To the Graduate Council:

I am submitting herewith a dissertation written by Jerome Fields Eastham, Jr. entitled "On the Finite Element Method in Anisotropic Sobolev Spaces." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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ON THE FINITE ELEMENT METHOD IN ANISOTROPIC SOBOLEV SPACES

A Dissertation

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Degree

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Jerome Fields Eastham, Jr.

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ABSTRACT

This thesis concerns the application of Galerkin methods to the numerical solution of some boundary value problems, essentially of elliptic type, which require the solution to possess different numbers of derivatives according to direction. The general theory of coercive bilinear forms on anisotropic Sobolev spaces is developed to prove existence and uniqueness for the solution of the continuous problem and abstract error estimates for the solution of the discrete problem. Using the tensor product of splines, we construct for these problems conforming finite element spaces in order to obtain concrete error estimates. Numerical results for a problem arising in the theory of high-speed gas centrifuges, the Onsager pancake equation, are given to illustrate the results obtained.

The anisotropic case differs from the isotropic case mainly in regard of the boundary. Anisotropic problems are usually posed on product domains, but isotropic problems are often treated under the assumption of a boundary without corners. For this reason the proof of a global regularity result for the problems considered here is of some independent interest.

As in the standard theory, the error of a Galerkin approximation is quasi-optimal when measured in the natural norm of the problem. An improved rate of convergence for the L_2-norm of the error is shown by the technique of Aubin-Nitsche. This estimate is not quasi-optimal in general, but is still best possible in the sense of agreement with experiments.

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INTRODUCTION

We consider various topics related to the theory and implementation of finite element methods for elliptic boundary value problems whose variational formulation is posed on an anisotropic Sobolev space. Although our results are applicable to a variety of elliptic problems, this thesis was motivated by a particular problem arising in the theory of high-speed gas centrifuges: the Onsager pancake equation.

For simplicity in regard to imposing boundary conditions, we will assume a rectangular domain:

$$\Omega \equiv (0,T) \times (0,1) . \tag{0.1}$$

However no use is made of results that are peculiar to two dimensions, and our methods carry over without change to product domains in higher dimensions.

Let A be a partial differential operator of the form:

$$Aw \equiv (-1)^m D_x^m (a D_x^m w) + (-1)^n D_y^{2n} w , \tag{0.2}$$

where the coefficient a is a sufficiently smooth strictly positive function of x alone. We will suppose that there exist positive constants C,c such that:

$$0 < c \le a(x) \le C , \quad 0 \le x \le T . \tag{0.3}$$

The reader is warned that we will use these constants generically, i.e. in a manner intended to suggest relationships among various estimates, but which is not always consistent. For example, the existence of the bounds (0.3) will be used to establish possibly different bounds for the bilinear form:

$$L(w,v) \equiv (a D_x^m w, D_x^m v) + (D_y^n w, D_y^n v). \tag{0.4}$$

The boundary value problems to be studied here have both a strong formulation:

$$Au = f \quad \text{in } \Omega, \tag{0.5}$$

$$Bu = g \quad \text{on } \partial\Omega, \tag{0.6}$$

where B represents a system of differential operators on each edge of the boundary (satisfying certain conditions to be specified later that will make the problem well-posed), and a weak or variational formulation involving the bilinear form L. For convenience we will adopt an abbreviated notation for boundary integrals and integration by parts:

$$\langle w,v\rangle_x \equiv \int_0^1 (wv|_{x=T} - wv|_{x=0})\,dy, \tag{0.7}$$

$$\langle w,v\rangle_y \equiv \int_0^T (wv|_{y=1} - wv|_{y=0})\,dx, \tag{0.8}$$

where the vertical slash represents the restriction of a function.

We have collected in Section I the results about anisotropic Sobolev spaces needed in the sequel. In particular we show that certain trace and inverse imbedding theorems imply the continuity of B and allows us to justify the use of an integration by parts formula for functions not necessarily in $C^{\infty}(\bar{\Omega})$, which as in [3] is the space of functions with derivatives in every direction of arbitrarily high order having unique continuous extension to the closure of Ω. We also lay the groundwork for a discussion of admissibility requirements for inhomogeneous boundary conditions.

In Section II we present the basic tools needed from functional analysis: the Lax-Milgram theorem and related ideas. These are by now well-known results, and the proofs are omitted.

Section III is devoted to the solution of the continuous variational problem, specifically the question of global regularity. For the case of homogeneous boundary conditions, we will show that the solution u satisfies estimates of the form:

$$\text{(0.9)} \qquad \|u\|_0, \|D_x^{2m}u\|_0, \|D_y^{2n}u\|_0 \leq C\|f\|_0$$

where $\|\cdot\|_0$ denotes the usual norm on $L_2(\Omega)$. A similar estimate is also given for the case of inhomogeneous boundary conditions satisfying an admissibility requirement. Such estimates will be needed for the homogeneous case to apply the technique of Aubin-Nitsche.

The main results of this thesis are contained in Section IV. In order to obtain concrete error estimates we show how to construct conforming finite element spaces using the tensor product of two one-dimensional spline spaces. If the discrete subspaces S_h are parameterized so that:

$$\inf_{w_h \in S_h} \|u - w_h\|_W \leq Ch(\|D_x u\|_W + \|D_y u\|_W), \tag{0.10}$$

where $\|\cdot\|_W$ is the natural norm of the problem, then by combining previous results we show:

$$\|u - u_h\|_W \leq Ch(\|D_x u\|_W + \|D_y u\|_W), \tag{0.11}$$

$$\|u - u_h\|_0 \leq Ch^2(\|D_x u\|_W + \|D_y u\|_W). \tag{0.12}$$

The Dirichlet problem for Poisson's equation is of the type considered here, but such problems for which $m \neq n$ are rare. Some related variational problems appear in least-squares methods for parabolic initial-boundary value problems investigated by King [5,6], who also considers the tensor product of splines in constructing finite elements for certain domains. Otherwise there seems to be little overlap with the present work in the numerical analysis literature.

In Section V we verify the hypotheses necessary to apply our results to the Onsager pancake approximation. Some modifications are needed to treat an oblique boundary condition and the subsequent addition of lower order terms. Several numerical experiments are summarized, and we close with a brief discussion of topics for future investigation.

We assume without further comment that all point functions here are real-valued and locally integrable. Expressions involving their derivatives are then known to exist in a distributional sense [16]. The identification of certain distributions with point functions is accomplished by the use of $L_2(\Omega)$ as a pivot space [1]. That is, we identify $L_2(\Omega)$ with its dual via representation of linear functionals in the usual L_2-inner product. Hence for a Sobolev space or other Hilbert space continuously imbedded in $L_2(\Omega)$, say:

$$W \subseteq L_2(\Omega) \quad , \tag{0.13}$$

we have the dual inclusion:

$$L_2(\Omega) \subseteq W' \quad , \tag{0.14}$$

where W' denotes the dual of W. We adopt the following notation for the operator norm on W', which can no longer be identified with W:

$$\|w\|_W^{op} \equiv \sup_{v \in W \setminus \{0\}} \frac{|(w,v)|}{\|v\|_W} \quad , \tag{0.15}$$

where $(\cdot,\cdot)$ is a duality pairing on W'xW that extends the inner product on $L_2(\Omega)$. There is an important caveat regarding this procedure, however. In practice one takes the indicated supremum over a closed subspace of W, namely W_o, the closure in W of $C_o^\infty(\Omega)$. The reason for this is that by duality W' is a closed subspace of W_o', and hence the latter is a more general setting.

I. ANISOTROPIC SOBOLEV SPACES

We will assume familiarity of the reader with the spaces of functions introduced by S. L. Sobolev for nonnegative integer m:

$$(1.1)\qquad W^m(\Omega) \equiv \{ w \in L_2(\Omega) \mid D_x^i D_y^j w \in L_2(\Omega),\ i,j \geq 0,\ i+j \leq m \},$$

which is a Hilbert spaces with inner product:

$$(1.2)\qquad (w,v)_m \equiv \sum_{i,j\geq 0}^{i+j\leq m} (D_x^i D_y^j w, D_x^i D_y^j v),$$

and their analogs for one-dimensional domains. These spaces may be said to be isotropic in the sense that they require equal orders of differentiability in every direction. Various generalizations of these spaces are known [9], but it is not our purpose to be thorough in surveying them. Rather we limit consideration to results needed in the sequel. For remarks concerning the history of these spaces, see [16, Sec. 4.3].

Def. (S. M. Nikol'skii, 1951) For nonnegative integers m,n we define the anisotropic Sobolev space:

$$(1.3)\qquad W^{m,n}(\Omega) \equiv \{ w \in L_2(\Omega) \mid D_x^m w, D_y^n w \in L_2(\Omega) \},$$

which is a Hilbert space with inner product:

$$(1.4)\qquad (w,v)_{m,n} \equiv (w,v) + (D_x^m w, D_x^m v) + (D_y^n w, D_y^n v).$$

We adopt the following notation for norms and seminorms associated with these spaces:

$$(1.5)\qquad \|w\|_m \equiv \Big(\sum_{i,j\geq 0}^{i+j\leq m} \|D_x^i D_y^j w\|_0^2\Big)^{1/2},$$

$$(1.6)\qquad |w|_{m,n} \equiv \big(\|D_x^m w\|_0^2 + \|D_y^n w\|_0^2\big)^{1/2},$$

$$(1.7)\qquad \|w\|_{m,n} \equiv \big(\|w\|_0^2 + |w|_{m,n}^2\big)^{1/2}.$$

The isotropic spaces are actually a special case of the anisotropic ones with m = n, but some work is required to show this. Note that the isotropic norm contains terms representing mixed and intermediate derivatives, while the only terms that appear explicitly in the anisotropic norm are unmixed derivatives of highest and lowest orders. The next result explains how these quantities are related.

<u>Prop</u>. 1 For fixed nonnegative integers m,n there exists a positive constant C such that for every positive ε, for all w in $W^{m,n}(\Omega)$:

$$(1.8)\qquad \|D_x^i w\|_0 \leq C\big(\varepsilon^{-i}\|w\|_0 + \varepsilon^{m-i}\|D_x^m w\|_0\big),\quad 0<i<m,$$

$$(1.9)\qquad \|D_y^j w\|_0 \leq C\big(\varepsilon^{-j}\|w\|_0 + \varepsilon^{n-j}\|D_y^n w\|_0\big),\quad 0<j<n,$$

$$(1.10)\qquad \|D_x^i D_y^j w\|_0 \leq C\|w\|_{m,n},\quad i,j\geq 0,\ \frac{i}{m}+\frac{j}{n}\leq 1.$$

<u>Proof</u>: For the case of a domain equal to the whole plane, see [16, Sec. 7.2]. Passage to the case of a bounded domain satisfying

an edge condition [16, Rmk. 4.3], e.g. a rectangle, can be made via the following result on extension with preservation of norms:

<u>Thm</u>. 1 There exists a bounded linear map $K: W^{m,n}(\Omega) \rightarrow W^{m,n}(R^2)$ such that for some positive constant C, for all w in $W^{m,n}(\Omega)$:

$$(1.11) \qquad Kw|_{\Omega} = w ,$$

$$(1.12) \qquad \|D_x^i Kw\|_0 \le C\|D_x^i w\|_0 , \quad 0 \le i \le m ,$$

$$(1.13) \qquad \|D_y^j Kw\|_0 \le C\|D_y^j w\|_0 , \quad 0 \le j \le n ,$$

where the norms in which K appears are for $L_2(R^2)$.

<u>Proof</u>: For the case of preserving a single norm, see [16, Thm. 2 of Rmk. 4.3]. The simultaneous nature of the extension is explicitly noted in [7, Thm. 6.2].

Let us now complete the proof of the previous Proposition. We have for all w in $W^{m,n}(\Omega)$, i = 0,1,...,m:

$$(1.14) \qquad \begin{aligned} \|D_x^i w\|_0 &\le \|D_x^i Kw\|_0 \\ &\le C(\varepsilon^{-i}\|Kw\|_0 + \varepsilon^{m-i}\|D_x^m Kw\|_0) \\ &\le C(\varepsilon^{-i}\|w\|_0 + \varepsilon^{m-i}\|D_x^m w\|_0) , \end{aligned}$$

which proves (1.8). Note that the first inequality depends merely on the fact that one cannot increase the L_2-norm of a function by taking its restriction to a subdomain. The estimates (1.9), (1.10) are also proven in this way.

Cor. 1 $W^m(\Omega) = W^{m,m}(\Omega)$.

Proof: By applying Prop. 1 with $\varepsilon = 1$ we see that the isotropic norm is equivalent to the anisotropic norm.

We will require at least once the definition of anisotropic Sobolev spaces for noninteger (positive) m,n. We cite [16, Sec. 9.2] for the case of an unbounded domain, where fractional derivatives are defined by the Fourier transform. We may then define $W^{m,n}(\Omega)$ as the restrictions of functions in $W^{m,n}(R^2)$ to Ω. See also the discussion in [16, Sec. 4.3] of the equivalence of various norms for fractional Sobolev spaces based on square-integrability (and their nonequivalence when this condition is replaced by a more general p-integrability).

For this reason we do not need any additional notation to distinguish between Sobolev spaces of integral and nonintegral orders. Moreover the results given here and below apply to the case of positive noninteger m,n (provided we replace norms of derivatives with suitable seminorms), and the references cited cover this case. But negative values of m,n are another matter, and we will avoid them entirely.

In order to achieve a variational formulation of the boundary value problem (0.5), (0.6), we will need to define certain boundary operators. Since the boundary of Ω is a set of measure zero, the restriction of a function to the boundary must be understood in a somewhat technical sense [16, Sec. 6.4]. Therefore we will make frequent use of the procedure of extending boundary operators defined on $C^\infty(\bar{\Omega})$ to a wider setting in which these operators are bounded, so long as $C^\infty(\bar{\Omega})$ is dense in that topology. Note that by [20, I.10] $C_0^\infty(R^2)$ is dense in $W^m(R^2)$ and similarly is dense in $W^{m,n}(R^2)$.

Prop. 2 $C^\infty(\bar{\Omega})$ is dense in $W^{m,n}(\Omega)$.

Proof: For any w in $W^{m,n}(\Omega)$ we choose v in $C_0^\infty(R^2)$ close to Kw in $W^{m,n}(R^2)$. Then the restriction of v to Ω is close to w in $W^{m,n}(\Omega)$.

Def. Let the trace operators be defined for w in $C^\infty(\bar{\Omega})$:

$$\tau_i^{(x)} w \equiv D_x^{i-1} w \big|_{x=0,T}, \quad i = 1, \ldots, m, \tag{1.15}$$

$$\tau_j^{(y)} w \equiv D_y^{j-1} w \big|_{y=0,1}, \quad j = 1, \ldots, n, \tag{1.16}$$

for all positive integers i,j.

Thm. 2 For fixed positive m,n there exists a positive constant C such that for all w in $C^\infty(\bar{\Omega})$:

$$\|\tau_i^{(x)} w\|_{\alpha_i n} \le C \|w\|_{m,n}, \quad \alpha_i \equiv 1 - \frac{i}{m} + \frac{1}{2m} > 0, \tag{1.17}$$

$$\|\tau_j^{(y)} w\|_{\beta_j m} \le C \|w\|_{m,n}, \quad \beta_j \equiv 1 - \frac{j}{n} + \frac{1}{2n} > 0. \tag{1.18}$$

Therefore the trace operators may be extended by continuity:

$$\tau_i^{(x)} : W^{m,n}(\Omega) \to W^{\alpha_i n}(\{0,T\} \times (0,1)), \tag{1.19}$$

$$\tau_j^{(y)} : W^{m,n}(\Omega) \to W^{\beta_j m}((0,T) \times \{0,1\}). \tag{1.20}$$

Proof: See [7, 15.1], [11, Thm. 2.1 of Chap. 4], or [16, Sec. 6.7].

The essential boundary conditions may be defined as those which involve boundary operators which are continuous on the space in which the solution is sought, say $W^{m,n}(\Omega)$. Assuming that they are linear, the essential boundary conditions prescribe a closed affine subset of $W^{m,n}(\Omega)$, and by an appropriate translation one reduces to the case of homogeneous essential boundary conditions which prescribe a closed subspace of $W^{m,n}(\Omega)$, namely the nullspace of the corresponding combination of boundary operators, say $W^{m,n}_{*}(\Omega)$. This notation is motivated by the observation [12, Chap. 4] that $W^{m,n}_{\circ}(\Omega)$, which we have defined to be the closure in $W^{m,n}(\Omega)$ of $C^{\infty}_{\circ}(\Omega)$, is precisely the nullspace of all trace operators on $W^{m,n}(\Omega)$ combined as a product of operators. For our purposes we will need only the simplest form of essential boundary conditions, in which the value of a single trace operator on the solution is determined.

Next we introduce certain supplementary boundary operators whose definitions depend on the choice of partial differential operator A. Our starting point is the following integration by parts formula:

<u>Lemma</u> 1 For all w,v in $C^{\infty}(\overline{\Omega})$ we have:

$$(1.21)\quad L(w,v) - (Aw,v) = \sum_{i=1}^{m} (-1)^{m-i} \langle D_x^{m-i} a D_x^m w, D_x^{i-1} v\rangle_x + \sum_{j=1}^{n} (-1)^{n-j} \langle D_y^{2n-j} w, D_y^{j-1} v\rangle_y .$$

<u>Proof</u>: For i = 1,...,m we can integrate by parts to obtain:

$$(1.22)\quad (-1)^{i-1}(D_x^{i-1} a D_x^m w, D_x^{m-i+1} v) - (-1)^{i}(D_x^{i} a D_x^m w, D_x^{m-i} v)$$

$$= (-1)^{i-1} \langle D_x^{i-1} a D_x^m w, D_x^{m-i} v \rangle .$$

By taking the indicated collapsing sum:

$$(1.23) \quad (a D_x^m w, D_x^m v) - (-1)^m (D_x^m a D_x^m w, v)$$
$$= \sum_{i=1}^{m} (-1)^{m-i} \langle D_x^{m-i} a D_x^m w, D_x^{i-1} v \rangle_x ,$$

and similarly:

$$(1.24) \quad (D_y^n w, D_y^n v) - (-1)^n (D_y^{2n} w, v)$$
$$= \sum_{j=1}^{n} (-1)^{n-j} \langle D_y^{2n-j} w, D_y^{j-1} v \rangle .$$

Combining (1.23) and (1.24) gives the desired result.

Def. Let the supplementary operators be defined for w in $C^\infty(\bar{\Omega})$:

$$(1.25) \quad \sigma_i^{(x)} w \equiv D_x^{m-i} a D_x^m w \Big|_{x=0,T} , \quad i = 1, \ldots, m ,$$

$$(1.26) \quad \sigma_j^{(y)} w \equiv D_y^{2n-j} w \Big|_{y=0,1} , \quad j = 1, \ldots, n .$$

Although one cannot continuously extend the supplementary operators to $W^{m,n}(\Omega)$, the following space is a convenient setting:

Def. For any nonnegative m,n we define:

$$(1.27) \quad W^{m,n}(\Omega; A) \equiv \{ w \in W^{m,n}(\Omega) \mid Aw \in L_2(\Omega) \},$$

which is a Hilbert space with the graph norm:

$$(1.28) \quad \|w\|^2_{m,n;A} \equiv \|w\|^2_{m,n} + \|Aw\|^2_0$$

Thm. 3 For fixed positive m,n there exists a positive constant C such that for all w in $C^\infty(\bar{\Omega})$:

$$(1.29) \quad \|\sigma_i^{(x)} w\|^{op}_{\alpha_i;n} \leq C\|w\|_{m,n;A}, \alpha_i = 1 - \frac{i}{m} + \frac{1}{2m}, \quad i = 1,\ldots,m,$$

$$(1.30) \quad \|\sigma_j^{(y)} w\|^{op}_{\beta_j;m} \leq C\|w\|_{m,n;A}, \beta_j = 1 - \frac{j}{n} + \frac{1}{2n}, \quad j = 1,\ldots,n.$$

Thus the supplementary operators may be extended by continuity:

$$(1.31) \quad \sigma_i^{(x)} : W^{m,n}(\Omega;A) \to W_0^{\alpha_i;n}(\{0,T\}\times(0,1))',$$

$$(1.32) \quad \sigma_j^{(y)} : W^{m,n}(\Omega;A) \to W_0^{\beta_j;m}((0,T)\times\{0,1\})'.$$

Proof: We have an estimate for the combined sum of the boundary integrals in the right-hand side of (1.21) by inspection of the left-hand side:

$$(1.33) \quad |L(w,v) - (Aw,v)| \leq C\|w\|_{m,n;A}\|v\|_{m,n}.$$

In order to estimate these separately, we need the following result, which is called an inverse imbedding theorem.

Lemma 2 For fixed positive m,n there exists a positive constant C such that for any g_i in $C_0^\infty(\{0,T\}\times(0,1))$, $i = 1,\ldots,m$, there exists v in $W^{m,n}(\Omega)$ such that:

(1.34) $$\tau_i^{(x)} v = g_i \quad , \quad i = 1,\ldots,m \quad ,$$

(1.35) $$\tau_j^{(y)} v = 0 \quad , \quad j = 1,\ldots,n \quad ,$$

(1.36) $$\|v\|_{m,n} \leq C\left(\sum_{i=1}^{m} \|g_i\|_{\alpha_i n}\right) \quad , \quad \alpha_i = 1 - \frac{i}{m} + \frac{1}{2m} \ .$$

Similarly, for any g_j in $C_0((0,T)\times\{0,1\})$, $j = 1,\ldots,n$, there exists v in $W^{m,n}(\Omega)$ such that:

(1.37) $$\tau_i^{(x)} v = 0 \quad , \quad i = 1,\ldots,m \quad ,$$

(1.38) $$\tau_j^{(y)} v = g_j \quad , \quad j = 1,\ldots,n \quad ,$$

(1.39) $$\|v\|_{m,n} \leq C\left(\sum_{j=1}^{n} \|g_j\|_{\beta_j m}\right) \quad , \quad \beta_j = 1 - \frac{j}{n} + \frac{1}{2n} \ .$$

Proof: It suffices to consider the case when g is identically zero except at a single index on one edge, say g_i at x=0. Individually the trace operators are continuously right invertible [16, Sec. 6.8], but not collectively so because they must satisfy the compatibility relations [11, Thm. 2.3 of Chap. 4] which characterize the range of the combined product of the trace operators. Briefly, these amount to the equality of some mixed derivatives at each corner. But this condition is met since all traces are to be zero near corners.

We now complete the proof of Theorem 3. Choose g in $C_0^\infty(0,1)$ not identically zero. For fixed $i = 1,\ldots,m$, we can find v in $W^{m,n}(\Omega)$ as above:

$$(1.40)\qquad \tau_i^{(x)} v\big|_{x=0} = g \ , \ \tau_i^{(x)} v\big|_{x=T} = 0 \ ,$$

where otherwise the traces of v vanish identically. Then by the estimates (1.33), (1.36) we have:

$$(1.41)\qquad \left|\int_0^1 \sigma_i^{(x)} w\big|_{x=0}\, g\, dy\right| = \left|\langle \sigma_i^{(x)} w, \tau_i^{(x)} v\rangle_x\right|$$

$$\leq C\, \|w\|_{m,n;A} \|v\|_{m,n}$$

$$\leq C\, \|w\|_{m,n;A} \|g\|_{\alpha_i n} \ ,$$

where a limiting argument has been used to apply (1.21) with w in $C^\infty(\bar{\Omega})$ and v in $W^{m,n}(\Omega)$. Dividing through by the norm of g and taking the supremum over all such g gives by duality:

$$(1.42)\qquad \|\sigma_i^{(x)} w\|_{\alpha_i n}^{op} \leq C\, \|w\|_{m,n;A} \ .$$

The other estimates are similar. It remains only to note that $C^\infty(\bar{\Omega})$ is dense in $W^{m,n}(\Omega;A)$ because it is already dense in the stronger topology of $W^{2m,2n}(\Omega)$ by Prop. 2.

We now obtain estimates for the boundary operators in some different norms.

Thm. 4 The following estimates hold:

(1.43) $\|\tau_i^{(x)} w\|_{(1+\alpha_i)n} \leq C\|w\|_{2m,2n}\ ,\ i = 1,\ldots,m\ ,$

(1.44) $\|\sigma_i^{(x)} w\|_{(1-\alpha_i)n} \leq C\|w\|_{2m,2n}\ ,\ i = 1,\ldots,m\ ,$

(1.45) $\|\tau_j^{(y)} w\|_{(1+\beta_j)n} \leq C\|w\|_{2m,2n}\ ,\ j = 1,\ldots,n\ ,$

(1.46) $\|\sigma_j^{(y)} w\|_{(1-\beta_j)n} \leq C\|w\|_{2m,2n}\ ,\ j = 1,\ldots,n\ .$

where C is a positive constant not depending on w in $W^{2m,2n}(\Omega)$. So the trace and supplementary operators may be restricted to $W^{2m,2n}(\Omega)$:

(1.47) $\tau_i^{(x)}: W^{2m,2n}(\Omega) \to W^{(1+\alpha_i)n}(\{0,T\}\times(0,1))\ ,\ i = 1,\ldots,m\ ,$

(1.48) $\sigma_i^{(x)}: W^{2m,2n}(\Omega) \to W^{(1-\alpha_i)n}(\{0,T\}\times(0,1))\ ,\ i = 1,\ldots,m\ ,$

(1.49) $\tau_j^{(y)}: W^{2m,2n}(\Omega) \to W^{(1+\beta_j)m}((0,T)\times\{0,1\})\ ,\ j = 1,\ldots,n\ ,$

(1.50) $\sigma_j^{(y)}: W^{2m,2n}(\Omega) \to W^{(1-\beta_j)m}((0,T)\times\{0,1\})\ ,\ j = 1,\ldots,n\ .$

Proof: By applying Thm. 2 for $W^{2m,2n}(\Omega)$ instead of $W^{m,n}(\Omega)$ we obtain additional trace operators, so that:

(1.51) $\|\tau_i^{(x)} w\|_{(1-\frac{i}{2m}+\frac{1}{4m})(2n)} \leq C\|w\|_{2m,2n}\ ,\ i = 1,\ldots,2m\ .$

Now (1.43) is immediate. Also by the product rule and smoothness of the coefficient a we write:

(1.52) $$\sigma_i^{(x)} w = \sum_{k=0}^{m-i} (D_x^k a) D_x^{2m-i-k} w \Big|_{x=0,T}$$

$$= \sum_{k=0}^{m-i} (D_x^k a) \tau_{2m-i-k+1}^{(x)} w, \quad i=1,\ldots,m .$$

The roughest of these terms is for k = 0 , so:

(1.53) $$\|\sigma_i^{(x)} w\|_{(1-\frac{2m-i+1}{2m}+\frac{1}{4m})(2n)} \le C \|w\|_{2m,2n}, \quad i=1,\ldots,m,$$

and (1.44) is immediate. Estimates (1.45), (1.46) are similar.

Now we will extend the boundary operators to:

(1.54) $$W^0(\Omega;A) \equiv \{ w \in L_2(\Omega) \mid Aw \in L_2(\Omega) \},$$

which is a Hilbert space with graph norm:

(1.55) $$\|w\|_{0;A}^2 \equiv \|w\|_0^2 + \|Aw\|_0^2 .$$

<u>Lemma</u> 3 For all w,v in $C^\infty(\overline{\Omega})$ we have:

(1.56) $$(Aw,v) - (w,Av) =$$

$$- \sum_{i=1}^{m} (-1)^{m-i} \langle \sigma_i^{(x)} w, \tau_i^{(x)} v \rangle_x - \sum_{j=1}^{n} (-1)^{n-j} \langle \sigma_j^{(y)} w, \tau_j^{(y)} v \rangle_y$$

$$+ \sum_{i=1}^{m} (-1)^{m-i} \langle \sigma_i^{(x)} v, \tau_i^{(x)} w \rangle_x - \sum_{j=1}^{n} (-1)^{n-j} \langle \sigma_j^{(y)} v, \tau_j^{(y)} w \rangle_y$$

Proof: Interchange w and v in Lemma 1 and subtract the respective versions of (1.21).

Thm. 5 The following estimates hold:

$$(1.57)\quad \| \tau_i^{(x)} w \|^{op}_{(1-\alpha_i)n} \leq C \| w \|_{0;A}\ , \quad i = 1,\dots,m\ ,$$

$$(1.58)\quad \| \sigma_i^{(x)} w \|^{op}_{(1+\alpha_i)n} \leq C \| w \|_{0;A}\ , \quad i = 1,\dots,m\ ,$$

$$(1.59)\quad \| \tau_j^{(y)} w \|^{op}_{(1-\beta_j)m} \leq C \| w \|_{0;A}\ , \quad j = 1,\dots,n\ ,$$

$$(1.60)\quad \| \sigma_j^{(y)} w \|^{op}_{(1+\beta_j)m} \leq C \| w \|_{0;A}\ , \quad j = 1,\dots,n\ ,$$

where C is a positive constant not depending on w in $C^{\infty}(\overline{\Omega})$. Hence the trace and supplementary operators may be continuously extended:

$$(1.61)\quad \tau_i^{(x)} : W^{0}(\Omega;A) \to W_0^{(1-\alpha_i)n}(\{0,T\}\times(0,1))'\ , \quad i=1,\dots,m,$$

$$(1.62)\quad \sigma_i^{(x)} : W^{0}(\Omega;A) \to W_0^{(1+\alpha_i)n}(\{0,T\}\times(0,1))'\ , \quad i=1,\dots,m,$$

$$(1.63)\quad \tau_j^{(y)} : W^{0}(\Omega;A) \to W_0^{(1-\beta_j)m}((0,T)\times\{0,1\})'\ , \quad j=1,\dots,n,$$

$$(1.64)\quad \sigma_j^{(y)} : W^{0}(\Omega;A) \to W_0^{(1+\beta_j)m}((0,T)\times\{0,1\})'\ , \quad j=1,\dots,n.$$

Proof: We imitate the methods used to prove Thm. 3 above. We note first that the combined boundary integrals in (1.56) satisfy an estimate of the form:

$$(1.65)\quad \Big| \sum_{i=1}^{m} (-1)^{m-i} \left(\langle \sigma_i^{(x)} w, \tau_i^{(x)} v \rangle_x - \langle \sigma_i^{(x)} v, \tau_i^{(x)} w \rangle_x \right) + \sum_{j=1}^{n} (-1)^{n-j} \left(\langle \sigma_j^{(y)} w, \tau_j^{(y)} v \rangle_y - \langle \sigma_j^{(y)} v, \tau_j^{(y)} w \rangle_y \right) \Big| \leq C \, \|w\|_{0;A} \, \|v\|_{2m,2n} \, .$$

Second we apply Lemma 2 to $W^{2m,2n}(\Omega)$ instead of $W^{m,n}(\Omega)$ to obtain functions with prescribed traces in $C_0^\infty(0,1)$ or $C_0^\infty(0,T)$. Since the coefficient a has a smooth reciprocal, we can solve the triangular system (1.52) in order to prescribe the values of the supplementary operators. The desired estimates will then follow by a duality argument. Finally we note that $C^\infty(\bar{\Omega})$ is dense in $W^0(\Omega;A)$ because it is already dense in $W^{2m,2n}(\Omega)$.

II. THE LAX-MILGRAM THEOREM AND COROLLARIES

This work depends throughout on estimates for solutions of variational problems. The results of this section are therefore essential to our treatment of both continuous and discrete versions of a problem.

Thm. 1 (Lax, Milgram, 1954) Let W be a Hilbert space and let $L(\cdot,\cdot)$ be a bilinear form on WxW. If there exist positive constants C,c such that for all w,v in W:

$$(2.1) \qquad |L(w,v)| \leq C\,\|w\|_W \|v\|_W \ ,$$

$$(2.2) \qquad L(w,w) \geq c\,\|w\|_W^2 \ ,$$

then there exists a continuous invertible map $A:W \rightarrow W'$ such that for all w,v in W:

$$(2.3) \qquad L(w,v) = (Aw,v) \ ,$$

where $(\cdot,\cdot)$ denotes a duality pairing on W'xW. Hence for any bounded linear functional F in W', the solution u in W of the variational problem:

$$(2.4) \qquad L(u,v) = F(v) \quad \text{for all } v \in W \ ,$$

is given by solving Au = F. We have estimates:

(2.5) $$c\|u\|_W \le \|F\|_W^{op} \le C\|u\|_W .$$

Proof: See [3, Thm. 5.8] or [20, III.7].

Cor. 1 (Galerkin, 1915) Let W, L, F be as above. For any closed subspace S of W, there exists a solution u_h in S of the variational problem:

(2.6) $$L(u_h, v_h) = F(v_h) \text{ for all } v_h \in S,$$

and we have the stability estimate:

(2.7) $$c\|u_h\|_W \le \|F\|_W^{op} .$$

Proof: Since S is a Hilbert space under the inner product inherited from W, and since (2.1), (2.2) are also inherited, we can apply the previous Thm. to find u_h in S satisfying (2.6). Hence we have the estimate:

(2.8) $$c\|u_h\|_S \le \|F\|_S^{op} \le \|F\|_W^{op} ,$$

and (2.7) follows since the norm on S is inherited from W.

Cor. 2 (Cea, 1964) Let W, L, F, S be as above. Then the solutions u, u_h of the variational problems (2.4), (2.6) respectively, satisfy the orthogonality condition:

$$(2.9) \qquad L(u-u_h, v_h) = 0 \text{ for all } v_h \in S .$$

Thus we have the following abstract error estimate:

$$(2.10) \qquad c\|u-u_h\|_W \leq C \inf_{w_h \in S} \|u-w_h\|_W .$$

Proof: First we remark that the estimates (2.5), (2.7) imply the uniqueness of solutions u, u_h to their respective problems. By subtracting (2.6) from (2.4) holding with v_h in S and using the fact that L is bilinear, we get (2.9). Hence:

$$(2.11) \qquad \begin{aligned} c\|u-u_h\|_W^2 &\leq L(u-u_h, u-u_h) \\ &= L(u-u_h, u-w_h) \\ &\leq C\|u-u_h\|_W\|u-w_h\|_W , \end{aligned}$$

for any w_h in S. The desired result follows by cancellation and taking the infimum over all w_h in S.

III. REGULARITY OF THE MODEL PROBLEM

In many studies of isotropic elliptic boundary value problems (see [3], [10]) the question of global regularity on a bounded domain is broken into two parts: the interior (local) estimate and an estimate at the boundary. If the boundary is sufficiently smooth, a change of coordinates can be made to reduce to the halfspace setting. Here finite difference or Fourier transform techniques can be applied.

Our treatment avoids the technical problems involved in a change of coordinates, which are exacerbated by the anisotropic properties to be preserved. We will obtain a full regularity estimate directly from repeated application of the Lax-Milgram theorem (Thm. II.1).

Prop. 1 Let $L(\cdot,\cdot)$ be given by (0.4). Then L is a bounded bilinear form on $W^{m,n}(\Omega)\mathrm{x}W^{m,n}(\Omega)$, i.e. there exists a positive constant C such that for all w,v in $W^{m,n}(\Omega)$:

$$|L(w,v)| \le C\,\|w\|_{m,n}\,\|v\|_{m,n} \tag{3.1}$$

Proof: Apply Holder's inequality to both terms in (0.4) and use the boundedness of the coefficient a:

$$\begin{aligned} |(aD_x^m w, D_x^m v)| &\le \|aD_x^m w\|_0\,\|D_x^m v\|_0 \\ &\le \|a\|_\infty\,\|w\|_{m,0}\,\|v\|_{m,0}\,, \end{aligned} \tag{3.2}$$

$$|(D_y^n w, D_y^n v)| \le \|w\|_{0,n}\,\|v\|_{0,n}\ . \tag{3.3}$$

Hence the desired result follows.

We now describe the possible combinations of boundary conditions to be allowed. On each edge x=0,T there will be m conditions. For the present we consider only the simplest case, in which exactly one boundary operator appears in each condition. We assume that for i = 1,...,m the conditions include the following:

$$(3.4)\quad \text{either } \sigma_i^{(x)}u\big|_{x=0} = g_i\big|_{x=0} \text{ or } \tau_i^{(x)}u\big|_{x=0} = 0\ ,$$

$$(3.5)\quad \text{either } \sigma_i^{(x)}u\big|_{x=T} = g_i\big|_{x=T} \text{ or } \tau_i^{(x)}u\big|_{x=T} = 0\ .$$

Similarly, at each edge y=0,1 there will be n conditions that include the following for j = 1,...,n :

$$(3.6)\quad \text{either } \sigma_j^{(y)}u\big|_{y=0} = g_j\big|_{y=0} \text{ or } \tau_j^{(y)}u\big|_{y=0} = 0,$$

$$(3.7)\quad \text{either } \sigma_j^{(y)}u\big|_{y=1} = g_j\big|_{y=1} \text{ or } \tau_j^{(y)}u\big|_{y=1} = 0\ .$$

We assume one further thing, that for i = 1,...,m the boundary conditions include the following:

$$(3.8)\quad \text{either } \tau_i^{(x)}u\big|_{x=0} = 0 \text{ or } \tau_i^{(x)}u\big|_{x=T} = 0\ .$$

<u>Prop</u>. 2 The essential boundary conditions, those which involve the trace operators, describe a closed subspace $W_*^{m,n}(\Omega)$ of $W^{m,n}(\Omega)$.

<u>Proof</u>: This follows from the continuity of the trace operators

on $W^{m,n}(\Omega)$ as in Thm. I.2, and the fact that we have assumed homogeneous data for the essential boundary conditions.

Prop. 3 If the boundary conditions satisfy the alternative (3.8), then L is coercive on $W^{m,n}_*(\Omega)$, i.e. there exists a positive constant c such that for all w in $W^{m,n}_*(\Omega)$:

$$L(w,w) \geq c\,\|w\|^2_{m,n} \quad . \tag{3.9}$$

Proof: We need the following Poincaré inequality:

Lemma 1 There exists a positive constant C such that for all w in $W^{1,0}(\Omega)$ satisfying:

$$w|_{x=0} = 0 \quad (\text{resp. } w|_{x=T} = 0) \quad , \tag{3.10}$$

we have the following inequality:

$$\|w\|_0 \leq C\,\|D_x w\|_0 \quad . \tag{3.11}$$

Proof: For definiteness we assume:

$$w(x,y) = \int_0^x D_x w(t,y)\,dt \quad . \tag{3.12}$$

Therefore:

$$\|w\|_0^2 = \int_0^1\int_0^T w^2\,dx\,dy \tag{3.13}$$

$$= \int_0^T \int_0^1 \left(\int_0^x D_x w \, dt \right)^2 dy \, dx$$

$$\leq \int_0^T \int_0^1 \int_0^T (D_x w)^2 \, dt \, dy \, dx$$

$$= T \| D_x w \|_0^2 \; .$$

To complete the proof of Proposition 3 we induct on Lemma 1 to get an estimate:

(3.14) $$\| w \|_0 \leq C \| D_x^m w \|_0 \; ,$$

for all w in $W_*^{m,n}(\Omega)$. Now (3.14) and the estimate:

(3.15) $$L(w,w) \geq c \| D_x^m w \|_0^2 + \| D_y^n w \|_0^2 \; ,$$

combine to yield:

(3.16) $$L(w,w) \geq c \| w \|_{m,n}^2 \; ,$$

as desired.

Thm. 1 Let f be in $L_2(\Omega)$. Then there exists unique u in $W_*^{m,n}(\Omega)$ such that:

(3.17) $$Au = f \text{ in } \Omega \; ,$$

(3.18) $$Bu = 0 \text{ on } \partial\Omega\ ,$$

provided that the alternatives (3.4)-(3.8) hold.

Proof: We have verified the hypotheses (2.1), (2.2) of the Lax-Milgram theorem (Thm. II.1) by the estimates (3.1), (3.9). So let u in $W_*^{m,n}(\Omega)$ be the solution of the continuous variational problem:

(3.19) $$L(u,v) = (f,v) \quad \text{for all } v \in W_*^{m,n}(\Omega).$$

Hence we have the existence and uniqueness of such a solution, together with the estimate:

(3.20) $$c\,\|u\|_{m,n} \leq \sup_{v \in W_*^{m,n}(\Omega)} \frac{|(f,v)|}{\|v\|_{m,n}} \leq C\,\|u\|_{m,n}.$$

But:

(3.21) $$|(f,v)| \leq \|f\|_0 \|v\|_{m,n}\ ,$$

so:

(3.22) $$c\,\|u\|_{m,n} \leq \|f\|_0\ .$$

Now by taking v in $C_0^\infty(\Omega)$ we see by comparing (3.19) with (1.21) that (3.17) holds in a weak sense, i.e. u is in $W^{m,n}(\Omega;A)$. Further comparison then shows that the boundary conditions (3.18) hold.

Def. Let $W_{**}^{m,n}(\Omega;A)$ be the closed subspace of $W^{m,n}(\Omega;A)$ given by the homogeneous natural and essential boundary conditions (3.18).

Cor. 1 A is an isomorphism from $W_{**}^{m,n}(\Omega;A)$ to $L_2(\Omega)$.

Proof: It is clear from (3.22) and the definition of the norm:

$$c\,\|u\|_{m,n;A} \leq \|f\|_0 \;. \tag{3.23}$$

But also:

$$\|f\|_0 \leq \|u\|_{m,n;A} \;. \tag{3.24}$$

Thm. 2 Let f be in $L_2(\Omega)$. Then there exists u in $W^{2m,2n}(\Omega)$ such that:

$$Au = f \text{ in } \Omega \,, \tag{3.25}$$

$$Bu = 0 \text{ on } \partial\Omega \;. \tag{3.26}$$

Moreover, the solution is unique in $W^0(\Omega;A)$.

Proof: We consider first the case that f is in $C_0^\infty(\Omega)$. By Thm. 1 we have that there exists a unique solution in $W_{**}^{m,n}(\Omega;A)$ of (3.25). Then:

$$AD_y^n u = D_y^n Au = D_y^n f \tag{3.27}$$

must hold in a weak sense, and $D_y{}^n u$ is in $W^0(\Omega;A)$. Next we determine the boundary conditions satisfied by $D_y^n u$. At x=0,T the tangential derivative is defined and commutes with the corresponding trace and supplementary operators:

$$\tau_i^{(x)} D_y^n u = D_y^n \tau_i^{(x)} u \quad , \tag{3.28}$$

$$\sigma_i^{(x)} D_y^n u = D_y^n \sigma_i^{(x)} u \quad . \tag{3.29}$$

Hence $D_y^n u$ satisfies the same homogeneous boundary conditions as u at x=0,T. By contrast there is a reversal of the roles of natural and essential boundary conditions at y=0,1. First we note:

$$\sigma_j^{(y)} u = \tau_{n-j+1}^{(y)} D_y^n u \quad . \tag{3.30}$$

Now because f has compact support, the following equality holds in a neighborhood of y=0,1:

$$D_y^{2n} u = (-1)^{m-n+1} D_x^m a D_x^m u \quad , \tag{3.31}$$

so that:

$$\begin{aligned} \sigma_j^{(y)} D_y^n u &= D_y^{3n-j} u \,\Big|_{y=0,1} \\ &= \tau_{n-j+1}^{(y)} D_y^{2n} u \\ &= (-1)^{m-n+1} D_x^m a D_x^m \tau_{n-j+1}^{(y)} u \,. \end{aligned} \tag{3.32}$$

Thus (3.30), (3.32) imply that:

$$\sigma_j^{(y)} u|_{y=0} = 0 \quad \Rightarrow \quad \tau_{n-j+1}^{(y)} D_y^n u|_{y=0} = 0, \tag{3.33}$$

$$\sigma_j^{(y)} u|_{y=1} = 0 \quad \Rightarrow \quad \tau_{n-j+1}^{(y)} D_y^n u|_{y=1} = 0, \tag{3.34}$$

$$\tau_j^{(y)} u|_{y=0} = 0 \quad \Rightarrow \quad \sigma_{n-j+1}^{(y)} D_y^n u|_{y=0} = 0, \tag{3.35}$$

$$\tau_j^{(y)} u|_{y=1} = 0 \quad \Rightarrow \quad \sigma_{n-j+1}^{(y)} D_y^n u|_{y=1} = 0, \tag{3.36}$$

which preserves the alternatives (3.6), (3.7). To summarize, we have shown that $D_y{}^n u$ is a solution in $W^0(\Omega;A)$ of (3.27) subject to suitable homogeneous boundary conditions. By Thm. 1 we know that the auxilary problem so introduced has a solution in $W^{m,n}(\Omega)$. But since the operator A satisfies:

$$c\|w\|_{m,n;A} \leq \|Aw\|_0 \leq C\|w\|_{m,n;A} \tag{3.37}$$

on $W_{**}^{m,n}(\Omega;A)$, we have by a density argument:

$$c\|w\|_{0;A} \leq \|Aw\|_0 \leq C\|w\|_{0;A} \tag{3.38}$$

on $W_{**}^0(\Omega;A)$ where we have imposed the homogeneous natural and essential boundary conditions of the auxilary problem. Hence the solution of the auxilary problem is unique in the larger setting, and we can conclude that $D_y{}^n u$ is in $W_{**}^{m,n}(\Omega;A)$, and we have the variational problem which it satisfies:

$$(3.39) \qquad L(D_y^n u, v) = (D_y^n f, v) \text{ for all } v \in W_*^{m,n}(\Omega).$$

But since f has compact support, we estimate:

$$(3.40) \qquad |(D_y^n f, v)| = |(f, D_y^n v)|$$

$$\leq \|f\|_0 \|v\|_{m,n} .$$

In this way we obtain from (3.20):

$$(3.41) \qquad c\|D_y^{2n} u\|_0 \leq c\|D_y^n u\|_{m,n} \leq \|f\|_0 .$$

But by applying the triangle inequality to (3.25):

$$(3.42) \qquad \|D_x^m a D_x^m u\|_0 \leq \|f\|_0 + \|D_y^{2n} u\|_0$$

$$\leq C\|f\|_0 .$$

From (3.22) we also have:

$$(3.43) \qquad \|a D_x^m u\|_0 \leq C\|D_x^m u\|_0$$

$$\leq C\|f\|_0 ,$$

which gives by the definition of the norm:

$$(3.44)\qquad \|aD_x^m u\|_{m,0} \leq C\|f\|_0 .$$

Multiplication by the reciprocal of the coefficient a is a bounded linear map on $W^{m,0}(\Omega)$, so:

$$(3.45)\qquad \|D_x^m u\|_{m,0} \leq C\|f\|_0 ,$$

which combines with (3.41) to yield the estimate:

$$(3.46)\qquad \|u\|_{2m,2n} \leq C\|f\|_0 .$$

For the general case of f in $L_2(\Omega)$ we can always find a sequence f_k in $C_0^\infty(\Omega)$ that converges to f in $L_2(\Omega)$. Hence the corresponding solutions u_k converge in $W_{**}^{m,n}(\Omega;A)$ by (3.23) and form a bounded sequence in $W^{2m,2n}(\Omega)$ by (3.46). Since we can always extract a weakly convergent subsequence from a bounded sequence in a Hilbert space [20], we have without loss of generality that u_k converges weakly to u in $W^{2m,2n}(\Omega)$. But the weak limit must equal the strong limit given by Thm. 1 because every bounded linear functional on $W^{m,n}(\Omega)$ is also a bounded linear functional on $W^{2m,2n}(\Omega)$. Since the weak limit satisfies:

$$(3.47)\qquad \|u\|_{2m,2n} \leq \lim_{k\to\infty} \|u_k\|_{2m,2n} ,$$

we have extended the estimate (3.46) to the general case. We note that we have shown the uniqueness of the solution in $W^{0}_{**}(\Omega;A)$ by the estimate (3.38).

Cor. 2 Let f be in $L_2(\Omega)$ and let g be such that for some w in $W^{2m,2n}(\Omega)$:

$$Bw = g \quad \text{on } \partial\Omega. \tag{3.48}$$

Then the problem (0.5), (0.6) has a solution u in $W^{2m,2n}(\Omega)$, and:

$$\|u\|_{2m,2n} \leq C(\|f\|_0 + \|w\|_{2m,2n}). \tag{3.49}$$

Proof: By the triangle inequality and by applying the previous result to:

$$A(u-w) = f - Aw \quad \text{in } \Omega, \tag{3.50}$$

$$B(u-w) = 0 \quad \text{on } \partial\Omega, \tag{3.51}$$

we have:

$$\begin{aligned} \|u\|_{2m,2n} &\leq \|u-w\|_{2m,2n} + \|w\|_{2m,2n} \\ &\leq C\|f-Aw\|_0 + \|w\|_{2m,2n} \\ &\leq C(\|f\|_0 + \|w\|_{2m,2n}). \end{aligned} \tag{3.52}$$

IV. THE FINITE ELEMENT METHOD IN ANISOTROPIC SOBOLEV SPACES

We now obtain some basic convergence results for the Galerkin approximation. In order to obtain concrete estimates we show how to construct conforming finite element spaces for the problems studied here. Consistent with our restriction to rectangular domains, we consider the tensor product of two one-dimensional spline spaces. Some results depending on a peculiarity of the tensor product space are also noted.

<u>Notation</u> Let S_h be a parameterized family of conforming finite element subspaces:

$$S_h \subseteq W \equiv W_*^{m,n}(\Omega), \tag{4.1}$$

and let V be a Hilbert space continuously imbedded in W such that for some positive constant C:

$$\inf_{w_h \in S_h} \| w - w_h \|_W \leq C h \| w \|_V . \tag{4.2}$$

<u>Thm</u>. 1 If the continuous solution u is in V, then the discrete solution u_h in S_h given by Corollary II.1 satisfies:

$$\| u - u_h \|_W \leq C h \| u \|_V , \tag{4.3}$$

where C is a positive constant not depending on u.

<u>Proof</u>: Immediate from Cor. II.2 and (4.2).

Cor. 1 If $U = W^{2m,2n}_{**}(\Omega)$ is continuously imbedded in V, then for f in $L_2(\Omega)$ and homogeneous boundary conditions:

$$\|u - u_h\|_W \leq Ch\|f\|_0 . \tag{4.4}$$

Proof: Immediate from (4.3) and Thm. III.2.

Next we show how to obtain an improved rate of convergence for the L_2-norm of the error via the technique of Aubin-Nitsche.

Thm. 2 If the solution u is in V and U is continuously imbedded in V, then:

$$\|u - u_h\|_0 \leq Ch^2\|u\|_V . \tag{4.5}$$

Proof: We introduce the solution v of the adjoint problem:

$$L(w, v) = (w, u - u_h) \text{ for all } w \in W . \tag{4.6}$$

Then by the orthogonality condition (2.9):

$$\begin{aligned} (u-u_h, u-u_h) &= L(u-u_h, v) \\ &= L(u-u_h, v-v_h) \end{aligned} \tag{4.7}$$

for any v_h in S_h. Using the boundedness of L and taking the infimum over all such v_h:

$$\|u-u_h\|_0^2 \leq C\|u-u_h\|_W \inf_{v_h \in S_h}\|v-v_h\|_W \tag{4.8}$$

But since L is symmetric, we have by Thm. III.2 that v is in U, and so is in V. Then by applying (4.2) to v:

$$\|u-u_h\|_0^2 \leq C\|u-u_h\|_W h \|u-u_h\|_0 . \tag{4.9}$$

The desired result follows by cancellation and substitution of (4.3).

Cor. 2 If U is continuously imbedded in V, then for f in $L_2(\Omega)$ and homogeneous boundary conditions:

$$\|u-u_h\|_0 \leq C h^2 \|f\|_0 . \tag{4.10}$$

Proof: Immediate from (4.5) and Thm. III.2.

Unlike the estimate of the W-norm of the error, the estimate of the L_2-norm of the error is not in general quasi-optimal. However the result is sharp, in the sense that $O(h^2)$ rates of convergence are observed computationally. Examples will be given in Section V.

Now let us consider the construction of a finite element space for the model problem using the tensor product construction.

Prop. 1 Let $S_h^{(x)}$, $S_h^{(y)}$ be one-dimensional spline spaces such that:

$$S_h^{(x)} \subseteq W^m(0,T) , \tag{4.11}$$

$$S_h^{(y)} \subseteq W^n(0,1) . \tag{4.12}$$

Then the tensor product of these spaces:

(4.13) $$S_h^{(x)} \otimes S_h^{(y)} \equiv \{ \sum \varphi_i(x) \psi_j(y) \mid \varphi_i \in S_h^{(x)}, \psi_j \in S_h^{(y)} \}$$

is contained in $W^{m,n}(\Omega)$.

Proof: It suffices to note that:

(4.14) $$D_x^m(\varphi_i(x)\psi_j(y)) = (D_x^m \varphi_i(x))\,\psi_j(y) \quad ,$$

(4.15) $$D_y^n(\varphi_i(x)\psi_j(y)) = \varphi_i(x)\,(D_y^n \psi_j(y)) \quad ,$$

because if φ is in $L_2(0,T)$, ψ in $L_2(0,1)$, then:

(4.16) $$\iint_\Omega (\varphi(x)\psi(y))^2 dx\,dy = \left(\int_0^T (\varphi(x))^2 dx\right)\left(\int_0^1 (\psi(y))^2 dy\right),$$

so that $\varphi(x)\psi(y)$ is in $L_2(\Omega)$.

Prop. 2 The tensor product space:

(4.17) $$S_h \equiv S_h^{(x)} \otimes S_h^{(y)} \quad ,$$

as above, is contained in W if and only if the essential boundary conditions are satisfied by $S_h^{(x)}$ at x=0,T and by $S_h^{(y)}$ at y=0,1.

Proof: Let φ be in $S_h^{(x)}$ and ψ in $S_h^{(y)}$, not identically zero. Then we have:

(4.18) $$\tau_i^{(x)}(\varphi(x)\psi(y)) = (\tau_i \varphi)\,\psi(y) \quad ,$$

$$\tau_j^{(y)}(\varphi(x)\psi(y)) = (\tau_j\psi)\varphi(x)\,, \tag{4.19}$$

and the desired result follows easily.

Prop. 3 Let one-dimensional interpolation operators be given:

$$I_h^{(x)}: C_*^\infty[0,T] \to S_h^{(x)} \subseteq W_*^m(0,T)\,, \tag{4.20}$$

$$I_h^{(y)}: C_*^\infty[0,1] \to S_h^{(y)} \subseteq W_*^n(0,1)\,, \tag{4.21}$$

where "*" indicates the imposition of essential boundary conditions at x=0,T (resp. y=0,1) such that for all w in $C_*[0,T]$ (resp. $C_*[0,1]$):

$$\| w - I_h^{(x)} w \|_i \leq C h^r |w|_{i+r}\,, \tag{4.22}$$

$$\| w - I_h^{(y)} w \|_j \leq C h^s |w|_{j+s}\,. \tag{4.23}$$

Then the natural extensions of these given by:

$$I_h^x: C_*^\infty(\bar{\Omega}) \to S_h^x\,, \tag{4.24}$$

$$(I_h^x w)(x,y) \equiv (I_h^{(x)} w(\cdot,y))(x)\,, \tag{4.25}$$

$$I_h^y: C_*^\infty(\bar{\Omega}) \to S_h^y\,, \tag{4.26}$$

$$(I_h^y w)(x,y) \equiv (I_h^{(y)} w(x,\cdot))(y)\,, \tag{4.27}$$

will satisfy the corresponding estimates:

(4.28) $$\| w - I_h^x w \|_{i,0} \le C h^r |w|_{i+r,0} ,$$

(4.29) $$\| w - I_h^y w \|_{0,j} \le C h^s |w|_{0,j+s} ,$$

for all w in $C_*^\infty(\bar{\Omega})$.

Proof: Since w in $C_*^\infty(\bar{\Omega})$ implies that for fixed y in [0,1] we have w(·,y) in $C_*^\infty[0,T]$:

(4.30) $$\begin{aligned} \| w - I_h^x w \|_{i,0}^2 &= \int_0^1 \| w(\cdot,y) - I_h^{(x)} w(\cdot,y) \|_i^2 \, dy \\ &\le (C h^r)^2 \int_0^1 |w(\cdot,y)|_{i+r}^2 \, dy \\ &= (C h^r |w|_{i+r,0})^2 . \end{aligned}$$

This establishes (4.28), and (4.29) is similar.

We will assume the following commutativity properties, for all w in $C_*^\infty(\bar{\Omega})$:

(4.31) $$I_h^x D_y w = D_y I_h^x w ,$$

(4.32) $$I_h^y D_x w = D_x I_h^y w ,$$

(4.33) $$I_h^x I_h^y w = I_h^y I_h^x w .$$

Technically the compositions in (4.33) are not defined since S_h^x, S_h^y are not contained in $C_*^\infty(\bar{\Omega})$. However the properties (4.31), (4.32) imply that for w in $C_*^\infty(\bar{\Omega})$:

(4.34) $$I_h^x w(\cdot,y) \in C_*^\infty[0,T], \quad 0 \le y \le 1,$$

(4.35) $$I_h^y w(x,\cdot) \in C_*^\infty[0,1], \quad 0 \le x \le T.$$

Hence for fixed (x,y) in $\bar{\Omega}$ we define:

(4.36) $$(I_h^x I_h^y w)(x,y) \equiv (I_h^{(x)}(I_h^{(y)} w(\cdot,y))(x),$$

(4.37) $$(I_h^y I_h^x w)(x,y) \equiv (I_h^{(y)}(I_h^{(x)} w(x,\cdot))(y).$$

The property (4.33) is to be understood as the agreement of expressions (4.36), (4.37). We then define:

(4.38) $$I_h : C_*^\infty(\bar{\Omega}) \to S_h,$$

(4.39) $$I_h w \equiv I_h^x I_h^y w.$$

Prop. 4 Let S_h be the range of I_h. Then:

(4.40) $$S_h = S_h^{(x)} \otimes S_h^{(y)}.$$

Proof: Fix w in $C_*^\infty(\bar{\Omega})$. For any x in [0,T]:

$$(4.41) \qquad I_h^y w(x,\cdot) = \sum c_j(x)\, \psi_j \quad ,$$

where the summation is taken over a basis ψ_j of $S_h^{(y)}$. Hence we have:

$$(4.42) \qquad (I_h w)(x,y) = (I_h^x(\sum c_j \psi_j(y)))(x)$$

$$= \sum (I_h^{(x)} c_j)(x)\, \psi_j(y) \, ,$$

by substitution of (4.41) in (4.36). Note that by (4.34) and the proof of Prop. 2 above, each $c_j(\cdot)$ is in $C_*^\infty[0,T]$. It follows that the range of I_h is contained in the tensor product space. The reverse inclusion is deduced from the surjectivity of $I_h^{(x)}$, $I_h^{(y)}$. For example, suppose:

$$(4.43) \qquad (I_h^{(x)} f)(x) = \varphi_i(x) \quad ,$$

$$(4.44) \qquad (I_h^{(y)} g)(y) = \psi_j(y) \quad .$$

Then clearly (4.42) implies:

$$(4.45) \qquad I_h(f(x)g(y)) = \varphi_i(x)\, \psi_j(y) \quad ,$$

so that the range of I_h contains a basis for the tensor product space. Thus the spaces are equal.

Thm. 3 Suppose (4.22) holds for $r = 1$ when $i = 0,m$ and for $r = 0$ when $i = m$, and that (4.23) holds for $s = 1$ when $j = 0,n$ and for $s = 0$ when $j = n$. Then the interpolation operator I_h satisfies:

$$(4.46) \qquad \|w - I_h w\|_{m,n} \leq Ch\left(|D_x^m w|_{1,1} + |D_y^n w|_{1,1}\right),$$

for all w in $C_*^\infty(\bar{\Omega})$.

Proof: We consider the term:

$$(4.47) \qquad \|w - I_h w\|_{m,0} \leq \|w - I_h^x w\|_{m,0} + \|w - I_h^y w\|_{m,0}$$

$$+ \|(w - I_h^y w) - I_h^x(w - I_h^y w)\|_{m,0}.$$

The first term in the right-hand side of (4.47) is estimated by (4.28) with $r = 1$, $i = m$:

$$(4.48) \qquad \|w - I_h^x w\|_{m,0} \leq Ch|w|_{m+1,0}$$

$$= Ch|D_x^m w|_{1,0} .$$

Concerning the second term in the right-hand side of (4.47), we have by a standard inequality:

$$(4.49) \qquad \|w - I_h^y w\|_{m,0} \leq \|w - I_h^y w\|_0 + \|D_x^m(w - I_h^y w)\|_0 .$$

The first term of the right-hand side of (4.49) can be estimated by taking (4.29) with s = 1 , j = n :

(4.50) $$\|w - I_h^y w\|_{0,n} \leq Ch|w|_{0,n+1}$$

$$= Ch|D_y^n w|_{0,1} .$$

To estimate the second term in the right-hand side of (4.49), we use (4.29) with s = 1 , j = 0 :

(4.51) $$\|D_x^m(w - I_h^y w)\|_0 = \|D_x^m w - I_h^y D_x^m w\|_0$$

$$\leq Ch|D_x^m w|_{0,1} .$$

It remains to estimate the third term of the right-hand side of (4.47) by applying (4.28) with r = 0 , i = m :

(4.52) $$\|(w - I_h^y w) - I_h^x(w - I_h^y w)\|_{m,0} \leq C\|w - I_h^y w\|_{m,0} .$$

Combining this with (4.49) above, which is in turn estimated by (4.50), (4.51), we have established:

(4.53) $$\|w - I_h w\|_{m,0} \leq Ch(|D_x^m w|_{1,1} + |D_y^n w|_{1,1}) .$$

Similarly, one can estimate:

(4.54) $$\|w - I_h w\|_{0,n} \leq Ch(|D_x^m w|_{1,1} + |D_y^n w|_{1,1}) .$$

We have thus shown the existence of a finite element subspace S_h as in (4.1), (4.2) where:

$$V \equiv \{ w \in W_*^{m+1,n+1}(\Omega) \mid D_{xy} w \in W^{m-1,n-1}(\Omega) \}, \tag{4.55}$$

which is a Hilbert space with the graph norm:

$$\|w\|_V^2 \equiv \|w\|_{m+1,n+1}^2 + \|D_{xy} w\|_{m-1,n-1}^2 . \tag{4.56}$$

Since m,n are at least 1, we have the proper continuous imbeddings needed to apply the earlier results of this section:

$$U \subset V \subset W . \tag{4.57}$$

In the following section we will use this construction with Lagrange interpolants, whose value depends only on the value of the function to be interpolated at certain points. In that case (4.33) is automatic because both underlying one-dimensional interpolants leave the value of the function unchanged at the points where it is to be evaluated.

For the remainder of this section we will consider a peculiarity of the tensor product construction which may make a contribution to the approximation of mixed derivatives for anisotropic problems. We will assume that m is greater than n , and that $S_h^{(y)}$ is contained in $W^m(0,1)$. This is the case, for example, if the same or similar underlying interpolation operators are used in both directions.

Def. With the same notation as above:

(4.58) $$S_h^j \equiv S_h^{(x)} \otimes \{D_y^j w \mid w \in S_h^{(y)}\}, \ 1 \leq j \leq m-n,$$

(4.59) $$T_h^j \equiv S_h^{(x)} \otimes \{w \mid D_y^j w \in S_h^{(y)}\}, \ 1 \leq j \leq m-n.$$

Prop. 5 For $j = 1,\ldots,m-n$, $S_h{}^j$, $T_h{}^j$ are contained in $W^{m,n}(\Omega)$. Moreover $S_h{}^j$, $T_h{}^j$ satisfy essential boundary conditions at $x=0,T$ that make L coercive on these finite-dimensional spaces.

Proof: Since by assumption $S_h^{(y)}$ is contained in $W^m(0,1)$, we have:

(4.60) $$\{D_y^j w \mid w \in S_h^{(y)}\} \subseteq W^{m-j}(0,1) \ ,$$

(4.61) $$\{w \mid D_y^j w \in S_h^{(y)}\} \subseteq W^{m+j}(0,1) \ .$$

But for $j = 1,\ldots,m-n$ these are contained in $W^n(0,1)$, so the first assertion of this Prop. follows from Prop. 1. The second assertion follows by noting that the alternative (3.8) is imposed on $S_h^{(x)}$.

Because of the above result it is credible that the derivatives with respect to y of the solution of the discrete variational problem will themselves satisfy some auxilary discrete variational problems, analogous to the auxilary continuous variational problem introduced in Section III. We conjecture that the use of the tensor product construction will lead in some circumstances to an auxilary Galerkin estimate for the error of approximation to corresponding derivatives of the solution with respect to y :

(4.62) $$\| D_y^j (u - u_h) \|_{m,n} \le C h \| u \|_k , \quad 1 \le j \le m-n ,$$

where k is a sufficiently large integer. We shall see an example in the next section where the mixed derivatives estimated by (4.62) are of physical interest.

V. THE ONSAGER PANCAKE APPROXIMATION

Centrifuges are often used to separate materials according to density through a process of radial diffusion. A gas centrifuge, used to separate isotopes of uranium, presents a somewhat different case because at the high speeds at which they rotate, almost all of the material (uranium hexaflouride) is confined to a thin, flat region against the outer radial wall. Practical enrichment design is based on development of a countercurrent in this layer, which induces a top to bottom separation effect (we assume a vertical axis of rotation) that makes continuous operation feasible.

The Onsager pancake approximation [19] models axially symmetric flow of a viscous compressible fluid rotating almost rigidly at a nearly constant radius. This should be understood in the sense that the non-dimensionalized radius η is set equal to unity wherever it appears algebraically, i.e. not as the argument of the solution. In some respects the pancake approximation resembles the multiple boundary layers of Stewartson [18].

Historically the methods of asymptotic expansions were used to solve the pancake equation by matching boundary conditions with combinations of the relevant eigenfunctions. We shall see, however, that the application of finite elements to this problem is fortuitous. Our treatment follows that of Gunzburger and Wood [4].

Onsager's pancake approximation, developed for the AEC in the early sixties and only recently declassified, involves a single partial differential equation:

(5.1) $$Au \equiv (e^x(e^x u_{xx})_{xx})_{xx} + au_{yy} = f(x,y),$$

where the constant a is positive, and boundary conditions:

(5.2) $$u_x = u_{xx} = 0 \quad \text{at} \quad x = 0\,,$$

(5.3) $$u = u_x = 0 \quad \text{at} \quad x = T\,,$$

(5.4) $$(e^x(e^x u_{xx})_{xx})_x = g(y) \quad \text{at} \quad x = 0\,,$$

(5.5) $$(e^x u_{xx})_x = 0 \quad \text{at} \quad x = T,$$

(5.6) $$-au_y = d(e^{x/2}u_x)_x + \gamma_0(x) \quad \text{at} \quad y = 0,$$

(5.7) $$au_y = d(e^{x/2}u_x)_x + \gamma_1(x) \quad \text{at} \quad y = 1,$$

where the constant d is positive. Before examining the aspects of the pancake approximation which differ from the problems studied in Sections III and IV, we discuss the physical interpretation of (5.1)-(5.7). A derivation of the problem is given in [19].

We assume the countercurrent is a small axially symmetric perturbation of isothermal rigid rotation of a perfect gas. This reference solution, whose pressure is given by:

(5.8) $$p(\eta) = e^{-c^2(1-\eta^2)}p(1)\,,$$

where C is a constant proportional to the rate of rotation, is used to linearize the equations of conservation and state. We see from (5.8) that for large values of C the density drops to near zero at a short distance from the radial wall. Strictly speaking continuum mechanics does not apply to the rarefied gas in the interior of the centrifuge, so we choose a cutoff T large enough to simulate the 'top of the atmosphere.' Actually, the variable x is a transformed radius:

$$x \equiv C^2(1-\eta^2) \tag{5.9}$$

so that x=0 is the radial wall. Axial distances are represented linearly by the variable y.

The solution u is called the master potential, and quantities of physical interest are related to its derivatives. For example, $D_x u$ is essentially the streamfunction of the perturbation. Boundary conditions at y=0,1 are derived from Ekman layers along the endplates. The function g in (5.4) gives the temperature gradient along the outer radial wall. The term f in (5.1) represents internal sources of mass, energy, momentum, etc. Frequently f is taken to be zero, and the solution u is then driven by the boundary conditions. Convection, as induced by g, and mass fluxes through the endplates, which can be modeled by the Ekman layers, are two of many ways in which a countercurrent can be produced.

The pancake approximation modifies in two ways the type of problem considered previously. First, the variable coefficients play a slightly different role in the partial differential operator A. This amounts to the addition of some lower order terms, and indeed the variational formulation will be coercive without resort to a Garding-type inequality. Therefore this difference is benign. Second, the boundary conditions at y=0,1 are oblique. That is, besides the normal derivative $D_y u$, which by itself would constitute a natural boundary condition, there appears a term representing tangential derivatives. This is handled by the inclusion of a boundary integral in the bilinear form on which the variational principle is based.

Def. Let $L(\cdot,\cdot)$ be the bilinear form:

$$L(w,v) \equiv ((e^x w_{xx})_x, (e^x v_{xx})_x) + (a w_y, v_y) + (d e^{x/2} w_x, v_x)_y, \tag{5.10}$$

where:

$$(w,v)_y = \int_0^T (wv|_{y=0} + wv|_{y=1})\,dx. \tag{5.11}$$

Prop. 1 Let $W = W^{3,1}_*(\Omega)$ be the closed subspace of $W^{3,1}(\Omega)$ defined by the essential boundary conditions (5.2), (5.3). Then L is bounded and coercive on WxW.

Proof: The surface integrals are clearly bounded on WxW, so we need only show that the boundary integral is. But by the trace theorem (Thm. I.2) we know that for w in W:

$$w|_{y=0,1} \in W^{3/2}((0,T)\times\{0,1\}), \tag{5.12}$$

(5.13) $$w_x\big|_{y=0,1} \in W^{1/2}((0,T)\times\{0,1\}) .$$

This verifies that the boundary integral term is bounded.

The coercivity of L on WxW is proved almost as in Prop. III.3. If w is in W, then $e^x w_{xx} = 0$ at x=0. Hence by Lemma III.1:

(5.14) $$\|w_{xx}\|_0 \le \|e^x w_{xx}\|_0 \le C\,\|(e^x w_{xx})_x\|_0 .$$

But by the triangle inequality:

(5.15) $$\begin{aligned}\|w_{xxx}\|_0 &\le \|e^x w_{xxx}\|_0 \\ &\le \|(e^x w_{xx})_x\|_0 + \|e^x w_{xx}\|_0 \\ &\le C\ \|(e^x w_{xx})_x\|_0 .\end{aligned}$$

Thus we can apply (3.14) to get:

(5.16) $$\begin{aligned}C\,L(w,w) &\ge C\,(\|(e^x w_{xx})_x\|_0^2 + a\|w_y\|_0) \\ &\ge \|w_{xxx}\|_0^2 + \|w_y\|_0^2 \\ &\ge c\,\|w\|_{3,1}^2 \text{ for } w \in W.\end{aligned}$$

Since the boundary integral is positive definite, it cannot offend.

Prop. 2 There exists unique u in W such that:

$$L(u,v) = -(f,v) - \int_0^1 gv|_{x=0}\,dy + (\gamma,v)_y\,, \tag{5.17}$$

for all v in W, provided the right-hand side denotes a bounded linear functional on W. If f is in $L_2(\Omega)$, such u solves the corresponding problem (5.1)-(5.7).

Proof: The first assertion is immediate from Thm. II.1 and Prop. 1 above. The second assertion follows from the development of the integration by parts formula:

$$\begin{aligned}(Aw,v) &= -((e^x(e^x w_{xx})_{xx})_x, v_x) + \langle (e^x(e^x w_{xx})_{xx})_x, v\rangle_x \\ &\quad - (aw_y, v_y) + \langle aw_y, v\rangle_y \\ &= (e^x(e^x w_{xx})_{xx}, v_{xx}) - \langle e^x(e^x w_{xx})_{xx}, v_x\rangle_x \\ &\quad + \langle (e^x(e^x w_{xx})_{xx})_x, v\rangle_x \\ &\quad - (aw_y, v_y) + \langle aw_y, v\rangle_y \\ &= -((e^x w_{xx})_x, (e^x v_{xx})_x) + \langle (e^x w_{xx})_x, e^x v_{xx}\rangle_x \\ &\quad - \langle e^x(e^x w_{xx})_{xx}, v_x\rangle_x + \langle (e^x(e^x w_{xx})_{xx})_x, v\rangle_x \\ &\quad - (aw_y, v_y) + \langle aw_y, v\rangle_y\,.\end{aligned} \tag{5.18}$$

valid for w in $W^{3,1}(\Omega;A)$ and v in $W^{3,1}(\Omega)$. Since for v in $C_0^\infty(\Omega)$:

$$(5.19)\qquad (Au,v) = -L(u,v) = (f,v)\ ,$$

we know that (5.1) holds and that u is in $W^{3,1}_*(\Omega;A)$. For v in W:

$$(5.20)\qquad (f,v) = -L(u,v) + (de^{x/2}u_x, v_x)_y - \int_0^1 (e^x(e^x u_{xx})_{xx})_x v\big|_{x=0}\,dy + \int_0^1 (e^x u_{xx})_x (e^x v_{xx})\big|_{x=T}\,dy + \langle au_y, v\rangle_y .$$

Comparison of (5.20) with (5.17) yields the following:

$$(5.21)\qquad \int_0^1 (e^x(e^x u_{xx})_{xx})_x v\big|_{x=0}\,dy = \int_0^1 gv\big|_{x=0}\,dy\ ,$$

$$(5.22)\qquad \int_0^1 (e^x u_{xx})_x (e^x v_{xx})\big|_{x=T}\,dy = 0\ ,$$

$$(5.23)\qquad \langle au_y, v\rangle_y + (de^{x/2}u_x, v_x)_y = (\gamma, v)_y\ .$$

The conditions (5.21), (5.22) imply (5.4), (5.5) respectively. Since u_x is zero at each corner, (5.23) implies:

$$(5.24)\qquad \langle au_y, v\rangle_y - (d(e^{x/2}u_x)_x, v)_y = (\gamma, v)_y\ .$$

Since the boundary integrals at y=0, y=1 are independent, we have:

(5.25) $$-au_y - d(e^{x/2}u_x)_x = \gamma_0 \text{ at } y=0,$$

(5.26) $$au_y - d(e^{x/2}u_x)_x = \gamma_1 \text{ at } y=1,$$

which verifies (5.6), (5.7).

Thm. 1 If f is in $L_2(\Omega)$ and g, γ_0, γ_1 are zero, then u given by (5.17) is in $W^{6,2}(\Omega)$ and:

(5.27) $$c\|u\|_{6,2} \leq \|f\|_0 \leq C\|u\|_{6,2} .$$

Proof: One proceeds as in Thm. III.2, with the introduction of an auxilary problem satisfied by u_y. We treat the oblique boundary conditions (5.6), (5.7) as nonhomogeneous essential boundary conditions for u_y, and in order for u_y to be in $W^{3,1}(\Omega)$ we require:

(5.28) $$u_{xx}|_{y=0,1} \in W^{3/2}(0,T),$$

so that the right-hand sides of (5.6), (5.7) will have the proper smoothness. We can find as in [11, Thm. 3 of Chap. 4] a function z in W satisfying:

(5.29) $$-az|_{y=0} = d(e^{x/2}u_x)_x ,$$

(5.30) $$az|_{y=1} = d(e^{x/2}u_x)_x ,$$

(5.31) $$\|z\|_{3,1} \leq C \, \|u_{xx}|_{y=0,1}\|_{3/2} \, .$$

Then by translation of u_y by z one obtains the estimate:

(5.32) $$\|u\|_{6,2} \leq C(\|f\|_0 + \|u_{xx}|_{y=0,1}\|_{3/2}) \, .$$

We now use fractional anisotropic Sobolev spaces to eliminate the unwanted term by interpolation of norms. The trace theorem (Thm. I.2) implies:

(5.33) $$\|u_{xx}|_{y=0,1}\|_{3/2} \leq C \, \|u\|_{5\frac{1}{2},1\frac{2}{3}} \quad ,$$

(5.34) $$\|u\|_{6,2} \leq C(\|f\|_0 + \|u\|_{5\frac{1}{2},1\frac{2}{3}}) \, .$$

Let the positive constant C be as in (5.34) for the remainder of this proof. By the inequalities of Prop. I.1 we can find positive constants ϵ,c such that:

(5.35) $$\|u\|_{5\frac{1}{2},1\frac{2}{3}} \leq c\,\|u\|_0 + \epsilon\,\|u\|_{6,2}$$

(5.36) $$\epsilon C < 1/2$$

By combining (5.34), (5.35) we have finally:

(5.37) $$1/2 \, \|u\|_{6,2} \leq (c+1)\,C\,\|f\|_0 \, .$$

This proves the first half of (5.27). The second half follows from the boundedness of A as a map from $W^{6,2}(\Omega)$ to $L_2(\Omega)$.

We consider two conforming finite element spaces for this problem. First we take the tensor product of cubic B-splines in x and piecewise linear functions in y. Second we use cubic B-splines in both directions. Each one-dimensional spline space admits a Lagrange interpolant. Consequently the interpolation operators commute with each other and with derivatives as in (4.31)-(4.33).

We note the following approximation properties of these spline spaces, assuming a regular subdivision of the interval [1, Chap. 4]:

$$\inf_{w_h \in S_h} \|w - w_h\|_i \le C h^s |w|_{i+s} \tag{5.38}$$

where for cubic B-splines we may take:

$$0 < i+s \le 4, \quad i = 0,1,2,3, \tag{5.39}$$

and for piecewise linear functions:

$$0 < i+s \le 2, \quad i = 0,1. \tag{5.40}$$

Thus for either tensor product space we have by Thm. IV.3 the sharp estimate:

$$\inf_{w_h \in S_h} \|w - w_h\|_{3,1} \le C h (|D_x^3 w|_{1,1} + |D_y w|_{1,1}). \tag{5.41}$$

Thm. 2 For either choice of a tensor product finite element space, we have the following error estimates of the Galerkin approximation:

$$\|u-u_h\|_{3,1} \leq Ch(|D_x^3 u|_{1,1} + |D_y u|_{1,1}), \tag{5.42}$$

$$\|u-u_h\|_0 \leq Ch^2(|D_x^3 u|_{1,1} + |D_y u|_{1,1}). \tag{5.43}$$

Proof: As in Thm. IV.1 and Thm. IV.2.

Cor. 1 For the case of homogeneous boundary conditions and f in $L_2(\Omega)$, we have:

$$\|u-u_h\|_{3,1} \leq Ch\|f\|_0 , \tag{5.44}$$

$$\|u-u_h\|_0 \leq Ch^2\|f\|_0 . \tag{5.45}$$

Proof: As in Cor. IV.1 and Cor. IV.2.

The reason that the same rate of convergence appears for both of the tensor product spaces is that a limiting factor in both cases is the approximation of $D_x{}^3 u$. We now describe some numerical experiments. A choice of basis for one of these spaces, say $\{\varphi_i\}_{i=1}^N$, leads to the following formulation of the discrete variational problem as a linear system:

$$M\vec{U} = \vec{K} , \tag{5.46}$$

where M is an NxN matrix defined by:

$$M_{ij} \equiv L(\varphi_i, \varphi_j) \quad , \tag{5.47}$$

where $\vec{K}$ is a vector of length N:

$$K_i \equiv (-f, \varphi_i) - \int_0^1 g\varphi_i|_{x=0}\, dy + (\gamma, \varphi_i)_y \quad , \tag{5.48}$$

and where $\vec{U}$ is a vector of length N representing the coefficients of the basis functions that occur in the discrete solution:

$$u_h = \sum_{i=1}^{N} U_i \varphi_i \quad . \tag{5.49}$$

Two computer programs were written in Fortran, one for each choice of tensor product finite element space. The integrals in (5.47), (5.48) were evaluated using a four-point Gaussian quadrature scheme in one dimension at a time. In solving (5.46) we took advantage of its special properties: banded, symmetric, and positive definite. All computations were performed in double precision on an IBM 370 at the University of Tennessee Computing Center.

A problem artificial in nature but having an exact closed form solution was used to make the first experiments. The L_2-norm of the error of approximation for several derivatives of the solution was estimated using the same four-point quadrature scheme. The results, expressed as a percentage of the relevant quantity, are summarized in

Tables 1 and 2. Note that in both cases the convergence is not limited to terms estimated by the natural norm of the problem. Thus we have computational evidence of the conjecture made in Section IV.

A further experiment with the cubic-cubic test functions is summarized in Table 3. Here we took f, to be zero and set g equal to one. This is similar to problems of physical interest and corresponds to the case of a countercurrent driven by a linear temperature profile along the outside wall. Since no exact closed form solution is available for this problem, the rates of convergence shown were estimated from the change in the L_2-norm of the discrete solution and various derivatives caused by a refinement of the mesh. Although the higher derivatives of the discrete solution seem not to converge at all, the behavior of the lower order terms is not much worse than for the case with exact solution.

We close this thesis with a brief mention of some topics for further investigation. First, we would like to have more elaborate regularity results for the continuous problem in order to apply the technique of Aubin-Nitsche for higher order derivatives. Second, we may ask if inverse estimates, which dominate higher order seminorms by lower order ones times negative powers of h, can be inferred for the tensor product spaces.

TABLE 1. Cubic-Linear Splines with Exact Solution[a]

	Quantity Observed (Normalized Error)					
	e_h	$D_x e_h$	$D_x^2 e_h$	$D_y e_h$	$D_{xy} e_h$	$D_{xxy} e_h$
Mesh Size						
11x11	13.78%	10.50%	8.00%	17.13%	13.63%	11.76%
13x13	9.61%	7.33%	5.60%	12.73%	10.40%	9.20%
16x16	6.17%	4.71%	3.60%	9.01%	7.62%	6.94%
Accuracy[b]	2.13	2.13	2.13	1.67	1.50	1.36

a $u = x^3(x-1)^2(x-1.3)\cos(\ \ y)$, $e_h = u - u_h$, $T = 1$

b Estimated accuracy = $\log(e_{11}/e_{16})/\log(h_{11}/h_{16})$

TABLE 2. Cubic-Cubic Splines with Exact Solution[c]

	Quantity Observed (Normalized Error)							
	e_h	$D_x e_h$	$D_x^2 e_h$	$D_y e_h$	$D_{xy} e_h$	$D_{xxy} e_h$	$D_{yy} e_h$	$D_{xy}^2 e_h$
Mesh Size								
11x11	8.38%	8.32%	7.43%	10.20%	10.96%	9.93%	49.50%	62.70%
13x13	5.82%	5.79%	5.19%	7.13%	7.72%	7.00%	38.42%	48.92%
16x16	3.73%	3.72%	3.34%	4.59%	4.99%	4.52%	27.17%	34.82%
Accuracy[d]	2.16	2.15	2.13	2.13	2.10	2.10	1.60	1.57

c $u = x^3(x-1)^2(x-1.3)\cos(\ \ y)$, $e_h = u - u_h$, $T = 1$

d Estimated accuracy = $\log(e_{11}/e_{16})/\log(h_{11}/h_{16})$

TABLE 3. Cubic-Cubic Splines without Exact Solution[e]

	Quantity Observed(Discrete Solution x 10^4)							
	u_h	$D_x u_h$	$D_x^2 u_h$	$D_y u_h$	$D_{xy} u_h$	$D_{xxy} u_h$	$D_{yy} u_h$	$D_{xy}^2 u_h$
Mesh Size								
8x8	5.913	10.73	35.54	5.236	10.70	42.00	30.72	605.4
11x11	6.140	11.17	36.71	5.476	11.60	49.40	37.36	1100.
16x16	6.263	11.40	37.34	5.610	12.31	57.78	47.94	2048.
Accuracy[f]	1.93	2.04	1.95	1.83	0.75	-0.39	-1.37	-2.04

e Solution driven by g = 1, T = 1, z = L_2- norm of quantity

f Estimated accuracy = $\log((z_{11}-z_8)/(z_{16}-z_{11})/\log(h_8/h_{11})$

LIST OF REFERENCES

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1. Aubin, Jean-Pierre, Approximation of Elliptic Boundary-Value Problems, New York: Wiley-Interscience, 1972

2. Aziz, A. K. (ed.), The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, New York: Academic Press, 1972

3. Gilbarg, David and Neil S. Trudinger, Elliptic Partial Differential Equations of Second Order, New York: Springer-Verlag, 1977

4. Gunzburger, Max D. and Houston G. Wood III, "Finite Element Methods for the Onsager Pancake Equation," (to appear)

5. King, J. Thomas, Least Squares Methods for Parabolic Initial-Boundary Value Problems, doctoral thesis, Cornell University, 1971

6. King, J. Thomas, "The Approximate Solution of Parabolic Initial Boundary Value Problems by Weighted Least Squares Methods," SIAM Journal of Numerical Analysis, v. 9, p. 215-229 (1972)

7. Kudrjavcev, L. D., Direct and Inverse Imbedding Theorems, Providence, R.I.: American Mathematical Society, 1974

8. Kudrjavcev, L. D. and S. M. Nikol'skii, "Imbedding Theorems and their Application to Solving Elliptic Equations," American Mathematical Society Translations, Series 2, v. 86, p. 199-228 (1970)

9. Kufner, Alois et al, Function Spaces, Prague: Noordhoff, 1977

10. Lions, Jacques-Louis and Enrico Magenes, Non-Homogeneous Boundary Value Problems and Applications, vol. I, New York: Springer-Verlag, 1972

11. Lions, Jacques-Louis and Enrico Magenes, Non-Homogeneous Boundary Value Problems and Applications, vol. II, New York: Springer-Verlag, 1972

12. Nikol'skii, S. M., "Boundary Properties of Functions Defined on a Region with Angular Points I," American Mathematical Society Translations, Series 2, v. 83, p. 101-120 (1969)

13. Nikol'skii, S. M., "Boundary Properties of Functions Defined on a Region with Angular Points II," American Mathematical Society Translations, Series 2, v. 83, p. 121-142 (1969)

14. Nikol'skii, S. M., "Boundary Properties of Functions Defined on a Region with Angular Points III," American Mathematical Society Translations, Series 2, v. 83, p. 143-158 (1969)

15. Nikol'skii, S. M., "Extension of Functions of Several Variables Preserving Differential Properties," American Mathematical Society Translations, Series 2, v. 83, p. 159-188 (1969)

16. Nikol'skii, S. M., Approximation of Functions of Several Variables and Imbedding Theorems, New York: Springer-Verlag, 1975

17. Sobolev, S. L., "On a Boundary Value Problem for Polyharmonic Equations," American Mathematical Society Translations, Series 2, v. 33, p. 1-40 (1963)

18. Stewartson, K., "On Almost Rigid Rotations," Journal of Fluid Mechanics, v. 3, p. 17-26 (1957)

19. Wood, Houston G. III and J. B. Morton, "Onsager's Pancake Approximation for the Fluid Dynamics of a Gas Centrifuge," Journal of Fluid Mechanics, v. 101, p. 1-31 (1980)

20. Yosida, Kosaku, Functional Analysis, fifth ed., New York: Springer-Verlag, 1978

VITA

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