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I am submitting herewith a dissertation written by Robert Wayne Rice, entitled "A Study of Rain Clutter Cancellation in a Monostatic Radar Using Circular Polarization." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Electrical Engineering.

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A STUDY OF RAIN CLUTTER CANCELLATION IN A MONOSTATIC RADAR USING CIRCULAR POLARIZATION

A Dissertation

Presented for the

Doctor of Philosophy

Degree

The University of Tennessee

Robert Wayne Rice

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ABSTRACT

The purpose of this investigation was to evaluate the performance of a rain clutter cancellation system which is a modified version of the traditional circularly polarized radar system. Determination of an accurate model for the rain particles was the first step in the analysis, and then the electromagnetic scattering parameters were determined for the model, an oblate spheroid.

Next certain assumptions were made relative to the statistical nature of the rain's spatial characteristics. These assumptions together with the equations for the scattered returns allowed the output noise power of the radar receiver to be evaluated in a statistical sense.

The result of the analysis was a set of curves showing the clutter suppression ability of the cancellation system relative to a linearly polarized monostatic system. Improvements in clutter cancellation on the order of 15 dB appear to be possible under a wide range of conditions; however, it should be noted that the cancellation system is found to be inferior to the ordinary circularly polarized system unless the latter experiences significant imbalances.

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1. INTRODUCTION

1.1. Historical Development

In spite of improved performance due to advances in detection methods, low-noise receivers, waveform optimization, etc., the modern radar system often gives a much lower level of performance because of rain clutter [1].

Early radar systems were generally designed to emit and receive linearly polarized radiation. There is nothing inherent in the structure of the linearly polarized system to suppress the clutter caused by precipitation. Recognition of this problem prompted investigation into different types of systems. In 1947 Ridenour [2] proposed a significant new concept for the basic radar system. Ridenour noted that if rain particles could be correctly modeled as spheres, then the use of circular polarization on both the transmitting and receiving antennas offered the possibility of significant clutter reduction. In the same discussion Ridenour was also able to offer experimental data that indicated that clutter reduction on the order of 26 dB was possible using circular polarization. Additionally Ridenour was able to point out that two limitations of this approach to clutter cancellation are rain particles which are not true spheres and radiation which is not exactly circular. In spite of the advantages offered by the use of circular polarization, it was not immediately adopted. In fact, circular polarization was rediscovered in 1954 by White [3] while at

the same time Hunter [4] reemphasized the limitations of the technique due to radiation which was not perfectly circular.

A necessary step in the process of correcting the errors introduced by noncircular polarization and nonspherical scatterers was that of obtaining an accurate model of the scattering particles and then obtaining a solution to the scattering of electromagnetic waves by an object whose shape is that of the model. From the literature one recognizes that interest in modeling rain particles preceded the development of radar technology by many years. A recent paper by Pruppacher and Beard [5] offers a review of the efforts along this line while another paper by Pruppacher and Pitter [6] claims to offer the most accurate model yet available for rain. In simple terms, Pruppacher and Pitter claim that the oblate spheroid is a good approximation to the true shape of rain particles in many cases. Additional detail concerning particle shape will be discussed in Section 3.

The scattering of electromagnetic waves by bodies of various shapes is a problem that has also received considerable attention for many years. Logan [7] notes that some of the earliest work done in this area involved scattering by a sphere and was done by Clebsch in 1861. In the same article, Logan offers an excellent outline of the historical development of the theory of scattering by spheres. Another paper deserving mention at this point is a survey of scattering literature by Corriher and Pyron [8]. It should be noted, however, that none of the sources referenced by the two previously mentioned works considers the problem of scattering by an oblate spheroid. This

problem was apparently first solved by Labrum [9] in 1952, and later Atlas, Kerker, and Hitschfield [10] offered concurring results.

1.2. Description of the Proposed Method

This work is an analysis of the system depicted in Figure 1.2-1. In the block diagram, the transmitter is represented in the left-hand portion of the figure, the propagating medium and scattering elements are shown in the center of the figure, and the receiver is depicted in the right-hand portion of the diagram.

The features which distinguish this system from the ordinary circular system are indicated in the block diagram as elements T, R, and G. Elements T and R account for the magnitude and phase imperfections which prevent the ordinary system from being truly circularly polarized. G is a dynamic gain element which has been added in the effort to reduce the performance limitations of the ordinary system caused by nonspherical rain and noncircular polarization. A detailed description of G will be given in Section 4.

Two propagation paths representing the orthogonal spatial components of the transmitted and received fields connect the transmitter and receiver. Each of the field components incident upon a clutter particle produces two components upon scattering. The reflected field is thus composed of two orthogonal components which are in turn comprised of two parts.

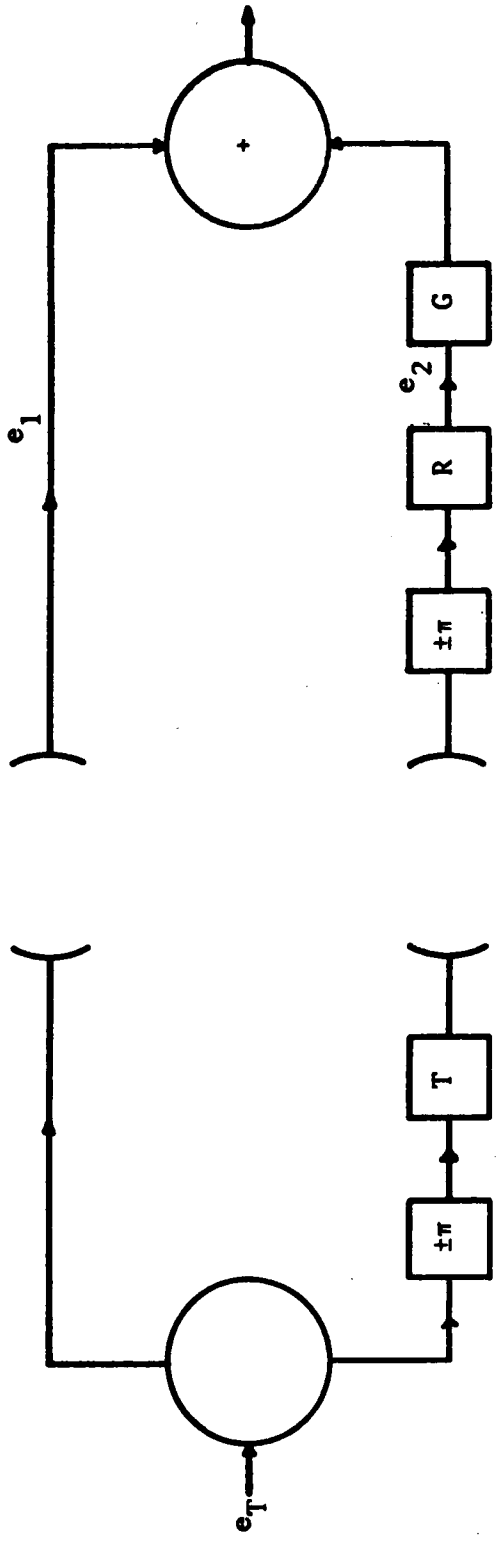


Figure 1.2-1. The clutter cancellation system.

2. BACKGROUND THEORY AND ASSUMPTIONS

2.1. Basic Assumptions

Prior to detailed development of the concept which gives rain clutter cancellation it is helpful, indeed necessary, to first set down certain definitions, assumptions and preliminary theory applicable to the whole work [1]. For example, to avoid confusion we shall use the work "signal" to imply either a waveform resulting from backscatter from a target or waveforms preceding target or rain illumination while the work "clutter" will apply to waveforms resulting from rain backscatter. Other definitions are made clear in the course of developments.

We make the following basic assumptions throughout the effort.

1. Only rain clutter is present. Thus, noise, ground clutter, multipath and other interference sources are neglected. This assumption is approximately satisfied for a low-noise, narrow-beam system directed several beamwidths above the horizon.
2. Only a single target exists within the radar beam.
3. The target behaves as a point reflector; that is, the target size is small relative to the pulse length in space.
4. There are no multiple reflections between rain particles or between the rain and target.
5. The radar is a monostatic system. Generalization to the bistatic system is relatively straightforward, and some of the

developments are formulated in such a manner prior to specialization to the monostatic case.

6. The system is capable of transmitting and receiving two linearly polarized waves that are approximately space-orthogonal.
7. The antenna patterns have narrow main lobes; that is, beam-widths are on the order of a few degrees or less in any plane containing the boresight axis.
8. Antenna pattern sidelobes fall off with angle such that the dominant contribution to the rain clutter power comes from angles sufficiently near the direction of the mainlobe maximum that small-angle approximations may be made.

2.2. Coordinate Definitions

Figures 2.2-1 and 2.2-2 define the coordinate system which interfaces the radar with its environment. The radar is at the origin of the X, Y, Z system and radiates in the Z, boresight, direction. Rain particles are assumed to fall, on the average, along a line parallel to O-x. This line is defined by angles ζ, called the rain heading, and ψ, called the rain fall angle. With these definitions the average wind vector lies in the O-x-Y plane; it also lies in the X-Z plane if the boresight axis is horizontal and the wind is entirely horizontal. In general, even for horizontal wind, the wind vector is tilted in the O-x-Y plane when the boresight axis is elevated.

The above definitions apply to the fall geometry for raindrops, which is determined by the wind. Locations of the drops are defined

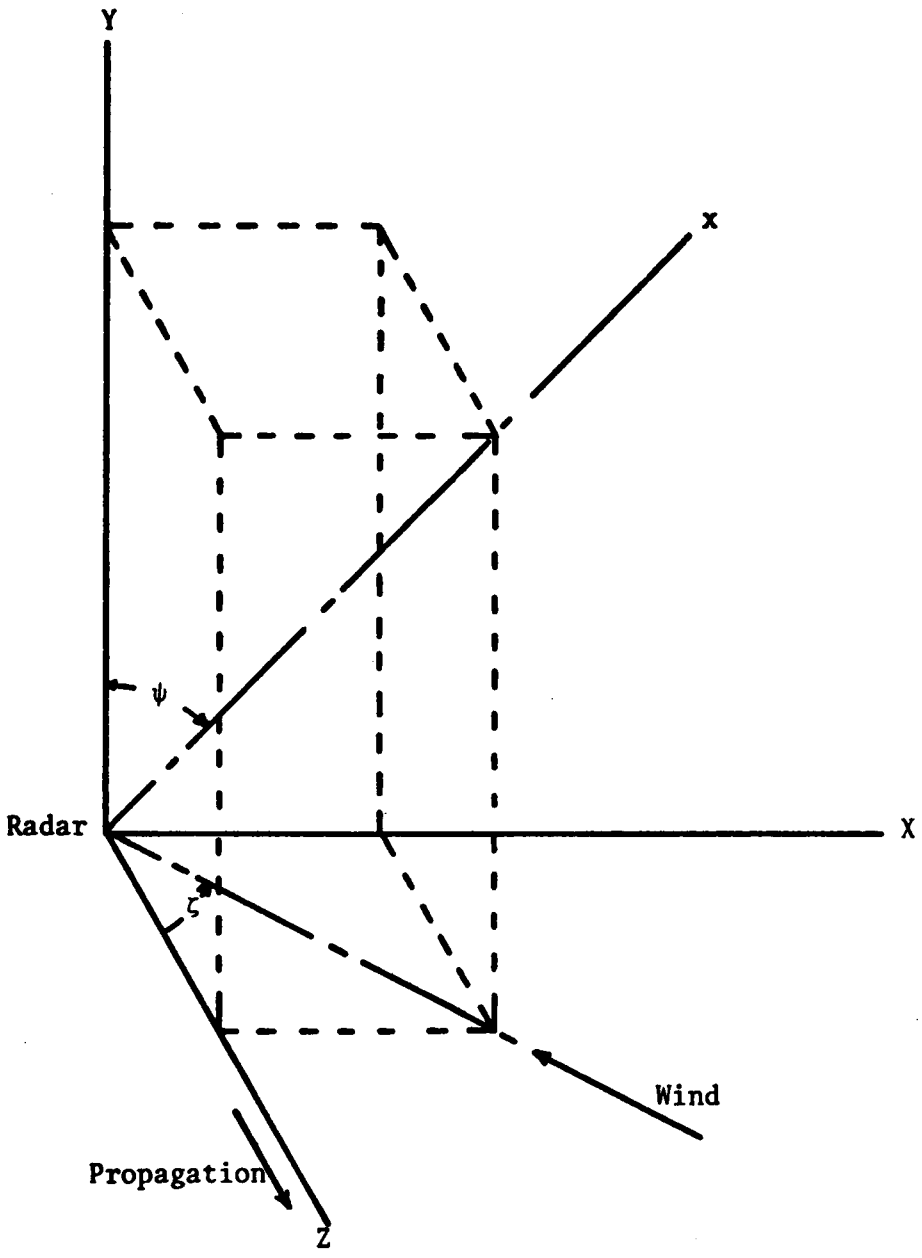


Figure 2.2-1. The coordinate system.

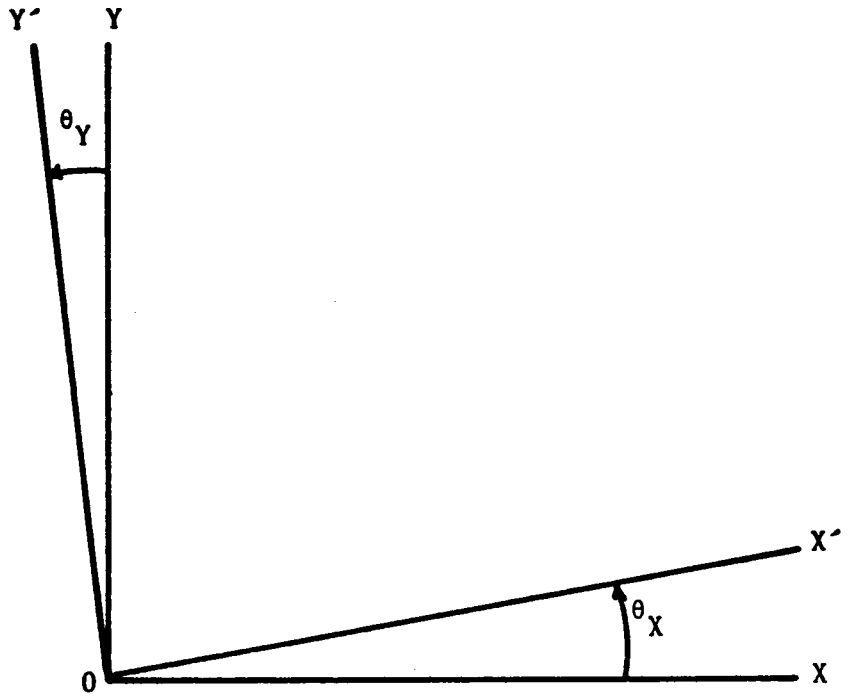


Figure 2.2-2. The geometry for nonorthogonal polarizations.

by spherical coordinates, radar-centered. The coordinates for the i^{th} drop, α_i and β_i, are analogous to ζ and ψ.

2.3. Scattering Theory

It is desirable to be able to write an equation relating the electric field incident upon and reflected by a scattering body such as a rain particle. The first step in the process of obtaining the desired equation is the specification of the nature of the scattering particle. For the purposes of the following calculations it shall be assumed that the scattering particle is a homogeneous oblate spheroid. Justification for the assumed shape is provided in Section 3.

N. R. Labrum [9] has provided a fundamental solution to the problem of electromagnetic scattering by an oblate spheroid. His approach to the problem is based upon work done by Stratton [11], which has shown that where the ratio

$$\frac{\text{particle radius}}{\text{radiation wavelength}}$$

is small, the dipole moment induced in a sphere is the same for both static and time-varying incident fields. Labrum's work relates the dipole moment induced in the scattering particle to the incident electric field. In turn, the reflected electric field may be related to the induced dipole moment, but this is unnecessary for the current work since the final equation will involve only ratios of this parameter which associates a reflected field with an incident field. Thus, the equations relating the reflected electric field to the incident electric field will be of the form

$$E^- = SE^+ \qquad (2.3\text{-}1)$$

where

E^- = reflected electric field omitting a proportionality constant, and

E^+ = incident electric field,

but S will not be unitless. As in Labrum's work, S will have the units

$$\frac{\text{dipole moment}}{\text{electric field intensity}} = \text{Farad - meter}^2 \ . \qquad (2.3\text{-}2)$$

Rice and Peebles [1] have extended Labrum's work to the case of circular polarization and homogeneous oblate spheroids. From the standpoint of notation, this extension is simplified if the field intensities are resolved into orthogonal components which may be entered as elements in a column matrix. Thus the following shall be used.

$$\begin{pmatrix} E_x^- \\ E_y^- \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} E_x^+ \\ E_y^+ \end{pmatrix} \ . \qquad (2.3\text{-}3)$$

For the particles assumed, Rice and Peebles develop the S_{ij} terms as follows:

$$\begin{aligned} S_{11} &= g_x \cos^2(\theta) + g_y \sin^2(\theta) \\ S_{12} &= (g_x - g_y) \cos(\theta) \cos(\psi) \\ S_{21} &= (g_x - g_y) \cos(\theta) \cos(\psi) \\ S_{22} &= g_x \cos^2(\psi) + g_y \sin^2(\psi) \end{aligned} \qquad (2.3\text{-}4)$$

$$g_x = \frac{8\pi\varepsilon_1}{3}\left[\frac{\rho(1-\rho^2)^{3/2}\,(\varepsilon_2-\varepsilon_1)b^3}{[2\sqrt{1-\rho^2} - \rho\pi + 2\rho\arctan\,(\rho/\sqrt{1-\rho^2})]\,(\varepsilon_2-\varepsilon_1) + 2\varepsilon_1(1-\rho^2)^{3/2}}\right]$$

$$g_y = \frac{8\pi\varepsilon_1}{3}\left[\frac{\rho(1-\rho^2)^{3/2}\,(\varepsilon_2-\varepsilon_1)b^3}{\rho[\frac{\pi}{2} - \rho\sqrt{1-\rho^2} - \arctan\,(\rho/\sqrt{1-\rho^2})]\,(\varepsilon_2-\varepsilon_1) + 2\varepsilon_1(1-\rho^2)^{3/2}}\right]$$

$\rho = a/b$.

In the above definitions for g_x and g_y the parameters ε_1, ε_2, ε_3, a, and b pertain to the situation shown in Figure 2.3-1. Labrum's scattering solution is for confocal ellipsoids having dielectric permitivities of ε_2 and ε_3 imbedded in a medium having a dielectric permitivity of ε_1. The quantities a and b are the semi-minor and semi-major axes of the outer ellipsoid.

In equation (2.3-4), θ is a particle orientation parameter which is used by Labrum and is related to the coordinate parameters used in this work as indicated in equation (2.3-5).

$$\cos(\theta) = \sin(\zeta)\,\sin(\psi)\ . \qquad (2.3\text{-}5)$$

The results of applying equation (2.3-5) to the definition of the S_{ij} terms given in equation (2.3-4) are provided in equation (2.3-6).

$$\begin{aligned} S_{11} &= g_y + (g_x - g_y)\sin^2(\zeta)\sin^2(\psi) \\ S_{12} &= (g_x - g_y)\sin(\zeta)\sin(\psi)\cos(\psi) \\ S_{22} &= g_x\cos^2(\psi) + g_y\sin^2(\psi) \end{aligned} \qquad (2.3\text{-}6)$$

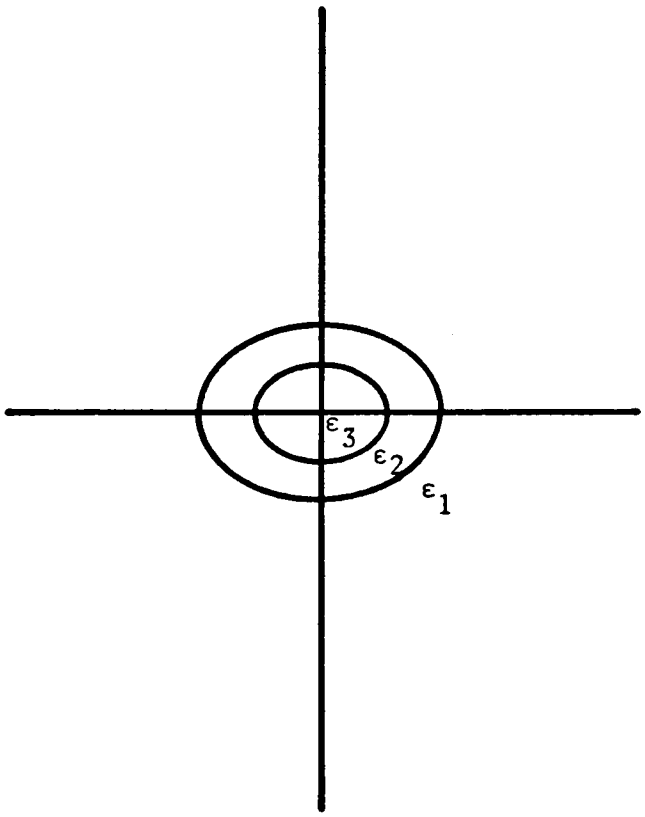


Figure 2.3-1. Particle and related dielectric quantities.

With the S matrix elements defined as in equation (2.3-6), it is now possible to express the components of the scattered electric field in terms of the components of the incident electric field as indicated in equation (2.3-3). Note that the equation in this form represents scattering from a single scattering particle. The effects of multi-particle scattering will be represented by a summation of the following form.

$$E_x^- = \sum_{i=1}^{N_k} (S_{11i}\, E_{xi}^+) + \sum_{i=1}^{N_k} (S_{12i}\, E_{yi}^+)$$

$$E_y^- = \sum_{i=1}^{N_k} (S_{21i}\, E_{xi}^+) + \sum_{i=1}^{N_k} (S_{22i}\, E_{yi}^+) \; . \qquad (2.3\text{-}7)$$

In equation (2.3-7), the total number of scattering particles within the scattering volume of interest is N_k.

2.4. The Reference System

What will be called the reference system in this work is presented in block diagram form in Figure 2.4-1. The system is a linearly polarized monostatic radar, and the performance of the ordinary circularly polarized system and the cancellation system will be compared to the performance of this reference system.

From the theory of linear antennas [11], it can be shown that the electric field intensity in the far zone of a dipole radiator, e^+, is related to the driving point potential, e_T, as shown in equation (2.4-1).



Figure 2.4-1. Reference system diagram.

$$e^{+} = \frac{K_1 e_T G_t}{Z} e^{-j\beta_0 Z} . \qquad (2.4\text{-}1)$$

In the preceding equation, K_1 is a complex constant, β_0 is the traditional phase constant, Z is the radial distance between the phase center of the antenna and the point where the field intensity is to be measured, and G_t is the voltage radiation characteristic of the transmitter antenna.

By the scattering theory of the preceding section, the scattered field at a distance Z' from the scattering body has an intensity, e^{-}, which is given by

$$e^{-} = \frac{S_{11} e^{+}}{Z'} e^{-j\beta_0 Z'} \qquad (2.4\text{-}2)$$

where S_{11} is an element of the previously defined scattering matrix.

From the theory of reciprocity as it applies to linear antennas, the received signal, e_r, is

$$e_r = K_2 G_r e^{-} = \frac{K_2 G_r S_{11} e^{+}}{Z'} e^{-j\beta_0 Z'}$$

$$= \frac{K_2 K_1 G_r G_t S_{11} e_T}{Z \cdot Z'} e^{-j\beta_0 (Z+Z')} . \qquad (2.4\text{-}3)$$

For a monostatic radar, $G_r = G_t = G$ and $Z = Z'$, and to simplify notation let

$$K = K_2 \cdot K_1 . \qquad (2.4\text{-}4)$$

Thus the signal received as a result of scattering from one scattering body is given by equation (2.4-5).

$$e_r = \frac{Ke_T S_{11}}{Z^2} G^2 e^{-j2\beta_0 Z} \quad . \tag{2.4-5}$$

Extending the analysis to cover multiparticle scattering involves only implementing a summation of returns from the individual scattering bodies. Such a summation is

$$e_r = \sum_{i=1}^{N_k} \frac{Ke_T S_{11i}(k) G_i^2}{Z_i^2} e^{-j2\beta_0 Z_i} \quad . \tag{2.4-6}$$

The receiver signal, e_r, indicated in equation (2.4-6) is a result of returns from all the scattering bodies lying within the range cell corresponding to the time at which e_r is evaluated. A convenient way of noting this fact is to specify the location of each particle with respect to the center of the range cell in which it lies. For the i^{th} particle in the k^{th} range cell this would be

$$Z_i(k) = Z(k) + \Delta Z_i \quad . \tag{2.4-7}$$

An approximation which will be used in the denominator of equation (2.4-6) is

$$Z_i(k) \approx Z(k) \quad . \tag{2.4-8}$$

This approximation is in error by no more than 0.5 percent for a transmitter pulsewidth of one microsecond and a range of 16 km. Application of the approximation to equation (2.4-6) results in the following. Note that N_k is the total number of scattering particles in the k^{th} range cell.

$$e_r(k) = \frac{K\, e_T}{Z^2(k)}\, e^{-j2\beta_0 Z(k)} \sum_{i=1}^{N_k} S_{11i}(k) G_i^2 e^{-j2\beta_0 \Delta Z_i} \quad . \qquad (2.4\text{-}9)$$

In summary, equation (2.4-9) is an expression for the output voltage resulting from clutter occurring in the k^{th} range cell for what will be called the reference system.

2.5. The Ordinary System

An ordinary system will be defined as a radar which radiates and receives the same-sense circular polarization to reduce rain clutter. Such a system is ideal if it radiates perfectly circular polarization of one sense and has zero response to a perfectly circular wave of the opposite sense. An imperfect ordinary system does not produce perfectly circular operation due to manufacturing or alignment errors.

By recognizing that a circularly polarized wave, indeed, even an elliptically polarized wave, may be considered to be comprised of two linearly polarized waves, we may model the transmission and reception operations of the system as shown in Figure 2.5-1. A signal e_T is power-split with one signal driving an antenna representing the x-polarized pattern. The second signal passes through a complex gain constant T, representing the error that the transmitting function

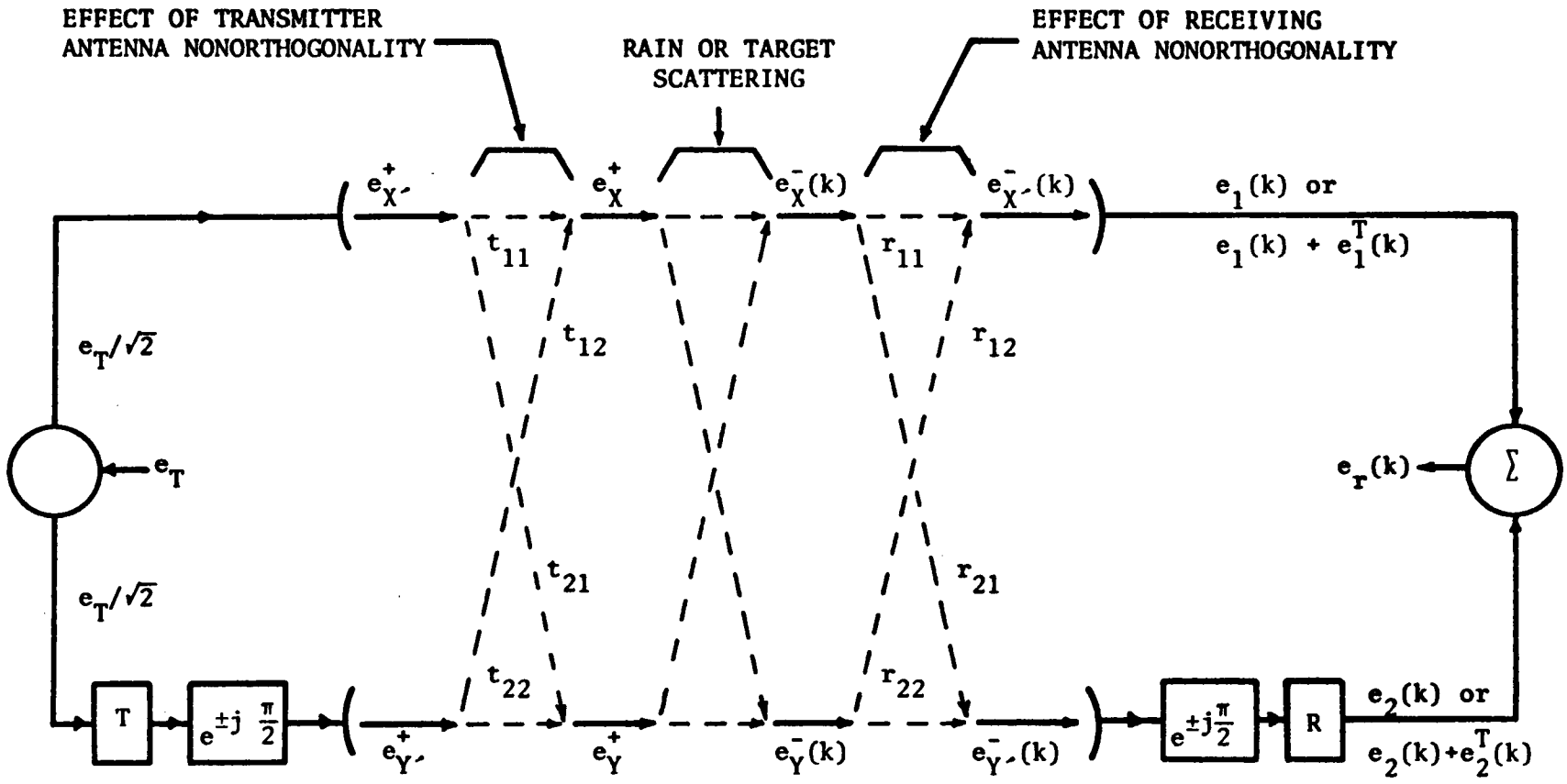


Figure 2.5-1. Ordinary system diagram.

introduces relative to true circular polarization. This signal then passes through a $\pm\pi/2$ phase shift to produce either left- or right-hand polarizations respectively when transmitted through an antenna representing the Y-polarized pattern. Similar, but reverse, operations exist in the receiver. The parameter R of the receiver equals T in a monostatic system.

If the true radiations $e^+_{X'}$ and $e^+_{Y'}$ are nonorthogonal, the components each couple to the desired space axes X and Y. The coupling may be accounted for by use of a coupling matrix with elements t_{ij}, i and j = 1 and 2. Thus,

$$\begin{bmatrix} e^+_X \\ e^+_Y \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} e^+_{X'} \\ e^+_{Y'} \end{bmatrix} . \qquad (2.5\text{-}1)$$

Using Figure 2.2-2, page 8, we may find that

$$[t] = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \begin{bmatrix} \cos(\theta_X) & -\sin(\theta_Y) \\ \sin(\theta_X) & \cos(\theta_Y) \end{bmatrix} . \qquad (2.5\text{-}2)$$

Another matrix accounts for either rain or target scattering. Still a third matrix accounts for the receiver nonorthogonal polarizations. For a monostatic radar the matrix, defined to have elements r_{ij}, i and j = 1 and 2, relates received fields at the antenna to reflections from range cell k as follows:

$$\begin{bmatrix} e^{-}_{X'}(k) \\ e^{-}_{Y'}(k) \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} e^{-}_{X}(k) \\ e^{-}_{Y}(k) \end{bmatrix} \tag{2.5-3}$$

where

$$[r] = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \frac{\begin{bmatrix} \cos(\theta_X) & \sin(\theta_X) \\ -\sin(\theta_Y) & \cos(\theta_Y) \end{bmatrix}}{\cos(\theta_X - \theta_Y)} = \frac{[t]^t}{\cos(\theta_X - \theta_Y)} . \tag{2.5-4}$$

Here $[\cdot]^t$ represents matrix transpose.

The received waveforms for the system of Figure 2.5-1, page 18, will now be calculated using the exponential, phasor, representation of the waveforms. The transmitter excitation is given by

$$e_T = E_T e^{j\omega_0 t} \tag{2.5-5}$$

where E_T is the magnitude of e_T and ω_0 is radian frequency. Let $G_t^X(\alpha, \beta)$ and $G_t^Y(\alpha, \beta)$ be the transmit patterns for X′ and Y′ polarization components evaluated in directions α and β. The electric fields $e^{+}_{X'}$ and $e^{+}_{Y'}$ near the antenna become

$$e^{+}_{X'} = K_1 G_t^X(\alpha, \beta)\ e_T/\sqrt{2} \tag{2.5-6a}$$

$$e^{+}_{Y'} = \pm jTK_1 G_t^Y(\alpha, \beta)\ e_T/\sqrt{2} \tag{2.5-6b}$$

where K_1 is a proportionality constant.

To continue further we first assume scattering is from a single rain particle. Results are then extended to multiple-particle rain and target scattering.

Let rain particle i be located in range cell k at a distance

$$Z_i(k) = Z(k) + \Delta Z_i \tag{2.5-7}$$

where $Z(k)$ is the distance to the center of the cell and ΔZ_i is particle position relative to cell center. By combining (2.5-5) with (2.5-1) the field components e_X^+ and e_Y^+ may be determined. The received fields due to range cell k, $e_X^-(k)$ and $e_Y^-(k)$, may be determined by use of the rain particle scattering matrix in conjunction with an attenuation factor due to distance.

$$\begin{bmatrix} e_X^-(k) \\ e_Y^-(k) \end{bmatrix} = \begin{bmatrix} S_{11i}(k) & S_{12i}(k) \\ S_{21i}(k) & S_{22i}(k) \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} G_t^X(\alpha_i, \beta_i) \\ \pm jTG_t^Y(\alpha_i, \beta_i) \end{bmatrix} \frac{K_1 e_T 2^{-j2\beta_0 Z_i(k)}}{Z_i^2(k)\sqrt{2}} . \tag{2.5-8}$$

Here the rain scattering matrix [S] has elements $S_{mni}(k)$ for cell k.

Next (2.5-3) and (2.5-8) are used to obtain the fields in the polarization planes of the receiving antenna. The received waveforms $e_1(k)$ and $e_2(k)$ are related to these fields, $e_{X'}^-(k)$ and $e_{Y'}^-(k)$, by

$$e_1(k) = K_2 G_r^X(\alpha_i, \beta_i)\, e_{X'}^-(k) \tag{2.5-9a}$$

$$e_2(k) = \pm jK_2 R G_r^Y(\alpha_i, \beta_i)\, e_{Y'}^-(k) \tag{2.5-9b}$$

where K_2 is a proportionality constant. Combining and recognizing that $R = T$, $G_t^X = G_r^X = G^X$ and $G_t^Y = G_r^Y = G^Y$ for a monostatic system gives

$$\begin{bmatrix} e_1(k) \\ e_2(k) \end{bmatrix} = \begin{bmatrix} G^X(\alpha_i,\beta_i) & 0 \\ 0 & \pm jTG^Y(\alpha_i,\beta_i) \end{bmatrix} [t]^t [S] [t] \begin{bmatrix} G^X(\alpha_i,\beta_i) \\ \pm jTG^T(\alpha_i,\beta_i) \end{bmatrix}$$

$$\cdot \frac{Ke_T e^{-j2\beta_0 Z_i(k)}}{Z_i^2(k)\sqrt{2}} \qquad (2.5\text{-}10)$$

where $K = K_1 \cdot K_2$.

To be more explicit, (2.5-10) may be expanded to obtain $e_1(k)$ and $e_2(k)$ as

$$\begin{aligned} e_1(k) = G^X(\alpha_i, \beta_i) \Big[& \{t_{11}[t_{11}S_{11i}(k) + t_{21}S_{12i}(k)] \\ & + t_{21}[t_{11}S_{21i}(k) + t_{21}S_{22i}(k)]\} G^X(\alpha_i, \beta_i) \\ & \pm jT\{t_{11}[t_{12}S_{11i}(k) + t_{22}S_{12i}(k)] \\ & + t_{21}[t_{12}S_{21i}(k) + t_{22}S_{22i}(k)]\} G^Y(\alpha_i, \beta_i) \Big] \quad \frac{Ke_T e^{-j2\beta_0 Z_i(k)}}{\sqrt{2}\, Z_i^2(k)} \end{aligned} \qquad (2.5\text{-}11a)$$

$$\begin{aligned} e_2(k) = \pm jTG^Y(\alpha_i, \beta_i) \Big[& \{t_{12}[t_{11}S_{11i}(k) + t_{21}S_{12i}(k)] \\ & + t_{22}[t_{11}S_{21i}(k) + t_{21}S_{22i}(k)]\} G^X(\alpha_i, \beta_i) \\ & \pm jT\{t_{12}[t_{12}S_{11i}(k) + t_{22}S_{12i}(k)] \end{aligned}$$

$$+ t_{22}[t_{12}S_{21i}(k) + t_{22}S_{22i}(k)]\} G^{Y}(\alpha_i, \beta_i) \Big] \frac{Ke_T e^{-j2\beta_0 Z_i(k)}}{\sqrt{2}\, Z_i^2(k)} .$$

(2.5-11b)

Certain special cases of these results are of interest. For example, in much of the ensuing analysis it is assumed that the systems' polarizations are orthogonal which means that θ_X and θ_Y in Figure 2.2-2, page 8, are zero and that $t_{11} = t_{22} = 1$ while $t_{12} = t_{21} = 0$. Hence,

$$e_1(k) = G^{X}(\alpha_i, \beta_i) \{S_{11i}(k)\, G^{X}(\alpha_i, \beta_i) \pm jTS_{12i}(k)\, G^{Y}(\alpha_i, \beta_i)\} \cdot \frac{Ke_T e^{-j2\beta_0 Z_i(k)}}{\sqrt{2}\, Z_i^2(k)} \qquad (2.5\text{-}12a)$$

and

$$e_2(k) = \pm jTG^{Y}(\alpha_i, \beta_i) \{S_{21i}(k)\, G^{X}(\alpha_i, \beta_i) \pm jTS_{22i}(k)\, G^{Y}(\alpha_i, \beta_i)\} \cdot \frac{Ke_T e^{-j2\beta_0 Z_i(k)}}{\sqrt{2}\, Z_i^2(k)} . \qquad (2.5\text{-}12b)$$

Another special case derives from an assumption that the patterns G^X and G^Y are identical.

$$G^{X}(\alpha_i, \beta_i) = G^{Y}(\alpha_i, \beta_i) = G(\alpha_i, \beta_i) \qquad (2.5\text{-}13)$$

$$e_1(k) = G^2(\alpha_i, \beta_i)\ \{S_{11i}(k) \pm jTS_{12i}(k)\}\ \frac{Ke_T e^{-j2\beta_0 Z_i(k)}}{\sqrt{2}\ Z_i^2(k)} \qquad (2.5\text{-}14a)$$

$$e_2(k) = \pm jTG^2(\alpha_i, \beta_i)\ \{S_{21i}(k) \pm jTS_{22i}(k)\}\ \frac{Ke_T e^{-j2\beta_0 Z_i(k)}}{\sqrt{2}\ Z_i^2(k)}\ . \qquad (2.5\text{-}14b)$$

Any of the single particle rain clutter results may be used for the multiple particle case by simply summing over all particles in cell k. The result of such a summation coupled with the simplification indicated in equation (2.4-8) is given below for equation (2.5-12).

$$e_1(k) = \frac{Ke_T e^{-j2\beta_0 Z(k)}}{\sqrt{2}\ Z^2(k)} \left[\sum_{i=1}^{N_k} S_{11i}(k)[G^X(\alpha_i, \beta_i)]^2\ e^{-j2\beta_0 \Delta Z_i} \right.$$

$$\left. \pm\ jT \sum_{i=1}^{N_k} S_{12i}(k) G^X(\alpha_i, \beta_i) G^Y(\alpha_i, \beta_i)\ e^{-j2\beta_0 \Delta Z_i} \right] \qquad (2.5\text{-}15a)$$

$$e_2(k) = \pm jT\ \frac{Ke_T e^{-j2\beta_0 Z(k)}}{\sqrt{2}\ Z^2(k)} \left[\sum_{i=1}^{N_k} S_{21i}(k) G^X(\alpha_i, \beta_i) G^Y(\alpha_i, \beta_i) e^{-j2\beta_0 \Delta Z_i} \right.$$

$$\left. \pm\ jT \sum_{i=1}^{N_k} S_{22i}(k)[G^Y(\alpha_i, \beta_i)]^2\ e^{-j2\beta_0 \Delta Z_i} \right]\ . \qquad (2.5\text{-}15b)$$

Similarly, (2.5-14) becomes

$$e_1(k) = \frac{Ke_T e^{-j2\beta_0 Z(k)}}{\sqrt{2}\ Z^2(k)} \sum_{i=1}^{N_k} G^2(\alpha_i, \beta_i)[S_{11i}(k) \pm jTS_{12i}(k)] e^{-j2\beta_0 \Delta Z_i} \qquad (2.5\text{-}16a)$$

$$e_2(k) = \pm jT \frac{Ke_T e^{-j2\beta_0 Z(k)}}{\sqrt{2}\, Z^2(k)} \sum_{i=1}^{N_k} G^2(\alpha_i, \beta_i)[S_{21i}(k) \pm jTS_{22i}(k)]e^{-j2\beta_0 \Delta Z_i} .$$

(2.5-16b)

If a target exists in cell k the received signals are given by (2.5-11) in general if the $S_{mni}(k)$ are replaced by the corresponding target scattering matrix elements $S^T_{mn}(k)$. For the special case of a monostatic system with orthogonal radiations

$$e_1^T(k) = \frac{Ke_T e^{-j2\beta_0 Z(k)}}{\sqrt{2}\, Z^2(k)} \left[[G^X(\alpha, \beta)]^2 S_{11}^T(k) \pm jTG^X(\alpha, \beta)G^Y(\alpha, \beta)S_{12}^T(k) \right]$$

(2.5-17a)

and

$$e_2^T(k) = \pm jT \frac{Ke_T e^{-j2\beta_0 Z(k)}}{\sqrt{2}\, Z^2(k)} \left[G^X(\alpha, \beta)G^Y(\alpha, \beta)S_{21}^T(k) \right.$$

$$\left. \pm jT[G^Y(\alpha, \beta)]^2 S_{22}^T(k) \right] . \qquad (2.5\text{-}17b)$$

If, in addition, all beam patterns are identical

$$e_1^T(k) = \frac{Ke_T e^{-j2\beta_0 Z(k)}}{\sqrt{2}\, Z^2(k)} G^2(\alpha, \beta)[S_{11}^T(k) \pm jTS_{12}^T(k)] \qquad (2.5\text{-}18a)$$

$$e_2^T(k) = \pm jT \frac{Ke_T e^{-j2\beta_0 Z(k)}}{\sqrt{2}\, Z^2(k)} G^2(\alpha, \beta)[S_{21}^T(k) \pm jTS_{22}^T(k)]. \qquad (2.5\text{-}18b)$$

In all these target signals it is assumed that the target is located on the beam axis so X = 0 and Y = 0.

3. THEORETICAL RAIN MODEL

In this section a rain model is presented which will be used in later sections for evaluating the performance of the ordinary and cancellation systems. The model has previously been described in a work by Rice and Peebles [1], and the material presented here is a summary of the pertinent results.

3.1. Drop Shape

The actual drop shape is shown in Figure 3.1-1a, while the shape assumed for analytical purposes is given in Figure 3.1-1b. The latter shape is chosen over the actual shape due to the availability of scattering coefficients for the oblate spheroid.

3.2. Drop Size Distribution

Many drop size distributions have been developed over the years; however, only two such distributions will be considered for the present effort. The choice of these two distributions was based upon both the ease of application to the analytical work and the accuracy of the distribution when compared with available data. The first, and probably most famous, distribution is that of Marshall and Palmer [12].

$$n(d_o) = A\, e^{-\lambda d_o} \quad m^{-3}\ mm^{-1} \tag{3.2-1}$$

$$A = 8000 \quad m^{-3}\ mm^{-1} \tag{3.2-2}$$

$$\lambda = 4.1\, I^{-0.21} \quad mm^{-1} \tag{3.2-3}$$

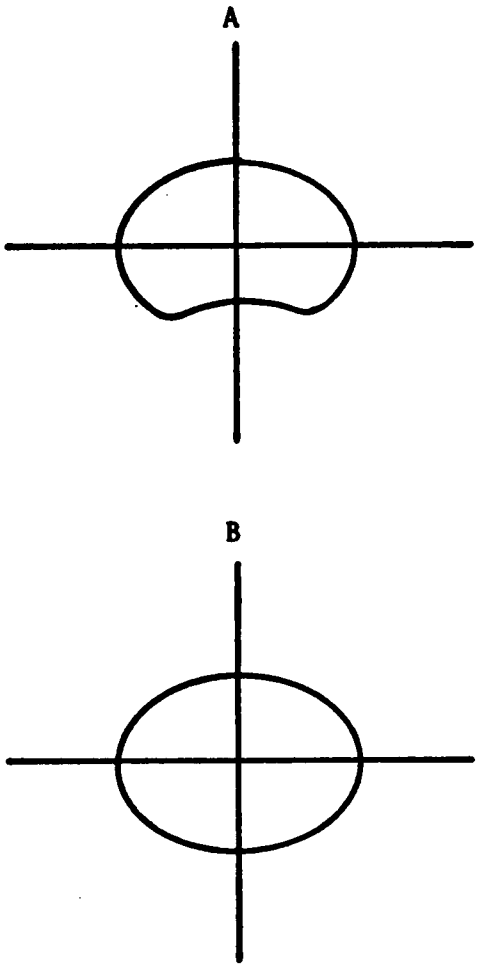


Figure 3.1.1. Raindrop cross sections.

Note that I is the precipitation intensity in mm/hr, and that $n(d_o)$ specifies the number of drops per unit volume having diameters ranging from d_o to $d_o + \delta d_o$. It should also be pointed out that d_o is an equivalent drop diameter, and as such represents the diameter of a sphere having the same volume as the actual drop which is represented as an oblate spheroid.

The second drop size distribution, developed by E. A. Polyakova and K. S. Shifrin and described by Litvinov [13], has the form

$$n(d_o) = A\, d_o^2\, e^{-\gamma d_o} . \qquad (3.2\text{-}4)$$

The quantities A and γ are defined below.

Type of Rain	$A(m^{-3}mm^{-3})$	$\gamma(mm^{-1})$
Thawing of Pellets (Hail)	$64{,}500\ I^{-0.50}$	$6.95\ I^{-0.27}$
Thawing of Granular Snow (Sleet)	$11{,}750\ I^{-0.29}$	$4.87\ I^{-0.20}$
Thawing of Nongranular Snow (Snow)	$2{,}820\ I^{-0.18}$	$4.01\ I^{-0.19}$

3.3. Probability Density Function for Drop Size

For the present work, the ultimate utility of the drop size distribution is its contribution in determining a probability density function for drop sizes. Again the results presented here come from work by Rice and Peebles [1]. The density functions are for the Marshall-Palmer and Polyakova-Shifrin models respectively.

$$p(d_o) = \lambda e^{-\lambda d_o} \qquad (3.3\text{-}1)$$

$$p(d_o) = \frac{\gamma^3}{2}\, d_o^2\, e^{-\gamma d_o} . \qquad (3.3\text{-}2)$$

4. PERFORMANCE OF THE CANCELLATION SYSTEM

The performance of the cancellation system and the ordinary system will be evaluated relative to the performance of the reference system by means of a cancellation ratio CR. For the cancellation system, CR takes the form

$$CR = P_{ref}/P_{can} , \tag{4.0-1}$$

and for the ordinary system, CR is given by

$$CR = P_{ref}/P_{ord} . \tag{4.0-2}$$

P_{ref} and P_{ord} are given by Rice and Peebles [1, page 54] as follows.

$$P_{ref} = |A|^2 \; 2 \; E \left[\sum_{i=1}^{N_k} \frac{|G(\alpha_i,\beta_i)|^4 \; |S_{11i}|^2}{2} \right] \tag{4.0-3}$$

$$P_{ord} = |A|^2 \; E \left[\sum_{i=1}^{N_k} \frac{|G(\alpha_i,\beta_i)|^4 \; |S_{11i} \pm j2TS_{12i} - T^2S_{22i}|^2}{2} \right] \tag{4.0-4}$$

$$A = \frac{Ke_T \; e^{-j2\beta Z}}{Z^2} . \tag{4.0-5}$$

4.1. The Rain Model

There are basically two major areas involved in specifying a rain model. The first area is concerned with describing the individual rain particle, and this aspect of rain modeling has been dealt with in

Section 3, where a major emphasis was on defining particle size and shape. Particle size and shape affect the radar system equations through the S matrix elements defined in equation (2.3-6), and nothing in this section will be done to alter the nature of the influence that particle size and shape have on the system equations.

The second aspect of rain modeling is concerned with the nature of large scale rain systems. Here individual particle characteristics are lost and only overall effects are noted. The following assumptions describe the rain system model in effect in the ensuing work.

1. The rain particles exhibit a uniform spatial distribution over the range cells.
2. Particle size and orientation will be treated as random variables, and it is assumed that they are statistically independent of the particle position parameters. The distribution of particle size presented in Section 3 will be used here.
3. The number of particles occurring in a range cell is strictly random, and the distribution function for the number of particles will be determined in later work. It is assumed that the number of particles is statistically independent of particle size, shape, orientation, and location.
4. Although several parameters relating to the rain are random in nature, it is assumed that the statistical distributions describing the various parameters are invariant in time and space.

5. It is assumed that the clutter returns from one range cell are statistically independent of the clutter returns from any other range cell.

4.2. System Equations

The physical system being analyzed here is the same as the one presented in Figure 1.2-1, page 4, and the definitions for $e_1(k)$, $e_2(k)$, and G(k) are given in equations (4.2-1), (4.2-2), and (4.2-3) respectively.

$$e_1(k) = \frac{A}{\sqrt{2}} \left[X_1(k) + jY_1(k) \right] \tag{4.2-1}$$

$$e_2(k) = \frac{\pm jTA}{\sqrt{2}} \left[X_2(k) + jY_2(k) \right] \tag{4.2-2}$$

$$G(k) = \left(\frac{-1}{2M+1} \right) \sum_{\ell=k-M}^{k+M} \frac{e_1(\ell)}{e_2(\ell)} = \left(\frac{-1}{2M+1} \right) \left[\sum_{\substack{\ell=k-M \\ \ell \neq k}}^{k+M} \frac{e_1(\ell)}{e_2(\ell)} + \frac{e_1(k)}{e_2(k)} \right]$$

$$= \left(\frac{-1}{2M+1} \right) \left[G'(k) + \frac{e_1(k)}{e_2(k)} \right] . \tag{4.2-3}$$

The quantities X_1, Y_1, X_2, and Y_2 are determined by the following.

$$S_{11i}(k) \pm jTS_{12i}(k) = \alpha_{1i}(k)\, e^{j\gamma_{1i}(k)} \tag{4.2-4}$$

$$S_{21i}(k) \pm jTS_{22i}(k) = \alpha_{2i}(k)\, e^{j\gamma_{2i}(k)} \tag{4.2-5}$$

$$2\beta_0 \Delta Z_i = \theta_i \tag{4.2-6}$$

$$X_1(k) = \sum_{i=1}^{N_k} G^2(\alpha_i,\beta_i)\alpha_{1i}(k) \cos\left[\gamma_{1i}(k) - \theta_i\right] \tag{4.2-7}$$

$$Y_1(k) = \sum_{i=1}^{N_k} G^2(\alpha_i,\beta_i)\alpha_{1i}(k) \sin\left[\gamma_{1i}(k) - \theta_i\right] \tag{4.2-8}$$

$$X_2(k) = \sum_{i=1}^{N_k} G^2(\alpha_i,\beta_i)\alpha_{2i}(k) \cos\left[\gamma_{2i}(k) - \theta_i\right] \tag{4.2-9}$$

$$Y_2(k) = \sum_{i=1}^{N_k} G^2(\alpha_i,\beta_i)\alpha_{2i}(k) \sin\left[\gamma_{2i}(k) - \theta_i\right] . \tag{4.2-10}$$

Equations (4.2-1) through (4.2-10) may be used to express the system output as well as the magnitude squared of the system output.

$$e_r(k) = e_1(k) + G(k) \cdot e_2(k) = e_1(k) + \left(\frac{-1}{2M+1}\right)\left[G'(k) + \frac{e_1(k)}{e_2(k)}\right]e_2(k)$$

$$= \left(\frac{2M}{2M+1}\right)e_1(k) - \frac{G'(k) \cdot e_2(k)}{(2M+1)} \tag{4.2-11}$$

$$|e_r(k)|^2 = e_r(k) \cdot e_r^*(k) = \left(\frac{2M}{2M+1}\right)^2 |e_1(k)|^2 - \frac{4M}{(2M+1)^2} \mathrm{Re}\left[e_1^*(k)e_2(k)G'(k)\right]$$

$$+ \frac{|G'(k)|^2 \, |e_2(k)|^2}{(2M+1)^2} \tag{4.2-12}$$

$$|e_1(k)|^2 = \frac{|A|^2}{2}\left[X_1^2(k) + Y_1^2(k)\right] \tag{4.2-13}$$

$$|e_2(k)|^2 = \frac{|T|^2\,|A|^2}{2}\left[X_2^2(k) + Y_2^2(k)\right] \tag{4.2-14}$$

$$e_1^*(k)\, e_2(k) = \frac{\pm jT|A|^2}{2}\left[\left(X_1(k)\, X_2(k) + Y_1(k)\, Y_2(k)\right) + j\left(X_k(k)\, Y_2(k) - X_2(k)\, Y_1(k)\right)\right] \tag{4.2-15}$$

$$G'(k) = \sum_{\substack{\ell=k-M \\ \ell\neq k}}^{k+M} \frac{[X_1(\ell) + jY_1(\ell)]}{(\pm jT)[X_2(\ell) + jY_2(\ell)]}$$

$$= \pm \sum_{\substack{\ell=k-M \\ \ell\neq k}}^{k+M} \frac{\mathrm{Re}(T)\left[X_2(\ell)Y_1(\ell) - X_1(\ell)Y_2(\ell)\right] - \mathrm{Im}(T)\left[X_1(\ell)X_2(\ell) + Y_1(\ell)Y_2(\ell)\right]}{|T|^2\left[X_2^2(\ell) + Y_2^2(\ell)\right]}$$

$$\mp j \sum_{\substack{\ell=k-M \\ \ell\neq k}}^{k+M} \frac{\mathrm{Re}(T)\left[X_1(\ell)X_2(\ell) + Y_1(\ell)Y_2(\ell)\right] + \mathrm{Im}(T)\left[X_2(\ell)Y_1(\ell) - X_1(\ell)Y_2(\ell)\right]}{|T|^2\left[X_2^2(\ell) + Y_2^2(\ell)\right]} \tag{4.2-16}$$

$$|G'(k)|^2 = \left[\sum_{\substack{\ell=k-M \\ \ell\neq k}}^{k+M}\left(\frac{\mathrm{Re}(T)\left[X_2(\ell)Y_1(\ell) - X_1(\ell)Y_2(\ell)\right] - \mathrm{Im}(T)\left[X_2(\ell)X_1(\ell) + Y_1(\ell)Y_2(\ell)\right]}{|T|^2\left[X_2^2(\ell) + Y_2^2(\ell)\right]}\right)\right]^2 + \left[\sum_{\substack{\ell=k-M \\ \ell\neq k}}^{k+M}\left(\frac{\mathrm{Re}(T)\left[X_1(\ell)X_2(\ell) + Y_1(\ell)Y_2(\ell)\right] + \mathrm{Im}(T)\left[X_2(\ell)Y_1(\ell) - X_1(\ell)Y_2(\ell)\right]}{|T|^2\left[X_2^2(\ell) + Y_2^2(\ell)\right]}\right)\right]^2 \tag{4.2-17}$$

The relationship given in equation (4.2-18) will be useful in later work with equation (4.2-17).

$$\left[\sum_{i=1}^{N} a_i\right]^2 = \sum_{i=1}^{N} a_i^2 + \sum_{\substack{i=1 \\ i\neq j}}^{N} \sum_{j=1}^{N} a_i a_j \qquad (4.2\text{-}18)$$

As a result of the random nature of the rain as outlined in the preceding section, much of the evaluation done in this section will be done in a statistical sense. This means that power will be defined as $E\{|x|^2\}$ where the operator $E\{\ \cdot\ \}$ is the expectation operator discussed in Thomas [14]. Thus, the output noise power of the cancellation system is

$$E\{|e_r(k)|^2\} = \left(\frac{2M}{2M+1}\right)^2 E\{|e_1(k)|^2\} - \frac{4M}{(2M+1)^2} E\{\mathrm{Re}\left[e_1^*(k)e_2(k)G'(k)\right]\}$$

$$+ \left(\frac{1}{2M+1}\right)^2 E\{|G'(k)|^2\ |e_2(k)|^2\}\ . \qquad (4.2\text{-}19)$$

Now consider, in order, the three expectation terms appearing on the right-hand side of the equality in equation (4.2-19).

$$E\{|e_1(k)|^2\} = E\left\{\frac{|A|^2}{2}\left[X_1^2(k) + Y_1^2(k)\right]\right\}\ . \qquad (4.2\text{-}20)$$

Examination of equation (4.0-5) reveals that A is not random in nature so that

$$E\{|e_1(k)|^2\} = \frac{|A|^2}{2}\left[E\{X_1^2(k)\} + E\{Y_1^2(k)\}\right]\ . \qquad (4.2\text{-}21)$$

From Appendix A,

$$E\{X_1^2(k)\} = E\{N_k\}\ E\left\{\frac{G^4(\alpha,\beta)}{2}\right\}\ E\{\alpha_1^2\}\ , \tag{4.2-22}$$

and

$$E\{Y_1^2(k)\} = E\{N_k\}\ E\left\{\frac{G^4(\alpha,\beta)}{2}\right\}\ E\{\alpha_1^2\}\ . \tag{4.2-23}$$

Later work will show that it is not necessary to evaluate $E\{N_k\}$ and $E\{G^4(\alpha,\beta)/2\}$ in order to calculate the cancellation ratio. Consider the remaining unknown factor $E\{\alpha_1^2\}$. From equation (4.2-4),

$$\alpha_1 = |S_{11} \pm jTS_{12}|\ . \tag{4.2-24}$$

Thus

$$\alpha_1^2 = S_{11}^2 + |T|^2\ S_{12}^2 \mp 2\mathrm{Im}\left[TS_{11}S_{12}\right]\ , \tag{4.2-25}$$

and

$$E\{\alpha_1^2\} = E\{S_{11}^2\} + |T|^2\ E\{S_{12}^2\} \mp 2\mathrm{Im}\left[E\{TS_{11}S_{12}\}\right]\ . \tag{4.2-26}$$

Consideration is now given to the second term,

$$\mathrm{Re}\left[E\{e_1^*(k)\ e_2(k)\ G'(k)\}\right]\ , \tag{4.2-27}$$

in equation (4.2-19). Application of condition five in Section 4.1 in conjunction with the development in Appendix B permits the preceding term to be expressed as

$$\mathrm{Re}\left[E\{e_1^*(k)\ e_2(k)\}\ E\{G'(k)\}\right]\ . \tag{4.2-28}$$

Equations (4.2-1) and (4.2-2) may be used to obtain an expression for $e_1^*(k)e_2(k)$.

$$e_1^*(k)e_2(k) = \frac{\mp|A|^2}{2}\left[\mathrm{Im}(T)\left[X_1(k)X_2(k) + Y_1(k)Y_2(k)\right] + \mathrm{Re}(T)\left[X_1(k)Y_2(k) - X_2(k)Y_1(k)\right]\right]$$

$$\pm j\frac{|A|^2}{2}\left[\mathrm{Re}(T)\left[X_1(k)X_2(k) + Y_1(k)Y_2(k)\right] - \mathrm{Im}(T)\left[X_1(k)Y_2(k) - X_2(k)Y_1(k)\right]\right] \qquad (4.2\text{-}29)$$

$$E\{e_1^*(k)e_2(k)\} = \frac{\mp|A|^2}{2}\left[\mathrm{Im}(T)\left[E\{X_1(k)X_2(k)\} + E\{Y_1(k)Y_2(k)\}\right] + \mathrm{Re}(T)\left[E\{X_1(k)Y_2(k)\} - E\{X_2(k)Y_1(k)\}\right]\right]$$

$$\pm\frac{j|A|^2}{2}\left[\mathrm{Re}(T)\left[E\{X_1(k)X_2(k)\} + E\{Y_1(k)Y_2(k)\}\right] - \mathrm{Im}(T)\left[E\{X_1(k)Y_2(k)\} - E\{X_2(k)Y_1(k)\}\right]\right]. \qquad (4.2\text{-}30)$$

The corss-correlation terms in the preceding equation are evaluated in Appendix C.

Now attention must be given to the evaluation of $E\{G'(k)\}$. From equation (4.2-3),

$$G'(k) = \sum_{\substack{\ell=k-M \\ \ell\neq k}}^{k+M} \frac{e_1(\ell)}{e_2(\ell)} \tag{4.2-31}$$

$$= \sum_{\substack{\ell=k-M \\ \ell\neq k}}^{k+M} \left(\frac{X_1(\ell) + jY_1(\ell)}{jT\left(X_2(\ell) + jY_2(\ell)\right)} \right) \tag{4.2-32}$$

$$= \mp \sum_{\substack{\ell=k-M \\ \ell\neq k}}^{k+M} \left(\frac{[X_1(\ell)Y_2(\ell)-X_2(\ell)Y_1(\ell)]Re(T)+[X_1(\ell)X_2(\ell)+Y_1(\ell)Y_2(\ell)]Im(T)}{|T|^2\ [X_2^2(\ell) + Y_2^2(\ell)]} \right)$$

$$\mp j \sum_{\substack{\ell=k-M \\ \ell\neq k}}^{k+M} \left(\frac{[X_1(\ell)X_2(\ell)+Y_1(\ell)Y_2(\ell)]Re(T)-[X_1(\ell)Y_2(\ell)-X_2(\ell)Y_1(\ell)]Im(T)}{|T|^2\ [X_2^2(\ell) + Y_2^2(\ell)]} \right) \tag{4.2-33}$$

The assumed uniformity in space and time of the statistical distributions allows the expectation of $G'(k)$ to be expressed in the following form.

$$E\{G'(k)\} = \frac{\mp 2M}{|T|^2} E\left(\frac{[X_1(\ell)Y_2(\ell)-X_2(\ell)Y_1(\ell)]Re(T)+[X_1(\ell)X_2(\ell)+Y_1(\ell)Y_2(\ell)]Im(T)}{[X_2^2(\ell) + Y_2^2(\ell)]} \right)$$

$$\frac{\mp j2M}{|T|^2} E\left(\frac{[X_1(\ell)X_2(\ell)+Y_1(\ell)Y_2(\ell)]Re(T)-[X_1(\ell)Y_2(\ell)-X_2(\ell)Y_1(\ell)]Im(T)}{[X_2^2(\ell) + Y_2^2(\ell)]} \right) \tag{4.2-34}$$

Evaluation of $E\{G'(k)\}$ requires that the expectation of the quantities

$$\frac{X_1(\ell)Y_2(\ell)}{X_2^2(\ell) + Y_2^2(\ell)}, \quad \frac{X_2(\ell)Y_1(\ell)}{X_2^2(\ell) + Y_2^2(\ell)}, \quad \frac{X_1(\ell)X_2(\ell)}{X_2^2(\ell) + Y_2^2(\ell)}, \text{ and } \frac{Y_1(\ell)Y_2(\ell)}{X_2^2(\ell) + Y_2^2(\ell)}$$

be determined. The required expectations are in the form of three-dimensional integrals, and Appendix D shows how the integrals may be reduced.

The third term of Equation (4.2-19) may now be expressed as

$$\frac{1}{(2M+1)^2} E\{|G'(k)|^2\} \cdot E\{|e_2(k)|^2 \quad ,$$

and $E\{|e_2(k)|^2\}$ is evaluated in the same manner as was $E\{|e_1(k)|^2\}$.

$$E\{|e_2(k)\}^2| = \frac{|T|^2 |A|^2}{2} [E\{X_2^2(k)\} + E\{Y_2^2(k)\}] \qquad (4.2\text{-}35)$$

From Appendix A

$$E\{X_2^2(k)\} = E\{N_k\} E\left\{\frac{G^4(\alpha,\beta)}{2}\right\} E\{\alpha_2^2\} \quad , \qquad (4.2\text{-}36)$$

and

$$E\{Y_2^2(k)\} = E\{N_k\} E\left\{\frac{G^4(\alpha,\beta)}{2}\right\} E\{\alpha_2^2\} \quad . \qquad (4.2\text{-}37)$$

Equation (4.2-5) may be used to show that

$$\alpha_2^2 = S_{21}^2 + |T|^2 S_{22}^2 \mp \mathrm{Im}(TS_{21}S_{22}) \quad . \qquad (4.2\text{-}38)$$

Thus,

$$E\{\alpha_2^2\} = E\{S_{21}^2\} + |T|^2 E\{S_{22}^2\} \mp 2\mathrm{Im}(T)\, E\{S_{21}S_{22}\} \quad . \qquad (4.2\text{-}39)$$

$|G'(k)|^2$ is indicated in equation (4.2-17) as being the sum of two squared quantities. Each of the two squared quantities may,

through application of equation (4.2-18), be shown to be as indicated in the following.

$$\sum_{\substack{\ell=k-M \\ \ell\neq k}}^{\ell+M} \left[\frac{\mathrm{Re}(T)\left[X_2(\ell)Y_1(\ell)-X_1(\ell)Y_2(\ell)\right]-\mathrm{Im}(T)\left[X_2(\ell)X_1(\ell)+Y_1(\ell)Y_2(\ell)\right]}{|T|^2\left[X_2^2(\ell)+Y_2^2(\ell)\right]} \right]^2 +$$

$$\sum_{\substack{\ell=k-M \\ \ell\neq k \\ \ell\neq j}}^{k+M} \sum_{\substack{j=k-M \\ j\neq k}}^{k+M} \frac{\mathrm{Re}(T)\left[X_2(\ell)Y_1(\ell)-X_1(\ell)Y_2(\ell)\right]-\mathrm{Im}(T)\left[X_2(\ell)X_1(\ell)+Y_1(\ell)Y_2(\ell)\right]}{|T|^2\left[X_2^2(\ell)+Y_2^2(\ell)\right]} \cdot$$

$$\frac{\mathrm{Re}(T)\left[X_2(j)Y_1(j)-X_1(j)Y_2(j)\right]-\mathrm{Im}(T)\left[X_2(j)X_1(j)+Y_1(j)Y_2(j)\right]}{|T|^2\left[X_2^2(j)+Y_2^2(j)\right]} \tag{4.2-40}$$

The above represents the first term of equation (4.2-17), and the following is another form for the second term in the same equation.

$$\sum_{\substack{\ell=k-M \\ \ell\neq k}}^{k+M} \left[\frac{\mathrm{Re}(T)\left[X_1(\ell)X_2(\ell)+Y_1(\ell)Y_2(\ell)\right]+\mathrm{Im}(T)\left[X_2(\ell)Y_1(\ell)-X_1(\ell)Y_2(\ell)\right]}{|T|^2\left[X_2^2(\ell)+Y_2^2(\ell)\right]} \right]^2 +$$

$$\sum_{\substack{\ell=k-M \\ \ell\neq k \\ \ell\neq j}}^{k+M} \sum_{\substack{j=k-M \\ j\neq k}}^{k+M} \left[\frac{\mathrm{Re}(T)\left[X_1(\ell)X_2(\ell)+Y_1(\ell)Y_2(\ell)\right]+\mathrm{Im}(T)\left[X_2(\ell)Y_1(\ell)-X_1(\ell)Y_2(\ell)\right]}{|T|^2\left[X_2^2(\ell)+Y_2^2(\ell)\right]} \right] \cdot$$

$$\left[\frac{\mathrm{Re}(T)\left[X_1(j)X_2(j)+Y_1(j)Y_2(j)\right]+\mathrm{Im}(T)\left[X_2(j)Y_1(j)-X_1(j)Y_2(j)\right]}{|T|^2\left[X_2^2(j)+Y_2^2(j)\right]} \right] \cdot \tag{4.2-41}$$

Note that when T is real the following simplified form results for $|G'(k)|^2$.

$$|G'(k)|^2 = \sum_{\substack{\ell=k-M \\ \ell\neq k}}^{k+M} \left[\frac{X_2(\ell)Y_1(\ell) - X_1(\ell)Y_2(\ell)}{T\left[X_2^2(\ell) + Y_2^2(\ell)\right]} \right]^2 +$$

$$\sum_{\substack{\ell=k-M \\ \ell\neq k \\ \ell\neq j}}^{k+M} \sum_{\substack{j=k-M \\ j\neq k}}^{k+M} \left[\frac{X_2(\ell)Y_1(\ell)-X_1(\ell)Y_2(\ell)}{T\left[X_2^2(\ell) + Y_2^2(\ell)\right]} \right] \left[\frac{X_2(j)Y_1(j)-X_1(j)Y_2(j)}{T\left[X_2^2(j)+Y_2^2(j)\right]} \right] +$$

$$\sum_{\substack{\ell=k-M \\ \ell\neq k}}^{k+M} \left[\frac{X_1(\ell)X_2(\ell) + Y_1(\ell)Y_2(\ell)}{T\left[X_2^2(\ell) + Y_2^2(\ell)\right]} \right]^2 +$$

$$\sum_{\substack{\ell=k-M \\ \ell\neq k \\ \ell\neq j}}^{k+M} \sum_{\substack{j=k-M \\ j\neq k}}^{k+M} \left[\frac{X_1(j)X_2(j)+Y_1(j)Y_2(j)}{T\left[X_2^2(j)+Y_2^2(j)\right]} \right] \left[\frac{X_1(\ell)X_2(\ell)+Y_1(\ell)Y_2(\ell)}{T\left[X_2^2(\ell) + Y_2^2(\ell)\right]} \right] . \qquad (4.2\text{-}42)$$

Evaluation of $E\{|G'(k)|^2\}$ may now proceed, and the assumed independence of returns from different range cells will play an important part in the evaluation process. For example, one term in the expression for $E\{|G'(k)|^2\}$ is of the following form.

$$E\left\{ \sum_{i=1}^{N_k} \sum_{j=1}^{N_k} a_i a_j \right\} = \sum_{i=1}^{N_k} \sum_{j=1}^{N_k} E\{a_i a_j\} . \qquad (4.2\text{-}43)$$

If the a_i and a_j have the same density function and are known to be independent then according to Beckmann [15, page 70]

$$\sum_{i=1}^{N_k}\sum_{j=1}^{N_k} E\{a_i a_j\} = (N_k^2 - 2N_k)\, E\{a_i\}\, E\{a_j\} = (N_k^2 - 2N_k)\, E^2\{a_i\} \,. \tag{4.2-44}$$

This same approach may be applied in the evaluation of $E\{|G'(k)|^2\}$.

$$\begin{aligned} E\{|G'(k)|^2 &= 2M\, E\left\{\frac{[X_2(\ell)Y_1(\ell)]^2 - 2[X_1(\ell)X_2(\ell)Y_1(\ell)Y_2(\ell)] + [X_1(\ell)Y_2(\ell)]^2}{T^2\left[X_2^2(\ell) + Y_2^2(\ell)\right]^2}\right\} \\ &+ 2M\, E\left\{\frac{[X_1(\ell)X_2(\ell)]^2 + 2[X_1(\ell)X_2(\ell)Y_1(\ell)Y_2(\ell)] + [Y_1(\ell)Y_2(\ell)]^2}{T^2\left[X_2^2(\ell) + Y_2^2(\ell)\right]^2}\right\} \\ &+ (4M^2 - 2M)\left\{E\left\{\frac{X_2(\ell)Y_1(\ell) - X_1(\ell)Y_2(\ell)}{T\left[X_2^2(\ell) + Y_2^2(\ell)\right]}\right\}\right\}^2 \\ &+ (4M^2 - 2M)\left\{E\left\{\frac{X_1(\ell)X_2(\ell) + Y_1(\ell)Y_2(\ell)}{T\left[X_2^2(\ell) + Y_2^2(\ell)\right]}\right\}\right\}^2 . \end{aligned} \tag{4.2-45}$$

Effectively, the last two terms of equation (4.2-45) were evaluated in the discussion of $E\{G'(k)\}$. Evaluation of the first two terms of equation (4.2-45) requires knowledge of

$$E\left\{\frac{\left[X_2(\ell)Y_1(\ell)\right]^2}{T^2[X_2^2(\ell) + Y_2^2(\ell)]^2}\right\} \quad , \quad E\left\{\frac{\left[X_1(\ell)Y_2(\ell)\right]^2}{T^2[X_2^2(\ell) + Y_2^2(\ell)]^2}\right\} \quad ,$$

$$E\left\{\frac{\left[X_1(\ell)X_2(\ell)\right]^2}{T^2[X_2^2(\ell) + Y_2^2(\ell)]^2}\right\} \quad , \text{ and } \quad E\left\{\frac{\left[Y_1(\ell)Y_2(\ell)\right]^2}{T^2[X_2^2(\ell) + Y_2^2(\ell)]^2}\right\} \,.$$

These quantities are analyzed in Appendix D.

4.3. Lower Bound on Performance

Rice and Peebles [1, page 59] have determined that the lower bound of performance of the ordinary system results under the following conditions. These same conditions will be applied to the equations describing the cancellation system, thereby providing a basis of comparison between the ordinary and cancellation systems.

$$\cos^2(\psi) \approx \left(\frac{V_\infty}{V_r}\right)^2 , \tag{4.3-1}$$

$$\sin^2(\psi) \approx 1 - \left(\frac{V_\infty}{V_r}\right)^2 , \tag{4.3-2}$$

$$\sin(\zeta) = 1 ,$$

and

$$\cos(\zeta) = 0 \tag{4.3-4}$$

where V_∞ is the drop terminal velocity, and V_r is the radial component of the wind velocity in a radar centered coordinate system. Under these conditions, the S matrix elements assume the following form.

$$S_{11} \approx g_x - (g_x - g_y)(V_\infty/V_r)^2 \tag{4.3-5}$$

$$S_{12} \approx (g_x - g_y)(V_\infty/V_r)\sqrt{1-(V_\infty/V_r)^2} \tag{4.3-6}$$

$$S_{22} \quad (g_x - g_y)(V_\infty/V_r)^2 + g_y . \tag{4.3-7}$$

The following, determined by Rice and Peebles [1, page 100], is useful in evaluating the S matrix elements.

$$g_x^2 = \left[\frac{\pi\varepsilon_1}{6 \times 10^9}\right]^2 d_0^6 \, [8.25 - 0.244d_0^2 + 0.023d_0^3] \qquad (4.3\text{-}8)$$

$$g_y^2 = \left[\frac{\pi\varepsilon_1}{6 \times 10^9}\right]^2 d_0^6 \, [8.33 - 0.112d_0^2] \qquad (4.3\text{-}9)$$

$$(g_x - g_y)^2 = \left[\frac{\pi\varepsilon_1}{6 \times 10^9}\right]^2 d_0^8 \, [0.0024 + 0.00415d_0^2] \qquad (4.3\text{-}10)$$

$$g_x g_y = \left[\frac{\pi\varepsilon_1}{6 \quad 10^9}\right]^2 d_0^6 \, [8.29 - 0.0672d_0^2 + 0.0115d_0^3 - 0.002075d_0^4] \qquad (4.3\text{-}11)$$

$$V_\infty = 10.105 \, [1 - e^{-d_0/2}] \, . \qquad (4.3\text{-}12)$$

Evaluation of the cancellation system's output equation requires knowledge of various moments of the S matrix elements. The required moments are given below.

$$E\{S_{11}^2\} = E\{g_x^2\} - 2E\{g_x(g_x - g_y)(V_\infty/V_r)^2\} + E\{(g_x - g_y)^2(V_\infty/V_r)^4\} \qquad (4.3\text{-}13)$$

$$E\{S_{12}^2\} = E\{(g_x - g_y)^2(V_\infty/V_r)^2\} - E\{(g_x - g_y)^2(V_\infty/V_r)^4\} \qquad (4.3\text{-}14)$$

$$E\{S_{22}^2\} = E\{(g_x - g_y)^2(V_\infty/V_r)^4\} + 2E\{g_y(g_x - g_y)(V_\infty/V_r)^2\} + E\{g_y^2\} \qquad (4.3\text{-}15)$$

$$E\{S_{11}S_{12}\} = E\{g_x(g_x-g_y)(V_\infty/V_r)\} - (\tfrac{1}{2})\, E\{g_x(g_x-g_y)(V_\infty/V_r)^3\}$$

$$- E\{(g_x-g_y)^2(V_\infty/V_r)^3\} + (\tfrac{1}{2})\, E\{(g_x-g_y)^2(V_\infty/V_r)^5\} \qquad (4.3\text{-}16)$$

$$E\{S_{11}S_{22}\} = E\{g_x(g_x-g_y)(V_\infty/V_r)^2\} - E\{(g_x-g_y)^2(V_\infty/V_r)^4\}$$

$$+ E\{g_xg_y)\} - E\{g_y(g_x-g_y)(V_\infty/V_r)^2\}\ . \qquad (4.3\text{-}17)$$

Evaluation of equations (4.3-13) through (4.3-17) is accomplished in Appendix E, and the following material shows how to apply the results obtained to determine the output noise for the cancellation system. The following assumes that T is real.

$$E\{|e_1(k)|^2\} = |A|^2\, E\{N_k\}\, E\{\frac{G^4(\alpha,\beta)}{2}\} \cdot [E\{S_{11}^2\} + T^2\, E\{S_{12}^2\}] \qquad (4.3\text{-}18)$$

$$E\{e_1^*(k)e_2(k)\} = \frac{\mp|A|^2T}{2}\left(\left(E\{X_1(k)Y_2(k)\} - E\{X_2(k)Y_1(k)\}\right)\right.$$

$$\left. -j\left(E\{X_1(k)X_2(k)\} + E\{Y_1(k)Y_2(k)\}\right)\right) \qquad (4.3\text{-}19)$$

$$= \frac{\mp|A|^2}{2}\, TE\{N_k\}\, E\{G^4(\alpha,\beta)\}\left(\pm(T)E\{S_{11}S_{22}-S_{12}^2\}\right.$$

$$\left. -j\, E\{S_{11}S_{21} + S_{12}S_{22}T^2\}\right) \qquad (4.3\text{-}20)$$

$$E\{|e_2(k)|^2\} = T^2\, |A|^2\, E\{N_k\}\, E\{\frac{G^4(\alpha,\beta)}{2}\} \cdot \left(E\{S_{21}^2\} + T^2E\{S_{22}^2\}\right) \qquad (4.3\text{-}21)$$

Quantities such as $E\{G'(k)\}$ and $E\{|G'(k)|^2$ are analyzed in Appendix D and may be used in calculating output noise power. The next step in the analysis process is one designed to simplify both the notational form of many parameters and the calculations involved in evaluating those parameters.

Each of the quantities defined in equations (4.3-18), (4.3-20), and (4.3-21) has a common multiplying factor of $|A|^2 E\{N_k\} E\{G^4(\alpha,\beta)/2\}$. As a result of the work in Appendix A this factor also appears in the expressions for P_{ref} and P_{ord} presented below.

$$P_{ref} = 2|A|^2 E\{N_k\} E\{G^4(\alpha,\beta)/2\} E\{S_{11}^2\} \qquad (4.3\text{-}22)$$

$$P_{ord} = |A|^2 E\{N_k\} E\{G^4(\alpha,\beta)/2\} E\{|S_{11} \pm jTS_{12} - T^2S_{22}|^2\} . \qquad (4.3\text{-}23)$$

The presence of this common factor suggests an algebraic cancellation for the cancellation ratios described in equations (4.0-1) and (4.0-2). The desired effect may be achieved mathematically by the following.

$$C^2 = E\{N_k\} E\{G^4(\alpha,\beta)/2\} . \qquad (4.3\text{-}24)$$

Now introduce the following change of variables.

$$\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{bmatrix} + \begin{bmatrix} 1/C & 0 & 0 & 0 \\ 0 & 1/C & 0 & 0 \\ 0 & 0 & 1/C & 0 \\ 0 & 0 & 0 & 1/C \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \end{bmatrix} \qquad (4.3\text{-}25)$$

The resulting Z variables are Gaussian since the X and Y variables are Gaussian and the transformation is linear [14, page 66]. It is also known that the covariance matrix of the Z variables is related to the covariance matrix of the X and Y variables by the factor $1/C^2$ [14, page 65]. In short, C^2 has now been eliminated from the calculations for the covariance matrix of Z.

Actually, the transformation introduced above eliminates C^2 from the realm of consideration since all terms in the cancellation ratio may be expressed as functions of Z rather than as functions of X and Y. Consider the following.

$$\frac{X_1(\ell)X_2(\ell)}{X_2^2(\ell)+Y_2^2(\ell)} = \frac{(CZ_1)\ (CZ_2)}{(CZ_2)^2+(CZ_4)^2} = \frac{Z_1\ Z_2}{Z_2^2 + Z_4^2}. \qquad (4.3\text{-}26)$$

Terms such as $E\{|e_1(k)|^2\}$ obviously do not involve ratios of X and Y parameters, and therefore, C^2 is not eliminated from this individual quantity. However, the cancellation ratio is a ratio of terms each of which has the multiplying factor C^2 so that the ratio as a whole shows no dependence upon C^2. The quantity $|A|^2$ is also removed from the cancellation ratio in the same manner.

At this point there is available sufficient information in usable forms to allow numerical computation. The quantity of fundamental interest is the cancellation ratio of equation (4.0-1) involving P_{ref}, defined in equation (4.3-22), and P_{can}, the analysis of which has been the center of attention of the present section. The numerator of the cancellation ratio is

$$2\,E\{S_{11}^2\}$$

while the denominator can be shown to be

$$\left(\frac{2M}{2M+1}\right)\left[E\{S_{11}^2\} + T^2\,E\{S_{12}^2\}\right] -$$

$$8\left(\frac{M}{2M+1}\right)^2\left[E\{S_{11}S_{21}\} + T^2\,E\{S_{12}S_{22}\}\right] E\left[\frac{Z_1\,Z_2 + Z_3\,Z_4}{Z_2^2 + Z_4^2}\right] +$$

$$\frac{2M}{(2M+1)^2}\left[E\{S_{21}^2\} + T^2\,E\{S_{22}^2\}\right]E\left[\frac{(Z_1Z_2)^2 + (Z_3Z_4)^2 + (Z_2Z_3)^2 + (Z_1Z_4)^2}{(Z_2^2 + Z_4^2)^2}\right] +$$

$$8\left(\frac{M}{2M+1}\right)^2\left[T\,E\{S_{12}S_{21}\} - T\,E\{S_{11}S_{22}\}\,E\left[\frac{Z_1Z_4 - Z_2Z_3}{Z_2^2 + Z_4^2}\right]\right. +$$

$$\frac{(4M^2-2M)}{(2M+1)^2}\left[E\{S_{21}^2\} + T^2\,E\{S_{22}^2\}\right]\left[E^2\left[\frac{Z_2Z_3 - Z_1Z_4}{Z_2^2 + Z_4^2}\right] + E^2\left[\frac{Z_1Z_2 + Z_3Z_4}{Z_2^2 + Z_4^2}\right]\right] .$$

Data has been obtained which describe the performance of the cancellation system with respect to the reference system and is presented in Figures 4.3-1 through 4.3-4. One new parameter appears in the figures, and it is defined below.

$$\text{IMPROVEMENT} = 10 \cdot \text{Log}_{10}\,(P_{ord}/P_{can}) . \qquad (4.3\text{-}27)$$

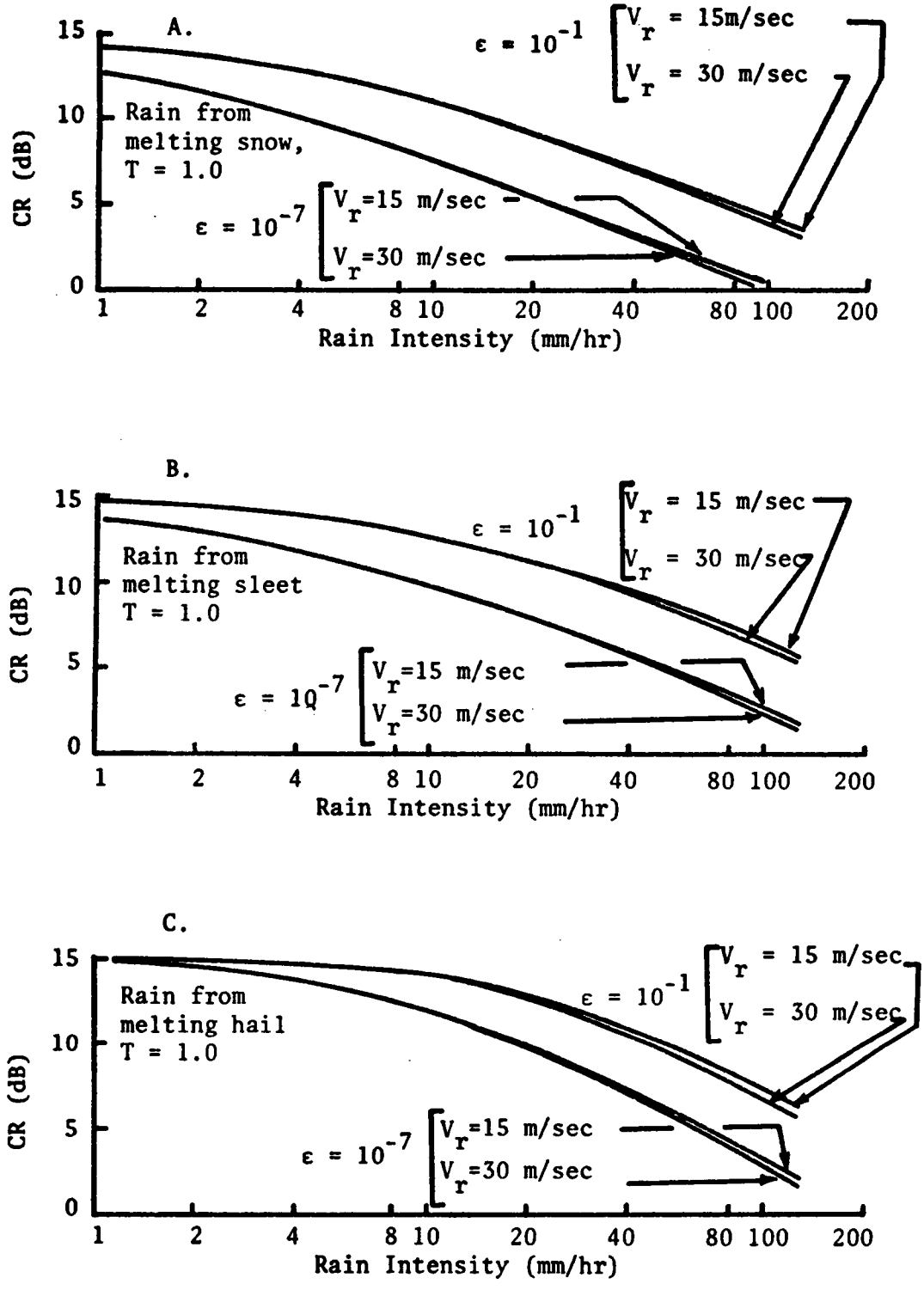


Figure 4.3-1. Lower bound of performance.

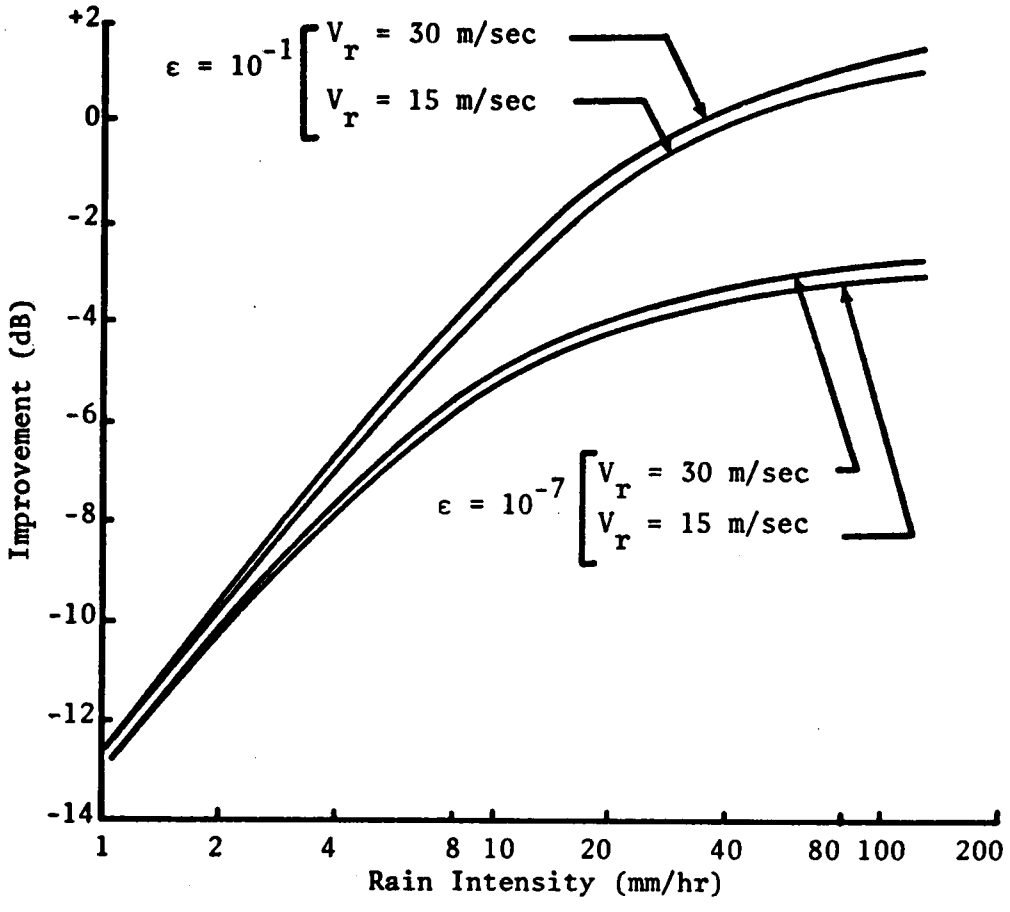


Figure 4.3-2. Lower bound of improvement for rain from melting hail and T = 1.0.

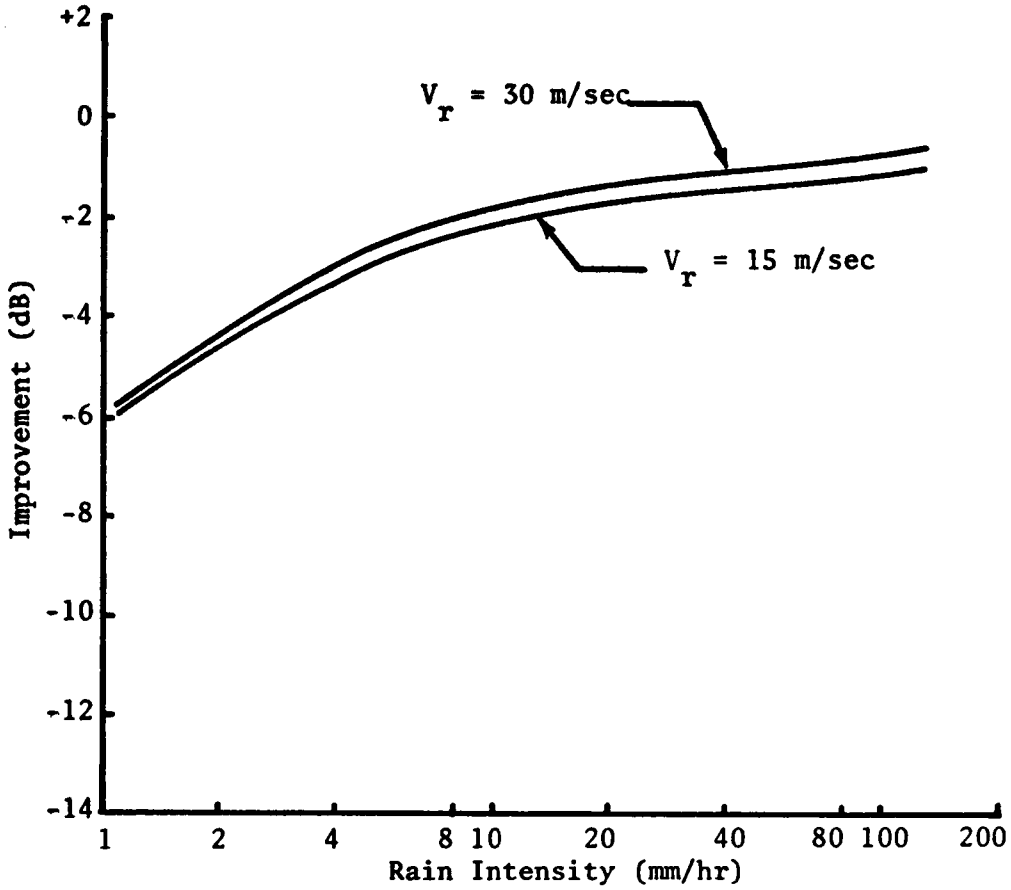


Figure 4.3-3. Lower bound of improvement for rain from melting sleet, $T = 1.0$, and $\varepsilon = 10^{-7}$.

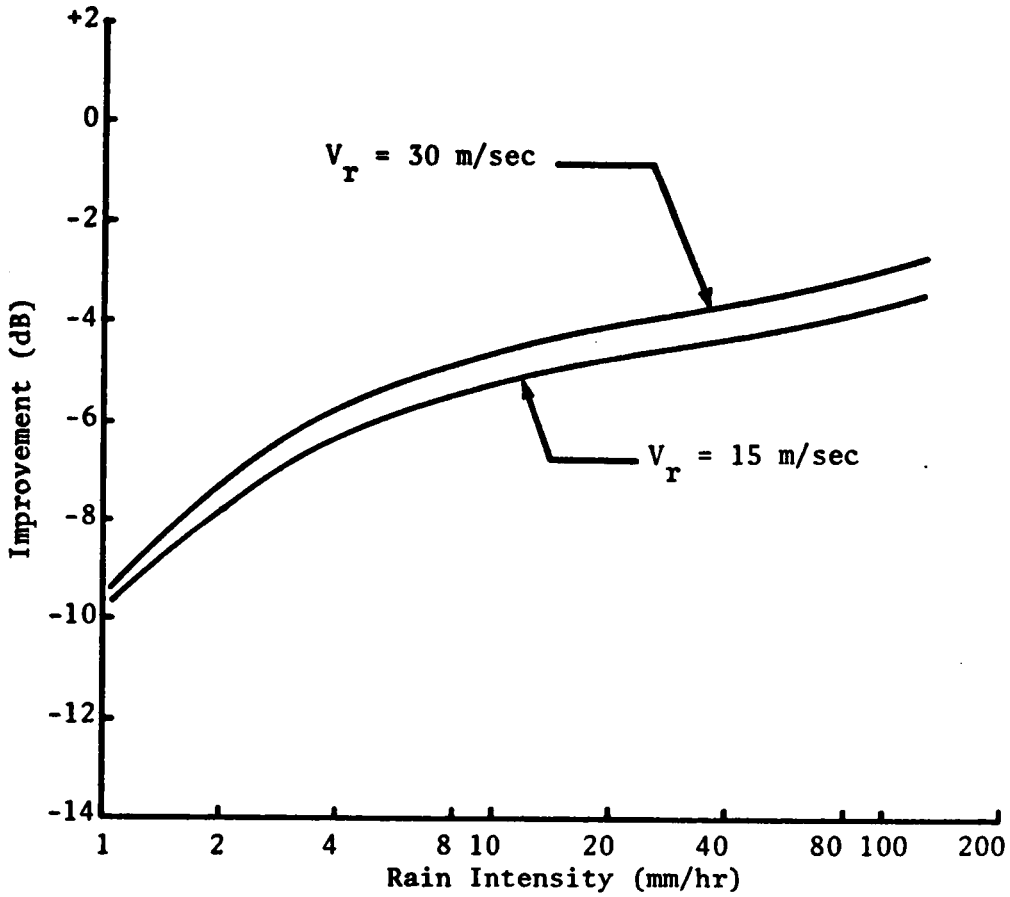


Figure 4.3-4. Lower bound of improvement for rain from melting snow, T = 1.0, and $\varepsilon = 10^{-7}$.

4.4. Upper Bound on Performance

The upper bound of performance of the ordinary system has been shown to occur under the following conditions [1, pages 108-109].

$$\sin^2(\zeta) = 0$$

$$\cos^2(\zeta) = 1$$

$$\cos^2(\psi) = (V_\infty/V_r)^2$$

$$\sin^2(\psi) = 1 - (V_\infty/V_r)^2 \ .$$

These conditions produce the following definitions for the S matrix elements [1, page 109].

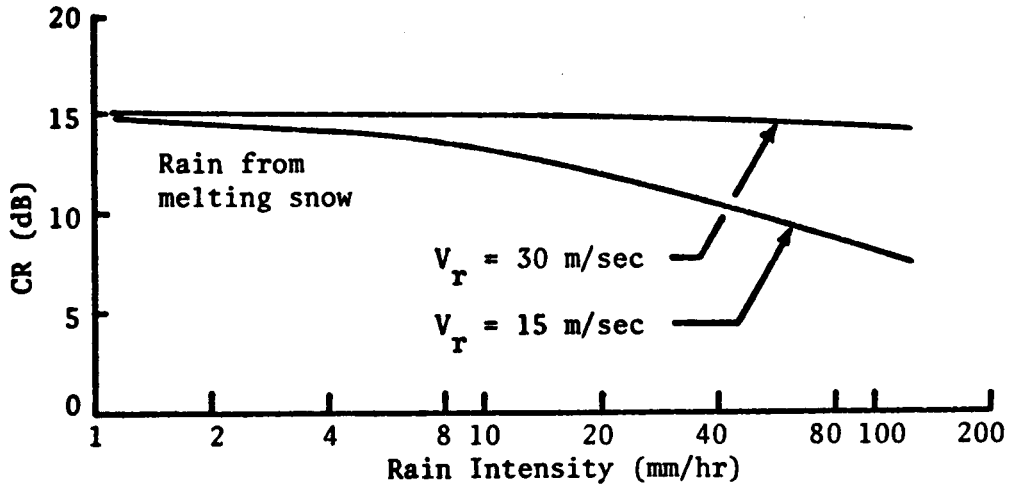
$$S_{11} = g_x$$

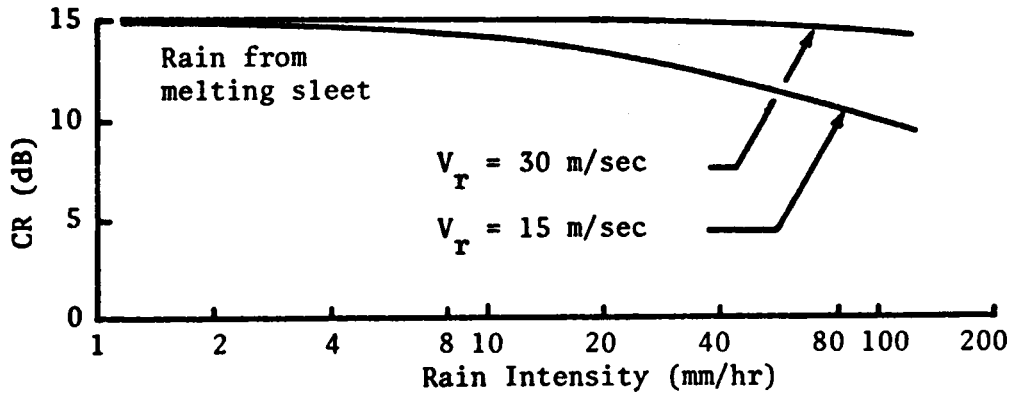
$$S_{12} = 0$$

$$S_{22} = g_y + (g_x - g_y)(V_\infty/V_r)^2 \ .$$

Evaluation of the cancellation system output equation is carried out in the same manner as was used in the analysis of the lower bound case with the exception that the above expressions for the S matrix elements are used rather than those given in Section 4.3. Numerical data for the upper bound conditions are presented in Figures 4.4-1 through 4.4-6.

The performance of the cancellation system may be summarized as follows. When the basic system is assumed to be perfect and the rain





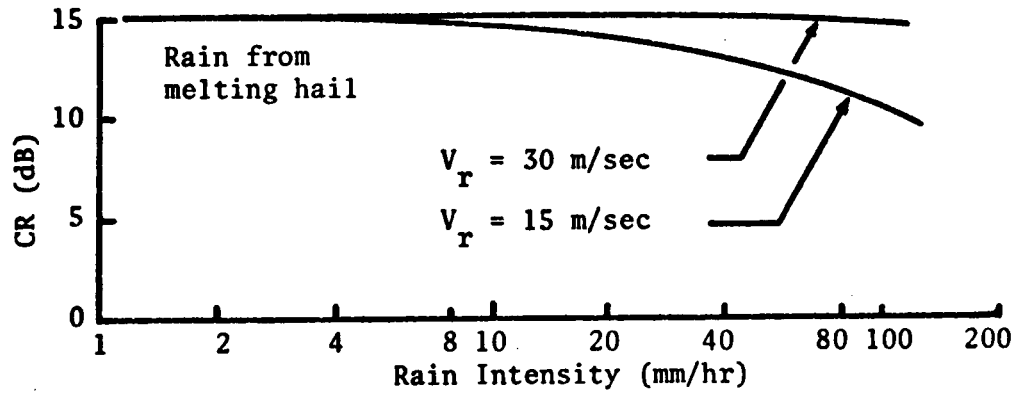


Figure 4.4-1. Upper bound of performance for T = 1.0 and $\varepsilon = 10^{-7}$.

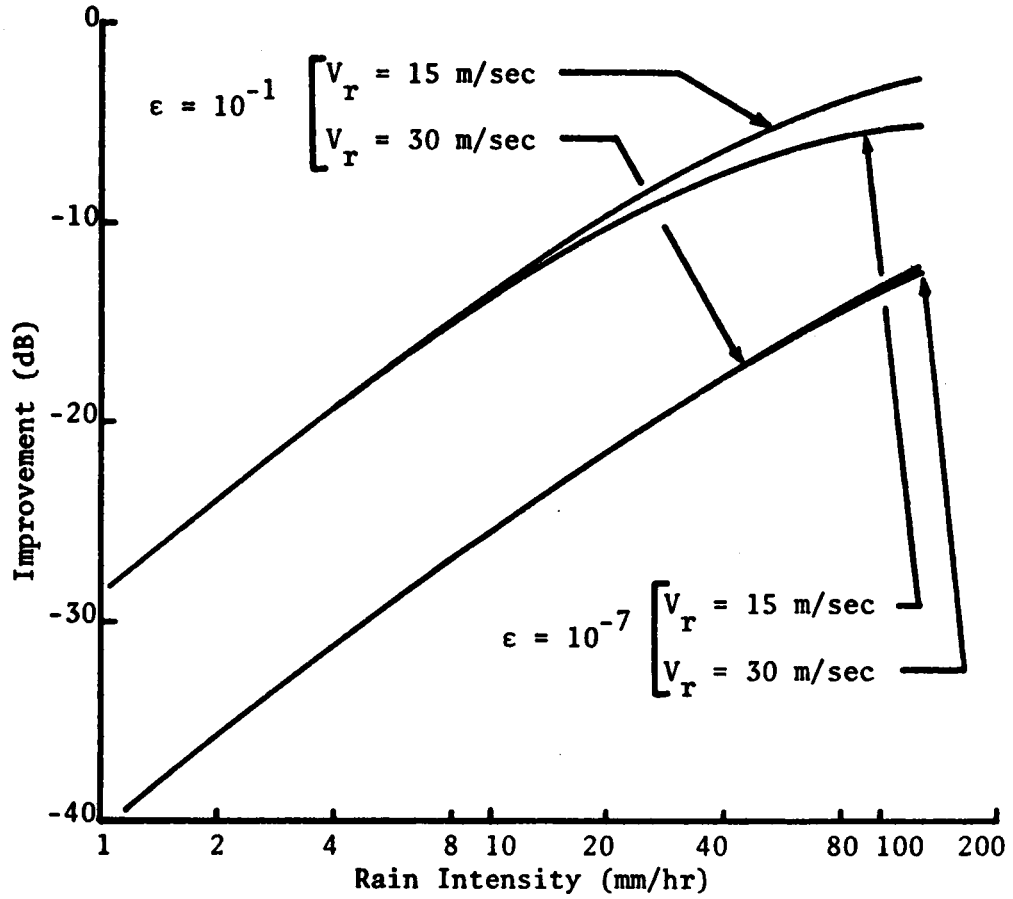


Figure 4.4-2. Upper bound of performance for rain from melting hail and T = 1.0.

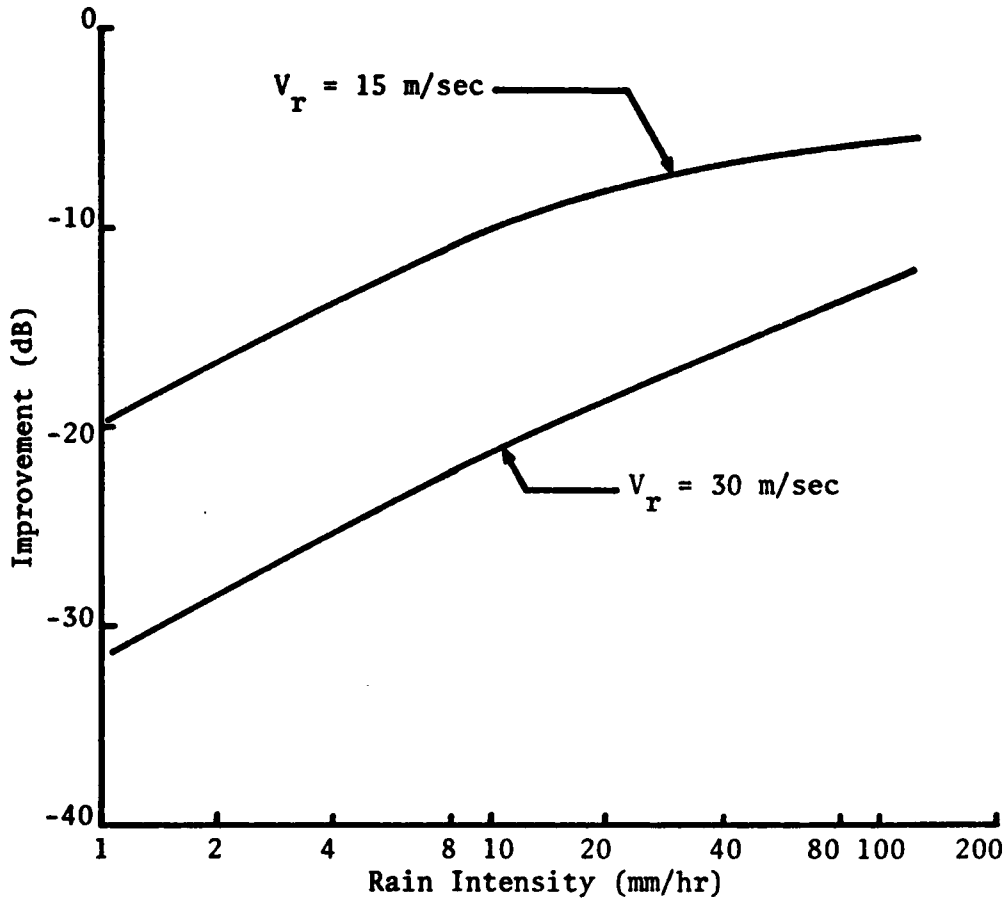


Figure 4.4-3. Upper bound of improvement for rain from melting sleet, T = 1.0, and $\varepsilon = 10^{-7}$.

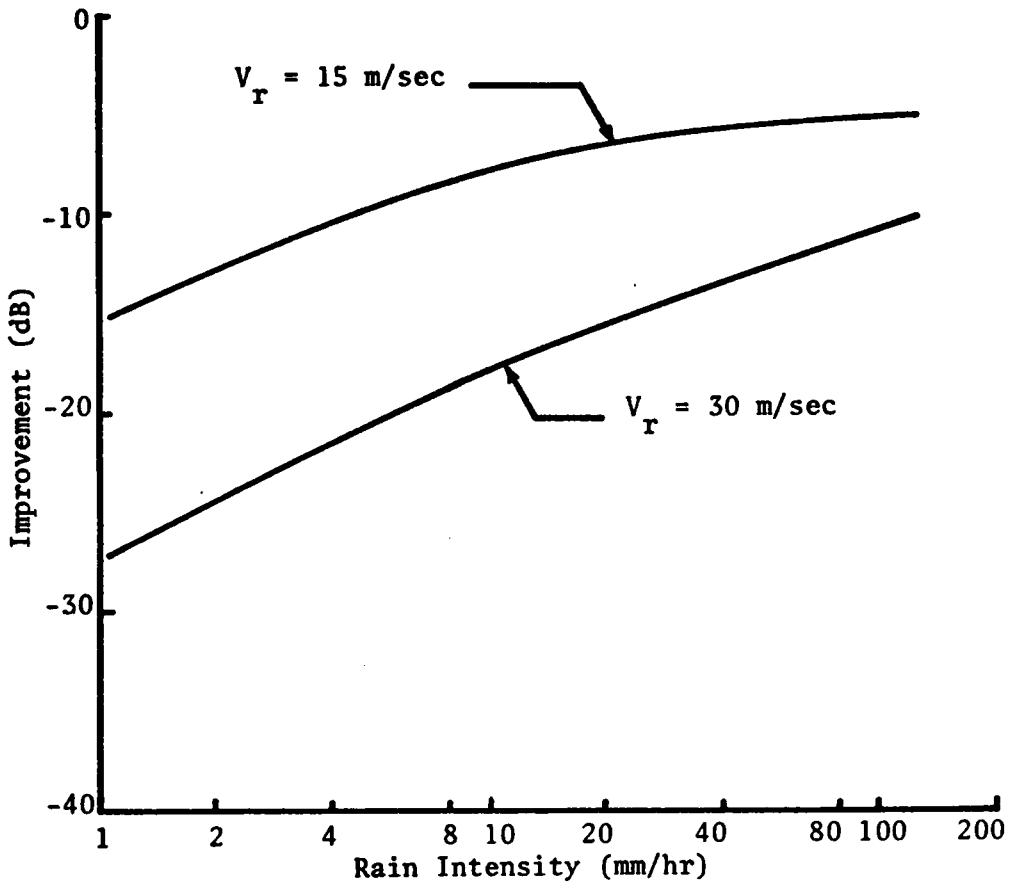


Figure 4.4-4. Upper bound of improvement for rain from melting snow, T = 1.0, and $\epsilon = 10^{-7}$.

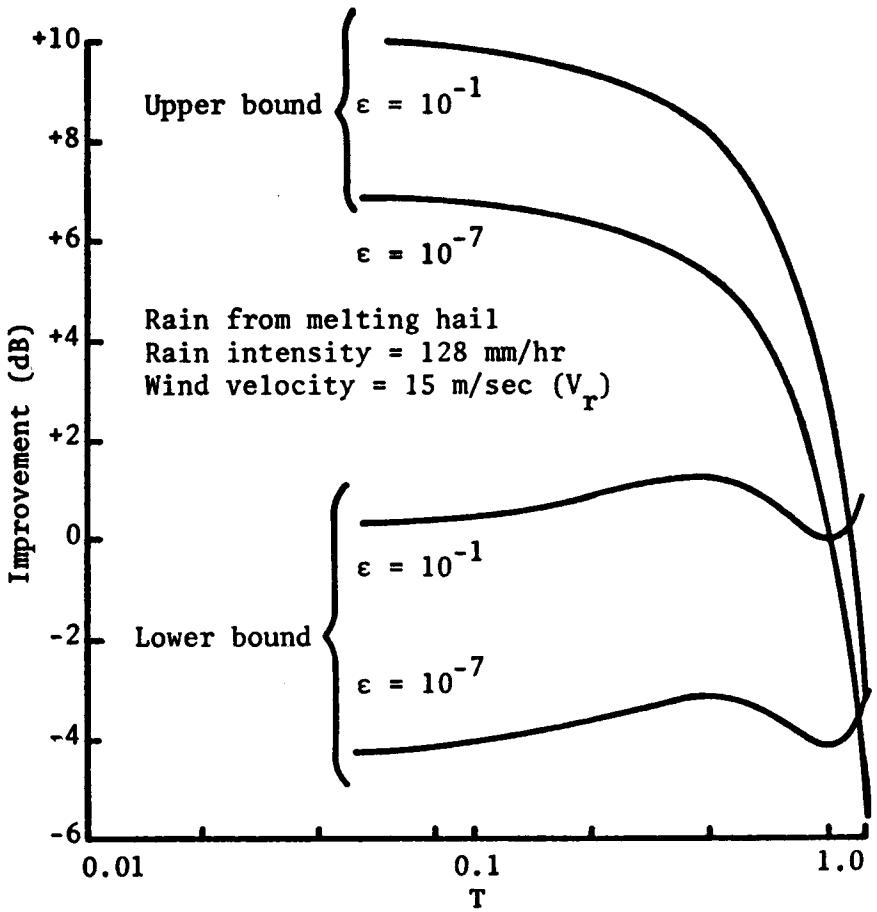


Figure 4.4-5. Improvement as a function of T and ε.

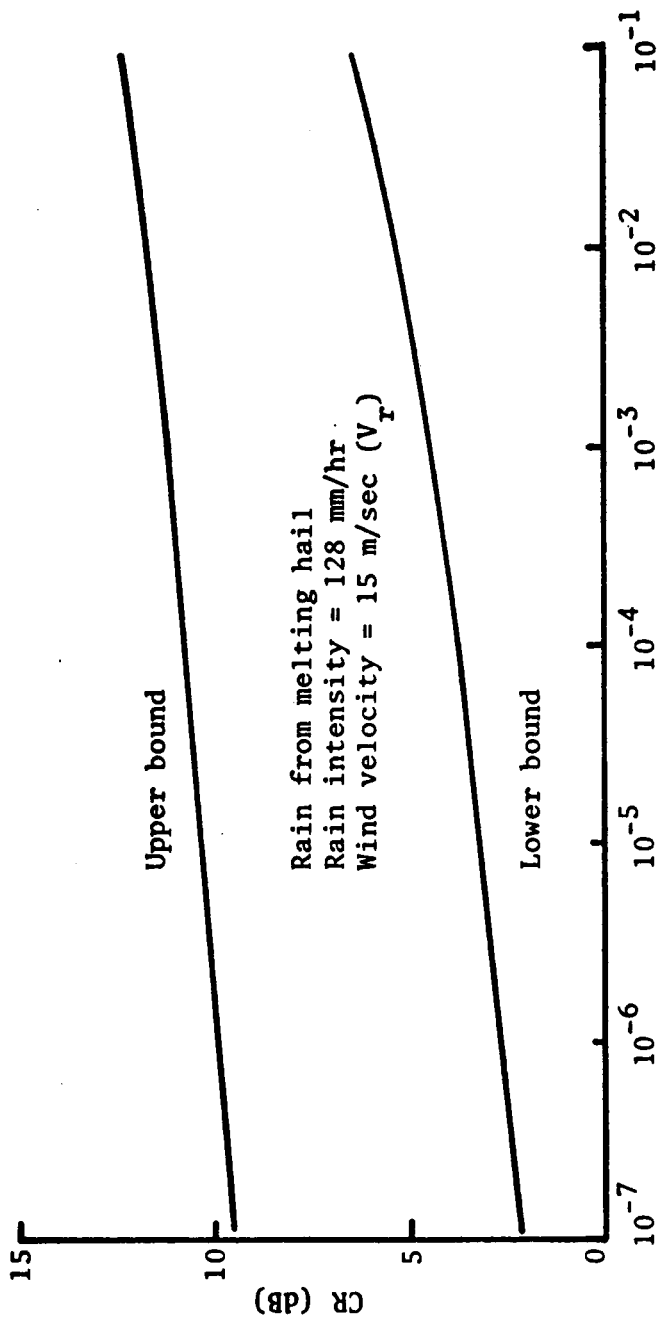


Figure 4.4-6. CR as a function of ε.

intensity is between one and ten mm/hr, the cancellation system offers no less than 8 dB and no more than 15 dB of clutter suppression compared with the reference system. However, under the same conditions the ordinary system offers as much as 40 dB of cancellation above that of the cancellation system.

For rain intensities between ten and 128 mm/hr, the cancellation offered by the cancellation system decreases but at a rate much slower than the decrease for the ordinary system. The net result is that the performance of the cancellation system comes within 3 dB of that of the ordinary system at the higher rain intensities. Additionally, if an imbalance in the ordinary system is allowed, $T \neq 1.0$, then the performance of the cancellation system may surpass that of the ordinary system by as much as 7 dB.

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APPENDIXES

APPENDIX A

ANALYSIS OF MOMENTS FOR RANDOM SUMS

Assume that it is desired to find the statistical moments of the random quantity

$$X = \sum_{i=1}^{n} U_i \tag{A-1}$$

where n and U_i are both random in nature. Beckmann [15, page 138] has shown that the density function of the variable X is

$$f(x) = \sum_{n=0}^{\infty} P(n)\ p(X^{(n)}) \tag{A-2}$$

where P(n) is the discrete density function of n and $p(X^{(n)})$ is the conditional density function of X given a specific value of n.

The mean value of X is calculated as follows.

$$E\{X\} = \int_{-\infty}^{\infty} X\ f(X)\ dX \tag{A-3}$$

$$= \int_{-\infty}^{\infty} X \sum_{n=0}^{\infty} P(n)\ p(X^{(n)})\ dX \tag{A-4}$$

$$= \sum_{n=0}^{\infty} P(n) \int_{-\infty}^{\infty} X^{(n)}\ p(X^{(n)})\ dX^{(n)}\ . \tag{A-5}$$

If $X^{(n)}$ is Gaussian with all U_i having the same distribution then from Thomas [14, page 50]

$$p(X^{(n)}) = \frac{1}{\sigma\sqrt{2\pi n}} \exp\left[\frac{-(X^{(n)} - na)^2}{2n\,\sigma^2}\right] \tag{A-6}$$

where $a = E\{U_i\}$ (A-7)

and $\sigma^2 = E\{(U_i - a)^2\}$. (A-8)

Returning to the mean value calculation,

$$E\{X\} = \sum_{n=0}^{\infty} P(n) \cdot n \cdot E\{U_i\} \tag{A-9}$$

$$= E\{U_i\} \cdot E\{n\} \; . \tag{A-10}$$

The mean squared value of x is

$$E\{X^2\} = \int_{-\infty}^{\infty} X^2 \, f(X) \, dX \tag{A-11}$$

$$= \int_{-\infty}^{\infty} X^2 \sum_{n=0}^{\infty} P(n) \, p(X^{(n)}) \, dX \tag{A-12}$$

$$= \sum_{n=0}^{\infty} P(n) \int_{-\infty}^{\infty} X^{(n)2} \, p(X^{(n)}) \, dX^{(n)} \; . \tag{A-13}$$

Again, if $X^{(n)}$ is Gaussian and the U_i all have the same distribution then

$$E\{X^2\} = \sum_{n=0}^{\infty} P(n) \cdot n \, \sigma^2 \tag{A-14}$$

$$= E\{n\} \cdot \sigma^2 \; . \tag{A-15}$$

The foregoing analysis may be applied to the evaluation of moments of $X_1(k)$, $X_2(k)$, $Y_1(k)$, and $Y_2(k)$. All four quantities are the result

of a summation of random terms, and the number of terms in the summation is also random. Beckmann [15, page 133] observes that terms having the form exhibited by $X_1(k)$, $X_2(k)$, $Y_1(k)$, and $Y_2(k)$ tend to be Gaussian in nature if the number of terms in the appropriate summations is large. The number of terms in the summations corresponds to the number of rain particles in a range cell, and the material of Section 3 indicates that the number of particles in a range cell is generally very large. Thus, the density function for $X_1(k)$ is

$$p(X_1^{(n)}(k)) = \frac{1}{\sigma\sqrt{2\pi n}} \exp \frac{-(X_1^{(n)}(k)-na)^2}{2n\sigma^2} \tag{A-16}$$

where

$$a = E\{G^2(\alpha_i, \beta_i)\alpha_{1i}(k) \cos(\gamma_{1i}(k) - \theta_i)\} \tag{A-17}$$

$$= E\{G^2(\alpha_i, \beta_i)\alpha_{1i}(k) \cos(\gamma_{1i}(k))\cos(\theta_i)\}$$

$$+ E\{G^2(\alpha_i, \beta_i)\alpha_{1i}(k) \sin(\gamma_{1i}(k))\sin(\theta_i)\} \tag{A-18}$$

and

$$\sigma^2 = E\{(U_i - a)^2\} . \tag{A-19}$$

The condition of uniform particle distribution means that θ_i, as defined in equation (4.2-6), is uniformly distributed between the symmetric extremes + θ_M and - θ_M where

$$\theta_M = 2\beta_0\Delta z_M . \tag{A-20}$$

Thus, the density function for θ_i is

$$p(\theta_i) = \frac{1}{2\,\theta_M} . \tag{A-21}$$

If

$$\theta_M = \pm m \ , \ m = 1, 2, 3, \ldots \tag{A-22}$$

then a in equation (A-17) equals zero, but another statement of the condition given in equation (A-22) is

$$2\,\Delta z_M = \frac{ct_p}{2} = \frac{\theta_M}{\beta_0} \tag{A-23}$$

where t_p is the transmitter pulse length and c is the speed of light. Equation (A-23) may be rearranged to obtain

$$t_p = \pm m/f_0 \tag{A-24}$$

where f_0 is the carrier frequency of the transmitter. In the following work it will be assumed that the condition of equation (A-24) is satisfied.

Application of conditions one and two as stated in Section 4.1 to equations (A-18) and (A-19) gives the following.

$$a = E\{G^2(\alpha, \beta)\alpha_1 \cos \gamma\}\, E\{\cos \theta\} + E\{G^2(\alpha, \beta)\alpha_1 \sin \gamma\}\, E\{\sin \theta\} . \tag{A-25}$$

$$= 0 \tag{A-26}$$

$$\sigma^2 = E\{U_i^2\} \tag{A-27}$$

$$= E\{G^4(\alpha, \beta)\alpha_1^2 \cos^2(\gamma_1 - \theta)\} \tag{A-28}$$

$$= \left(\frac{1}{2}\right)\left[E\{G^4(\alpha, \beta)\alpha_1^2\}\right.$$

$$\left. + E\{G^4(\alpha, \beta)\alpha_1^2 \cos(2\gamma_1 - 2\theta)\}\right] \quad \text{(A-29)}$$

$$= \left(\frac{1}{2}\right)E\{G^4(\alpha, \beta)\}\, E\{\alpha_1^2\} \; . \quad \text{(A-30)}$$

Thus, from equations (A-10) and (A-15)

$$E\{X_1(k)\} = 0 \quad \text{(A-31)}$$

$$E\{X_1^2(k)\} = E\{N_k\}\, E\left\{\frac{G^4(\alpha, \beta)}{2}\right\}\, E\{\alpha_1^2\} \; . \quad \text{(A-32)}$$

The moments for $X_2(k)$, $Y_1(k)$, and $Y_2(k)$ are evaluated in a manner similar to that used above for $X_1(k)$, and the results are presented below.

$$E\{X_2(k)\} = 0 \quad \text{(A-33)}$$

$$E\{X_2^2(k)\} = E\{N_k\}\, E\left\{\frac{G^4(\alpha, \beta)}{2}\right\}\, E\{\alpha_2^2\} \quad \text{(A-34)}$$

$$E\{Y_1(k)\} = 0 \quad \text{(A-35)}$$

$$E\{Y_1^2(k)\} = E\{N_k\}\, E\left\{\frac{G^4(\alpha, \beta)}{2}\right\}\, E\{\alpha_1^2\} \quad \text{(A-36)}$$

$$E\{Y_2(k)\} = 0 \quad \text{(A-37)}$$

$$E\{Y_2^2(k)\} = E\{N_k\}\, E\left\{\frac{G^4(\alpha, \beta)}{2}\right\}\, E\{\alpha_2^2\} \; . \quad \text{(A-38)}$$

APPENDIX B

STATISTICAL CONSIDERATIONS OF A PRODUCT

It is assumed that several random processes exist and may be divided into two groups, x and y, such that the members of the groups exhibit the characteristics that x_i and x_j are statistically independent if $i \neq j$, that y_i and y_j are statistically independent if $i \neq j$, and that x_i and y_j are statistically independent if $i \neq j$. Under the stated conditions, one may represent the joint density function of elements of the two sets as follows by application of the concept of conditional probability [15, page 67].

$$p(x_1 , \ldots, x_n, y_1, \ldots y_n) =$$

$$p(x_1,y_1)p(x_2, \ldots x_n, y_2, \ldots y_n \mid x_k, y_1) \qquad \text{(B-1)}$$

The assumed independence requires that the individual or joint outcomes of x_1 and y_1 in no way influence the realizations observed for the other x and y elements [15, page 72]. Thus

$$p(x_2, \ldots, x_n, y_2, \ldots, y_n \mid x_1, y_1) =$$

$$p(x_2, \ldots, x_n, y_2, \ldots y_n), \text{ and} \qquad \text{(B-2)}$$

$$p(x_1, \ldots, x_n, y_1, \ldots, y_n) =$$

$$p(x_1, y_1)p(x_2, \ldots, x_n, y_2, \ldots, y_n) . \qquad \text{(B-3)}$$

Let a function f exist such that

$$f = x_1 \cdot y_1 \cdot g \text{ where} \tag{B-4}$$

$$g = g(x_2, \ldots, x_n, y_2, \ldots, y_n) . \tag{B-5}$$

Then according to Thomas [14, page 54]

$$E\{f\} = \int_{2n} \ldots \int x_1 y_1 g(x_2, \ldots, y_n) p(x_1, y_1) \cdot p(x_2, \ldots, y_n) dx_1 \ldots dy_n \tag{B-6}$$

$$= \int_{2(n-1)} \ldots \int g(x_2, \ldots, y_n) p(x_2, \ldots, y_n) dx_2 \ldots dx_n dy_2 \ldots dy_n \cdot \iint x_1 y_1 \, p(x_1, y_1) dx_1 dy_1 \tag{B-7}$$

$$= E\{g\} \cdot E\{x_1 y_1\} . \tag{B-8}$$

APPENDIX C

THE CROSS CORRELATION OF $X_1(k)$, $X_2(k)$, $Y_1(k)$, AND $Y_2(k)$

The cross correlation of the four parameters $X_1(k)$, $X_2(k)$, $Y_1(k)$, and $Y_2(k)$ is fully described by the six quantities $E\{X_1(k)Y_1(k)\}$, $E\{X_2(k)Y_2(k)\}$, $E\{X_1(k)X_2(k)\}$, $E\{Y_1(k)Y_2(k)\}$, $E\{X_1(k)Y_2(k)\}$, and $E\{X_2(k)Y_1(k)\}$. These quantities will be evaluated in the stated order, and it should be noted that all conditions imposed or assumed in Section 4.1 are applied to the following calculations.

$$X_1(k)Y_1(k) = \sum_{i=1}^{N_k} G^2(\alpha_i, \beta_i)\, \alpha_{1i}(k) \cos\left[\gamma_{1i}(k) - \theta_i\right]$$

$$\cdot \sum_{i=1}^{N_k} G^2(\alpha_i, \beta_i)\, \alpha_{1i}(k) \sin\left[\gamma_{1i}(k) - \theta_i\right] \qquad \text{(C-1)}$$

$$= \sum_{i=1}^{N_k} G^4(\alpha_i, \beta_i)\, \alpha_{1i}^2(k) \sin\left[\gamma_{1i}(k) - \theta_i\right] \cos\left[\gamma_{1i}(k) - \theta_i\right]$$

$$+ \sum_{\substack{i=1 \\ i\neq\ell}}^{N_k} \sum_{\ell=1}^{N_k} G^2(\alpha_i, \beta_i)\, G^2(\alpha_\ell, \beta_\ell) \alpha_{1i}(k)\alpha_{1\ell}(k) \cos\left[\gamma_{1i}(k) - \theta_i\right]$$

$$\cdot \sin\left[\gamma_{1\ell}(k) - \theta_\ell\right] . \qquad \text{(C-2)}$$

Now the results of Appendix A are applied to the expectation operation.

$$E\{X_1(k)Y_1(k)\} = E\{N_k\}\, E\{G^4(\alpha,\beta)\left[E\left\{\frac{\alpha_1^2}{2}\sin 2\gamma_1\right\} \cdot E\{\cos 2\theta\}\right.$$

$$\left. + E\left\{\frac{\alpha_1^2}{2}\cos 2\gamma_1\right\} \cdot E\{\sin 2\theta\}\right] \qquad \text{(C-3)}$$

$$= 0\ . \qquad \text{(C-4)}$$

Several important points should be stressed. First it has been assumed that the beam parameters α and β, the particle orientation parameters α_1 and γ_1, and the particle location parameter θ are jointly independent in the statistical sense. The appearance of a product of expectations rather than an expectation of a product in equation (C-3) is a result of the assumed joint independence. Secondly, the zero result of equation (C-4) is an outgrowth of the definition of t_p in Appendix A since under the stated constraint

$$E\{\cos(n\theta)\} = E\{\sin(n\theta)\} = 0.$$

$$X_2(k)Y_2(k) = \sum_{i=1}^{N_k} G^2(\alpha_i,\beta_i)\alpha_{2i}(k)\cos\left[\gamma_{2i}(k) - \theta_i\right]$$

$$\cdot \sum_{i=1}^{N_k} G^2(\alpha_i,\beta_i)\alpha_{2i}(k)\sin\left[\gamma_{2i}(k) - \theta_i\right] \qquad \text{(C-5)}$$

$$= \sum_{i=1}^{N_k} G^4(\alpha_i,\beta_i)\alpha_{2i}^2(k)\cos\left[\gamma_{2i}(k) - \theta_i\right]\sin\left[\gamma_{2i}(k) - \theta_i\right] +$$

$$\sum_{i=1}^{N_k}\sum_{\substack{\ell=1\\ \ell\neq i}}^{N_k} G^2(\alpha_i,\beta_i)G^2(\alpha_\ell,\beta_\ell)\gamma_{2i}(k)\alpha_{2\ell}(k)\cos\left(\gamma_{2i}(k)-\theta_i\right)\sin\left(\gamma_{2\ell}(k)-\theta_\ell\right) \tag{C-6}$$

$$E\{X_2(k)Y_2(k)\} = E\{N_k\}\ E\{G^4(\alpha,\beta)\}\left[E\left\{\frac{\alpha_2^2}{2}\sin 2\gamma_2\right\}\cdot E\{\cos 2\theta\}\right.$$

$$\left. + E\left\{\frac{\alpha_2^2}{2}\cos 2\gamma_2\right\}\cdot E\{\sin 2\theta\}\right] \tag{C-7}$$

$$= 0 \tag{C-8}$$

$$X_1(k)X_2(k) = \sum_{i=1}^{N_k} G^2(\alpha_i,\beta_i)\alpha_{1i}(k)\cos\left(\gamma_{1i}(k) - \theta_i\right)$$

$$\cdot \sum_{i=1}^{N_k} G^2(\alpha_i,\beta_i)\alpha_{2i}(k)\cos\left(\gamma_{2i}(k) - \theta_i\right) \tag{C-9}$$

$$= \sum_{i=1}^{N_k} G^4(\alpha_i,\beta_i)\,\frac{\alpha_{1i}(k)\alpha_{2i}(k)}{2}\left[\cos\left(\gamma_{1i}(k) + \gamma_{2i}(k) - 2\theta_i\right)\right.$$

$$\left. + \cos\left(\gamma_{1i}(k) - \gamma_{2i}(k)\right)\right] +$$

$$\sum_{i=1}^{N_k}\sum_{\substack{\ell=1\\ i\neq\ell}}^{N_k} G^2(\alpha_i,\beta_i)G^2(\alpha_\ell,\beta_\ell)\alpha_{1i}(k)\alpha_{2\ell}(k)\cos\left(\gamma_{1\ell}(k)-\theta_\ell\right)\cos\left(\gamma_{2\ell}(k)-\theta_\ell\right) \tag{C-10}$$

$$E\{X_1(k)X_2(k)\} = E\{N_k\}\ E\{G^4(\alpha,\beta)\}\left[E\left\{\frac{\alpha_1\alpha_2}{2}\cos(\gamma_1+\gamma_2)\right\}\ E\{\cos 2\theta\}\right.$$

$$\left. + E\left\{\frac{\alpha_1\alpha_2}{2}\sin(\gamma_1+\gamma_2)\right\}\ E\{\sin 2\theta\} + E\left\{\frac{\alpha_1\alpha_2}{2}\cos(\gamma_1-\gamma_2)\right\}\right] \tag{C-11}$$

$$= E\{N_k\}\ E\{G^4(\alpha,\beta)\}\ E\left\{\frac{\alpha_1\alpha_2}{2}\cos(\gamma_1-\gamma_2)\right\} \qquad \text{(C-12)}$$

The following expressions, obtained from equations (4.2-4) and (4.2-5), may be applied to equation (C-12).

$$\cos\gamma_1 = \frac{S_{11} \pm S_{12}\ \mathrm{Im}(T)}{\alpha_1} \qquad \text{(C-13)}$$

$$\sin\gamma_1 = \frac{\pm S_{12}\ R_e(T)}{\alpha_1} \qquad \text{(C-14)}$$

$$\cos\gamma_2 = \frac{S_{21} \pm S_{22}\ \mathrm{Im}(T)}{\alpha_2} \qquad \text{(C-15)}$$

$$\sin\gamma_2 = \frac{\pm S_{22}\ R_e(T)}{\alpha_2} \qquad \text{(C-16)}$$

Thus equation (C-12) may be rewritten as

$$E\{X_1(k)X_2(k)\} = E\{N_k\}\ E\{G^4(\alpha,\beta)\}\left\{E\left\{\frac{\alpha_1\alpha_2}{2}\cos\gamma_1\cos\gamma_2\right\}\right.$$

$$\left. + E\left\{\frac{\alpha_1\alpha_2}{2}\sin\gamma_1\sin\gamma_2\right\}\right\} \qquad \text{(C-17)}$$

$$= E\{N_k\}\ E\{G^4(\alpha,\beta)\}\left\{\left(\tfrac{1}{2}\right)\ E\left\{\left(S_{11} \pm S_{12}\mathrm{Im}(T)\right)\left(S_{21} \pm S_{22}\mathrm{Im}(T)\right)\right.\right.$$

$$\left.\left. + S_{12}S_{22}\mathrm{Re}^2(T)\right\}\right\} . \qquad \text{(C-18)}$$

The fourth term to be evaluated is $E\{Y_1(k)Y_2(k)\}$.

$$Y_1(k)Y_2(k) = \sum_{i=1}^{N_k} G^2(\alpha_i,\alpha_i)\alpha_{1i}(k)\sin\left(\gamma_{1i}(k)-\theta_i\right)$$

$$\cdot \sum_{i=1}^{N_k} G^2(\alpha_i,\beta_i)\alpha_{2i}(k)\sin\left(\gamma_{2i}(k)-\theta_i\right) \tag{C-19}$$

$$= \sum_{i=1}^{N_k} G^4(\alpha_i,\beta_i)\,\frac{\alpha_{1i}(k)\alpha_{2i}(k)}{2}\left[\cos\left(\gamma_1(k)-\gamma_2(k)\right)\right.$$

$$\left. - \cos\left(\gamma_1(k)-\gamma_2(k)-2\theta_i\right)\right] +$$

$$\sum_{\substack{i=1\\ i\neq\ell}}^{N_k}\sum_{\ell=1}^{N_k} G^2(\alpha_i,\beta_i)G^2(\alpha_\ell,\beta_\ell)\,\frac{\alpha_{1i}(k)\alpha_{2\ell}(k)}{2}\left[\cos\left(\gamma_{1i}(k)-\theta_i-\gamma_{2\ell}(k)-\theta_\ell\right)\right.$$

$$\left. - \cos\left(\gamma_{1i}(k)-\theta_i + \gamma_{2\ell}(k)-\theta_\ell\right)\right] \tag{C-20}$$

$$E\{Y_1(k)Y_2(k)\} = E\{N_k\}\,E\{G^4(\alpha,\beta)\}\,E\left[\frac{\alpha_1\alpha_2}{2}\cos(\gamma_1-\gamma_2)\right]$$

$$= E\{X_1(k)X_2(k)\} \tag{C-21}$$

$$X_1(k)Y_2(k) = \sum_{i=1}^{N_k} G^2(\alpha_i,\beta_i)\alpha_{1i}(k)\cos\left(\gamma_{1i}(k)-\theta_i\right)\cdot$$

$$\sum_{i=1}^{N_k} G^2(\alpha_i,\beta_i)\alpha_{2i}(k)\sin\left(\gamma_{2i}(k)-\theta_i\right) \tag{C-22}$$

$$= \sum_{i=1}^{N_k} G^4(\alpha_i,\beta_i)\,\frac{\alpha_{1i}(k)\alpha_{2i}(k)}{2}\left[\sin\left(\gamma_{1i}(k)+\gamma_{2i}(k)-2\theta_i\right)\right.$$

$$\left. + \sin\left(\gamma_{2i}(k)-\gamma_{1i}(k)\right)\right] +$$

$$\sum_{\substack{i=1 \\ i\neq\ell}}^{N_k} \sum_{\ell=1}^{N_k} G^2(\alpha_i,\beta_i)G^2(\alpha_\ell,\beta_\ell) \frac{\alpha_{1i}(k)\alpha_{2\ell}(k)}{2}\left[\sin\left(\gamma_{1i}(k)+\gamma_{2\ell}(k)-\theta_i-\theta_\ell\right)\right.$$

$$\left. + \sin\left(\gamma_{2\ell}(k)-\gamma_{1i}(k)-\theta_\ell+\theta_i\right)\right] \tag{C-23}$$

$$E\{X_1(k)Y_2(k)\} = E\{N_k\}\ E\{G^4(\alpha,\beta)\}\ E\left[\frac{\alpha_1\alpha_2}{2}\sin(\gamma_2-\gamma_1)\right] \tag{C-24}$$

$$= \pm\ E\{N_k\}\ E\{G^4(\alpha,\beta)\}\ \left(\tfrac{1}{2}\right)\ E\left[S_{22}\left(S_{11} \pm S_{12}\mathrm{Im}(T)\right)\right.$$

$$\left. \cdot\ \mathrm{Re}(T) - S_{12}\left(S_{21} \pm S_{22}\mathrm{Im}(T)\right)\ \mathrm{Re}(T)\right] \tag{C-25}$$

$$= \pm\ E\{N_k\} \cdot E\{G^4(\alpha,\beta)\} \cdot \frac{\mathrm{Re}(T)}{2}\ E\{S_{22}[S_{11} \mp S_{12}\mathrm{Im}(T)]$$

$$- S_{12}[S_{21} \mp S_{22}\mathrm{Im}(T)]\} \tag{C-26}$$

$$X_2(k)Y_1(k) = \sum_{i=1}^{N_k} G^2(\alpha_i,\beta_i)\alpha_{2i}(k)\cos\left(\gamma_{2i}(k)-\theta_i\right)$$

$$\cdot \sum_{i=1}^{N_k} G^2(\alpha_i,\beta_i)\alpha_{1i}(k)\sin\left(\gamma_{1i}(k)-\theta_i\right) \tag{C-27}$$

$$= \sum_{i=1}^{N_k} G^4(\alpha_i,\beta_i)\ \frac{\alpha_{1i}(k)\alpha_{2i}(k)}{2}\left[\sin\left(\gamma_{1i}(k)+\gamma_{2i}(k)-2\theta_i\right)\right.$$

$$\left. + \sin\left(\gamma_{1i}(k)-\gamma_{2i}(k)\right)\right] +$$

$$\sum_{\substack{i=1 \\ i\neq\ell}}^{N_k} \sum_{\ell=1}^{N_k} G^2(\alpha_i,\beta_i)G^2(\alpha_\ell\ \beta_\ell)\ \frac{\alpha_{2i}(k)\alpha_{1\ell}(k)}{2}\left\{\sin\left(\gamma_{1\ell}(k)+\gamma_{2i}(k)-\theta_i-\theta_\ell\right)\right.$$

$$\left. + \sin\left(\gamma_{1\ell}(k)-\gamma_{2i}(k)-\theta_\ell+\theta_i\right)\right\} \qquad \text{(C-28)}$$

$$E\{X_2(k)Y_1(k)\} = E\{N_k\}\ E\{G^4(\alpha,\beta)\}\ E\left\{\frac{\alpha_1\alpha_2}{2}\sin(\gamma_1-\gamma_2)\right\} \qquad \text{(C-29)}$$

$$= -E\{N_k\}\ E\{G^4(\alpha,\beta)\}\ E\left\{\frac{\alpha_1\alpha_2}{2}\sin(\gamma_2-\gamma_1)\right\} \qquad \text{(C-30)}$$

$$= -E\{X_1(k)Y_2(k)\}\ . \qquad \text{(C-31)}$$

APPENDIX D

THE EXPECTATION OF G'(k) AND $|G'(k)|^2$

Evaluation of $E\{G'(k)\}$ requires the determination of

$$E\left\{\frac{X_1(\ell)X_2(\ell)}{X_2^2(\ell)+Y_2^2(\ell)}\right\}, \quad E\left\{\frac{Y_1(\ell)Y_2(\ell)}{X_2^2(\ell)+Y_2^2(\ell)}\right\}, \quad E\left\{\frac{X_1(\ell)Y_2(\ell)}{X_2^2(\ell)+Y_2^2(\ell)}\right\}, \text{ and } E\left\{\frac{X_2(\ell)Y_1(\ell)}{X_2^2(\ell)+Y_2^2(\ell)}\right\}$$

where

$$E\left\{\frac{X_1(\ell)X_2(\ell)}{X_2^2(\ell)+Y_2^2(\ell)}\right\} = \iiiint \frac{X_1(\ell)X_2(\ell)}{X_2^2(\ell)+Y_2^2(\ell)}\, p\left\{X_1(\ell),X_2(\ell),Y_1(\ell),Y_2(\ell)\right\} dX_1(\ell)\ldots dY_2(\ell). \tag{D-1}$$

Wilkes [16, page 168] indicates that integration of the density function, $p(\cdot)$, with respect to $Y_1(\ell)$ will result in a marginal density function of three variables with the same means and variances for $X_2(\ell)$, $X_1(\ell)$, and $Y_2(\ell)$. Thus,

$$E\left\{\frac{X_1(\ell)X_2(\ell)}{X_2^2(\ell)+Y_2^2(\ell)}\right\} = \iiint \frac{X_1(\ell)X_2(\ell)}{X_2^2(\ell)+Y_2^2(\ell)}\, p\left\{X_1(\ell),\ X_2(\ell),Y_2(\ell)\right\} dX_1(\ell)\ldots dY_2(\ell). \tag{D-2}$$

Appendix A points out that the X and Y parameters are zero mean Gaussian variates, and thus the joint density functions may be expressed in the form

$$p\left\{X_1(\ell),X_2(\ell),Y_2(\ell)\right\} = K_1 e^{-[C_1X_1^2 + C_2X_1 + C_3]}. \tag{D-3}$$

The quadratic form appearing in the exponential is known to be positive definite [17, page 233], and thus C_1 is known to be positive [18, page 416]. Therefore, after completing the square in the exponent and integrating with respect to $X_1(\ell)$, the following result is obtained [19, page 349].

$$E\left[\frac{X_1(\ell)X_2(\ell)}{X_2^2(\ell)+Y_2^2(\ell)}\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[\frac{-K_1C_2\sqrt{\pi}}{2C_1\sqrt{C_1}}\right]\left[\frac{X_2(\ell)}{X_2^2(\ell)+Y_2^2(\ell)}\right] e^{-[C_3 - \frac{C_2^2}{4C_1}]}\, dX_2(\ell)dY_2(\ell) \tag{D-4}$$

where the inverse of the covariance matrix of the variables $X_1(\ell)$, $X_2(\ell)$, and $Y_2(\ell)$ is

$$\Lambda_B^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \tag{D-5}$$

and

$$C_1 = a/2 \tag{D-6}$$

$$C_2 = [(b+d)X_2(\ell)+(c+g)Y_2(\ell)]/2 \tag{D-7}$$

$$C_3 = [eX_2^2(\ell)+(f+h)X_2(\ell)Y_2(\ell)+iY_2^2(\ell)]/2 \tag{D-8}$$

$$K_1 = \frac{1}{(2\pi)^{3/2} \cdot \sqrt{\det(\Lambda_B)}} . \tag{D-9}$$

Equation (D-4) may be rewritten as follows by applying the definitions in equations (D-6) through (D-8).

$$E\left\{\frac{X_1(\ell)X_2(\ell)}{X_2^2(\ell)+Y_2^2(\ell)}\right\} = \frac{-K_1\sqrt{\pi}}{4C_1\sqrt{C_1}} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{X_2(\ell)[m_4X_2(\ell)+m_5Y_2(\ell)]}{X_2^2(\ell)+Y_2^2(\ell)}\, e^{-n}\, dX_2(\ell)dY_2(\ell)$$

(D-10)

$$n = m_1X_2^2(\ell) + m_2X_2(\ell)Y_2(\ell) + m_3Y_2^2(\ell) \quad \text{(D-11)}$$

$$m_1 = \frac{e}{2} - \frac{(b+d)^2}{8a} \quad \text{(D-12)}$$

$$m_2 = \frac{(f+h)}{2} - \frac{(b+d)(c+g)}{4a} \quad \text{(D-13)}$$

$$m_3 = \frac{i}{2} - \frac{(c+g)^2}{8a} \quad \text{(D-14)}$$

$$m_4 = b + d \quad \text{(D-15)}$$

$$m_5 = c + g\ . \quad \text{(D-16)}$$

Application of the results of Appendix C reveals that the covariance matrix Λ_B has the form

$$\Lambda_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ B_2 & B_4 & 0 \\ B_3 & 0 & B_4 \end{pmatrix} \quad \text{(D-17)}$$

and thus it is possible to show that $m_1 = m_3$ and that $m_2 = 0$.

The above findings for m_1, m_2, and m_3 used in conjunction with the change of variables

$$X_2(\ell) = r\cos(\theta) \quad \text{(D-18)}$$

and

$$Y_2(\ell) = r\sin(\theta) \quad \text{(D-19)}$$

produce the following integral.

$$E\left\{\frac{X_1(\ell)X_2(\ell)}{X_2^2(\ell)+Y_2^2(\ell)}\right\} = \frac{-K_1\sqrt{\pi}}{4C_1\sqrt{C_1}} \int_0^{2\pi}\int_0^{\infty} [m_4\cos^2(\theta)+m_5\sin(\theta)\cos(\theta)]e^{-m_1r^2} r\,dr\,d\theta \tag{D-20}$$

$$= \frac{-K_1\sqrt{\pi}}{4C_1\sqrt{C_1}} \int_0^{2\pi} [m_4\cos^2(\theta)+m_5\sin(\theta)\cos(\theta)]\left[\int_0^{\infty} r\, e^{-m_1r^2}\, dr\right]d\theta \; . \tag{D-21}$$

Consider first the integration with respect to r. It is possible to show that

$$m_1 = 1/(2B_4) \tag{D-22}$$

where

$$B_4 = E\{X_2^2(\ell)\} \; . \tag{D-23}$$

Since B_4 is positive [14, page 101], m_1 must also be positive, and the integration in r becomes [19, page 349]

$$\int_0^{\infty} r\, e^{-m_1r^2}\, dr = \frac{1}{2m_1} \; . \tag{D-24}$$

The integration with respect to θ is even simpler [19, page 347].

$$\int_0^{2\pi} [m_4\cos^2(\theta) + m_5\sin(\theta)\cos(\theta)]d\theta = \pi m_4 \; . \tag{D-25}$$

The results of equations (D-24) and (D-25) may be combined to produce the following.

$$E\left\{\frac{X_1(\ell)X_2(\ell)}{X_2^2(\ell)+Y_2^2(\ell)}\right\} = \frac{-K_1\, m_4}{8m_1}\left[\frac{\pi}{C_1}\right]^{3/2} \; . \tag{D-26}$$

A similar approach is used in the evaluation of the other expectation terms. The results are presented below.

$$E\left[\frac{X_1(\ell)Y_2(\ell)}{X_2^2(\ell)+Y_2^2(\ell)}\right] = \frac{-K_1\ m_5}{8m_1}\left[\frac{\pi}{C_1}\right]^{3/2} \tag{D-27}$$

$$E\left[\frac{Y_1(\ell)Y_2(\ell)}{X_2^2(\ell)+Y_2^2(\ell)}\right] = \frac{-K_2\ m_4}{8m_1}\left[\frac{\pi}{C_4}\right]^{3/2} \tag{D-28}$$

$$E\left[\frac{Y_1(\ell)X_2(\ell)}{X_2^2(\ell)+Y_2^2(\ell)}\right] = \frac{-K_2\ m_5}{8m_1}\left[\frac{\pi}{C_4}\right]^{3/2} . \tag{D-29}$$

In the last two expressions, the quantities m_1, m_4, m_5, C_4, and K_2 are derived from the covariance matrix Λ_F of the variables $Y_1(\ell)$, $Y_2(\ell)$, and $X_2(\ell)$.

$$\Lambda_F^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \tag{D-30}$$

$$C_4 = a/2 \tag{D-31}$$

$$m_1 = \frac{e}{2} - \frac{(b+d)^2}{8a} \tag{D-32}$$

$$m_4 = b + d \tag{D-33}$$

$$m_5 = c + g \tag{D-34}$$

$$K_2 = \frac{1}{(2\pi)^{3/2}\sqrt{\det(\Lambda_F)}} \tag{D-35}$$

The evaluation of $E\{|G'(k)|^2\}$ involves such terms as

$$E\left\{\frac{X_1^2(\ell)X_2^2(\ell)}{[X_2^2(\ell)+Y_2^2(\ell)]^2}\right\} .$$

By using the density function of equation (D-3), this expectation may be represented as follows.

$$E\left\{\frac{X_1^2(\ell)X_2^2(\ell)}{[X_2^2(\ell)+Y_2^2(\ell)]^2}\right\} = \iiint \frac{X_1^2(\ell)X_2^2(\ell)}{[X_2^2(\ell)+Y_2^2(\ell)]^2} p(X_1(\ell),X_2(\ell),Y_2(\ell)) \cdot$$

$$dX_1(\ell)dX_2(\ell)dY_2(\ell) . \qquad \text{(D-36)}$$

The integration with respect to $X_1(\ell)$ is of the following form [19, page 349].

$$\int_{-\infty}^{+\infty} X_1^2(\ell)\ K_1\ e^{-(C_1X_1^2(\ell)+C_2X_1(\ell)+C_3)}\, dX_1(\ell) =$$

$$K_1\ e^{-[C_3-(C_2^2/4C_1)]}\ \frac{\sqrt{\pi}}{2C_1\sqrt{C_1}}\left[1+\frac{C_2^2}{2\ C_1}\right] \qquad \text{(D-37)}$$

In the above expression the quantities K_1, C_1, C_2, and C_3 are given by equations (D-6) through (D-9). The expectation may now be represented as a two dimensional integral.

$$E\left\{\frac{X_1^2(\ell)X_2^2(\ell)}{[X_2^2(\ell)+Y_2^2(\ell)]^2}\right\} = \frac{K_1\sqrt{\pi}}{2C_1\ C_1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{X_2^2(\ell)}{[X_2^2(\ell)+Y_2^2(\ell)]^2}\left[1+\frac{C_2^2}{2C_1}\right]\cdot$$

$$e^{-[C_3 - (C_2^2/2C_1)]} dX_2(\ell)dY_2(\ell) \quad . \tag{D-38}$$

The following change of variables is now made in the integration.

$$X_2(\ell) = r\cos(\theta) \tag{D-39}$$

$$Y_2(\ell) = r\sin(\theta) \tag{D-40}$$

$$E\left\{\frac{X_1^2(\ell)X_2^2(\ell)}{[X_2^2(\ell)+Y_2^2(\ell)]^2}\right\} = \frac{K_1\sqrt{\pi}}{2C_1\sqrt{C_1}} \int_0^{2\pi}\int_0^{\infty}\left\{1 + \frac{(m_4\cos(\theta)+m_5\sin(\theta))^2r^2}{8C_1}\right\} \cdot$$

$$\frac{\cos^2(\theta)}{r} e^{-vr^2} \, drd\theta \tag{D-41}$$

$$v = m_1\cos^2(\theta) + m_2\sin(\theta)\cos(\theta) + m_3\sin^2(\theta) \quad . \tag{D-42}$$

Recalling that $m_1 = m_3$ and that $m_2 = 0$, allows one to show that

$$v = m_1 = m_3 \quad . \tag{D-43}$$

Integration with respect to r has two parts in the above expectation. The first part is similar to that encountered before.

$$\int_0^{\infty} r\, e^{-vr^2} dr = \frac{1}{2v} \tag{D-44}$$

The second integral in r is of the following form.

$$\int_0^{\infty} \frac{e^{-vr^2}}{r} \, dr = (1/2) \int_0^{\infty} \frac{e^{-vu}}{u} \, du \tag{D-45}$$

The latter form in equation (D-45) is recognized as the exponential integral [20, page 228], and when the lower bound of integration is zero

the value of the integral becomes infinite. It may be shown that this leads to an infinite value for $E\{|G'(k)|^2\}$ which in turn leads to an infinite output noise power.

Obviously an infinite output noise power is undesirable and represents a degradation in system performance, but it is important to realize how this state came about. When r in equation (D-45) is allowed to have a lower bound of zero then $G'(k)$ is required to have an upper bound of infinity since r is effectively the magnitude of $e_2(k)$, and $G'(k)$ is inversely proportional to $e_2(k)$. Since no realizable processor, $G'(k)$, can have an infinite dynamic range it is reasonable to consider the possibility of a nonzero lower bound on r.

The choice of a nonzero lower bound for r has two possible interpretations. Either it reflects the impossibility of a return in channel two with a magnitude less than the prescribed lower bound, ε, or it indicates that the processor, $G'(k)$, ignores such returns. In any event the following results apply if a lower bound of ε is chosen for r.

$$E\left[\frac{X_1(\ell)X_2(\ell)}{X_2^2(\ell)+Y_2^2(\ell)}\right] = \frac{-K_1\, m_4\, e^{-m_1\varepsilon^2}}{8m_1}\left(\frac{\pi}{C_1}\right)^{3/2} \tag{D-46}$$

$$E\left[\frac{X_1(\ell)Y_2(\ell)}{X_2^2(\ell)+Y_2^2(\ell)}\right] = \frac{-K_1\, m_5\, e^{-m_1\varepsilon^2}}{8m_1}\left(\frac{\pi}{C_1}\right)^{3/2} \tag{D-47}$$

$$E\left[\frac{X_1^2(\ell)X_2^2(\ell)+X_1^2(\ell)Y_2^2(\ell)}{[X_2^2(\ell)+Y_2^2(\ell)]^2}\right] = \frac{K_1}{4}\left(\frac{\pi}{C_1}\right)^{3/2}\left[2E_1(m_1\varepsilon^2) + \left(\frac{7}{32}\right)\left(\frac{m_4^2+m_5^2}{C_1 m_1}\right)\right]. \tag{D-48}$$

In the above equations, (D-46), (D-47), and (D-48), K_1, m_1, m_4, m_5, and C_1 are determined from the elements of Λ_B while in the following the corresponding quantities are evaluated by elements from Λ_F.

$$E\left[\frac{Y_1(\ell)Y_2(\ell)}{X_2^2(\ell)+Y_2^2(\ell)}\right] = \frac{-K_2\, m_4\, e^{-m_1\epsilon^2}}{8m_1}\left[\frac{\pi}{C_4}\right]^{3/2} \tag{D-49}$$

$$E\left[\frac{Y_1(\ell)X_2(\ell)}{X_2^2(\ell)+Y_2^2(\ell)}\right] = \frac{-K_2\, m_5\, e^{-m_1\epsilon^2}}{8m_1}\left[\frac{\pi}{C_4}\right]^{3/2} \tag{D-50}$$

$$E\left[\frac{Y_1^2(\ell)Y_2^2(\ell)+Y_1^2(\ell)X_2^2(\ell)}{[X_2^2(\ell)+Y_2^2(\ell)]^2}\right] = \frac{K_2}{4}\left[\frac{\pi}{C_4}\right]^{3/2}\left[2E_1(m_1\epsilon^2) + \left[\frac{7}{32}\right]\left[\frac{m_4^2 + m_5^2}{C_1\, m_1}\right]\right]. \tag{D-51}$$

One final observation is now made on the significance of choosing a nonzero lower bound for the integral in equation (D-45). The integral in question may be rewritten so that its structure exactly parallels the expectation of $1/r^2$ where r has the Rayleigh envelope distribution [15, page 119]. The choice of ϵ rather than zero as a lower bound of integration is then seen to be equivalent to ignoring an event, $0 \leq r \leq \epsilon$, which has a probability [14, page 25] of

$$1 - e^{-v\epsilon^2} .$$

APPENDIX E

MOMENTS OF THE S MATRIX ELEMENTS

Evaluation of the output noise power of the various systems being considered requires the ability to specify certain moments of the S matrix elements. In the lower bound case, as specified in Section 4.3, the S matrix elements are defined by equations (4.3-5) through (4.3-7) while the required moments are designated by equations (4.3-13) through (4.3-17). Equations (4.3-8) through (4.3-11) shows that the S matrix elements and their moments are functions of the effective drop diameter d_o, and thus the drop size distribution must be specified before the work may proceed. In the following work the Polyakova-Shifrin distribution given in equation (3.3-2) will be used.

Consider S_{11}^2.

$$S_{11}^2 = g_x^2 - 2g_x(g_x-g_y)\left(\frac{V_\infty}{V_r}\right)^2 + (g_x-g_y)^2\left(\frac{V_\infty}{V_r}\right)^4 \tag{E-1}$$

$$E\{S_{11}^2\} = E\{g_x^2\} - 2E\left[g_x(g_x-g_y)\left(\frac{V_\infty}{V_r}\right)^2\right] + E\left[(g_x-g_y)^2\left(\frac{V_\infty}{V_r}\right)^4\right] \tag{E-2}$$

Each of the expectations in equation (E-2) is an integral of the form

$$\int_0^\infty x^m e^{-ax} dx ,$$

and it may be shown that [19, page 345]

$$\int_0^\infty x^m e^{-ax} dx = \frac{\Gamma(m+1)}{(a-1)^{m+1}}$$

where $a > 0$ and $m > -1$.

In accordance with the above, the expectation operations proceed as follows.

$$E\{g_x^2\} = \int_0^\infty d_0^6 \left[8.25 - 0.244 d_0^2 + 0.023 d_0^3\right] \frac{\gamma^3}{2} d_0^2 \, d^{-\gamma d_0} \, dd_0$$

$$= \left(\frac{\gamma^3}{2}\right)\left(\frac{3.25 \times 8!}{\gamma^9} - \frac{0.244 \times 10!}{\gamma^{11}} + \frac{0.023 \times 11!}{\gamma^{12}}\right) \tag{E-3}$$

$$E\left[g_x(g_x - g_y)\left(\frac{V_\infty}{V_r}\right)^2\right] = E\left[g_x^2\left(\frac{V_\infty}{V_r}\right)^2\right] - E\left[g_x g_y \left(\frac{V_\infty}{V_r}\right)^2\right] \tag{E-4}$$

$$E\left[g_x^2\left(\frac{V_\infty}{V_r}\right)^2\right] = \left(\frac{\gamma^3}{2}\right)\left(\frac{10.105}{V_r}\right)^2 \left[\left(\frac{8.25 \times 8!}{\gamma^9} - \frac{0.244 \times 10!}{\gamma^{11}} + \frac{0.023 \times 11!}{\gamma^{12}}\right)\right.$$

$$- 2\left(\frac{8.25 \times 8!}{\left(\frac{1}{2} + \gamma\right)^9} - \frac{0.244 \times 10!}{\left(\frac{1}{2} + \gamma\right)^{11}} + \frac{0.023 \times 11!}{\left(\frac{1}{2} + \gamma\right)^{12}}\right)$$

$$\left. + \left(\frac{8.25 \times 8!}{(1+\gamma)^9} - \frac{0.244 \times 10!}{(1+\gamma)^{11}} + \frac{0.023 \times 11!}{(1+\gamma)^{12}}\right)\right] \tag{E-5}$$

$$E\left[g_x g_y \left(\frac{V_\infty}{V_r}\right)^2\right] = \left(\frac{\gamma^3}{2}\right)\left(\frac{10.105}{V_r}\right)^2$$

$$\cdot \left[\left(\frac{8.29 \times 8!}{\gamma^9} - \frac{0.0672 \times 10!}{\gamma^{11}} + \frac{0.0115 \times 11!}{\gamma^{12}} - \frac{0.002075 \times 12!}{\gamma^{13}}\right)\right.$$

$$- 2\left[\frac{8.29 \times 8!}{(\frac{1}{2}+\gamma)^9} - \frac{0.0672 \times 10!}{(\frac{1}{2}+\gamma)^{11}} + \frac{0.0115 \times 11!}{(\frac{1}{2}+\gamma)^{12}} - \frac{0.002075 \times 12!}{(\frac{1}{2}+\gamma)^{13}}\right]$$

$$+ \left[\frac{8.29 \times 8!}{(1+\gamma)^9} - \frac{0.0672 \times 10!}{(1+\gamma)^{11}} + \frac{0.0115 \times 11!}{(1+\gamma)^{12}} - \frac{0.002075 \times 12!}{(1+\gamma)^{13}}\right]\Bigg] \qquad \text{(E-6)}$$

$$E\left[(g_x - g_y)^2 \left(\frac{V_\infty}{V_r}\right)^4\right] = (\frac{\gamma^3}{2})(\frac{10.105}{V_r})^4$$

$$\cdot \left[\left[\frac{0.0024 \times 10!}{\gamma^{11}} + \frac{0.00415 \times 12!}{\gamma^{13}}\right] - 4\left[\frac{0.0024 \times 10!}{(\frac{1}{2}+\gamma)^{11}} + \frac{0.00415 \times 12!}{(\frac{1}{2}+\gamma)^{13}}\right]\right.$$

$$+ 6\left[\frac{0.0024 \times 10!}{(1+\gamma)^{11}} + \frac{0.00415 \times 12!}{(1+\gamma)^{13}}\right] - 4\left[\frac{0.0024 \times 10!}{(\frac{3}{2}+\gamma)^{11}} + \frac{0.00415 \times 12!}{(\frac{3}{2}+\gamma)^{13}}\right]$$

$$\left. + \left[\frac{0.0024 \times 10!}{(2+\gamma)^{11}} + \frac{0.00415 \times 12!}{(2+\gamma)^{13}}\right]\right] \qquad \text{(E-7)}$$

The following notation is adopted in order to simplify representation of the following work.

$A_1 = \gamma^3/2$

$A_2 = 8.25 \times 8!$

$A_3 = 0.244 \times 10!$

$A_4 = 0.023 \times 11!$

$A_5 = (10.105/V_r)^2$

$A_6 = 8.29 \times 8!$

$A_7 = 0.0672 \times 10$

$A_8 = 0.0115 \times 11!$

$A_9 = 0.002075 \times 12!$

$A_{10} = 0.0024 \times 10!$

$A_{11} = 0.00415 \times 12!$

$A_{12} = 8.33 \times 8!$

$A_{13} = 0.112 \times 10!$

$G_1 = \gamma^9$

$G_2 = \gamma^{11}$

$G_3 = \gamma^{12}$

$G_4 = \gamma^{13}$

$G_5 = (\frac{1}{2}+\gamma)^9$

$G_6 = (\frac{1}{2}+\gamma)^{11}$

$G_7 = (\frac{1}{2}+\gamma)^{12}$

$G_8 = (\frac{1}{2}+\gamma)^{13}$

$G_9 = (1+\gamma)^9$

$G_{10} = (1+\gamma)^{11}$

$G_{11} = (1+\gamma)^{12}$

$G_{12} = (1+\gamma)^{13}$

$G_{13} = (\frac{3}{2}+\gamma)^9$

$G_{14} = (\frac{3}{2}+\gamma)^{11}$

$G_{15} = (\frac{3}{2}+\gamma)^{12}$

$G_{16} = (\frac{3}{2}+\gamma)^{13}$

$G_{17} = (2+\gamma)^9$

$G_{18} = (2+\gamma)^{11}$ $\quad G_{20} = (2+\gamma)^{13}$ $\quad G_{22} = (\frac{5}{2}+\gamma)^{13}$

$G_{19} = (2+\gamma)^{12}$ $\quad G_{21} = (\frac{5}{2}+\gamma)^{11}$

$$S_{12}^2 = (g_x-g_y)^2 \left(\frac{V_\infty}{V_r}\right)^2 - (g_x-g_y)^2 \left(\frac{V_\infty}{V_r}\right)^4 \tag{E-8}$$

$$E\{S_{12}^2\} = E\left[(g_x-g_y)^2 \left(\frac{V_\infty}{V_r}\right)^2\right] - E\left[(g_x-g_y)^2 \left(\frac{V_\infty}{V_r}\right)^4\right] \tag{E-9}$$

$$E\left[(g_x-g_y)^2 \left(\frac{V_\infty}{V_r}\right)^2\right] = A_1A_5 \left[\left(\frac{A_{10}}{G_2} + \frac{A_{11}}{G_4}\right) - 2\left(\frac{A_{10}}{G_6} + \frac{A_{11}}{G_8}\right) + \left(\frac{A_{10}}{G_{10}} + \frac{A_{11}}{G_{12}}\right)\right] \tag{E-10}$$

$E\left[(g_x-g_y)^2 (V_\infty/V_r)^4\right]$ is evaluated in equation (E-7).

$$S_{22}^2 = (g_x-g_y)^2 \left(\frac{V_\infty}{V_r}\right)^4 + 2g_y(g_x-g_y) \left(\frac{V_\infty}{V_r}\right)^2 + g_y^2 \tag{E-11}$$

$$E\{S_{22}^2\} = E\left[(g_x-g_y)^2 \left(\frac{V_\infty}{V_r}\right)^4\right] + 2E\left[g_y(g_x-g_y) \left(\frac{V_\infty}{V_r}\right)^2\right] + E\{g_y^2\} \tag{E-12}$$

$$E\left[g_y(g_x-g_y) \left(\frac{V_\infty}{V_r}\right)^2\right] = E\left[g_xg_x \left(\frac{V_\infty}{V_r}\right)^2\right] - E\left[g_y^2 \left(\frac{V_\infty}{V_r}\right)^2\right] \tag{E-13}$$

$E\left[g_xg_y \left(\frac{V_\infty}{V_r}\right)^2\right]$ is evaluated in equation (E-6).

$$E\left[g_y^2\left(\frac{V_\infty}{V_r}\right)^2\right] = A_1A_5\left[\left(\frac{A_{12}}{G_1}+\frac{A_{13}}{G_2}\right) - 2\left(\frac{A_{12}}{G_5}+\frac{A_{13}}{G_6}\right) + \left(\frac{A_{12}}{G_{13}}+\frac{A_{13}}{G_{14}}\right)\right] \quad \text{(E-14)}$$

$$S_{11}S_{12} = g_x(g_x-g_y)\left(\frac{V_\infty}{V_r}\right) - \left(\tfrac{1}{2}\right) g_x(g_x-g_y)\left(\frac{V_\infty}{V_r}\right)^3$$

$$- (g_x-g_y)^2\left(\frac{V_\infty}{V_r}\right)^3 + \left(\tfrac{1}{2}\right)(g_x-g_y)^2\left(\frac{V_\infty}{V_r}\right)^5 \quad \text{(E-15)}$$

$$E\{S_{11}S_{12}\} = E\left[g_x(g_x-g_y)\left(\frac{V_\infty}{V_r}\right)\right] - \left(\tfrac{1}{2}\right) E\left[g_x(g_x-g_y)\left(\frac{V_\infty}{V_r}\right)^3\right]$$

$$- E\left[(g_x-g_y)^2\left(\frac{V_\infty}{V_r}\right)^3\right] + \left(\tfrac{1}{2}\right) E\left[(g_x-g_y)^2\left(\frac{V_\infty}{V_r}\right)^5\right] \quad \text{(E-16)}$$

$$E\left[g_x(g_x-g_y)\left(\frac{V_\infty}{V_r}\right)\right] = E\left[g_x^2\left(\frac{V_\infty}{V_r}\right)\right] - E\left[g_xg_y\left(\frac{V_\infty}{V_r}\right)\right] \quad \text{(E-17)}$$

$$E\left[g_x^2\left(\frac{V_\infty}{V_r}\right)\right] = A_1\sqrt{A_5}\left[\left(\frac{A_2}{G_1}-\frac{A_3}{G_2}+\frac{A_4}{G_3}\right) - \left(\frac{A_2}{G_5}-\frac{A_3}{G_6}+\frac{A_4}{G_7}\right)\right] \quad \text{(E-18)}$$

$$E\left[g_xg_y\left(\frac{V_\infty}{V_r}\right)\right] = A_1\sqrt{A_5}\left[\left(\frac{A_6}{G_1}-\frac{A_7}{G_2}+\frac{A_8}{G_3}-\frac{A_9}{G_4}\right) - \left(\frac{A_6}{G_5}-\frac{A_7}{G_6}+\frac{A_8}{G_7}-\frac{A_9}{G_8}\right)\right] \quad \text{(E-19)}$$

$$E\left[g_x(g_x-g_y)\left(\frac{V_\infty}{V_r}\right)^3\right] = E\left[g_x^2\left(\frac{V_\infty}{V_r}\right)^3\right] - E\left[g_xg_y\left(\frac{V_\infty}{V_r}\right)^3\right] \quad \text{(E-20)}$$

$$E\left[g_x^2\left(\frac{V_\infty}{V_r}\right)^3\right] = A_1A_5^{3/2}\left[\left(\frac{A_2}{G_1}-\frac{A_3}{G_2}+\frac{A_4}{G_3}\right) - 3\left(\frac{A_2}{G_5}-\frac{A_3}{G_6}+\frac{A_4}{G_7}\right)\right.$$

$$+ 3\left(\frac{A_2}{G_9} - \frac{A_3}{G_{10}} + \frac{A_4}{G_{11}}\right) - \left(\frac{A_3}{G_{13}} - \frac{A_3}{G_{14}} + \frac{A_4}{G_{15}}\right)\Bigg) \tag{E-21}$$

$$E\left(g_x g_y \left(\frac{V_\infty}{V_r}\right)^3\right) = A_1 A_5^{3/2} \left(\left(\frac{A_6}{G_1} - \frac{A_7}{G_2} + \frac{A_8}{G_3} - \frac{A_9}{G_4}\right) - 3\left(\frac{A_6}{G_5} - \frac{A_7}{G_6} + \frac{A_8}{G_7} - \frac{A_9}{G_8}\right)\right.$$

$$\left. + 3\left(\frac{A_6}{G_9} - \frac{A_7}{G_{10}} + \frac{A_8}{G_{11}} - \frac{A_9}{G_{12}}\right) - \left(\frac{A_6}{G_{13}} - \frac{A_7}{G_{14}} + \frac{A_8}{G_{15}} - \frac{A_9}{G_{16}}\right)\right) \tag{E-22}$$

$$E\left((g_x-g_y)^2 \left(\frac{V_\infty}{V_r}\right)^3\right) = A_1 A_5^{3/2} \left(\left(\frac{A_{10}}{G_2} + \frac{A_{11}}{G_4}\right) - 3\left(\frac{A_{10}}{G_6} + \frac{A_{11}}{G_8}\right)\right.$$

$$\left. + 3\left(\frac{A_{10}}{G_{10}} + \frac{A_{11}}{G_{12}}\right) - \left(\frac{A_{10}}{G_{14}} + \frac{A_{11}}{G_{16}}\right)\right) \tag{E-23}$$

$$E\left((g_x-g_y)^2 \left(\frac{V_\infty}{V_r}\right)^5\right) = A_1 A_5^{5/2} \left(\left(\frac{A_{10}}{G_2} + \frac{A_{11}}{G_4}\right) - 5\left(\frac{A_{10}}{G_6} + \frac{A_{11}}{G_8}\right) + 10\left(\frac{A_{10}}{G_{10}} + \frac{A_{11}}{G_{12}}\right)\right.$$

$$\left. - 10\left(\frac{A_{10}}{G_{14}} + \frac{A_{11}}{G_{16}}\right) + 5\left(\frac{A_{10}}{G_{18}} + \frac{A_{11}}{G_{20}}\right) - \left(\frac{A_{10}}{G_{21}} + \frac{A_{11}}{G_{22}}\right)\right) \tag{E-24}$$

$$S_{11}S_{22} = g_x(g_x-g_y)\left(\frac{V_\infty}{V_r}\right)^2 - (g_x-g_y)^2\left(\frac{V_\infty}{V_r}\right)^4 + g_x g_y - g_y(g_x-g_y)\left(\frac{V_\infty}{V_r}\right)^2 \tag{E-25}$$

$$E\{S_{11}S_{22}\} = E\left(g_x(g_x-g_y)\left(\frac{V_\infty}{V_r}\right)^2\right) - E\left((g_x-g_y)^2\left(\frac{V_\infty}{V_r}\right)^4\right)$$

$$+ E\{g_x g_y\} - E\left(g_y(g_x-g_y)\left(\frac{V_\infty}{V_r}\right)^2\right) \tag{E-26}$$

$E\left[g_x(g_x-g_y)\left(\frac{V_\infty}{V_r}\right)^2\right]$ is given in equations (E-4), (E-5), and (E-6),

and $E\left[(g_x-g_y)^2\left(\frac{V_\infty}{V_r}\right)^4\right]$ is evaluated in equation (E-7).

$$E\{g_x g_y\} = A_1\left(\frac{A_6}{G_1} - \frac{A_7}{G_2} + \frac{A_8}{G_3} - \frac{A_9}{G_4}\right) \qquad \text{(E-27)}$$

Note also that $E\left[g_y(g_x-g_y)\left(\frac{V_\infty}{V_r}\right)^2\right]$ is expressed in equations (E-6) and (E-14).

VITA

Robert Wayne Rice is the son of Walter and Frances Rice of Oak Ridge, Tennessee. It was there that he grew up, and he graduated from Oak Ridge High School in 1966. His formal education continued at The University of Tennessee where he received a B.S. degree in electrical engineering in 1970, an M.S. degree in electrical engineering in 1972, and a Ph.D. degree in electrical engineering in 1975.

Robert married Beverly V. Stout, also of Oak Ridge, in 1971, and he wishes to express his gratitude for the understanding, patience, and encouragement that she provided during the often times trying period of graduate education.

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