 0$ and $\int_0^1 (y_0')^2\, dt \geq 0$. If $y_0' \equiv 0$, then $y_0 \equiv c$ where c is any nonzero constant, so $\int_0^1 q y_0^2\, dt = c^2 \int_0^1 q\, dt = c^2 M > 0$. Hence, substituting into the above equality, we know that $\lambda_0(q)$ is strictly positive and we can choose $\lambda = 0$ so that $\lambda_0(q) > \lambda$. The necessary condition (2.15) implies that $q(t)$ and $\theta(t)$ need to have the form given in Figure 2.2. Thus the function $q(t)$ defined by

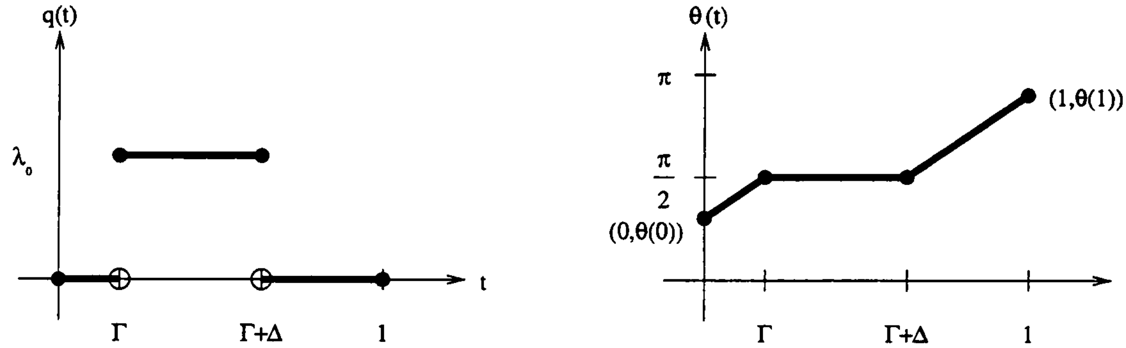

Figure 2.2: Graph of functions $q(t)$ and $\theta(t)$ in case A of the general boundary conditions

Figure 2.2 will give equality in (2.4) provided the following four requirements can be satisfied:

$$\lambda_0 \Delta = M, \tag{2.20}$$

$$\lambda_0 = \frac{1}{4}\left[\theta(1) - \theta(0) + \sqrt{[\theta(1) - \theta(0)]^2 + 4M}\right]^2, \tag{2.21}$$

$$\sqrt{\lambda_0}\,\Gamma = \frac{\pi}{2} - \theta(0), \tag{2.22}$$

$$\sqrt{\lambda_0}(1 - \Gamma - \Delta) = \theta(1) - \frac{\pi}{2}. \tag{2.23}$$

In this case, $\frac{\pi}{2} \leq \theta(1) \leq \pi$ so that $\theta(1) - \frac{\pi}{2} \geq 0$; thus, by algebraically rearranging (2.23), we have

$$\Gamma + \Delta = -\frac{1}{\sqrt{\lambda_0}}\left[\theta(1) - \frac{\pi}{2}\right] + 1$$

which assures us that $\Gamma + \Delta \leq 1$.

The above four equations in three unknowns are not overdetermined since

Lemma 2.1 *Requirements (2.20), (2.21), and (2.22) imply (2.23).*

Proof: First, if we solve (2.21) for M, we have

$$M = \lambda_0 - \sqrt{\lambda_0}\,[\theta(1) - \theta(0)]\,.$$

Then, using (2.20) and (2.22), we get

$$\begin{aligned}
\sqrt{\lambda_0}(1 - \Gamma - \Delta) &= \sqrt{\lambda_0}\left(1 - \frac{\frac{\pi}{2} - \theta(0)}{\sqrt{\lambda_0}} - \frac{M}{\lambda_0}\right) \\
&= \sqrt{\lambda_0} - \frac{\pi}{2} + \theta(0) - \frac{1}{\sqrt{\lambda_0}}\left[\lambda_0 - \sqrt{\lambda_0}[\theta(1) - \theta(0)]\right] \\
&= \theta(1) - \frac{\pi}{2}
\end{aligned}$$

which gives us (2.23). ■

If $\alpha = 0$ and $\beta = \pi$, we are in the Dirichlet boundary condition case which we treated in section 2.3. Thus, we assume that either $\alpha \neq 0$ or $\beta \neq \pi$ (or both); in this case, either $\theta(0) = \cot^{-1}\left(\frac{\cot\alpha}{\sqrt{\lambda_0}}\right)$ or $\theta(1) = \cot^{-1}\left(\frac{\cot\beta}{\sqrt{\lambda_0}}\right)$ (or both) and the equation (2.21) is transcendental.

Lemma 2.2 *If $M > 0$,*

$$\theta(0) = \begin{cases} 0, & \alpha = 0 \\ \cot^{-1}\left(\frac{\cot\alpha}{\sqrt{\lambda_0}}\right), & 0 < \alpha \leq \frac{\pi}{2} \end{cases} \tag{2.24}$$

and

$$\theta(1) = \begin{cases} \pi, & \beta = \pi \\ \cot^{-1}\left(\frac{\cot\beta}{\sqrt{\lambda_0}}\right), & \frac{\pi}{2} \leq \beta < \pi \end{cases} \tag{2.25}$$

the equation (2.21) has a unique solution, which we denote by λ_0^*.

Proof: If $\alpha = 0$ and $\beta = \pi$, the right hand side of (2.21) is constant. Assume $\alpha \neq 0$ or $\beta \neq \pi$. We can take the square root of both sides of equation (2.21) to get

$$2\sqrt{\lambda_0} = \theta(1) - \theta(0) + \sqrt{[\theta(1) - \theta(0)]^2 + 4M}.$$

Now, taking the term $\theta(1) - \theta(0)$ to the left hand side and squaring both sides of the above equation, we get

$$4\lambda_0 - 4\sqrt{\lambda_0}[\theta(1) - \theta(0)] + [\theta(1) - \theta(0)]^2 = [\theta(1) - \theta(0)]^2 + 4M.$$

Finally, dividing by $4\sqrt{\lambda_0}$ and rearranging terms, we have

$$\sqrt{\lambda_0} = \frac{M}{\sqrt{\lambda_0}} + [\theta(1) - \theta(0)].$$

Let $y(\lambda_0) = \sqrt{\lambda_0}$, $y_1(\lambda_0) = \frac{M}{\sqrt{\lambda_0}}$ and $y_2(\lambda_0) = \theta(1) - \theta(0)$ and let $\tilde{y}(\lambda_0) = y_1(\lambda_0) + y_2(\lambda_0)$. Clearly, $y(\lambda_0) = \sqrt{\lambda_0}$ is strictly increasing with $y(0) = 0$ and $\lim_{\lambda_0 \to \infty} y(\lambda_0) = \infty$. Also, $y_1(\lambda_0) = \frac{M}{\sqrt{\lambda_0}}$ is strictly decreasing with $\lim_{\lambda_0 \to 0^+} y_1(\lambda_0) = \infty$ and $\lim_{\lambda_0 \to \infty} y_1(\lambda_0) = 0$.

Now, let $\alpha \neq 0$ and $\beta \neq \pi$ and consider the first derivative of $y_2(\lambda_0)$:

$$\begin{aligned}
\frac{d}{d\lambda_0}(y_2(\lambda_0)) &= \frac{d}{d\lambda_0}\left(\cot^{-1}\left(\frac{\cot\beta}{\sqrt{\lambda_0}}\right) - \cot^{-1}\left(\frac{\cot\alpha}{\sqrt{\lambda_0}}\right)\right) \qquad (2.26)\\
&= \left[-\frac{1}{1+\left(\frac{\cot\beta}{\sqrt{\lambda_0}}\right)^2}\right]\left(\frac{-\cot\beta}{2\lambda_0\sqrt{\lambda_0}}\right) - \left[-\frac{1}{1+\left(\frac{\cot\alpha}{\sqrt{\lambda_0}}\right)^2}\right]\left(\frac{-\cot\alpha}{2\lambda_0\sqrt{\lambda_0}}\right)\\
&= \frac{\cot\beta}{2\sqrt{\lambda_0}(\lambda_0 + \cot^2\beta)} - \frac{\cot\alpha}{2\sqrt{\lambda_0}(\lambda_0 + \cot^2\alpha)}\\
&\leq 0
\end{aligned}$$

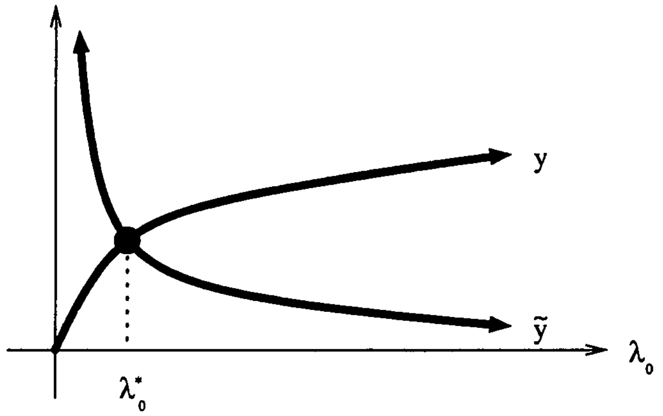

Figure 2.3: Existence of point λ_0^*

since $\lambda_0 > 0$. Hence, $y_2(\lambda_0)$ is decreasing with $\lim_{\lambda_0 \to 0^+} y_2(\lambda_0) = \pi$ and $\lim_{\lambda_0 \to \infty} y_2(\lambda_0) = 0$, and $\tilde{y}(\lambda_0) = y_1(\lambda_0) + y_2(\lambda_0)$ is strictly decreasing with $\lim_{\lambda_0 \to 0^+} \tilde{y}(\lambda_0) = \infty$ and $\lim_{\lambda_0 \to \infty} \tilde{y}(\lambda_0) = 0$. Hence, there must be an intersection point of the two curves $y(\lambda_0)$ and $\tilde{y}(\lambda_0)$ (see Figure 2.3), and thus the transcendental equation (2.21) has a unique solution, which we denote λ_0^*.

Note that if $\alpha = 0$ and $\beta \neq \pi$, then $\frac{d}{d\lambda_0}(y_2(\lambda_0)) = \frac{\cot \beta}{2\sqrt{\lambda_0}(\lambda_0 + \cot^2 \beta)} \leq 0$. If $\alpha \neq 0$ and $\beta = \pi$, then $\frac{d}{d\lambda_0}(y_2(\lambda_0)) = -\frac{\cot \alpha}{2\sqrt{\lambda_0}(\lambda_0 + \cot^2 \alpha)} \leq 0$. In both cases, the proof follows as above. ■

Theorem 2.3 *If $M > 0$, $0 \leq \alpha \leq \frac{\pi}{2}$, $\frac{\pi}{2} \leq \beta \leq \pi$, $\theta(0)$ and $\theta(1)$ are as in (2.24) and (2.25) with $\lambda_0 = \lambda_0^*$,*

$$\lambda_0^* \text{ is the positive root of } \lambda_0 = \frac{1}{4}\left[\theta(1) - \theta(0) + \sqrt{[\theta(1) - \theta(0)]^2 + 4\mathrm{M}}\right]^2,$$

$$\Delta = \frac{M}{\lambda_0^*},$$

$$\Gamma = \frac{[\frac{\pi}{2} - \theta(0)]}{\sqrt{\lambda_0^*}},$$

$$q^*(t) = \begin{cases} 0, & 0 \le t < \Gamma \\ \lambda_0^*, & \Gamma \le t \le \Gamma + \Delta \\ 0, & \Gamma + \Delta < t \le 1, \end{cases}$$

$$\theta(t) = \begin{cases} \sqrt{\lambda_0^*}\, t + \theta(0), & 0 \le t < \Gamma \\ \frac{\pi}{2}, & \Gamma \le t \le \Gamma + \Delta \\ \sqrt{\lambda_0^*}\, t + (\theta(1) - \sqrt{\lambda_0^*}), & \Gamma + \Delta < t \le 1, \end{cases}$$

and $r(t)$ solves

$$r' = r\left(\sqrt{\lambda_0^*} + \frac{q^* - \lambda_0^*}{\sqrt{\lambda_0^*}}\right)\cos\theta\sin\theta$$

$$r(0) = 1$$

then $\phi_0 = r\sin\theta$ satisfies

$$-\phi_0'' + q^*\phi_0 = \lambda_0^*\phi_0$$

$$(\cos\alpha)\phi_0(0) - (\sin\alpha)\phi_0'(0) = 0$$

$$(\cos\beta)\phi_0(1) - (\sin\beta)\phi_0'(1) = 0$$

except at points of discontinuity of $q^*(t)$.

Proof: The proof is similar to the proof of Theorem 2.2. ■

Again, Theorem 2.1 establishes that λ_0^* is an upper bound for all the eigenvalues $\lambda_0(q)$ of the Sturm-Liouville problem (2.1) and (2.2) on $[0,1]$ with $0 \le \alpha \le \frac{\pi}{2}$ and $\frac{\pi}{2} \le \beta \le \pi$. Theorem 2.3 shows furthermore that $\lambda_0^* = \lambda_0(q^*)$ is an eigenvalue of (2.1) and (2.2) where $0 \leq \alpha \leq \frac{\pi}{2}$ and $\frac{\pi}{2} \leq \beta \leq \pi$ and $\int_0^1 q^*(t)\,dt = M$ so that

$$\lambda_0^* \in \{\lambda_0(q) : \int_0^1 q(t)dt = M\}$$

and, hence, we have the following:

Corollary 2.3.1 *Given the hypotheses of Theorem 2.3, the eigenvalue λ_0^* given in Theorem 2.3 is the maximum of the set $S = \{\lambda_0(q) : \int_0^1 q(t)dt = M\}$, i.e.,*

$$\lambda_0^{\#} = \lambda_0^*,$$

and the eigenvalue $\lambda_0(q^*)$, which corresponds to q^* in Theorem 2.3, is the value $\lambda_0^{\#}$.

Case B: Let $\frac{\pi}{2} < \alpha < \pi$ and $\frac{\pi}{2} \leq \beta \leq \pi$. Here, $\frac{\pi}{2} < \theta(0) < \pi$ and $\frac{\pi}{2} \leq \theta(1) \leq \pi$. In this case, the necessary condition (2.15) forces $\theta(t)$ to have a jump discontinuity at $t = 0$ and suggests that $q(t)$ has the form $q(t) = m\delta(t) + \tilde{q}(t)$ where m is to be determined, $\delta(t)$ is the delta function, and $\tilde{q}(t)$ looks like the graph given by Figure 2.4. Thus, for such q, the graph of $\theta(t)$ has the form also given in Figure 2.4. We use a heuristic argument to show that $m = -\cot\alpha$. First, we approximate $q(t)$ by the function $q_\epsilon(t)$ and $\theta(t)$ by the function $\theta_\epsilon(t)$ as demonstrated in Figure 2.5. Notice that as $\epsilon \to 0$, $\frac{m}{\epsilon} \to \infty$ to simulate the effect of the delta function. By equation (2.8) (assuming $\lambda = 0$), we have, on the interval $[0, \epsilon]$,

$$\theta_\epsilon'(t) = \sqrt{\lambda_0} - \frac{q_\epsilon(t)\sin^2\theta_\epsilon(t)}{\sqrt{\lambda_0}} = \sqrt{\lambda_0} - \frac{m\sin^2\theta_\epsilon(t)}{\epsilon\sqrt{\lambda_0}}.$$

Integrating over $[0, \epsilon]$ and taking the limit as ϵ approaches 0 from the right hand

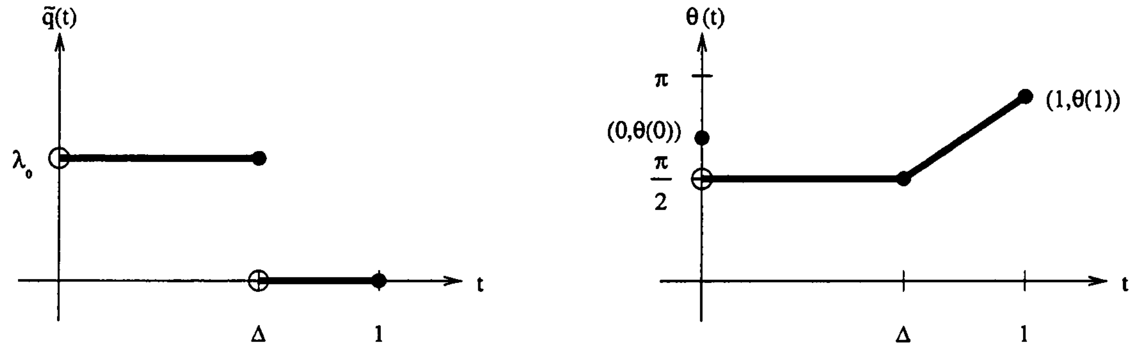

Figure 2.4: Graph of functions $\tilde{q}(t)$ and $\theta(t)$ in case B of the general boundary conditions

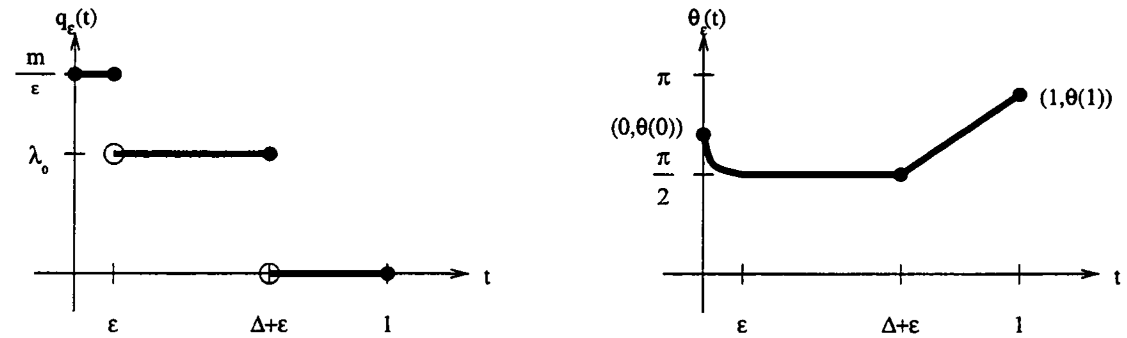

Figure 2.5: Graph of functions $q_\epsilon(t)$ and $\theta_\epsilon(t)$ in case B of the general boundary conditions

side, we have:

$$\lim_{\epsilon\to 0+}\int_0^{\epsilon}\theta_\epsilon'(t)\,dt = \lim_{\epsilon\to 0+}\left[\sqrt{\lambda_0}(\epsilon-0)-\frac{m}{\epsilon\sqrt{\lambda_0}}\int_0^{\epsilon}\sin^2\theta_\epsilon(t)\,dt\right]$$
$$= \lim_{\epsilon\to 0+}\left[-\frac{m}{\epsilon\sqrt{\lambda_0}}\int_0^{\epsilon}\sin^2\theta_\epsilon(t)\,dt\right]. \tag{2.27}$$

Thus, on the interval $[0,\epsilon]$,

$$\theta_\epsilon'(t)\approx-\frac{m}{\epsilon\sqrt{\lambda_0}}\sin^2\theta_\epsilon(t)$$

or

$$\frac{\theta_\epsilon'(t)}{\sin^2\theta_\epsilon(t)}\approx-\frac{m}{\epsilon\sqrt{\lambda_0}}.$$

Now, integrating over $[0,\epsilon]$, we can solve the left hand side explicitly by the change of variables $\theta_\epsilon=\theta_\epsilon(t)$ and $d\theta_\epsilon=\theta_\epsilon'(t)dt$:

$$\int_0^{\epsilon}\frac{\theta_\epsilon'(t)}{\sin^2\theta_\epsilon(t)}\,dt \approx \int_0^{\epsilon}-\frac{m}{\epsilon\sqrt{\lambda_0}}\,dt,$$
$$\int_{\theta_\epsilon(0)=\theta(0)}^{\theta_\epsilon(\epsilon)=\frac{\pi}{2}}\frac{d\theta_\epsilon}{\sin^2\theta_\epsilon} \approx -\frac{m}{\sqrt{\lambda_0}}.$$

Hence, on $[0,\epsilon]$, we have

$$-\cot\theta_\epsilon\big|_{\theta(0)}^{\frac{\pi}{2}} \approx -\frac{m}{\sqrt{\lambda_0}},$$
$$-\cot\left(\frac{\pi}{2}\right)+\cot\theta(0) \approx -\frac{m}{\sqrt{\lambda_0}},$$
$$\frac{\cot\alpha}{\sqrt{\lambda_0}} \approx -\frac{m}{\sqrt{\lambda_0}}.$$

Thus, we assume $m = -\cot\alpha$ and that $q(t) = (-\cot\alpha)\delta(t) + \tilde{q}(t)$ (which, we note, is no longer piecewise continuous.) From equation (2.8), recall that we have

$$\theta_\epsilon'(t) = \sqrt{\lambda_0} - \frac{q_\epsilon(t)\sin^2\theta_\epsilon(t)}{\sqrt{\lambda_0}}.$$

This time, integrating over $[0,1]$, we have

$$\begin{aligned}\int_0^1 \theta_\epsilon'(t)\,dt &= \sqrt{\lambda_0} - \int_0^\epsilon \frac{-(\cot\alpha)\sin^2\theta_\epsilon(t)}{\epsilon\sqrt{\lambda_0}}\,dt - \int_\epsilon^{\Delta+\epsilon} \frac{\lambda_0\sin^2(\frac{\pi}{2})}{\sqrt{\lambda_0}}\,dt \\ &= \sqrt{\lambda_0} + \frac{\cot\alpha}{\sqrt{\lambda_0}}\left(\frac{1}{\epsilon}\int_0^\epsilon \sin^2\theta_\epsilon(t)\,dt\right) - \sqrt{\lambda_0}(\Delta+\epsilon-\epsilon).\end{aligned}$$

Now, taking the limit as ϵ approaches 0 from the right hand side,

$$\lim_{\epsilon\to 0^+}\int_0^1 \theta_\epsilon'(t)\,dt = \sqrt{\lambda_0} + \frac{\cot\alpha}{\sqrt{\lambda_0}}\left(\lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\int_0^\epsilon \sin^2\theta_\epsilon(t)\,dt\right) - \Delta\sqrt{\lambda_0}. \qquad (2.28)$$

But, from equation (2.27),

$$\begin{aligned}\lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\int_0^\epsilon \sin^2\theta_\epsilon(t)\,dt &= -\frac{\sqrt{\lambda_0}}{m}\lim_{\epsilon\to 0^+}\int_0^\epsilon \theta_\epsilon'(t)\,dt \\ &= \frac{\sqrt{\lambda_0}}{\cot\alpha}\lim_{\epsilon\to 0^+}\left[\theta_\epsilon(\epsilon) - \theta_\epsilon(0)\right] \\ &= \frac{\sqrt{\lambda_0}}{\cot\alpha}\lim_{\epsilon\to 0^+}\left[\frac{\pi}{2} - \theta(0)\right] \\ &= \frac{\sqrt{\lambda_0}}{\cot\alpha}\left[\frac{\pi}{2} - \theta(0)\right].\end{aligned}$$

Also,

$$M = \int_0^1 q(t)\,dt = \int_0^1 \left[(-\cot\alpha)\delta(t) + \tilde{q}(t)\right]dt = -\cot\alpha + \Delta\lambda_0$$

so that $M > -\cot\alpha$ and $\Delta = \frac{M+\cot\alpha}{\lambda_0}$. Thus, substituting into (2.28), we have

$$\lim_{\epsilon\to 0^+}\int_0^1 \theta_\epsilon'(t)\,dt = \sqrt{\lambda_0} + \frac{\cot\alpha}{\sqrt{\lambda_0}}\left(\frac{\sqrt{\lambda_0}}{\cot\alpha}\left(\frac{\pi}{2} - \theta(0)\right)\right) - \left(\frac{M+\cot\alpha}{\lambda_0}\right)\sqrt{\lambda_0}$$

or

$$\theta(1) - \theta(0) = \sqrt{\lambda_0} + \left(\frac{\pi}{2} - \theta(0)\right) - \left(\frac{M+\cot\alpha}{\sqrt{\lambda_0}}\right).$$

After rearrangement, we have

$$\lambda_0 + \left(\frac{\pi}{2} - \theta(1)\right)\sqrt{\lambda_0} - (M+\cot\alpha) = 0$$

which is quadratic in $\sqrt{\lambda_0}$ so that

$$\sqrt{\lambda_0} = \frac{1}{2}\left(\left[\theta(1) - \frac{\pi}{2}\right] + \sqrt{\left[\theta(1) - \frac{\pi}{2}\right]^2 + 4[M+\cot\alpha]}\right)$$

or

$$\lambda_0 = \frac{1}{4}\left(\left[\theta(1) - \frac{\pi}{2}\right] + \sqrt{\left[\theta(1) - \frac{\pi}{2}\right]^2 + 4[M+\cot\alpha]}\right)^2.$$

Now, we still must show that the following three requirements can be satisfied simultaneously:

$$\lambda_0\Delta = M + \cot\alpha, \tag{2.29}$$

$$\lambda_0 = \frac{1}{4}\left[\left(\theta(1) - \frac{\pi}{2}\right) + \sqrt{\left(\theta(1) - \frac{\pi}{2}\right)^2 + 4(M+\cot\alpha)}\right]^2, \tag{2.30}$$

$$\sqrt{\lambda_0}(1-\Delta) = \theta(1) - \frac{\pi}{2}. \tag{2.31}$$

Although we have three equations in two unknowns, this system is not overdetermined since

Lemma 2.3 *Requirements (2.29) and (2.30) imply (2.31).*

Proof: First we solve (2.30) for M:

$$M = \lambda_0 - \sqrt{\lambda_0}\left[\theta(1) - \frac{\pi}{2}\right] - \cot\alpha.$$

Then, using (2.29), we have

$$\begin{aligned}
\sqrt{\lambda_0}(1-\Delta) &= \sqrt{\lambda_0}\left(1 - \frac{M + \cot\alpha}{\lambda_0}\right) \\
&= \sqrt{\lambda_0}\left(1 - \frac{\lambda_0 - \sqrt{\lambda_0}[\theta(1) - \frac{\pi}{2}] - \cot\alpha}{\lambda_0} - \frac{\cot\alpha}{\lambda_0}\right) \\
&= \sqrt{\lambda_0}\left(\frac{\theta(1) - \frac{\pi}{2}}{\sqrt{\lambda_0}}\right) \\
&= \theta(1) - \frac{\pi}{2}
\end{aligned}$$

which gives us (2.31). ■

Notice that, in this case, $\frac{\pi}{2} \leq \theta(1) \leq \pi$ so that $\theta(1) - \frac{\pi}{2} \geq 0$; thus, if we algebraically rearrange (2.31), we have $\Delta = 1 - \frac{1}{\sqrt{\lambda_0}}\left[\theta(1) - \frac{\pi}{2}\right]$ which assures us that $\Delta \leq 1$.

Now, if $\beta = \pi$, the right hand side of equation (2.30) is constant, but if $\beta \neq \pi$, then $\theta(1) = \cot^{-1}\left(\frac{\cot\beta}{\sqrt{\lambda_0}}\right)$ and equation (2.30) is therefore transcendental; it remains to show that

Lemma 2.4 *If $M > -\cot\alpha$, $\frac{\pi}{2} < \alpha < \pi$, and $\theta(1)$ is as in (2.25), then the equation (2.30) has a unique solution, which we denote λ_0^{**}.*

Proof: Assume $\frac{\pi}{2} \leq \beta < \pi$, since the equation (2.30) clearly has a solution if $\beta = \pi$.

We can rearrange equation (2.30) to show that it is equivalent to

$$\sqrt{\lambda_0} = \frac{(M + \cot\alpha)}{\sqrt{\lambda_0}} + \left[\theta(1) - \frac{\pi}{2}\right].$$

Let $y(\lambda_0) = \sqrt{\lambda_0}$, $y_1(\lambda_0) = \frac{(M+\cot\alpha)}{\sqrt{\lambda_0}}$ and $y_2(\lambda_0) = \theta(1) - \frac{\pi}{2}$ and let $\tilde{y}(\lambda_0) = y_1(\lambda_0) + y_2(\lambda_0)$. Clearly, $y(\lambda_0) = \sqrt{\lambda_0}$ is strictly increasing with $y(0) = 0$ and $\lim_{\lambda_0\to\infty} y(\lambda_0) = \infty$. Also, since $M > -\cot\alpha$, $y_1(\lambda_0) = \frac{(M+\cot\alpha)}{\sqrt{\lambda_0}}$ is strictly decreasing with $\lim_{\lambda_0\to 0^+} y_1(\lambda_0) = \infty$ and $\lim_{\lambda_0\to\infty} y_1(\lambda_0) = 0$. Now, recalling (2.26), consider the first derivative of y_2:

$$\begin{aligned}
\frac{d}{d\lambda_0}\left(y_2(\lambda_0)\right) &= \frac{d}{d\lambda_0}\left(\cot^{-1}\left(\frac{\cot\beta}{\sqrt{\lambda_0}}\right) - \frac{\pi}{2}\right) \\
&= \frac{\cot\beta}{2\sqrt{\lambda_0}(\lambda_0 + \cot^2\beta)} \\
&\le 0
\end{aligned}$$

when $\lambda_0 > 0$ and $\frac{\pi}{2} \le \beta < \pi$. Hence, $y_2(\lambda_0)$ is decreasing with $\lim_{\lambda_0\to 0^+} y_2(\lambda_0) = \frac{\pi}{2}$ and $\lim_{\lambda_0\to\infty} y_2(\lambda_0) = 0$ and $\tilde{y}(\lambda_0)$ is strictly decreasing with $\lim_{\lambda_0\to 0^+} \tilde{y}(\lambda_0) = \infty$ and $\lim_{\lambda_0\to\infty} \tilde{y}(\lambda_0) = 0$. Thus, the two curves $y(\lambda_0)$ and $\tilde{y}(\lambda_0)$ intersect uniquely (see Figure 2.6) so that the transcendental equation (2.30) has a unique solution, say λ_0^{**}. ■

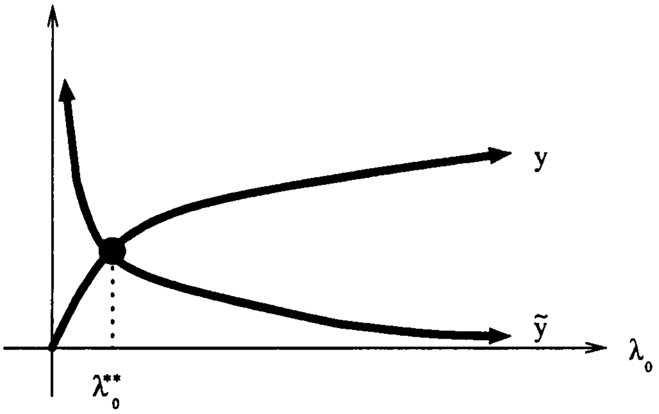

Figure 2.6: Existence of point λ_0^{**}

Theorem 2.4 *If* $M > -\cot\alpha$, $\frac{\pi}{2} < \alpha < \pi$, $\frac{\pi}{2} \le \beta \le \pi$, $\theta(1)$ *is as in (2.25) with* $\lambda_0 = \lambda_0^{**}$,

λ_0^{**} is the positive root of $\lambda_0 = \dfrac{1}{4}\left[\left(\theta(1) - \dfrac{\pi}{2}\right) + \sqrt{\left(\theta(1) - \dfrac{\pi}{2}\right)^2 + 4(\mathrm{M} + \cot\alpha)}\right]^2$,

$$\Delta = \frac{M + \cot\alpha}{\lambda_0^{**}},$$

$$q^{**}(t) = (-\cot\alpha)\delta(t) + \tilde{q}^{**}(t)$$

where

$$\tilde{q}^{**}(t) = \begin{cases} \lambda_0^{**}, & 0 < t \le \Delta \\ 0, & \Delta < t \le 1, \end{cases} \tag{2.32}$$

$$\theta(t) = \begin{cases} \cot^{-1}\left(\frac{\cot\alpha}{\sqrt{\lambda_0^{**}}}\right), & t = 0 \\ \frac{\pi}{2}, & 0 < t \le \Delta \\ \sqrt{\lambda_0^{**}}\, t + (\theta(1) - \sqrt{\lambda_0^{**}}), & \Delta < t \le 1, \end{cases} \tag{2.33}$$

and $r(t)$ solves, on $0 < t \le 1$,

$$r' = r\left(\sqrt{\lambda_0^{**}} + \frac{q^{**} - \lambda_0^{**}}{\sqrt{\lambda_0^{**}}}\right)\cos\theta\sin\theta$$

$$r(1) = 1$$

then $\phi_0 = r\sin\theta$ satisfies, on $0 < t \le 1$,

$$-\phi_0'' + q^{**}\phi_0 = \lambda_0^{**}\phi_0$$

$$(\cos\beta)\phi_0(1) - (\sin\beta)\phi_0'(1) = 0.$$

Further, if ϕ_0 is extended to $t = 0$ by continuity and $\phi_0'(0)$ is defined to be $(\cot\alpha)\phi_0(0)$, then ϕ_0 satisfies

$$\int_0^x -\phi_0''(t)\,dt + \int_0^x \{(-\cot\alpha)\delta(t) + \tilde{q}^{**}(t)\}\phi_0(t)\,dt = \int_0^x \lambda_0^{**}\phi_0(t)\,dt,$$

i.e., for $0 < x \le 1$,

$$-\phi_0'(x) + \phi_0'(0) - (\cot\alpha)\phi_0(0) + \int_0^x \tilde{q}^{**}(t)\phi_0(t)\,dt = \lambda_0^{**}\int_0^x \phi_0(t)\,dt.$$

Proof: The proof of this theorem is similar to the proof of Theorem 2.2. ■

Remark: Note that θ is continuous at $t = \Delta$ since

$$\begin{aligned}\lim_{t\to\Delta^+}\theta(t) &= \lim_{t\to\Delta^+}\left(\sqrt{\lambda_0^{**}}t + (\theta(1) - \sqrt{\lambda_0^{**}})\right)\\ &= \lim_{t\to\Delta^+}\left(\sqrt{\lambda_0^{**}}t + \left(\frac{\pi}{2} - \sqrt{\lambda_0^{**}}\Delta\right)\right)\end{aligned}$$

by (2.31) with $\lambda_0 = \lambda_0^{**}$. Hence, $\lim_{t\to\Delta^+}\theta(t) = \frac{\pi}{2} = \theta(\Delta)$.

Also, by the definitions of q^{**} and $\theta(t)$, $r(t) = 1$ on $(0, 1]$ and

$$\phi_0(t) = \begin{cases} 1, & 0 \le t \le \Delta \\ \sin\theta(t), & \Delta \le t \le 1. \end{cases} \tag{2.34}$$

Finally, note that $\lim_{x \to 0^+} \phi_0'(x) = 0$ so that ϕ_0' has a discontinuity at 0. However, ϕ_0 is continuous and, since $\phi_0'(0) - (\cot\alpha)\phi_0(0) = 0$, then ϕ_0 satisfies the following equation:

$$-\phi_0(x) + \phi_0(0) + \int_0^x (x-t)\tilde{q}^{**}(t)\phi_0(t)\,dt = \lambda_0^{**}\int_0^x (x-t)\phi_0(t)\,dt$$

for all $x \in [0, 1]$. •

This theorem shows that $\lambda_0^{**} = \lambda_0(q^{**})$ is an eigenvalue (with eigenfunction ϕ_0) of (2.1) and (2.2) on $[0, 1]$ in the distributional sense where $\frac{\pi}{2} < \alpha < \pi$ and $\frac{\pi}{2} \le \beta \le \pi$, but, here, q^{**} is no longer piecewise continuous! We will show, however, in section 2.6 that λ_0^{**} *is* an upper bound for the set $S := \{\lambda_0(q) : \int_0^1 q(t)\,dt = M\}$; that is, $\lambda_0^{**} \ge \lambda_0^{\#}$. Yet, even then, we will not have shown that $\lambda_0^{**} = \lambda_0^{\#}$ since we lack the inequality in (2.30) which we need to be certain that λ_0^{**} is the <u>least</u> upper bound of all the eigenvalues. What we do know is that λ_0^{**} is less than the upper bound of S established by Theorem 2.1; that is, $\lambda_0^{**} < \bar{\lambda_0}$ where $\bar{\lambda_0}$ is a solution of

$$\lambda_0 = \frac{1}{4}\left[\theta(1) - \theta(0) + \sqrt{[\theta(1) - \theta(0)]^2 + 4M}\right]^2$$

with $\theta(1)$ as in (2.25) and $\theta(0) = \cot^{-1}\left(\frac{\cot\alpha}{\sqrt{\lambda_0}}\right)$ where $\frac{\pi}{2} < \alpha < \pi$. We can show, in a modification of the proof of Lemma 2.2, that $\bar{\lambda_0}$ is an intersection point of the curves $y(\lambda_0) = \sqrt{\lambda_0}$ and $y_1(\lambda_0) = \frac{M}{\sqrt{\lambda_0}} + \theta(1) - \theta(0)$. From Lemma 2.4, we know that λ_0^{**} is the unique intersection point of the curves $y(\lambda_0) = \sqrt{\lambda_0}$ and $y_2(\lambda_0) =$

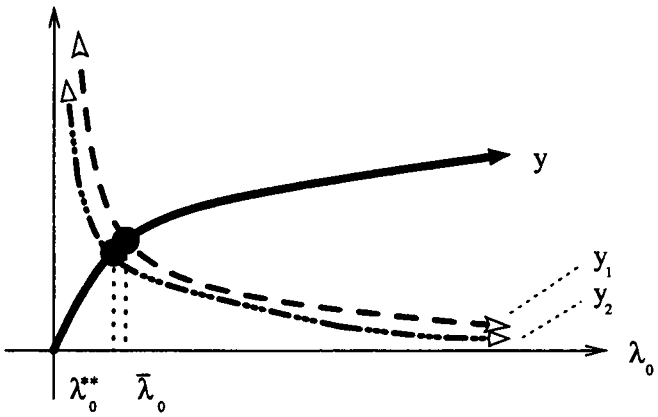

Figure 2.7: Graph where $\bar{\lambda_0} > \lambda_0^{**}$
Here, y_1 is assumed to be strictly decreasing.

$\frac{M+\cot\alpha}{\sqrt{\lambda_0}} + \theta(1) - \frac{\pi}{2}$. By a few manipulations, we show below that $y_1(\lambda_0) > y_2(\lambda_0)$ for all $\lambda_0 > 0$ and hence, an intersection point of y and y_1, say $\bar{\lambda_0}$, is greater than the intersection point of y and y_2, namely λ_0^{**} (see Figure 2.7 for an example where the function y_1 is strictly decreasing). First, since $\frac{\pi}{2} < \theta(0) < \pi$, we know that

$$\cot\theta(0) < \frac{\pi}{2} - \theta(0)$$

where $\cot\theta(0) = \frac{\cot\alpha}{\sqrt{\lambda_0}}$ so that, by substitution,

$$\frac{\cot\alpha}{\sqrt{\lambda_0}} < \frac{\pi}{2} - \theta(0).$$

Now, rearranging terms and adding $\frac{M}{\sqrt{\lambda_0}} + \theta(1)$ to both sides, we have that

$$\frac{M}{\sqrt{\lambda_0}} + \theta(1) - \theta(0) > \frac{M+\cot\alpha}{\sqrt{\lambda_0}} + \theta(1) - \frac{\pi}{2}$$

or

$$y_1(\lambda_0) > y_2(\lambda_0).$$

2.5 Examples

In this section, we consider the case presented by Theorem 2.4 and give examples that support our conjecture that $\lambda_0^{\#} = \lambda_0^{**}$.

Given the hypotheses of Theorem 2.4, λ_0^{**} is an eigenvalue of

$$-y''(t) + q^{**}(t)y(t) = \lambda_0^{**}y(t)$$
$$(\cos\alpha)y(0) - (\sin\alpha)y'(0) = 0$$
$$(\cos\beta)y(1) - (\sin\beta)y'(1) = 0.$$

For $\epsilon > 0$, let $\lambda_0(q_\epsilon^{**})$ represent the first eigenvalue of

$$-y''(t) + q_\epsilon^{**}(t)y(t) = \lambda y(t)$$
$$(\cos\alpha)y(0) - (\sin\alpha)y'(0) = 0$$
$$(\cos\beta)y(1) - (\sin\beta)y'(1) = 0$$

where

$$q_\epsilon^{**}(t) = \begin{cases} -\frac{\cot\alpha}{\epsilon}, & 0 \le t \le \epsilon \\ \lambda_0^{**}, & \epsilon < t \le \Delta + \epsilon \\ 0, & \Delta + \epsilon < t \le 1. \end{cases} \tag{2.35}$$

Note that q_ϵ^{**} satisfies the property $\int_0^1 q_\epsilon^{**}dt = M$ from the definition of Δ and λ_0^{**} in Theorem 2.4. We demonstrate numerically that, as ϵ approaches 0, the eigenvalues $\lambda_0(q_\epsilon^{**})$ approach the eigenvalue λ_0^{**}. A shooting method is used to determine the eigenvalue $\lambda_0(q_\epsilon^{**})$ for each value of ϵ in the following manner. By [10], we know that $\theta(t,\lambda)$ is a strictly increasing function in λ for the standard Prüfer transformation, i.e., $\phi = r\sin\theta$, $\phi' = r\cos\theta$. That is, if $\lambda_1 < \lambda_2$, then $\theta(t,\lambda_1) < \theta(t,\lambda_2)$ and conversely, if $\theta(t,\lambda_1) < \theta(t,\lambda_2)$, then $\lambda_1 < \lambda_2$.

Thus, we can choose a value $\bar{\lambda}$, calculate $y(1, \bar{\lambda})$ using a numerical integrator (in our case, MAPLE) and if, for $\beta = \pi$,

$$y(1, \bar{\lambda}) > y(1, \lambda_0(q_\epsilon^{**})) = 0,$$

then

$$\theta(1, \bar{\lambda}) < \theta(1, \lambda_0(q_\epsilon^{**})) = \pi$$

and thus

$$\bar{\lambda} < \lambda_0(q_\epsilon^{**}).$$

Similarly, if

$$y(1, \bar{\lambda}) < y(1, \lambda_0(q_\epsilon^{**})) = 0,$$

then

$$\theta(1, \bar{\lambda}) > \theta(1, \lambda_0(q_\epsilon^{**})) = \pi$$

so that

$$\bar{\lambda} > \lambda_0(q_\epsilon^{**}).$$

In this manner, we can determine $\lambda_0(q_\epsilon^{**})$ as precisely as desired by varying the choice of $\bar{\lambda}$.

For both of the following examples, we choose $\alpha = \frac{3\pi}{4}$ and $\beta = \pi$. The boundary conditions, in this case, become $y(0) = -y'(0)$ and $y(1) = 0$. Furthermore, without loss of generality, we choose $y(0) = 1$ (which implies $y'(0) = -1$.) Thus, the eigenfunction $y(t, \lambda_0(q_\epsilon^{**}))$ which we seek has initial value 1, initial slope -1, and terminal value 0. The values for the sequence $\{\lambda_0(q_{\epsilon_n}^{**})\}$ below were determined by evaluating the calculated value of $y(1, \bar{\lambda})$ and comparing this value with the terminal condition $y(1) = 0$.

Example 2.1 *Choose $M = 2 > -\cot(\frac{3\pi}{4}) = 1$. In this case,*

$$\lambda_0^{**} = \frac{1}{4}\left[\pi - \frac{\pi}{2} + \sqrt{\left[\pi - \frac{\pi}{2}\right]^2 + 4(2-1)}\right]^2 \approx 4.2310533.$$

For small values of $\epsilon > 0$, we generate a sequence of eigenvalues $\{\lambda_0(q_{\epsilon_n}^{**})\}$ associated with the problems

$$-y''(t) + q_{\epsilon_n}^{**}(t)y(t) = \lambda y(t)$$
$$y(0) = -y'(0) = 1, y(1) = 0$$

Choice of ϵ	Eigenvalue
$\epsilon_1 = .01$	$\lambda_0(q_{\epsilon_1}^{**}) \approx 4.2246159$
$\epsilon_2 = .001$	$\lambda_0(q_{\epsilon_2}^{**}) \approx 4.2305037$
$\epsilon_3 = .0001$	$\lambda_0(q_{\epsilon_3}^{**}) \approx 4.2309993$
$\epsilon_4 = .00001$	$\lambda_0(q_{\epsilon_4}^{**}) \approx 4.2310478$

Example 2.2 *Choose $M = 5 > -\cot(\frac{3\pi}{4}) = 1$. In this case,*

$$\lambda_0^{**} = \frac{1}{4}\left[\pi - \frac{\pi}{2} + \sqrt{\left[\pi - \frac{\pi}{2}\right]^2 + 4(5-1)}\right]^2 \approx 8.608848.$$

Again, for small values of $\epsilon > 0$, we generate a sequence of eigenvalues $\{\lambda_0(q^{**}_{\epsilon_n})\}$ associated with the problems

$$-y''(t) + q^{**}_{\epsilon_n}(t)y(t) = \lambda y(t)$$
$$y(0) = -y'(0) = 1, y(1) = 0$$

Choice of ϵ	Bounds on the Eigenvalue
$\epsilon_1 = .01$	$\lambda_0(q^{**}_{\epsilon_1}) \approx 8.598829$
$\epsilon_2 = .001$	$\lambda_0(q^{**}_{\epsilon_2}) \approx 8.608340$
$\epsilon_3 = .0001$	$\lambda_0(q^{**}_{\epsilon_3}) \approx 8.608803$
$\epsilon_4 = .00001$	$\lambda_0(q^{**}_{\epsilon_4}) \approx 8.608844$

Note that, in both examples, the value λ_0^{**} appears to be an upper bound for the eigenvalues $\{\lambda_0(q^{**}_{\epsilon_n})\}$ and, moreover, the eigenvalues $\lambda_0(q^{**}_{\epsilon_n})$ appear to approach λ_0^{**} from below with a high degree of accuracy. This supports our hypothesis that

Conjecture 1

$$\lambda_0^{**} = \lambda_0^{\#} := \sup\left\{\lambda_0(q) : \int_0^1 q(t)\,dt = M\right\}. \tag{2.36}$$

2.6 An Improved Upper Bound for Case B of the General Boundary Conditions

In this section, we will verify that the first of the two above observations about the examples is correct: namely, that λ_0^{**} is an upper bound of the eigenvalues $\lambda_0(q)$ where $\int_0^1 q(t)\,dt = M$ in case B of the general boundary conditions. Moreover, from the last paragraph of section 2.4, we know that λ_0^{**} is an improved upper bound of the set $\{\lambda_0(q) : \int_0^1 q(t)\,dt = M\}$ since λ_0^{**} is smaller than the upper bound established by Theorem 2.1. (The second of the above observations will be proved in chapter 4.)

Theorem 2.5 *Given the hypothesis of Theorem 2.4, λ_0^{**} is an upper bound of the set of eigenvalues*

$$S := \{\lambda_0(q) : \int_0^1 q(t)\,dt = M\}.$$

Proof: From the Rayleigh quotient characterization of the first eigenvalue, we have

$$\lambda_0(q) = \min_{y\in\mathcal{D}} \frac{Q(y)}{\int_0^1 y^2 dt}$$

where

$$\begin{aligned} Q(y) &:= \int_0^1 (-y'' + qy)y\,dt \\ &= -y'(1)y(1) + y'(0)y(0) + \int_0^1 \left[(y')^2 + qy^2\right] dt \\ &= (-\cot\beta)(y(1))^2 + (\cot\alpha)(y(0))^2 + \int_0^1 \left[(y')^2 + qy^2\right] dt \end{aligned}$$

and the set

$$\mathcal{D} := \{y \in AC[0,1] : \int_0^1 (y')^2 dt < \infty\}.$$

Moreover, if $\beta = \pi$, we add the condition $y(1) = 0$ to $\mathcal{D}$ and delete the term $(-\cot\beta)y(1)^2$ from $Q(y)$.

Now, if $q_\epsilon^{**}(t)$ is defined by (2.35), then, from our numerical examples, the eigenfunctions y_ϵ of the eigenvalue problem

$$-y'' + q_\epsilon^{**} y = \lambda y$$

$$y(0) = 1, y'(0) = \cot\alpha \qquad \Leftrightarrow \qquad (\cos\alpha)y(0) - (\sin\alpha)y'(0) = 0$$

$$y'(1) = (\cot\beta)y(1) \qquad \Leftrightarrow \qquad (\cos\beta)y(1) - (\sin\beta)y'(1) = 0$$

appear to approach a function $\tilde{y}$ which is identical to the eigenfunction $\phi_0(t)$ in (2.34), namely,

$$\tilde{y}(t) = \begin{cases} 1, & 0 \le t \le \Delta \\ \sin\theta(t), & \Delta \le t \le 1. \end{cases}$$

(Recall, from (2.33), that $\theta(\Delta) = \frac{\pi}{2}$ and, if $\beta = \pi$, then $\theta(1) = \pi$; otherwise, if $\frac{\pi}{2} \le \beta < \pi$, then $\theta(1) = \cot^{-1}\left(\frac{\cot\beta}{\sqrt{\lambda_0^{**}}}\right)$.) Notice that $\tilde{y} \in \mathcal{D}$ and hence

$$\lambda_0(q) = \min_{y\in\mathcal{D}} \frac{Q(y)}{\int_0^1 y^2 dt} \le \frac{Q(\tilde{y})}{\int_0^1 \tilde{y}^2 dt}.$$

Assume $\beta \ne \pi$. First, we calculate $\int_0^1 \tilde{y}^2\,dt$:

$$\begin{aligned}
\int_0^1 \tilde{y}^2\,dt &= \int_0^\Delta 1^2 dt + \int_\Delta^1 \sin^2\theta(t)dt \\
&= \Delta + \frac{1}{2}\int_\Delta^1 [1 - \cos(2\theta(t))]\,dt \\
&= \Delta + \frac{1}{2}(1-\Delta) - \frac{1}{4\sqrt{\lambda_0^{**}}}\sin(2\theta(1)) \\
&= \frac{\Delta+1}{2} - \frac{1}{2\sqrt{\lambda_0^{**}}}\sin\theta(1)\cos\theta(1).
\end{aligned}$$

Now, we calculate $Q(\tilde{y})$ using $\cot\alpha + M = \lambda_0^{**}\Delta$ and $\cot\beta = \sqrt{\lambda_0^{**}}\cot\theta(1)$.

$$\begin{aligned}
Q(\tilde{y}) &= -\tilde{y}'(1)\tilde{y}(1) + \tilde{y}'(0)\tilde{y}(0) + \int_0^1 \left[(\tilde{y}')^2 + q\tilde{y}^2\right] dt \\
&= (-\cot\beta)(\tilde{y}(1))^2 + \cot\alpha + \int_0^\Delta q\,dt \\
&\qquad + \int_\Delta^1 \left[\left(\sqrt{\lambda_0^{**}}\cos\theta(t)\right)^2 + q\sin^2\theta(t)\right] dt \\
&\le (-\cot\beta)(\tilde{y}(1))^2 + \cot\alpha + \int_0^\Delta q\,dt + \lambda_0^{**}\left(\frac{1}{2}\right)\int_\Delta^1 [1 + \cos(2\theta(t))]\,dt \\
&\qquad + \int_\Delta^1 q\,dt \\
&= (-\cot\beta)(\tilde{y}(1))^2 + \cot\alpha + M + \lambda_0^{**}\left(\frac{1}{2}\right)\left[1 - \Delta + \frac{\sin(2\theta(1))}{2\sqrt{\lambda_0^{**}}}\right] \\
&= (-\cot\beta)\sin^2\theta(1) + \lambda_0^{**}\Delta + \lambda_0^{**}\left(\frac{1}{2}\right)[1-\Delta] + \frac{1}{4}\sqrt{\lambda_0^{**}}\sin(2\theta(1)) \\
&= \lambda_0^{**}\left(\frac{\Delta+1}{2}\right) + (-\cot\beta)\sin^2\theta(1) + \frac{1}{2}\sqrt{\lambda_0^{**}}\sin\theta(1)\cos\theta(1) \\
&= \lambda_0^{**}\left(\frac{\Delta+1}{2}\right) - \sqrt{\lambda_0^{**}}\cos\theta(1)\sin\theta(1) + \frac{1}{2}\sqrt{\lambda_0^{**}}\sin\theta(1)\cos\theta(1) \\
&= \lambda_0^{**}\left(\frac{\Delta+1}{2}\right) - \frac{1}{2}\sqrt{\lambda_0^{**}}\cos\theta(1)\sin\theta(1) \\
&= \lambda_0^{**}\left[\frac{\Delta+1}{2} - \frac{1}{2\sqrt{\lambda_0^{**}}}\sin\theta(1)\cos\theta(1)\right].
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\lambda_0(q) = \min_{y\in\mathcal{D}} \frac{Q(y)}{\int_0^1 y^2 dt} &\le \frac{Q(\tilde{y})}{\int_0^1 \tilde{y}^2 dt} \\
&\le \frac{\lambda_0^{**}\left[\frac{\Delta+1}{2} - \frac{1}{2\sqrt{\lambda_0^{**}}}\sin\theta(1)\cos\theta(1)\right]}{\left[\frac{\Delta+1}{2} - \frac{1}{2\sqrt{\lambda_0^{**}}}\sin\theta(1)\cos\theta(1)\right]} \\
&= \lambda_0^{**}.
\end{aligned}$$

If $\beta = \pi$, then $Q(\tilde{y}) = \lambda_0^{**}\left(\frac{\Delta+1}{2}\right)$ and $\int_0^1 \tilde{y}^2\,dt = \frac{\Delta+1}{2}$ so that

$$\lambda_0(q) \le \frac{\lambda_0^{**}\left(\frac{\Delta+1}{2}\right)}{\left(\frac{\Delta+1}{2}\right)} = \lambda_0^{**}.$$

Therefore, we have shown that λ_0^{**} is an upper bound for the set $S := \{\lambda_0(q) : \int_0^1 q(t)dt = M\}$. ■

CHAPTER III

THE GENERALIZED STURM-LIOUVILLE PROBLEM

3.1 Definitions and Useful Theorems

We begin this section with a short discussion of important definitions and previous results by D.B.Hinton [6].

Let N be a non-degenerate ring. Suppose that $\|\cdot\|$ is a norm for N which satisfies the submultiplicative property, i.e., for all x and y in N, $\| xy \| \leq \|x\| \cdot \|y\|$, and that N is complete with respect to this norm. (For our purposes, N is the ring of all 2×2 matrices and if $A = [a_{ij}] \in N$, then $\|A\| = \sum_{i,j} |a_{ij}|$.)

A function $F : [a, b] \to N$ is called *quasi-continuous* if it has left and right limits at each interior point of $[a, b]$ and has one-sided limits at a and b.

A function $F : [a, b] \to N$ is said to be of *bounded variation* on $[a, b]$ if there is a constant K such that for any partition $a = t_0 < t_1 < ... < t_{n-1} < t_n = b$,

$$\sum_{i=1}^{n} \|F(t_i) - F(t_{i-1})\| \leq K.$$

The greatest lower bound of all the constants K is called the *total variation* of F on $[a, b]$ and is denoted $\bigvee_a^b F$ or

$$\bigvee_a^b F.$$

Let $\mathcal{F}$ be the set of all functions $F : [a, b] \times [a, b] \to N$ such that

(i) $F(x, x) = I$ for all x,

(ii) F is quasi-continuous with respect to its first place, and

(iii) there is a real nondecreasing function g on $[a, b]$ such that $g(a) = 0$ and $\|F(t, x) - F(t, y)\| \leq |g(x) - g(y)|$ for all t, x, and y.

The function g is called a *super function* for F.

We say that a sequence $\{s_i\}_0^n$ is a *chain* from x to y if $\{s_i\}_0^n$ is a monotone sequence such that $s_0 = x$ and $s_n = y$. The chain $\{t_i\}_0^m$ is a *refinement* of the chain $\{s_i\}_0^n$ if $\{s_i\}_0^n$ is a subsequence of $\{t_i\}_0^m$.

We can now define the Cauchy-left and Cauchy-right integrals as follows. For $F : [a, b] \to N$ and $G : [a, b] \to N$, the *Cauchy-left integral* $(L) \int_a^b dF(x)G(x)$ denotes an element Y of N with the following property: for each positive number ϵ, there is a chain $\{s_i\}_0^n$ from a to b such that if $\{t_i\}_0^m$ is a refinement of $\{s_i\}_0^n$, then

$$\left\| Y - \sum_{i=1}^{m} [F(t_i) - F(t_{i-1})]G(t_{i-1}) \right\| < \epsilon.$$

The *Cauchy-right integral* $(R) \int_a^b dF(x)G(x)$ denotes an element Y of N with the following property: for each positive number ϵ, there is a chain $\{s_i\}_0^n$ from a to b such that if $\{t_i\}_0^m$ is a refinement of s, then

$$\left\| Y - \sum_{i=1}^{m} [F(t_i) - F(t_{i-1})]G(t_i) \right\| < \epsilon.$$

Similarly, we can define the Cauchy-left integral $(L) \int_a^b G(x)dF(x)$ and the Cauchy-right integral $(R) \int_a^b G(x)dF(x)$. The Cauchy integrals are known to exist if F is of bounded variation and G is quasi-continuous, and classical arguments show that, if the integral exists, it is unique.

Moreover, a Cauchy-left integral $(L)\int_a^t dF(s)G(s)$ or a Cauchy-right integral $(R)\int_a^t dF(s)G(s)$ is equal to an ordinary Stieltjes integral $\int_a^t dF(s)G(s)$ if, under the above circumstances, either F is continuous or G is continuous.

We make use of the following theorems in this chapter, which we state here without proof. The first theorem establishes a bound on the norm of a Cauchy-left integral and the second theorem provides a variation of the well-known Gronwall's inequality. The third theorem establishes conditions under which a Cauchy-left integral is quasi-continuous or even continuous. The last two theorems assure us that solutions exist to certain integral equations.

Theorem 3.1 *If F is of bounded variation from $[a,b]$ to N, G is quasi-continuous from $[a,b]$ to N, f and g are real functions on $[a,b]$ such that, for $x \le y$, $\bigvee_x^y F \le f(y) - f(x)$, and $g(x) := \|G(x)\|$, then*

$$\left\|(L)\int_a^b dF(s)G(s)\right\| \le \left|(L)\int_a^b df(s)g(s)\right| \le \left(\bigvee_a^b f\right)|g|_{[a,b]} \tag{3.1}$$

where $|g|_{[a,b]} := \sup\{|g(x)| : x \in [a,b]\}$.

Theorem 3.2 *If $K \ge 0$, h is a real nondecreasing function on $[a,b]$, and m is a real function on $[a,b]$ bounded above by $T > 0$ and such that, for each x,*

$$m(x) \le K + (L)\int_a^x m(s)dh(s),$$

then

$$m(x) \le Ke^{[h(x)-h(a)]}$$

for each x.

Theorem 3.3 *If $F \in \mathcal{F}$, $Q : [a,b] \to N$ is quasi-continuous, and P is defined on $[a,b]$ by*

$$P(t) = (L) \int_a^t d_s F(t,s) Q(s),$$

then P is quasi-continuous. Moreover, if F is continuous with respect to its first variable, then P is continuous.

Theorem 3.4 *Given $F \in \mathcal{F}$, there is one and only one function M that is quasi-continuous with respect to its first variable and that is a solution of*

$$M(t,x) = I + (L) \int_x^t d_s F(t,s) M(s,x)$$

for all t and x. Moreover, if F is continuous with respect to its first place, then so is M.

Theorem 3.5 *If $F \in \mathcal{F}$ and G is a quasi-continuous function from $[a,b]$ to N, then there is a unique quasi-continuous function Y on $[a,b]$ such that*

$$Y(t) = G(t) + (L) \int_a^t d_s F(t,s) Y(s).$$

Finally, we state two convergence theorems for Stieltjes integrals (see [7] and [5] for references) which we use later.

Theorem 3.6 (Helly's Convergence Theorem) *Let f be a continuous function on $[a,b]$ and let $\{g_n\}$ be a sequence of functions of bounded variation on $[a,b]$ converging to a function g at every point of $[a,b]$. If there exists $M > 0$ such that $\bigvee_a^b g_n \le M$ for all n, then*

$$\lim_{n\to\infty} \int_a^b f dg_n = \int_a^b f dg.$$

Theorem 3.7 *Let g be a function of bounded variation on $[a,b]$ and let $\{f_n\}$ be a sequence of functions which is uniformly bounded and converges pointwise to a function f on $[a,b]$. If $\int_a^b f_n dg$ and $\int_a^b f dg$ exist, then*

$$\lim_{n\to\infty}\int_a^b f_n dg = \int_a^b f dg.$$

3.2 The Initial Value Problem

We wish to consider the system of equations

$$\begin{aligned} dy &= (dP)\, z \\ dz &= (dQ - \lambda dW) y \end{aligned} \tag{3.2}$$

by which we mean the integral equation

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} y(a) \\ z(a) \end{bmatrix} + \int_a^t \begin{bmatrix} 0 & dP(s) \\ dQ(s) - \lambda dW(s) & 0 \end{bmatrix} \begin{bmatrix} y(s) \\ z(s) \end{bmatrix}. \tag{3.3}$$

Here, we require that $\lambda \in \mathbb{C}$ and that the following basic conditions hold on $a \le t \le b$:

$$\begin{aligned} &P(t) \ge 0, W(t) \ge 0; \\ &P(a) = W(a) = Q(a) = 0; \\ &P(t), W(t) \text{ are nondecreasing;} \\ &P(t) \text{ is continuous; and,} \\ &Q(t) \text{ is of bounded variation.} \end{aligned} \tag{3.4}$$

Note that in the case where $P(t) = W(t) = t - a$ and $Q(t) = \int_a^t q(\tau)\, d\tau$, then the system (3.2) is equivalent to

$$\begin{aligned} y' &= z \\ z' &= (q(t) - \lambda)y \end{aligned}$$

or

$$-y'' + qy = \lambda y$$

which is equation (2.1).

For now, we consider the integral in (3.3) to be a Cauchy-left integral, but as a result of the proof of Theorem 3.9, we will have established the conditions that show that the integral is an ordinary Stieltjes integral.

Theorem 3.8 *A quasi-continuous solution to* (3.3) *(interpreted as a Cauchy-left integral) subject to* (3.4) *exists and is unique.*

Proof: Let

$$F(t) := \begin{bmatrix} 0 & P(t) \\ Q(t) - \lambda W(t) & 0 \end{bmatrix} \tag{3.5}$$

and note that F is quasi-continuous and of bounded variation. Furthermore, if we let $\tilde{F}(t,s) := I + F(s) - F(t)$, then (3.3) has the form of the non-homogeneous integral equation

$$Y(t) = Y(a) + (L)\int_a^t d_s\tilde{F}(t,s)Y(s)$$

where $Y(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$. Here, $\tilde{F}(t,t) = I$ for all t and $\tilde{F}$ is quasi-continuous with respect to its first variable since F is quasi-continuous.

Now, if we define

$$f(t) := P(t) + \bigvee_a^t (Q + \mid \lambda \mid W), \tag{3.6}$$

then $f(t)$ is a real nondecreasing function such that $f(a) = 0$ and, for $x > y$ and any $t \in [a, b]$,

$$\begin{aligned}
&\|\tilde{F}(t,x) - \tilde{F}(t,y)\| \\
&\quad = \|F(x) - F(y)\| \\
&\quad = \mid P(x) - P(y) \mid + \mid [Q(x) - \lambda W(x)] - [Q(y) - \lambda W(y)] \mid \\
&\quad \leq \mid P(x) - P(y) \mid + \mid [Q(x) - Q(y)] + \mid \lambda \mid [W(x) - W(y)] \mid \\
&\quad = \mid P(x) - P(y) \mid + \mid [Q(x) + \mid \lambda \mid W(x)] - [Q(y) + \mid \lambda \mid W(y)] \mid \\
&\quad \leq P(x) - P(y) + \bigvee_y^x (Q + \mid \lambda \mid W) \\
&\quad = [P(x) + \bigvee_a^x (Q + \mid \lambda \mid W)] - [P(y) + \bigvee_a^y (Q + \mid \lambda \mid W)] \\
&\quad = f(x) - f(y) \\
&\quad = \mid f(x) - f(y) \mid .
\end{aligned}$$

The case where $x < y$ is proved similarly; this verifies that f is a super function for $\tilde{F}$ and hence $\tilde{F} \in \mathcal{F}$.

Thus, by Theorem 3.5, there exists a unique solution $Y(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$ which is quasi-continuous. ■

Theorem 3.9 *The quasi-continuous solution* $\begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$ *of* (3.3) *subject to* (3.4) *has the following properties on* $[a, b]$*:*

1. $y(t)$ is continuous and of bounded variation, and

2. $z(t)$ is quasi-continuous and of bounded variation.

Proof: (For this proof, we consider N to be the ring of all real numbers $\mathbb{R}$ and let the norm $\|\cdot\|$ be the scalar norm.)

From Theorem 3.8, $y(t)$ and $z(t)$ are both quasi-continuous. We define a new function $\tilde{P}(t,s) = I + P(s) - P(t)$ and note that $d_s\tilde{P} = dP$ so that

$$y(t) = y(a) + (L)\int_a^t d_s\tilde{P}(t,s)z(s).$$

Here, $\tilde{P}(t,t) = I$ for all t and $\tilde{P}$ is continuous with respect to its first variable since P is continuous. Now, $P(t)$ is a real nondecreasing function such that $P(a) = 0$ and for all t, x, and y in $[a,b]$,

$$\|\tilde{P}(t,x) - \tilde{P}(t,y)\| = | P(x) - P(y) |$$

which verifies that P is a super function for $\tilde{P}$. Hence, by Theorem 3.3, we know that the integral $(L)\int_a^t d_s\tilde{P}(t,s)z(s)$ is continuous and thus $y(t)$ is continuous.

Next, let F be as in (3.5) and let

$$I(t) := (L)\int_a^t dF(s)\begin{bmatrix} y(s) \\ z(s) \end{bmatrix}$$

and define a partition of $[a,b]$ by $\{t_0, t_1, ..., t_n\}$ where $t_0 = a$ and $t_n = b$. Set

$$g(t) := \left\| \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \right\| \tag{3.7}$$

and recall the notation $|g|_{[t_{i-1},t_i]} = \sup\{|g(s)| : s \in [t_{i-1}, t_i]\}$. As before, define $f(t)$ as in (3.6). Then, for $x \leq y$, we know that

$$\bigvee_x^y F \leq \bigvee_x^y P + \bigvee_x^y (Q + | \lambda | W) = f(y) - f(x).$$

Thus, applying Theorem 3.1, we have

$$\begin{aligned}
\|I(t_i) - I(t_{i-1})\| &= \left\| (L)\int_{t_{i-1}}^{t_i} dF(s) \begin{bmatrix} y(s) \\ z(s) \end{bmatrix} \right\| \\
&\leq \left| (L)\int_{t_{i-1}}^{t_i} df(s)g(s) \right| \\
&\leq \left(\bigvee_{t_{i-1}}^{t_i} f \right) \mid g \mid_{[t_{i-1},t_i]}
\end{aligned}$$

so that finally

$$\begin{aligned}
\sum_{i=1}^{n} \|I(t_i) - I(t_{i-1})\| &\leq \sum_{i=1}^{n} \left(\bigvee_{t_{i-1}}^{t_i} f \right) \mid g \mid_{[t_{i-1},t_i]} \\
&\leq \mid g \mid_{[a,b]} \sum_{i=1}^{n} \left(\bigvee_{t_{i-1}}^{t_i} f \right) \\
&\leq \left(\bigvee_{a}^{b} f \right) \mid g \mid_{[a,b]} .
\end{aligned}$$

This proves that $I(t)$ (and hence both $y(t)$ and $z(t)$) are of bounded variation. ■

Remark: Since P and y are continuous, then the Cauchy-left integral interpretation of Theorem 3.8 can be dropped. •

Lemma 3.1 *If* $\begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$ *is the solution of (3.3) subject to (3.4), then the norm* $\left\| \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \right\|$ *is bounded above by a constant which depends only on* λ, P, Q, W, *and the initial conditions.*

Proof: Again, if we let F be as in (3.5), f be as in (3.6), and g be as in (3.7), then by Theorem 3.1 we have

$$\begin{aligned} \left\| \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \right\| &\leq \left\| \begin{bmatrix} y(a) \\ z(a) \end{bmatrix} \right\| + \left\| (L)\int_a^t dF(s) \begin{bmatrix} y(s) \\ z(s) \end{bmatrix} \right\| \\ &\leq \left\| \begin{bmatrix} y(a) \\ z(a) \end{bmatrix} \right\| + \left| (L)\int_a^t df(s)g(s) \right|. \end{aligned}$$

Notice that $g(t)$ is a real continuous function on a compact interval and hence is bounded above. Also, $f(t)$ is a real nondecreasing function on $[a, b]$.

Hence, applying Theorem 3.2, we have

$$\begin{aligned} \left\| \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \right\| &\leq \left\| \begin{bmatrix} y(a) \\ z(a) \end{bmatrix} \right\| e^{f(t)} \\ &\leq \left\| \begin{bmatrix} y(a) \\ z(a) \end{bmatrix} \right\| e^{f(b)}. \end{aligned}$$

Here, the constant $C := \left\| \begin{bmatrix} y(a) \\ z(a) \end{bmatrix} \right\| e^{f(b)}$ and depends only on P,Q,W,λ, and the initial conditions. ■

Theorem 3.10 *Let $y^{(k)}$ and $z^{(k)}$ denote the kth iterates in the successive approximations to (3.3), i.e.,*

$$\begin{bmatrix} y^{(k+1)}(t) \\ z^{(k+1)}(t) \end{bmatrix} = \begin{bmatrix} y(a) \\ z(a) \end{bmatrix} + \int_a^t \begin{bmatrix} 0 & dP(s) \\ dQ(s) - \lambda dW(s) & 0 \end{bmatrix} \begin{bmatrix} y^{(k)}(s) \\ z^{(k)}(s) \end{bmatrix}$$

and

$$\begin{bmatrix} y^{(0)}(t) \\ z^{(0)}(t) \end{bmatrix} = \begin{bmatrix} y(a) \\ z(a) \end{bmatrix},$$

where $\lambda \in \mathbb{C}$ and P, Q, and W satisfy (3.4). *Then $y^{(k)} \to y$ and $z^{(k)} \to z$ uniformly on $[a, b]$ as $k \to \infty$ where $\begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$ is the solution of (3.3) subject to (3.4). Moreover, $|\ y(t) - y^{(k)}(t)\ |$ and $|\ z(t) - z^{(k)}(t)\ |$ are bounded by constants that depend only on P, Q, W, λ, the initial conditions, and k (and which tend to zero as $k \to \infty$).*

Proof: Let $L := \left\| \begin{bmatrix} y(a) \\ z(a) \end{bmatrix} \right\|$ and let f be as in (3.6). Since f is a nondecreasing function, then, by [6, p.318], we know that

$$(L)\int_a^t df(s) \cdot f(s)^p \leq \frac{[f(t)]^{p+1}}{p+1}.$$

First, we calculate

$$\begin{aligned} &\left\| \begin{bmatrix} y^{(1)}(t) \\ z^{(1)}(t) \end{bmatrix} - \begin{bmatrix} y^{(0)}(t) \\ z^{(0)}(t) \end{bmatrix} \right\| \\ = &\left\| \int_a^t \begin{bmatrix} 0 & dP(s) \\ dQ(s) - \lambda dW(s) & 0 \end{bmatrix} \begin{bmatrix} y(a) \\ z(a) \end{bmatrix} \right\| \\ \leq &\ Lf(t). \end{aligned}$$

Next, we have

$$\begin{aligned} &\left\| \begin{bmatrix} y^{(2)}(t) \\ z^{(2)}(t) \end{bmatrix} - \begin{bmatrix} y^{(1)}(t) \\ z^{(1)}(t) \end{bmatrix} \right\| \\ &= \left\| \int_a^t \begin{bmatrix} 0 & dP(s) \\ dQ(s) - \lambda dW(s) & 0 \end{bmatrix} \left(\begin{bmatrix} y^{(1)}(s) \\ z^{(1)}(s) \end{bmatrix} - \begin{bmatrix} y^{(0)}(s) \\ z^{(0)}(s) \end{bmatrix} \right) \right\| \\ &\leq \left| (L)\int_a^t df(s) Lf(s) \right| \\ &\leq L\frac{[f(t)]^2}{2}. \end{aligned}$$

In general, induction shows that, for $k > 0$,

$$\left\| \begin{bmatrix} y^{(k)}(t) \\ z^{(k)}(t) \end{bmatrix} - \begin{bmatrix} y^{(k-1)}(t) \\ z^{(k-1)}(t) \end{bmatrix} \right\| \leq L\frac{[f(t)]^k}{k!}$$

and, for $p > k$ and $a \leq t \leq b$,

$$\begin{aligned}
\left\| \begin{bmatrix} y^{(p)}(t) \\ z^{(p)}(t) \end{bmatrix} - \begin{bmatrix} y^{(k)}(t) \\ z^{(k)}(t) \end{bmatrix} \right\| &\leq \sum_{i=k+1}^{p} \left\| \begin{bmatrix} y^{(i)}(t) \\ z^{(i)}(t) \end{bmatrix} - \begin{bmatrix} y^{(i-1)}(t) \\ z^{(i-1)}(t) \end{bmatrix} \right\| \\
&\leq \sum_{i=k+1}^{p} L\frac{[f(t)]^i}{i!} \\
&\leq L \sum_{i=k+1}^{p} \frac{f(b)^i}{i!} \\
&= L \left(\sum_{i=0}^{p} \frac{f(b)^i}{i!} - \sum_{i=0}^{k} \frac{f(b)^i}{i!} \right).
\end{aligned}$$

Recalling that

$$\lim_{m \to \infty} \sum_{i=0}^{m} \frac{f(b)^i}{i!} = e^{f(b)},$$

we have shown that $\left\{ \begin{bmatrix} y^{(k)}(t) \\ z^{(k)}(t) \end{bmatrix} \right\}_{k=0}^{\infty}$ is uniformly Cauchy on $[a, b]$ and hence converges uniformly on $[a, b]$ to

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} y(a) \\ z(a) \end{bmatrix} + \int_a^t \begin{bmatrix} 0 & dP(s) \\ dQ(s) - \lambda dW(s) & 0 \end{bmatrix} \begin{bmatrix} y(s) \\ z(s) \end{bmatrix}.$$

Moreover,

$$\left\| \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} - \begin{bmatrix} y^{(k)}(t) \\ z^{(k)}(t) \end{bmatrix} \right\| \leq L \sum_{i=k+1}^{\infty} \frac{f(b)^i}{i!}$$
$$\leq \frac{Le^{f(b)} f(b)^{(k+1)}}{(k+1)!}$$

and

$$\frac{Le^{f(b)} f(b)^{(k+1)}}{(k+1)!} \to 0$$

as $k \to \infty$. ■

3.3 The Eigenvalue Problem

Now, under the basic conditions given by (3.4), let us define the eigenvalue problem by attaching general boundary conditions to (3.3):

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} y(a) \\ z(a) \end{bmatrix} + \int_a^t \begin{bmatrix} 0 & dP(s) \\ dQ(s) - \lambda dW(s) & 0 \end{bmatrix} \begin{bmatrix} y(s) \\ z(s) \end{bmatrix} \tag{3.8}$$

and

$$A \begin{bmatrix} y(a) \\ z(a) \end{bmatrix} + B \begin{bmatrix} y(b) \\ z(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.9}$$

where A and B are (possibly complex) 2×2 matrices and the 2×4 matrix $[A \mid B]$ has full rank. (Self-adjointness in the differential equations case (2.1) and (2.2) would also require that $AEA^* = BEB^*$ where $E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.)

In this section, we can show that the eigenvalues of this problem are the roots of a certain entire function and that, in the case of separated boundary conditions, the eigenvalues are, in fact, real and simple.

We define Φ to be the 2×2 matrix such that the columns of $\Phi(t, \lambda)$ satisfy (3.8) (which is the same equation as (3.3)) and $\Phi(a, \lambda) = I$. The existence and uniqueness of $\Phi(t, \lambda)$ is guaranteed by Theorem 3.4 where $\Phi(t, \lambda) = M(t, a)$. As is easily verified, the solution of (3.8) is given by $\Phi(t, \lambda) \begin{bmatrix} y(a) \\ z(a) \end{bmatrix}$.

Theorem 3.11 *The eigenvalues of problem* (3.8) *and* (3.9) *are solutions of*

$$\det[A + B\Phi(b, \lambda)] = 0.$$

Proof: We need to show that the problem composed of (3.8) and (3.9) has a nontrivial solution $\phi(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$ if and only if $\det[A + B\Phi(b, \lambda)] = 0$.

Now, there exists a nontrivial solution $\phi(t)$ of problem (3.8) and (3.9) if and only if there exists $\phi(a)$, not the zero vector, such that $\phi(t) = \Phi(t, \lambda)\phi(a)$ and $A\phi(a) + B\phi(b) = 0$. By substitution, we have

$$0 = A\phi(a) + B\Phi(b, \lambda)\phi(a) = [A + B\Phi(b, \lambda)]\phi(a).$$

If $\phi(a)$ is not the zero vector, we must have $\det[A + B\Phi(b, \lambda)] = 0$, and conversely if $\det[A + B\Phi(b, \lambda)] = 0$, then $[A + B\Phi(b, \lambda)]\phi(a) = 0$ for some nonzero vector $\phi(a)$. ■

Lemma 3.2 *Define* $d : \mathbb{C} \to \mathbb{R}$ *by*

$$d(\lambda) = \det[A + B\Phi(b, \lambda)]. \tag{3.10}$$

Then $d(\lambda)$ *is an entire function.*

Proof: To show that $d(\lambda)$ is an entire function, we first define the successive approximations for the integral equation

$$\Phi(t,\lambda) = I + \int_a^t \begin{bmatrix} 0 & dP(s) \\ dQ(s) - \lambda dW(s) & 0 \end{bmatrix} \Phi(s,\lambda)$$

and then show that, for $\lambda \in K \subset \mathbb{C}$ (where K is a compact set of $\mathbb{C}$), $\det[A + B\Phi^{(k)}(b,\lambda)] \to \det[A + B\Phi(b,\lambda)]$ uniformly as $k \to \infty$ and $\det[A + B\Phi^{(k)}(b,\lambda)]$ is a polynomial in λ (and is hence analytic). Thus, by basic theorems of complex analysis, $d(\lambda) = \det[A + B\Phi(b,\lambda)]$ is also analytic.

Now, fix $\lambda \in K$ and let $\Phi^{(k)}(t,\lambda)$ be defined by

$$\Phi^{(k+1)}(t,\lambda) = I + \int_a^t \begin{bmatrix} 0 & dP(s) \\ dQ(s) - \lambda dW(s) & 0 \end{bmatrix} \Phi^{(k)}(s,\lambda), \qquad (3.11)$$

$$\Phi^{(0)}(t,\lambda) = I.$$

By the same argument as in Theorem 3.10, the sequence $\left\{\Phi^{(k)}(t,\lambda)\right\}_{k=0}^{\infty}$ is uniformly Cauchy on $[a,b] \times K$ and hence converges uniformly on $[a,b] \times K$ as $k \to \infty$ to

$$\Phi(t,\lambda) = I + \int_a^t \begin{bmatrix} 0 & dP(s) \\ dQ(s) - \lambda dW(s) & 0 \end{bmatrix} \Phi(s,\lambda).$$

In particular, $\Phi^{(k)}(b,\lambda) \to \Phi(b,\lambda)$ uniformly as $k \to \infty$ for $\lambda \in K$ and hence $\det[A + B\Phi^{(k)}(b,\lambda)] \to \det[A + B\Phi(b,\lambda)]$ uniformly as $k \to \infty$ for $\lambda \in K$. Now, if $d(\lambda) = \det[A + B\Phi(b,\lambda)]$ is the uniform limit of a sequence of analytic functions, then $d(\lambda)$ is analytic and is an entire function (since K is an arbitrary compact set in $\mathbb{C}$.) Hence, it suffices to finish the proof by showing that $\det[A + B\Phi^{(k)}(t,\lambda)]$ is a polynomial in λ, i.e., a finite sum of the form $\sum_{i=0}^{n} a_i(t)\lambda^i$. First, note that any expression of the form $\det[A + B\Psi(\lambda)]$ is a polynomial in λ if every component of

$\Psi(\lambda)$ is also a polynomial in λ. Thus, we show by induction that the components of $\Phi^{(k)}(t,\lambda)$ are polynomial in λ, i.e., each has the form of a finite sum $\sum_{i=0}^{m} b_i(t)\lambda^i$.

First, the components of $\Phi^{(0)}(t,\lambda)$ are constant in λ and the components of

$$\Phi^{(1)}(t,\lambda) = I + \int_a^t \begin{bmatrix} 0 & dP(s) \\ dQ(s) - \lambda dW(s) & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

are linear in λ.

Now, suppose that the components of $\Phi^{(k)}(t,\lambda)$ are polynomial in λ of at most order k and designate

$$\Phi^{(k)}(t,\lambda) = \begin{bmatrix} \sum_{i=0}^{k} c_{1i}(t)\lambda^i & \sum_{i=0}^{k} c_{2i}(t)\lambda^i \\ \sum_{i=0}^{k} c_{3i}(t)\lambda^i & \sum_{i=0}^{k} c_{4i}(t)\lambda^i \end{bmatrix}.$$

From the iteration (3.11), it is clear that each of the components of $\Phi^{(k+1)}(t,\lambda)$ is polynomial in λ of at most order $k+1$, which completes the induction and the proof. ■

Lemma 3.3 *For separated boundary conditions, i.e.,*

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ \cos\beta & -\sin\beta \end{bmatrix}$$

where $0 \le \alpha < \pi$ *and* $0 < \beta \le \pi$, *the eigenvalues of* (3.8) *and* (3.9) *are real and simple provided that* $W(t)$ *is constant on no subinterval and* $P(t)$ *is not the zero function.*

Proof: Using (3.2), we have, for an eigenfunction $\begin{bmatrix} y \\ z \end{bmatrix}$ with corresponding eigenvalue λ,

$$\begin{aligned}\int_a^b [dQ - \lambda dW] y\bar{y} &= \int_a^b (dz)\bar{y} \\ &= z(b)\bar{y}(b) - z(a)\bar{y}(a) - \int_a^b z d\bar{y}.\end{aligned}$$

If $\alpha \neq 0$ and $\beta \neq \pi$, then $z(a) = (\cot\alpha)y(a)$ and $z(b) = (\cot\beta)y(b)$. Thus, by substitution and (3.2), we have

$$\int_a^b dQ|y|^2 - \lambda \int_a^b dW|y|^2 = (\cot\beta)|y(b)|^2 - (\cot\alpha)|y(a)|^2 - \int_a^b dP|z|^2$$

so that

$$\lambda = \frac{-(\cot\beta)|y(b)|^2 + (\cot\alpha)|y(a)|^2 + \int_a^b dP|z|^2 + \int_a^b dQ|y|^2}{\int_a^b dW|y|^2}. \tag{3.12}$$

First, we will show that $\int_a^b dW|y|^2 > 0$. Suppose that $\int_a^b dW|y|^2 = 0$; since W is constant on no subinterval, this implies that $y \equiv 0$. Now, if $y \equiv 0$, then from (3.8), z is a constant function, and from above, this forces $\int_a^b dP|z|^2 = 0$ so that P must be the zero function, which contradicts the hypothesis. Thus, $\int_a^b dW|y|^2 > 0$ and hence λ is real.

If $\alpha = 0$, then $y(a) = 0$ so that

$$\lambda = \frac{-(\cot\beta)|y(b)|^2 + \int_a^b dP|z|^2 + \int_a^b dQ|y|^2}{\int_a^b dW|y|^2}.$$

If $\beta = \pi$, then $y(b) = 0$ so that

$$\lambda = \frac{(\cot\alpha)|y(a)|^2 + \int_a^b dP|z|^2 + \int_a^b dQ|y|^2}{\int_a^b dW|y|^2}.$$

Again, as above, since W is constant on no subinterval and P is not the zero function, then λ is real.

Next we show that the eigenvalues of (3.8) and (3.9) are simple by considering two cases.

First, if $\alpha = \frac{\pi}{2}$, then $z(a) = 0$. Let $\begin{bmatrix} \phi(t) \\ \psi(t) \end{bmatrix}$ be the eigenfunction corresponding to the eigenvalue λ of (3.8) and (3.9) with initial condition

$$\begin{bmatrix} y(a) \\ z(a) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

If we now examine the eigenvalue problem (3.8) and (3.9) with the initial condition $\begin{bmatrix} y(a) \\ z(a) \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix}$ where c is any non-zero real number, then by the property of uniqueness of initial value problems, the eigenfunction $\begin{bmatrix} \tilde{\phi}(t) \\ \tilde{\psi}(t) \end{bmatrix}$ corresponding to the new initial condition must be a multiple of the original eigenfunction, i.e.,

$$\begin{bmatrix} \tilde{\phi}(t) \\ \tilde{\psi}(t) \end{bmatrix} = c \begin{bmatrix} \phi(t) \\ \psi(t) \end{bmatrix}.$$

Hence, in the case where $\alpha = \frac{\pi}{2}$, the eigenvalue λ has a one-dimensional family of eigenfunctions.

Next, if $\alpha \neq \frac{\pi}{2}$, then $\frac{z(a)}{y(a)} = \cot\alpha$. Here, we let $\begin{bmatrix} \phi(t) \\ \psi(t) \end{bmatrix}$ represent the eigenfunction corresponding to the eigenvalue λ of (3.8) and (3.9) with initial condition

$$\begin{bmatrix} y(a) \\ z(a) \end{bmatrix} = \begin{bmatrix} 1 \\ \cot\alpha \end{bmatrix}.$$

Again, by the uniqueness property of initial value problems, the eigenfunction $\begin{bmatrix} \tilde{\phi}(t) \\ \tilde{\psi}(t) \end{bmatrix}$ for the problem (3.8) and (3.9) with new initial conditions $\begin{bmatrix} y(a) \\ z(a) \end{bmatrix} =$

$\begin{bmatrix} c \\ c\cot\alpha \end{bmatrix}$ where c is any non-zero real number must be

$$\begin{bmatrix} \tilde{\phi}(t) \\ \tilde{\psi}(t) \end{bmatrix} = c \begin{bmatrix} \phi(t) \\ \psi(t) \end{bmatrix}.$$

Thus, in both cases, the eigenvalue λ has a one-dimensional family of eigenfunctions and hence λ is simple. ■

3.4 A Sequence of Initial Value Problems

Now, we can define a sequence of initial value problems of the form

$$\begin{bmatrix} y_n(t) \\ z_n(t) \end{bmatrix} = \begin{bmatrix} y_n(a) \\ z_n(a) \end{bmatrix} + \int_a^t \begin{bmatrix} 0 & dP_n(s) \\ dQ_n(s) - \lambda dW_n(s) & 0 \end{bmatrix} \begin{bmatrix} y_n(s) \\ z_n(s) \end{bmatrix} \tag{3.13}$$

where

$$\begin{bmatrix} y_n(a) \\ z_n(a) \end{bmatrix} \to \begin{bmatrix} y(a) \\ z(a) \end{bmatrix} \tag{3.14}$$

as $n \to \infty$ and where the terms of $\{P_n\}$, $\{W_n\}$, and $\{Q_n\}$ satisfy the basic conditions given previously in (3.4), namely, for all n and $a \le t \le b$,

$$\begin{aligned} &P_n(t) \text{ and } W_n(t) \text{ are non-negative functions;} \\ &P_n(a) = W_n(a) = Q_n(a) = 0; \\ &P_n(t) \text{ and } W_n(t) \text{ are nondecreasing;} \\ &P_n(t) \text{ is continuous;} \\ &Q_n(t) \text{ is of bounded variation,} \end{aligned} \tag{3.15}$$

and furthermore the following properties hold on $[a,b]$:

$$P_n(t) \to P(t) \text{ uniformly;} \tag{3.16}$$

$$W_n(t) \to W(t) \text{ pointwise;}$$

$$Q_n(t) \to Q(t) \text{ pointwise; and,}$$

there exists a real non-negative number M_Q

$$\text{such that} \bigvee_a^b Q \leq M_Q \text{ and } \bigvee_a^b Q_n \leq M_Q \text{ for all } n.$$

Note that Theorems 3.8, 3.9, and 3.10 and Lemma 3.1 can also be applied to (3.13) and (3.14) subject to (3.15) if we replace $\begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$ by $\begin{bmatrix} y_n(t) \\ z_n(t) \end{bmatrix}$ and P, Q, and W by P_n, Q_n, and W_n, respectively. In (3.13), y_n and z_n are functions of λ as well, but we suppress the λ except in the statement of the next theorem. The proof of the uniform convergence of y_n to y in the next theorem is a modification of a proof by Reid in [12]. In his paper, he proves uniform convergence of y_n to y where the first equation of his generalized differential system is $y' = A(t)y + B(t)z$ where A and B are Lebesgue-integrable functions on $[a,b]$; we replace this equation with the generalized equation $dy = (dP)z$.

Theorem 3.12 *Given the sequence of initial value problems consisting of* (3.13) *through* (3.16) *where* $\lambda \in K \subset \mathbb{C}$ *(compact),*

1. $y_n(t,\lambda) \to y(t,\lambda)$ *uniformly on* $[a,b] \times K$, *and*

2. $z_n(t,\lambda) \to z(t,\lambda)$ *pointwise on* $[a,b] \times K$

where $\begin{bmatrix} y(t,\lambda) \\ z(t,\lambda) \end{bmatrix}$ *is the solution of* (3.3) *subject to* (3.4).

Proof: Let $L_n := \left\| \begin{bmatrix} y_n(a) \\ z_n(a) \end{bmatrix} \right\|$ and $L := \left\| \begin{bmatrix} y(a) \\ z(a) \end{bmatrix} \right\|$ and note that since $L_n \to L$ as $n \to \infty$, then L_n is uniformly bounded by $\max_i\{L_i, L\}$.

Fix $\lambda \in K$ and let

$$f_n(t) := P_n(t) + \bigvee_a^t (Q_n + |\lambda| W_n). \tag{3.17}$$

Then $\{f_n(b)\}$ is uniformly bounded in n since $P_n(b) \to P(b)$ and $W_n(b) \to W(b)$ as $n \to \infty$ so that

$$\begin{aligned} f_n(b) &= P_n(b) + \bigvee_a^b Q_n + |\lambda| \cdot \bigvee_a^b W_n \\ &\leq \max_i\{P_i(b), P(b)\} + M_Q + \max_K |\lambda| \cdot \max_i\{W_i(b), W(b)\}. \end{aligned}$$

Therefore, we can apply Theorem 3.10 to conclude that $|y_n(t) - y_n^{(k)}(t)|$ and $|z_n(t) - z_n^{(k)}(t)|$ are bounded by constants that do not depend on n or t and that tend to zero as $k \to \infty$.

Hence, given $\epsilon > 0$, there exists K_1 sufficiently large enough that for $k > K_1$ and $t \in [a, b]$,

$$\begin{aligned} |y(t) - y^{(k)}(t)| &< \frac{\epsilon}{2}, \\ |y_n(t) - y_n^{(k)}(t)| &< \frac{\epsilon}{2}, \\ |z(t) - z^{(k)}(t)| &< \frac{\epsilon}{2}, \\ |z_n(t) - z_n^{(k)}(t)| &< \frac{\epsilon}{2}. \end{aligned}$$

Thus, for $k > K_1$,

$$\begin{aligned} |y(t) - y_n(t)| &\leq |y(t) - y^{(k)}(t)| + |y^{(k)}(t) - y_n^{(k)}(t)| + |y_n^{(k)}(t) - y_n(t)| \\ &< \epsilon + |y^{(k)}(t) - y_n^{(k)}(t)| \end{aligned}$$

and

$$\begin{aligned} |z(t) - z_n(t)| &\leq |z(t) - z^{(k)}(t)| + |z^{(k)}(t) - z_n^{(k)}(t)| + |z_n^{(k)}(t) - z_n(t)| \\ &< \epsilon + |z^{(k)}(t) - z_n^{(k)}(t)|. \end{aligned}$$

Claim 1 • $|y^{(k)}(t) - y_n^{(k)}(t)| \to 0$ *uniformly on* $[a,b]$ *as* $n \to \infty$ *for* $k \in \mathbb{N} \cup \{0\}$.

• $|z^{(k)}(t) - z_n^{(k)}(t)| \to 0$ *pointwise for every* $t \in [a,b]$ *as* $n \to \infty$ *for* $k \in \mathbb{N} \cup \{0\}$.

The proof of this claim is long, but notice that the claim, together with the inequalities above, will complete the proof of the theorem.

Proof of Claim 1: If $k = 0$, then $|y^{(0)}(t) - y_n^{(0)}(t)| = |y(a) - y_n(a)| \to 0$ as $n \to \infty$ and $|z^{(0)}(t) - z_n^{(0)}(t)| = |z(a) - z_n(a)| \to 0$ as $n \to \infty$ by assumption (3.14).

Assume the claim is true for some k.

Now,

$$\begin{aligned} &y^{(k+1)}(t) - y_n^{(k+1)}(t) \\ &\quad = y(a) + \int_a^t dP(s) z^{(k)}(s) - y_n(a) - \int_a^t dP_n(s) z_n^{(k)}(s) \\ &\quad = y(a) - y_n(a) + \int_a^t [dP(s) - dP_n(s)] z^{(k)}(s) \\ &\qquad\quad - \int_a^t dP_n(s) [z_n^{(k)}(s) - z^{(k)}(s)] \end{aligned}$$

and

$$
\begin{aligned}
&z^{(k+1)}(t) - z_n^{(k+1)}(t) \\
&\quad = z(a) + \int_a^t [dQ(s) - \lambda dW(s)]y^{(k)}(s) - z_n(a) \\
&\qquad\qquad - \int_a^t [dQ_n(s) - \lambda dW_n(s)]y_n^{(k)}(s) \\
&\quad = z(a) - z_n(a) + \int_a^t ([dQ(s) - \lambda dW(s)] - [dQ_n(s) - \lambda dW_n(s)])\, y^{(k)}(s) \\
&\qquad\qquad - \int_a^t [dQ_n(s) - \lambda dW_n(s)][y_n^{(k)}(s) - y^{(k)}(s)].
\end{aligned}
$$

Let's examine $|z^{(k+1)}(t) - z_n^{(k+1)}(t)|$ first.

By (3.14), $|z(a) - z_n(a)| \to 0$ as $n \to \infty$. We know that $y^{(k)}(t)$ is continuous by Theorem 3.9 and that Q_n and W_n are of bounded variation and there exists an M_Q such that $\bigvee_a^b Q_n \leq M_Q$ for all n by (3.15) and (3.16). Furthermore, $\bigvee_a^b W_n$ is uniformly bounded for all n by $\bigvee_a^b W_n \leq \max_i\{W_i(b), W(b)\}$. Finally, by (3.16), $Q_n(t) \to Q(t)$ and $W_n(t) \to W(t)$ pointwise for all $t \in [a, b]$ as $n \to \infty$ so that by Theorem 3.6 we have

$$\left|\int_a^t [dQ(s) - dQ_n(s)]y^{(k)}(s)\right| \to 0$$

as $n \to \infty$ and

$$\left|\int_a^t \lambda[dW(s) - dW_n(s)]y^{(k)}(s)\right| \to 0$$

as $n \to \infty$.

For the last term of $|z^{(k+1)}(t) - z_n^{(k+1)}(t)|$, we can obtain an upper bound via

$$\left|\int_a^t [dQ_n(s) - \lambda dW_n(s)][y_n^{(k)}(s) - y^{(k)}(s)]\right|$$

$$\leq \bigvee_a^b (Q_n + |\lambda| W_n) \max_{a \leq t \leq b} |y_n^{(k)}(t) - y^{(k)}(t)|.$$

Recall that $\bigvee_a^b (Q_n + |\lambda| W_n)$ is bounded independent of n and, by the induction hypothesis, $\max_{a \leq t \leq b} |y_n^{(k)}(t) - y^{(k)}(t)| \to 0$ as $n \to \infty$. Hence, $|z^{(k+1)}(t) - z_n^{(k+1)}(t)| \to 0$ pointwise for every $t \in [a, b]$ as $n \to \infty$.

To prove the other part of the claim, we consider

$$\begin{aligned} |y^{(k+1)}(t) - y_n^{(k+1)}(t)| \;&\leq\; |y(a) - y_n(a)| + \left|\int_a^t [dP(s) - dP_n(s)] z^{(k)}(s)\right| \\ &\quad + \left|\int_a^t dP_n(s)[z_n^{(k)}(s) - z^{(k)}(s)]\right|. \end{aligned}$$

Again, $|y(a) - y_n(a)| \to 0$ as $n \to \infty$ by assumption (3.14).

We can integrate by parts on $\int_a^t [dP(s) - dP_n(s)] z^{(k)}(s)$ to obtain

$$\begin{aligned} &\int_a^t [dP(s) - dP_n(s)] z^{(k)}(s) \\ &\quad = \; -\int_a^t [P(s) - P_n(s)] dz^{(k)}(s) + [P(t) - P_n(t)] z^{(k)}(t). \end{aligned}$$

Thus,

$$\begin{aligned} &\left|\int_a^t [dP(s) - dP_n(s)] z^{(k)}(s)\right| \\ &\quad \leq \; \left(\bigvee_a^b z^{(k)}\right) \left(\sup_{a \leq t \leq b} |P(t) - P_n(t)|\right) + |P(t) - P_n(t)| \cdot |z^{(k)}(t)|. \end{aligned}$$

Here, $\bigvee_a^b z^{(k)}$ is bounded independent of k (by the proof of Theorem 3.9), $|z^{(k)}(t)|$ is bounded independent of k and t by Theorem 3.10, and by (3.16), $P_n(t) \to P(t)$ uniformly so that $\int_a^t [dP(s) - P_n(s)]z^{(k)}(s) \to 0$ uniformly as $n \to \infty$.

All that remains to be shown is that $\int_a^t dP_n(s)[z_n^{(k)}(s) - z^{(k)}(s)] \to 0$ uniformly as $n \to \infty$ and the claim will be complete.

First, we rewrite this integral as

$$\int_a^t dP_n(s)[z_n^{(k)}(s) - z^{(k)}(s)]$$

$$= \int_a^t [dP_n(s) - dP(s)][z_n^{(k)}(s) - z^{(k)}(s)]$$

$$+ \int_a^t dP(s)[z_n^{(k)}(s) - z^{(k)}(s)].$$

We know that $|z_n^{(k)}(t) - z^{(k)}(t)| \leq |z_n^{(k)}(t)| + |z^{(k)}(t)| \leq 2M$ for some $M > 0$ since $z_n^{(k)}(t)$ and $z^{(k)}(t)$ are bounded independent of k and n for all $t \in [a, b]$ by Theorem 3.10. Thus, using the same technique as above, we can show that

$$\left| \int_a^t [dP_n(s) - dP(s)][z_n^{(k)}(s) - z^{(k)}(s)] \right|$$

$$\leq \left(\bigvee_a^b [z_n^{(k)} - z^{(k)}] \right) \left(\sup_{a \leq t \leq b} |P(t) - P_n(t)| \right)$$

$$+ |P_n(t) - P(t)| \cdot |z_n^{(k)}(t) - z^{(k)}(t)|$$

and hence $|\int_a^t [dP_n(s) - dP(s)][z_n^{(k)}(s) - z^{(k)}(s)]| \to 0$ uniformly as $n \to \infty$.

Finally, we know that $P(t)$ is a function of bounded variation on $[a, b]$ and the sequence $\{z_n^{(k)}(t)\}_{n=1}^{\infty}$ is uniformly bounded on $[a, b]$ and (by induction) converges pointwise to $z^{(k)}(t)$ on $[a, b]$. The integrals $\int_a^t dP(s)z_n^{(k)}(s)$ and $\int_a^t dP(s)z^{(k)}(s)$ do exist so that, by Theorem 3.7, we have $\int_a^t dP(s)[z_n^{(k)}(s) - z^{(k)}(s)] \to 0$ pointwise as $n \to \infty$. Now for the harder part: we have to show uniform convergence of this sequence. Since $|\int_a^t dP(s)[z_n^{(k)}(s) - z^{(k)}(s)]| \leq \left(\bigvee_a^b P\right) (\sup_{a\leq t\leq b} |z_n^{(k)}(t) - z^{(k)}(t)|)$, the sequence $\{\int_a^t dP(s)[z_n^{(k)}(s) - z^{(k)}(s)]\}$ is uniformly bounded on $[a,b]$. Moreover, the sequence is equicontinuous on $[a,b]$ since

$$\begin{aligned}
\left|\int_a^{t_2} dP(s)[z_n^{(k)}(s) - z^{(k)}(s)] - \int_a^{t_1} dP(s)[z_n^{(k)}(s) - z^{(k)}(s)]\right| \\
&= \left|\int_{t_1}^{t_2} dP(s)[z_n^{(k)}(s) - z^{(k)}(s)]\right| \\
&\leq \left(\bigvee_{t_1}^{t_2} P\right) \left(\sup_{a\leq t\leq b} |z_n^{(k)}(t) - z^{(k)}(t)|\right)
\end{aligned}$$

where P is continuous on $[a,b]$ so that $\bigvee_a^t P = P(t)$ is continuous and thus uniformly continuous on the compact interval $[a,b]$. Therefore, by Ascoli-Arzela, we know that $\int_a^t dP(s)[z_n^{(k)}(s) - z^{(k)}(s)] \to 0$ uniformly on a subsequence of $\{\int_a^t dP(s)[z_n^{(k)}(s) - z^{(k)}(s)]\}_n$ (which, in fact, is the original sequence as we already have pointwise convergence of this sequence.) This completes the induction step, i.e.,

$$|y^{(k+1)}(t) - y_n^{(k+1)}(t)| \to 0$$

uniformly as $n \to \infty$, and therefore verifies the claim. □

This also completes the proof of the theorem. ■

Remark: If we have a sequence of complex numbers $\{\lambda^{(n)}\}$ such that $\lambda^{(n)} \to \lambda$ as $n \to \infty$, then a slight modification of the proof of Theorem 3.12 shows that

1. $y_n(t, \lambda_n) \to y(t, \lambda)$ uniformly on $[a,b]$.

2. $z_n(t, \lambda_n) \to z(t, \lambda)$ pointwise on $[a, b]$.

We will use this result in chapter 4. •

3.5 A Sequence of Eigenvalue Problems

In this section, we define a sequence of eigenvalue problems

$$\begin{aligned} dy_n &= (dP_n)\, z_n \\ dz_n &= (dQ_n - \lambda^{(n)} W_n) y_n \end{aligned} \tag{3.18}$$

with boundary conditions, for each n, given by

$$A \begin{bmatrix} y_n(a) \\ z_n(a) \end{bmatrix} + B \begin{bmatrix} y_n(b) \\ z_n(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.19}$$

where A and B are (possibly complex) 2×2 matrices and the 2×4 matrix $(A|B)$ has full rank. Recall that (3.18) is equivalent to (3.13) with λ replaced by $\lambda^{(n)}$; as before, we require the terms of $\{P_n\}$, $\{Q_n\}$, and $\{W_n\}$ to satisfy (3.15) and (3.16) for all n and for all $t \in [a, b]$.

As previously for (3.8), we define, for each n, $\Phi_n(t, \lambda)$ to be a 2×2 matrix such that the columns of Φ_n satisfy (3.18) and $\Phi_n(a, \lambda) = I$. (Note that the solution of (3.18) is given by $\Phi_n(t, \lambda) \begin{bmatrix} y_n(a) \\ z_n(a) \end{bmatrix}$.) We also define

$$d_n(\lambda) := \det[A + B\Phi_n(b, \lambda)]. \tag{3.20}$$

(Note that Theorem 3.11 and Lemmas 3.2 and 3.3 can be applied to (3.18) and (3.19) if we make the appropriate replacements, i.e., $\begin{bmatrix} y_n \\ z_n \end{bmatrix}$ for $\begin{bmatrix} y \\ z \end{bmatrix}$, Φ_n for Φ, d_n for d, and f_n for f.)

Theorem 3.13 *$d_n(\lambda) \to d(\lambda)$ uniformly as $n \to \infty$ on compact subsets $K \subset \mathbb{C}$.*

Proof: Let K be an arbitrary compact subset contained in $\mathbb{C}$ and let $\lambda \in K$.

We begin by verifying that $\{d_n(\lambda)\}_n$ is a uniformly bounded sequence of analytic functions on K. We know that the columns of $\Phi_n(t, \lambda)$ satisfy (3.18) so that, by Lemma 3.1 and letting $c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$$\begin{aligned} \|\Phi_n(t, \lambda)\, c\| &\leq \|\Phi_n(a, \lambda)\, c\|\, e^{f_n(b)} \\ &\leq e^M \end{aligned}$$

where M, independent of n, is such that $f_n(b) \leq M$ (as shown in the proof of Theorem 3.12.) Hence, $\{d_n(\lambda)\}$ is uniformly bounded on K. Moreover, by the method used in the proof of Lemma 3.2, $d_n(\lambda)$ is analytic for each n as d_n is the uniform limit of a sequence of polynomials in λ on compact subsets of $\mathbb{C}$. Therefore, by Montel's Theorem, there exists a subsequence $\{d_{n_i}(\lambda)\}_{n_i}$ such that $d_{n_i}(\lambda)$ converges uniformly on every compact subset of $\mathbb{C}$. From Theorem 3.12, we already have pointwise convergence of the sequence $\Phi_n(t, \lambda) \to \Phi(t, \lambda)$ as $n \to \infty$, which guarantees that $d_n(\lambda) \to d(\lambda)$ uniformly as $n \to \infty$ on compact subsets of $\mathbb{C}$. ∎

Lemma 3.4 *Suppose that P and P_n are not zero functions for all n and suppose, furthermore, that there exists $\tilde{W}$, strictly increasing on $[a, b]$, such that*

$$W(x) - W(t) \geq \tilde{W}(x) - \tilde{W}(t)$$

and

$$W_n(x) - W_n(t) \geq \tilde{W}(x) - \tilde{W}(t)$$

for all $a \leq t < x \leq b$ and for all n.

Then, for separated boundary conditions, i.e.,

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ 0 & 0 \end{bmatrix} \textit{ and } B = \begin{bmatrix} 0 & 0 \\ \cos\beta & -\sin\beta \end{bmatrix}$$

where $0 \leq \alpha < \pi$ and $0 < \beta \leq \pi$, there exists $M_E < 0$ such that all the eigenvalues of (3.18) and (3.19) subject to (3.15) and (3.16) are bounded below by M_E.

Proof: For each n, let $\begin{bmatrix} y_n \\ z_n \end{bmatrix}$ be an eigenfunction, corresponding to an eigenvalue $\lambda^{(n)}$, that is the solution of (3.18) and (3.19) subject to (3.15) and (3.16). By Theorem 3.9, y_n is a continuous function on $[a, b]$ so we can let t_n^* denote a value on $[a, b]$ such that

$$|y_n(t_n^*)| = \max_{a \leq t \leq b} |y_n(t)|. \tag{3.21}$$

Also, for each n, let I_n be an interval denoted by $I_n = [a_n, b_n]$ such that $a \leq a_n < b_n \leq b$ and $t_n^* \in [a_n, b_n]$ (where a_n and b_n are to be determined later).

Our goal is to develop an upper bound on $|y_n(t_n^*)|$. By Theorem 3.9, each y_n is a continuous function on a compact interval so its minimum value exists; for each n, let c_n^* denote a value in I_n such that

$$|y_n(c_n^*)| = \min_{a_n \leq t \leq b_n} |y_n(t)|.$$

Since $\int_{a_n}^{b_n} dW_n\, y_n^2 \geq [y_n(c_n^*)]^2[W_n(b_n) - W_n(a_n)]$ and W_n is constant on no subinterval of $[a_n, b_n]$, we have an upper bound on $y_n(c_n^*)$, namely,

$$|y_n(c_n^*)| \leq \left[\frac{\int_{a_n}^{b_n} dW_n\, y_n^2}{W_n(b_n) - W_n(a_n)}\right]^{\frac{1}{2}}.$$

Using $dy_n = dP_n z_n$, we can write

$$y_n(t_n^*) = y_n(c_n^*) + \int_{c_n^*}^{t_n^*} dP_n z_n$$

so that

$$\begin{aligned}
|y_n(t_n^*)| &\leq |y_n(c_n^*)| + \int_{a_n}^{b_n} dP_n |z_n| \\
&\leq |y_n(c_n^*)| + \left(\int_{a_n}^{b_n} dP_n\right)^{\frac{1}{2}} \left(\int_{a_n}^{b_n} dP_n z_n^2\right)^{\frac{1}{2}} \\
&\leq \left[\frac{\int_{a_n}^{b_n} dW_n\, y_n^2}{W_n(b_n) - W_n(a_n)}\right]^{\frac{1}{2}} + [P_n(b_n) - P_n(a_n)]^{\frac{1}{2}} \left(\int_{a_n}^{b_n} dP_n z_n^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Using the fact that $(A+B)^2 \leq 2(A^2 + B^2)$, we have

$$[y_n(t_n^*)]^2 \leq 2\left[\frac{\int_{a_n}^{b_n} dW_n\, y_n^2}{|W_n(b_n) - W_n(a_n)|} + |P_n(b_n) - P_n(a_n)| \int_{a_n}^{b_n} dP_n z_n^2\right].$$

Let $\epsilon > 0$ be given. By the uniform continuity of P, there exists $\delta > 0$ such that $|P(x) - P(t)| < \frac{\epsilon}{6}$ if $|x - t| < \delta$. Also, by the uniform convergence of $P_n \to P$, there exists N such that for all $n \geq N$ and $t \in [a, b]$, $|P_n(t) - P(t)| < \frac{\epsilon}{6}$. Hence,

$$\begin{aligned}
|P_n(x) - P_n(t)| &\leq |P_n(x) - P(x)| + |P(x) - P(t)| + |P(t) - P_n(t)| \\
&< \frac{\epsilon}{2}
\end{aligned}$$

for all $n \geq N$ and for all x and t such that $|x - t| < \delta$.

Now, for each n, choose I_n such that $a \leq a_n < b_n \leq b$, $t_n^* \in [a_n, b_n]$, and $|b_n - a_n| = \frac{\delta}{2}$.

Claim 2 *There exists $M_{\tilde{W}} = M_{\tilde{W}}(\delta) > 0$ such that*

$$\inf_{a \le t \le b - \frac{\delta}{2}} \left[\tilde{W}\left(t + \frac{\delta}{2}\right) - \tilde{W}(t)\right] = M_{\tilde{W}}.$$

Then, for any n,

$$\frac{1}{|W_n(b_n) - W_n(a_n)|} \le \frac{1}{|\tilde{W}(b_n) - \tilde{W}(a_n)|} \le \frac{1}{M_{\tilde{W}}} < \infty$$

and, hence, we obtain an upper bound for $|y_n(t_n^*)|$:

$$[y_n(t_n^*)]^2 \ \le \ \frac{2}{M_{\tilde{W}}} \int_a^b dW_n \, y_n^2 + \epsilon \int_a^b dP_n z_n^2 \tag{3.22}$$

for all $n \ge N$.

Now, we can show that a lower bound exists for all the eigenvalues of (3.18) and (3.19) subject to (3.15) and (3.16). From the proof of Lemma 3.3 applied to (3.18) and (3.19), where $\begin{bmatrix} y_n \\ z_n \end{bmatrix}$ is the eigenfunction corresponding to $\lambda^{(n)}$, we have that

$$\lambda^{(n)} = \frac{(-\cot\beta)[y_n(b)]^2 + (\cot\alpha)[y_n(a)]^2 + \int_a^b dP_n {z_n}^2 + \int_a^b dQ_n {y_n}^2}{\int_a^b dW_n {y_n}^2}$$

provided that $\alpha \ne 0$ and $\beta \ne \pi$.

Without loss of generality, we assume $\int_a^b dW_n {y_n}^2 = 1$ since we can always normalize y_n.

Thus,

$$\lambda^{(n)} \ge -|\cot\beta|[y_n(t_n^*)]^2 - |\cot\alpha|[y_n(t_n^*)]^2 + \int_a^b dP_n {z_n}^2 - [y_n(t_n^*)]^2 \bigvee_a^b Q_n.$$

Recall that, by property (3.16), there exists M_Q such that $\bigvee_a^b Q_n \leq M_Q$ for all n; if we then let $C := |\cot\alpha| + |\cot\beta| + M_Q$, we have

$$\lambda^{(n)} \geq -C[y_n(t_n^*)]^2 + \int_a^b dP_n {z_n}^2$$

and, substituting from inequality (3.22) above, we obtain

$$\begin{aligned}\lambda^{(n)} &\geq -C\left[\frac{2}{M_{\tilde{W}}} + \epsilon\int_a^b dP_n {z_n}^2\right] + \int_a^b dP_n {z_n}^2 \\ &= -\frac{2C}{M_{\tilde{W}}} + (1 - C\epsilon)\int_a^b dP_n {z_n}^2.\end{aligned}$$

Since the choice of ϵ is arbitrary, we can choose $\epsilon < \frac{1}{C}$, so that we finally obtain

$$\lambda^{(n)} \geq -\frac{2C}{M_{\tilde{W}}}$$

for all $n \geq N$.

Thus, if $\alpha \neq 0$, $\beta \neq \pi$ and $n \geq N$, all the eigenvalues of (3.18) and (3.19) subject to (3.15) and (3.16) are bounded below by $M_E := -\frac{2C}{M_{\tilde{W}}}$.

If $\alpha = 0$, then $y_n(a) = 0$; in this case, we define C by $C := |\cot\beta| + M_Q$. If $\beta = \pi$, then $y_n(b) = 0$ and, in this case, we define C by $C := |\cot\alpha| + M_Q$. In both cases, the rest of the proof follows as above. ■

Proof of Claim 2: By definition of strictly increasing, $\tilde{W}(t + \frac{\delta}{2}) - \tilde{W}(t) > 0$ for all $t \in [a, b - \frac{\delta}{2}]$. Hence,

$$\inf_{a \leq t \leq b - \frac{\delta}{2}} \left[\tilde{W}\left(t + \frac{\delta}{2}\right) - \tilde{W}(t)\right] \geq 0.$$

If $\inf_{a\leq t\leq b-\frac{\pi}{2}}[\tilde{W}(t+\frac{\delta}{2})-\tilde{W}(t)]>0$, we are done. Now, suppose $\inf_{a\leq t\leq b-\frac{\delta}{2}}[\tilde{W}(t+\frac{\delta}{2})-\tilde{W}(t)]=0$. Then there exists a sequence $\{t_n\}$, $t_n\in[a,b]$, such that $t_n\to\tau$ and $\tilde{W}(t_n+\frac{\delta}{2})-\tilde{W}(t_n)\to 0$ as $n\to\infty$. Without loss of generality, assume t_n approaches τ monotonically from the right and let

$$\tilde{W}_+\left(\tau+\frac{\delta}{2}\right):=\lim_{t_n\to\tau^+}\tilde{W}\left(t_n+\frac{\delta}{2}\right),\tilde{W}_+(\tau):=\lim_{t_n\to\tau^+}\tilde{W}(t_n).$$

Let $L:=\tilde{W}_+(\tau+\frac{\delta}{2})=\tilde{W}_+(\tau)$. Given $\tilde{\epsilon}>0$, there exists $0<\tilde{\delta}<\frac{\delta}{2}$ such that if $t-\tau<\tilde{\delta}$, then $\tilde{W}(t)-L<\tilde{\epsilon}$. Let $\tilde{t}$ be an element in $(\tau,\tau+\tilde{\delta})$ and let $\epsilon^*<\tilde{W}(\tilde{t})-L$. Then there exists $\delta^*>0$ such that if $t-\tau=(t+\frac{\delta}{2})-(\tau+\frac{\delta}{2})<\delta^*$, then $\tilde{W}(t+\frac{\delta}{2})-L<\epsilon^*$. Let $t^*+\frac{\delta}{2}$ be an element in $(\tau+\frac{\delta}{2},\tau+\frac{\delta}{2}+\delta^*)$. Then,

$$\tilde{W}\left(t^*+\frac{\delta}{2}\right)-L<\epsilon^*<\tilde{W}(\tilde{t})-L$$

or

$$\tilde{W}\left(t^*+\frac{\delta}{2}\right)<\tilde{W}(\tilde{t}).$$

But, since $\tilde{t}<\tau+\tilde{\delta}<\tau+\frac{\delta}{2}<t^*+\frac{\delta}{2}$, this contradicts the fact that $\tilde{W}$ is strictly increasing.

Hence,

$$\inf_{a\leq t\leq b-\frac{\delta}{2}}\left[\tilde{W}\left(t+\frac{\delta}{2}\right)-\tilde{W}(t)\right]>0$$

and the claim is proved. □

Remark: Clearly this theorem can also be applied to the eigenvalue problem in section 3.3; i.e., if λ is an eigenvalue of problem (3.8) and (3.9) subject to (3.4), then λ is bounded below by M_E. •

Now, we are finally prepared to show that, under separated boundary conditions, the eigenvalues of problems (3.18) and (3.19) converge to the eigenvalues of the original problem (3.8) and (3.9) and that this convergence respects order, i.e., $\lambda_0^{(n)} \to \lambda_0$ as $n \to \infty$, $\lambda_1^{(n)} \to \lambda_1$ as $n \to \infty$, etc.

Theorem 3.14 *Given the sequence of eigenvalue problems* (3.18) *and* (3.19) *with separated boundary conditions and subject to* (3.15) *and* (3.16), *if the hypothesis of Lemma 3.4 holds and if* $\{\lambda_i\}_{i=0}^{\infty}$ *and* $\{\lambda_i^{(n)}\}_{i=0}^{\infty}$ *are the roots of* $d(\lambda) = 0$ *and* $d_n(\lambda) = 0$, *respectively, then*

$$\lambda_i^{(n)} \to \lambda_i$$

as $n \to \infty$ *for each* $i \in \mathbb{N} \cup \{0\}$.

Proof: By Theorem 3.11 and Lemma 3.2, $\{\lambda_i\}_{i=0}^{\infty}$ are the eigenvalues of the problem (3.8) and (3.9), and $\{\lambda_i^{(n)}\}_{i=0}^{\infty}$ are the eigenvalues of the problem (3.18) and (3.19).

Consider $i = 0$. Since the eigenvalue λ_0 is real and simple by Lemma 3.3, we can choose a symmetric convex contour C_0 that encloses the lower bound M_E of all the eigenvalues and λ_0, but no other eigenvalues $\{\lambda_i : i > 0\}$ are on C_0 or are enclosed by it. This is possible since the eigenvalues are zeros of an entire function and have no finite point of accumulation. The points $\lambda \in C_0$ form a compact set such that $m_0 := \min\{|d(\lambda)| : \lambda \in C_0\} > 0$.

Let $0 < \epsilon_0 < m_0$ be given. By Theorem 3.13, since $d_n(\lambda) \to d(\lambda)$ uniformly on compact sets, there exists N_0 such that for all $n \geq N_0$

$$|d_n(\lambda) - d(\lambda)| < \epsilon_0 < |d(\lambda)|$$

for $\lambda \in C_0$.

Note that $d(\lambda)$ and $d_n(\lambda)$ are both entire functions (by Lemma 3.2) and are therefore analytic inside and on C_0.

By Rouche's Theorem, for $n \geq N_0$, $d_n(\lambda) = d(\lambda) + [d_n(\lambda) - d(\lambda)]$ must have a single root inside C_0 and, due to the lower bound M_E, this root must be the eigenvalue $\lambda_0^{(n)}$.

We can then choose $0 < r_0 < \min\{|\lambda - \lambda_0| : \lambda \in C_0\}$ and form a new contour $\tilde{C}_0$ which is a circle with center λ_0 and radius r_0. By the same argument above, there exists $\tilde{N}_0$ such that $|\lambda_0 - \lambda_0^{(n)}| < r_0$ for all $n \geq \tilde{N}_0$. Hence, $\lambda_0^{(n)} \to \lambda_0$ as $n \to \infty$.

Let p_0 be the point of intersection of C_0 and $\mathbb{R}$ such that $p_0 > \lambda_0$. We can construct another symmetric convex contour C_1 which passes through p_0, encloses the eigenvalue λ_1, and does not enclose or contain any other eigenvalues $\{\lambda_i : i > 1\}$. Again, the points $\lambda \in C_1$ form a compact set such that $m_1 := \min\{|d(\lambda)| : \lambda \in C_1\} > 0$.

Let $0 < \epsilon_1 < m_1$ be given. By Theorem 3.13, there exists $N_1 \geq N_0$ such that

$$|d_n(\lambda) - d(\lambda)| < \epsilon_1 < |d(\lambda)|$$

for all $\lambda \in C_1$ and for all $n \geq N_1$.

Again, $d(\lambda)$ and $d_n(\lambda)$ are analytic inside and on C_1, so that, for $n \geq N_1$, by Rouche's Theorem, $d_n(\lambda)$ must have a single root inside C_1 and, by the construction of the contour C_1, this root must be the eigenvalue $\lambda_1^{(n)}$.

As before, we can choose $0 < r_1 < \min\{|\lambda - \lambda_1| : \lambda \in C_1\}$ and form a new contour $\tilde{C}_1$ which is a circle with center λ_1 and radius r_1. By the same argument above, there exists $\tilde{N}_1$ such that $|\lambda_1 - \lambda_1^{(n)}| < r_1$ for all $n \geq \tilde{N}_1$. Hence, $\lambda_1^{(n)} \to \lambda_1$

as $n \to \infty$.

By induction, we can repeat this process to prove that for each $i \in \mathbb{N} \cup \{0\}$,

$$\lambda_i^{(n)} \to \lambda_i$$

as $n \to \infty$. ■

CHAPTER IV

FINAL RESULTS

From Theorem 2.4, we know that λ_0^{**} is an eigenvalue (with corresponding eigenfunction ϕ_0) of the following generalized boundary value problem of the form given in (3.8) with separated boundary conditions for (3.9), namely,

$$\begin{bmatrix} \phi_0(t) \\ \phi_0'(t) \end{bmatrix} = \begin{bmatrix} \phi_0(0) \\ \phi_0'(0) \end{bmatrix} + \int_0^t \begin{bmatrix} 0 & dP(s) \\ dQ(s) - \lambda_0^{**} dW(s) & 0 \end{bmatrix} \begin{bmatrix} \phi_0(s) \\ \phi_0'(s) \end{bmatrix}$$

and

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_0(0) \\ \phi_0'(0) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \cos\beta & -\sin\beta \end{bmatrix} \begin{bmatrix} \phi_0(1) \\ \phi_0'(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where $P(t) = W(t) = t$ and

$$Q(t) = \begin{cases} 0, & t = 0 \\ -\cot\alpha + \int_0^t \tilde{q}^{**}(s)ds, & 0 < t \le 1. \end{cases}$$

Notice that $P(t)$, $Q(t)$, and $W(t)$ satisfy (3.4) on $[0, 1]$.

From section 2.5, let $\epsilon = \frac{1}{n}$ and let $q_n(t) := q_\epsilon^{**}(t)$ as defined by (2.35). For each n, let $\{\lambda_k^{(n)}\}_k$ be the eigenvalues (with corresponding eigenfunctions $\{\phi_k^{(n)}\}_k$) of (2.1) and (2.2) on $[0, 1]$ which is equivalent to the generalized boundary value problem of the form given in (3.18) with separated boundary conditions for (3.19), namely,

$$\begin{bmatrix} \phi_k^{(n)}(t) \\ (\phi_k^{(n)})'(t) \end{bmatrix} = \begin{bmatrix} \phi_k^{(n)}(0) \\ (\phi_k^{(n)})'(0) \end{bmatrix} + \int_0^t \begin{bmatrix} 0 & dP_n(s) \\ dQ_n(s) - \lambda_k^{(n)} dW_n(s) & 0 \end{bmatrix} \begin{bmatrix} \phi_k^{(n)}(s) \\ (\phi_k^{(n)})'(s) \end{bmatrix}$$

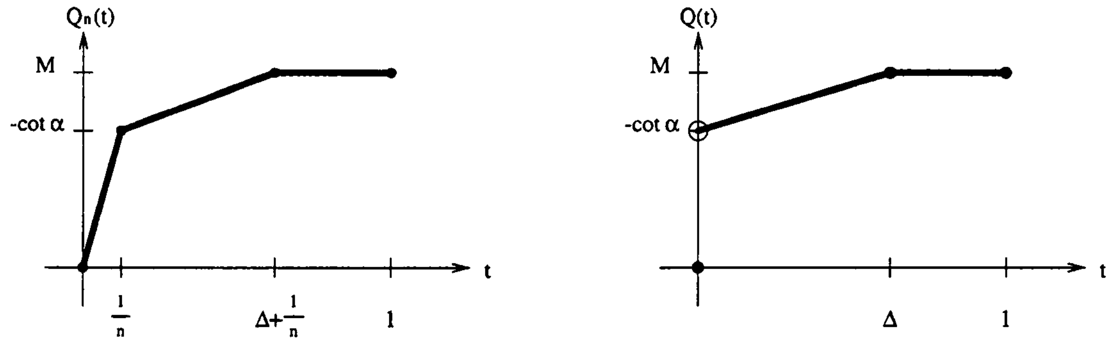

Figure 4.1: Graph of functions $Q_n(t)$ and $Q(t)$ in case B of the general boundary conditions

and

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_k^{(n)}(0) \\ (\phi_k^{(n)})'(0) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \cos\beta & -\sin\beta \end{bmatrix} \begin{bmatrix} \phi_k^{(n)}(1) \\ (\phi_k^{(n)})'(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where $P_n(t) = W_n(t) = t$ and $Q_n(t) = \int_0^t q_n(s)\,ds$. Again, for each n, $P_n(t)$, $Q_n(t)$, and $W_n(t)$ satisfy (3.15) on $[0, 1]$.

Theorem 4.1 *Given the hypothesis of Theorem 2.4 and the above definitions of the eigenvalues* $\{\lambda_k^{(n)}\}_k$ *and their corresponding eigenfunctions* $\{\phi_k^{(n)}\}_k$, *we know that there exists* $\tilde{k}$ *such that*

$$\lambda_{\tilde{k}}^{(n)} \to \lambda_0^{**}$$

as $n \to \infty$.

Proof: $Q_n(t) \to Q(t)$ pointwise (see Figures 4.1), $\bigvee_0^1 Q \leq Q(1)$ and $\bigvee_0^1 Q_n \leq Q(1)$ for all n, so that (3.16) is satisfied. We can let $\tilde{W}(t) = t$ in order to satisfy the hypothesis of Lemma 3.4. Thus, by Theorem 3.14, there exists $\tilde{k}$ such that $\lambda_{\tilde{k}}^{(n)} \to \lambda_0^{**}$ as $n \to \infty$. ■

Corollary 4.1.1 *Given the hypothesis of Theorem 4.1,*

$$\phi_{\tilde{k}}^{(n)}(t, \lambda_{\tilde{k}}^{(n)}) \to \phi_0(t, \lambda_0^{**}) \text{ uniformly on } [0,1] \text{ and}$$

$$(\phi_{\tilde{k}}^{(n)})'(t, \lambda_{\tilde{k}}^{(n)}) \to \phi_0'(t, \lambda_0^{**}) \text{ pointwise on } [0,1]$$

as $n \to \infty$ provided that $\phi_{\tilde{k}}^{(n)}(0, \lambda_{\tilde{k}}^{(n)}) \to \phi_0(0, \lambda_0^{**})$ and $(\phi_{\tilde{k}}^{(n)})'(0, \lambda_{\tilde{k}}^{(n)}) \to \phi_0'(0, \lambda_0^{**})$.

Proof: This corollary follows from Theorem 4.1, Theorem 3.12 and the remark directly following Theorem 3.12. ■

Finally, for case B of the general boundary conditions presented in section 2.4, we have completed the goal which was set forth in (2.3); that is, for this case, we can determine $\lambda_0^{\#}$ which was defined as the supremum of the set of all first eigenvalues subject to the constraint $\int_0^1 q(t)\,dt = M$. This theorem corresponds to Corollary 2.2.1 in section 2.3 (the Dirichlet boundary condition case) and to Corollary 2.3.1 in section 2.4 (case A of the general boundary conditions.)

Theorem 4.2 *Given the hypothesis of Theorem 4.1, the eigenvalue λ_0^{**} is the supremum of the set $S := \{\lambda_0(q) : \int_0^1 q(t)\,dt = M\}$, i.e.,*

$$\lambda_0^{\#} = \lambda_0^{**},$$

and the eigenvalue $\lambda_0(q^{**})$, which corresponds to q^{**} in Theorem 2.4, is the value $\lambda_0^{\#}$.

Proof: We need to show that $\tilde{k} = 0$ in Theorem 4.1.

If $\beta \neq \pi$, then $\phi_0(1) > 0$ and $\phi_0(t)$ has no zeros on $[0,1]$ (see (2.34)). Also, $\phi_{\tilde{k}}^{(n)}(t) \to \phi_0(t)$ uniformly on $[0,1]$. Thus, for a given $\epsilon > 0$, there exists N such that $|\phi_{\tilde{k}}^{(n)}(t) - \phi_0(t)| < \epsilon$ for all $n \geq N$ and $t \in [0,1]$. Hence, $\phi_{\tilde{k}}^{(n)}(t)$ has no zeros on

$[0,1]$ for $n \geq N$ and must be the first eigenfunction of the problem by the classical result on zeros of eigenfunctions of (2.1) and (2.2) [10]; that is, $\tilde{k} = 0$.

If $\beta = \pi$, then by (2.34),

$$\phi_0(t) = \begin{cases} 1, & 0 \leq t \leq \Delta \\ \sin(\sqrt{\lambda_0^{**}}(1-t)), & \Delta \leq t \leq 1. \end{cases}$$

Note that $\phi_0(1) = 0$ and $\phi_0(t)$ has no zeros on $[0,1)$. Choose $x_0 \in (\Delta, 1)$. Then there exists N such that for all $n \geq N$, $q_n \equiv q^{**} \equiv 0$ on $[x_0, 1]$. On $[x_0, 1]$, $\phi_0(t) = \sin(\sqrt{\lambda_0^{**}}(1-t))$ and $\phi_{\tilde{k}}^{(n)}(t) = c_n \sin\left(\sqrt{\lambda_{\tilde{k}}^{(n)}}(1-t)\right)$ where c_n is a nonzero constant. Since $\phi_{\tilde{k}}^{(n)}(1) = 0 = \phi_0(1)$, $\lambda_{\tilde{k}}^{(n)} \to \lambda_0^{**}$ and $\phi_{\tilde{k}}^{(n)}(t) \to \phi_0(t)$ uniformly, then $c_n \to 1$ and $\phi_{\tilde{k}}^{(n)}(t)$ has no zeros on $[x_0, 1)$ (for x_0 sufficiently close to 1) since the distance from the zero at $\phi_{\tilde{k}}^{(n)}(1)$ to the next zero of $\phi_{\tilde{k}}^{(n)}(t)$ is $\frac{\pi}{\sqrt{\lambda_{\tilde{k}}^{(n)}}}$. Moreover, since $\phi_{\tilde{k}}^{(n)}(t) \to \phi_0(t)$ uniformly on $[0, x_0]$ and $\phi_0(t)$ has no zeros on $[0, x_0]$, then as in the case for $\beta \neq \pi$, $\phi_{\tilde{k}}^{(n)}(t)$ has no zeros on $[0, x_0]$. Thus, $\phi_{\tilde{k}}^{(n)}(t)$ has no zeros on $[0,1)$ and must be the first eigenfunction, again by the classical result on zeros of eigenfunctions of (2.1) and (2.2) [10]; that is, $\tilde{k} = 0$.

We have already shown that λ_0^{**} is an upper bound of S in Theorem 2.5 and now we have shown that there exists a sequence $\{\lambda_0^{(n)}\}$ in S that converges to λ_0^{**} so λ_0^{**} is the least upper bound of S, i.e.,

$$\lambda_0^{\#} = \lambda_0^{**}.$$

(Note that Theorem 2.4 showed previously that $\lambda_0(q^{**}) = \lambda_0^{**}$ which, by this theorem, equals $\lambda_0^{\#}$.) ■

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VITA

Charlotte A. Knotts-Zides was born in Charleston, West Virginia on June 13, 1968. She graduated as valedictorian from Buckhannon-Upshur High School, Buckhannon, West Virginia in June, 1986, and entered Guilford College, Greensboro, North Carolina in August of 1986. During the summers of 1988 and 1989, she was chosen to attend the Science Alliance-National Science Foundation Research Experience for Undergraduates at the University of Tennessee-Knoxville where she worked with Robert McConnel; a paper of the results of their research entitled "Properties of Commutative Semigroups of Idempotents" was published in the Journal of Undergraduate Mathematics in 1989. She received a Bachelor of Science in Mathematics with a minor in Spanish in May, 1990. In July of 1990, she flew to Germany as a participant in the Seventh Congress-Bundestag Youth Exchange Program for Young Professionals, a program organized by the Carl Duisberg Society; during the spring of 1991, she was employed by Siemens as an intern programmer. Immediately thereafter, she completed a year of graduate studies in computer science at Cornell University, Ithaca, New York, before transferring to graduate studies in the Department of Mathematics at the University of Tennessee in August, 1992. She served as president and treasurer of the departmental Graduate Student Committee from 1996 until 1998; three times she received the Eaves Outstanding Graduate Student Teacher Award. In 1997, she received a graduate research award from the Scholarly Activities Research Incentive Fund and, in 1998, she placed first in the Sigma Xi Graduate Student Paper Competition. She received her Master's degree in May, 1997, and her doctoral degree in August, 1999.

She is presently employed as an Assistant Professor of Mathematics at Wofford College in Spartanburg, South Carolina.