To the Graduate Council:

I am submitting herewith a dissertation written by Philip L. Bowers entitled "Applications of General Position Properties of Dendrites to Hilbert Space Topology." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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The Graduate School

APPLICATIONS OF GENERAL POSITION PROPERTIES OF DENDRITES TO HILBERT SPACE TOPOLOGY

A Dissertation

Presented for the

Doctor of Philosophy

Degree

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Philip L. Bowers

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To my parents, Harry and Mary, and

to Kristina.

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ABSTRACT

Let $\overline{A}$ be a dendrite whose endpoints are dense and let A be the complement in $\overline{A}$ of a dense σ-compact collection of endpoints of $\overline{A}$. We investigate the general position properties that products of $\overline{A}$ and A possess and apply these to Hilbert space topology. In particular, it is shown that $\overline{A}^n \times [-1, 1]$ is a compact $(n+1)$-dimensional AR that satisfies the disjoint n-cells property, $\overline{A}^{n+1}$ is a compact $(n+1)$-dimensional AR that satisfies the stronger general position property that maps of n-dimensional compacta into $\overline{A}^{n+1}$ are approximable by Z-maps, and A^{n+1} is a nowhere locally compact topologically complete $(n+1)$-dimensional AR that satisfies the discrete n-cells property. As for applications, we use A to build a hierarchy of examples of fake boundary sets in the Hilbert cube that satisfy higher and higher orders of local connectivity and whose complements, though not homeomorphic to the pseudo-interior s of the Hilbert cube, share many of the topological properties of s. Also, it is shown that A stabilizes those complete separable ANR's that are ℓ_2-manifolds off of compact subsets with infinite codimension and this theorem applies to stabilize the fake ℓ_2-manifolds that are constructed herein using two different techniques, one using A.

Finally, some partial results on characterizing ℓ_2-manifolds based on their homological structure are included.

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CHAPTER I

INTRODUCTION

This treatise presents a study of the role of dendrites (uniquely arcwise connected Peano spaces) in Hilbert space manifold theory. Of particular interests are results that parallel previously known results in both Hilbert cube manifold theory and finite dimensional manifold theory; however, we do not limit our study to the world of infinite dimensional topology. Indeed, in studying properties of dendrites and subsets of dendrites that are useful in the study of Hilbert space topology, we obtain results of independent interests in the finite dimensional world, in particular, results concerning finite products of dendrites and finite dimensional versions of Torunczyk's discrete approximation property (see Chapter II).

Our investigations begin in Chapter II with a study of various disjoint and discrete cells properties satisfied by finite products of dendrites and of special subsets of dendrites. There we obtain somewhat surprising examples of compact $(n+1)$-dimensional absolute retracts that satisfy the disjoint n-cells property and nowhere locally compact $(n+1)$-dimensional absolute retracts that satisfy the discrete n-cells property. This latter property, the discrete n-cells property, is a finite dimensional version of Torunczyk's discrete approximation property that is so important in the study of Hilbert space manifolds.

Chapter III contains several applications of dendrite topology to Hilbert space topology. Let $\overline{A}$ be a dendrite whose endpoints are dense and let A be the complement in $\overline{A}$ of a dense σ-compact collection of the endpoints of $\overline{A}$. We use A to obtain three distinct results in Hilbert space manifold theory. First we answer a question of Curtis [Cu_2, Question 1)] in the negative by producing a hierarchy of examples of fake boundary sets in the Hilbert cube that admit small maps from the Hilbert cube and are locally continuum-connected. Then we use A to obtain examples of decompositions of Hilbert space ℓ_2 that are analogous to various examples of nonshrinkable cell-like upper semicontinuous decompositions of both the Hilbert cube and finite dimensional Euclidean spaces. Like their counterparts in Hilbert cube manifold theory and finite dimensional manifold theory, the decomposition spaces associated with these decompositions of ℓ_2 are not manifolds, but they stabilize to manifolds upon multiplication by a 1-dimensional absolute retract, in this case A rather than $(0, 1)$. The final result of the chapter is a stabilization theorem for Hilbert space manifolds which roughly states that if a topologically complete separable ANR X is an ℓ_2-manifold off of a "homologically small" compact subset, then $X \times A$ is an ℓ_2-manifold. This corresponds to a similar result of Torunczyk that states that if a topologically complete separable ANR X is an ℓ_2-manifold off of a "homotopically small" closed subset, then X is an ℓ_2-manifold [To_3]. The examples that we construct previous to the stabilization theorem show that the conclusion of our stabilization theorem cannot be strengthened to say that X is an ℓ_2-manifold.

We include a final chapter, Chapter IV, where we attempt to produce a characterization of Hilbert space manifolds analogous to the Daverman-Walsh characterization of Hilbert cube manifolds found in [DW] that combines a minimal amount of general positioning, the disjoint disks property, with an algebraic condition to characterize Hilbert cube manifolds among locally compact ANR's. Their characterization is based on Torunczyk's entirely geometric characterization of Hilbert cube manifolds [To_4] and we attempt a characterization of Hilbert space manifolds in the spirit of Daverman-Walsh based on Torunczyk's entirely geometric characterization of Hilbert space manifolds [To_5]. We come close to a characterization but, unfortunately, are unable to supply a complete proof of the conjectured characterization. We end with a discussion of the difficulties involved in proving the conjectured characterization.

Definitions and Notation

If X is a metric space, we usually use ρ to denote any metric on X compatible with the topology on X. Sometimes, for emphasis, we write (X, ρ) for the metric space X with metric ρ. Absolute neighborhood retracts, abbreviated ANR, and absolute retracts, abbreviated AR, are understood to be absolute neighborhood retracts and absolute retracts, respectively, for the class of metrizable spaces. If K is an abstract simplicial complex, we use standard abuse of notation and also write K for the standard geometric realization of K and $K^{(n)}$ for both the abstract n-skeleton of K and its standard geometric realization. If $\mathcal{V}$ is an open cover of X and f and g are maps of Y into X, then f is $\mathcal{V}$-close to g provided for every $y \in Y$, $\{f(y), g(y)\} \subset V$ for some $V \in \mathcal{V}$. If X is a metric space and $\varepsilon : X \to (0, \infty)$ is a map, then f and g are ε-close if $\rho(f(y), g(y)) < \varepsilon(f(y))$ for every $y \in Y$. We say that a map $f : Y \to X$ can be arbitrarily closely approximated by maps g provided for every open cover $\mathcal{V}$ of X, there exists a map $g : Y \to X$ such that f and g are $\mathcal{V}$-close. If X is metric, this is equivalent to saying that for every map $\varepsilon : X \to (0, \infty)$, there exists such a g with f and g ε-close and if, in addition, Y is compact, this is equivalent to saying that for every constant $\varepsilon > 0$, there exists such a g with $\rho(f, g) < \varepsilon$. $X \approx Y$ means that X is homeomorphic to Y. $\mathbb{N}$ denotes $\{1, 2, 3, \ldots\}$, the set of positive integers.

The separable Hilbert space of square summable sequences is denoted by ℓ_2 and, by [An$_2$], ℓ_2 can be identified with the pseudo-interior $s = \Pi_{n \in \mathbb{N}} (-1, 1)_n$ of the Hilbert cube $I^\infty = \Pi_{n \in \mathbb{N}} [-1, 1]_n$. I denotes the interval $[-1, 1]$. I^n for $n \in \mathbb{N}$ is the standard n-cell, the n-fold product of I with itself. If H is either I^∞ or ℓ_2, a manifold modeled on H, briefly, an H-manifold, is a metric space admitting an open cover by sets homeomorphic to open subsets of H. Accordingly, each connected H-manifold is a topologically complete separable ANR since H is a topologically complete separable ANR [BP]. If $H = I^\infty$, we use the more standard Q-manifold in place of I^∞-manifold and we note that in the literature, ℓ_2-manifolds also are referred to as s-manifolds.

CHAPTER II

DISJOINT AND DISCRETE CELLS PROPERTIES SATISFIED BY PRODUCTS OF DENDRITES

A dendrite is a nondegenerate uniquely arcwise connected Peano space. A point x in the dendrite X is an endpoint if $X - \{x\}$ is connected and a cutpoint if $X - \{x\}$ is not connected. $\overline{\mathscr{A}}$ denotes the class of all dendrites whose endpoints are dense and $\overline{A}$ denotes a typical element of $\overline{\mathscr{A}}$. $\mathscr{A}$ denotes the following class of spaces: $A \in \mathscr{A}$ if and only if $A = \overline{A} - F$ for some $\overline{A} \in \overline{\mathscr{A}}$ where F is a dense σ-compact subset of endpoints of $\overline{A}$. Since each $A \in \mathscr{A}$ is a G_δ subset of a complete space, each such A is completely metrizable. Endpoints and cutpoints also are defined for elements of $\mathscr{A}$ and if $X \in \overline{\mathscr{A}}$ or $X \in \mathscr{A}$, then $\mathcal{E}(X)$ denotes the collection of all the endpoints of X.

The following proposition exposes several elementary properties that elements of $\overline{\mathscr{A}}$ possess. The proofs are straightforward and therefore are omitted and the reader is referred to Chapter V of [Wh] for further facts about dendrites.

Proposition 2.0. *Let $\overline{A} \in \overline{\mathscr{A}}$.*

(i) *If $\alpha : [0, 1] \to \overline{A}$ is an arc, then $\alpha((0, 1)) \cap \mathcal{E}(\overline{A}) = \phi$.*

(ii) *Every subcontinuum of $\overline{A}$ is a dendrite.*

(iii) *The intersection of any two connected subsets of $\overline{A}$ is connected.*

(iv) *Every connected subset of $\overline{A}$ is arcwise connected and contractible in itself.*

(v) $\bar{A} - \mathcal{E}(\bar{A})$ *is a dense* 1-*dimensional* σ-*compact subset of* $\bar{A}$ *and* $\mathcal{E}(\bar{A})$ *is a dense* 0-*dimensional* G_δ *subset of* $\bar{A}$.

(vi) *If* F *is a* σ-*compact subset of* $\mathcal{E}(\bar{A})$, *then* $\mathcal{E}(\bar{A}) - F$ *is dense in* $\bar{A}$.

(vii) *If* U *is an open neighborhood of an endpoint* e *in* $\bar{A}$, *then there exists a cutpoint* a *in* $\bar{A}$ *such that* $e \in C \subset U$ *where* C *is the component of* $\bar{A} - \{a\}$ *containing* e .

Let X denote either an element of $\bar{\mathcal{A}}$ or an element of $\mathcal{A}$. Since X is locally connected, (iv) and (v) combine to show that X is finite dimensional and locally contractible and thus, since X is contractible, X is an AR [Bo]. Thus each $\bar{A} \in \bar{\mathcal{A}}$ is a compact 1-dimensional AR and each $A \in \mathcal{A}$ is a nowhere locally compact completely metrizable 1-dimensional AR .

Disjoint Cells Properties and Products of Dendrites

In this section we investigate the disjoint cells properties satisfied by various products of dendrites. Our investigation leads to somewhat surprising examples of compact $(n+1)$-dimensional AR's that satisfy the disjoint n-cells property. The number $n+1$ cannot be improved since no topologically complete n-dimensional ANR can satisfy the disjoint n-cells property, for it would contain uncountably many pairwise disjoint embedded n-cells and thereby violate a result of Sieklucki [Sk].

A metric space X is said to satisfy the *disjoint n-cells property* if whenever $f_1 , f_2 : I^n \to X$ are maps of the n-cell I^n into X and $\varepsilon > 0$, then there exist maps $f_1' , f_2' : I^n \to X$ such that $f_1'(I^n) \cap f_2'(I^n) = \phi$ and $\rho(f_i\ , f_i') < \varepsilon$ for $i = 1, 2$. According to [To_4], if X is a topologically complete ANR that satisfies the disjoint n-cells property, then every map of a compact n-dimensional metric space into X can be arbitrarily closely approximated by embeddings. The following theorem provides examples of $(n+1)$-dimensional AR's that are universal for n-dimensional compacta in the sense of the previous sentence.

Theorem 2.1. *Let* X *denote either* $I \times \overline{A}^n$, $I \times A^n$, $\overline{A}^{n+1}$, *or* A^{n+1} *where* $\overline{A} \in \overline{\mathscr{A}}$, $A \in \mathscr{A}$, *and* $n \in \mathbb{N}$. *Then every map of a compact* n-*dimensional metric space into* X *can be arbitrarily closely approximated by embeddings*.

Proof. X is a topologically complete ANR and according to Corollary 2.3 below, X satisfies the disjoint n-cells property.

In [Da], R. J. Daverman proves that if a space X satisfies the disjoint 1-cells property, then $X \times I^2$ satisfies the disjoint 2-cells property. The following proposition illustrates the usefullness of replacing the interval I by a dendrite $\overline{A}$ whose endpoints are dense.

Proposition 2.2. *If for some* $n \in \mathbb{N} \cup \{0\}$, X *is an* LC^n *metric space that satisfies the disjoint* n-*cells property and* $\overline{A} \in \overline{\mathscr{A}}$ *or* $A \in \mathscr{A}$, *then* $X \times \overline{A}$ *and* $X \times A$ *satisfy the disjoint* $(n+1)$-*cells property*.

The main fact that allows us to prove Proposition 2.2 is that both $\overline{A}$ and A contain a dense collection of endpoints (Proposition 2.0. (vi)). Since the proofs for both $\overline{A}$ and A are exactly the same, we prove the proposition only for $\overline{A}$.

<u>Proof</u>. Let $\varepsilon > 0$ and let f and g be maps of I^{n+1} into $X \times \overline{A}$. Since X is LC^n and $\overline{A}$ contains a dense set of endpoints, we can choose a finite triangulation T of I^{n+1} and for each $(n+1)$-simplex σ in T, open sets U_σ and V_σ in X of diameter less than $\frac{\varepsilon}{2}$ and B_σ and C_σ in $\overline{A}$ such that $f(\sigma) \subset U_\sigma \times B_\sigma$, $g(\sigma) \subset V_\sigma \times C_\sigma$, maps of the n-sphere S^n into U_σ and V_σ are $\frac{\varepsilon}{2}$ - homotopic to constant maps, B_σ (respectively, C_σ) is $\frac{\varepsilon}{2}$ - contractible to an endpoint $b(\sigma)$ (respectively, $c(\sigma)$) in $\overline{A}$, and $\{b(\sigma) | \sigma \in T^{(n+1)}\} \cap \{c(\sigma) | \sigma \in T^{(n+1)}\} = \phi$. Let $f_1 = p_X \circ f|P$ and $f_2 = p_{\overline{A}} \circ f|P$ where P denotes the n-skeleton of T and $p_X : X \times \overline{A} \to X$ and $p_{\overline{A}} : X \times \overline{A} \to \overline{A}$ are the projection mappings, and define g_1 and g_2 similarly by replacing f by g. Choose $\frac{\varepsilon}{2}$ - approximations μ and ν to f_1 and g_1, respectively, such that $\mu(P) \cap \nu(P) = \phi$ and assume that μ is so close to f_1 that $\mu(\partial\sigma) \subset U_\sigma$ for each $\sigma \in T^{(n+1)}$ and similarly for ν. It is easy to construct a retraction h of $\overline{A}$ onto a subset of $\overline{A}$ so that h is $\frac{\varepsilon}{2}$ - close to $id_{\overline{A}}$, $h(B_\sigma) \subset B_\sigma$ and $h(C_\sigma) \subset C_\sigma$ for each $\sigma \in T^{(n+1)}$, and $b(\sigma) \notin h(\overline{A})$ and $c(\sigma) \notin h(\overline{A})$ for each $\sigma \in T^{(n+1)}$. Let $f' = (\mu, h \circ f_2)$ and $g' = (\nu, h \circ g_2)$ and observe that $f', g' : P \to X \times \overline{A}$ are maps which satisfy $f'(P) \cap g'(P) = \phi$, and $f'(\partial\sigma) \subset U_\sigma \times B_\sigma$ and $g'(\partial\sigma) \subset V_\sigma \times C_\sigma$ for each $\sigma \in T^{(n+1)}$.

We now define f' and g' over the $(n+1)$-skeleton of T. Let σ be an $(n+1)$-simplex and write σ as σ $\frac{\partial\sigma \times [0,2]}{\partial\sigma \times \{2\}}$. Since $\mu(\partial\sigma) \subset U_\sigma$, there is an $\frac{\varepsilon}{2}$- homotopy $\psi : \partial\sigma \times [1,2] \to X$ of $\mu|\partial\sigma$ to a point in X. Since $h \circ f_2(\partial\sigma) \subset B_\sigma$, there is an $\frac{\varepsilon}{2}$- homotopy $\theta : \partial\sigma \times [0,1] \to \overline{A}$ of $h \circ f_2|\partial\sigma$ to the endpoint $b(\sigma)$ of $\overline{A}$. Since for each $\tau \in T^{(n+1)}$, $c(\tau) \notin h \circ f_2(\partial\sigma)$, it can be arranged that $c(\tau) \notin \theta(\partial\sigma \times [0,1])$. Use $[s,t]$ for $(s,t) \in \partial\sigma \times [0,2]$ to denote a point of σ and define f' on σ by

$$f'([s,t]) = \begin{cases} (\mu(s),\ \theta(s,t)) & 0 \leq t \leq 1 \\ (\psi(s,t),\ b(\sigma)) & 1 \leq t \leq 2 . \end{cases}$$

f' is well-defined and continuous on σ and extends $f'|\partial\sigma$. Similarly, with ν, V_σ, $h \circ g_2$, C_σ, and $c(\sigma)$ respectively in place of μ, U_σ, $h \circ f_2$, B_σ, and $b(\sigma)$, define g' on σ. In this way we obtain maps $f', g' : I^{n+1} \to X \times \overline{A}$ and it is easy to see that $p_X \circ f'$ is ε-close to $p_X \circ f$ and $p_{\overline{A}} \circ f'$ is ε-close to $p_{\overline{A}} \circ f$ and similarly for g'. Also, it is straightforward to verify that $f'(I^{n+1}) \cap g'(I^{n+1}) = \phi$ and thus $X \times \overline{A}$ satisfies the disjoint $(n+1)$-cells property.

The following corollary provides examples of $(n+1)$-dimensional AR's that satisfy the disjoint n-cells property and also completes the proof of Theorem 2.1.

Corollary 2.3. If $\overline{A} \in \overline{\mathscr{A}}$, $A \in \mathscr{A}$, and $n \in \mathbb{N}$, then $I \times \overline{A}^n$, $I \times A^n$, $\overline{A}^{n+1}$, and A^{n+1} satisfy the disjoint n-cells property.

Proof. I, $\bar{A}$, and A satisfy the disjoint 0-cells property. Apply Proposition 2.2 n times.

In [DW], R. J. Daverman and J. J. Walsh prove that if $X \times Y$ is a Q-manifold for a locally compact ANR X and a finite dimensional space Y, then $X \times I^2$ is a Q-manifold. Proposition 2.2 provides an improvement of this result by replacing the 2-dimensional ANR I^2 by a 1-dimensional ANR $\bar{A}$.

Corollary 2.4. If $X \times Y$ is a Q-manifold for a locally compact ANR X and a finite dimensional space Y and $\bar{A} \in \mathscr{A}$, then $X \times \bar{A}$ is a Q-manifold.

Proof. Apply Proposition 2.2 above along with Corollaries 6.3 and 6.4 of [DW].

The $(n+1)$-dimensional AR's $\bar{A}^{n+1}$ and A^{n+1} satisfy a stronger disjointness property than the disjoint n-cells property, namely, maps of I^n into $\bar{A}^{n+1}$ and A^{n+1} can be approximated by maps with Z-set images. This is proved by showing that both $\bar{A}^{n+1}$ and A^{n+1} contain σ-Z-sets that absorb n-cells. The remainder of this section outlines a proof and several corollaries of this result.

A subset F of a space X is a Z_n-*set* in X for some $n \in \mathbb{N} \cup \{0, \infty\}$ if each map $f : I^n \to X$ can be arbitrarily closely approximated by maps into $X - F$. It is easy to see that F is a Z_∞-set in X if and only if X is a Z_n-set for all $n \in \mathbb{N}$. A Z-*set* is a closed Z_∞-set and a σ-Z-*set* is a countable union of Z-sets. A map $f : Y \to X$ is a Z-*map* if $f(Y)$ is a Z-set in X.

It is well-known that a closed subset F of a separable ANR X is a Z-set if and only if the identity map on X can be arbitrarily closely approximated by maps into $X - F$. Also, a standard argument shows that if F is a σ-Z-set in a topologically complete separable ANR X, then the identity map on X can be arbitrarily closely approximated by maps into $X - F$.

A subset F of a space X is said to <u>absorb n-complexes</u> for some nonnegative integer n if each map $f : K \to X$ of a compact n-complex K into X can be arbitrarily closely approximated by maps whose images lie in F. If F is a σ-Z-set in X that absorbs n-complexes, then every map of a compact n-complex into X can be arbitrarily closely approximated by Z-maps.

Given $\overline{A} \in \bar{\mathcal{A}}$ and a dense σ-compact subset F_0 of $E(\overline{A})$, let $F_1 = (F_0 \times \overline{A}) \cup (\overline{A} \times F_0) \subset \overline{A}^2$ and, inductively, let $F_n = (F_{n-1} \times \overline{A}) \cup (\overline{A}^n \times F_0) \subset \overline{A}^{n+1}$. Note that $\overline{A} - F_0 = A \in \mathcal{A}$ and $F_n = \overline{A}^{n+1} - A^{n+1}$.

<u>Lemma 2.5</u>. <u>If for some</u> $n \in \mathbb{N} \cup \{0\}$, F <u>is a subset of an</u> LC^n <u>metric space</u> X <u>that absorbs n-complexes, then</u> $(F \times \overline{A}) \cup (X \times F_0) \subset X \times \overline{A}$ <u>absorbs</u> $(n+1)$-<u>complexes</u>.

<u>Proof</u>. Let $f : K \to X \times \overline{A}$ be a map of a compact $(n+1)$-complex K into $X \times \overline{A}$ and let f_1 and f_2 be f followed by projection to X and $\overline{A}$, respectively. Name a triangulation T of K having small mesh and let P denote its n-skeleton. If the mesh of T is small enough, we can approximate $f_1|P$ by a map $\mu : P \to F$

so that for each $\sigma \in T^{(n+1)}$, $\mu|\partial\sigma$ is homotopic via a small homotopy to a constant map and $f_2|\partial\sigma$ is homotopic via a small homotopy to an endpoint $b(\sigma)$ in F_0 (for details, see the proof of Proposition 2.2). Now extend $(\mu, f_2|P)$ to each $(n+1)$-simplex σ in T exactly as in the proof of Proposition 2.2 and thereby obtain a map f' which approximates f. If $z \in \sigma \in T^{(n+1)}$, then either $p_X \circ f'(z) = \mu(z) \in F$ or $p_{\overline{A}} \circ f'(z) = b(\sigma) \in F_0$ where p_X and $p_{\overline{A}}$ are the obvious projections. Thus $f'(K) \subset (F \times \overline{A}) \cup (X \times F_0)$ and $(F \times \overline{A}) \cup (X \times F_0)$ absorbs $(n+1)$-complexes.

Proposition 2.6. *For $n \in \mathbb{N} \cup \{0\}$, F_n is a σ-compact σ-Z-set in $\overline{A}^{n+1}$ that absorbs n-complexes.*

Proof. It is an easy matter to verify that a compact subset of $E(\overline{A})$ is a Z-set in $\overline{A}$. Thus F_0 is a σ-Z-set in $\overline{A}$ and since F_0 is dense in $\overline{A}$, F_0 absorbs 0-complexes. Assume that F_{n-1} satisfies the conclusion of the proposition. Then easily $F_n = (F_{n-1} \times \overline{A}) \cup (\overline{A}^n \times F_0)$ is a σ-Z-set in $\overline{A}^{n+1}$ and an application of Lemma 2.5 with $X = \overline{A}^n$ and $F = F_{n-1}$ implies that F_n absorbs n-complexes.

Almost immediately we have the following corollary.

Corollary 2.7. *If $\overline{A} \in \mathcal{A}$, then for each $n \in \mathbb{N} \cup \{0\}$, every map of a compact n-dimensional metric space into $\overline{A}^{n+1}$ can be arbitrarily closely approximated by a Z-map.*

Proof. First, Proposition 2.6 shows that the corollary is true for maps of compact n-complexes into $\overline{A}^{n+1}$. Since $\overline{A}^{n+1}$ is an

ANR, every map of a compact n-dimensional metric space into $\overline{A}^{n+1}$ can be factored, arbitrarily closely, through a compact n-complex.

Corollary 2.8. *The previous corollary is true with* $A \in \mathcal{A}$ *in place of* $\overline{A} \in \overline{\mathcal{A}}$.

Proof. Let $F_0' \subset E(A)$ be a dense σ-compact subset of A (Proposition 2.0.(vi)) and define F_n' analogously to F_n . The proofs of Lemma 2.5 and Proposition 2.6 apply to show that F_n' is a σ-Z-set in A^{n+1} that absorbs n-complexes.

Compact subsets of the pseudo-interior s of the Hilbert cube I^∞ are Z-sets in both s and I^∞ . The following corollary points out that the pair $(\overline{A}^{n+1}, A^{n+1})$ satisfies the analogous finite dimensional property.

Corollary 2.9. *Let* $A \in \mathcal{A}$ *and let* $\overline{A}$ *be the corresponding element of* $\overline{\mathcal{A}}$. *If* $n \in \mathbb{N} \cup \{0\}$, *then compact subsets of* A^{n+1} *are* Z_n*-sets in both* A^{n+1} *and* $\overline{A}^{n+1}$.

The proof is immediate from Proposition 2.6.

Discrete Cells Properties

The strong discrete approximation property was introduced by H. Torunczyk in his study of Hilbert space manifolds [To_5] and this extremely useful approximation property subsequently has been studied and applied by R. D. Anderson, D. W. Curtis, and J. van Mill in [ACM], D. W. Curtis in [Cu_1], and T. Dobrowolski and H. Torunczyk in [DT]. The purpose of this section is to define finite dimensional versions of the strong discrete approximation property and to supply sufficient conditions on a dense σ-compact subset F of a locally compact ANR X to ensure that $X - F$ satisfies various ones of these finite dimensional versions. In the next section, we use the results of this section to investigate the various discrete cells properties satisfied by finite products of elements of $\mathcal{A}$.

A collection $\mathcal{D}$ of subsets of a space X is <u>discrete</u> if each point of X has a neighborhood that meets at most one element of $\mathcal{D}$. $\oplus_i X_i$ denotes the free union of countably many spaces $X_1, X_2, \ldots$ and for any positive integers i and n, I_i^∞ denotes a Hilbert cube and I_i^n denotes an n-cell.

In [ACM], a metric space X is said to satisfy the <u>strong discrete approximation property</u> if for each map $f : \oplus_i I_i^\infty \to X$ and each open cover $\mathcal{U}$ of X, there exists a map $g : \oplus_i I_i^\infty \to X$ such that f and g are $\mathcal{U}$-close and $\{g(I_i^\infty)\}_{i \in \mathbb{N}}$ is discrete. This easily is seen to be equivalent to the property we obtain by replacing $\oplus_i I_i^\infty$ by $\oplus_i I^i$ in the above definition. Throughout the remainder of the dissertation we delete the adjective <u>strong</u> and refer to either of these equivalent properties as the <u>discrete approximation property</u>. Introduced below is a hierarchy of discrete approximation properties, namely the discrete n-cells property for $n = 0, 1, 2, \ldots$. These properties along with the discrete approximation property are known collectively as discrete cells properties. A metric space X is said to satisfy the <u>discrete</u> n-<u>cells property</u>

if for each map $f : \oplus_i I_i^n \to X$ and each open cover $\mathcal{U}$ of X, there exists a map $g : \oplus_i I_i^n \to X$ such that f and g are $\mathcal{U}$-close and $\{g(I_i^n)\}_{i \in \mathbb{N}}$ is discrete. Obviously, if X satisfies the discrete n-cells property, then X satisfies the discrete k-cells property for each nonnegative integer $k < n$ and if X satisfies the discrete approximation property, then X satisfies the discrete n-cells property for each $n \in \mathbb{N} \cup \{0\}$. It is unknown whether the latter implication in the previous sentence can be reversed.

Each of these discrete cells properties can be stated in various equivalent ways by varying the method used to measure the closeness of maps. We illustrate three equivalent ways to define these properties by stating only the discrete approximation property in each of these ways:

<u>2.10</u>. For each map $f : \oplus_i I_i^\infty \to X$ and each open cover $\mathcal{U}$ of X, there exists a map $g : \oplus_i I_i^\infty \to X$ such that f and g are $\mathcal{U}$-close and $\{g(I_i^\infty)\}_{i \in \mathbb{N}}$ is discrete.

<u>2.11</u>. There exists a metric ρ on X such that for each map $f : \oplus_i I_i^\infty \to X$ and each map $\varepsilon : X \to (0, \infty)$, there exists a map $g : \oplus_i I_i^\infty \to X$ such that $\rho(f(y), g(y)) < \varepsilon(f(y))$ for each y and $\{g(I_i^\infty)\}_{i \in \mathbb{N}}$ is discrete.

<u>2.12</u>. For <u>every</u> metric ρ on X, for each map $f : \oplus_i I_i^\infty \to X$ and each $\varepsilon > 0$, there exists a map $g : \oplus_i I_i^\infty \to X$ such that $\rho(f(y), g(y)) < \varepsilon$ for each y and $\{g(I_i^\infty)\}_{i \in \mathbb{N}}$ is discrete.

The equivalence of (2.10) through (2.12) is well-known and we subsequently use these interchangeably depending on which best suits our needs. An example described in [ACM] shows that the discrete approximation property cannot be relaxed by considering only positive constants ε and a fixed metric ρ on X in (2.12).

The following theorem provides a characterization of those σ-Z-sets in a locally compact separable metric space whose complements satisfy one of the discrete cells properties. The reader should note that if $X - F$ satisfies either of the equivalent properties in the theorem, then necessarily F is dense in X.

Theorem 2.13. *Let F be a σ-Z-set in a locally compact separable metric space (X, ρ) and let $n \in \mathbb{N} \cup \{0, \infty\}$. The following statements are equivalent:*

(1.n) *$X - F$ satisfies the discrete n-cells property (discrete ∞-cells property = discrete approximation property)*

(2.n) *for each map $\varepsilon : X - F \to (0, \infty)$, there exists a closed in X subset $J \subset F$ such that for every map $f : I^n \to X - F$ and every neighborhood $N(J)$ of J in X, there exists a map $f' : I^n \to N(J)$ such that $\rho(f, f') < \varepsilon \circ f$.*

Proof. (2.n) implies (1.n): Let $\varepsilon : X - F \to (0, \infty)$ and $f : \oplus_i D_i \to X - F$ be maps where for each $i \in \mathbb{N}$, $D_i \approx I^n$, and set $f_i = f|D_i$. Let J be as hypothesized in (2.n) for the map $\frac{1}{2}\varepsilon$ and let $\beta(1) = \frac{1}{4}\rho(g_1(D_1), J)$ where $g_1 = f_1$. According to (2.n), we can choose a map $f_2' : D_2 \to N_{\beta(1)}(J)$ (for a positive number β , $N_\beta(J)$ denotes the β-neighborhood of J in X) such that $\rho(f_2, f_2') < \frac{1}{2}\varepsilon \circ f_2$. Since F is a σ-Z-set in the locally compact space X , the argument given in [Ch_2, Theorem 3.1.(3)] applies to give a map $G_2 : X \to X - F$ which is γ_2-close to id_X where $\gamma_2 = \frac{1}{2}\min[\{\beta(1)\} \cup \varepsilon(f_2(D_2))]$. Let $g_2 = G_2 \circ f_2' : D_2 \to X - F$ and note that $g_2(D_2) \subset N_{2\beta(1)}(J)$ and for each $x \in D_2$, $\rho(f_2(x), g_2(x)) < \varepsilon(f_2(x))$. Inductively we let $\beta(i) = \frac{1}{4}\rho(g_i(D_i), J)$ and $\gamma_{i+1} = \frac{1}{2}\min[\{\beta(i)\} \cup \varepsilon(f_{i+1}(D_{i+1}))]$ and obtain a map $g_{i+1} : D_{i+1} \to X - F$ which satisfies $g_{i+1}(D_{i+1}) \subset N_{2\beta(i)}(J)$ and for each $x \in D_{i+1}$, $\rho(f_{i+1}(x), g_{i+1}(x)) < \varepsilon(f_{i+1}(x))$. Let $g = \oplus_i g_i : \oplus_i D_i \to X - F$ and observe that $\rho(f, g) < \varepsilon \circ f$. Since $\beta(i+1) < \frac{1}{2}\beta(i)$, $\beta(i) \to 0$ as $i \to \infty$ and, since $g(D_{i+1}) \subset N_{2\beta(i)}(J)$ and J is closed in X , it follows easily from the definition of $\beta(i)$ that $\{g(D_i)\}_{i \in \mathbb{N}}$ is discrete in $X - F$.

(1.n) implies (2.n) : Let $\varepsilon : X - F \to (0, \infty)$ be a map. Choose a cover of $X - F$ by relatively compact sets open in X and let Y denote the union of this cover. Let $\mathcal{W}$ be a locally finite refinement of a double star refinement of this cover of Y and for each $x \in Y$, let $g(x) = \rho(x, Y - st(x, \mathcal{W}))$ where $\rho(x, \phi)$ is defined to be ∞ . Thus $N_{g(x)}(x)$ is the largest ball about x contained in $st(x, \mathcal{W})$. Clearly $0 < g(x)$ for each $x \in Y$ and we show that g is a lower semicontinuous function of Y into the extended reals. Let $x \in Y$ so that $g(x) > \alpha > 0$ and let $\beta\pm = \frac{g(x) \pm \alpha}{2}$ ($\beta\pm$ may be ∞) . $V = \cap\{W \in \mathcal{W} \mid x \in W\}$ is open in

Y since $\mathcal{W}$ is locally finite. Check that for each $y \in V \cap N_{\beta-}(x)$, $N_{\beta+}(y) \subset N_{g(x)}(x) \subset st(x, \mathcal{W}) \subset st(y, \mathcal{W})$ so that $g(y) \geq \beta+ > \alpha$ and thus g is lower semicontinuous. Since $0 < g$, Theorem 4.3, page 171 of [Du] applies to give a continuous function $\overline{\varepsilon} : Y \to (0, \infty)$ such that $0 < \overline{\varepsilon} < g$. Let $\delta : X - F \to (0, \infty)$ be the map given by $\delta(x) = \min\{\varepsilon(x), \overline{\varepsilon}(x)\}$ and observe that $N_{\delta(x)}(x) \subset st(x, \mathcal{W})$ for each $x \in X - F$. We leave it to the reader to show that $N_\delta(C) = \cup \{N_{\delta(c)}(c) \mid c \in C\}$ is a relatively compact subset of Y for each compact $C \subset X - F$.

Since the function space $C(I^n, X - F)$ with metric topology is separable [Du; Theorem 5.2, p. 265 and Theorem 8.2(3), p. 270], we can choose a countable dense collection $\{f_i\}_{i \in \mathbb{N}}$ of maps of I^n into $X - F$. For each $i, j \in \mathbb{N}$, define $f_{i,j} = f_i$ and use the fact that $X - F$ satisfies the discrete n-cells property to obtain maps $g_{i,j} : I^n \to X - F$ such that $\rho(f_i, g_{i,j}) < \frac{1}{2}\delta \circ f_i$ and $\{g_{i,j}(I^n) \mid i,j \in \mathbb{N}\}$ is discrete in $X - F$. Let J be the "limit points" of the maps $g_{i,j}$ in X, that is, $x \in X$ is in J if and only if there are points $x(i, j) \in g_{i,j}(I^n)$ such that every neighborhood of x in X contains infinitely many of the $x(i, j)$'s. Since $\{g_{i,j}(I^n)\}$ is discrete in $X - F$, $J \subset F$ and it is easy to see that J is closed in X.

Let $f : I^n \to X - F$ be a map and let $N(J)$ be a neighborhood of J in X. Since $\{f_i\}_{i \in \mathbb{N}}$ is dense in $C(I^n, X - F)$ and I^n is compact, there exists an $i \in \mathbb{N}$ such that $|\delta \circ f - \delta \circ f_i| < \frac{1}{2}\delta_f$ [Du, Theorem 2.1(2), page 259] and $\rho(f, f_i) < \frac{1}{4}\delta_f$ where $\delta_f = \min[\delta(f(I^n))]$. Observe that for each $j \in \mathbb{N}$ and $x \in I^n$,

$$\rho(f(x), g_{i,j}(x)) \leq \rho(f(x), f_i(x)) + \rho(f_i(x), g_{i,j}(x))$$

$$\leq \frac{1}{4}\delta_f + \frac{1}{2}\delta(f_i(x)) \leq \frac{1}{4}\delta_f + \frac{1}{2}(\frac{1}{2}\delta_f + \delta(f(x))) \leq \delta(f(x)) .$$

Thus all we need do is show that there exists a $j \in \mathbb{N}$ such that $g_{i,j}(I^n) \subset N(J)$. But this follows easily from the fact that $N_\delta(f_i(I^n))$ is relatively compact in X and contains $g_{i,j}(I^n)$ for each $j \in \mathbb{N}$, for otherwise there are "limit points" of $\{g_{i,j}\}_{j \in \mathbb{N}}$ which are not in $N(J)$.

For an ANR X , local compactness in Theorem 2.13 is used only for the implication (1.n) implies (2.n). The other implication is true in the setting of topologically complete separable ANR's and thus provides a method for detecting those σ-Z-sets in this more general setting whose complements satisfy a discrete cells property.

Corollary 2.14. *(2.n) implies (1.n) in Theorem 2.13 provided X is a topologically complete separable ANR.*

Proof. The proof is exactly the proof that (2.n) implies (1.n) in Theorem 2.13 except that G_i exists because X is a topologically complete separable ANR.

In [Cu_1], D. W. Curtis provides various other characterizations of those σ-Z-sets in locally compact separable metric spaces whose complements satisfy one of the discrete cells properties. Each of the following conditions on a subset F of a metric space X is stated for each $n \in \mathbb{N} \cup \{\infty\}$ and is due to Curtis, and these conditions are used in various ones of the previously mentioned characterizations that are stated later.

(C1.n) For every $\varepsilon > 0$, there exists a closed in X subset $J \subset F$ such that for every map $f : I^n \to X - F$ and every neighborhood $N(J)$ of J in X, there exists a map $f' : I^n \to N(J)$ such that $\rho(f, f') < \varepsilon$.

(C2.n) For every $x \in F$ and neighborhood U of x in X, there exists a neighborhood V of x in X such that, for every compactum $K \subset V \cap F$, there exists a compactum $\tilde{K} \subset U \cap F$ such that, for every neighborhood $N(\tilde{K})$ of $\tilde{K}$ in X, there exists a neighborhood $N(K)$ of K in X such that every map $f : S^i \to N(K)$ for $0 \leq i < n$ is null-homotopic in $N(\tilde{K})$.

(C3.n) For every $x \in X$ and neighborhood U of x in X, there exists a neighborhood V of x in X such that, for every compactum $K \subset V \cap F$, there exists a compactum $\tilde{K} \subset U \cap F$ such that, for every neighborhood $N(\tilde{K})$ of $\tilde{K}$ in X, there exists a neighborhood $N(K)$ of K in X such that every map $f : S^i \to N(K)$ for $0 \leq i < n$ is null-homotopic in $N(\tilde{K})$.

Notice that condition (C3.n) is exactly condition (C2.n) except that the phrase "For every $x \in F$" is replaced by the phrase "For every $x \in X$." Remember that a metric space X is LC^n for some

$n \in \mathbb{N} \cup \{0, \infty\}$ if for every $x \in X$ and neighborhood U of x, there exists a neighborhood V of x such that every map $f : S^i \to V$ for $0 \leq i < n+1$ is null-homotopic in U.

Theorem 2.15. Let F be a dense σ-Z-set in a locally compact separable metric space (X, ρ). If $n \in \mathbb{N} \cup \{\infty\}$ and X is LC^{n-1}, then the following statements hold:

(i) Condition (1.n) is equivalent to condition (2.n).

(ii) [Curtis] Condition (1.n) is equivalent to conditions (C1.n) and (C2.n) together.

(iii) Condition (2.n) implies condition (C3.n) and if n is finite, condition (2.n) is equivalent to condition (C3.n).

Discussion of Proof. (i) This is Theorem 2.13.

(ii) This is proved in [Cu_1]. Note, however, that one implication is easy given (iii). Indeed, (1.n) implies (2.n) which in turn implies (C3.n) according to (iii). But (2.n) and (C3.n) trivially imply (C1.n) and (C2.n), respectively. The other implication is not needed in what follows and therefore its proof is omitted.

(iii) (2.n) implies (C3.n). Let $x \in X$ and let U be a neighborhood of x in X. Choose a δ-ball N_δ about x such that $Cl_X(N_{3\delta})$ is compact and contained in U where $N_{3\delta}$ is the 3δ-neighborhood of x, and choose a neighborhood V of x in X such that maps of S^i for $0 \leq i < n$ into V are null-homotopic in N_δ. Let $K \subset V \cap F$ be compact and let $\varepsilon : X \to [0, \frac{1}{2}\delta]$ be a map such that $\varepsilon^{-1}(0) = K$. Let J be as promised in (2.n) for the map $\varepsilon|X - F$ and assume that $K \subset J$. Define $\tilde{K} = J \cap Cl_X(N_{3\delta})$, a compact subset of $U \cap F$, and let $N(\tilde{K})$ be a neighborhood of $\tilde{K}$ in X .

Since X is LC^{n-1} and locally compact and $\varepsilon^{-1}(0) = K$, we can choose γ with $\delta > \gamma > 0$, a neighborhood $N(K)$ of K in X , and a relatively compact open set W in X such that $N(K) \subset W \subset Cl_X W \subset N(\tilde{K}) \cap V$, $\varepsilon(x) < \gamma$ for each $x \in N(K)$, and the γ-neighborhood of $N(K)$ is contained in W , and γ-close maps into W of spheres of dimension less than n are homotopic in $N(\tilde{K})$.

Let $0 \leq i < n$ and let $f: S^i \to N(K)$ be a map. Since F is a σ-Z-set in X and X is LC^{n-1} , f is homotopic in $N(\tilde{K})$ to a map $g: S^i \to N(K) - F$ and since $N(K) \subset V$, there exists an extension $G: I^{i+1} \to N_\delta$ of g . Again since F is a σ-Z-set and since $g(S^i) \cap F = \phi$, we may assume that $G(I^{i+1}) \cap F = \phi$. Let $\eta = \min[\delta, \rho(\tilde{K}, X - N(\tilde{K}))]$. According to (2.n), there exists a map $f': I^{i+1} \to N_\eta(J)$ such that $\rho(G, f') < \varepsilon \circ G$. Check that $f'(I^{i+1}) \subset N_\eta(J) \cap N_{2\delta}$ and therefore $f'(I^{i+1}) \subset N_\eta(\tilde{K}) \subset N(\tilde{K})$. Also, $f'|S^i$ is γ-close to $G|S^i$ and both $f'|S^i$ and $G|S^i$ are homotopic in $N(\tilde{K})$ and this shows that f is null-homotopic in $N(\tilde{K})$.

_If_ n _is finite_, (C3.n) _implies_ (2.n). The proof of this uses techniques found in [Cu_1]. Since we never apply (iii), we delete its proof; however, since [Cu_1] is not yet available in print, we prove a less general theorem than (iii), Theorem 2.16, that supplies a condition on F sufficient to ensure that $X - F$ satisfies a finite dimensional discrete cells property. Theorem 2.16 is sufficient for our purposes and is employed in the next section to show that finite products of elements of $\mathcal{A}$ satisfy various discrete cells properties and in the next chapter in our study of fake boundary sets in the Hilbert cube.

Theorem 2.16, unlike the above theorem, does not characterize those σ-Z-sets whose complements satisfy various discrete cells properties. Indeed, an example constructed by J. van Mill [vM] of a boundary set in the Hilbert cube (see Chapter III) that contains no arcs shows that condition (†) of Theorem 2.16 need not be satisfied even though $X - F$ satisfies the discrete n-cells property.

Theorem 2.16. _Let_ F _be a dense_ σ-Z-_set in a locally compact separable metric space_ (X, ρ). $X - F$ _satisfies the discrete_ n-_cells property for some_ $n \in \mathbb{N}$ _if the following condition is satisfied_:

(†) _for every_ $x \in X$ _and neighborhood_ U _of_ x _in_ X, _there exists a neighborhood_ V _of_ x _in_ X _such that every map_ $f : S^i \to V \cap F$ _for_ $0 \leq i < n$ _is null-homotopic in_ $U \cap F$.

Notice that in Theorem 2.16, condition (†) is a sufficient condition to ensure that $X - F$ satisfies the discrete n-cells property only whenever n is finite. Whenver $n = \infty$, (†) implies that $X - F$ satisfies the discrete k-cells property for each positive integer k , however, it is unknown whether or not this is sufficient to ensure that $X - F$ satisfies the discrete approximation property. Still, a proof similar to Curtis' proof [Cu_1] of (ii) in the previous theorem shows that if we assume condition (C1.∞) in addition to condition ($\dagger$) whenever $n = \infty$, then $X - F$ does satisfy the discrete approximation property.

The proof that (C3.n) implies (2.n) for finite n in Theorem 2.15 is a straightforward, though somewhat technical, modification of the proof of Theorem 2.16 to which the remainder of this section is devoted. The idea of the proof of Theorem 2.16 is to use ($\dagger$) to construct for a given map $\varepsilon : X - F \to (0, \infty)$, a closed subset J of X contained in F that "ε-absorbs" n-cells from $X - F$. This J then satisfies condition (2.n).

Let $\bar{\varepsilon} : X - F \to (0, \infty)$ be a map. We use the following lemma to replace $\bar{\varepsilon}$ by a map ε that is defined on all of X .

Lemma 2.17. There is a map $\varepsilon : X \to [0, 1]$ such that $0 < \varepsilon(x) < \bar{\varepsilon}(x)$ for each $x \in X - F$ and $\varepsilon^{-1}(0) \subset F$.

Proof. For each $x \in X - F$, choose an open neighborhood U_x of x in X such that $\bar{\varepsilon}(U_x - F)$ is bounded away from 0 and let $\varepsilon_x = \inf \bar{\varepsilon}(U_x - F) > 0$. Let $Y = \cup \{U_x \mid x \in X - F\}$, an open subset of X , and note that $X - Y \subset F$. For each $x \in X - F$, define $f_x : Y \to [0, 1]$ via $f_x(y) = 0$ if $y \notin U_x$ and $f_x(y) = \min\{\varepsilon_x, 1\}$ if $y \in U_x$. Each f_x is a lower semi-continuous function on Y and thus the function

$f(y) = \sup\{f_x(y) \mid x \in X - F\}$ is lower semicontinuous on Y [Wi; 7K.1]. Note that $0 < f(y) \leq 1$ for each $y \in Y$ and $f(x) \leq \overline{\varepsilon}(x)$ for each $x \in X - F$. Theorem 4.3, page 171 of [Du] applies to give a continuous function $\phi : Y \to [0,1]$ so that $0 < \phi(y) < f(y)$ for each $y \in Y$.

Assuming $X \neq Y$, since $X - Y$ is a closed G_δ subset of X, there is a Urysohn function $\psi : X \to [0,1]$ with $\psi^{-1}(0) = X - Y$. Define $\varepsilon : X \to [0,1]$ by

$$\varepsilon(x) = \begin{cases} \psi(x) \cdot \phi(x) & x \in Y \\ 0 & x \in X - Y . \end{cases}$$

Clearly, $0 < \varepsilon(x) < \overline{\varepsilon}(x)$ for each $x \in X - F$ and $\varepsilon^{-1}(0) = X - Y \subset F$ and it is easy to check that ε is continuous on X.

Throughout the remainder of this section, ε denotes the map of Lemma 2.17 and Y denotes the open subset of X containing $X - F$ that appears in the proof of Lemma 2.17. Note that $\varepsilon(y) > 0$ for each $y \in Y$, $\varepsilon^{-1}(0) = X - Y$, and, since Y is an open subset of the locally compact separable metric space X, Y is a locally compact separable metric space and as such is also σ-compact.

Write $Y = \cup_i \sigma_i$ where each $\sigma_i = \mathrm{Cl}_Y(\tau_i)$ is compact where τ_i is an open subset of Y and, for each $i \in \mathbb{N}$, $\sigma_i \subset \tau_{i+1}$. Let $\sigma_0 = \phi$. For each $i \in \mathbb{N}$, let $\varepsilon_i = 2^{-i}\min[\varepsilon(\sigma_i)]$ and note that $\varepsilon_1 > \varepsilon_2 > \ldots > 0$ and $\varepsilon_i \to 0$ as $i \to \infty$. Let $\delta_i(n+1) = \varepsilon_i$ and choose $\delta_i(n) > 0$ so small that if $\alpha : \partial I^n \to \tau_i \cap F$ and diam $\alpha(\partial I^n) < \delta_i(n)$, then there exists an extension $\beta : I^n \to \tau_{i+1} \cap F$ of α so that diam $\beta(I^n) < \frac{1}{3}\delta_{i+2}(n+1)$. Force the $\delta_i(n)$'s to have the further property that $\delta_i(n) > \delta_{i+1}(n)$ and notice that $\delta_i(n) \to 0$ as $i \to \infty$. Continue in this manner and obtain positive constants $\delta_i(j)$ for $i \in \mathbb{N}$ and $j = 1, \ldots, n+1$ that satisfy:

(1) $\delta_i(j) \to 0$ as $i \to \infty$ for each $j = 1, \ldots, n+1$,

(2) $\delta_i(j-1) < \frac{1}{3}\delta_{i+2}(j)$ for $i = 2, 3, \ldots$ and $j = 2, \ldots,$

(*) $n+1$,

(3) $\delta_i(n+1) = \varepsilon_i \leq \frac{1}{2}\min[\varepsilon(\sigma_i)]$ for $i \in \mathbb{N}$,

(4) if $\alpha : \partial I^j \to \tau_i \cap F$ and diam $\alpha(\partial I^j) < \delta_i(j)$, then there exists an extension $\beta : I^j \to \tau_{i+1} \cap F$ of α so that diam $\beta(I^j) < \frac{1}{3}\delta_{i+2}(j+1)$ for $j = 1, \ldots, n$.

For each $i \in \mathbb{N}$, choose a finite collection of points $S_i \subset (\tau_i - \sigma_{i-1}) \cap F$ such that every point x of $\sigma_i - \tau_{i-1}$ lies within $\frac{1}{3}\delta_{i+1}(1)$ of some element s of $S_i - \{x\}$ and let $S_0 = \phi$.

Let K be the collection of simplices whose vertices are the points of $\cup_i S_i$ and such that $\langle s_0, \ldots, s_p\rangle \in K$ for vertices $s_0, \ldots, s_p$ provided the following properties are satisfied:

(i) $p \leq n$,

(ii) $\{s_0, \ldots, s_p\} \subset S_{m-1} \cup S_m$ with some $s_i \in S_m$ for some $m \in \mathbb{N}$,

(iii) $\rho(s_i, s_j) < \delta_m(1)$ for $0 \leq i, j \leq p$ and m as in (ii).

The reader may verify that (i) through (iii) define K as a locally finite n-dimensional abstract simplicial complex.

There is an obvious map $\nu_0 : K^{(0)} \to J_0 \subset F$ onto the closed subset $J_0 = \cup_i S_i$ of Y. Given $\langle s_0, s_1 \rangle \in K^{(1)}$ suppose that $s_0, s_1 \in S_{m-1} \cup S_m$ with one of s_0 or s_1 in S_m. Then $\rho(s_0, s_1) < \delta_m(1)$ and $s_0, s_1 \in \tau_m \cap F$ so that (*)(4) applies to produce a map $\beta : \langle s_0, s_1 \rangle \to \tau_{m+1} \cap F$ so that $\operatorname{diam} \beta(\langle s_0, s_1 \rangle) < \frac{1}{3}\delta_{m+2}(2)$ and $\beta(s_i) = s_i$ for $i = 1, 2$. In this way extend ν_0 to obtain $\nu_1 : K^{(1)} \to F$. Note that if $\langle s_0, s_1, s_2 \rangle \in K^{(2)}$ then $\operatorname{diam} \nu_1(\langle s_i, s_j \rangle) < \frac{1}{3}\delta_{m+2}(2) < \frac{1}{3}\delta_{m+1}(2)$ for $i, j \in \{0, 1, 2\}$ where $s_0, s_1, s_2 \in S_{m-1} \cup S_m$ and at least one of s_0, s_1, s_2 is in S_m. Thus $\operatorname{diam} \nu_1(\partial\langle s_0, s_1, s_2 \rangle) < \delta_{m+1}(2)$ and $\nu_1(\partial\langle s_0, s_1, s_2 \rangle) \subset \tau_{m+1} \cap F$. (*)(4) applies again to give an extension ν_2 of $\nu_1 | \partial\langle s_0, s_1, s_2 \rangle$ to $\langle s_0, s_1, s_2 \rangle$ so that $\operatorname{diam} \nu_2(\langle s_0, s_1, s_2 \rangle) < \frac{1}{3}\delta_{m+3}(3)$. We continue through (n - 1)-steps and obtain a map $\nu_n : K^{(n)} \to F$ so that $\nu_n(\langle s_0, \dots, s_n \rangle)$ has diameter less than $\frac{1}{3}\delta_{m+n+1}(n+1) < \varepsilon_{m+n+1}$ if $\{s_0, \dots, s_n\} \subset S_{m-1} \cup S_m$ and at least one of $s_0, \dots, s_n$ is in S_m. Let $\nu = \nu_n$.

We claim that $\nu(K)$ is a closed subset of Y. Indeed, for each $m \in \mathbb{N}$, denote by K_m the finite subcomplex of K which consists of all simplices of K all of whose vertices lie in $S_1 \cup S_2 \cup \dots \cup S_{m-1}$. Given a positive integer j, choose $m > j + 1$ so large that $\varepsilon_{m+n+1} < \frac{1}{2}\rho(\sigma_j, Y - \tau_{j+1})$ and suppose that $\langle s_0, \dots, s_p \rangle \in K - K_m$ and $\{s_0, \dots, s_p\} \subset S_{k-1} \cup S_k$ with say $s_0 \in S_k$ where $k \geq m$. Then $\operatorname{diam} \nu(\langle s_0, \dots, s_p \rangle) < \varepsilon_{k+n+1}$

$\leq \varepsilon_{m+n+1} < \frac{1}{2}\rho(\sigma_j, Y - \tau_{j+1}) \leq \frac{1}{2}\rho(\sigma_j, Y - \tau_{k-1})$ and since $\nu(s_0) = s_0 \in Y - \tau_{k-1}$, $\nu(\langle s_0, \ldots, s_p\rangle) \cap \sigma_j = \phi$. We have shown that $\nu(K) \cap \sigma_j = \nu(K_m) \cap \sigma_j$, which is compact since K_m is compact and this implies that $\nu(K)$ is closed in Y .

Let $J = Cl_X(\nu(K))$. Since $\nu(K) \subset F \cap Y$ is closed in Y and $X - Y \subset F$, $J \subset F$. In fact, it follows easily that $J = (X - Y) \cup \nu(K)$. The next lemma shows that J "ε-absorbs" n-cells from $X - F$ and this completes the proof of Theorem 2.16.

Lemma 2.18. Let $f : I^n \to X - F$ be a map. Then there exists a map $g : I^n \to J$ such that $\rho(f, g) < \varepsilon \circ f$.

Proof. Choose m so large that $f(I^n) \subset \tau_m$ and choose a triangulation T of I^n such that each $\lambda \in T$ satisfies $\operatorname{diam} f(\lambda) < \frac{1}{3}\delta_{m+1}(1)$ and $f(\lambda) \subset \tau_k - \sigma_{k-2}$ for some integer k with $m \geq k \geq 2$. For each vertex $t \in T^{(0)}$, there is a unique integer k with $m \geq k \geq 1$ such that $f(t) \in \tau_k - \tau_{k-1}$. For each such vertex t , choose $s(t) \in S_k$ which lies within $\frac{1}{3}\delta_{k+1}(1)$ of $f(t)$ (see the definition of S_k) and define $\theta : T^{(0)} \to K^{(0)}$ via $\theta(t) = s(t)$. Suppose that $\lambda = \langle t_0, \ldots, t_p\rangle \in T$ where, of course, $p \leq n$. There is an integer k with $m \geq k \geq 2$ such that $\{f(t_0), \ldots, f(t_p)\} \subset \tau_k - \sigma_{k-2}$ and thus, $\{\theta(t_0), \ldots, \theta(t_p)\} \subset S_{k-1} \cup S_k$ and we may assume without loss of generality that $\theta(t_0) \in S_k$. For i and j in $\{0, \ldots, p\}$, we have

$$\rho(\theta(t_i), \theta(t_j)) \leq \rho(\theta(t_i), f(t_i)) + \rho(f(t_i)f(t_j)) + \rho(f(t_j), \theta(t_j))$$

$$\leq \frac{1}{3}\delta_k(1) + \frac{1}{3}\delta_{m+1}(1) + \frac{1}{3}\delta_k(1) \leq \delta_k(1) .$$

Thus $\theta(t_0), \ldots, \theta(t_p)$ satisfy (i) through (iii) of the definition of K and therefore $\langle\theta(t_o), \ldots, \theta(t_p)\rangle \in K$. This shows that θ sends the vertices of any simplex of T to the vertices of some simplex of K and therefore, we may extend θ linearly to obtain a simplicial map (still called θ) $\theta : T \to K$.

Let $g : I^n \to J$ be given by $g = \nu \circ \theta$ and let $x \in \lambda \in T$ where λ is an n-simplex. There is an integer k with $m \geq k \geq 2$ such that $f(\lambda) \subset \tau_k - \sigma_{k-2}$ and therefore diam $g(\lambda) =$ diam $\nu(\theta(\lambda))$ $< \varepsilon_{k+n}$ since $\theta(\lambda^{(0)}) \subset S_{k-1} \cup S_k$. Let $t \in \lambda^{(0)}$. Then

$$\rho(f(x), g(x)) \leq \rho(f(x), f(t)) + \rho(f(t), g(t)) + \rho(g(t), g(x))$$

$$\leq \frac{1}{3}\delta_{m+1}(1) + \frac{1}{3}\delta_k(1) + \varepsilon_{k+n}.$$

Properties (*)(1) through (*)(3) along with the fact that $k \leq m$ apply to show that $\delta_{m+1}(1) < \delta_{m+1}(n+1) = \varepsilon_{m+1} < \varepsilon_k$, $\delta_k(1) < \delta_k(n+1)$ $= \varepsilon_k$, and $\varepsilon_{k+n} < \varepsilon_k$. Thus property (*)(3) shows that $\rho(f(x), g(x)) < 2\varepsilon_k < \min[\varepsilon(\sigma_k)] \leq \varepsilon(f(x))$.

Discrete Cells Properties and Products of Dendrites

Throughout this section A denotes an arbitrary element of $\mathcal{A}$ and $\overline{A}$ denotes the element of $\overline{\mathcal{A}}$ from which A arises. In the sense described below, $\overline{A}^{n+1}$ can be thought of as a "finite dimensional Hilbert cube." Both $\overline{A}^{n+1}$ and I^∞ are compact absolute retracts and both A^{n+1} and s are nowhere locally compact topologically complete absolute retracts. A^{n+1} arises as the complement of a dense σ-Z-set in $\overline{A}^{n+1}$ as does s in I^∞. The pair $(\overline{A}^{n+1}, A^{n+1})$ satisfies finite dimensional versions of several properties that the pair (I^∞, s) satisfies. Compact subsets of A^{n+1} are Z_n-sets in both A^{n+1} and $\overline{A}^{n+1}$ and compact subsets of s are Z_∞-sets in both s and I^∞. $\overline{A}^{n+1}$ satisfies the disjoint n-cells property while I^∞ satisfies the disjoint Hilbert cubes property (maps of the Hilbert cube into I^∞ can be made disjoint by small moves) and A^{n+1} satisfies the discrete n-cells property (Proposition 2.19) while s satisfies the discrete approximation property. Also, $\overline{A}^\infty \approx I^\infty$ and $A^\infty \approx s$. For these reasons we call A^{n+1} the <u>pseudo-interior</u> of $\overline{A}^{n+1}$ and $\overline{A}^{n+1} - A^{n+1}$ the <u>pseudo-boundary</u> of $\overline{A}^{n+1}$.

As before, let $F_0 = \overline{A} - A$ and for each positive integer n, let $F_n = (F_{n-1} \times \overline{A}) \cup (\overline{A}^n \times F_0) = \overline{A}^{n+1} - A^{n+1}$. Recall that F_n is a σ-compact σ-Z-set in $\overline{A}^{n+1}$ that absorbs n-complexes (Proposition 2.6) and A is a complete metrizable AR whose endpoints are dense (Proposition 2.0(vi)).

<u>Proposition 2.19</u>. <u>If</u> $n \in \mathbb{N}$, <u>then</u> A^{n+1} <u>satisfies the discrete n-cells property</u>.

<u>Proof</u>. The proof for $n = 0$ is left to the reader as every nowhere locally compact space satisfies the discrete 0-cells property. If W and B are connected and therefore path connected open subsets of $\overline{A}$, then since F_0 is dense in $\overline{A}$, $(W \times B) \cap F_1$ is path connected and therefore $\pi_0((W \times B) \cap F_1) = 0$. Therefore statement (†) of Theorem 2.16 is satisfied with $n = 1$, $F = F_1$, and $X = \overline{A}^2$, and $\overline{A}^2 - F_1 = A^2$ satisfies the discrete 1-cells property.

Assume that statement (†) of Theorem 2.16 is satisfied with $n = n-1$, $F = F_{n-1}$, and $X = \overline{A}^n$ (and thus A^n satisfies the discrete $(n-1)$-cells property). Let W be a connected open subset of $\overline{A}^n$ and B a contractible open subset of $\overline{A}$. According to Theorem 2.16, it is sufficient to show that $\pi_j((W \times B) \cap F_n) = 0$ for each $j = 0, \ldots, n-1$ and we show this for $j = n-1$, the remaining cases being similar.

Let $f : \partial I^n \to (W \times B) \cap F_n$ be a map and $p_1 : \overline{A}^n \times \overline{A} \to \overline{A}^n$ and $p_2 : \overline{A}^n \times \overline{A} \to \overline{A}$ be the projections. Let $g : I^n \to W \times B$ be any extension of f to the n-cell I^n and choose collections of <u>product</u> open subsets of $W \times B$, say $\mathcal{V}_0, \mathcal{U}_0, \ldots, \mathcal{V}_{n-2}, \mathcal{U}_{n-2}, \mathcal{U}$, such that the following hold:

(i) $\mathcal{V}_0$ is a cover of $g(I^n) - g(\partial I^n)$,

(ii) $\mathcal{V}_0 \prec \mathcal{U}_0 \overset{st}{\prec} \mathcal{V}_1 \prec \cdots \overset{st}{\prec} \mathcal{V}_{n-2} \prec \mathcal{U}_{n-2} \overset{st}{\prec} \mathcal{U}$ where $\overset{st}{\prec}$ denotes star refinement,

(iii) for each $j = 0, \ldots, n-2$ and for each $V \in \mathcal{V}_j$, there exists a $U \in \mathcal{U}_j$ containing V such that $\pi_j(p_1(V) \cap F_{n-1}) \to \pi_j(p_1(U) \cap F_{n-1})$ is the zero homomorphism, and for each $U' \in \mathcal{U}_{n-2}$, there exists a $U \in \mathcal{U}$ containing U' such that $\pi_{n-1}(p_1(U')) \to \pi_{n-1}(p_1(U))$ is the zero homomorphism.

(iv) for each $U \in \mathcal{U}$, $\rho(g(\partial I^n), U) > \operatorname{diam} U$ where ρ denotes a metric for $\overline{A}^{n+1}$.

Let T be a locally finite triangulation of $I^n - g^{-1}g(\partial I^n)$ such that T refines $g^{-1}(\mathcal{V}_0)$ and such that diameters of simplices go to zero near $g^{-1}g(\partial I^n)$. For each simplex $\sigma \in T$, choose $V_\sigma \in \mathcal{V}_0$ such that $g(\sigma) \subset V_\sigma$ and for each vertex $t \in T^{(0)}$, choose a point $\theta(t)$ in F_{n-1} which lies in $\cap\{p_1(V_\sigma) | t \in \sigma^{(0)}\}$ and within $\rho(g(t), g(\partial I^n))$ of $p_1(g(t))$. Thus obtain a map $\theta : T^{(0)} \to W \cap F_{n-1}$ such that if $t_i \in T^{(0)}$ and $t_i \to x \in g^{-1}g(\partial I^n)$, then $\theta(t_i) \to p_1(g(x))$.

Since for each $\sigma \in T^{(1)}$, $\theta(\partial\sigma) \subset p_1(V)$ for some $V \in \mathcal{V}_0$, (iii) allows us to extend θ to $T^{(1)}$ so that by (ii), if $\sigma \in T^{(2)}$, then $\theta(\partial\sigma) \subset p_1(V)$ for some $V \in \mathcal{V}_1$. We continue to extend θ on higher dimensional skeleta of T until we obtain a map $\theta : T^{(n-1)} \to W \cap F_{n-1}$ that is limited by $p_1(\mathcal{U})$, that is, for each n-simplex Δ in T, $\theta(\partial\Delta)$ is contained in $p_1(U)$ for some $U \in \mathcal{U}$. Let $\psi : T^{(n-1)} \to W \times B$ be the map $\psi = (\theta, p_2 \circ g|T^{(n-1)})$ and extend ψ to T by extending on each n-simplex Δ of T using (iii) via the trick used in the proof of Proposition 2.2 so that $\psi(\Delta) \subset U$ for some $U \in \mathcal{U}$ and $p_1\psi(c(\partial\Delta)) = p_1(\psi(\partial\Delta)) \subset F_{n-1}$ and $p_2\psi(\Delta - c(\partial\Delta)) \in F_0$ where $c(\partial\Delta)$ denotes a collar on $\partial\Delta$. Since if $t_i \in T^{(0)}$ and $t_i \to x \in g^{-1}g(\partial I^n)$, $\psi(t_i) = (\theta(t_i), p_2g(t_i)) \to g(x)$, and since each ψ is limited by $\mathcal{U}$, (iv) allows us to extend ψ to $g^{-1}g(\partial I^n)$ via g to obtain a map $h : I^n \to W \times B$ given by $h = g$ on $g^{-1}g(\partial I^n)$ and $h = \psi$ otherwise. Notice that $h = f$ on ∂I^n.

Given $x \in I^n$, there are three possibilities: either $h(x) = g(x) \in F_n$ (for $x \in g^{-1}g(\partial I^n)$), $p_1h(x) = p_1\psi(x) \in F_{n-1}$

(for $x \in c(\partial\Delta)$), or $p_2h(x) = p_2\psi(x) \in F_0$ (for $x \in \Delta - c(\partial\Delta)$). Thus $h(I^n) \subset F_n$ and this implies since $h|\partial I^n = f$ that $[f]$ is zero in $\pi_{n-1}((W \times B) \cap F_n)$.

We actually have proved a stronger result than Proposition 2.19. This appears as the following corollary to the proof of Proposition 2.19 and is used in Chapter III to prove that certain spaces satisfy discrete cells properties.

Corollary 2.20. *Let X, F, and $n \geq 1$ be as in the statement of Theorem 2.16 and assume that (†) holds for X, F, and n . If in addition X is LC^{n+k-1} for some positive integer k , then $(X - F) \times A^k$ satisfies the discrete $(n+k)$-cells property.*

Proof. The proof is exactly the proof of Proposition 2.19 with X, F, and n in place of $\overline{A}^n$, F_{n-1}, and $n-1$, respectively, in the inductive hypothesis, except that open sets W and W' in X are chosen so that maps of spheres of dimension less than $n+k-1$ into W are null-homotopic in W', and it is proved that maps of spheres of the correct dimension into $(W \times B) \cap \overline{F}$ are null-homotopic in $(W' \times B) \cap \overline{F}$, where $\overline{F} = (F \times \overline{A}) \cup (X \times F_0)$.

Corollary 2.21. *A^∞ satisfies the discrete n-cells property for each $n \in \mathbb{N}$.*

In fact, A^∞ satisfies the discrete approximation property. Indeed, with $X = \overline{A}^\infty$ and $F = \overline{A}^\infty - A^\infty$, condition (C1.∞) trivially is satisfied and all the ingredients to prove that condition (†)

whenever $n = \infty$ is satisfied appear in the proof of Proposition 2.19. According to Torunczyk's characterization of Hilbert space [To_5, Corollary 3.2], $A^\infty \approx s$ and , since $\overline{A}^\infty$ easily satisfies the disjoint Hilbert cubes property, Torunczyk's characterization of the Hilbert cube [To_4, Theorem 1] implies that $\overline{A}^\infty \approx I^\infty$.

CHAPTER III

APPLICATIONS OF DENDRITE TOPOLOGY TO HILBERT SPACE TOPOLOGY

In this chapter, we use elements of $\mathcal{A}$ to obtain various results about Hilbert space topology. First, we produce examples of fake boundary sets in the Hilbert cube whose complements satisfy various finite dimensional discrete cells properties, but not the discrete approximation property. In particular, these fake boundary sets are continuum-connected and this answers a question of Curtis [Cu_2] in the negative. We then use elements of $\mathcal{A}$ to obtain results in Hilbert space topology that parallel various known results in Hilbert cube topology. The elements of $\mathcal{A}$ in our study of ℓ_2-manifold theory take the place of the interval I in previous studies of Q-manifold theory. Specifically, we "thread" an element of $\mathcal{A}$ through a wild closed 0-dimensional subset of ℓ_2 to obtain an example of a decomposition of ℓ_2 whose associated decomposition space is a topologically complete separable AR that is an ℓ_2-manifold off of a point, yet the decomposition space is not ℓ_2. The corresponding result in Q-manifold theory is the example of a nonshrinkable cell-like decomposition of the Hilbert cube that one obtains by threading an arc through a wild Cantor set in I^∞. We also exhibit a very simple method for producing fake ℓ_2-manifolds that are ℓ_2-manifolds off of "homologically small" compact subsets. Then, in the final section of the chapter, we prove a stabilization theorem for ℓ_2-manifolds that shows that the aforementioned examples become ℓ_2-manifolds upon multiplication by elements of $\mathcal{A}$.

The proof of the above mentioned stabilization theorem relies on H. Torunczyk's extremely useful characterization of ℓ_2-manifolds that parallels his characterization of Q-manifolds found in [To_4]. His elegant characterization has become the standard method for recognizing ℓ_2-manifolds, in situations ranging from hyperspaces to infinite products of topological groups (see [Cu_1], [Cu_3], [To_5], [DT]), and appears below as Theorem 3.0, the proof of which is found in [To_5].

Theorem 3.0. *A topologically complete separable* ANR X *is an* ℓ_2*-manifold if and only if* X *satisfies the discrete approximation property*.

Fake Boundary Sets in the Hilbert Cube

A *boundary set* in the Hilbert cube is a σ-Z-set $B \subset I^\infty$ for which $I^\infty - B \approx s \approx \ell_2$. In [$Cu_1$], D. W. Curtis introduces the notion of boundary sets and gives various necessary and sufficient conditions for a σ-Z-set in I^∞ to be a boundary set. Recognizing boundary sets can be a rather subtle problem, as evidenced by the following example of R. D. Anderson, D. W. Curtis, and J. van Mill [ACM].

Example 3.1. *There exists a* σ*-Z-set* $B_0 \subset I^\infty$ *that is not a boundary set though it satisfies:*

(i) $I^\infty - B_0$ *is an infinite dimensional topologically complete separable* AR *that embeds as a convex subset of* ℓ_2,

(ii) *every compact subset of* $I^\infty - B_0$ *is a* Z*-set,*

(iii) $(I^\infty - B_0) \times (I^\infty - B_0) \approx s \approx \ell_2$,

(iv) B_0 __admits small maps__ $I^\infty \to B_0$, __that is, for every__ $\varepsilon > 0$, __the identity map on__ I^∞ __is__ ε-__close to a map on__ I^∞ __whose image lies in__ B_0 .

Since the complement of every boundary set satisfies the discrete 1-cells property (Theorem 3.0), every boundary set is connected and locally continuum-connected. The "fake" boundary set B_0 of the example, though connected, is not locally continuum-connected.

In this section, we use elements of $\mathcal{A}$ to help produce a hierarchy of examples of fake boundary sets, all of which satisfy (i) through (iv) of Example 3.1. Specifically, for each positive integer n , we produce a fake boundary set B_n whose complement satisfies the discrete n-cells property and each of our examples, though not a boundary set, is a locally continuum-connected σ-Z-set in I^∞ that admits small maps from the Hilbert cube into itself. This answers question 1) of [Cu_2] in the negative.

For each $i \in \mathbb{N}$, let $W_i = \prod_{j \neq i} [-1 + 2^{-i}, 1 - 2^{-i}]_j \times \{1\}_i \subset I^\infty$. W_i is a "shrunken endface" in the ith coordinate direction and is a Z-set in I^∞ . $B_0 = \cup_i W_i$ is the Anderson, Curtis, van Mill σ-Z-set of Example 3.1. Theorems 3.1 through 3.5 of [ACM] show that B_0 possesses properties (i) through (iv) of Example 3.1 and Sierpinski's Theorem [Sr] shows that B_0 is not locally continuum-connected, and therefore not a boundary set. Observe that $I^\infty - B_0$ satisfies the discrete 0-cells property since $I^\infty - B_0$ is nowhere locally compact.

Let $\overline{A} \in \bar{\mathcal{A}}$ and, as usual, let F_0 be a dense σ-compact collection of endpoints of $\overline{A}$ and, for each $n \in \mathbb{N}$, F_n the corresponding σ-Z-set in $\overline{A}^{n+1}$. For each $n \in \mathbb{N}$, define B_n to be the σ-Z-set $(B_0 \times \overline{A}^n) \cup (I^\infty \times F_{n-1})$ in $I^\infty \times \overline{A}^n \approx I^\infty$. Routine set arguments and induction reveal that $B_{n+1} = (B_n \times \overline{A}) \cup (I^\infty \times \overline{A}^n \times F_0)$ $\subset I^\infty \times \overline{A}^{n+1} \approx I^\infty$ and $(I^\infty \times \overline{A}^n) - B_n = (I^\infty - B_0) \times A^n$ where $A = \overline{A} - F_0 \in \mathcal{A}$ (remember that $A^n = \overline{A}^n - F_{n-1}$).

Example 3.2. For $n = 0, 1, 2, \ldots$, B_n is a σ-Z-set in $I^\infty (\approx I^\infty \times \overline{A}^n)$ that is not a boundary set though it satisfies (i) through (iv) of Example 3.1 and in addition satisfies:

(v) $I^\infty - B_n$ satisfies the discrete n-cells property.

In particular, if $n > 0$, (v) shows that B_n is locally continuum-connected.

It is easy to verify that each B_n satisfies (iv) and Theorem 3.1 through 3.5 of [ACM] apply to verify that each B_n satisfies (i) through (iii). We already have observed that $I^\infty - B_0$ satisfies the discrete 0-cells property and the verification of (v) for $n > 0$ appears below.

$I^\infty - B_n$ satisfies the discrete n-cells property: If U and V are connected and therefore path connected open subsets of I^∞ and $\overline{A}$, respectively, then since B_0 is dense in I^∞ and F_0 is dense in $\overline{A}$, $(U \times V) \cap B_1$ is path connected and therefore $\pi_0((U \times V) \cap B_1)$ $= 0$. This says that condition $(\dagger)$ of Theorem 2.16 is satisfied with $n = 1$, $F = B_1$, and $X = I^\infty \times \overline{A}$, and therefore $(I^\infty \times \overline{A}) - B_1$

satisfies the discrete 1-cells property. But $I^\infty - B_1 = (I^\infty \times \overline{A}) - B_1$ under the identification of Example 3.2, namely, $I^\infty \times \overline{A}$ is identified with I^∞ .

Since $I^\infty \times \overline{A}$ is LC^k for each positive integer k , Corollary 2.20 applies to show that $[(I^\infty \times \overline{A}) - B_1] \times A^k$ satisfies the discrete $(k+1)$-cells property. But $[(I^\infty \times \overline{A}) - B_1] \times A^k =$ $[(I^\infty - B_0) \times A] \times A^k = (I^\infty - B_0) \times A^{k+1} = (I^\infty \times \overline{A}^{k+1}) - B_{k+1}$. This shows, under the identification $I^\infty \times \overline{A}^n \approx I^\infty$ of Example 3.2, that $I^\infty - B_n$ satisfies the discrete n-cells property for $n > 1$.

We have left only to show that each B_n is not a boundary set. Since each B_n for $n > 0$ is locally continuum-connected, we cannot use the lack of this property to detect that B_n is not a boundary set. Instead, we use a local homological property that boundary sets satisfy in order to detect that B_n is not a boundary set. $\tilde{H}_n$ denotes reduced singular homology.

A subset B of I^∞ <u>satisfies</u> $H(n)$ for some nonnegative integer n if:

> for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, for every compact set $S \subset B$ with $\operatorname{diam} S < \delta$, there exists a compact set $K \subset B$ containing S with $\operatorname{diam} K < \varepsilon$ such that, for every neighborhood $N(K)$ of K in X , there exists a neighborhood $N(S)$ of S in X contained in $N(K)$ such that the inclusion induced homomorphism $\tilde{H}_n(N(S)) \to \tilde{H}_n(N(K))$ is the zero homomorphism.

This is a uniform homological version of Curtis' condition (C2.n) of Chapter II that he uses in [Cu_1] in his characterization of boundary sets in the Hilbert cube. We use this homological version rather than Curtis' original homotopic version because of the resulting ease in working with products. Each boundary set in I^∞ satisfies $H(n)$ for every nonnegative integer n. This is a trivial consequence of another Curtis characterization of boundary sets [ACM, Theorem 7.1] which in turn is a consequence of the fact that s satisfies the discrete approximation property (Theorem 3.0). B_0 does not satisfy $H(0)$ since $H(0)$ implies that B is locally continuum-connected. The next proposition is used to show that B_n for $n > 0$ does not satisfy $H(n)$, and therefore, B_n is not a boundary set.

Proposition 3.3. Let $B \subset I^\infty$ and define $B' = (B \times \overline{A}) \cup (I^\infty \times F_0) \subset I^\infty \times \overline{A} \approx I^\infty$. For $n \in \mathbb{N}$, if B' satisfies $H(n+1)$ in $I^\infty \times \overline{A}$, then B satisfies $H(n)$ in I^∞.

Armed with Proposition 3.3, it is easy to show that B_n does not satisfy $H(n)$. Indeed, since for each nonnegative integer i, $B_{i+1} = (B_i \times \overline{A}) \cup (I^\infty \times F_0)$ where we have identified $I^\infty \times \overline{A}^i$ with I^∞, successive applications of Proposition 3.3 show that if B_n satisfies $H(n)$ for some $n > 0$, then B_i satisfies $H(i)$ for $0 \leq i \leq n$ and, in particular, B_0 is locally continuum-connected, a contradiction.

Proof. Let $\varepsilon > 0$ and let δ be the delta that works in $H(n+1)$ for ε and $B' \subset I^\infty \times \bar{A}$. Choose a cutpoint $a \in \bar{A}$ and identify I^∞ with $I^\infty \times \{a\} \subset I^\infty \times \bar{A}$, and let S be a compact subset of B of diameter less than $\frac{\delta}{2}$. Choose a small arc L in $\bar{A}$ containing a in its interior and having endpoints e_1 and e_2 in F_0 and let D be a compact contractible subset of I^∞ containing S of diameter less than $\frac{\delta}{2}$. Assume that L has small enough diameter that the compact subset $S' = (S \times L) \cup (D \times \{e_1, e_2\})$ of B' has diameter less than δ and let $K' \subset B'$ be the compact subset of diameter less than ε that is promised in $H(n+1)$. Since $K' \subset B'$ and $a \notin F_0$, $K = K' \cap I^\infty = K' \cap (I^\infty \times \{a\})$ is contained in B. Note that $S \subset K$ and K has diameter less than ε.

Let U be an arbitrary neighborhood of K in I^∞ and let U' be a neighborhood of K' given by $U' = (U \times \bar{A}) \cup \bar{U}$ where $\bar{U}$ is an open subset of $I^\infty \times \bar{A}$ that contains $K' - (U \times \bar{A})$ and misses $I^\infty \times \{a\}$. Write $\bar{A} = \bar{A}_1 \cup \bar{A}_2$ where $\bar{A}_i$ is a closed subset of $\bar{A}$ containing e_i for $i = 1, 2$ and $\bar{A}_1 \cap \bar{A}_2 = \{a\}$. For an arbitrary subset Y of $I^\infty \times \bar{A}$, define $Y_i = Y \cap (I^\infty \times \bar{A}_i)$ for $i = 1, 2$ and note that $\{U_1', U_2'\}$ is an excisive couple of subsets [Sp].

$H(n+1)$ promises a neighborhood W of S' contained in U' so that $\tilde{H}_{n+1}(W) \to \tilde{H}_{n+1}(U')$ is zero. For $i = 1, 2$, since S_i' is a contractible subset of W_i, there exists a contractible neighborhood V_i' of S_i' in W_i. Let $V' = V_1' \cup V_2'$ and $V = V_1' \cap V_2'$ and note that $\{V_1', V_2'\}$ is an excisive couple of subsets and since

$V' \subset W$, $\tilde{H}_{n+1}(V') \to \tilde{H}_{n+1}(U')$ is zero. Also, V is a neighborhood of S in I^∞ contained in U .

Since $\{U_1', U_2'\}$ and $\{V_1', V_2'\}$ are excisive couples of subsets we have the following commutative diagram, the rows being appropriate Mayer-Vietoris sequences and therefore exact, and the columns being inclusion induced maps.

$$\begin{array}{ccccc} \tilde{H}_{n+1}(V') & \xrightarrow{\tau} & \tilde{H}_n(V) & \longrightarrow & \tilde{H}_n(V_1') \oplus \tilde{H}_n(V_2') \\ {\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle i_*} & & \\ \tilde{H}_{n+1}(U') & \to & \tilde{H}_n(U) & & \end{array}$$

Since V_1' and V_2' are contractible, τ is an epimorphism and therefore, since $j_* = 0$, i_* must be the zero homomorphism. This completes the proof the the proposition.

<u>Some Examples of Fake Hilbert Space Manifolds</u>

In this section, we use two different techniques to produce examples of ANR's that are ℓ_2-manifolds off of "homologically small" compact subsets. The first technique corresponds to a standard technique used to produce examples of nonshrinkable cell-like but not cellular decompositions of both Q-manifolds and finite dimensional manifolds that produce ANR decomposition spaces that are manifolds precisely off of a point. These examples are constructed by threading an arc through a wild Cantor set and using the resulting decomposition whose only nondegenerate element is that arc, and these examples stabilize, that is, become manifolds, upon multiplication by the open interval $(0, 1)$. Since there are no wild Cantor sets in ℓ_2 [Ch_2, Theorem 7.1], we cannot use the above technique to obtain examples of spaces that are ℓ_2-manifolds precisely off of a point. Moreover, since J. Mogilski has shown in [Mo_1] that cell-like decompositions of ℓ_2-manifolds yield ℓ_2-manifold decomposition spaces if the decomposition spaces are ANR's, we cannot find any cell-like decomposition of ℓ_2 that yields the desired example. Instead, we start with a wild closed 0-dimensional subset C of ℓ_2 and we thread an element A of $\mathscr{A}$ through C and obtain a decomposition of ℓ_2 whose only non-degenerate element is A. Since the quotient topology on the decomposition space ℓ_2/A is not metrizable, we describe a countable neighborhood base about A that allows us to put a metrizable topology on ℓ_2/A that exhibits ℓ_2/A as a topologically complete separable AR that is an ℓ_2-manifold precisely off of the "bad point" A. Moreover, the main result of the last section of this chapter shows that this example stabilizes upon multiplication by the 1-dimensional AR A just as the corresponding examples in Q-manifold and finite dimensional manifold theory stabilize upon multiplication by the 1-dimensional AR $(0, 1)$.

The second technique that we employ is a very simple technique that allows us to produce examples of ANR's that are ℓ_2-manifolds off of certain compact subsets that have infinite codimension, and these examples also stabilize upon multiplication by elements of $\mathscr{A}$. Though this technique is purely infinite dimensional, one can use this to produce examples of decompositions of ℓ_2 that are analogous to the various generalizations to the Hilbert cube and higher dimensional Euclidean spaces of Bing's example of a nonshrinkable decomposition of E^3 into a Cantor set's worth of tame arcs [Bi]. These examples are built "backwards" in the sense that one first builds the decomposition space X so that X is an ℓ_2-manifold off of a Cantor set C. Then one uses the fact that $X \times A$ is homeomorphic to ℓ_2 where $A \in \mathcal{A}$ to decompose ℓ_2 into points and a Cantor set's worth of copies of A, each of whose image under the decomposition map is a Z-set in X though the image C of the nondegeneracy set is not a Z-set in X. This is accomplished by carefully adjusting the projection map $X \times A \to X$ to obtain a map π from $X \times A \approx \ell_2$ onto X that is one to one over $X - C$.

We begin with the following extension theorem for embeddings that allows us to "thread" an element of $\mathcal{A}$ through a closed 0-dimensional subset of ℓ_2. An embedding into a space X is a closed Z_n-embedding provided its image is a closed Z_n-set in X.

Theorem 3.4. *Let X be a topologically complete separable ANR that satisfies the discrete n-cells property for some $n \in \mathbb{N}$. Let f be a map of a topologically complete separable n-dimensional space Y into X that is a closed Z_n-embedding on the closed subset C of Y. Then f can be arbitrarily closely approximated by closed Z_n-embeddings g such that $g|C = f|C$.*

Outline of Proof. For this proof only, we use the notation of [To_5, Section 1] and any terms left undefined here can be found there. $C(Y, X)$ denotes the set of all maps from Y to X topologized by the limitation topology. This topology coincides with the topology of uniform convergence with respect to all metrics on X and a typical basic open set containing f consists of the collection of all maps g in $C(Y, X)$ that are $\mathcal{V}$-close to f for some given open cover $\mathcal{V}$ of X. For a given open cover $\mathcal{U}$ of Y, a map $g : Y \to X$ is said to be a $\mathcal{U}$-map if there exists an open cover $\mathcal{V}$ of X with $f^{-1}(\mathcal{V})$ refining $\mathcal{U}$. Given a subset F of $C(Y, X)$ and a bounded metric ρ on X, the ρ-closure F_ρ of F in $C(Y, X)$ is the closure of F in $C(Y, X)$ topologized by the sup-norm metric induced by ρ.

Let d (respectively, ρ) be a complete metric on Y (respectively, X) and let $\{D_i\}_{i \in \mathbb{N}}$ be a countable dense pairwise disjoint collection of singular n-cells in X, each of which misses $f(C)$. For each $k \in \mathbb{N}$, let $\mathcal{U}_k$ be an open cover of Y by sets of d-diameter less than 2^{-k} and define V_k to be the collection of all $\mathcal{U}_k$-maps of Y into X that miss D_k. According to [To_5, Lemma 4], each V_k is open in $C(Y, X)$ and a map $g : Y \to X$ is a closed embedding if and only if g is a $\mathcal{U}_k$-map for all $k \in \mathbb{N}$. This implies that $V = \bigcap_{k\in\mathbb{N}} V_k$ is a G_δ-collection of closed Z_n-embeddings.

Let F be the subspace of $C(Y, X)$ that consists of all maps $g : Y \to X$ such that $g|C = f|C$ and observe that the ρ-closure F_ρ of F is exactly F. Since X is topologically complete and $F_\rho = F$, Lemma 1.1 of [To_5] implies that if $V_k \cap F$ is dense in F

for each $k \in \mathbb{N}$, then maps in F can be arbitrarily closely approximated by elements of $V \cap F$. In particular, f can be arbitrarily closely approximated by closed Z_n-embeddings g such that $g|C = f|C$.

It remains to show that $V_k \cap F$ is dense in F for each $k \in \mathbb{N}$. This is where we use the fact that Y is a separable n-dimensional metric space and X is an ANR that satisfies the discrete n-cells property. Since the details involved in the argument are rather cumbersome, we provide a brief outline of the argument and leave to the reader the task of supplying the details.

Step 1. The first move is to replace Y by a space $C \cup K$ where K is a locally finite n-complex whose simplices have small mesh near C . This is accomplished by choosing a cover $\mathcal{W}_k$ of $Y - C$ refining $\mathcal{U}_k$ such that $\mathcal{W}_k$ has order n and the diameters of elements of $\mathcal{W}_k$ go to zero near C . Then $K = N(\mathcal{W}_k)$, the nerve of $\mathcal{W}_k$, where we insist that simplices that arise from elements of $\mathcal{W}_k$ near C have small mesh. We then can "attach" C to K and obtain the space $C \cup K$ and a natural map $\mu : Y \to C \cup K$ that satisfies $\mu|C = id_C$. If $\mathcal{W}_k$ is small enough, then μ will be a $\mathcal{U}_k$-map.

Step 2. Given a map $h \in F$, use the fact that X is an ANR to obtain a map $\theta : C \cup K \to X$ so that $\theta \circ \mu$ is close to h and satisfies $\theta \circ \mu|C = \theta|C = h|C = f|C$. (The closeness of $\theta \circ \mu$ to h depends on the cover $\mathcal{W}_k$ of Step 1.)

Step 3. Use the fact that X satisfies the discrete n-cells property and that $\{D_i\}$ is pairwise disjoint to adjust θ to obtain a closed embedding $\psi : C \cup K \to X$ such that $\psi(C \cup K)$ is closed in X and misses D_k, and $\psi|C = \theta|C$. Since ψ is a closed embedding, $\psi \circ \mu$ is a $\mathcal{U}_k$-map and obviously its image misses D_k.

By taking care in steps one through three, we can make $\psi \circ \mu$ as close to h as we wish and this shows that $V_k \cap F$ is dense in F.

We now are ready to describe our first example of a fake ℓ_2-manifold that is an ℓ_2-manifold precisely off of a point. Choose an element A of $\mathcal{A}$ and let $\overline{A}$ be the corresponding element of $\overline{\mathcal{A}}$, and let $\overline{C}$ be a wild Cantor set in the Hilbert cube whose compliment in I^∞ is not simply connected (see [La] and [Wo]). Identify ℓ_2 with the pseudo-interior s of the Hilbert cube [An_2] and let $C = \overline{C} \cap \ell_2$, a closed 0-dimensional subset of ℓ_2. Note that the complement of C in ℓ_2 is not simply connected. Since $E(\overline{A})$ is a universal 0-dimensional space, there exists an embedding e of $\overline{C}$ into $E(\overline{A})$ and, since $\overline{C} - C$ is σ-compact, we may assume that $e(C) = e(\overline{C}) \cap A$, and thus $e(C)$ is a closed subset of A. Let $f : A \to \ell_2$ be any extension of $e^{-1}|e(C)$ to A and notice that f is a closed Z_1-embedding on the closed subset $e(C)$ of A. Use Theorem 3.4 to obtain a closed Z_1-embedding g of A into ℓ_2 such that $g|e(C) = f|e(C) = e^{-1}|e(C)$. Now identify A with $g(A)$ so that the AR A is a closed Z_1-set in ℓ_2 whose complement in ℓ_2 is not simply connected. Let ℓ_2/A denote the partition of ℓ_2 into points and the single nondegenerate set A .

The quotient topology on ℓ_2/A is too large for our purposes since it does not yield a metrizable space, and thus certainly not an AR. Instead, we topologize ℓ_2/A as follows. First we require that the projection map $\pi : \ell_2 \to \ell_2/A$ be a homeomorphism on $\ell_2 - A$. We describe a local base at $A \in \ell_2/A$ by choosing a sequence $\{U_n\}_{n\in\mathbb{N}}$ of open neighborhoods of A in ℓ_2 such that $\bigcap_{n\in\mathbb{N}} U_n = A$ and, for each $n \in \mathbb{N}$, $\mathrm{Cl}_{\ell_2} U_{n+1}$ strong deformation retracts to A in U_n . Such a sequence of neighborhoods exists since A is an AR that is embedded as a closed subset of the AR ℓ_2 . We let $\{\pi(U_n)\}_{n\in\mathbb{N}}$ be a local base at $A \in \ell_2/A$ and from now on, whenever we write ℓ_2/A , we mean ℓ_2/A topologized in this way.

<u>Example 3.5</u>. ℓ_2/A <u>is a topologically complete separable</u> AR <u>that is an</u> ℓ_2-<u>manifold precisely on</u> $(\ell_2/A) - A$.

<u>Proof</u>. ℓ_2/A is an ℓ_2-manifold on $(\ell_2/A) - A$ since $(\ell_2/A) - A$ is homeomorphic to the open subset $\ell_2 - A$ of ℓ_2 . ℓ_2/A is not an ℓ_2-manifold since by construction, A is not a Z-set in ℓ_2/A (in fact, not even a Z_2-set in ℓ_2/A) . It remains to prove that ℓ_2/A is a topologically complete separable AR. It is easy to show that ℓ_2/A is a second countable regular space and as such is a separable metric space. Moreover, we can write ℓ_2/A as $(\ell_2 - A) \cup A$ and both the open subset $\ell_2 - A$ of ℓ_2 and the <u>point</u> A are topologically complete spaces. Thus, if we embed the metric space ℓ_2/A

in a complete space M , then both $\ell_2 - A$ and A are G_δ subsets of M and consequently, so is ℓ_2/A . This implies that ℓ_2/A is topologically complete. By our choice of local base at A in ℓ_2/A , ℓ_2/A strong deformation retracts to the point A in ℓ_2/A . This along with the fact that $\ell_2 - A$ is an ANR allows us to apply [KR] or [Hy] to conclude that ℓ_2/A is an AR. Note also that Theorem 5 of [Ko] can be used to show that ℓ_2/A is an AR. Indeed, by our choice of local base $\{U_n\}_{n\in\mathbb{N}}$ for A , given an open cover $\mathcal{U}$ of ℓ_2/A , we can construct a map $g : \ell_2/A \to \ell_2$ such that $g \circ \pi$ is $\pi^{-1}(\mathcal{U})$-homotopic to the identity on ℓ_2 . Theorem 5 of [Ko] applies to show that ℓ_2/A is an ANR and, since ℓ_2/A is contractible, an AR.

We note that depending on different choices of strong deformation retractions of ℓ_2 to A , there are different ways of choosing a local base at A so that ℓ_2/A satisfies the conclusions of Example 3.5. Though these different ways may yield equivalent topologies on ℓ_2/A , in general the identity function on ℓ_2/A will not be a homeomorphism.

We now turn to our second technique for constructing examples of fake ℓ_2-manifolds. If C is a compact subset of the Hilbert cube, we use s_C to denote C union the pseudo-interior s of the Hilbert cube with the subspace topology. Notice that $s_C - C$ is simply connected if and only if $I^\infty - C$ is simply connected.

<u>Example 3.6</u>. <u>Let</u> C <u>be any compact subset of the Hilbert cube whose complement is not simply connected</u>. <u>Then</u> s_C <u>is a</u>

topologically complete separable AR _that is not an_ ℓ_2-_manifold though it is an_ ℓ_2-_manifold at each point of_ $s_C - C$.

Proof. Since $F = (I^\infty - s) - C = I^\infty - s_C$ is a σ-compact subset of the pseudo-boundary of the Hilbert cube, F is a σ-Z-set in I^∞ . As such, Theorem 3.1 of [To_3] applies to show that s_C is an ANR and since obviously s_C is contractible, s_C is an AR. Also, s_C is a topologically complete separable space and, since $s_C - C$ is open in s and $s \approx \ell_2$ [An_2], s_C is an ℓ_2-manifold at each point of $s_C - C$. It remains to prove that s_C is not an ℓ_2-manifold. If it were, since s_C is contractible, s_C would be homeomorphic to ℓ_2 [He] and as such, compact subsets of s_C would be negligible in s_C [An_4], that is, the complement of a compact subset of s_C would be homeomorphic to s_C . In particular, $s_C - C$ would be homeomorphic to s_C and therefore simply connected, a contradiction.

A closed subset C of an ANR X has _infinite codimension_ in X provided $H_q(U, U - C) = 0$ for all integers $q \geq 0$ and all open sets U of X . In the last section of this chapter, we prove that if X is a topologically complete separable ANR that is an ℓ_2-manifold on the compliment of a compact subset of X with infinite codimension, then $X \times A$ is an ℓ_2-manifold for any $A \in \mathscr{A}$. Examples 3.5 and 3.6 provide a source of examples of such spaces that are ℓ_2-manifolds off of compact subsets with infinite codimension.

Example 3.7. ℓ_2/A of Example 3.5 and, if C is finite dimensional, s_C of Example 3.6 are topologically complete separable AR's that are not ℓ_2-manifolds though they are ℓ_2-manifolds off of compact subsets with infinite codimension.

In particular, if C is a wild Cantor set in the Hilbert cube with non-simply connected complement, then s_C satisfies Example 3.7.

Proof. All we need prove is that the point A has infinite codimension in ℓ_2/A and that $C \subset s_C$ has infinite codimension in s_C. The latter statement is true because points in s_C have infinite codimension in s_C and Corollary 2.5 of [DW] then implies that each finite dimensional closed subset of s_C (in particular, C) has infinite codimension in s_C.

The fact that A has infinite codimension in ℓ_2/A is a consequence of the topology that we have placed on ℓ_2/A. First note that since A is a finite dimensional closed subset of ℓ_2, Corollary 2.5 of [DW] again applies to show that A has infinite codimension in ℓ_2. Let U be an open subset of ℓ_2/A and let $z \in H_q(U, U - A)$ and let $(P, \partial P)$ be a compact pair carrying z, that is, $(P, \partial P)$ is a compact pair contained in $(U, U - A)$ such that

$$z \in \operatorname{im}[\, H_q(P, \partial P) \xrightarrow{i_*} H_q(U, U - A)\,]$$

where i_* is the inclusion induced homomorphism. Remember that $\pi : \ell_2 \to \ell_2/A$ denotes the natural projection map. By our choice of local base at A in ℓ_2/A, we can choose a map $g : \ell_2/A \to \ell_2$ such that $\pi \circ g$ is so close to $\mathrm{id}_{\ell_2/A}$ that $\pi \circ g|P$ is homotopic to the inclusion of P into U where the action of the homotopy takes place in U, and in fact, we may require that the image of ∂P under the homotopy miss A. The following diagram where i_* and j_* are inclusion induced and g_* and π_* are induced by g and π, respectively, commutes since $\pi \circ g|P$ and i are homotopic as maps of the pair $(P, \partial P)$ into $(U, U - A)$ and since $(g(P), g(\partial P)) \subset (\pi^{-1}(U), \pi^{-1}(U - A))$.

$$\begin{tikzcd}
H_q(P, \partial P) \arrow[r, "i_*"] \arrow[d, "g_*"'] & H_q(U, U - A) \\
H_q(g(P), g(\partial P)) \arrow[r, "j_*"'] \arrow[ur, "\pi_*"] & H_q(\pi^{-1}(U), \pi^{-1}(U - A)) \arrow[u, "\pi_*"']
\end{tikzcd}$$

Let $z' \in i_*^{-1}(z) \subset H_q(P, \partial P)$ and observe that $z = i_*(z') = \pi_* g_*(z') = \pi_* j_* g_*(z')$. Since A has infinite codimension in ℓ_2 and $\pi^{-1}(U)$ is open and $\pi^{-1}(U - A) = \pi^{-1}(U) - A$, the group $H_q(\pi^{-1}(U), \pi^{-1}(U - A))$ is zero and therefore $z = \pi_* j_* g_*(z') = \pi_*(0) = 0$. This implies that $H_q(U, U - A) = 0$ and A has infinite codimension in ℓ_2/A.

Philosophically, we view ℓ_2/A as the decomposition space associated with a "cell-like" decomposition of ℓ_2 that is not "cellular" or "point-like," much like the corresponding decomposition of the Hilbert cube that we obtain by threading an arc through a wild Cantor set. Of course, A is not a cell-like set since it is not compact, but A does behave like a cell-like set in the sense that whenever A is embedded as a closed subset of an ANR, then A contracts to a point in every neighborhood of itself (remember, A itself is contractible). Moreover, it is easy to re-embed A into ℓ_2 so that A acts like a "cellular" or "pointlike" subset. Indeed, just embed A so that it is a Z-set in ℓ_2. Then $\ell_2 - A$ is homeomorphic to ℓ_2 [An_4] and, since the point A is a Z-set in ℓ_2/A, ℓ_2/A is homeomorphic to ℓ_2 (by [To_3, (B)]). In this example, it is not important that we use A as our nondegenerate decomposition element. It is important that we use a space B that can be embedded as a closed subset of ℓ_2 containing the wild 0-dimensional set C such that B has infinite codimension and contracts to a point in every neighborhood of itself. This allows us to put a "correct" topology on the decomposition space ℓ_2/B, that is, a topology that first exhibits ℓ_2/B as a topologically complete ANR, and second allows us to conclude that the decomposition space has the local homology of an ℓ_2-manifold, meaning, of course, that points in ℓ_2/B have infinite codimension. This is a condition that must be satisfied if there is to be hope of proving that such decomposition spaces stabilize to manifolds upon multiplication by finite dimensional spaces.

In the locally compact setting of both Hilbert cube manifolds and finite dimensional manifolds, the correct decomposition theory is that of cell-like upper semicontinuous decompositions with quotient topologies. This is enough to ensure that the decomposition spaces have the local homology of the corresponding manifold. In this setting, the fact that the decomposition spaces look like manifolds in terms of local homology is a consequence of the Vietoris-Begle mapping theorem [Be], which implies that for open subsets $V \subset U$ in the decomposition space, the map $\pi : (\pi^{-1}(U), \pi^{-1}(V)) \to (U, V)$ induces homology isomorphisms in all dimensions where π is the decomposition map. In the nonlocally compact setting of ℓ_2-manifolds, the decomposition theory of cell-like upper semicontinuous decompositions that yield ANR decomposition spaces is a trivial theory, since all such decompositions yield ℓ_2-manifold decomposition spaces [Mo_1]. To obtain interesting decompositions that yield ANR's, we must not require our cell-like sets to be compact. But once we allow non-compact decomposition elements, we must relinquish the use of the quotient topology if we wish to obtain metrizable decomposition spaces. In describing our example ℓ_2/A, we have shown in a special case how to topologize the decomposition space associated with an upper semicontinuous "non-compact cell-like" decomposition of ℓ_2 so as to obtain a "generalized ℓ_2-manifold," that is, a nowhere locally compact metric ANR that has the local homology of ℓ_2. This technique shows promise of providing an interesting decomposition theory in the nonlocally compact world of ℓ_2-manifolds.

A Lemma Concerning Pairs of Discrete Families

In the last section of this chapter we prove a stabilization theorem for ℓ_2-manifolds that applies to show that the fake ℓ_2-manifolds of the previous section produce ℓ_2-manifolds upon multiplication by elements of $\mathscr{A}$. In the proof of the stabilization theorem it is important to know when, given two discrete collections $C = \{C_i\}_{i \in \mathbb{N}}$ and $\mathcal{D} = \{D_i\}_{i \in \mathbb{N}}$ of closed sets in a separable ANR X , we can move one collection $\mathcal{D}$ by small moves to produce a collection $\mathcal{D}' = \{D_i'\}_{i \in \mathbb{N}}$ so that the collection $\{C_i \cup D_i'\}_{i \in \mathbb{N}}$ is discrete. This section is concerned with proving that the above move is possible provided the collection C consists of Z-sets and the collection $\mathcal{D}$ consists of compacta. This result also is important in Chapter IV in our study of the relationship between the discrete carriers property and discrete cells properties. We begin with a lemma that states a well-known property of Z-sets in ANR's.

Lemma 3.8. *Let Z be a Z-set in a separable ANR X and let C be a closed subset of X that misses Z . Then id_X can be arbitrarily closely approximated by maps g such that $g(X) \cap Z = \phi$ and $g|C = id_C$,*

Proof. Since Z is a Z-set and X is a separable ANR, there exists a small homotopy $G : X \times [0, 1] \to X$ from id_X to a map G_1 whose image misses Z and since C is a closed set that misses Z , G may be chosen so that $G(C \times [0, 1]) \cap Z = \phi$. Let U be an open set in X containing C such that $G(U \times [0, 1]) \cap Z = \phi$

and let ψ be a Urysohn function on X such that $\psi(C) = 0$ and $\psi(X - U) = 1$. Define $g : X \to X$ by $g(x) = G(x, \psi(x))$ and observe that $g(X) \cap Z = \phi$ and $g|C = id_C$.

The next lemma is the promised result of this section. It is important for applications to note that the map g_i in Lemma 3.9 is chosen so that $g_i|C_i = id_{C_i}$

<u>Lemma 3.9</u>. <u>Let</u> $\mathcal{C} = \{C_i\}_{i\in\mathbb{N}}$ <u>and</u> $\mathcal{D} = \{D_i\}_{i\in\mathbb{N}}$ <u>be discrete collections of closed subsets of a separable</u> ANR X. <u>If each</u> C_i <u>in</u> $\mathcal{C}$ <u>is a</u> Z-<u>set and each</u> D_i <u>in</u> $\mathcal{D}$ <u>is a compactum, then there exist maps</u> $g_i : X \to X$ <u>arbitrarily close to</u> id_X <u>such that</u> $g_i|C_i = id_{C_i}$ <u>and</u> $\{g_i(C_i \cup D_i)\}_{i\in\mathbb{N}} = \{C_i \cup g_i(D_i)\}_{i\in\mathbb{N}}$ <u>forms a discrete family</u>.

<u>Proof</u>. First note the following two facts: (i) given $x_i \in C_i \cup D_i$, $\{x_i\}_{i\in\mathbb{N}}$ has no convergent subsequence, (ii) for each $i \in \mathbb{N}$, D_i meets at most finitely many elements of $\mathcal{C}$. (i) follows because both $\mathcal{C}$ and $\mathcal{D}$ are discrete and (ii) follows because each D_i is compact and $\mathcal{C}$ is discrete. By (ii), there is a positive integer $m(1)$ such that $D_1 \cap C_i = \phi$ for all $i > m(1)$. Let $C_1 = \cup\{C_i | i > m(1)\}$ and $\mathcal{D}_1 = \cup\{D_i | i > 1\}$. Since $\mathcal{C}$ and $\mathcal{D}$ are discrete, C_1 and $\mathcal{D}_1$ are closed in X and since C_1 and $\mathcal{D}_1$ are disjoint from the compact set D_1, $\varepsilon_1 = \frac{1}{2}\rho(D_1, C_1 \cup \mathcal{D}_1) > 0$. Let $Z_1 = C_2 \cup \ldots \cup C_{m(1)}$, a Z-set in X, and let $\mathcal{U}_1$ be an arbitrary open cover of X. Since $C_1 \cap Z_1 = \phi$ and C_1 is closed in X, Lemma 3.8 applies to give a

map $g_1 : X \to X$ that is ε_1-close, $\frac{1}{2}$-close, and $\mathcal{U}_1$-close to id_X that satisfies $g_1|C_1 = id_{C_1}$ and $g_1(X) \cap Z_1 = \phi$. It is clear that $g_1(D_1) \cap (C_i \cup D_i) = \phi$ for all $i > 1$.

There is a positive integer $m(2)$ such that $D_2 \cap C_i = \phi$ for $i > m(2)$. Let $\mathcal{C}_2 = \cup\{C_i | i > m(2)\}$, $\mathcal{D}_2 = g_1(D_1) \cup D_3 \cup D_4 \cup \ldots$, and $Z_2 = C_1 \cup C_3 \cup \ldots \cup C_{m(2)}$, a Z-set in X. Then $\varepsilon_2 = \frac{1}{2}\rho(D_2, \mathcal{C}_2 \cup \mathcal{D}_2) > 0$ and if $\mathcal{U}_2$ is an arbitrary open cover of X, then there exists a map $g_2 : X \to X$ that is ε_2-close, $\frac{1}{2^2}$-close, and $\mathcal{U}_2$-close to id_X that satisfies $g_2|C_2 = id_{C_2}$ and $g_2(X) \cap Z_2 = \phi$. It is clear, again, that $g_2(D_2) \cap (C_i \cup D_i) = \phi$ for $i > 2$ and $g_2(D_2) \cap (C_1 \cup g_1(D_1)) = \phi$.

Continue in this manner to obtain for each $i \in \mathbb{N}$, a map $g_i : X \to X$ that is $\frac{1}{2^i}$-close and $\mathcal{U}_i$-close to id_X, where $\mathcal{U}_i$ is any preassigned open cover of X, such that $g_i|C_i = id_{C_i}$ and $g_i(D_i) \cap (C_j \cup g_j(D_j)) = \phi$ for all $j \neq i$. Then $\{C_i \cup g_i(D_i)\}_{i\in\mathbb{N}}$ is a pairwise disjoint collection and since g_i is $\frac{1}{2^i}$-close to id_X, the collection $\{C_i \cup g_i(D_i)\}_{i\in\mathbb{N}}$ also satisfies (i). But this implies that $\{C_i \cup g_i(D_i)\}_{i\in\mathbb{N}}$ is a discrete collection.

Z-sets in Non-locally Compact ANR's

Several times we have used the fact that if F is a Z-set in a separable ANR X, then the identity map on X can be arbitrarily closely approximated by maps into $X - F$. This is proved by approximately factoring id_X through a locally finite countable complex K and then pushing the image of K off of F one simplex at a time. If X is in addition locally compact, the map f

into $X - F$ approximating id_X can be chosen so close to id_X that it is a proper map, and therefore its image $f(X)$ is closed in X [Ch_2, Theorem 4.1]. Thus, in the locally compact setting, X can be "uniformly pushed off of F" in the sense that id_X can be approximated by a map f such that $[Cl_X f(X)] \cap F = \phi$. It is not clear at all that this can be accomplished in the general setting of separable ANR's; however, we need such a result in our proof of the stabilization theorem for ℓ_2-manifolds found in the next section. We prove such a result in the setting of topologically complete separable ANR's and this result, Theorem 3.10, and its corollary suffice for our purposes.

Theorem 3.10. *Let F be a Z-set in the topologically complete separable ANR X. Then for every open cover $\mathcal{U}$ of X, there exists a map $f : X \to X - F$ such that f is $\mathcal{U}$-close to id_X and $[Cl_X f(X)] \cap F = \phi$.*

The idea of the proof of Theorem 3.10 was suggested to the author by J. J. Walsh and the proof heavily depends on the results of Torunczyk concerning ℓ_2-manifolds found in [To_3]. Before we prove Theorem 3.10, we state and prove its important corollary.

Corollary 3.11. *Let F be a σ-Z-set in the topologically complete separable ANR X. Then for every open cover $\mathcal{U}$ of X, there exists a map $f : X \to X - F$ such that f is $\mathcal{U}$-close to id_X and $[Cl_X f(X)] \cap F = \phi$.*

Proof. Write $F = \bigcup_{i \in \mathbb{N}} F_i$ where each F_i is a Z-set in X and choose a map $f_1 : X \to X - F_1$ and a closed neighborhood $\overline{N}_1$ of F_1 so that $[\mathrm{Cl}_X f_1(X)] \cap \overline{N}_1 = \phi$. Choose $f_2 : X \to X - F_2$ so close to f_1 that $[\mathrm{Cl}_X f_2(f_1(X))] \cap \overline{N}_1 = \phi$ and a closed neighborhood $\overline{N}_2$ of F_2 so that $[\mathrm{Cl}_X f_2(X)] \cap \overline{N}_2 = \phi$. Continue in this manner and obtain maps $f_i : X \to X - F_i$ and closed neighborhoods $\overline{N}_i$ of F_i such that $[\mathrm{Cl}_X f_n \circ \cdots \circ f_1(X)] \cap [\overline{N}_n \cup \cdots \cup \overline{N}_1] = \phi$. Since X is topologically complete, the f_i may be chosen so that the sequence $\{f_n \circ \cdots \circ f_1\}_{n \in \mathbb{N}}$ converges to a map f and the f_i also may be chosen so close to id_X that the limit map f is $\mathcal{U}$-close to id_X for any preassigned open cover $\mathcal{U}$ of X . It is easy to verify that $f(X) \cap N = \phi$ where N is the open set $N = \bigcup_{i \in \mathbb{N}} \mathrm{Int}_X \overline{N}_i$ containing F . Then $[\mathrm{Cl}_X f(X)] \cap F = \phi$.

The remainder of this section is devoted to a proof of Theorem 3.10. Throughout the proof, d denotes a metric on X and d_∞ denotes the metric on the Hilbert cube I^∞ given by

$$d_\infty((x_i), (y_i)) = \max\left\{\frac{|x_i - y_i|}{i} \;\middle|\; i \in \mathbb{N}\right\} .$$

ρ denotes the metric on the product $X \times I^\infty$ given by $\rho((x, y), (x', y')) = \max\{d(x, x'), d_\infty(y, y')\}$. We also use ρ to denote the restriction of the metric ρ on $X \times I^\infty$ to any given subspace of $X \times I^\infty$. $p_j : I^\infty \to I_j = [-1, 1]$ denotes the j-th projection map.

Since F is a Z-set in X, F is closed and the continuous real-valued function r on X defined by $r(x) = \frac{d(x, F)}{1 + d(x, F)}$ satisfies $r^{-1}(0) = F$. Let $X \times_r I^\infty$ and $X \times_r s$ be the variable products with respect to r of X with I^∞ and s, respectively, that is,

$$X \times_r I^\infty = \cup \{\{x\} \times \Pi_{i \in \mathbf{N}} [-r(x), r(x)]_i | x \in X\}$$

$$X \times_r s = \cup \{\{x\} \times \Pi_{i \in \mathbf{N}} (-r(x), r(x))_i | x \in X\}$$

where the respective topologies are induced by $X \times_r s \subset X \times_r I^\infty \subset X \times I^\infty$, or, equivalently, by the respective restrictions of ρ to $X \times_r I^\infty$ and $X \times_r s$. The following facts are used in the proof of Theorem 3.10 and are verified following its proof.

(i) $X \times_r s$ <u>is a completely metrizable separable</u> ANR.

(ii) $F = F \times \{0\} \subset X \times_r s$ <u>is a</u> Z-<u>set in</u> $X \times_r s$.

(iii) $X \times_r s$ <u>is an</u> ℓ_2-<u>manifold</u>.

(iv) <u>Let</u> K <u>be a closed subset of</u> $X \times_r s$ <u>with</u> $K \cap F = \phi$ <u>and let</u> $\pi : X \times I^\infty \to X$ <u>be the projection</u>. <u>Then</u> $[\mathrm{Cl}_X \pi(K)] \cap F = \phi$.

(v) <u>Let</u> K <u>be a closed subset of a metric space</u> M <u>and let</u> $(f_t) : M \to M$ <u>be a homotopy such that</u> $f_0 = \mathrm{id}_M$, $\cup_{t>0} f_t(M) = M - K$, <u>and</u> $(x, t) \to (f_t(x), t)$ <u>is a closed embedding of</u> $M \times [0, 1]$ <u>into</u> $M \times [0, 1]$. <u>Then given a map</u> $\varepsilon : M \to (0, \infty)$, <u>there exists a map</u> $h : M \to M - K$ <u>such that</u> h <u>is</u> ε-<u>close to</u> id_M <u>and</u> $h(M)$ <u>is closed in</u> M.

<u>Proof of Theorem 3.10</u>. Let $\overline{\varepsilon} : X \to (0, \infty)$ be a map such that $\{N_{\overline{\varepsilon}(x)}(x) | x \in X\} \prec \mathcal{U}$. By (ii) and (iii), F is a Z-set in the ℓ_2-manifold $X \times_r s$ and therefore, according to [To$_3$, Lemma 1°, p. 104], there exists a homotopy $(f_t) : X \times_r s \to X \times_r s$ such that $f_0 = \mathrm{id}_{X \times_r s}$, $\bigcup_{t>0} f_t(X \times_r s) = (X \times_r s) - F$, and $(x, t) \to (f_t(x), t)$ is a closed embedding of $(X \times_r s) \times [0, 1]$ into itself. Define $\varepsilon : X \times_r s \to (0, \infty)$ via $\varepsilon(x, y) = \overline{\varepsilon}(x)$. According to (v), there exists a map $h : X \times_r s \to (X \times_r s) - F$ that is ε-close to $\mathrm{id}_{X \times_r s}$ and whose image is closed in $X \times_r s$. Since $h(X \times_r s)$ is a closed subset of $X \times_r s$ that misses F, (iv) implies that $[\mathrm{Cl}_X\, \pi \circ h(X \times_r s)] \cap F = \phi$ where $\pi : X \times I^\infty \to X$ is the projection. Let $f = \pi \circ h | X \times \{0\}$. It is clear that $[\mathrm{Cl}_X\, f(X)] \cap F = \phi$ and since for each $x \in X$, $d(x, f(x)) = d(x, \pi h(x, 0)) \leq \rho((x, 0), h(x, 0)) < \varepsilon(x, 0) = \overline{\varepsilon}(x)$, f is $\mathcal{U}$-close to id_X. This completes the proof of Theorem 3.10, except for the verification of (i) through (v).

<u>Proof of (i)</u>. Given a number δ with $0 \leq \delta < 1$, let $R_\delta : I^\infty \to I^\infty$ be defined by

$$p_j R_\delta(q) = \mathrm{sign}(p_j(q)) \cdot \min\{\delta, |p_j(q)|\},$$

obviously a continuous function. Define $R : X \times I^\infty \to X \times I^\infty$ by $R(x, q) = (x, R_{r(x)}(q))$. The reader may verify that $R = \mathrm{id}$ on $X \times_r I^\infty$, $R(X \times I^\infty) = X \times_r I^\infty$, and R is continuous, and therefore R exhibits $X \times_r I^\infty$ as a retract of the ANR $X \times I^\infty$. This implies that $X \times_r I^\infty$ is an ANR and a closed subset of the topologically complete separable space $X \times I^\infty$, and therefore $X \times_r I^\infty$ is a topologically complete separable ANR.

Write $X - F = \cup_{i \in \mathbb{N}} C_i$ where each C_i is closed in X and $I^\infty - s = \cup_{i \in \mathbb{N}} D_i$ where each D_i is closed in I^∞. It is a straightforward, though somewhat tedious exercise to show that $L = \cup \{R(C_i \times D_j) | i, j \in \mathbb{N}\}$ is a σ-Z-set in $X \times_r I^\infty$ and that $X \times_r s$ is the compliment of L in $X \times_r I^\infty$. Since $X \times_r s$ is then a G_δ subset of a topologically complete separable space, $X \times_r s$ is topologically complete and separable, and since L is a σ-Z-set in a topologically complete separable ANR, Theorem 3.1 of [To_3] implies that $X \times_r s$ is an ANR.

Proof of (ii). Clearly $F = F \times \{0\} \subset X \times_r s$ is a closed subset of $X \times I^\infty$ and therefore is closed in $X \times_r s$. Let $\varepsilon > 0$ and let $f = (f_1, f_2) : I^\infty \to X \times_r s$ be a map and use the fact that F is a Z-set in X to produce a map $g_1 : I^\infty \to X - F$ such that $d(g_1, f_1) < \varepsilon$. If g_1 is close enough to f_1, we can find $g_2 : I^\infty \to s$ so that $g = (g_1, g_2)$ is ε-close to f and has image in $X \times_r s$.

Proof of (iii). First observe that $(X \times_r s) - F$ is homeomorphic to $(X - F) \times s$ via the homeomorphism $h(x, q) = (x, \frac{q}{r(x)})$ defined on $(X \times_r s) - F$. Since $X - F$ is an ANR, Theorem 4.3 of [To_1] implies that $(X - F) \times s$ is an ℓ_2-manifold. Since $X \times_r s$ is a topologically complete separable ANR, F is a Z-set in $X \times_r s$, and $(X \times_r s) - F$ is an ℓ_2-manifold, Proposition 5.1 of [To_3] implies that $X \times_r s$ is an ℓ_2-manifold.

Proof of (iv). Let $\{x_k\}$ be a sequence in $\pi(K)$ that converges to a point x in X and for each k, choose $y_k \in I^\infty$ such that $(x_k, y_k) \in K$. If $x \in F$, then $r(x) = 0$ and since $r(x_k) \to r(x) = 0$ and $y_k \in \Pi_{i\in\mathbb{N}} (-r(x_k), r(x_k))_i$, $y_k \to 0$ in s. But then $(x_k, y_k) \to (x, 0) \in X \times_r s$ and since K is closed in $X \times_r s$, $(x, 0) \in K$. This implies that $K \cap F \neq \phi$ since $(x, 0) \in F$ and this contradicts the hypothesis. Therefore, $x \notin F$ and $[\mathrm{Cl}_X\, \pi(K)] \cap F = \phi$.

Proof of (v). Let d_M denote a metric on M and for each $x \in M$, define $\psi(x)$ by

$$\psi(x) = \begin{cases} 1 & \text{if } d_M(x, f_t(x)) < \varepsilon(x) \quad \text{for all } t > 0 \\ \inf\{t \mid d_M(x, f_t(x)) \geq \varepsilon(x)\} & \text{otherwise .} \end{cases}$$

The reader should verify that ψ is a lower semicontinuous function on M and since $0 < \psi$, Theorem 4.3, page 171 of [Du] applies to give a continuous function $\theta : M \to [0, 1]$ such that $0 < \theta < \psi$.

Define $h : M \to M$ by $h(x) = f_{\theta(x)}(x)$, certainly a continuous function on M. Since $\theta(x) < \psi(x)$ for $x \in M$, $d_M(x, h(x)) < \varepsilon(x)$ and since $0 < \theta$, $h(x) \in M - K$. It remains to show that $h(M)$ is closed in M. Let $\{y_k = h(x_k)\}_{k\in\mathbb{N}}$ be a sequence in $h(M)$ and assume that $y_k \to y$. Without loss of generality, we assume that $\theta(x_k) \to t$ for some t in $[0, 1]$. Let e be the closed embedding of $M \times [0, 1]$ into itself that is promised in the hypothesis. Obviously, $e(x_k, \theta(x_k)) = (f_{\theta(x_k)}(x_k), \theta(x_k)) = (h(x_k), \theta(x_k))$

$= (y_k, \theta(x_k)) \to (y, t)$. Since $e(M \times [0, 1])$ is closed, $(y, t) = e(x, t)$ for some $x \in M$. Since e is an embedding and $e(x_k, \theta(x_k)) \to e(x, t)$, we must have $x_k \to x$. But then $h(x_k) \to h(x)$ and since $h(x_k) = y_k \to y$, we have $y = h(x) \in h(M)$. This implies that $h(M)$ is closed in M .

A Stabilization Theorem for ℓ_2-manifolds

We prove a stabilization theorem for ℓ_2-manifolds that allows us to conclude that the examples of "generalized ℓ_2-manifolds" that we constructed in Example 3.7 stabilize, that is, become ℓ_2-manifolds, upon multiplication by an element A of $\mathscr{A}$. This result appears as the following theorem.

Theorem 3.12. *If X is a topologically complete separable ANR and K is a compact subset of X with infinite codimension such that $X - K$ is an ℓ_2-manifold, then $X \times A$ is an ℓ_2-manifold for any $A \in \mathscr{A}$.*

We view Theorem 3.12 as a generalization to spaces that are ℓ_2-manifolds off of homologically small (infinite codimension) compact subsets of the corresponding result of Torunczyk concerning spaces that are ℓ_2-manifolds off of homotopically small (Z-sets) closed subsets.

Theorem. [Torunczyk; To_3, Proposition 5.1] *If X is a topologically complete separable ANR and K is a Z-set in X such that $X - K$ is an ℓ_2-manifold, then X is an ℓ_2-manifold.*

Our examples of Example 3.7 show that the conclusion of Theorem 3.12 cannot be strengthened to say that X is an ℓ_2-manifold.

The proof of Theorem 3.12 consists of showing that $X \times A$ satisfies the discrete approximation property so that Trounczyk's characterization of ℓ_2-manifolds (Theorem 3.0) applies to conclude that $X \times A$ is an ℓ_2-manifold. In the proof, it is important to know that the space that we multiply X by, A, satisfies the discrete 0-cells property. In particular, we use the fact that A satisfies the conclusion of the following lemma.

Lemma 3.13. *If $A \in \mathcal{A}$ and L is a closed subset of A contained in $E(A)$ and $\mathcal{U}$ is a collection of open subsets of A that covers L, then there exists a countable collection $\mathcal{V}$ of (pairwise disjoint) open subsets of A that refines $\mathcal{U}$ and covers L such that each element of $\mathcal{V}$ is a component of $A - \{*\}$ for some point $* \in A - E(A)$ and $\overline{\mathcal{V}} = \{\mathrm{Cl}_A V \mid V \in \mathcal{V}\}$ is discrete.*

Proof. First, use the fact that $E(A)$ is 0-dimensional to obtain a countable pairwise disjoint collection $\mathcal{W}$ of connected open subsets of A that refines $\mathcal{U}$ and covers L, and assume that each element of $\mathcal{W}$ meets L. For each W in $\mathcal{W}$, choose an open subset $\overline{W}$ in $\overline{A}$, where $\overline{A}$ corresponds to A, such that $\overline{W} \cap A = W$, and let w be a cutpoint of A in W. Let $[w]$ denote the union of all arcs from w to points in the frontier of $\overline{W}$ in $\overline{A}$, and show that $[w]$ is closed in $\overline{A}$. Thus, $W - [w]$ is open in A, and therefore, each component of $W - [w]$ is open in A. Show that each component C of $W - [w]$ is a component of $A - \{a\}$ for some cutpoint a of A and use the fact that L is closed in A to find a cutpoint a_C in A such that $L \cap C$ is contained in a component of $A - \{a_C\}$ contained in W and $a_C \neq a_D$ if $C \neq D$. In this way, obtain a countable pairwise disjoint collection $\mathcal{V}$ of connected open subsets of A that refines $\mathcal{W}$ and covers L such that each V in $\mathcal{V}$ meets L and is a component of $A - \{*\}$ for some point $*$ in A and $\overline{\mathcal{V}}$ is pairwise disjoint. Now observe that $\overline{\mathcal{V}}$ is discrete since the fact that $\overline{A}$ is locally connected and uniquely arcwise connected implies that only finitely many elements of $\overline{\mathcal{V}}$ can have diameter greater than any given positive number.

To prove that $X \times A$ satisfies the discrete approximation property where X is as in the statement of Theorem 3.12, we first push the "limit points" of any given map of $\oplus_{i \in \mathbb{N}} I^i$ into $X \times A$ off of $K \times (A - E(A))$. Then we use Lemma 3.13 to adjust the map near $K \times E(A)$ so that the images of the I^i's form a discrete family near $K \times A$. We then adjust the map away from $K \times A$ to obtain discreteness there without destroying discreteness near $K \times A$ and we then use Lemma 3.9 to match up the map near $K \times A$ to the map away from $K \times A$ in order to obtain an approximation such that the images of the I^i's form a discrete family.

Recall that K has infinite codimension in X means that $H_q(U, U - K)$ is zero for all integers $q \geq 0$ and all open sets U of X.

Proof of Theorem 3.12. $X \times A$ is a topologically complete separable ANR. According to Theorem 3.0 (Torunczyk), it suffices to show that $X \times A$ satisfies the discrete approximation property.

We first show that compact subsets of $X \times A$ are Z-sets in $X \times A$. Observe that $K \times A$ has infinite codimension in $X \times A$, the proof of which is exactly the proof of [DW, Lemma 2.2] with the exception that $\mathcal{B}$ denotes the basis of $X \times A$ consisting of all sets $U \times J$ where U is open in X and J is connected and open in A. (The important property of this basis is that if J and J' are connected and open in A, then so is $J' \cap J$ (Proposition 2.0. (iii)), and that each such J is contractible in itself (Proposition 2.0 (iv)). See [DW, Lemma 2.2].) Let $C \subset X \times A$ be compact and write $C = C_0 \cup C_1 \cup C_2 \cup \ldots$ where $C_0 = C \cap (K \times A)$ and $C - C_0 = \cup_{k \in \mathbb{N}} C_k$ where C_k is compact for each $k \in \mathbb{N}$. C_0 has infinite codimension in $X \times A$ by [DW, Lemma 2.1] and it is easy to verify that C_0 is a 1-LCC subset of $X \times A$. This means that given a point $c \in C_0$ and a neighborhood U of c in $X \times A$, there exists a neighborhood V of c in $X \times A$ such that loops in $V - C_0$ are contractible in $U - C_0$. The Hurewicz Theorem then implies that C_0 is a Z-set in $X \times A$ (see [DW, Proposition 4.2]). Since $(X - K) \times A$ is an ℓ_2-manifold, each C_k for $k \in \mathbb{N}$ is a Z-set in $(X - K) \times A$ and thus, since $K \times A$ is closed in $X \times A$, each such C_k is a Z-set in $X \times A$. Thus, C is a σ-Z-set in $X \times A$ and since C is closed in $X \times A$, C is a Z-set in $X \times A$.

Let $\mathcal{U}$ be an open cover of $X \times A$ consisting of product open sets $V \times W$ where V and W are open in X and A, respectively, and let $f : \oplus_{n \in \mathbb{N}} I^n \to X \times A$ be a map. Since $A - E(A)$ is σ-compact, $K \times (A - E(A))$ is a σ-Z-set in $X \times A$ and Corollary 3.11 applies to produce a map $u : X \times A \to X \times A$ that is $\mathcal{U}$-close to $id_{X \times A}$ that satisfies $[Cl_{X \times A} u(X \times A)] \cap [K \times (A - E(A))] = \phi$. Let $L(u \circ f)$ denote the <u>limit points of the map</u> $u \circ f$, that is, $L(u \circ f)$ consists of all points x in $X \times A$ such that there exists points $x_i \in u \circ f(I^i)$ for $i \in \mathbb{N}$ such that every neighborhood of x in $X \times A$ contains infinitely many elements of $\{x_i\}_{i \in \mathbb{N}}$. Easily, $L(u \circ f)$ is a closed subset of $X \times A$ and, since K is compact, this implies that $L = p_A((K \times A) \cap L(u \circ f))$ is a closed subset of A where p_A denotes the obvious projection map. By our choice of u, L consists only of endpoints of A, that is, $L \subset E(A)$.

For the moment, fix $a \in L$. Choose finitely many sets $V_1 \times W_1, \ldots, V_k \times W_k$ in $\mathcal{U}$ so that $(K \times \{a\}) \cap (V_i \times W_i) \neq \phi$ for each i and $K \times \{a\} \subset \cup_{i=1}^{k} V_i \times W_i$. Let $V_a = V_1 \cup \ldots \cup V_k$ and $W_a = W_1 \cap \ldots \cap W_k$ and observe that $K \times \{a\} \subset V_a \times W_a$ and if $v \in V_a$ and $W \subset W_a$, then $\{v\} \times W \subset U$ for some $U \in \mathcal{U}$. Obtain such sets V_a and W_a for each element a of L.

Since $L \subset E(A)$ is closed in A and $\{W_a \mid a \in L\}$ is a cover of L by sets open in A, Lemma 3.13 applies to give a refinement $\{W_\gamma \mid \gamma \in \mathbb{N}\}$ of $\{W_a \mid a \in L\}$ by pairwise disjoint open sets in A that cover L and so that each W_γ is a component of $A - \{a_\gamma\}$ for some $a_\gamma \in A - E(A)$. Also, if $\overline{W}_\gamma = W_\gamma \cup \{a_\gamma\} = Cl_A W_\gamma$, then $\{\overline{W}_\gamma \mid \gamma \in \mathbb{N}\}$ is discrete in A. For each $\gamma \in \mathbb{N}$, choose some W_a

so that $W_\gamma \subset W_a$ and define V_γ to be V_a. Then the collection $\{\{v\} \times W_\gamma \mid v \in V_\gamma, \gamma \in \mathbb{N}\}$ refines $\mathcal{U}$ and the collection $\{V_\gamma \times W_\gamma \mid \gamma \in \mathbb{N}\}$ is an open cover of $(K \times A) \cap L(u \circ f)$ by pairwise disjoint open subsets of $X \times A$.

For each $\gamma \in \mathbb{N}$, choose a sequence $\{b_\gamma(i)\}_{i \in \mathbb{N}}$ of points in W_γ such that $\{b_\gamma(i)\}_{i \in \mathbb{N}}$ is discrete in A. Since $\{\overline{W}_\gamma \mid \gamma \in \mathbb{N}\}$ is discrete, $\{b_\gamma(i) \mid i, \gamma \in \mathbb{N}\}$ forms a discrete collection in A. Fix $\gamma \in \mathbb{N}$ and for each $i \in \mathbb{N}$, let $H_i : W_\gamma \times [0, 1] \to W_\gamma$ be a contraction of W_γ to the point $b_\gamma(i)$ that keeps $b_\gamma(i)$ fixed. Choose open subsets $S_0(\gamma)$ and $S_1(\gamma)$ in X such that

(i) $K \subset S_0(\gamma) \subset \overline{S}_0(\gamma) = Cl_X S_0(\gamma) \subset S_1(\gamma) \subset \overline{S}_1(\gamma) = Cl_X S_1(\gamma) \subset V_\gamma$

(ii) $[\overline{S}_1(\gamma) \times \{a_\gamma\}] \cap [Cl_{X \times A}\, u(X \times A)] = \phi$.

Let $C = (X \times A) - (S_1(\gamma) \times W_\gamma)$ and $D = \overline{S}_0(\gamma) \times \overline{W}_\gamma$ and let $C' = (uf)^{-1}(C)$ and $D' = (uf)^{-1}(D)$. For each $i \in \mathbb{N}$, the sets $C_i' = C' \cap I^i$ and $D_i' = D' \cap I^i$ are closed subsets of I^i and since $C \cap D = \overline{S}_0(\gamma) \times \{a_\gamma\}$ misses $u \circ f(I^i)$ (by (ii)), $C_i' \cap D_i' = \phi$. Let $\theta_i : I^i \to [0, 1]$ be a Urysohn function such that $\theta_i(C_i') = 0$ and $\theta_i(D_i') = 1$ and define f_γ on I^i by the formula

$$f_\gamma(x) = (p_X \circ u \circ f(x), H_i(p_A \circ u \circ f(x), \theta_i(x)))$$

where p_X and p_A are the obvious projections. In this way, we obtain a map $f_\gamma : \oplus_{i \in \mathbb{N}} I^i \to X \times A$ that is $\mathcal{U}$-close to $u \circ f$ since f_γ differs from $u \circ f$ only in a movement of the second coordinate that takes place in W_γ. We obtain such a map f_γ for each $\gamma \in \mathbb{N}$.

Define a function $g : \oplus_{i \in \mathbb{N}} I^i \to X \times A$ via $g(x) = f_\gamma(x)$ if $uf(x) \in V_\gamma \times W_\gamma$ and $g(x) = uf(x)$ if $uf(x)$ is not in any $\overline{S}_1(\gamma) \times \overline{W}_\gamma$. We make three claims about g .

1) g <u>is well-defined and $\mathcal{U}$-close to</u> $u \circ f$. First of all, since $(V_\gamma \times W_\gamma) \cap (V_{\gamma'} \times W_{\gamma'}) = \phi$ if $\gamma \neq \gamma'$, there is at most one value $g(x)$ assigned to any $x \in \oplus_{i \in \mathbb{N}} I^i$ and since $uf(x)$ is in some $V_\gamma \times W_\gamma$ if it is in $\overline{S}_1(\gamma) \times \overline{W}_\gamma$, there is at least one value $g(x)$ assigned to any such x . g is $\mathcal{U}$-close to $u \circ f$ since each f_γ is $\mathcal{U}$-close to $u \circ f$.

2) g <u>is continuous</u>. Since $\{\overline{W}_\gamma \mid \gamma \in \mathbb{N}\}$ is discrete in A , $T = \cup \{\overline{S}_1(\gamma) \times \overline{W}_\gamma \mid \gamma \in \mathbb{N}\}$ is a closed subset of $X \times A$ and the set $B = (uf)^{-1}((X \times A) - T)$ is an open subset of $\oplus_{i \in \mathbb{N}} I^i$. Also, $B_\gamma = (uf)^{-1}(V_\gamma \times W_\gamma)$ is open in $\oplus_{i \in \mathbb{N}} I^i$ for each $\gamma \in \mathbb{N}$ and since $u(X \times A) \cap (\overline{S}_1(\gamma) \times \{a_\gamma\}) = \phi$, $\{B\} \cup \{B_\gamma\}_{\gamma \in \mathbb{N}}$ covers $\oplus_{i \in \mathbb{N}} I^i$. Also, $g \mid B_\gamma = f_\gamma \mid B_\gamma$ which is continuous and $g \mid B = uf \mid B$ which is continuous and for each γ , $f_\gamma = uf$ on $B_\gamma \cap B$ and if $\gamma \neq \gamma'$, then $B_\gamma \cap B_{\gamma'} = \phi$. This implies that g is continuous.

3) $L(g) \cap (K \times A) = \phi$ <u>where</u> $L(g)$ <u>denotes the limit points of the map</u> g . Remember that $L(g)$ consists of all points p in $X \times A$ each of whose neighborhoods meets infinitely many elements of $\{g(I^i)\}_{i \in \mathbb{N}}$. Suppose that $p_i \to p$ where $p_i \in g(I^{k(i)})$ and $k(i) \to \infty$. If $p \in (V_\gamma \times W_\gamma) \cap (K \times A)$ for some $\gamma \in \mathbb{N}$, then eventually $p_i \in S_0(\gamma) \times W_\gamma$ and there is an x_i in $I^{k(i)}$ such that $p_i = g(x_i) = f_\gamma(x_i)$, and since $p_A(f_\gamma(x_i)) = b_\gamma(i)$ in this case, $b_\gamma(i) \to p_A(p)$ as $i \to \infty$. But this contradicts the fact that

$\{b_\gamma(j) \mid j, \gamma \in \mathbb{N}\}$ is discrete in A. Therefore, if $p \in V_\gamma \times W_\gamma$, then $p \notin K \times A$. Now suppose that $p \in (K \times A) - \cup\{V_\gamma \times W_\gamma \mid \gamma \in \mathbb{N}\}$ and $p_i = g(x_i)$ for $x_i \in I^{k(i)}$. Without loss of generality, we may assume that $p_i \in \overline{S}_1(\gamma(i)) \times \overline{W}_{\gamma(i)}$ for some $\gamma(i) \in \mathbb{N}$ for otherwise $p_i = g(x_i) = uf(x_i)$ so that $p \in L(u \circ f)$ and this is impossible since $(K \times A) \cap L(u \circ f)$ is covered by $\{V_\gamma \times W_\gamma \mid \gamma \in \mathbb{N}\}$. Suppose $\{\gamma(i) \mid i \in \mathbb{N}\}$ is unbounded. Then since $p_i \to p$ and $p_A(p_i) \in \overline{W}_{\gamma(i)}$, we reach a contradiction of the fact that $\{\overline{W}_\gamma \mid \gamma \in \mathbb{N}\}$ is discrete. Therefore, we may assume that p_i, $p \in \overline{S}_1(\gamma) \times \overline{W}_\gamma$ for some $\gamma \in \mathbb{N}$ and all $i \in \mathbb{N}$. Remember that $p_i = g(x_i)$ and that $p_X \circ g = p_X \circ (uf)$ so that $p_X(uf(x_i)) \in \overline{S}_1(\gamma)$. By (ii), $uf(x_i) \in \overline{S}_1(\gamma) \times W_\gamma$ and this implies that $p_i = g(x_i) \in \overline{S}_1(\gamma) \times W_\gamma$. Since $p_i \to p$, eventually $p_i \in S_0(\gamma) \times W_\gamma$ and this implies that $p_A(p_i) = p_A(g(x_i)) = b_\gamma(i)$ and therefore $b_\gamma(i) \to p_A(p)$ as $i \to \infty$. This contradicts the fact that $\{b_\gamma(j) \mid j \in \mathbb{N}\}$ is discrete. Therefore, $p \notin K \times A$ and $L(g) \cap (K \times A) = \phi$.

Since $L(g)$ and $K \times A$ are disjoint closed subsets of $X \times A$, there are open sets Y_1 and Y_2 in $X \times A$ that satisfy $K \times A \subset Y_1 \subset \overline{Y}_1 \subset Y_2 \subset \overline{Y}_2 \subset (X \times A) - L(g)$ where $\overline{Y}_i$ denotes $\mathrm{Cl}_{X \times A} Y_i$ for $i = 1, 2$. Either by construction of g or by the fact that compact subsets of $X \times A$ are Z-sets, since $\overline{Y}_2 \cap L(g) = \phi$, we may assume that $\{g(I^i) \cap \overline{Y}_2 \mid i \in \mathbb{N}\}$ is discrete in $X \times A$ (either carefully choose Y_2 based on the construction of g or assume that $\{g(I^i) \mid i \in \mathbb{N}\}$ is pairwise disjoint). Let $\mathcal{U}^*$ be an open cover of $X \times A$ with the following properties.

a) $\mathcal{U}^*$ refines $\mathcal{U}$

b) If $U \in \mathcal{U}^*$ and $U \cap [(X \times A) - Y_2] \neq \phi$, then $U \cap \overline{Y}_1 = \phi$

c) $\{\mathrm{Cl}_{X \times A} [\mathrm{st}(g(I^i) \cap \overline{Y}_2, \mathcal{U}^*)] \mid i \in \mathbb{N}\}$ is discrete in $X \times A$.

Remember that the controlled version of the homotopy extension theorem states that if g_0 and g_1 are maps from a closed subset B of a space C that are $\mathcal{U}^*$-homotopic and g_0 extends to a map G_0 of C into $X \times A$, then g_1 extends to a map G_1 of C into $X \times A$ that is $\mathcal{U}^*$-homotopic to G_0. Let $\mathcal{V}$ be a refinement of $\mathcal{U}^*$ so that $\mathcal{V}$-close maps are $\mathcal{U}^*$-homotopic and for each $i \in \mathbb{N}$, let $g_i = g \mid I^i$ and $B_i = g_i^{-1}((X \times A) - Y_2)$.

Since $\{g_i \mid B_i\}_{i \in \mathbb{N}}$ is a collection of maps of finite dimensional compacta into the ℓ_2-manifold $(X - K) \times A$, there are maps $h_i \mid B_i$ that are $\mathcal{V}$-close to $g_i \mid B_i$ and such that $\{h_i(B_i)\}_{i \in \mathbb{N}}$ forms a discrete family in $(X - K) \times A$ (by Theorem 3.0). Thus $h_i \mid B_i$ is $\mathcal{U}^*$-homotopic to $g_i \mid B_i$ and since $g_i \mid B_i$ extends to the map g_i of I^i into $X \times A$, $h_i \mid B_i$ extends to a map h_i of I^i into $X \times A$ that is $\mathcal{U}^*$-close to g_i. Let h denote the map on $\oplus_{i \in \mathbb{N}} I^i$ defined by $h \mid I^i = h_i$ and let $C_i = h_i(\mathrm{Cl}_I i\ (I^i - B_i))$ and $D_i = h_i(B_i)$ for each $i \in \mathbb{N}$. Since h_i is $\mathcal{U}^*$-close to g_i and $g_i(\mathrm{Cl}_I i(I^i - B_i)) \subset \overline{Y}_2 \cap g(I^i)$, c) shows that $\mathcal{C} = \{C_i\}_{i \in \mathbb{N}}$ is a discrete collection of compacta in $X \times A$. Also, b) along with the fact that $\mathcal{D} = \{h_i(B_i)\}_{i \in \mathbb{N}}$ is discrete in $(X - K) \times A$ implies that $\mathcal{D}$ is also a discrete collection of compacta in $X \times A$. Since each C_i is compact, each C_i is a Z-set in $X \times A$.

According to Lemma 3.9, there exist maps $v_i : X \times A \to X \times A$ such that $v_i \mid C_i = \mathrm{id}_{C_i}$, $v_i \mid D_i$ is $\mathcal{U}$-close to the inclusion of D_i into $X \times A$, and $\{v_i(C_i \cup D_i)\}_{i \in \mathbb{N}}$ is discrete in $X \times A$. Define $v : \oplus_{i \in \mathbb{N}} I^i \to X \times A$ by $v \mid I^i = v_i \circ h_i$. Obviously v is continuous and $\mathcal{U}$-close to h and since $v(I^i) = v_i \circ h(I^i) = v_i(C_i \cup D_i)$, $\{v(I^i)\}_{i \in \mathbb{N}}$ forms a discrete family in $X \times A$. We have

$$f \underset{\mathcal{U}}{\sim} u \circ f \underset{\mathcal{U}}{\sim} g \underset{\mathcal{U}^*}{\sim} h \underset{\mathcal{U}}{\sim} v$$

and thus f is $\mathrm{st}^3\,\mathcal{U}$-close to v. We have shown that f is approximable by maps v such that $\{v(I^i)\}_{i \in \mathbb{N}}$ is discrete. Thus $X \times A$ satisfies the discrete approximation property and $X \times A$ is an ℓ_2-manifold.

CHAPTER IV

THE DISCRETE CARRIERS PROPERTY AND CHARACTERIZATIONS OF HILBERT SPACE TOPOLOGY

H. Torunczyk characterizes Q-manifolds as precisely those locally compact ANR's that satisfy the disjoint k-cells property for each $k \geq 0$ [To_4]. R. J. Daverman and J. J. Walsh refine Torunczyk's characterization by showing that Q-manifolds are characterized among locally compact ANR's by combining the disjoint 2-cells property with a disjoint Čech carriers property; roughly, the latter property is that, at the level of Čech homology, homology elements can be made disjoint [DW]. The Daverman-Walsh characterization is an attempt to parallel the corresponding result in finite dimensional manifold theory where the work of R. D. Edwards [Ed] and F. Quinn [Qu] combines to show that n-dimensional manifolds for $n \geq 5$ are characterized among ANR homology manifolds by the disjoint 2-cells property. Both the Daverman-Walsh and the Edwards-Quinn approach to characterizing manifolds are successful attempts at characterizing manifolds by combining a minimal amount of general positioning, the disjoint 2-cells property, with a hopefully minimal algebraic condition. In the case of Daverman-Walsh, it is unknown whether or not their algebraic condition, the disjoint Čech carriers property, is minimal. It may be that this condition can be replaced by the weaker condition that points have infinite codimension thus making the parallel between the Q-manifold characterization and the finite dimensional manifold characterization exact.

This chapter is a first attempt at a characterization of ℓ_2-manifolds in the spirit of Daverman-Walsh and Edwards-Quinn. We make the following conjecture.

Conjecture 4.0. *A topologically complete separable* ANR X *is an* ℓ_2*-manifold if and only if* X *satisfies the discrete* 2-*cells property and the discrete carriers property*.

We define the discrete carriers property in the next section.

Not surprisingly, as in the Daverman-Walsh program for Q-manifolds, our approach attempts to make use of Torunczyk's entirely geometric characterization of ℓ_2-manifolds [To_5] (see Theorem 3.0) by attempting to show that the discrete 2-cells property combines with the discrete carriers property to imply the discrete approximation property. However, our program falls short of this goal and the closest approach we are able to make to this goal appears as Theorem 4.1.

Theorem 4.1. *If* X *is a topologically complete separable* ANR *that satisfies the discrete* 2-*cells property and the discrete carriers property, then* X *satisfies the discrete* k-*cells property for each* $k \geq 0$.

Herein lies the difficulty. We do not know whether or not the discrete k-cells property for all $k \geq 0$ is equivalent to the discrete approximation property. If so, Conjecture 4.0 is true and we have our characterization. If not, though the conjecture still may be true, it might be that one needs some extra condition, either geometric or algebraic, in order to obtain a characterization of ℓ_2-manifolds in the spirit of Daverman-Walsh and Edwards-Quinn.

The Discrete Carriers Property and Its Relationship with Discrete Cells Properties

First, we define a hierarchy of discrete carriers properties. For notational convenience, we use $B^{\mathcal{U}}$ to denote $\mathrm{st}(B, \mathcal{U})$ whenever B is a subset of a space X and $\mathcal{U}$ is any collection of subsets of X. H_* denotes singular homology with integer coefficients and i_* and j_* always denote appropriate inclusion-induced homomorphisms on homology.

A (singular) carrier for an element $z \in H_q(U, V)$ for $V \subset U$ subsets of a space X and integer $q \geq 0$ is a pair of compact sets $\partial C \subset C$ with $(C, \partial C) \subset (U, V)$ such that $z \in \mathrm{im}[i_*: H_q(C, \partial C) \to H_q(U, V)]$. For example, if $c = \sum_{i=1}^{k} n_i f_i \in z$ is a singular chain and the support of c, denoted by $|c|$, is defined to be $\bigcup_{i=1}^{k} \mathrm{im}\, f_i$, then $C = |c|$ and $\partial C = |\partial c|$ form a carrier for z. The next lemma is a consequence of the definitions and appears in [DW] as Lemma 3.1.

Lemma 4.2. A closed subset F of an ANR X has infinite codimension if and only if, for pairs of open sets $V \subset U$ and integer $q \geq 0$, each element $z \in H_q(U, V)$ has a carrier $(C, \partial C)$ with $C \cap F = \phi$.

A space X is said to satisfy the discrete k-carriers property for some non-negative integer k provided for every open cover $\mathcal{U}$ of X and sequence $\{z_i \in H_{q(i)}(U_i, V_i)\}_{i \in \mathbb{N}}$ where $q(i) \leq k$ and $V_i \subset U_i$ are open in X, there exist carriers $(C_i, \partial C_i)$ for $i_*(z_i)$ where $i_*: H_{q(i)}(U_i, V_i) \to H_{q(i)}(U_i^{\mathcal{U}}, V_i^{\mathcal{U}})$ such that $\{C_i\}_{i \in \mathbb{N}}$ forms a discrete family in X . In words, X satisfies the discrete k-carriers property provided any sequence of k-dimensional homology elements can be made discrete by cover close moves. If we remove the restriction $q(i) \leq k$ on the sequence $\{z_i\}_{i \in \mathbb{N}}$ of homology elements in the definition of the discrete k-carriers property, we arrive at the definition of the <u>discrete carriers property</u>.

The remainder of this section is devoted to a proof of Theorem 4.1. The first lemma states that if an ANR X satisfies the discrete 2-cells property and the discrete carriers property, then X acts like ℓ_2 in the sense that compact subsets of X are Z-sets.

<u>Lemma 4.3</u>. <u>If</u> X <u>is an</u> ANR <u>that satisfies the discrete</u> 2-<u>cells property and the discrete carriers property</u>, <u>then compact subsets of</u> X <u>are</u> Z-<u>sets in</u> X .

<u>Proof</u>. Let D be a compact subset of X and U an open subset of X . For any element $z \in H_q(U, U - D)$ where q is a nonnegative integer, let $z_i = z$ for each $i \in \mathbb{N}$ and use the fact that X satisfies the discrete carriers property to obtain carriers $(C_i, \partial C_i)$ for z_i such that $\{C_i\}_{i \in \mathbb{N}}$ is discrete. Since D is compact and $\{C_i\}_{i \in \mathbb{N}}$ is discrete, all but finitely many C_i miss D . Thus there exists a carrier $(C, \partial C)$ for z such that $C \cap D = \phi$ and this implies that $z = 0$ in $H_q(U, U - D)$. Therefore, D has infinite codimension in X . A similar argument using the compactness of D in conjunction with the discrete 2-cells property shows that D is a 1-LCC subset of X . The Hurewicz theorem (see [DW, Proposition 4.2]) then implies that D is a Z-set in X .

Given a collection $\{f_i\}_{i\in\mathbb{N}}$ of maps of a space Y into X and an open cover $\mathcal{U}$ of X, we say that the maps f_i <u>are limited by</u> $\mathcal{U}$ provided $\{f_i(Y)\}_{i\in\mathbb{N}}$ refines $\mathcal{U}$, that is, each $f_i(Y)$ is contained in some element of $\mathcal{U}$. The "guts" of the proof of Theorem 4.1 appear below as Lemma 4.4, which is in the spirit of Hurewicz in that we obtain $(n+1)$-dimensional homotopy data from n-dimensional homotopy data (discrete n-cells property) coupled with $(n+1)$-dimensional homology data (discrete $(n+1)$-carriers property).

<u>Lemma 4.4</u>. <u>Let</u> X <u>be a separable</u> ANR <u>in which compact subsets are</u> Z-<u>sets and suppose that for some integer</u> $n > 1$, X <u>satisfies the discrete</u> n-<u>cells property and the discrete</u> $(n+1)$-<u>carriers property</u>. <u>Then for each open cover</u> $\mathcal{U}$ <u>of</u> X, <u>there exists an open cover</u> $\mathcal{W}$ <u>of</u> X <u>refining</u> $\mathcal{U}$ <u>such that if</u> $f_i : I^{n+1} \to X$ <u>for</u> $i \in \mathbb{N}$ <u>are maps limited by</u> $\mathcal{W}$ <u>and</u> $\{f_i(\partial I^{n+1})\}_{i\in\mathbb{N}}$ <u>forms a discrete family, then there exist maps</u> $g_i : I^{n+1} \to X$ <u>for</u> $i \in \mathbb{N}$ <u>that are limited by</u> $\mathcal{U}$ <u>and satisfy</u> $g_i|\partial I^{n+1} = f_i|\partial I^{n+1}$ <u>and</u> $\{g_i(I^{n+1})\}_{i\in\mathbb{N}}$ <u>forms a discrete family</u>.

<u>Proof</u>. Let $\mathcal{U}'$ be a refinement of $\mathcal{U}$ by contractible in $\mathcal{U}$ open sets, that is, each element of $\mathcal{U}'$ contracts in some element of $\mathcal{U}$ to a point, and let $\mathcal{W}$ be a locally finite star refinement of $\mathcal{U}'$. For each $i \in \mathbb{N}$, let $f_i : I^{n+1} \to X$ be a map such that $f_i(I^{n+1}) \subset W_i$ for some $W_i \in \mathcal{W}$ and $\{f_i(\partial I^{n+1})\}_{i\in\mathbb{N}}$ forms a discrete family. Choose open sets V_i and V_i' in X such that $f_i(\partial I^{n+1}) \subset V_i \subset \mathrm{Cl}_X V_i \subset V_i' \subset W_i$ and such that $\{\mathrm{Cl}_X V_i'\}_{i\in\mathbb{N}}$ is discrete. Let $\mathcal{V}$ be a re-

finement of $\mathcal{W}$ such that if $V \in \mathcal{V}$ and $V \cap (\mathrm{Cl}_X V_i) \neq \phi$, then $V \subset V_i'$.

For each $i \in \mathbb{N}$, f_i represents an element $z_i = [f_i]$ in $H_{n+1}(W_i, V_i)$. Since X satisfies the discrete $(n+1)$-carriers property, there exist carriers $(C_i, \partial C_i) \subset (W_i^{\mathcal{V}}, V_i^{\mathcal{V}})$ for $i_*(z_i)$ where $i_* : H_{n+1}(W_i, V_i) \to H_{n+1}(W_i^{\mathcal{V}}, V_i^{\mathcal{V}})$ such that $\{C_i\}_{i \in \mathbb{N}}$ forms a discrete family. Since $W_i^{\mathcal{V}} \subset W_i^{\mathcal{W}} \subset U_i'$ for some $U_i' \in \mathcal{U}'$ and $V_i^{\mathcal{V}} \subset V_i'$, $(C_i, \partial C_i)$ is also a carrier for $i_*'(z)$ where $i_*' : H_{n+1}(W_i, V_i) \to H_{n+1}(U_i', V_i')$. Let $j_* : H_{n+1}(C_i, \partial C_i) \to H_{n+1}(U_i', V_i')$. There is a $z_i^* \in H_{n+1}(C_i, \partial C_i)$ with $j_*(z_i^*) = i_*'(z) = [f_i] \in H_{n+1}(U_i', V_i')$. Let b_i be a relative $(n+1)$-chain in $(C_i, \partial C_i)$ representing z_i^*, that is, $b_i \in z_i^*$ and $j_*(z_i^*) = [b_i] \in H_{n+1}(U_i', V_i')$. Since $[b_i] = [f_i]$ in $H_{n+1}(U_i', V_i')$, $f_i - b_i \sim 0 \mod V_i'$, that is, there exists an $(n+1)$-chain a_i in V_i' such that $f_i - b_i - a_i = \partial d_i$ for some $(n+2)$-chain d_i in U_i'. Let $c_i = b_i + a_i$ so that $\partial c_i = \partial f_i = f_i | \partial I^{n+1}$. Then c_i can be represented by a map $\psi_i : (N_i, \partial N_i) \to (U_i', V_i')$ where N_i is a compact connected $(n+1)$-dimensional CW-complex with boundary $\partial N_i = \partial I^{n+1}$ and such that $\psi_i | \partial N_i = f_i | \partial I^{n+1}$ and $\psi_i(N_i) = |c_i| = |b_i + a_i| \subset C_i \cup |a_i|$. Our first task is to modify ψ_i so as to make $\{\psi_i(N_i)\}_{i \in \mathbb{N}}$ discrete.

Since each $|a_i|$ is compact, each $|a_i|$ is a Z-set and since $|a_i| \subset \mathrm{Cl}_X V_i'$, $\{|a_i|\}_{i \in \mathbb{N}}$ is discrete. Also, $\{C_i\}_{i \in \mathbb{N}}$ is a discrete family of compacta and for each i, $C_i \cup |a_i| \subset U_i'$ According to Lemma 3.9, there exist maps $q_i : X \to X$ such that $q_i | |a_i| = \mathrm{id}_{|a_i|}$, $q_i(C_i) \subset U_i'$, and $\{q_i(|a_i| \cup C_i)\}_{i \in \mathbb{N}}$ is discrete. For each i, let $\phi_i = q_i \circ \psi_i$ and obtain maps $\phi_i : N_i \to X$ such that $\phi_i(N_i) \subset U_i'$ and $\phi_i|\partial N_i = q_i \circ f_i|\partial I^{n+1} = f_i|\partial I^{n+1}$ since $f_i(\partial I^{n+1}) \subset |a_i|$. Since $\phi_i(N_i) \subset q_i(|a_i| \cup C_i)$, $\{\phi_i(N_i)\}_{i \in \mathbb{N}}$ is discrete.

Let $cN_i^{(n-1)}$ denote the cone of the $(n-1)$-skeleton $N_i^{(n-1)}$ of N_i and let M_i be the union of N_i and $cN_i^{(n-1)}$ along $N_i^{(n-1)}$. Since the $(n-1)$-skeleton of M_i is contractible, $\pi_q(M_i, *) = 0$ for $q < n$ and the Hurewicz homomorphism $\pi_n(M_i, *) \to H_n(M_i, *)$ is an isomorphism [Sp, Theorem 7.5.5]. We wish to extend ϕ_i to a map, still called ϕ_i , of M_i into X such that ϕ_i is limited by $\mathcal{U}$ and $\{\phi_i(M_i)\}_{i \in \mathbb{N}}$ forms a discrete family.

First, since each N_i is path connected, we may assume that ϕ_i is defined on $M_i^{(1)}$ so that $\phi_i(N_i \cup M_i^{(1)}) = \phi_i(N_i) \subset U_i'$ and, since each U_i' contracts to a point in some element U_i of $\mathcal{U}$, we may assume further that ϕ_i is defined on M_i so that $\phi_i(M_i) \subset U_i$. Observe that $\{\phi_i(N_i \cup M_i^{(1)})\}_{i \in \mathbb{N}}$ is discrete and assume that $\{\phi_i(N_i \cup M_i^{(k)})\}_{i \in \mathbb{N}}$ is discrete for some integer k , $1 < k < n$. For each i , let S_i be a closed neighborhood of $\phi_i(N_i \cup M_i^{(k)})$ contained in U_i such that $\{S_i\}_{i \in \mathbb{N}}$ is discrete and use the fact that X is an ANR and satisfies the discrete n-cells property to obtain for each $(k+1)$-cell Δ in M_i a map $\mu_\Delta : \Delta \to U_i$ such that $\mu_\Delta|\partial\Delta$ is homotopic in S_i to $\phi_i|\partial\Delta$ and $\{\mu_\Delta(\Delta) \mid \Delta$ is a $(k+1)$-cell in some $M_i\}$ is discrete. Use the homotopies connecting $\mu_\Delta|\partial\Delta$ and $\phi_i|\partial\Delta$ to extend ϕ_i to small collars on $\partial\Delta$ for each Δ and let C_i denote $\phi_i(N_i \cup M_i^{(k)} \cup (\text{collars on } \partial\Delta))$ and let D_i denote $\cup\{\mu_\Delta(\Delta) \mid \Delta \subset M_i\}$. Since $\{C_i\}_{i \in \mathbb{N}}$ and $\{D_i\}_{i \in \mathbb{N}}$ are discrete families of compacta, we can use Lemma 3.9 to assume that $\{C_i \cup D_i\}_{i\in\mathbb{N}}$ is discrete and then use μ_Δ to extend ϕ_i to each $\Delta \subset M_i$. Notice then that $\phi_i(N_i \cup M_i^{(k+1)}) = C_i \cup D_i$ so that $\{\phi_i(N_i \cup M_i^{(k+1)})\}_{i\in\mathbb{N}}$ is discrete. Since X is an ANR, we may assume that all moves made to make $\{\phi_i(N_i \cup M_i^{(k+1)})\}_{i\in\mathbb{N}}$ discrete are so small that, since ϕ_i was originally defined on M_i so that $\phi_i(M_i) \subset U_i$, the homotopy extension theorem applies to ensure that $\phi_i|N_i \cup M_i^{(k+1)}$ extends to ϕ_i so that still $\phi_i(M_i) \subset U_i$. In this way, we eventually obtain maps $\phi_i : M_i \to U_i$ such that $\{\phi_i(M_i)\}_{i\in\mathbb{N}}$ is discrete and $\phi_i|\partial M_i = \phi_i|\partial N_i = f_i|\partial I^{n+1}$.

Recall that the Hurewicz homomorphism $\pi_n(M_i, *) \to H_n(M_i, *)$ is an isomorphism and observe that the inclusion map $\partial I^{n+1} = \partial M_i \to M_i$ represents the zero element of $H_n(M_i, *)$ since ∂M_i homologically bounds the chain represented by N_i. Therefore, $\partial I^{n+1} \to M_i$ represents the zero element of $\pi_n(M_i, *)$ and there exist a map $h_i : I^{n+1} \to M_i$ such that $h_i|\partial I^{n+1}$ is the inclusion. Let $g_i = \phi_i \circ h_i$ and observe that the maps g_i are limited by $\mathcal{U}$, $g_i|\partial I^{n+1} = \phi_i|\partial M_i = f_i|\partial I^{n+1}$, and $\{g_i(I^{n+1})\}_{i\in\mathbb{N}}$ is discrete since $\{\phi_i(M_i)\}_{i\in\mathbb{N}}$ is discrete and $g_i(I^{n+1}) \subset \phi_i(M_i)$ for each $i \in \mathbb{N}$.

Lemma 4.5. *Let X be a separable ANR in which compact subsets are Z-sets and let $n > 1$ be an integer. If X satisfies the discrete n-cells property and the discrete $(n+1)$-carriers property, then X satisfies the discrete $(n+1)$-cells property.*

Proof. Let $\mathcal{U}$ be an open cover of X and $f_i : I^{n+1} \to X$ a sequence of maps. Let $\mathcal{W}$ be a cover of X as promised in Lemma 4.4

and choose a refinement $\mathcal{V}$ of $\mathcal{W}$ by contractible in $\mathcal{W}$ open sets. For each $i \in \mathbb{N}$, choose a finite triangulation T_i of I^{n+1} so fine that for each $\sigma \in T_i$, $f_i(\sigma)$ is contained in some element V of $\mathcal{V}$. Since X satisfies the discrete n-cells property, without loss of generality we assume that $\{f_i(T_i^{(n)})\}_{i\in\mathbb{N}}$ is a discrete collection of compacta in X and for each $i \in \mathbb{N}$, we choose a closed neighborhood N_i of $f_i(T_i^{(n)})$ such that $\{N_i\}_{i\in\mathbb{N}}$ forms a discrete family. For each $(n+1)$-simplex σ in T_i, let $c(\sigma) = \partial\sigma \times [0, 1]$ be a collar in σ on $\partial\sigma$ and write $\sigma = c(\sigma) \cup \hat{\sigma}$ where $c(\sigma) \cap \hat{\sigma} = \partial\sigma \times \{1\}$. Assume that $c(\sigma)$ is chosen so that $f_i(c(\sigma)) \subset \operatorname{Int} N_i$ and use the fact that X is an ANR that satisfies the discrete n-cells property to obtain maps $g_{i,\sigma} : \partial\sigma \times \{1\} \to \operatorname{Int} N_i$ where σ is an $(n+1)$-simplex in T_i such that $\{g_{i,\sigma}(\partial\sigma \times \{1\}) \mid \text{all } i \text{ and } \sigma\}$ is discrete and $g_{i,\sigma}$ is homotopic to $f_i|\partial\sigma \times \{0\}$ by a homotopy limited by $N_i \cap \mathcal{V}$. Use these homotopies to extend $f_i|T_i^{(n)}$ to maps g_i defined on $T_i^{(n)} \cup \{c(\sigma) \mid \sigma \in T_i^{(n+1)}\}$ such that the image of g_i is contained in N_i, each $g_i(c(\sigma))$ is contained in some element V of $\mathcal{V}$, $g_i|\partial\sigma \times \{1\} = g_{i,\sigma}$, and $g_i|T_i^{(n)} = f_i|T_i^{(n)}$. Since $\operatorname{Im} g_i \subset N_i$, $\{\operatorname{Im} g_i\}_{i\in\mathbb{N}}$ forms a discrete family. Use the fact that each V in $\mathcal{V}$ is contractible in some W in $\mathcal{W}$ to extend $g_i|\partial\sigma \times \{1\}$ and then use the previous lemma to obtain extensions of the g_i to each $\hat{\sigma}$ such that $\{g_i(\hat{\sigma}) \mid \text{all } i \text{ and } \sigma\}$ is discrete and refines $\mathcal{U}$. For each $i \in \mathbb{N}$, let $C_i = g_i(T_i^{(n)} \cup \{c(\sigma) \mid \sigma \in T_i^{(n+1)}\})$ and $D_i = \cup \{g_i(\hat{\sigma}) \mid \sigma \in T_i^{(n+1)}\}$. Then $\{C_i\}_{i\in\mathbb{N}}$ and $\{D_i\}_{i\in\mathbb{N}}$ are discrete and Lemma 3.9 allows us to assume that

$\{C_i \cup D_i\}_{i\in\mathbb{N}}$ is discrete via a small move of the D_i rel the C_i. We obtain maps, still called g_i, such that $\{g_i(I^{n+1}) = C_i \cup D_i\}_{i\in\mathbb{N}}$ is discrete and g_i is $\mathcal{U}$-close to f_i.

The proof of Theorem 4.1 now follows easily. Since X satisfies the discrete 2-cells property and the discrete carriers property, Lemma 4.3 guarantees that compact subsets of X are Z-sets so that Lemma 4.5 applies. Then induction and Lemma 4.5 reveal that X satisfies the discrete n-cells property for each $n \in \mathbb{N}$.

The difficulty in proving that X satisfies the discrete approximation property given the hypotheses of Theorem 4.1 lies in the use of induction in the proof of Lemma 4.5. There, given countably many maps of n-cells into X, we can inductively assume that the maps are discrete on fine $(n-1)$-skeletons of the n-cells and then use Lemma 4.4 to adjust the maps to make them discrete. If we are given countably many maps of finite dimensional cells into X with no upper bound for the dimensions of the cells, then for each $n \in \mathbb{N}$, we can inductively assume that the maps are discrete on fine n-skeletons of the cells and, as before, we can use Lemma 4.4 to adjust the maps so that they are discrete on $(n+1)$-skeletons. In this way, we can obtain maps such that for each $n \in \mathbb{N}$, the maps are discrete on the n-skeletons of some predetermined triangulations of the cells; however, we cannot conclude that the maps are discrete. It is possible that there is unwanted convergence that arises by choosing points from higher and higher dimensional simplices from the various triangulations.

Notice that in assuming that X satisfies the discrete carriers property, we are assuming something stronger than that X satisfies the discrete n-carriers property for all $n \in \mathbb{N}$, which is all that is used in the above paragraph and in the proof of Theorem 4.1. The discrete carriers property is a "uniform" version of the simultaneous possession of the discrete n-carriers property for all $n \in \mathbb{N}$ in the same way that the discrete approximation property or uniform UV^{∞} properties are "uniform" versions of the simultaneous possession of the corresponding properties for all $n \in \mathbb{N}$, and a space satisfying these uniform versions does not necessarily follow from a space simultaneously satisfying each finite dimensional version. It is possible then that we must use more of the strength of the discrete carriers property in order to prove that the space X of Theorem 4.1 satisfies the discrete approximation property. Given countably many maps of finite dimensional cells of arbitrary dimensions into X, if the maps are discrete on the boundaries of the cells, then the discrete carriers property does imply that the given maps can be adjusted so as to obtain maps whose images form a discrete family. However, the problem of adjusting arbitrary maps so that boundaries go in discretely is equivalent to the problem of showing that X satisfies the discrete approximation property.

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