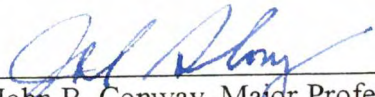
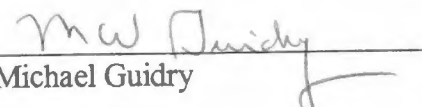


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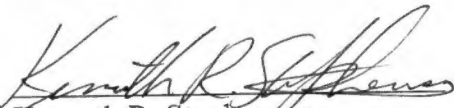
I am submitting herewith a dissertation written by Nathan Stephen Feldman entitled "The Self - Commutator of a Subnormal Operator." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

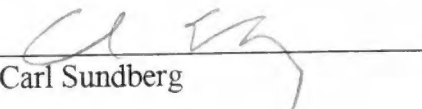

John B. Conway, Major Professor

We have read this dissertation
and recommend its acceptance.


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Associate Vice Chancellor and
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THE SELF - COMMUTATOR OF A SUBNORMAL OPERATOR

A Dissertation
Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

Nathan Stephen Feldman
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Dedication

This dissertation is dedicated to my wife

Raelyn

for her patience, understanding, and encouragement.

Acknowledgments

I would like to thank my major professor, John B. Conway for all his time, help and support. I would also like to thank my other committee members: Mike Guidry, Stefan Richter, Kenneth R. Stephenson, and Carl Sundberg for all their help, time, and many useful comments during this project; and over the past many years. Finally, I would like to thank my parents, my wife, and my son for all the encouragement, patience and interest they have shown.

Abstract

This work is directed toward a study of the self-commutator of a subnormal operator. This can lead one in several directions. One place the self-commutator plays an important role is in the C^* -algebra generated by a subnormal operator. We shall do a systematic study of the C^* -algebra generated by a subnormal operator. We shall see the importance of having compact self-commutator and shall use C^* -algebra techniques to get estimates on the essential norm of self-commutators. We also use these methods to show certain operators have diagonalizable self-commutators and to show that compactness of the self-commutator is preserved under similarity for (essentially) subnormal operators. These methods naturally lead into a new class of operators, namely essentially subnormal operators. We shall characterize these as those operators that have an essentially normal extension.

We give a sharp form of the Berger-Shaw Theorem for cyclic subnormal operators S and characterize those operators in the commutant of S that have trace class self-commutator.

We shall also study subnormal operators that have zero as an eigenvalue for there self-commutators. These are natural generalizations of operators with finite rank self-commutator and shall lead into generalized quadrature domains and some interesting approximation questions. In short we study questions involving the self-commutator.

Preface

For a given operator T , there are some natural self-adjoint operators associated with it: the real and imaginary parts of T , the absolute value of T , and the self-commutator of T . If one claims to understand an operator, or a class of operators, then one should understand these naturally associated operators. In addition to being naturally associated with an operator, the self-commutator is part of a unitary invariant for pure subnormal operators. This is illustrated in the following result by Xia [1987]; also see Martin and Putinar [1989], p.46.

Theorem If S is a pure subnormal operator and $\mathcal{M} = \text{cl}\{\text{range}[S^*, S]\}$, then the pair $(S^*|_{\mathcal{M}}, [S^*, S]|_{\mathcal{M}})$ is a complete unitary invariant for S within the class of pure subnormal operators.

The present work is directed toward a better understanding of the self-commutator of a subnormal operator and the various techniques used in its study. Two of the most fundamental results about self-commutators are Putnam's Inequality and the Berger-Shaw Theorem (see Putnam [1970] and Berger and Shaw [1973]). Also see Conway [1991], p.152 and p. 156.

Putnam's Inequality If T is hyponormal, then $\|[T^*, T]\| \leq \frac{1}{\pi} \text{Area}[\sigma(T)]$.

The Berger-Shaw Theorem (special case) If T is a rationally cyclic hyponormal operator, then $[T^*, T]$ is trace class and $\text{tr}[T^*, T] \leq \frac{1}{\pi} \text{Area}[\sigma(T)]$.

It is an easy exercise to deduce Putnam's Inequality from the Berger-Shaw Theorem. A sharper form of Putnam's Inequality for subnormal operators was given in Axler and Shapiro [1985]. They proved the following.

Theorem If S is a subnormal operator, then $\|[S^*, S]\| \leq \text{dist}[\bar{z}, R(\sigma(S))]^2$.

It is well known that $\text{dist}[\bar{z}, R(K)]^2 \leq \frac{1}{\pi} \text{Area}(K)$ for any compact set $K \subseteq \mathbb{C}$ (see Alexander [1973] or Conway[1991], p. 176). One of the main results in this work establishes a version of Axler and Shapiro's inequality for the essential norm of the self-commutator of a subnormal operator. This inequality was first proven in Aleman[1996]. We give a C^* -algebra approach to this inequality. Also our method applies not only to subnormal operators, but to any essentially subnormal operator (see Section II.7).

Theorem If S is an essentially subnormal operator, then $\|[S^*, S]\|_e \leq \text{dist}[\bar{z}, R(\sigma_e(S))]^2$.

See Theorem II.7.5(a) for a proof. See Theorem III.1.5 and its corollaries for a version of this inequality that includes functions of the operator S . Our method of proof also gives an essential version of Putnam's Inequality for essentially hyponormal operators.

Theorem If T is an essentially hyponormal operator, then $\|[T^*, T]\|_e \leq \frac{1}{\pi} \text{Area}[\sigma_e(T)]$.

Also see Theorem III.1.10 for a characterization of when $[S^*, S]$ is compact. The above C^* -algebra approach is also used to show that compactness of the self-commutator is a similarity invariant for essentially subnormal operators (see Theorem III.1.14).

Another major result in this work establishes a sharp form of the Berger-Shaw Theorem for cyclic subnormal operators. See Corollary III.2.7.

Theorem If S is a cyclic subnormal operator, then $[S^*, S]$ is trace class and

$$\text{tr}[S^*, S] = \frac{1}{\pi} \text{Area}[\sigma(S) - \sigma_e(S)].$$

We shall deduce this as a corollary to a more general result. It is known that every cyclic subnormal operator can be represented as $S = M_Z$ on $P^2(\mu)$, where $P^2(\mu)$ denotes the closure of the (analytic) polynomials in $L^2(\mu)$. If $f \in P^2(\mu) \cap L^\infty(\mu)$, then $f(S)$ denotes the operator of multiplication by f on $P^2(\mu)$. Let G be the set of analytic bounded point evaluations for $P^2(\mu)$. If $f \in P^2(\mu)$, then $\hat{f}$ denotes the analytic extension of f to G .

Theorem If S is a pure cyclic subnormal operator and $f \in P^2(\mu) \cap L^\infty(\mu)$, then $[f(S)^*, f(S)]$ is trace class if and only if $\hat{f}$ is in the Dirichlet space on G . In this case,

$$\text{tr}[f(S)^*, f(S)] = \frac{1}{\pi} \int_G |\hat{f}'|^2 dA.$$

There are several results in the literature where the trace of the self-commutator of a multiplication operator on a Hilbert space of analytic functions is computed as the Dirichlet integral of the symbol. This was first done for multiplication operators on the Bergman space of an arbitrary bounded region by Berger and Shaw. It was a fundamental part of the original proof of the Berger-Shaw Theorem (see Berger and Shaw [Preprint]). In Arazy, Fisher, Janson and Peetre [1990], they establish a similar result for multiplication operators on $L_a^2(G, \mu)$ where μ is any finite measure carried by G such that the inclusion

$L_a^2(G, \mu) \rightarrow \text{Hol}(G)$ is continuous. They also ask if similar results hold when μ is allowed to put measure on ∂G . The first non-trivial instance of this appears in Aleman[1996]. He computes the trace of the self-commutator of a multiplication operator on the Hardy space $H^2(G)$ for an arbitrary bounded region G . As one expects, it is the Dirichlet integral of the symbol. Our Theorem for cyclic subnormal operators allows for many more non-trivial examples where μ is allowed to put measure on ∂G .

The above C^* -algebra approach, used in obtaining the essential norm estimates on self-commutators, raises several natural questions about C^* -algebras and subnormal operators. If S and T are operators, then by $C^*(S) \approx C^*(T)$ we shall mean that there exists a $*$ -isomorphism $\phi: C^*(S) \rightarrow C^*(T)$ with $\phi(S) = T$. A basic and natural question is when is $C^*(S) \approx C^*(T)$? We say that two operators S and T are *strongly approximately equivalent* if there exist unitaries U_n such that $U_n^* S U_n - T$ is compact and converges to zero in norm. This is denoted by $S \sim_{sa} T$. In Corollary II.2.4 it is shown that the “algebraic equivalence” of C^* -algebras is the same as an “approximate equivalence” of S and T .

Theorem If S and T are non-degenerate operators, then $C^*(S) \approx C^*(T)$ if and only if $S^{(\infty)} \sim_{sa} T^{(\infty)}$.

This is a simple corollary of Voiculescu’s Theorem (see Theorem I.1.13) and is well known by experts. For subnormal operators, the question of equivalence of C^* -algebras was answered for the unilateral shift in Coburn [1967] and for bundle shifts with finite

multiplicity by Abrahamse and Douglas [1976]. We state their results since they are special cases of our work.

Theorem (Coburn) If S is the unilateral shift, then an operator T satisfies $C^*(T) \approx C^*(S)$ if and only if $T \cong S^{(n)} \oplus U$ where $n \in \{\infty, 1, 2, \dots\}$ and U is any unitary.

Theorem (Abrahamse and Douglas) If S and T are bundle shifts with compact self-commutator, then $C^*(S) \approx C^*(T)$ if and only if $S^{(\infty)} \cong T^{(\infty)}$.

By $\text{mne}S$ we mean the minimal normal extension of S . Our first main result about C^* -algebras generated by subnormals is the following (see Corollaries II.3.5 and II.3.9).

Theorem If S and T are subnormal operators, then the following hold.

- (a) If $C^*(S) \approx C^*(T)$, then $C^*(\text{mne}S) \approx C^*(\text{mne}T)$.
- (b) If $W^*(S) \approx W^*(T)$, then $W^*(\text{mne}S) \approx W^*(\text{mne}T)$.

We now state two of our main results about the equivalence of C^* -algebras. They naturally generalize the work of Coburn and Abrahamse and Douglas. For the first result see Theorem II.6.14 and for the second see Theorem II.6.7.

Theorem If S is an irreducible subnormal operator with compact self-commutator, then an operator T satisfies $C^*(T) \approx C^*(S)$ if and only if $T \cong S^{(n)} \oplus N$ where $n \in \{\infty, 1, 2, \dots\}$ and N is any normal operator with $\sigma(N) \subseteq \sigma_{\text{ap}}(S)$.

Theorem If S and T are pure essentially normal operators, then $C^*(S) \approx C^*(T)$ if and only if $S^{(\infty)} \cong T^{(\infty)}$.

A nice corollary is the following. See Corollaries II.6.10 and II.3.11. Here $\text{svsm}S$ denotes the scalar-valued spectral measure, and $\mathcal{R}(S)$ denotes the restriction algebra.

Corollary If S and T are pure essentially normal subnormal operators and $C^*(S) \approx C^*(T)$, then $\text{svsm}S \approx \text{svsm}T$, $W^*(S) \approx W^*(T)$, $\mathcal{R}(S) = \mathcal{R}(T)$, and $C^*(\text{dual}S) \approx C^*(\text{dual}T)$.

In Section II.4 the notion of essential normal spectrum is defined and several basic properties are proven. In particular for a subnormal operator S , Theorem II.4.10(a) gives a useful characterization of the essential normal spectrum of S . Namely, if f is a continuous function on the normal spectrum of S , then the Toeplitz operator S_f is compact if and only if f vanishes on the essential normal spectrum. This is useful for computing the essential normal spectrum.

In Section II.5 the properties developed in Section II.4 for the essential normal spectrum are used to prove that certain operators have diagonalizable self-commutators. In particular, this is done for the restriction of the Bergman shift to an invariant subspace with infinite index (see Corollary II.5.11). It is also shown that this operator is unitarily equivalent modulo the compacts to the unilateral shift of infinite multiplicity (see Example II. 5.6); this answers a question in Conway and Putnam [1985], p. 297.

In Section II.7 the class of essentially subnormal operators is introduced and discussed. The main result of this section characterizes these operators as exactly those with essentially normal extensions (see Theorem II.7.3).

In Chapter IV a general study of subnormal operators with $0 \in \sigma_p([S^*, S])$ is made. These operators naturally generalize operators with finite rank self-commutator. The cyclic subnormals with this property make a connection with generalized quadrature domains. See Theorem IV.2.1, Corollary IV.3.6 and Theorem IV.3.7.

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CHAPTER I

Preliminaries

In this Chapter we shall present some preliminary material that will be used throughout. Proofs will not be given if there are standard references. All Hilbert spaces are separable and complex.

1. C^* -Algebras

We want to discuss the *hereditary functional calculus* for an arbitrary operator. That is we want to specify a convention of how to apply a polynomial in z and $\bar{z}$ to a given operator. For an operator S , if S is not normal, then S and S^* do not commute. Hence we need to set down a useful convention. If $p(z, \bar{z}) = z^n \bar{z}^k$ and S is any operator, then let's agree that $p(S) = S^{*k} S^n$. So, "the adjoints" are all grouped together on the left. Thus if p is any polynomial in z and $\bar{z}$, then we may define $p(S)$ by first writing p as a linear combination of terms like $z^n \bar{z}^k$ and then applying the above convention. This is called the *hereditary functional calculus* for the operator S see Agler[1988]. It is easy to check that if S is a subnormal operator and p is any polynomial in z and $\bar{z}$, then $p(S) = S_p$, the Toeplitz operator with symbol p . Thus if S is a subnormal operator and p is any

polynomial in z and $\bar{z}$ such that $p \geq 0$ on $\sigma(S)$, then $p(S) \geq 0$. We now see that a converse of this is true.

Theorem 1.1 Let S be an operator and Δ a closed disk such that $\sigma(S) \subseteq \Delta$. If $p \geq 0$ on Δ implies that $p(S) \geq 0$ for every polynomial p in z and $\bar{z}$, then S is a subnormal operator.

We shall prove this since it follows easily from a dilation theorem that we also need.

If $\kappa: \mathcal{A} \rightarrow \mathcal{B}$ is a linear map between C^* -algebras, then κ is said to be positive if $\kappa(a) \geq 0$ whenever $a \geq 0$. For a positive integer n , we may consider $M_n(\mathcal{A})$, the C^* -algebra of $n \times n$ matrices with entries from $\mathcal{A}$. The linear map $\kappa: \mathcal{A} \rightarrow \mathcal{B}$ may be extended to a linear map $\kappa_n: M_n(\mathcal{A}) \rightarrow M_n(\mathcal{B})$ in a natural way, simply $\kappa_n([a_{ij}]) = [\kappa(a_{ij})]$. We say that κ is completely positive if κ_n is positive for all $n \geq 1$. We shall need two results of Stinespring, see Davidson [1996], p. 260.

Theorem 1.2 If X is a compact Hausdorff space and $\mathcal{A}$ is a C^* -algebra, then every positive linear map from $C(X)$ into $\mathcal{A}$ is completely positive.

Theorem 1.3 If $\mathcal{A}$ is a C^* -algebra with identity and $\kappa: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ is a completely positive linear map, then there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a $*$ -homomorphism $\varphi: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{K})$ such that $\kappa(A) = P\varphi(A)|_{\mathcal{H}}$ for each $A \in \mathcal{A}$, where P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$.

Proof of Theorem 1.1 Suppose S acts on $\mathcal{H}$. We want to define a positive linear map $\kappa: C(\text{cl}\Delta) \rightarrow \mathcal{B}(\mathcal{H})$. If p is a polynomial in z and $\bar{z}$, then define $\kappa(p) = p(S)$ where we use the hereditary functional calculus to define $p(S)$. It is easy to check that every positive linear map is bounded with norm at most 2 (see Paulsen [1986], p. 9). Since we are

assuming that κ is positive, κ extends to a positive linear map on all of $C(\text{cl}\Delta)$. By Theorem 1.2 and Theorem 1.3, there exists a $*$ -homomorphism $\varphi: C(\text{cl}\Delta) \rightarrow \mathcal{B}(\mathcal{K})$ on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ such that $\kappa(f) = P\varphi(f)|_{\mathcal{H}}$ for each $f \in C(\text{cl}\Delta)$, where P is the projection of $\mathcal{K}$ onto $\mathcal{H}$. If $N = \varphi(z)$ on $\mathcal{K}$, then N is a normal operator whose compression to $\mathcal{H}$ equals S . If we can show that $\mathcal{H}$ is actually invariant for N , then N will be a normal extension of S . Hence S will be subnormal. Suppose that $N = \begin{bmatrix} S & * \\ A & * \end{bmatrix}$ with respect to $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$. We need to show that $A = 0$. Notice that $\kappa(z\bar{z}) = S^*S$ and thus the compression of N^*N to $\mathcal{H}$ should be S^*S . However, $N^*N = \begin{bmatrix} S^*S + A^*A & * \\ * & * \end{bmatrix}$. Thus, it follows that $A^*A = 0$. Hence $A = 0$ and so $\mathcal{H}$ is invariant for N . $\square$

It follows from Theorem 1.1 that subnormal operators are defined in terms of positivity conditions. As such it is a concept that may be discussed in any C^* -algebra. If $\mathcal{A}$ is any C^* -algebra, then an element $s \in \mathcal{A}$ is said to be subnormal if for some disk Δ containing $\sigma(s)$ we have $p(s) \geq 0$ for every polynomial p in z and $\bar{z}$ that is non-negative on Δ .

Corollary 1.4 Suppose $\mathcal{A}$ and $\mathcal{B}$ are C^* -algebras and $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ is a positive linear map. If S is subnormal in $\mathcal{A}$, $\varphi(S) = T$, and $\varphi(S^*S) = T^*T$, then T is subnormal in $\mathcal{B}$.

The next result follows from the well known Gelfand-Naimark-Segal construction.

Theorem 1.5 If $\mathcal{A}$ is any separable C^* -algebra, then there exists a C^* -algebra $\mathcal{B}$ of operators acting on a separable Hilbert space such that $\mathcal{A}$ is isomorphic to $\mathcal{B}$.

That is, every separable C^* -algebra admits a faithful representation on a separable Hilbert space, see Conway [1985], p. 259.

Theorem 1.6 If $\mathcal{I}$ is an ideal in a C^* -algebra $\mathcal{A}$, X is a compact Hausdorff space and $\kappa:C(X) \rightarrow \mathcal{A}/\mathcal{I}$ is a positive linear map, then there exists a positive linear map $\rho:C(X) \rightarrow \mathcal{A}$ such that $\kappa = \pi \circ \rho$, where $\pi:\mathcal{A} \rightarrow \mathcal{A}/\mathcal{I}$ is the natural quotient map.

See Davidson [1996], p. 266 for a proof.

For a C^* -algebra A , the commutator ideal of A is the ideal generated by all commutators, $[a,b] = ab-ba$. The next proposition is elementary, see Arveson[1969], p. 194.

Proposition 1.7 If $\mathcal{A}$ is a C^* -algebra and S is any self-adjoint subset such that S together with the identity generate $\mathcal{A}$, then the commutator ideal of $\mathcal{A}$ is generated by all commutators from S .

Corollary 1.8 If T is an operator, then the commutator ideal of $C^*(T)$ is the ideal generated by the self-commutator of T , $[T^*,T] = T^*T - TT^*$.

Proof. The set $S = \{T, T^*\}$ is self-adjoint and together with I , generates $C^*(T)$. $\square$

For any operator, let $\text{Comm}(T)$ denote the commutator ideal of $C^*(T)$. Whenever one studies a C^* -algebra A , one should always begin by first identifying its abelianization, that is $A / \text{Comm}(A)$. This was done for the C^* -algebra generated by a hyponormal operator in Bunce [1970] (see also Conway[1991], p. 88).

Theorem 1.9 If T is a hyponormal operator, then $C^*(T)/\text{Comm}(T)$ is $*$ -isomorphic to $C(\sigma_{\text{ap}}(T))$, under the map that sends T to z .

If S and T are operators, then by $C^*(S) \approx C^*(T)$ we shall mean that there exists a $*$ -isomorphism $\varphi: C^*(S) \rightarrow C^*(T)$ such that $\varphi(S) = T$.

Corollary 1.10 If S and T are hyponormal operators and $C^*(S) \approx C^*(T)$, then $\sigma_{ap}(S) = \sigma_{ap}(T)$.

Proof. Briefly, $C(\sigma_{ap}(S)) \approx C^*(S)/\text{Comm}(S) \approx C^*(T)/\text{Comm}(T) \approx C^*(\sigma_{ap}(T))$ the isomorphisms mapping $z \rightarrow S, S \rightarrow T$ and $T \rightarrow z$. Hence $\sigma_{ap}(S) = \sigma_{ap}(T)$. $\square$

Corollary 1.11 If T is any hyponormal operator and N is a normal operator such that $\sigma(N) = \sigma_{ap}(T)$, then there exists a $*$ -homomorphism $\varphi: C^*(T) \rightarrow C^*(N)$ such that $\varphi(T) = N$ and $\ker(\varphi) = \text{Comm}(T)$.

The next result is a general version of the classical *Weyl-Von Neumann-Berg Theorem*, see Davidson [1996], p. 60. We shall say that two operators S and T are strongly approximately equivalent, denoted $S \sim_{sa} T$, if there exists a sequence of unitary operators U_n such that $U_n^* S U_n$ converges to T in norm and $U_n^* S U_n - T$ is compact for each n . If the compactness condition is not required, then S and T are said to be approximately equivalent, $S \sim_a T$. In fact the two notions coincide, see Davidson[1996], p. 70.

Theorem 1.12 If N and M are normal operators, then the following are equivalent:

- (a) $N \sim_{sa} M$,
- (b) $N \sim_a M$,
- (c) $\sigma_e(N) = \sigma_e(M)$ and the isolated eigenvalues of N and M are the same with the same multiplicities.

We shall also need a non-commutative version of the Weyl-Von Neumann-Berg Theorem, namely *Voiculescu's Theorem*. If $\mathcal{A}$ is any algebra of operators on a Hilbert space $\mathcal{H}$, we say that $\mathcal{A}$ acts *non-degenerately* on $\mathcal{H}$ if $\{Ax : A \in \mathcal{A} \text{ and } x \in \mathcal{H}\}$ is dense in $\mathcal{H}$.

Theorem 1.13 If $\mathcal{A}$ be a separable C^* -algebra of operators and ρ is a non-degenerate representation of $\mathcal{A}$ such that $\mathcal{A} \cap \mathcal{B}_0 \subseteq \ker(\rho)$, then $\text{id} \sim_{\text{sa}} \text{id} \oplus \rho$.

This version of Voiculescu's Theorem will be very useful in Chapter II.2. A consequence of Voiculescu's Theorem that more closely resembles the classical Weyl-Von Neumann-Berg Theorem is as follows.

Theorem 1.14 Every representation of a separable C^* -algebra on a separable Hilbert space is strongly approximately equivalent to a direct sum of irreducible representations.

Applying this to the identity representation of $C^*(T)$ for an operator T gives the following.

Corollary 1.15 Every operator is strongly approximately equivalent to a direct sum of irreducible operators.

Observe that Corollary 1.15 is exactly the classical Weyl-Von Neumann-Berg Theorem when the operator is normal.

If S and T are operators on separable Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, then $S \otimes T$ is a bounded linear operator on $\mathcal{H} \otimes \mathcal{K}$. The elements of the form $\{h \otimes k : h \in \mathcal{H} \text{ and } k \in \mathcal{K}\}$ have dense linear span in $\mathcal{H} \otimes \mathcal{K}$. Furthermore, $(S \otimes T)(h \otimes k) = Sh \otimes Tk$. This completely

determines $S \otimes T$. If S has a matrix representation $\{a_{ij}\}$ on $\mathcal{H}$ with respect to some basis, then $S \otimes T$ has a matrix representation $\{a_{ij}T\}$ acting on $\mathcal{K}^{(\infty)}$. In particular, if U is the

unilateral shift, then $U = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \dots \end{bmatrix}$ with respect to the shifted basis. Thus if S is any

operator acting on $\mathcal{K}$ then $U \otimes S$ can be represented as the operator $\begin{bmatrix} 0 & 0 & 0 & 0 \\ S & 0 & 0 & 0 \\ 0 & S & 0 & 0 \\ 0 & 0 & S & \dots \end{bmatrix}$

acting on $\mathcal{K}^{(\infty)}$. We say that an operator S is *quasinormal* if S commutes with S^*S . Or equivalently, if S commutes with $|S|$. It is known that every quasinormal operator is subnormal, see Conway[1991], p. 29 and the pure ones may be classified as the tensor product of the unilateral shift with a one-to-one positive operator.

Theorem 1.16 An operator S is a pure quasinormal operator if and only if S is unitarily equivalent to $U \otimes A$ where U is the unilateral shift and A is a positive one-to-one operator.

See Conway[1991], p. 44 for this and more about quasinormal operators.

We now need a few results about C^* -algebras of compact operators. See Davidson[1996], p. 36 - 39. We shall denote by $\mathcal{B}_0$ the ideal of compact operators.

Theorem 1.17 If an irreducible C^* -algebra contains a non-zero compact operator then it contains all the compact operators.

Corollary 1.18 If T is an irreducible operator on $\mathcal{H}$ with compact self-commutator, then the commutator ideal of $C^*(T)$ is the ideal of all compact operators on $\mathcal{H}$. Furthermore, the commutator ideal is a minimal ideal.

Proof. Clearly, we may assume $\dim \mathcal{H} > 1$. If $\text{Comm}(T)$ denotes the commutator ideal, then $[T^*, T]$ is a compact operator in $\text{Comm}(T)$. Since T is not normal, $[T^*, T]$ is a non-zero operator. Since T is irreducible, so is $\text{Comm}(T)$. Thus by Theorem 1.17, $\text{Comm}(T)$ contains all compact operators. On the other hand, if one considers $C^*(T)/\mathcal{B}_0$, then this algebra is abelian since it has a normal generator. Thus $\text{Comm}(T) \subseteq \mathcal{B}_0$. Hence $\text{Comm}(T)$ equals $\mathcal{B}_0$. Now, if $\mathcal{J}$ is any ideal in $\text{Comm}(T) = \mathcal{B}_0$, then $\mathcal{J}$ is irreducible (since $\mathcal{B}_0$ is) and thus Theorem 1.17 says that either $\mathcal{J} = (0)$ or $\mathcal{J} = \mathcal{B}_0 = \text{Comm}(T)$. So, $\text{Comm}(T)$ is a minimal ideal. $\square$

Theorem 1.19 If $\mathcal{J}$ is a C^* -algebra of compact operators on a Hilbert space $\mathcal{H}$, then there are reducing subspaces $\{\mathcal{H}_n: n \geq 0\}$ for $\mathcal{J}$ such that $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$, $\mathcal{J}|_{\mathcal{H}_0} = (0)$ and $\mathcal{J}|_{\mathcal{H}_n} = \mathcal{B}_0(\mathcal{H}_n)$.

Theorem 1.20 A non-degenerate representation of a non-degenerate C^* -algebra of compact operators has an irreducible subrepresentation unitarily equivalent to a subrepresentation of the identity representation.

Corollary 1.21 Every irreducible representation of $\mathcal{B}_0$ is equivalent to the identity representation.

See Davidson[1996], p. 19-22 for the next two results. The following is von Neumann's Double Commutant Theorem.

Theorem 1.22 If $\mathcal{A}$ is a C^* -algebra of operators, then $(\text{WOT})\text{-cl}(\mathcal{A}) = (\text{SOT})\text{-cl}(\mathcal{A}) = \mathcal{A}''$.

The next result is Kaplansky's Density Theorem.

Theorem 1.23 If $\mathcal{A}$ is a C^* -algebra of operators and A is a self-adjoint operator in $(\text{SOT})\text{-cl}(\mathcal{A})$, then there is a sequence of self-adjoint operators A_n in $\mathcal{A}$ such that $A_n \rightarrow A$ (SOT) and $\|A_n\| \leq \|A\|$ for every n .

2. Subnormal Operators

The ultimate reference on subnormal operators is Conway [1991]. We shall simply collect a few results here for easy reference. The first one is a well known result about normal operators, see Conway [1985], p. 286.

Theorem 2.1 If N_1 and N_2 be normal operators, Y is a one-to-one operator with dense range and $YN_1 = N_2Y$, then N_1 is unitarily equivalent to N_2 .

The next result is a generalization of the well-known fact that the unilateral shift is the only subnormal contraction with rank one self-commutator (see Halmos [1982], problem 237). We shall use this result in Theorem II.9.11.

Theorem 2.2 If S is a pure subnormal operator such that $\|S\| \leq 1$ and $[S^*, S]$ is a projection, then S is an isometry.

Proof. Suppose S acts on $\mathcal{H}$. Let $\mathcal{M} = \text{Range}[S^*, S]$. If $x \in \mathcal{M}$, then since $[S^*, S]x = x$ we have $\|x\|^2 = \langle [S^*, S]x, x \rangle = \|Sx\|^2 - \|S^*x\|^2 \leq \|Sx\|^2 \leq \|x\|^2$. Thus equality holds throughout. In particular we see $S^*\mathcal{M} = (0)$, $S|_{\mathcal{M}}$ is an isometry. Also, since $S^*\mathcal{M} = (0)$, this implies that $I = [S^*, S]|_{\mathcal{M}} = S^*S|_{\mathcal{M}}$. Now, If $x, y \in \mathcal{M}$ and $n \geq 1$, then $\langle S^n x, y \rangle = \langle S^{n-1} x, S^* y \rangle = \langle S^{n-1} x, 0 \rangle = 0$. Thus, $S^n \mathcal{M} \perp \mathcal{M}$ for all $n \geq 1$. So, if $n \geq 1$, then $S^n \mathcal{M} \subseteq \ker[S^*, S]$. However $\ker[S^*, S] = \{ h \in \mathcal{H} : N^* h \in \mathcal{H} \}$ where $N = mneS$ (see Proposition IV.1.1) Thus, if $x \in \ker[S^*, S]$, then $S^*x = N^*x$. So, if $n \geq 0$ and $x \in \mathcal{M}$, then since $S^{n+1}x \in \ker[S^*, S]$ we have $S^*S(S^n x) = N^*N^{n+1}x = N^n N^* N x = N^n S^* S x = N^n x = S^n x$. That is, if $x \in \mathcal{M}$ and $n \geq 0$, then $S^*S(S^n x) = S^n x$. Hence, S^*S is the identity on the invariant subspace $\mathcal{L}$ for S generated by $\mathcal{M}$. However since S is pure it will follow that $\mathcal{L} = \mathcal{H}$. To see this, we shall show that $\mathcal{L}$ is a reducing subspace for S . We know it is invariant for S . So, if $n \geq 0$, we need to check that $S^*(S^n x) \in \mathcal{L}$. If $n = 0$, then $S^*x = 0 \in \mathcal{L}$. If $n \geq 1$, then $S^*(S^n x) = S^*S(S^{n-1} x) = S^{n-1} x \in \mathcal{L}$. Hence $\mathcal{L}$ is reducing for S . Since $\text{Range}[S^*, S] = \mathcal{M} \subseteq \mathcal{L}$, it follows that $S|_{\mathcal{L}^\perp}$ is a normal operator. But since S is pure we must have $\mathcal{L} = \mathcal{H}$. Hence $S^*S = I$ on $\mathcal{H}$. So, S is an isometry. $\square$

We now state some results about cyclic subnormal operators that have appeared since Thomson's Theorem. One, allows us to pull these operators back to the disk. The other allows us to replace the measure μ on $G = \text{abpe}(\mu)$ by a discrete sum of point masses and still have a similar operator.

Let S_μ be an irreducible cyclic subnormal operator. Set $G = \text{abpe}(\mu)$ and let $\phi: \mathbf{D} \rightarrow G$ be a Riemann map. Also let $\psi: G \rightarrow \mathbf{D}$ be the inverse of ϕ . Thomson's Theorem says that the natural extension map $\wedge: P^2(\mu) \cap L^\infty(\mu) \rightarrow H^\infty(G)$ is a dual algebra isomorphism. Let $\sim: H^\infty(G) \rightarrow P^2(\mu) \cap L^\infty(\mu)$ be the inverse of $\wedge$. So, $\tilde{\psi} \in P^2(\mu) \cap L^\infty(\mu)$. Define a measure ν on $\text{cl}\mathbf{D}$ by $\nu = \mu \circ \tilde{\psi}^{-1}$. The following nice result is proven in Olin and Yang [1995]. Keep the above notation.

Theorem 2.3 S_ν is irreducible and $\text{abpe}(\nu) = \mathbf{D}$. The densely defined map $U: P^2(\mu) \rightarrow P^2(\nu)$ given by $U(p) = p \circ \tilde{\phi}$ extends to an onto isometry and $U^* \tilde{\phi}(S_\nu)U = S_\mu$.

The following result was shown to me for weighted Bergman spaces by Stefan Richter. This is an easy adaptation of that to cyclic operators that makes use of Theorem 2.3.

Theorem 2.4 If S_μ is an irreducible cyclic subnormal operator, $G = \text{abpe}(\mu)$ and $\mu|\partial G = 0$, then there exists a measure $\nu = \sum_{n=1}^{\infty} w_n \delta_{a_n}$ such that $a_n \in G \cap \text{supp}(\mu)$ for all n and the identity map on polynomials extends to a similarity between S_μ and S_ν .

Proof. By Theorem 2.3 it suffices to prove this when $\text{abpe}(\mu) = \mathbf{D}$. Choose a sequence $\{r_n\}$ of positive numbers increasing to one. Let $A_n = \{z : r_{n-1} \leq |z| < r_n\}$. If $z \in \mathbf{D}$, then the map $f \rightarrow \hat{f}'(z)$ is a continuous linear functional, denote it by D_z , the derivative at z . It follows from the principle of uniform boundedness that D_z is norm bounded for z in a

compact set in $\mathbf{D}$. So, for each n , there is a constant C_n such that $|\hat{f}(z) - \hat{f}(w)| \leq C_n |z - w|$

for all $z, w \in A_n$ and $f \in P^2(\mu)$ with $\|f\| \leq 1$. Let $\{\Delta_{nk} : k \geq 1\}$ be a partition of A_n into small "circular squares" of the form $\{re^{i\theta} : 0 \leq a \leq r < b < 1 \text{ and } \theta_1 \leq \theta < \theta_2\}$ such that

$$|\hat{f}(z) - \hat{f}(w)| \leq \frac{1}{2\|\mu\|^{1/2}} \text{ for all } z, w \in \Delta_{nk} \text{ and } f \in P^2(\mu) \text{ with } \|f\| \leq 1. \text{ Now re-index the set}$$

of circular squares $\{\Delta_{nk} : n \geq 1 \text{ and } k \geq 1\}$ as $\{\Delta_j : j \geq 1\}$. Next, for each j such that $\mu(\Delta_j) > 0$, choose a point $z_j \in \Delta_j \cap \text{supp}(\mu)$. So, if $\|f\| \leq 1$, then we have

$$\left\| \hat{f} - \sum_{j=1}^{\infty} \hat{f}(z_j) \chi_{\Delta_j} \right\|_{\mu}^2 = \int_{\mathbf{D}} \left| \hat{f} - \sum_{j=1}^{\infty} \hat{f}(z_j) \chi_{\Delta_j} \right|^2 d\mu = \sum_{j=1}^{\infty} \int_{\Delta_j} |\hat{f} - \hat{f}(z_j)|^2 d\mu \leq \sum_{j=1}^{\infty} \frac{1}{4\|\mu\|} \mu(\Delta_j) = \frac{1}{4}.$$

Notice here that since μ puts no measure on $\partial\mathbf{D}$, the $L^2(\mu)$ norm of f and of $\hat{f}$ are the same. Now, if f does not have norm one, then by normalizing and applying the above inequalities we see that

$$(*) \quad \left\| \hat{f} - \sum_{j=1}^{\infty} \hat{f}(z_j) \chi_{\Delta_j} \right\|_{\mu} \leq \frac{1}{2} \|\hat{f}\|_{\mu}$$

for all $f \in P^2(\mu)$. Now, let $\nu = \sum_{j=1}^{\infty} \mu(\Delta_j) \delta_{z_j}$. Notice that $\|\hat{f}\|_{\nu} = \left\| \sum_{j=1}^{\infty} \hat{f}(z_j) \chi_{\Delta_j} \right\|_{\mu}$. So,

$$\text{the triangle inequality gives } \|\hat{f}\|_{\mu} \leq \left\| \hat{f} - \sum_{j=1}^{\infty} \hat{f}(z_j) \chi_{\Delta_j} \right\|_{\mu} + \left\| \sum_{j=1}^{\infty} \hat{f}(z_j) \chi_{\Delta_j} \right\|_{\mu} \leq \frac{1}{2} \|\hat{f}\|_{\mu} + \|\hat{f}\|_{\nu}.$$

$$\text{Thus, } \|\hat{f}\|_{\mu} \leq 2\|\hat{f}\|_{\nu}. \text{ Likewise, } \|\hat{f}\|_{\nu} \leq \frac{1}{2} \|\hat{f}\|_{\mu} + \|\hat{f}\|_{\mu} = \frac{3}{2} \|\hat{f}\|_{\mu}. \text{ So, } \frac{1}{2} \|\hat{f}\|_{\mu} \leq \|\hat{f}\|_{\nu} \leq \frac{3}{2} \|\hat{f}\|_{\mu}.$$

Thus, S_{μ} and S_{ν} are similar. $\square$

CHAPTER II

The C^* -Algebra Generated By A Subnormal Operator

In this chapter we study the C^* -algebra generated by a subnormal operator. This study is interesting and important in its own right. Several new things about subnormal operators and Toeplitz operators shall be obtained as well as some applications to understanding various properties of the self-commutator of a subnormal operator. The basic question is when do two subnormal operators generate isomorphic C^* -algebras. Also, if two operators have isomorphic C^* -algebras what other properties do they have in common? Another basic C^* -algebra is the one generated by S in the Calkin algebra, that is if π is the canonical map into the Calkin algebra, we want to study $C^*(\pi(S))$. As we shall see, it is an important property for $C^*(S)$ to contain a compact operator. If S has compact self-commutator then $C^*(S)$ will contain a compact operator, namely the self-commutator, and hence several of the above questions shall have complete answers in this case.

Finally, let's note that if S is a subnormal operator, then the C^* -algebra generated by S is not completely unrelated to the self-commutator of S . In fact every operator in the C^* -algebra generated by S is the sum of a Toeplitz operator with a continuous symbol and

something from the commutator ideal. Furthermore, the commutator ideal is generated by just one operator, the self-commutator.

1. Basic Properties

In this section we prove some basic properties for the C^* -algebra generated by a subnormal operator. We begin with a simple description of what types of operators are in $C^*(S)$ for a subnormal operator S . Recall, all Hilbert spaces are separable and complex.

If S is a subnormal operator on $\mathcal{H}$ with minimal normal extension (mne) N , then for $f \in C(\sigma(N))$ we define the Toeplitz operator S_f to be the compression of $f(N)$ to $\mathcal{H}$. That is, $S_f = P f(N)|_{\mathcal{H}}$. One easily checks that $S_{\bar{z}^n z^k} = S^{*n} S^k$. Thus, if p is any polynomial in z and $\bar{z}$ then S_p belongs to $C^*(S)$. Now, a simple limiting argument gives that $S_f \in C^*(S)$ for every $f \in C(\sigma(N))$.

Theorem 1.1 If S is a subnormal operator and $N = \text{mne} S$, then $C^*(S) = \{ S_f + C : f \in C(\sigma(N)) \text{ and } C \in \text{Comm}(S) \}$, where $\text{Comm}(S)$ denotes the commutator ideal in $C^*(S)$.

Proof. It follows from a result of Bunce that $C^*(S)/\text{Comm}(S) \approx C(\sigma_{\text{ap}}(S))$ (see Theorem I.1.9). Thus there is a $*$ -homomorphism $\varphi: C^*(S) \rightarrow C(\sigma_{\text{ap}}(S))$ so that $\varphi(S) = z$ and $\ker(\varphi) = \text{Comm}(S)$. Using the fact that $S_{\bar{z}^n z^k} = S^{*n} S^k$ and $\sigma_{\text{ap}}(S) \subseteq \sigma(N)$, we see that $\varphi(S_f) = f|_{\sigma_{\text{ap}}(S)}$ for each $f \in C(\sigma(N))$. Now, if $T \in C^*(S)$, $g = \varphi(T) \in C(\sigma_{\text{ap}}(S))$ and f is

any continuous extension of g to $\sigma(N)$ then $\varphi(T - S_f) = 0$, hence $T - S_f \in \ker(\varphi) = \text{Comm}(S)$. Thus, $T = S_f + (T - S_f)$. So T has the required form. $\square$

It's easy to see that if S is any irreducible operator with compact self-commutator, then $\text{Comm}(S) = \mathcal{B}_0$ the ideal of all compact operators. Hence in this case Theorem 1.1 becomes very useful.

If $\mathcal{A}$ is any algebra of operators on a Hilbert space $\mathcal{H}$, we say that $\mathcal{A}$ acts *non-degenerately* on $\mathcal{H}$ if $\{Ax : A \in \mathcal{A} \text{ and } x \in \mathcal{H}\}$ is dense in $\mathcal{H}$. We shall be interested in ideals in $C^*(S)$ that act non-degenerately on $\mathcal{H}$.

Proposition 1.2 Let S be an operator on $\mathcal{H}$ and $\mathcal{J}, \mathcal{K}$ non-zero ideals in $C^*(S)$.

- (a) If $\mathcal{J}$ acts non-degenerately on $\mathcal{H}$, then $\mathcal{J} \cap \mathcal{K} \neq (0)$.
- (b) If $\mathcal{J}$ and $\mathcal{K}$ act non-degenerately on $\mathcal{H}$, then $\mathcal{J} \cap \mathcal{K}$ also acts non-degenerately on $\mathcal{H}$.
- (c) The commutator ideal, $\text{Comm}(S)$, acts non-degenerately on $\mathcal{H}$ if and only if S is pure.

Proof. (a) If $K \in \mathcal{K}$ is a non-zero operator, then $KA \in \mathcal{J} \cap \mathcal{K}$ for all $A \in \mathcal{J}$. Now, since $\{Ax : A \in \mathcal{J} \text{ and } x \in \mathcal{H}\}$ is dense in $\mathcal{H}$, if $KA = 0$ for all $A \in \mathcal{J}$, then $K = 0$. Thus as $K \neq 0$, there exists an $A \in \mathcal{J}$ such that $KA \neq 0$. Hence $\mathcal{J} \cap \mathcal{K} \neq (0)$.

(b) Since $\{Ax : A \in \mathcal{J} \text{ and } x \in \mathcal{H}\}$ is dense in $\mathcal{H}$ and $\mathcal{K}$ acts non-degenerately we see that $\{KAx : A \in \mathcal{J}, K \in \mathcal{K} \text{ and } x \in \mathcal{H}\}$ is also dense in $\mathcal{H}$. Thus since $\{KA : A \in \mathcal{J}, K \in \mathcal{K}\} \subseteq \mathcal{J} \cap \mathcal{K}$, it follows that $\mathcal{J} \cap \mathcal{K}$ acts non-degenerately on $\mathcal{H}$.

(c) It is clear that if S is not pure, then $\text{Comm}(S)$ acts degenerately. For the converse, if S is pure, then let $\mathcal{M} = \text{cl}\{Ax : A \in \text{Comm}(S) \text{ and } x \in \mathcal{H}\}$. Clearly $\mathcal{M}$ is a reducing subspace for $\text{Comm}(S)$. Since $\text{Comm}(S)$ is an ideal in $C^*(S)$, $\mathcal{M}$ is also a reducing subspace for S . Furthermore, if $T = S|_{\mathcal{M}^\perp}$, then $[T^*, T] = 0$. As S is pure, we must have $\mathcal{M}^\perp = (0)$. Hence $\mathcal{M} = \mathcal{H}$ and it follows that $\text{Comm}(S)$ acts non-degenerately. $\square$

Next we see when the C^* -algebra generated by $S \oplus T$ splits. Recall that $\sigma_n(S)$ denotes the normal spectrum of S .

Theorem 1.3 Let S and T be subnormal operators.

- (a) If $C^*(S \oplus T) = C^*(S) \oplus C^*(T)$, then $\sigma_{\text{ap}}(S) \cap \sigma_{\text{ap}}(T) = \emptyset$.
- (b) If $\sigma_n(S) \cap \sigma_n(T) = \emptyset$, then $C^*(S \oplus T) = C^*(S) \oplus C^*(T)$.
- (c) If S is pure and T is irreducible with compact self-commutator, then $C^*(S \oplus T) = C^*(S) \oplus C^*(T)$ if and only if $\sigma_{\text{ap}}(S) \cap \sigma_{\text{ap}}(T) = \emptyset$.

Proof. (a) If $C^*(S \oplus T) = C^*(S) \oplus C^*(T)$, then Corollary I.1.11 gives that there is a $*$ -homomorphism $\varphi: C^*(S \oplus T) \rightarrow C^*(N)$ with $\varphi(S \oplus T) = N$ such that N is normal with $\sigma(N) = \sigma_{\text{ap}}(S) \cup \sigma_{\text{ap}}(T)$ and whose kernel is exactly the commutator ideal. Since $I \oplus 0$ and $0 \oplus I$ are complementary projections in $C^*(S \oplus T)$, if we define $P = \varphi(I \oplus 0)$ and $Q = \varphi(0 \oplus I)$, then P and Q are complementary projections in $C^*(N)$. If we define $N_1 = PNP$ and $N_2 = QNQ$, then $N = N_1 \oplus N_2$ and $C^*(N) = C^*(N_1 \oplus N_2) = C^*(N_1) \oplus C^*(N_2)$. Since N_1 and N_2 are normal we see that $\sigma(N_1) \cap \sigma(N_2) = \emptyset$. Now considering $\varphi[C^*(S) \oplus 0]$ and $\varphi[0 \oplus C^*(T)]$ we see that $\sigma(N_1) = \sigma_{\text{ap}}(S)$ and $\sigma(N_2) = \sigma_{\text{ap}}(T)$. So, (a) follows.

(b) If $\sigma_n(S) \cap \sigma_n(T) = \emptyset$, then let f be the characteristic function of $\sigma_n(S)$. It follows that f is continuous on $\sigma_n(S) \cup \sigma_n(T) = \sigma_n(S \oplus T)$. Hence the Toeplitz operator $(S \oplus T)_f$ is well defined and equal to $I \oplus 0$. Thus $I \oplus 0$ belongs to $C^*(S \oplus T)$. So the C^* -algebra splits.

(c) Suppose S is pure and T is an irreducible subnormal with compact self-commutator. If S and T are acting on $\mathcal{H}$ and $\mathcal{K}$ respectively, then $\text{Comm}(T) = \mathcal{B}_0(\mathcal{K})$ and $\text{Comm}(T)$ is a minimal ideal in $C^*(T)$, (see Corollary I.1.18).

Claim: $\text{Comm}(S \oplus T) = \text{Comm}(S) \oplus \text{Comm}(T)$.

Given the Claim, let's finish the proof. If $\sigma_{ap}(S) \cap \sigma_{ap}(T) = \emptyset$, then let f be any continuous function on $\sigma_n(S \oplus T)$ that is one on $\sigma_{ap}(S)$ and zero on $\sigma_{ap}(T)$. Now if $(S \oplus T)_f$ is the Toeplitz operator with symbol f , then $(S \oplus T)_f = S_f \oplus T_f$. Also by Theorem I.1.9, $S_f - I \in \text{Comm}(S)$ and $T_f \in \text{Comm}(T)$. So, if the claim is true, then $(S_f - I) \oplus T_f \in \text{Comm}(S \oplus T)$. It follows from Theorem 1.1 that $I \oplus 0 = (S \oplus T)_f - (S_f - I) \oplus T_f \in C^*(S \oplus T)$. So, the C^* -algebra splits.

We now prove the Claim. For this it suffices to suppose that $\sigma_{ap}(T) \not\subset \sigma_{ap}(S)$. Let $\pi_1: C^*(S \oplus T) \rightarrow C^*(S)$ be the natural restriction map; that is, $\pi_1(A) = A|_{\mathcal{H}}$. Let π_2 be defined similarly. If π_1 is one-to-one, then since the approximate point spectrum is an invariant for $*$ -isomorphisms (see Corollary I.1.10) we have $\sigma_{ap}(T) \subseteq \sigma_{ap}(S \oplus T) = \sigma_{ap}(S)$. Thus since $\sigma_{ap}(T) \not\subset \sigma_{ap}(S)$ we have that $\ker(\pi_1)$ is a non-zero ideal in $C^*(S \oplus T)$. Thus if we set $\mathcal{J} = \ker(\pi_1) \cap \text{Comm}(S \oplus T)$, this is also an ideal in $C^*(S \oplus T)$. Since $S \oplus T$ is pure

we have that $\text{Comm}(S \oplus T)$ acts non-degenerately on $\mathcal{H} \oplus \mathcal{K}$. It follows that $\mathcal{I} \neq (0)$, hence $\pi_2(\mathcal{I})$ is a non-zero ideal contained in $\text{Comm}(T)$. Since $\text{Comm}(T)$ is a minimal ideal we have $\pi_2(\mathcal{I}) = \text{Comm}(T)$. Unraveling we see, $0 \oplus \text{Comm}(T) \subseteq \text{Comm}(S \oplus T)$. So, $0 \oplus [T^*, T] \in \text{Comm}(S \oplus T)$. Hence also $[S^*, S] \oplus 0 \in \text{Comm}(S \oplus T)$. So, $[S^*, S] \in \pi_1[\ker(\pi_2) \cap \text{Comm}(S \oplus T)]$, implying that $\pi_1[\ker(\pi_2) \cap \text{Comm}(S \oplus T)] = \text{Comm}(S)$. Hence it follows that $\text{Comm}(S) \oplus 0 \subseteq \text{Comm}(S \oplus T)$. So the claim follows. $\square$

Next we consider the problem of determining for a given subnormal operator S , not necessarily pure, when there exists a pure subnormal operator T with $C^*(S)$ isomorphic to $C^*(T)$. In what follows, when we say ideal we always mean closed two-sided ideals.

Theorem 1.4 If S is a subnormal operator and $S = S_p \oplus N$ with S_p pure and N normal, then there is a pure subnormal operator T and a $*$ -isomorphism $\varphi: C^*(S) \rightarrow C^*(T)$ with $\varphi(S) = T$ if and only if $\sigma(N) \subseteq \sigma_{ap}(S_p)$. Also, if such a T exists it may be chosen to be S_p .

Lemma 1.5 If S is any subnormal operator and N is any normal operator with $\sigma(N) \subseteq \sigma_{ap}(S)$, then there is a $*$ -isomorphism $\varphi: C^*(S) \rightarrow C^*(S \oplus N)$ with $\varphi(S) = S \oplus N$.

Proof. From Theorem 1.1 we have $C^*(S) = \{ S_f + C : f \in C(\sigma_n(S)) \text{ and } C \in \text{Comm}(S) \}$. Define $\varphi: C^*(S) \rightarrow C^*(S \oplus N)$ by $\varphi(S_f + C) = (S_f + C) \oplus f(N)$. By Theorem 1.1.9, $C^*(S)/\text{Comm}(S) \cong C(\sigma_{ap}(S))$. Hence given $T_f + C$ the values of f are well defined on $\sigma_{ap}(S)$, independent of C . So φ is easily seen to be a $*$ -isomorphism of $C^*(S)$ onto $C^*(S \oplus N)$ with $\varphi(S) = S \oplus N$. $\square$

Lemma 1.6 If T is a pure subnormal operator on $\mathcal{H}$, then there are no non-zero ideals of $C^*(T)$ contained in the center of $C^*(T)$.

Proof. Let Z be the center of $C^*(T)$; so $Z = C^*(T) \cap \{T, T^*\}'$. Suppose $\mathcal{J}$ is an ideal in $C^*(T)$ and $\mathcal{J} \subseteq Z$. If $J \in \mathcal{J}$, then $JT \in \mathcal{J}$. So both J and TJ commute with T and T^* .

It follows that:

$$(JT)T^* = T^*(JT)$$

$$J(TT^*) = T^*(TJ)$$

$$(TT^*)J = (T^*T)J.$$

So we get that $[T^*, T]J = 0$ for each $J \in \mathcal{J}$. Now let $\mathcal{M}$ be the closure of the subspace $\{Jx: x \in \mathcal{H} \text{ and } J \in \mathcal{J}\}$. Then as $\mathcal{J}$ is an ideal we see that $\mathcal{M}$ is a reducing subspace of T on which the self-commutator is zero. Hence T is normal on this reducing subspace. But this contradicts the purity of T , so $\mathcal{J} = 0$. $\square$

Proof of Theorem 1.4. If $\sigma(N) \subseteq \sigma_{ap}(S_p)$, then since $S = S_p \oplus N$, Lemma 1.5 applies to give that $T = S_p$ is a pure subnormal operator and $C^*(S)$ is $*$ -isomorphic to $C^*(T)$ under a map which sends S to T .

Conversely, suppose there is a $*$ -isomorphism $\phi: C^*(S) \rightarrow C^*(T)$ with $\phi(S) = T$ where T is a pure subnormal operator. Also, if $S = S_p \oplus N$ is acting on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, then let $\pi: C^*(S) \rightarrow C^*(S_p)$ be given by $\pi(A) = A|_{\mathcal{H}_1}$. So π is a $*$ -homomorphism and $\pi(S) = S_p$. Now the kernel of π , $\ker(\pi)$, is an ideal in $C^*(S)$ which is also contained in the center Z of $C^*(S)$. Thus $\phi(\ker(\pi))$ is an ideal in $C^*(T)$ which must be contained in the center of $C^*(T)$.

Hence $\varphi(\ker(\pi)) = (0)$, so we see that $\ker(\pi) = (0)$. Hence π is injective, thus a $*$ -isomorphism of $C^*(S)$ onto $C^*(S_p)$. So by Corollary I.1.10, $\sigma_{ap}(S) = \sigma_{ap}(S_p)$. Thus we get that $\sigma(N) \subseteq \sigma_{ap}(S_p)$ because it is easy to see that $\sigma(N) = \sigma_{ap}(N) \subseteq \sigma_{ap}(S) = \sigma_{ap}(S_p)$. $\square$

Theorem 1.4 and the following corollary will be useful in later sections. The next result was first proven by Coburn [1967].

Corollary 1.7 If S is the unilateral shift and T is any isometry that is not a unitary, then there is a $*$ -isomorphism $\varphi: C^*(S) \rightarrow C^*(T)$ with $\varphi(S) = T$.

Proof. It follows by the von Neumann-Wold decomposition that $T \cong S^{(n)} \oplus U$ where U is a unitary. Thus since $\sigma(U) \subseteq \partial\mathbf{D} = \sigma_{ap}(S^{(n)})$, Theorem 1.4 applies to give that $C^*(T) \approx C^*(S^{(n)}) \approx C^*(S)$. $\square$

Corollary 1.7 will be generalized later so that S may be taken to be any irreducible subnormal operator with compact self-commutator and a completely analogous result holds. See Theorem 6.14.

A tool that is useful and fundamental in what follows is knowing when an operator T may be written as the direct sum of irreducible operators. Not all subnormal operators are direct sums of irreducible ones, and the next result shows what happens in general. If T is an operator we shall say that T is *completely reducible* if T is reducible and every reducing subspace for T contains another non-trivial reducing subspace for T . That is, the lattice of

reducing subspaces for T has no non-trivial minimal elements. For example, a normal operator is completely reducible if and only if it has no eigenvalues. The following is a result that generalizes a well known fact about operators with compact self-commutator. If $\mathcal{H}$ is any Hilbert space, then $\mathcal{B}_0(\mathcal{H})$ denotes the ideal of compact operators on $\mathcal{H}$.

Theorem 1.8 If T is an operator, then $T = S_0 \oplus \left(\bigoplus_{n=1}^{\infty} S_n \right) \oplus \left(\bigoplus_{n=1}^{\infty} T_n \right)$ where

- (a) S_0 is completely reducible and $C^*(S_0) \cap \mathcal{B}_0 = (0)$,
- (b) for $n \geq 1$, S_n is irreducible and $C^*(S_n) \cap \mathcal{B}_0 = (0)$,
- (c) T_n is irreducible and $\mathcal{B}_0 \subseteq C^*(T_n)$.

Naturally, in Theorem 1.8, not all the summands need to appear. The point is that every operator is the direct sum of three types of operators. Also notice that the decomposition in Theorem 1.8 is unique.

Lemma 1.9 If T is an operator and $C^*(T)$ contains a non-zero compact operator, then T has a non-zero irreducible orthogonal summand S such that $\mathcal{B}_0 \subseteq C^*(S)$.

Proof. If $\mathcal{J} = C^*(T) \cap \mathcal{B}_0$, then $\mathcal{J}$ is a non-zero C^* -algebra of compact operators. By Theorem I.1.19 there are reducing subspaces $\mathcal{H}_n$ for $\mathcal{J}$ such that $\mathcal{J}|_{\mathcal{H}_n} = \mathcal{B}_0(\mathcal{H}_n)$. If n is an integer, then $\mathcal{H}_n$ is a reducing subspace for T and $\mathcal{B}_0(\mathcal{H}_n) = \mathcal{J}|_{\mathcal{H}_n} \subseteq C^*(T|_{\mathcal{H}_n})$. So, if we set $S = T|_{\mathcal{H}_n}$ for some n , then S is irreducible and $\mathcal{B}_0 \subseteq C^*(S)$. $\square$

Notice that if T is irreducible in Lemma 1.9 and there is a compact operator in $C^*(T)$, then the operator S must equal T , so $\mathcal{B}_0 \subseteq C^*(T)$. Also if T is completely reducible, then Lemma 1.9 gives that $C^*(T)$ contains no compact operators.

Proof of Theorem 1.8. If T is irreducible, then by Lemma 1.9 T is either of the form in (b) or (c). Likewise if T has no minimal reducing subspaces, then it is completely reducible so by Lemma 1.9 we are finished. Otherwise T is reducible and has some minimal reducing subspaces. If T acts on $\mathcal{H}$, then let $\{\mathcal{H}_n : n \geq 1\}$ be a maximal collection of pairwise orthogonal minimal reducing subspaces for T . This gives that $T = S_0 \oplus \bigoplus_{n=1}^{\infty} B_n$ where $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$, $B_n = T|_{\mathcal{H}_n}$ for $n \geq 1$ and $S_0 = T|_{\mathcal{H}_0}$. Clearly B_n is irreducible and S_0 is completely reducible, by the maximal property of the sequence $\{\mathcal{H}_n\}$. Observe that $C^*(S_0)$ cannot contain a compact operator; otherwise Lemma 1.9 would give the existence of an irreducible summand of S_0 , contradicting the complete reducibility of S_0 . Also, for each $n \geq 1$, Lemma 1.9 implies that $C^*(B_n)$ either contains all the compact operators or none of them. Thus each B_n is one of the operators in (b) or (c). So the result follows. $\square$

The next corollary gives a nice property of operators with diagonalizable self-commutators.

Corollary 1.10 If T is any operator such that $[T^*, T]$ is diagonalizable and the non-zero eigenvalues of $[T^*, T]$ do not belong to $\sigma_e([T^*, T])$, then $T = N \oplus \bigoplus_{n=1}^{\infty} T_n$ where N is normal, T_n is irreducible and $\mathcal{B}_0(\mathcal{H}_n) \subseteq C^*(T_n)$.

Proof. As in Theorem 1.8, $T = S_0 \oplus \left(\bigoplus_{n=1}^{\infty} S_n \right) \oplus \left(\bigoplus_{n=1}^{\infty} T_n \right)$. If we set $T_0 = S_0 \oplus \left(\bigoplus_{n=1}^{\infty} S_n \right)$,

then $T = T_0 \oplus \bigoplus_{n=1}^{\infty} T_n$ where T_0 satisfies $C^*(T_0) \cap \mathcal{B}_0(\mathcal{H}_0) = (0)$ and the operators T_n for n

≥ 1 have the required properties. Now, since $[T^*, T]$ is diagonalizable and reductive, every invariant subspace for $[T^*, T]$ is the span of eigenvectors (see Wermer [1952] or Conway[1991], p.315). Thus, since $[T_0^*, T_0]$ is an orthogonal summand of $[T^*, T]$, $[T_0^*, T_0]$ is diagonalizable and the non-zero eigenvalues do not belong to $\sigma_e([T_0^*, T_0])$.

We claim that $[T_0^*, T_0] = 0$. For if $[T_0^*, T_0] \neq 0$, then $C^*(T_0)$ contains a non-zero compact operator; namely the spectral projection for one of its non-zero eigenvalues (notice that each non-zero eigenvalue is isolated since it does not belong to $\sigma_e([T^*, T])$). So $[T_0^*, T_0]$

$= 0$. Thus T_0 is normal and $T = N \oplus \bigoplus_{n=1}^{\infty} T_n$. $\square$

The following is due to Behncke[1968]. A more general result is due to Gilfeather [1971]. See also Conway [1991], p. 159. Theorem 1.8 easily recaptures both results.

Corollary 1.11 If T is any operator with compact self-commutator, then $T = N \oplus \bigoplus_{n=1}^{\infty} T_n$ where N is normal, T_n is irreducible with compact self-commutator and $\mathcal{B}_0(\mathcal{H}_n) \subseteq C^*(T_n)$.

Theorem 1.8 shows that there are three different basic types of operators. We shall show that there exist pure subnormal operators of all three types, although, those in part (b) of Theorem 1.8 are still very mysterious. However we shall give an example of one below.

Example 1.12 If φ is an analytic universal covering map of the unit disk $\mathbf{D}$ onto a non-degenerate annulus, then the Toeplitz operator T_φ acting on $H^2(\mathbf{D})$ is completely reducible. Thus, T_φ cannot be written as a direct sum of irreducible operators.

This follows from a remark in Abrahamse and Ball [1976] where they use the theory of bundle shifts to analyze some analytic Toeplitz operators. The next result characterizes the quasinormal operators that are completely reducible.

Proposition 1.13 If U is the unilateral shift and A is a positive one-to-one operator, then $U \otimes A$ is completely reducible if and only if A has no eigenvalues.

Proof. Let $S = U \otimes A$. One easily checks that the lattice of reducing subspaces for S is the inflation of the lattice of reducing subspaces for A . So S is completely reducible if and only if A is also. Since normal operators are completely reducible exactly when they have no eigenvalues, the result follows. $\square$

Proposition 1.14 If S is an irreducible operator and N is a one-to-one normal operator, then $S \otimes N$ may be written as the direct sum of irreducible operators if and only if N is diagonalizable.

Proof. If N is diagonalizable, say N has eigenvalues $\{\lambda_k : k \geq 1\}$, then $S \otimes N \cong \bigoplus_{k=1}^{\infty} \lambda_k S$.

Since S is irreducible, so is any non-zero multiple of S . Hence $S \otimes N$ is a direct sum of irreducible operators. Conversely, if $S \otimes N$ is a direct sum of irreducible operators, then the commutant of its von Neumann algebra, $W^*(S \otimes N)'$, is generated by a countable number of pairwise orthogonal minimal projections $\{P_n\}$. In particular every projection in

$W^*(S \otimes N)'$ is the (SOT)-sum of a subsequence of $\{P_n\}$. Hence any family of strictly decreasing reducing subspaces for $S \otimes N$ is countable. However, if N is not diagonalizable, then it has an uncountable number of strictly decreasing reducing subspaces and each corresponds to a projection in $W^*(N)'$. Since $I \otimes W^*(N)'$ is contained in $W^*(S \otimes N)'$, we see that N must be diagonalizable. $\square$

Corollary 1.15 If S is an irreducible operator with diagonalizable self-commutator and A is a positive one-to-one operator, then $S \otimes A$ may be written as a direct sum of irreducible operators if and only if its self-commutator is diagonalizable.

Proof. If $T = S \otimes A$, then $[T^*, T] = [S^*, S] \otimes A^2$. Now, since $[S^*, S]$ is diagonalizable, we see that A is diagonalizable if and only if $[T^*, T]$ is diagonalizable. $\square$

Example 1.16 If S is the unilateral shift and $A = M_1$ on $L^2(0,1)$, then $S \otimes A$ cannot be written as the direct sum of irreducible operators.

We now show that *there exists* an irreducible subnormal operator S such that $C^*(S)$ contains no compact operators. That is, there is a subnormal operator as in Theorem 1.8(b). A C^* -algebra $\mathcal{A}$ is called GCR if every irreducible representation of $\mathcal{A}$ contains a non-zero compact operator (and hence all compact operators) in its range. See Sakai [1971], p. 221. Thus we are claiming that $C^*(S)$ is *not* always a GCR algebra for a subnormal operator S . Now Sakai [1966] also showed that a C^* -algebra $\mathcal{A}$ is GCR if and only if for *every* representation ρ of $\mathcal{A}$, the von Neumann algebra generated by $\rho(\mathcal{A})$ is of

type I. However, Behncke [1970] has shown that there exist subnormal operators that generate a von Neumann algebra that is not type I. So, if S is a subnormal operator such that $W^*(S)$ is not of type I, then necessarily $C^*(S)$ is not a GCR algebra. Hence there is an irreducible representation ρ of $C^*(S)$ so that $\rho(C^*(S))$ contains no compact operators. Thus if we let $T = \rho(S)$, then T is an irreducible subnormal operator and $C^*(T)$ contains no compact operators.

It can be shown using the above arguments that there exists an irreducible bundle shift that has no compact operators in its C^* -algebra. The author believes that it may be possible to explicitly write one down.

For completeness we shall state a result due to O'Donovan [1975] characterizing those unilateral weighted shifts S with $C^*(S) \cap \mathcal{B}_0 = (0)$. It is known that if S has positive weights, then it is irreducible.

Theorem 1.17 If S is a unilateral weighted shift with positive weights $\{w_n : n \geq 0\}$, then $C^*(S) \cap \mathcal{B}_0 = (0)$ if and only if there exists a sequence $n_j \rightarrow \infty$ such that $w_{k+n_j} \rightarrow w_k$ if $k \geq 0$ and $w_{k+n_j} \rightarrow 0$ if $k < 0$.

In particular, notice that a subsequence of the weights $\{w_n\}$ must converge to zero. We now show, using Theorem 1.17, how to construct an irreducible subnormal operator with no compact operators in its C^* -algebra. The example is an operator valued weighted

shift. If A and B are positive operators on a separable Hilbert space H such that $0 < A \leq$

B , then define an operator S on $H^{(\infty)}$ by $S = \begin{bmatrix} 0 & 0 & 0 & 0 \\ A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & B & \dots \end{bmatrix}$. It is easy to check that S is

a subnormal operator with normal extension N acting on $H^{(\infty)} \oplus (H \oplus H) \oplus H^{(\infty)}$. Where

$$N = \left[\begin{array}{c|cc|ccc} & A & 0 & & & & \\ & 0 & C & & & & \\ S & 0 & 0 & & & & 0 \\ & \vdots & \vdots & & & & \\ \hline & 0 & 0 & B & & & \\ & C & 0 & 0 & & & \\ \hline & & & 0 & B & & \\ & & & & 0 & B & \\ 0 & & 0 & & & 0 & \dots \\ & & & & & & \dots \end{array} \right]$$

and $C = (B^2 - A^2)^{1/2}$. Also note that $[S^*, S] = \begin{bmatrix} A^2 & & & \\ & B^2 - A^2 & & \\ & & 0 & \\ & & & \dots \end{bmatrix}$. Since A is one-

to-one, both A and B have dense ranges. It follows that the smallest invariant subspace for S that contains the range of $[S^*, S]$ is all of $H^{(\infty)}$. Thus S is pure. One also easily checks that N is actually the minimal normal extension of S . Recall that if $\mathcal{A}$ is a C^* -algebra of operators on H , then $\mathcal{A} \otimes B(H)$ is the C^* -algebra of operators on $H^{(\infty)}$ whose matrix representations all have entries from $\mathcal{A}$.

Proposition 1.18 $C^*(S) \subseteq C^*(A, B) \otimes B(H)$.

Proof. It is clear that for any polynomial p with two non-commuting variables each entry in the matrix for $p(S, S^*)$ is some polynomial in A and B . Thus upon taking limits, each entry of each operator in $C^*(S)$ belongs to $C^*(A, B)$. $\square$

Corollary 1.19 If $C^*(S)$ contains a non-zero compact operator, then so does $C^*(A, B)$.

Proof. If K is a non-zero compact operator in $C^*(S)$, then by Proposition 1.18 each entry of K belongs to $C^*(A, B)$. Since K is non-zero, some entry is non-zero and necessarily compact. Hence $C^*(A, B)$ contains a compact operator. $\square$

For any operator T , let $\text{Red}(T)$ denote the lattice of reducing subspaces for T . For a pair (A, B) let $\text{Lat}(A, B)$ denote the common invariant subspaces of A and B . It is shown in Behncke [1970] that $W^*(S) = W^*(A, B) \otimes B(H)$. This implies the next result.

Proposition 1.20 $\text{Red}(S)$ is lattice isomorphic to $\text{Lat}(A, B)$.

If we define $T = A + iB$, then $\text{Lat}(A, B) = \text{Red}(T)$ and $C^*(A, B) = C^*(T)$. Now suppose T is an irreducible operator such that $C^*(T)$ contains no compact operators (see Theorem 1.17). By translating T if necessary, we may suppose that $0 < \text{Re}T < \text{Im}T$. If we construct S as above from the real and imaginary parts of T , then S will be an irreducible subnormal operator such that $C^*(S)$ contains no compact operators.

2. Approximately Equivalent Operators

In this section we give a result which shows for *any* two operators S and T exactly when $C^*(S)$ is $*$ -isomorphic to $C^*(T)$ under a map which sends S to T . This follows from Voiculescu's Theorem, a non-commutative version of the Weyl-von Neumann Theorem.

We now state a corollary of Voiculescu's Theorem which we need. First, recall that if A and B are any two operators, then we say that A is *strongly approximately equivalent* to B , $A \sim_{sa} B$, if there is a sequence of unitaries U_n so that $U_n^*AU_n - B \in \mathcal{B}_0$ for each n and $\|U_n^*AU_n - B\| \rightarrow 0$ as $n \rightarrow \infty$. If S and T are two operators, then by $C^*(S) \approx C^*(T)$ we

mean that there exists a $*$ -isomorphism $\phi: C^*(S) \rightarrow C^*(T)$ with $\phi(S) = T$.

Voiculescu's Theorem If $\mathcal{A}$ is a separable C^* -algebra of operators and ρ is a non-degenerate representation of $\mathcal{A}$ such that $\mathcal{A} \cap \mathcal{B}_0 \subseteq \ker(\rho)$, then $\text{id} \sim_{sa} \text{id} \oplus \rho$.

Let's begin with an easy result, not requiring Voiculescu's Theorem. Say that S is *approximately equivalent* to T if T is in the closure of the unitary orbit of S ; that is, if there are unitary operators U_n such that $U_n^*SU_n$ converges to T in norm as $n \rightarrow \infty$.

Theorem 2.1 If S and T are any operators and S is approximately equivalent to T , then $C^*(S) \approx C^*(T)$.

Proof. If there are unitaries U_n so that $U_n^*SU_n \rightarrow T$ in norm as $n \rightarrow \infty$, then it is easy to see that for any polynomial $p(z, \bar{z})$ in two non-commuting variables we still have that

$U_n^*p(S, S^*)U_n \rightarrow p(T, T^*)$, in norm, as $n \rightarrow \infty$. Finally an easy $\varepsilon/3$ argument shows that for any $A \in C^*(S)$ we have that $U_n^*AU_n$ is a norm convergent sequence, whose limit will lie in $C^*(T)$. So if we define a map $\varphi: C^*(S) \rightarrow C^*(T)$ by $\varphi(A) = \lim U_n^*AU_n$, then one can easily check that φ is a $*$ -homomorphism. Furthermore as $U_nTU_n^* \rightarrow S$ in norm as $n \rightarrow \infty$ we can construct an inverse to φ in exactly the same manner. Hence φ is a $*$ -isomorphism of $C^*(S)$ onto $C^*(T)$ and $\varphi(S) = T$. $\square$

The next result again shows the importance of having a compact operator in $C^*(S)$. Recall from Theorem 1.1.17 that if S is irreducible and there exists one non-zero compact operator in $C^*(S)$, then in fact $C^*(S)$ contains all the compact operators.

Theorem 2.2 If S and T are irreducible operators and $\mathcal{B}_0 \subseteq C^*(S)$, then $C^*(S) \approx C^*(T)$ if and only if S is unitarily equivalent to T .

Proof. Let $\varphi: C^*(S) \rightarrow C^*(T)$ be a $*$ -isomorphism, suppose S acts on $\mathcal{H}$ and T acts on $\mathcal{K}$. Consider $\varphi|_{\mathcal{B}_0(\mathcal{H})}$. This is an irreducible representation of $\mathcal{B}_0(\mathcal{H})$ and hence is equivalent to the identity representation (see Corollary 1.1.21). That is, there is a unitary $U: \mathcal{K} \rightarrow \mathcal{H}$ so that $\varphi(K) = U^*KU$ for each $K \in \mathcal{B}_0(\mathcal{H})$. So for every $K \in \mathcal{B}_0(\mathcal{H})$ we have that $T\varphi(K) = \varphi(SK) = U^*SKU = U^*SU U^*KU = U^*SU \varphi(K)$. But $\varphi(\mathcal{B}_0(\mathcal{H}))$ is an ideal in $C^*(T)$. Hence it has no reducing subspaces (as T is irreducible), so $\{\varphi(K)x : x \in \mathcal{K}, K \in \mathcal{B}_0(\mathcal{H})\}$ is dense in $\mathcal{K}$. Thus we see that $T = U^*SU$. So $S \cong T$. $\square$

We shall now use Voiculescu's Theorem to show that a converse to Theorem 2.1 is true. We shall say that an operator T is *non-degenerate* if $C^*(T)$ acts non-degenerately (see Proposition 1.2). That is T is non-degenerate if and only if T does not have the zero operator as a direct summand.

Theorem 2.3 If S and T are any operators and T is non-degenerate, then there is a $*$ -homomorphism $\varphi: C^*(S) \rightarrow C^*(T)$ with $\varphi(S) = T$ if and only if $S^{(\infty)} \sim_{sa} S^{(\infty)} \oplus T$.

Proof. If φ exists, then let $\rho: C^*(S^{(\infty)}) \rightarrow C^*(T)$ be given by $\rho(A^{(\infty)}) = \varphi(A)$ where $A \in C^*(S)$. Then ρ is a non-degenerate (since T is non-degenerate) representation and, as $C^*(S^{(\infty)})$ contains no compact operators, Voiculescu's Theorem gives that $S^{(\infty)} \sim_{sa} S^{(\infty)} \oplus T$.

Conversely, if $S^{(\infty)} \sim_{sa} S^{(\infty)} \oplus T$, then by Theorem 2.1 there is a $*$ -isomorphism $\psi: C^*(S^{(\infty)}) \rightarrow C^*(S^{(\infty)} \oplus T)$ with $\psi(S^{(\infty)}) = S^{(\infty)} \oplus T$. Clearly the map $\varphi: C^*(S) \rightarrow C^*(T)$ that sends A in $C^*(S)$ to $\psi(A^{(\infty)})$ restricted to the reducing subspace on which T acts is the required $*$ -homomorphism. $\square$

Corollary 2.4 If S and T are non-degenerate operators, then $C^*(S) \approx C^*(T)$ if and only if $S^{(\infty)} \sim_{sa} T^{(\infty)}$.

Proof. If $S^{(\infty)} \sim_{sa} T^{(\infty)}$, then an application of Theorem 2.1 gives the result. Now, if $\varphi: C^*(S) \rightarrow C^*(T)$ exists with $\varphi(S) = T$, then clearly there is a $*$ -homomorphism $\psi: C^*(S)$

$\rightarrow C^*(T^{(\infty)})$ given by $\psi(A) = \varphi(A)^{(\infty)}$. Since, $\psi(S) = T^{(\infty)}$ Theorem 2.2 gives that $S^{(\infty)} \sim_{sa} S^{(\infty)} \oplus T^{(\infty)}$. Similarly by considering φ^{-1} we also have that $T^{(\infty)} \sim_{sa} S^{(\infty)} \oplus T^{(\infty)}$. Hence $S^{(\infty)} \sim_{sa} T^{(\infty)}$. $\square$

Corollary 2.5 If S, T are irreducible operators, then $C^*(S) \approx C^*(T)$ if and only if $S \sim_{sa} T$.

Proof. One direction follows from Theorem 2.1. So suppose that there exists a $\varphi: C^*(S) \rightarrow C^*(T)$, with $\varphi(S) = T$. If $\mathcal{B}_0 \subseteq C^*(S)$, then Theorem 2.2 says that φ is given by a unitary, so $S \cong T$. On the other hand if $\mathcal{B}_0 \not\subseteq C^*(S)$, then, as S is irreducible, $\mathcal{B}_0 \cap C^*(S) = (0)$. So, Voiculescu's Theorem applies to give that $S \sim_{sa} S \oplus T$. Likewise, by considering φ^{-1} , if $\mathcal{B}_0 \subseteq C^*(T)$, then $S \cong T$. Otherwise as above $T \sim_{sa} T \oplus S$. In either case $S \sim_{sa} T$. $\square$

We now show exactly when $C^*(S)$ contains a compact operator for any non-degenerate operator S . Notice if $C^*(S)$ contains no compact operators, then, letting $\rho(A) = A^{(\infty)}$, Voiculescu's Theorem above applies and gives that $S \sim_{sa} S^{(\infty)}$. We now show the converse.

Theorem 2.6 If S is a non-degenerate operator, then the following are equivalent.

- (a) $C^*(S) \cap \mathcal{B}_0 = (0)$;
- (b) $S \sim_{sa} S^{(\infty)}$;
- (c) $S \sim_{sa} S \oplus S$.

Proof. We noted above that Voiculescu's Theorem gives that (a) implies (b). It's easy to see that (b) implies (c). So if $S \sim_{sa} S \oplus S$, then we must show that $C^*(S)$ can contain no compact operators. Suppose U_n are unitaries so that $U_n^* S U_n - S \oplus S$ is compact and converges to zero. It's easy to see as in the proof of Theorem 2.1 that the map $\varphi: C^*(S)$

$\rightarrow C^*(S \oplus S)$ given by $\varphi(A) = \lim_n U_n^* A U_n$ is a C^* -isomorphism. But also, the map $\psi: C^*(S) \rightarrow C^*(S \oplus S)$ given by $\psi(A) = A \oplus A$ is a C^* -isomorphism. As the action of a $*$ -homomorphism is determined by its value on a generator, we have that $\varphi = \psi$. That is, for $A \in C^*(S)$, $\lim_n U_n^* A U_n = A \oplus A$; i.e., $A \sim_{sa} A \oplus A$. Now if there is a non-zero compact operator K in $C^*(S)$, then $|K|$ is a non-zero compact positive operator in $C^*(S)$. Hence we must have that $|K| \sim_{sa} |K| \oplus |K|$. But this cannot happen because the multiplicity of the non-zero eigenvalues of $|K|$ is different from that of $|K| \oplus |K|$ and this multiplicity is preserved under approximate equivalence (see Theorem I.1.12). So $C^*(S)$ contains no non-zero compact operators. $\square$

Hence if $C^*(S)$ contains no compact operators, then, in some generalized sense, see Theorem 2.6, S has uniform infinite multiplicity. We now use the above results to determine exactly when quasinormal operators have isomorphic C^* -algebras.

Proposition 2.7 If S is any operator and T_1, T_2 are operators with $C^*(T_1) \approx C^*(T_2)$, then $C^*(S \otimes T_1) \approx C^*(S \otimes T_2)$.

Proof. It follows from Corollary 2.4 that $T_1^{(\infty)}$ is approximately unitarily equivalent to $T_2^{(\infty)}$. That is, there is a sequence of unitary operators U_n such that $U_n^* T_1^{(\infty)} U_n$ converges in norm to $T_2^{(\infty)}$. Now, set $S_1 = S \otimes T_1$ and $S_2 = S \otimes T_2$. Notice that $S_1^{(\infty)} = S \otimes T_1^{(\infty)}$ and $S_2^{(\infty)} = S \otimes T_2^{(\infty)}$. If we set $V_n = I \otimes U_n$, then $V_n^* S_1^{(\infty)} V_n = S \otimes U_n^* T_1^{(\infty)} U_n$. Since

$\|A \otimes B\| = \|A\| \|B\|$ for all operators A and B , $V_n^* S_1^{(\infty)} V_n$ converges in norm to $S \otimes T_2^{(\infty)} = S_2^{(\infty)}$. Thus $S_1^{(\infty)}$ is approximately equivalent to $S_2^{(\infty)}$. Hence by Theorem 2.1, $S_1^{(\infty)}$ and $S_2^{(\infty)}$ have isomorphic C^* -algebras. Thus S_1 and S_2 will also. $\square$

Corollary 2.8 If S is any operator and N, M are normal operators with $\sigma(N) = \sigma(M)$, then $C^*(S \otimes N) \approx C^*(S \otimes M)$.

Observe that Corollary 2.8 can also be proven using a version of the Weyl - von Neumann - Berg Theorem. In particular Voiculescu's Theorem is not necessary here. We now specialize Corollary 2.8 to quasinormal operators.

Corollary 2.9 If U is the unilateral shift and A, B are positive one-to-one operators, then $C^*(U \otimes A) \approx C^*(U \otimes B)$ if and only if $\sigma(A) = \sigma(B)$.

Proof. Set $S = U \otimes A$ and $T = U \otimes B$. Note that Corollary 2.8 provides one direction. So if $\phi: C^*(S) \rightarrow C^*(T)$ is a $*$ -isomorphism, then $\phi(S^*S) = T^*T$. Hence $\sigma(S^*S) = \sigma(T^*T)$. But since U is an isometry, $S^*S = I \otimes A^2$ and $T^*T = I \otimes B^2$. Now, since $\sigma(C \otimes D) = \sigma(C)\sigma(D)$ for any operators C and D , and A, B are positive it follows that $\sigma(A) = \sigma(B)$. $\square$

Recall that operators of the form $U \otimes A$ as in Corollary 2.9 are all subnormal operators (see Theorem I.1.16). Thus, Corollary 2.9 makes it possible to construct subnormal operators with isomorphic C^* -algebras having various properties. See Example 3.12.

3. The Normal Spectrum and The Spectral Measure

If S is a subnormal operator on $\mathcal{H}$ with minimal normal extension (mne) N , then the *normal spectrum* of S , $\sigma_n(S)$, is defined to be $\sigma(N)$. We now show that if two subnormal operators have isomorphic C^* -algebras, then they have the same normal spectrum. Recall, that if $f \in C(\sigma_n(S))$, then the Toeplitz operator with symbol f will be denoted by S_f and is defined as the compression of $f(N)$ to the subspace $\mathcal{H}$.

Theorem 3.1 If S and T are subnormal operators and $\varphi: C^*(S) \rightarrow C^*(T)$ is a positive linear map such that $\varphi(S) = T$ and $\varphi(S^*S) = T^*T$, then $\sigma_n(T) \subseteq \sigma_n(S)$.

Proof. If T acts on the Hilbert space $\mathcal{H}$, then we define $\Phi: C(\sigma_n(S)) \rightarrow C^*(T)$ by $\Phi(f) = \varphi(S_f)$. Since Φ is a positive linear map on $C(\sigma_n(S))$, Theorem I.1.2 says that Φ is completely positive. Hence by Theorem I.1.3 we can dilate Φ . So there exists a larger Hilbert space $\mathcal{K}$ which contains $\mathcal{H}$ and a $*$ -homomorphism $\Psi: C(\sigma_n(S)) \rightarrow \mathcal{B}(\mathcal{K})$ so that $\Phi(f) = P\Psi(f)|_{\mathcal{H}}$ for each f in $C(\sigma_n(S))$, where P is the projection of $\mathcal{K}$ onto $\mathcal{H}$. Now let $N = \Psi(z)$. So N is a normal operator on $\mathcal{K}$ and clearly $\sigma(N) \subseteq \sigma_n(S)$. Also $T = \Phi(z) = P\Psi(z)|_{\mathcal{H}} = PN|_{\mathcal{H}}$. We claim that N is a normal extension of T . Once we show this to be true, then we will have that $\sigma_n(T) \subseteq \sigma(N) \subseteq \sigma_n(S)$.

To show that N is an extension of T , write $N = \Psi(z) = \begin{bmatrix} T & * \\ A & * \end{bmatrix}$, with respect to the

decomposition $\mathcal{H} \oplus \mathcal{H}^\perp$ of $\mathcal{K}$. Since Ψ is a $*$ -homomorphism, $\Psi(\bar{z}z) = \Psi(z)^*\Psi(z)$. But

since $\varphi(S^*S) = T^*T$, we have $\Psi(\bar{z}z) = \begin{bmatrix} T^*T & * \\ * & * \end{bmatrix}$ and $\Psi(z)^*\Psi(z) = \begin{bmatrix} T^*T + A^*A & * \\ * & * \end{bmatrix}$.

So we see that $A^*A = 0$. Thus $A = 0$ and N is in fact a normal extension of T . $\square$

Corollary 3.2 If S and T are subnormal operators and $\varphi:C^*(S) \rightarrow C^*(T)$ is a $*$ -homomorphism such that $\varphi(S) = T$, then $\sigma_n(T) \subseteq \sigma_n(S)$.

Corollary 3.3 If S and T are subnormal operators and $C^*(S) \approx C^*(T)$, then $\sigma_n(S) = \sigma_n(T)$.

Corollary 3.4 If S and T are subnormal operators and $\varphi:C^*(S) \rightarrow C^*(T)$ is a $*$ -homomorphism such that $\varphi(S) = T$, then for any $f \in C(\sigma_n(S))$, S_f and T_f are well defined and $\varphi(S_f) = T_f$.

Proof. If $\varphi:C^*(S) \rightarrow C^*(T)$ is a $*$ -homomorphism such that $\varphi(S) = T$ and $f \in C(\sigma_n(S))$, then S_f is well defined and belongs to $C^*(S)$. Further by Corollary 3.2, $\sigma_n(T) \subseteq \sigma_n(S)$, so $f \in C(\sigma_n(T))$ and, thus, T_f is also well defined and belongs to $C^*(T)$. Now, if p is a polynomial in z and $\bar{z}$, then $\varphi(S_p) = T_p$. In general, for $f \in C(\sigma_n(S))$, if polynomials p_n in z and $\bar{z}$ approximate f uniformly on $\sigma_n(S)$, then they also approximate f uniformly on $\sigma_n(T)$. Now, since $\|S_f - S_{p_n}\| \leq \|f - p_n\|_{\sigma_n(S)}$ it follows that $S_{p_n} \rightarrow S_f$, and likewise for T . It follows that $\varphi(S_f) = \lim \varphi(S_{p_n}) = \lim T_{p_n} = T_f$. $\square$

It's now shown that every $*$ -homomorphism between the C^* -algebras generated by subnormal operators "lifts" to a $*$ -homomorphism between their normal extensions. For any subnormal operator S with $N = mneS$, there is a *natural map* $\theta_S:C^*(N) \rightarrow C^*(S)$

given by $\theta_S(f(N)) = S_f$. Observe that θ_S is not a homomorphism, but is a positive linear map.

Corollary 3.5 If S and T are subnormal operators with $\text{mne}S = N$ and $\text{mne}T = M$ and there is a $*$ -homomorphism $\phi:C^*(S) \rightarrow C^*(T)$ with $\phi(S) = T$, then there is a $*$ -homomorphism $\psi:C^*(N) \rightarrow C^*(M)$ with $\psi(N) = M$ and such that the following diagram commutes:

$$\begin{array}{ccc} C^*(N) & \xrightarrow{\psi} & C^*(M) \\ \downarrow \theta_S & & \downarrow \theta_T \\ C^*(S) & \xrightarrow{\phi} & C^*(T) \end{array}$$

Proof. It follows from Corollary 3.2 that $\sigma(M) = \sigma_n(T) \subseteq \sigma_n(S) = \sigma(N)$, thus there is a natural $*$ -homomorphism $\psi:C^*(N) \rightarrow C^*(M)$ defined as “restriction”. That is, $\psi(f(N)) = f(M)$ for every $f \in C(\sigma(N))$. Observe that since $\sigma(M) \subseteq \sigma(N)$, $f(M)$ is well defined and thus ψ is also. The result now follows immediately from Corollary 3.4. $\square$

We now want to consider when $*$ -homomorphisms are weak*-continuous. Let’s say that a $*$ -homomorphism $\phi:W^*(S) \rightarrow W^*(T)$ is a W^* -homomorphism if ϕ is weak* - continuous. Equivalently, $\phi:(W^*(S), \text{wk}^*) \rightarrow (W^*(T), \text{WOT})$ is sequentially continuous.

We now use Corollary 3.4 to show that if $\phi:C^*(S) \rightarrow C^*(T)$ is a $*$ -homomorphism with $\phi(S) = T$ and a weak*-continuous extension of ϕ to the von Neumann algebra generated by S exists, then the scalar valued spectral measures (svsm) behave as you would expect.

Theorem 3.6 If S and T are subnormal operators, $\mu = \text{svsm}S$, $\nu = \text{svsm}T$ and there is a W^* -homomorphism $\varphi: W^*(S) \rightarrow W^*(T)$ with $\varphi(S) = T$, then ν is absolutely continuous with respect to μ .

Lemma 3.7 Let X be a compact metric space and denote by χ_A the characteristic function of a set A in X . If K is a compact set in X , then there exists a sequence of continuous functions $\{f_n\}$ on X with $0 \leq f_n \leq 1$ and $f_n \rightarrow \chi_K$ pointwise everywhere on X .

Proof. If $U_n = \{x : \text{dist}(x, K) < 1/n\}$, then U_n is a sequence of open sets whose intersection equals K . Now Urysohn's Lemma gives the existence of continuous functions f_n on X with $0 \leq f_n \leq 1$, $f_n = 1$ on K and $f_n = 0$ off of U_n . This is the required sequence. $\square$

Lemma 3.8 If T is a subnormal operator with $\text{svsm}T = \mu$ and $f \in L^\infty(\mu)$, then the Toeplitz operator $T_f = 0$ if and only if $f = 0$ μ -almost everywhere.

Proof. If $N = \text{mne}T$ acts on $\mathcal{K}$, T acts on $\mathcal{H}$ and $T_f = 0$, then $f(N)\mathcal{H} \subseteq \mathcal{H}^\perp$. But also for any integer $k \geq 0$ and $h \in \mathcal{H}$ we have $f(N)N^{*k}h = N^{*k}f(N)h \in N^{*k}\mathcal{H}^\perp \subseteq \mathcal{H}^\perp$. So as $\{N^{*k}h : k \geq 0 \text{ and } h \in \mathcal{H}\}$ has dense linear span in $\mathcal{K}$, we see that $f(N)\mathcal{K} \subseteq \mathcal{H}^\perp$. Thus we

get that $f(N) = \begin{bmatrix} T_f & B \\ A & C \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ A & C \end{bmatrix}$. Since $f(N)$ is normal we have that $f(N)f(N)^* =$

$f(N)^*f(N)$, i.e., $\begin{bmatrix} 0 & 0 \\ A & C \end{bmatrix} \begin{bmatrix} 0 & A^* \\ 0 & C^* \end{bmatrix} = \begin{bmatrix} 0 & A^* \\ 0 & C^* \end{bmatrix} \begin{bmatrix} 0 & 0 \\ A & C \end{bmatrix}$. So by considering the (1,1) entry

of both sides we see that $A^*A = 0$. So, $A = 0$. Thus, $f(N)\mathcal{H} = (0)$. Hence repeating the argument above we see that in fact $f(N)\mathcal{K} = (0)$. So $f(N) = 0$ and this is equivalent to $f = 0$ a.e. $[\mu]$. $\square$

Proof of Theorem 3.6. Assume $\varphi: W^*(S) \rightarrow W^*(T)$ is a W^* -homomorphism with $\varphi(S) = T$. One easily checks that the restriction of φ to $C^*(S)$ gives a $*$ -homomorphism of $C^*(S)$ onto $C^*(T)$. So Corollary 3.2 gives us that $\sigma_n(T) \subseteq \sigma_n(S)$. Now if K is any compact subset of $\sigma_n(T)$, then Lemma 3.7 gives continuous functions $\{f_n\}$ on $\sigma_n(S)$ so that $0 \leq f_n \leq 1$ and $f_n \rightarrow \chi_K$ pointwise everywhere on $\sigma_n(S)$. Hence $f_n \rightarrow \chi_K$ weak* in both $L^\infty(\mu)$ and $L^\infty(\nu)$. So $S_{f_n} \rightarrow S_{\chi_K}$ (wk*) and $T_{f_n} \rightarrow T_{\chi_K}$ (wk*). In view of Corollary 3.4, $\varphi(S_{\chi_K}) = \lim \varphi(S_{f_n}) = \lim T_{f_n} = T_{\chi_K}$.

Thus for any compact set $K \subseteq \sigma_n(T)$ we have $\varphi(S_{\chi_K}) = T_{\chi_K}$. Now if $K \subseteq \sigma_n(T)$ is any compact set with $\nu(K) > 0$, then $T_{\chi_K} \neq 0$ by Lemma 3.8. Because $\varphi(S_{\chi_K}) = T_{\chi_K}$ we must have $S_{\chi_K} \neq 0$. Thus $\mu(K) > 0$. Now using the regularity of the measures we easily see that $\nu \ll \mu$. $\square$

Corollary 3.9 If S and T are subnormal operators, $\mu = \text{svsm}S$, $\nu = \text{svsm}T$ and there exists a W^* -isomorphism $\varphi: W^*(S) \rightarrow W^*(T)$ with $\varphi(S) = T$, then $\mu \approx \nu$.

Corollary 3.10 If S and T are subnormal operators, $\mu = \text{svsm}S$ and $\varphi: W^*(S) \rightarrow W^*(T)$ is a W^* -homomorphism, then for $f \in L^\infty(\mu)$ both S_f and T_f are well defined and $\varphi(S_f) = T_f$.

If S is any subnormal operator on $\mathcal{H}$ with $\text{mne}S = N$ and $\text{svsm}N = \mu$, then the restriction algebra $\mathcal{R}(S, \mu)$ consists of all those functions $f \in L^\infty(\mu)$ for which $f(N)$ leaves $\mathcal{H}$ invariant.

The following result is interesting.

Corollary 3.11 If S and T are subnormal operators and $\varphi: W^*(S) \rightarrow W^*(T)$ is a W^* -isomorphism with $\varphi(S) = T$, then $\mathcal{R}(S, \mu) = \mathcal{R}(T, \nu)$.

Proof. Since φ is a W^* -isomorphism, Corollary 3.9 gives that $\mu \approx \nu$. Hence $L^\infty(\mu) = L^\infty(\nu)$. Now if $f \in L^\infty(\mu)$, then using a matrices argument one can show that $f \in \mathcal{R}(S, \mu)$ if and only if $S_{|f|^2} = S_f S_f^*$. From Corollary 3.10 we see that $S_{|f|^2} = S_f S_f^*$ holds if and only if $T_{|f|^2} = T_f T_f^*$ holds. Thus $f \in \mathcal{R}(S, \mu)$ if and only if $f \in \mathcal{R}(T, \nu)$. $\square$

If we assume that $\varphi: C^*(S) \rightarrow C^*(T)$ is a $*$ -homomorphism, then we can still conclude as in Corollary 3.4 that $\mathcal{R}(S, \mu) \cap C(\sigma_n(S)) \subseteq \mathcal{R}(T, \nu) \cap C(\sigma_n(T))$. Thus the continuous elements in the respective restriction algebras are exactly the same when $C^*(S) \approx C^*(T)$.

It is natural to ask whether every $*$ -homomorphism $\varphi: C^*(S) \rightarrow C^*(T)$ with $\varphi(S) = T$ extends to a W^* -homomorphism $\tilde{\varphi}: W^*(S) \rightarrow W^*(T)$. It will be shown latter that this is true when S and T are pure and essentially normal. If this were always true for pure subnormal operators it would be a powerful tool. But as we shall show, and as you may expect, it is not always true. It would be interesting to know exactly when it is true!

Example 3.12 There exist pure subnormal operators S, T and a $*$ -isomorphism $\varphi: C^*(S) \rightarrow C^*(T)$ with $\varphi(S) = T$ such that φ does not extend to a W^* -homomorphism from $W^*(S)$ into $W^*(T)$. In fact, $\text{svsm}S \perp \text{svsm}T$.

Proof. Let $A = M_l$ on $L^2(\nu)$ and let $B = M_l$ on $L^2(\lambda)$ where $\nu(0) = \lambda(0) = 0$, $\text{supp}(\nu) = \text{supp}(\lambda) \subseteq [0, \infty)$ but $\nu \perp \lambda$. For example, we could choose ν = Lebesgue measure on the interval $(0,1)$ and λ = a weighted sum of point masses at the rational points of $(0,1)$. Now, if $S = U \otimes A$ and $T = U \otimes B$, where U is the unilateral shift, then S and T are pure subnormal operators. By Corollary 2.9 there exists a $*$ -isomorphism $\varphi: C^*(S) \rightarrow C^*(T)$ with $\varphi(S) = T$. However it is easy to check (see Conway[1991], p. 45) that $\text{svsm}S = d\theta d\nu$ is mutually singular with $\text{svsm}T = d\theta d\lambda$. Thus, in view of Theorem 3.6, this $*$ -isomorphism φ cannot extend to be a W^* -homomorphism of $W^*(S)$ into $W^*(T)$. $\square$

Notice that by choosing both ν and λ as weighted sums of point masses we may take both S and T in Example 3.12 as direct sums of irreducible subnormal operators.

Example 3.13 There exist pure subnormal operators S and T such that $C^*(S) \approx C^*(T)$ and $\text{svsm}S \approx \text{svsm}T$ and yet $W^*(S)$ is not W^* -isomorphic to $W^*(T)$.

Proof. The operators S and T will be chosen to be operator valued weighted shifts. See the remarks before Proposition 1.18. The operator S will be determined by a pair (A,B) and T will be determined by the pair (C,D) . Choose $B = D$ = identity on an infinite dimensional space. Let A and C be diagonalizable operators such that A and C have no

common eigenvalues, yet $\sigma(A) = \sigma(B) = [0, \frac{1}{2}]$. By Theorem I.1.12 the pair (A,B) is approximately equivalent to the pair (C,D) . Thus a matrix argument gives that S is approximately equivalent to T . So, Theorem 2.1 implies that $C^*(S) \approx C^*(T)$. Since the pair (A,B) is diagonalizable with respect to a single basis, the operator S splits as the direct sum of weighted shifts. These weighted shifts all have spectral measures equivalent to arc length measure on $\partial\mathbf{D}$ plus a point mass at 0. So, $\text{svsm}S \approx d\theta + \delta_0$. Likewise for T . So, $\text{svsm}S \approx \text{svsm}T$. However, if $W^*(S) \approx W^*(T)$, then $W^*([S^*,S]) \approx W^*([T^*,T])$. But $[S^*,S] = A^2 \oplus (I - A^2) \oplus 0$ and $[T^*,T] = B^2 \oplus (I - B^2) \oplus 0$ have singular spectral measures. $\square$

Another place one may try to find operators as in Example 3.13 is among the bundle shifts. Let R be a finitely connected region bounded by a $(n+1)$ of smooth Jordan curves and $\varphi:\mathbf{D} \rightarrow R$ a analytic covering map. Fix n generators $(g_1, \dots, g_n)$ of the group of covering transformations of φ . If K is a Hilbert space and $(U_1, \dots, U_n)$ is an n -tuple of unitaries on K , then define $M = \{ f \in H^2_K(\mathbf{D}) : f(g_i(z)) = U_i f(z) \text{ for } i = 1, \dots, n \text{ and } z \in \mathbf{D} \}$; where $H^2_K(\mathbf{D})$ is the vector valued Hardy space. We may define a subnormal operator S (called a bundle shift over R) as multiplication by φ acting on the subspace M . Hence with φ and $(g_1, \dots, g_n)$ fixed, S only depends on the n -tuple of unitaries. For more on bundle shifts, including the following result, see Abrahamse and Douglas [1976].

Theorem 3.14 If S is a bundle shift over R determined by $(U_1, \dots, U_n)$, then $W^*(S)$ is unitarily equivalent to $W^*(U_1, \dots, U_n) \otimes B(H^2(R))$.

Thus it may be possible to use bundle shifts to construct more examples as in Example 3.12 and 3.13.

4. The Essential Normal Spectrum

In this section we will begin to study the C^* -algebra generated by a subnormal operator S in the Calkin algebra. We shall denote by $\pi: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}/\mathcal{B}_0$ the natural quotient map into the Calkin algebra. The basic technique will be as follows: If S is a subnormal operator, then consider a *faithful* representation ρ of $C^*(\pi(S))$. If $T = \rho(\pi(S))$, then T is also a subnormal operator and we have the following *Fundamental Diagram*:

$$\begin{array}{ccc} C^*(S) & & \\ \downarrow \pi & & \\ C^*(\pi(S)) & \xrightarrow[\rho]{} & C^*(T) \end{array}$$

Thus $\phi = \rho \circ \pi: C^*(S) \rightarrow C^*(T)$ is a $*$ -homomorphism with $\phi(S) = T$ whose kernel is exactly $C^*(S) \cap \mathcal{B}_0$. In trying to understand $C^*(\pi(S))$ we shall often choose a faithful representation ρ (always onto a separable Hilbert space) as above and try to identify the resulting operator T or properties of T that are independent of ρ . In general, if T has a certain property P , then we say that S has property essentially P .

In this section, the above notation and Fundamental Diagram will remain fixed, although notice that ρ is not unique. Now, if S is any subnormal operator, then the operator T in the Fundamental Diagram is also subnormal; see Corollary 1.1.4. It follows from Corollary 3.3 that the normal spectrum of T is well defined independent of the *faithful* representation ρ . Hence we define the *essential normal spectrum* of S , denoted by $\sigma_n(\pi(S))$, to be the normal spectrum of T . This new subset of $\sigma(S)$ will be the topic of investigation for the remainder of this section.

Lemma 4.1 If K, L are any two compact sets in the complex plane $\mathbb{C}$ with $\partial L \subseteq K \subseteq L$, then L equals K union some bounded components of $\mathbb{C} - K$.

Proof. If U is any bounded component of $\mathbb{C} - K$, then consider $U \cap L$ and $U - L$. So, $U = [U \cap L] \cup [U - L]$. Now, since $\partial L \subseteq K \subseteq \mathbb{C} - U$, $L \cap U = \text{int} L \cap U$ is an open set and clearly $U - L$ is an open set. Thus U is the union of two disjoint open sets. Since U is connected one set is empty and the other equals U . Hence, either $U \subseteq L$ or $U \cap L = \emptyset$. Thus the Lemma follows. $\square$

Recall that for a subnormal operator S with $\mu = \text{svsm} S$, $\mathcal{R}(S, \mu)$ denotes the restriction algebra for S . See the remarks before Corollary 3.11.

Proposition 4.2 If S is a subnormal operator and $\mu = \text{svsm} S$, then:

- (a) $\sigma_{le}(S) \subseteq \sigma_n(\pi(S)) \subseteq \sigma_e(S) \cap \sigma_n(S)$,
- (b) $\sigma_e(S) = \sigma_n(\pi(S)) \cup \{\text{some bounded components of } \mathbb{C} - \sigma_n(\pi(S))\}$,
- (c) If $f \in C(\sigma_n(S)) \cap \mathcal{R}(S, \mu)$, then $\sigma_n(\pi(f(S))) = f(\sigma_n(\pi(S)))$.

Proof. (a) Using the Fundamental Diagram, we see that $\sigma_{le}(S) = \sigma_l(T) = \sigma_{ap}(T) \subseteq \sigma_n(T) = \sigma_n(\pi(S))$. The fact that $\sigma_n(\pi(S)) \subseteq \sigma_e(S)$ follows from a result of Halmos saying that $\sigma_n(T) \subseteq \sigma(T)$. Finally, the fact that $\sigma_n(\pi(S)) \subseteq \sigma_n(S)$ follows by applying Corollary 3.2 to the Fundamental Diagram.

(b) Since, $\sigma_n(\pi(S)) \subseteq \sigma_e(S)$ by part (a) and $\partial \sigma_e(S) = \partial \sigma(T) \subseteq \sigma_{ap}(T) \subseteq \sigma_n(T) = \sigma_n(\pi(S))$, the result follows immediately from Lemma 4.1.

(c) If $f \in C(\sigma_n(S)) \cap \mathcal{R}(S, \mu)$ and $\varphi: C^*(S) \rightarrow C^*(T)$ is the $*$ -homomorphism in the Fundamental Diagram, then $f(S) = S_f \in C^*(S)$ and $\varphi(S_f) = T_f$ by Corollary 3.4. Also from the remarks after Corollary 3.11 we see that $f \in \mathcal{R}(T, \text{svsm}T) \cap C(\sigma_n(T))$, so $\sigma_n(\pi(f(S))) = \sigma_n(f(T)) = f(\sigma_n(T)) = f(\sigma_n(\pi(S)))$. See Conway [1991], page 42. $\square$

Corollary 4.3 (a) If S is subnormal with compact self-commutator, then $\sigma_n(\pi(S)) = \sigma_e(S)$.
(b) If S is subnormal and $\sigma_e(S)$ has no interior, then $\sigma_n(\pi(S)) = \sigma_e(S)$.

Proof. (a) If S has compact self-commutator, then in the Fundamental Diagram, T is normal. Thus, $\sigma_n(\pi(S)) = \sigma_n(T) = \sigma(T) = \sigma_e(S)$. (b) See Proposition 4.2(b). $\square$

Example 4.4 If S is any subnormal operator, then $\sigma_n(\pi(S^{(\infty)})) = \sigma_n(S)$.

Proof. Consider the Fundamental Diagram for $S^{(\infty)}$. Since $C^*(S^{(\infty)})$ contains no compact operators, π is one-to-one, hence we may choose ρ to be π^{-1} . Thus T may be chosen to be $S^{(\infty)}$ (note T may also be chosen to be S). Thus $\sigma_n(\pi(S^{(\infty)})) = \sigma_n(T) = \sigma_n(S^{(\infty)}) = \sigma_n(S)$. $\square$

We now give an example of a subnormal operator showing that the essential normal spectrum can be quite arbitrary, subject only to the conditions of Proposition 4.2.

Example 4.5 If K_1 and K_2 are two compact subsets of $\text{cl}\mathbf{D}$ with $\partial\mathbf{D} \subseteq K_1 \subseteq K_2$ then there is a pure subnormal operator S such that $\sigma(S) = \sigma_e(S) = \text{cl}\mathbf{D}$, $\sigma_{\text{ap}}(S) = \partial\mathbf{D}$, $\sigma_n(\pi(S)) = K_1$ and $\sigma_n(S) = K_2$.

Before giving this example, we shall require another useful result.

Proposition 4.6 If S_i , $i = 1, 2$ are two operators and κ_i is a faithful representation of $C^*(\pi(S_i))$, then there is a faithful representation κ of $C^*(\pi(S_1 \oplus S_2))$ so that $\kappa(\pi(S_1 \oplus S_2)) = \kappa_1(\pi(S_1)) \oplus \kappa_2(\pi(S_2))$.

Proof. Simply consider the following diagram:

$$\begin{array}{ccc} \mathcal{B}(\mathcal{H}_1) \oplus \mathcal{B}(\mathcal{H}_2) & \longrightarrow & \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2) \\ \downarrow & & \downarrow \\ \mathcal{B}(\mathcal{H}_1) / \mathcal{B}_0(\mathcal{H}_1) \oplus \mathcal{B}(\mathcal{H}_2) / \mathcal{B}_0(\mathcal{H}_2) & \xrightarrow{\varphi} & \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2) / \mathcal{B}_0(\mathcal{H}_1 \oplus \mathcal{H}_2) \end{array}$$

One easily checks that φ is a C^* -monomorphism into $\mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2) / \mathcal{B}_0(\mathcal{H}_1 \oplus \mathcal{H}_2)$ that sends $\pi(S_1) \oplus \pi(S_2)$ to $\pi(S_1 \oplus S_2)$. So φ may be restricted to be a C^* -isomorphism of $C^*(\pi(S_1) \oplus \pi(S_2))$ onto $C^*(\pi(S_1 \oplus S_2))$. Hence if $\kappa := (\kappa_1 \oplus \kappa_2) \circ \varphi^{-1}$, then κ is the required faithful representation. Notice here that $C^*(\pi(S_1) \oplus \pi(S_2)) \subseteq C^*(\pi(S_1)) \oplus C^*(\pi(S_2))$ and $(\kappa_1 \oplus \kappa_2)$ is defined on this larger algebra. Hence κ is well defined. $\square$

Corollary 4.7 If S, T are subnormal operators, then $\sigma_n(\pi(S \oplus T)) = \sigma_n(\pi(S)) \cup \sigma_n(\pi(T))$.

Proof of Example 4.5. Let $S_\mu = M_z$ on $P^2(\mu)$ and $S_\nu = M_z$ on $P^2(\nu)$ where μ is arc length measure on the circle plus a weighted sum of point masses that form a dense set in K_1 , and, likewise, ν is arc length measure on the circle plus a weighted sum of point masses that form a dense set in K_2 . Now let $S = S_\mu^{(\infty)} \oplus S_\nu$. Notice that Proposition 4.6 and Example 4.4 give that there exists a faithful representation ρ of $C^*(\pi(S))$ so that $\rho(\pi(S)) = S_\mu \oplus N$ where N is any normal operator with $\sigma(N) = \partial\mathbf{D}$. Thus, $\sigma_n(\pi(S)) = \sigma_n(S_\mu \oplus N) = K_1$. The other spectral properties follow immediately. $\square$

The following immediate results will be useful.

Proposition 4.8 If S and T are operators and $S \equiv T \pmod{\mathcal{B}_0}$, then $C^*(\pi(S)) \approx C^*(\pi(T))$.

Corollary 4.9 If S and T are subnormal and $S \equiv T \pmod{\mathcal{B}_0}$, then $\sigma_n(\pi(S)) = \sigma_n(\pi(T))$.

If S is any subnormal operator and $f \in C(\sigma_n(S))$, then Olin and Thomson [1982] have shown that the Toeplitz operator S_f is in the commutator ideal of $C^*(S)$, $\text{Comm}(S)$, if and only if $f = 0$ on $\sigma_{ap}(S)$. We now show exactly when S_f is compact. This also gives a characterization of the essential normal spectrum of a subnormal operator that can be used without any reference to the Calkin algebra. This will be useful in proving certain properties for the essential normal spectrum.

Theorem 4.10 If S is a subnormal operator and $f \in C(\sigma_n(S))$, then

- (a) S_f is compact if and only if $f = 0$ on $\sigma_n(\pi(S))$,
- (b) $\|S_f\|_e \leq \|f\|_{\sigma_n(\pi(S))}$.

Proof. (a) If $\varphi: C^*(S) \rightarrow C^*(T)$ is the $*$ -homomorphism in the Fundamental Diagram, then, since $S_f \in C^*(S)$ and $\ker(\varphi) = C^*(S) \cap \mathcal{B}_0$, we see that S_f is compact if and only if $\varphi(S_f) = 0$. But Corollary 3.4, says that $\varphi(S_f) = T_f$ and Lemma 3.8 says that $T_f = 0$ if and only if $f = 0$ on $\sigma_n(T)$. Thus we see that S_f is compact if and only if $f = 0$ on $\sigma_n(\pi(S))$.

(b) Since $\varphi(S_f) = T_f$, we have $\|S_f\|_e = \|T_f\| \leq \|f\|_{\sigma_n(T)} = \|f\|_{\sigma_n(\pi(S))}$. $\square$

Thus we see that there is a non-zero compact Toeplitz operator with a continuous symbol if and only if $\sigma_n(\pi(S)) \neq \sigma_n(S)$. In more familiar terms, Proposition 4.2(a) implies that if $\sigma_n(S) \subsetneq \sigma_e(S)$, then $\sigma_n(\pi(S)) \neq \sigma_n(S)$. Hence there will exist a non-zero compact Toeplitz operator with continuous symbol. However, even if $\sigma_n(S) \subseteq \sigma_e(S)$, there may exist compact Toeplitz operators with continuous symbols; see Example 4.5.

We shall now use Theorem 4.10(a) to prove some nice regularity properties for the essential normal spectrum of a subnormal operator. Notice that Corollary 4.7 and Corollary 4.9 follow almost immediately from Theorem 4.10(a) without any mention of the Calkin algebra. The following theorem will be important.

Theorem 4.11 If S is a subnormal operator and $\mathcal{M} \in \text{Lat} S$, then $\sigma_n(\pi(S|_{\mathcal{M}})) \subseteq \sigma_n(\pi(S))$.

Lemma 4.12 Let S be a subnormal operator on $\mathcal{H}$, $\mathcal{M} \in \text{Lat} S$ and $T = S|_{\mathcal{M}}$. If $f \in C(\sigma_n(T))$ and $\tilde{f}$ is any continuous extension of f to $\sigma_n(S)$, then $T_f = P_{\mathcal{M}} S_{\tilde{f}}|_{\mathcal{M}}$, where $P_{\mathcal{M}}$ is the projection of $\mathcal{H}$ onto $\mathcal{M}$.

Proof. Let $N = mneS$ on $\mathcal{K}$ and $M = mneT$ on $\mathcal{K}_1 \subseteq \mathcal{K}$. Note that $M = N|_{\mathcal{K}_1}$. If $f \in C(\sigma_n(T))$ and $\tilde{f}$ is any continuous extension of f to $\sigma_n(S)$, then clearly $\tilde{f}(N)|_{\mathcal{K}_1} = f(M)$ (approximate $\tilde{f}$ by polynomials). Now if P is the projection of $\mathcal{K}_1$ onto $\mathcal{M}$, then by definition of T_f we get $T_f = Pf(M)|_{\mathcal{M}} = P\tilde{f}(N)|_{\mathcal{M}} = P_{\mathcal{M}}(P_{\mathcal{H}}\tilde{f}(N)|_{\mathcal{H}})|_{\mathcal{M}} = P_{\mathcal{M}}S_{\tilde{f}}|_{\mathcal{M}}$, where $P_{\mathcal{H}}$ is the projection of $\mathcal{K}$ onto $\mathcal{H}$. $\square$

Proof of Theorem 4.11. If $T = S|_{\mathcal{M}}$, then Proposition 4.2(a) implies that $\sigma_n(\pi(T)) \subseteq \sigma_n(T) \subseteq \sigma_n(S)$. If there is a point p in $\sigma_n(\pi(T))$ which is not in $\sigma_n(\pi(S))$, then p is in $\sigma_n(S)$ and not in $\sigma_n(\pi(S))$. If $f \in C(\sigma_n(S))$ with $f = 0$ on $\sigma_n(\pi(S))$ but $f(p) \neq 0$, then Theorem 4.10(a) says that S_f is compact, but not zero (see Lemma 3.8). Now f is not identically zero on $\sigma_n(T)$ because $p \in \sigma_n(T)$, so T_f is not zero. Also by Lemma 4.12, T_f is the compression of S_f ; hence T_f is compact. So, by Theorem 4.10(a), $f = 0$ on $\sigma_n(\pi(T))$. But this contradicts the fact that p is in $\sigma_n(\pi(T))$ and $f(p) \neq 0$. Hence no such point p exists. Thus $\sigma_n(\pi(T)) \subseteq \sigma_n(\pi(S))$. $\square$

If S is a pure subnormal operator on $\mathcal{H}$ with $N = mneS$, then $T = N^*|_{\mathcal{H}^\perp}$ is called the *dual* of S . We now want to show a natural relation between $\sigma_n(\pi(S))$ and $\sigma_n(\pi(T))$. The relation is well known when S has compact self-commutator, in which case the essential normal spectrum is simply the essential spectrum. In what follows, if K is a compact set in $\mathbb{C}$, then $K^* = \{\bar{z} : z \in K\}$.

Theorem 4.13 If S is a pure subnormal operator, $N = mneS$ and $T = \text{dual}S$, then $\sigma_e(N) = \sigma_n(\pi(S)) \cup \sigma_n(\pi(T))^*$.

Proof. We shall use Proposition 4.2 and Theorem 4.10 repeatedly. If $f \in C(\sigma_n(S))$ and $f = 0$ on $\sigma_e(N)$, then $f(N)$ is compact. Hence the Toeplitz operator S_f is also compact, so $\sigma_n(\pi(S)) \subseteq \sigma_e(mneS)$. Likewise $\sigma_n(\pi(T)) \subseteq \sigma_e(mneT)$. Since S is pure, $mneT = N^*$ and $\sigma_e(N^*) = \sigma_e(N)^*$. Thus we have $\sigma_n(\pi(S)) \cup \sigma_n(\pi(T))^* \subseteq \sigma_e(N)$. For the other inclusion

we will show that if $f \in C(\sigma(N))$ and $f = 0$ on $\sigma_n(\pi(S)) \cup \sigma_n(\pi(T))^*$, then $f(N)$ is compact.

Hence $f = 0$ on $\sigma_e(N)$ and by varying f one may see that $\sigma_n(\pi(S)) \cup \sigma_n(\pi(T))^* \supseteq \sigma_e(N)$.

So, suppose $f \in C(\sigma(N))$ and $f = 0$ on $\sigma_n(\pi(S)) \cup \sigma_n(\pi(T))^*$. Now, $f(N) = \begin{bmatrix} S_f & H_f^* \\ H_f & T_{f^\#}^* \end{bmatrix}$

where S_f is the Toeplitz operator, H_f is the Hankel operator and $T_{f^\#}$ is the Toeplitz operator for T with symbol $f^\#(z) = \overline{f(\bar{z})}$. As $f = 0$ on $\sigma_n(\pi(S)) \cup \sigma_n(\pi(T))^*$, Theorem 4.10 gives that S_f and $T_{f^\#}$ are both compact. So, to show that $f(N)$ is compact we must show that the off diagonal entries are compact, and these are simply the Hankel operators constructed from S and T . Hence the following claim will finish the proof.

Claim: For any subnormal operator S and $f \in C(\sigma_n(S))$, if $S_f \in \mathcal{B}_0$ then $H_f \in \mathcal{B}_0$.

Using the well known identity $S_{gf} - S_g S_f = H_g^* H_f$ that holds for any $f, g \in L^\infty(\text{svsm} S)$, one sees that if $g = \bar{f}$, then one gets $S_{f^2} - S_{\bar{f}} S_f = H_f^* H_f$. As we are given that S_f is compact, in order to show that H_f is compact it suffices to show that S_{f^2} is compact. But this follows immediately by first applying Theorem 4.10 to S_f to conclude that $f = 0$ on $\sigma_n(\pi(S))$; then another application of Theorem 4.10 gives that S_{f^2} is compact (because $|f|^2 = 0$ on $\sigma_n(\pi(S))$). Thus the claim follows and hence the Theorem also. $\square$

5. Diagonalizable Self-Commutators and Compact Perturbations

We now consider some examples and *applications* of the essential normal spectrum. In particular we shall see some results (Corollaries 5.5 and 5.10) whose statements make no reference to the essential normal spectrum yet it will be a fundamental part of their proofs. For example, the essential normal spectrum can be used to show that certain subnormal operators (that are *not* essentially normal) are unitarily equivalent modulo the compact operators. We shall also use it to show that a particular class of subnormal operators have diagonalizable self-commutators. Also see Proposition 5.2 for a nice application of operator theory to function theory.

We begin with an easy example illustrating Theorem 4.10(a). Although it is elementary it has an interesting consequence.

Example 5.1 If $f \in H^\infty(\mathbf{D})$ and T_f is multiplication by f on $H^2(\mathbf{D})$, then $\sigma_n(\pi(T_f)) = \sigma_n(T_f)$.

Proof. It is known that $\sigma_n(T_f)$ is the essential range of f on $\partial\mathbf{D}$. If $g \in C(\sigma_n(T_f))$, then $g \circ f$ is well defined and it is easy to check that the Toeplitz operator $(T_f)_g = T_{g \circ f}$. Since there are no compact Toeplitz operators on $H^2(\mathbf{D})$, the Toeplitz operators constructed from the subnormal operator T_f are compact if and only if they are zero. Thus Theorem 4.10(a) and Lemma 3.8 imply that $\sigma_n(\pi(T_f)) = \sigma_n(T_f)$. $\square$

In Example 5.1 we see the power of Theorem 4.10. Namely, we can compute the essential normal spectrum without having to consider the Fundamental Diagram. That is important because we still do not fully understand the Fundamental Diagrams associated with the general analytic Toeplitz operator.

Although the above example is not surprising, it has the following nice consequence. Let $\text{clu}(f)$ denote the cluster set of f on $\partial\mathbf{D}$ and $\text{ess-ran}(f)$ denote the essential range of f .

Proposition 5.2 (a) If $f \in H^\infty(\mathbf{D})$ and $\bar{f} \in H^\infty(\mathbf{D}) + C(\partial\mathbf{D})$, then $\text{ess-ran}(f) = \text{clu}(f)$.
(b) If $f \in H^\infty(\mathbf{D})$, then $\text{clu}(f) = \text{ess-ran}(f) \cup \{\text{some bounded components of } \mathbf{C} - \text{ess-ran}(f)\}$.

Proof. The idea is as follows. It is known that $\sigma_n(T_f) = \text{ess-ran}(f)$ and $\sigma_e(T_f) = \text{clu}(f)$. See Conway[1991], p. 42 for the first equality and p. 334 for the second. Also see Conway [1987] for the second equality. Hence we need to compare $\sigma_n(T_f)$ with $\sigma_e(T_f)$.

(a) If $\bar{f} \in H^\infty(\mathbf{D}) + C(\partial\mathbf{D})$, then one easily checks that T_f has compact self-commutator. Hence by Proposition 4.3(a), $\sigma_e(T_f) = \sigma_n(\pi(T_f))$. Also by Example 5.1 we have that $\sigma_n(\pi(T_f)) = \sigma_n(T_f)$. Thus, $\sigma_n(T_f) = \sigma_e(T_f)$ and hence $\text{ess-ran}(f) = \text{clu}(f)$. (b) This follows immediately from Proposition 4.2(b). $\square$

J. Shapiro [1987] proved that $\text{clu}(f) = \text{ess-ran}(f)$ for $f \in \text{VMO}$. See also Axler and Shields [1990] for another proof. Since $f \in \text{VMO} \cap L^\infty(\partial\mathbf{D})$ if and only if $f, \bar{f} \in H^\infty + C(\partial\mathbf{D})$; our part (a) recaptures their result when $f \in \text{VMOA} \cap L^\infty(\partial\mathbf{D})$.

We now characterize the subnormal operators that are compact perturbations of isometries.

Theorem 5.3 If S is a pure subnormal operator, then $\sigma_n(\pi(S)) = \partial\mathbf{D}$ if and only if S is unitarily equivalent (mod $\mathcal{B}_0$) to the unilateral shift of multiplicity n , where $n = -\text{ind}S$.

Notice that $n = \infty$ is allowed in the above result and as $\sigma_{\text{ap}}(S) \subseteq \sigma_n(\pi(S)) = \partial\mathbf{D}$, the index of S is well defined. In particular Theorem 5.3 allows us to identify subnormal operators that are unitarily equivalent modulo the compact operators to the unilateral shift of infinite multiplicity.

Proof. Let ρ be a faithful representation of $C^*(\pi(S))$. The assumption that $\sigma_n(\pi(S)) = \partial\mathbf{D}$ gives that $T = \rho(\pi(S))$ is an isometry. Let $S = W|S|$ be the polar decomposition of S . Since $T^*T = I$ we get that $\pi(S^*S) = I$, hence also $\pi(|S|) = I$. That is $|S| = I + K$ for some compact operator K . Thus, $S = W + C$ for the compact operator $C = WK$. But since S is one-to-one, W is an isometry; and because $S = W + C$, we get $\text{ind}W = \text{ind}S = (-n) \leq -1$.

Now if U is the unilateral shift, then the Wold decomposition gives $W \cong U^{(n)} \oplus V$ for some unitary V . But then we still have (even if $n = \infty$)

$$W \cong U^{(n)} \oplus V \cong U^{(n-1)} \oplus (U \oplus V) \cong U^{(n-1)} \oplus U \pmod{\mathcal{B}_0} \cong U^{(n)}$$

where we have used the classification of essentially unitary operators (or B-D-F Theory - see Davidson[1996], p. 275) to get that $(U \oplus V) \cong U \pmod{\mathcal{B}_0}$. Thus $W \cong U^{(n)} \pmod{\mathcal{B}_0}$.

Hence $S \cong U^{(n)} \pmod{\mathcal{B}_0}$ and one direction is proved. The converse follows from Corollary 4.9 and Example 4.4. $\square$

The next corollary is a natural generalization of Theorem 5.3. In what follows, if G is any bounded open set, then S_G denotes M_z on $H^2(G)$; the Hardy space on G .

Corollary 5.4 If Γ is any Jordan curve and $G = \text{ins}\Gamma$, then a pure subnormal operator S has $\sigma_n(\pi(S)) = \Gamma$ if and only if $S \cong S_G^{(n)} \pmod{\mathcal{B}_0}$, where $n = -\text{ind}(S - \lambda)$ for $\lambda \in G$.

Proof. If $S \cong S_G^{(n)} \pmod{\mathcal{B}_0}$ and $n < \infty$, then S has compact self-commutator and $\sigma_n(\pi(S)) = \sigma_e(S) = \sigma_e(S_G) = \partial G$. If $n = \infty$, then Example 4.4 and Corollary 4.9 show that $\sigma_n(\pi(S)) = \sigma_n(S_G) = \partial G$. Conversely, if $\sigma_n(\pi(S)) = \partial G$, then $\sigma(S) = \text{cl}G$ (because S is pure and $R(\partial G) = C(\partial G)$). Now, if $\varphi: \mathbf{D} \rightarrow G$ is a Riemann map, then, since ∂G is a Jordan curve, φ is a homeomorphism on $\partial \mathbf{D}$. Thus, by Mergelyan's Theorem (see Conway[1991], p. 253) $\varphi^{-1} \in R(\sigma(S))$. By Proposition 4.2(c) we have $\sigma_n(\pi(\varphi^{-1}(S))) = \varphi^{-1}(\sigma_n(\pi(S))) = \varphi^{-1}(\partial G) = \partial \mathbf{D}$. Thus Theorem 5.3 says that $\varphi^{-1}(S) \cong U^{(n)} \pmod{\mathcal{B}_0}$, where U is the unilateral shift. Now, since φ is continuous on $\text{cl}\mathbf{D}$, and hence a uniform limit of polynomials, we see that $S \cong \varphi(\varphi^{-1}(S)) \cong \varphi(U^{(n)}) \pmod{\mathcal{B}_0} \cong S_G^{(n)}$. $\square$

Corollary 5.5 If Γ is a Jordan curve, $G = \text{ins}\Gamma$ and S is a pure subnormal operator with $\sigma_e(S) = \Gamma$, then for each $\mathcal{M} \in \text{Lat}S$, $S|_{\mathcal{M}} \cong S_G^{(n)} \pmod{\mathcal{B}_0}$, where $n = -\text{ind}(S|_{\mathcal{M}})$.

Proof. Set $T = S|_{\mathcal{M}}$. It suffices to check that $\sigma_n(\pi(T)) = \Gamma$, for then Corollary 5.4 applies.

It follows from Proposition 4.2 that $\Gamma = \sigma_{ap}(T) \subseteq \sigma_n(\pi(T))$. Also, Theorem 4.11 gives that $\sigma_n(\pi(T)) \subseteq \sigma_n(\pi(S))$, but $\sigma_n(\pi(S)) \subseteq \sigma_e(S) = \Gamma$. Hence $\sigma_n(\pi(T)) = \Gamma$. $\square$

Example 5.6 If Γ is a Jordan curve and $G = \text{ins}\Gamma$, then let $B = M_z$ on $L_a^2(G)$ be the Bergman operator on G . If $\mathcal{M} \in \text{Lat}B$ is such that $\text{ind}(B|_{\mathcal{M}} - \lambda) = -\infty$ for all $\lambda \in G$, then $B|_{\mathcal{M}} \cong S_G^{(\infty)} \pmod{\mathcal{B}_0}$.

It's known that there are invariant subspaces for the Bergman operator on G with the property that $\text{ind}(B|_{\mathcal{M}}) = -\infty$, see C. Apostol, H. Bercovici, C. Foias and C. Pearcy [1985].

One may wonder in view of Example 5.6 if $B|_{\mathcal{M}}$ is also unitarily equivalent $\pmod{\mathcal{B}_0}$ to the Bergman operator on G of infinite multiplicity. We now show that this is not the case.

Theorem 5.7 If S and T are any operators with $\mathcal{B}_0 \subseteq C^*(S)$ and T irreducible, then the following are equivalent:

- (a) $S \cong T$,
- (b) $S^{(\infty)} \cong T^{(\infty)}$,
- (c) $S^{(\infty)} \cong T^{(\infty)} \pmod{\mathcal{B}_0}$,
- (d) $C^*(S) \approx C^*(T)$.

Proof. Clearly (a) $\Rightarrow$ (b) $\Rightarrow$ (c). So, if $S^{(\infty)} \cong T^{(\infty)} \pmod{\mathcal{B}_0}$, then by Proposition 4.8 we have that there exists a $*$ -isomorphism $\varphi: C^*(\pi(S^{(\infty)})) \rightarrow C^*(\pi(T^{(\infty)}))$ with $\varphi(\pi(S^{(\infty)})) = \pi(T^{(\infty)})$. But clearly $C^*(\pi(S^{(\infty)})) \approx C^*(S)$ and $C^*(\pi(T^{(\infty)})) \approx C^*(T)$. So, $C^*(S) \approx C^*(T)$. Hence (d) follows. Now, Theorem 2.2 says that (d) implies (a). $\square$

In particular, we see that for a given region G , $S_G^{(\infty)}$ is not unitarily equivalent (mod $\mathcal{B}_0$) to the Bergman shift on G of infinite multiplicity. The next result will allow us to determine that certain operators have a diagonalizable self-commutator.

Theorem 5.8 Suppose S is a pure subnormal operator and $P^\infty(S) \approx H^\infty(G)$ where G is a bounded region. If $\sigma_n(\pi(S)) = \partial G$ and $\sigma_e(S) = \text{cl}G$, then there is a faithful representation ρ of $C^*(\pi(S))$ such that $\rho(\pi(S)) = S_G$.

Proof. If ρ is any faithful representation of $C^*(\pi(S))$, then let $T = \rho(\pi(S))$. We need to show that $C^*(T) \approx C^*(S_G)$. If $T = T_p \oplus N$ is the decomposition of T into its pure and normal parts, then $\sigma(T) = \text{cl}G$ and $\sigma_n(T) = \partial G$. Since $\sigma(N) \subseteq \sigma_n(T) = \partial G$, it follows that $\sigma(T_p) = \text{cl}G$. The hypothesis on G implies that G is the interior of its closure, hence, $\partial G = \partial[\text{cl}G] = \partial\sigma(T_p) \subseteq \sigma_{\text{ap}}(T_p)$. Thus, $\sigma(N) \subseteq \sigma_n(T) = \partial G \subseteq \sigma_{\text{ap}}(T_p)$. Hence Theorem 1.4 says that $C^*(T) \approx C^*(T_p)$. So T_p is a pure subnormal operator with $\sigma(T_p) = \text{cl}G$ and $\sigma_n(T_p) = \partial G$. If $K = \text{cl}G$, then $R(K)$ is a Dirichlet algebra (see Conway [1991], p. 303). Thus since S is pure and $\mu = \text{svsm}T_p$ is carried by ∂K , it follows that $R^\infty(K, \mu) \approx H^\infty(G)$ (see Conway[1991], p. 295). If $\varphi: \mathbf{D} \rightarrow G$ is a Riemann map, then $\varphi^{-1} \in H^\infty(G) \approx R^\infty(K, \mu)$ and thus $\varphi^{-1}(T_p)$ is a well-defined pure subnormal operator. Since $\sigma_n(\varphi^{-1}(T_p)) = \varphi^{-1}(\sigma_n(T_p)) = \varphi^{-1}(\partial G) = \partial \mathbf{D}$ (Conway[1991], p. 42), $\varphi^{-1}(T_p)$ is a pure isometry. Thus $\varphi^{-1}(T_p) \cong U^{(n)}$ for some $n \geq 1$, where U is the unilateral shift. Hence $T_p \cong \varphi(\varphi^{-1}(T_p)) \cong \varphi(U^{(n)}) \cong S_G^{(n)}$. Thus $C^*(T) \approx C^*(T_p) \approx C^*(S_G)$. $\square$

The operators in Corollary 5.5 (with $n = \infty$) and Example 5.6 are easy examples of irreducible subnormal operators that do not have compact self-commutator. However the next result says that their self-commutators are still very well behaved.

Proposition 5.9 If S is any operator and there exists a faithful representation ρ of $C^*(\pi(S))$ such that $T = \rho(\pi(S))$ has compact self-commutator, then $[S^*, S]$ is diagonalizable with $\sigma_e([S^*, S]) = \sigma([T^*, T])$.

Proof. Clearly, $\sigma_e([S^*, S]) = \sigma([T^*, T])$. Since $[T^*, T]$ is compact, its spectrum is either finite or a sequence converging to zero. Since $\sigma([S^*, S])$ is countable, we must have that the scalar valued spectral measure for $[S^*, S]$ is purely atomic. Hence $[S^*, S]$ is diagonalizable. $\square$

In fact, in Proposition 5.9 it suffices to assume that $\sigma([T^*, T])$ is countable.

Corollary 5.10 Suppose S is a pure subnormal operator with $P^\infty(S) \approx H^\infty(G)$ and $\sigma_e(S) = \partial G$. If $\mathcal{M} \in \text{Lat} S$ with $\text{ind}(S|_{\mathcal{M}} - \lambda) = -\infty$ for $\lambda \in G$ and $T = S|_{\mathcal{M}}$, then $[T^*, T]$ is diagonalizable with $\sigma_e([T^*, T]) = \sigma([S_G^*, S_G])$.

Proof. We may assume that G is connected. We want to apply Theorem 5.8 along with Proposition 5.9. So, we need $P^\infty(T) \approx H^\infty(G)$, $\sigma_n(\pi(T)) = \partial G$ and $\sigma_e(T) = \text{cl} G$. First note that $\sigma_{\text{ap}}(T) \subseteq \sigma_{\text{ap}}(S) = \partial G$, so $\text{ind}(T - \lambda)$ is well defined for all $\lambda \in G$. Since, $\text{ind}(T - \lambda) = -\infty$ for $\lambda \in G$, we have $\text{cl} G \subseteq \sigma(T)$. Since $\mathcal{M} \in \text{Lat} S = \text{Lat} P^\infty(S)$ and $R(\text{cl} G) \subseteq P^\infty(\mu)$, $\mu = \text{svsm} S$, we see that $\mathcal{M}$ is rationally invariant for S . Hence $\sigma(T) \subseteq \text{cl} G$. Thus $\sigma(T) = \text{cl} G$.

It follows from Corollary 4.3(b) that $\sigma_n(\pi(S)) = \sigma_e(S) = \partial G$. Hence Theorem 4.11 gives that $\sigma_n(\pi(T)) \subseteq \sigma_n(\pi(S)) = \partial G$. Thus, $\partial G = \partial[\text{cl}G] = \partial\sigma(T) \subseteq \sigma_{\text{ap}}(T) \subseteq \sigma_n(\pi(T)) \subseteq \partial G$. So $\sigma_n(\pi(T)) = \partial G$. Also, $\sigma_{\text{ap}}(T) = \partial G$ and, because of the index assumption on T , $\sigma_e(T) = \text{cl}G$. It remains to show that $P^\infty(T) \approx H^\infty(G)$. Let K be the Sarason hull of T (see Conway[1991], p. 306). So, $P^\infty(T) \approx H^\infty(\text{int}K)$. It is always true that $(\text{cl}G =) \sigma(T) \subseteq K$. However since $\text{svsm}T \ll \text{svsm}S$, we see that there is a natural weak*-continuous (restriction) map $\rho: P^\infty(S) \rightarrow P^\infty(T)$. Hence the Sarason hull for T is contained in the Sarason Hull for S , so $K \subseteq \text{cl}G$. Thus, $P^\infty(T) \approx H^\infty(G)$. $\square$

It is interesting that in Corollary 5.10, $\sigma_e([T^*, T])$ depends only on G and not on S or $\mathcal{M}$.

Corollary 5.11 Suppose $B = M_z$ on $L_a^2(\mathbf{D})$ is the Bergman shift, $\mathcal{M} \in \text{Lat}B$ and $S = B|_{\mathcal{M}}$. If $\text{ind}(S) = -\infty$, then $[S^*, S]$ is diagonalizable and $\sigma_e([S^*, S]) = \{0, 1\}$. Furthermore, the points $\{0, 1\}$ are not eigenvalues of $[S^*, S]$ and every eigenvalue has finite multiplicity.

Proof. By Corollary 5.10 and the fact that every point in $\sigma([S^*, S]) - \sigma_e([S^*, S])$ is an eigenvalue of finite multiplicity, it suffices to establish that 0 and 1 are not eigenvalues for $[S^*, S]$. So, if $[S^*, S]f = f$ for some f , then $\|f\|^2 = \langle [S^*, S]f, f \rangle = \|Sf\|^2 - \|S^*f\|^2 \leq \|Sf\|^2 \leq \|f\|^2$. Hence $\|Sf\| = \|f\|$. But the Bergman shift is a strict contraction. So no such f exists. Thus, the point 1 is not an eigenvalue for $[S^*, S]$. Also, it is easy to check that if $f \in \ker[S^*, S]$, then $\bar{z}f \in L_a^2(\mathbf{D})$ (see Proposition IV.1.1); hence no such f exists. Thus 0 is not an eigenvalue for $[S^*, S]$. $\square$

The operator S in Corollary 5.11 can be used (with Corollary 5.5) to answer a question in Conway and Putnam [1985], p.297.

6. Essentially Normal Operators

In this section we shall see some non-trivial consequences of essential normality. In particular, we will sharpen many of the results of previous sections for essentially normal operators; that is, those with compact self-commutator. We shall see that when two subnormal operators S, T with compact self-commutators satisfy $C^*(S) \approx C^*(T)$, then more structure is preserved. In particular, $\text{svsm}S \approx \text{svsm}T$ and $C^*(\text{dual}S) \approx C^*(\text{dual}T)$.

Recall the *Fundamental Diagram* from Section 4.

$$\begin{array}{ccc} C^*(S) & & \\ \downarrow \pi & & \\ C^*(\pi(S)) & \xrightarrow[\rho]{} & C^*(T) \end{array}$$

The homomorphism π is the natural map into the Calkin algebra and ρ is a faithful representation (hence an isometry) on a separable Hilbert space. Thus $\varphi = \rho \circ \pi : C^*(S) \rightarrow C^*(T)$ is a $*$ -homomorphism with $\varphi(S) = T$ whose kernel is exactly $C^*(S) \cap \mathcal{B}_0$.

Observe that $\varphi([S^*, S]) = [T^*, T]$. Thus, since $\ker(\varphi) = C^*(S) \cap \mathcal{B}_0$, we have that $[S^*, S]$ is compact if and only if $[T^*, T] = 0$. That is, S is essentially normal if and only if T is normal. Theorem 6.1(c),(d) and (e) were proven by Olin and Thomson [1982] when S is an *irreducible* subnormal operator with compact self-commutator.

Observe also that Theorem 6.1 is a refinement of Theorem 4.10 for subnormal operators that are essentially normal.

Theorem 6.1 If S is a subnormal operator that is essentially normal and $f \in C(\sigma_n(S))$, then:

- (a) $\sigma_n(\pi(S)) = \sigma_e(S)$;
- (b) $\text{Comm}(S) = C^*(S) \cap \mathcal{B}_0$;
- (c) S_f is compact if and only if $f = 0$ on $\sigma_e(S)$;
- (d) $\sigma_e(S_f) = f(\sigma_e(S))$;
- (e) $\|S_f\|_e = \|f\|_{\infty, \sigma_e(S)}$.

Proof. Consider the Fundamental Diagram for S ,

$$\begin{array}{ccc} C^*(S) & & \\ \downarrow \pi & & \\ C^*(\pi(S)) & \xrightarrow[\rho]{} & C^*(N) \end{array}$$

Let $\varphi = \rho \circ \pi : C^*(S) \rightarrow C^*(N)$. Since S is essentially normal, N is normal. Clearly, $\sigma_n(\pi(S)) = \sigma_n(N) = \sigma(N) = \sigma_e(S)$. By Corollary 3.4 we know that $\varphi(S_f) = f(N)$ for each $f \in C(\sigma_n(S))$. Thus, since ρ is an isometry, $\|S_f\|_e = \|\varphi(S_f)\| = \|f(N)\| = \|f\|_{\infty, \sigma_e(S)}$. Hence we see that S_f is compact if and only if $f = 0$ on $\sigma_e(S)$. Also, $\sigma_e(S_f) = \sigma(\varphi(S_f)) = \sigma(f(N)) = f(\sigma(N)) = f(\sigma_e(S))$. Finally (b) follows from Corollary 6.4 below. $\square$

If $S = \bigoplus_{n=1}^{\infty} S_n$ on $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$, and one is considering $C^*(S)$, then one may often assume that

$S_n \cong S_k$ if and only if $n = k$. For if this is not the case, then by restricting the operators in $C^*(S)$ to a reducing subspace, one obtains a new operator that has this property and whose C^* -algebra is isomorphic to $C^*(S)$.

If $S = \bigoplus_{n=1}^{\infty} S_n$ is the direct sum of irreducible operators on $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$, then we shall denote

by π_n the $*$ -homomorphism $C^*(S) \rightarrow C^*(S_n)$ given by $\pi_n(A) = A|_{\mathcal{H}_n}$. Also, we shall say that S is *multiplicity free* if $S_n \cong S_k$ implies that $n = k$. That is, no two irreducible summands of S are unitarily equivalent. If X_n is a sequence of Banach spaces, then their

c_0 -direct sum, denoted by $\bigoplus_0^{\infty} X_n$, consists of all sequences $\{x_n\}$ such that $x_n \in X_n$ and

$\|x_n\| \rightarrow 0$. Endowed with the sup-norm, this is a Banach space. The following result is

very similar in nature to one in Arveson [1976], p. 21.

Theorem 6.2 Let $S = \bigoplus_{n=1}^{\infty} S_n$ be a multiplicity free operator on $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$ such that $\mathcal{B}_0(\mathcal{H}_n) \subseteq C^*(S_n)$ for all n . If $C^*(S) \cap \mathcal{B}_0(\mathcal{H})$ acts non-degenerately on $\mathcal{H}$ and $\mathcal{J}$ is any ideal in $C^*(S)$, then either $\mathcal{J} \subseteq \ker(\pi_n)$ for some n or $\bigoplus_0^{\infty} \mathcal{B}_0(\mathcal{H}_n) \subseteq \mathcal{J}$.

Corollary 6.3 Let $S = \bigoplus_{n=1}^{\infty} S_n$ be a multiplicity free operator on $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$ such that $\mathcal{B}_0(\mathcal{H}_n) \subseteq C^*(S_n)$ for all n . If $C^*(S) \cap \mathcal{B}_0(\mathcal{H})$ acts non-degenerately on $\mathcal{H}$, then $C^*(S) \cap \mathcal{B}_0(\mathcal{H}) = \bigoplus_0^{\infty} \mathcal{B}_0(\mathcal{H}_n)$.

Observe that if $C^*(S) \cap \mathcal{B}_0(\mathcal{H})$ acts non-degenerately on $\mathcal{H}$, then Theorem 1.8 says that S

necessarily has the form $\bigoplus_{n=1}^{\infty} S_n$ and $\mathcal{B}_0(\mathcal{H}_n) \subseteq C^*(S_n)$ for all n . Also, another way to

phrase Theorem 6.2 is that if $C^*(S) \cap \mathcal{B}_0(\mathcal{H})$ acts non-degenerately, then it is the unique minimal non-degenerate ideal.

Proof. If $\mathcal{J}$ is an ideal in $C^*(S)$ and $\mathcal{J}_n = \mathcal{J}|_{\mathcal{H}_n} \neq (0)$ for each n , then, since S_n is irreducible with $\mathcal{B}_0(\mathcal{H}_n) \subseteq C^*(S_n)$, it follows from Proposition 1.2(a) that $\mathcal{J}_n$ must contain a non-zero compact operator. Hence $\mathcal{B}_0(\mathcal{H}_n) \subseteq \mathcal{J}_n$. In particular $\mathcal{J}$ acts non-degenerately on $\mathcal{H}$. Since $C^*(S) \cap \mathcal{B}_0(\mathcal{H})$ also acts non-degenerately on $\mathcal{H}$, Proposition 1.2(b) implies that $\mathcal{J} \cap \mathcal{B}_0(\mathcal{H})$ acts non-degenerately. Thus by considering $\mathcal{J} \cap \mathcal{B}_0(\mathcal{H})$ we may suppose that $\mathcal{J} \subseteq \mathcal{B}_0(\mathcal{H})$, $\mathcal{J}_n = \mathcal{J}|_{\mathcal{H}_n} \neq (0)$ for each n and thus $\mathcal{J}_n = \mathcal{B}_0(\mathcal{H}_n)$.

So the following is what we need to prove.

Claim: If $K_n \in \mathcal{B}_0(\mathcal{H}_n)$ for each n and $\|K_n\| \rightarrow 0$ as $n \rightarrow \infty$, then there is a unique operator $K \in \mathcal{J}$ such that $K|_{\mathcal{H}_n} = K_n$.

Now for each n , $\pi_n: \mathcal{J} / \ker(\pi_n|_{\mathcal{J}}) \rightarrow \mathcal{B}_0(\mathcal{H}_n)$ is a $*$ -isomorphism, hence an isometry. So for the given $K_n \in \mathcal{B}_0(\mathcal{H}_n) = \pi_n(\mathcal{J})$ we can find an operator $A_n \in \mathcal{J}$ so that $\|A_n\| \leq \|K_n\| + 1/n$ and $\pi_n(A_n) = K_n$, so in particular $\|A_n\| \rightarrow 0$ as $n \rightarrow \infty$. Let $\mathcal{J}''$ be the double commutant of $\mathcal{J}$.

Suppose: $P_n \in \mathcal{J}''$ for each n , where P_n is the projection of $\mathcal{H}$ onto $\mathcal{H}_n$.

Given this, by Kaplansky's Density Theorem and von Neumann's Double Commutant Theorem, for each n there is a sequence of self-adjoint operators $\{B_k\} \subseteq \mathcal{J} \subseteq \mathcal{B}_0(\mathcal{H})$ with $\|B_k\| \leq 1$ and $B_k \rightarrow P_n$ (SOT) as $k \rightarrow \infty$. So if $J \in \mathcal{J} \subseteq \mathcal{B}_0(\mathcal{H})$, then $B_k J \rightarrow P_n J$ in norm

(because J is compact). Thus we see that $P_n J \in \mathcal{J}$ for all $J \in \mathcal{J}$. Thus, for each $J \in \mathcal{J}$, $P_n J P_n = \lim_k B_k J B_k \in \mathcal{J}$. So for the operators $A_n \in \mathcal{J}$ we have $P_n A_n P_n \in \mathcal{J}$ for all n . Now define $K = \sum_{n=1}^{\infty} P_n A_n P_n = \bigoplus_{n=1}^{\infty} P_n A_n P_n$. Notice that this sum converges in norm because $\|A_n\| \rightarrow 0$ and the finite sums all belong to $\mathcal{J}$. Thus $K \in \mathcal{J}$ and $K|_{\mathcal{H}_n} = K_n$ for all n . So the claim follows.

Now we finish by showing that $P_n \in \mathcal{J}''$ for each n . To do this let $\mathcal{M}_n = \{Ax : x \in \mathcal{H}_n \text{ and } A \in \mathcal{J}'\}$. Now its easy to see that $\mathcal{M}_n$ is a reducing subspace for both $\mathcal{J}$ and $\mathcal{J}'$. Thus, the projection onto $\mathcal{M}_n$ will commute with $\mathcal{J}'$ and hence it belongs to $\mathcal{J}''$. So it suffices to show that $\mathcal{M}_n = \mathcal{H}_n$. Clearly $\mathcal{H}_n \subseteq \mathcal{M}_n$ because the identity $I \in \mathcal{J}'$. Now suppose there is an operator $A \in \mathcal{J}'$ with $A(\mathcal{H}_n) \not\subseteq \mathcal{H}_n$, then there is an index $k \neq n$ such that $P_k A P_n \neq 0$. Also, because $\{P_n\} \subseteq \mathcal{J}'$ we see that $P_k A P_n \in \mathcal{J}'$. If U is the partial isometry that appears in the polar decomposition of $P_k A P_n$, then $U \in \mathcal{J}'$ because $\mathcal{J}'$ is a von Neumann algebra. But one easily sees that the initial and final space of U are reducing subspaces for $\mathcal{J}$ (because U^*U and UU^* are clearly in $\mathcal{J}'$ and are the projections onto the initial and final space of U) which are contained in $\mathcal{H}_n$ and $\mathcal{H}_k$ respectively. Since $\pi_n(\mathcal{J}) = \mathcal{B}_0(\mathcal{H}_n)$ we see that $U|_{\mathcal{H}_n}$ is a unitary of $\mathcal{H}_n$ onto $\mathcal{H}_k$ and $U(SJ) = (SJ)U$ for each $J \in \mathcal{J}$. Because $U^*U = P_n$ we get that $(U^*U)SJ = U^*SJU = (U^*SU)(U^*JU) = (U^*SU)(U^*UJ) = (U^*SU)(P_n J)$. That is $P_n(SJ) = (U^*SU)(P_n J)$ for all $J \in \mathcal{J}$. Thus, we get that if $x \in \mathcal{H}_n$, then $S(Jx) = (U^*SU)(JP_n x) = (U^*SU)(Jx)$. But $\{Jx : J \in \mathcal{J} \text{ and } x \in \mathcal{H}_n\}$ is dense in $\mathcal{H}_n$. So that gives

us that $S(h) = U^*SU(h)$ for each $h \in \mathcal{H}_n$. Now if $h \in \mathcal{H}_n$, then $S(h) = S_n(h)$ and, because $U(h) \in \mathcal{H}_k$, $SU(h) = S_kU(h)$. Hence we have that $S_n = U^*S_kU$ on $\mathcal{H}_n$. Thus $S_n \cong S_k$. Since S is assumed to be multiplicity free, we have $n = k$. But this contradicts our choice of k . It follows that $\mathcal{M}_n = \{Ax : x \in \mathcal{H}_n \text{ and } A \in \mathcal{J}'\} \subseteq \mathcal{H}_n$. So $\mathcal{M}_n = \mathcal{H}_n$ and thus the projection $P_n \in \mathcal{J}''$ as promised. $\square$

Recall that Corollary 1.11 implies that pure essentially normal operators may be written as direct sums of irreducible operators.

Corollary 6.4 If S is a pure essentially normal operator, then $\text{Comm}(S) = C^*(S) \cap \mathcal{B}_0$.

Proof. We may assume that S is multiplicity free. Otherwise we may restrict S to a reducing subspace and obtain a multiplicity free operator T , so that each orthogonal summand of S is unitarily equivalent to exactly one summand of T . If $\rho: C^*(S) \rightarrow C^*(T)$ is the restriction map, then one easily checks that $\rho(A)$ is compact if and only if A is compact and also $\rho(A) \in \text{Comm}(T)$ if and only if $A \in \text{Comm}(S)$. Now, since S is pure and essentially normal, it is clear that $\text{Comm}(S) \subseteq C^*(S) \cap \mathcal{B}_0$. Since S is pure, Proposition 1.2(c) implies that $\text{Comm}(S)$ acts non-degenerately. Hence $C^*(S) \cap \mathcal{B}_0$ does also. So the hypothesis of Theorem 6.2 is satisfied. In particular, if $\mathcal{J} = \text{Comm}(S)$, then Theorem 6.2 gives that $C^*(S) \cap \mathcal{B}_0 \subseteq \text{Comm}(S)$. $\square$

Corollary 6.5 Let $S = \bigoplus_{n=1}^{\infty} S_n$ be a pure essentially normal multiplicity-free operator. If $\mathcal{J}$ is an ideal in $C^*(S)$, then either $\mathcal{J} \subseteq \ker(\pi_n)$ for some n or $\text{Comm}(S) \subseteq \mathcal{J}$.

Proof. Apply Proposition 1.2(c), Theorem 6.2 and Corollary 6.4. $\square$

In what follows, if S is a pure essentially normal operator, then by Corollary 1.11 we may

write $S = \bigoplus_{n=1}^{\infty} S_n$ where each operator S_n is irreducible and essentially normal. Hence $\mathcal{B}_0 \subseteq$

$C^*(S_n)$ for all n .

Theorem 6.6 If $S = \bigoplus_{n=1}^{\infty} S_n$ is a pure essentially normal operator and T is a pure operator with $\mathcal{B}_0 \subseteq C^*(T)$, then there exists a $*$ -homomorphism $\varphi: C^*(S) \rightarrow C^*(T)$ with $\varphi(S) = T$ if and only if $T \cong S_n$ for some n .

Proof. If $T \cong S_n$ for some n , then φ can be obtained by compression to the subspace $\mathcal{H}_n$ followed by conjugating by a unitary. For the converse we may suppose that S is multiplicity free. If we are given a $*$ -homomorphism $\varphi: C^*(S) \rightarrow C^*(T)$ with $\varphi(S) = T$, then $\ker(\varphi)$ is an ideal in $C^*(S)$. If $\text{Comm}(S) \subseteq \ker(\varphi)$, then $[T^*, T] = \varphi([S^*, S]) = 0$, and hence T is normal. But this contradicts the fact that T is pure. By Corollary 6.5 we know that there is an integer n with $\ker(\varphi) \subseteq \ker(\pi_n)$. Thus we can define a natural $*$ -homomorphism $\kappa: C^*(S)/\ker(\varphi) \rightarrow C^*(S)/\ker(\pi_n)$. This gives the following,

$$C^*(T) \approx C^*(S)/\ker(\varphi) \xrightarrow{\kappa} C^*(S)/\ker(\pi_n) \approx C^*(S_n).$$

Here $C^*(S)/\ker(\varphi)$ is naturally $*$ -isomorphic to $C^*(T)$ and $C^*(S)/\ker(\pi_n)$ is naturally isomorphic with $C^*(S_n)$. Thus we get a $*$ -homomorphism $\psi: C^*(T) \rightarrow C^*(S_n)$ such that $\psi(T) = S_n$. Since $\mathcal{B}_0 \subseteq C^*(T)$, T is irreducible so Theorem 2.2 gives that $T \cong S_n$. $\square$

Theorem 6.7 If S and T are pure essentially normal operators, then $C^*(S) \approx C^*(T)$ if and only if $S^{(\infty)} \cong T^{(\infty)}$.

Theorem 6.7 should be compared to Corollary 2.4. We see that in the presence of operators with compact self-commutator, much nicer results can be obtained. Thus except for “multiplicity”, S and T should have exactly the same summands in order to have isomorphic C^* -algebras.

Proof. If $S^{(\infty)} \cong T^{(\infty)}$, then $C^*(S) \approx C^*(S^{(\infty)}) \cong C^*(T^{(\infty)}) \approx C^*(T)$. Conversely if $\varphi: C^*(S) \rightarrow C^*(T)$ is a $*$ -isomorphism with $\varphi(S) = T$, then for each n consider $\pi_n \circ \varphi: C^*(S) \rightarrow C^*(T_n)$ and apply Theorem 6.6 to conclude $T_n \cong S_j$ for some j . Hence every summand of T is unitarily equivalent to a summand of S . Similarly, we may do the same for the inverse of φ , giving that every summand of S is a summand of T . Thus, $S^{(\infty)} \cong T^{(\infty)}$. $\square$

A simple consequence of Theorem 6.7 is that if S and T are “multiplicity free”, then the C^* -algebra completely determines the operator!

Corollary 6.8 If S and T are pure essentially normal operators, then there exists a $*$ -homomorphism $\varphi: C^*(S) \rightarrow C^*(T)$ with $\varphi(S) = T$ if and only if $T^{(\infty)}$ is unitarily equivalent to an orthogonal summand of $S^{(\infty)}$.

We now give some nice applications of the above results. Recall that if S is a subnormal operator on $\mathcal{H}$ and $N = mneS$, then $\text{dual}S = N^*|_{\mathcal{H}^\perp}$.

Theorem 6.9 Let S and T be pure subnormal operators with compact self-commutators.

- (a) If $\varphi: C^*(S) \rightarrow C^*(T)$ is a $*$ -homomorphism with $\varphi(S) = T$, then φ has an extension to a weak*-continuous $*$ -homomorphism $\varphi: W^*(S) \rightarrow W^*(T)$.
- (b) If $\varphi: C^*(S) \rightarrow C^*(T)$ is a $*$ -homomorphism with $\varphi(S) = T$, then $\text{svsm}T \ll \text{svsm}S$.
- (c) If $\varphi: C^*(S) \rightarrow C^*(T)$ is a $*$ -homomorphism with $\varphi(S) = T$, then there exists a $*$ -homomorphism $\varphi_d: C^*(\text{dual}S) \rightarrow C^*(\text{dual}T)$ with $\varphi_d(\text{dual}S) = \text{dual}T$.

Proof. If S and T are pure subnormal operators with compact self-commutator and $\varphi: C^*(S) \rightarrow C^*(T)$ is a $*$ -homomorphism such that $\varphi(S) = T$, then by Corollary 6.8, we see that $T^{(\infty)}$ is unitarily equivalent to an orthogonal summand of $S^{(\infty)}$.

(a) If $\mathcal{M}$ is a reducing subspace for $S^{(\infty)}$ such that $S^{(\infty)}|_{\mathcal{M}} \cong T^{(\infty)}$ and $U^*(S^{(\infty)}|_{\mathcal{M}})U = T^{(\infty)}$ for some unitary U , then φ is given by $\varphi(A) = U^*(A^{(\infty)}|_{\mathcal{M}})U$ for $A \in C^*(S)$. Hence φ is given by an inflation then compression to a reducing subspace and, lastly, conjugation by a unitary. Since all these operations are weak*-continuous, φ is also. Thus φ has a natural extension to $W^*(S)$.

(b) If $N = mneS$ and $M = mneT$, then, since $T^{(\infty)}$ is unitarily equivalent to an orthogonal summand of $S^{(\infty)}$, we have that $M^{(\infty)} \cong mne(T^{(\infty)})$ is unitarily equivalent to an orthogonal summand of $mne(S^{(\infty)}) \cong N^{(\infty)}$. Hence $\text{svsm}M \approx \text{svsm}M^{(\infty)} \ll \text{svsm}N^{(\infty)} \approx \text{svsm}N$.

(c) Since $T^{(\infty)}$ is a summand of $S^{(\infty)}$, we see that $\text{dual}(T^{(\infty)})$ is a summand of $\text{dual}(S^{(\infty)})$. But $\text{dual}(T)^{(\infty)} \cong \text{dual}(T^{(\infty)})$ and $\text{dual}(S)^{(\infty)} \cong \text{dual}(S^{(\infty)})$. So, $\text{dual}(T)^{(\infty)}$ is a summand of $\text{dual}(S)^{(\infty)}$, thus there exists a *-homomorphism $\phi_d: C^*(\text{dual}S) \rightarrow C^*(\text{dual}T)$ with $\phi_d(\text{dual}S) = \text{dual}T$. $\square$

Observe that (b) also follows from (a) and Theorem 3.6. Also note that by Example 3.12, properties (a) and (b) of Theorem 6.9 need not hold if S and T do not have compact self-commutators. However, it is not known if (c) holds in general or not.

Corollary 6.10 Let S and T be pure subnormal operators with compact self-commutators. If $C^*(S) \approx C^*(T)$, then the following hold:

- (a) $W^*(S) \approx W^*(T)$;
- (b) $\text{svsm}S \approx \text{svsm}T$;
- (c) $C^*(\text{dual}S) \approx C^*(\text{dual}T)$;
- (d) $\mathcal{R}(S) = \mathcal{R}(T)$.

Question 6.11 If S and T are pure subnormal operators and $C^*(S) \approx C^*(T)$, then is it true that $C^*(\text{dual}S) \approx C^*(\text{dual}T)$?

We see in Theorem 6.7 that the C^* -algebra generated by a pure subnormal operator with compact self-commutator carries all the information about the operator except for its multiplicity. We now give an example to show that this multiplicity can be hard to detect.

Example 6.12 There exists pure subnormal operators S, T with compact self-commutators such that the following hold:

- (a) $C^*(S) \approx C^*(T)$;
- (b) $\text{mne} S \cong \text{mne} T$;
- (c) $[S^*, S] \cong [T^*, T]$;
- (d) S and T have the same Fredholm index function.

Nevertheless, S is *not* unitarily equivalent to T .

Proof. If $\mu = m|\partial\mathbf{D} + a\delta_0$, $a > 0$, and $S_\mu = M_z$ on $P^2(\mu)$, then $[S_\mu^*, S_\mu]$ has rank two self-commutator and $\text{dual} S_\mu \cong S_\nu$ where $\nu = m|\partial\mathbf{D} + (1/a)\delta_0$. Thus if $a \neq 1$, then S_μ is not unitarily equivalent to S_ν . However, since $\text{dual} S_\mu \cong S_\nu$ and $[S_\mu^*, S_\mu]$ is finite rank Corollary IV.1.5 gives $[S_\mu^*, S_\mu] \cong [S_\nu^*, S_\nu]$. Also, clearly $\text{mne} S_\mu \cong \text{mne} S_\nu$. Thus if we set $S = S_\mu \oplus S_\mu \oplus S_\nu$ and $T = S_\mu \oplus S_\nu \oplus S_\nu$, then S and T have the required properties. $\square$

Recall Theorem 1.8 where every operator is decomposed as the direct sum of three types of operators. We now finish this section by showing that if there exists a *-homomorphism $\varphi: C^*(S) \rightarrow C^*(T)$ with $\varphi(S) = T$ and S has compact self-commutator, then T cannot be completely reducible or irreducible with no compact operators in its C^* -algebra. Thus T must be a direct sum of operators as in Theorem 1.8(c) and Theorem 6.6 applies.

Theorem 6.13 If S is a pure essentially normal operator and T is either a pure completely reducible operator or irreducible with $C^*(T) \cap \mathcal{B}_0 = (0)$, then there does not exist a *-homomorphism $\varphi: C^*(S) \rightarrow C^*(T)$ with $\varphi(S) = T$.

Proof. If φ exists, then, since T is pure, φ is a non-degenerate representation. By Theorem I.1.20, T must have a reducing subspace $\mathcal{M}$ such that $T|_{\mathcal{M}}$ is unitarily equivalent

to an irreducible summand of S . However if S_n is the irreducible summand of S , then $\mathcal{B}_0 \subseteq C^*(S_n)$. But this contradicts the fact that T is completely reducible or that $C^*(T) \cap \mathcal{B}_0 = (0)$. $\square$

Theorem 6.14 If S is an irreducible essentially normal subnormal operator, then an operator T satisfies $C^*(T) \approx C^*(S)$ if and only if $T \cong S^{(n)} \oplus N$ where $n \in \{\infty, 1, 2, 3, \dots\}$ and N is a normal operator with $\sigma(N) \subseteq \sigma_{ap}(S)$.

Proof. If $C^*(S) \approx C^*(T)$, then Theorem 6.13 and Theorem 1.8 imply that the pure part of T is a direct sum of irreducible operators T_n each satisfying $\mathcal{B}_0 \subseteq C^*(T_n)$. Hence Theorem 6.6 applies to give that $T \cong S^{(n)} \oplus N$ where $n \geq 1$ and N is a normal operator. Also $\sigma(N) \subseteq \sigma_{ap}(T)$ and a result of Bunce (see Corollary I.1.10) gives that $\sigma_{ap}(T) = \sigma_{ap}(S)$. So $\sigma(N) \subseteq \sigma_{ap}(S)$. For the converse, notice that Lemma 1.5 gives that $C^*(S \oplus N) \approx C^*(S)$. Hence this easily gives that $C^*(S^{(n)} \oplus N) \approx C^*(S)$ for any n . $\square$

The next result follows in a similar way.

Theorem 6.15 If S is a pure essentially normal subnormal operator, then an operator T satisfies $C^*(T) \approx C^*(S)$ if and only if $T^{(\infty)} \cong S^{(\infty)} \oplus N$ where N is a normal operator with $\sigma(N) \subseteq \sigma_{ap}(S)$.

Recall that in Corollary 1.7 it was shown that if S is the unilateral shift, then an operator T satisfies $C^*(T) \approx C^*(S)$ if and only if T is an isometry that is not a unitary. In view of the von Neumann - Wold decomposition for isometries, this is a special case of Theorem 6.14.

7. Essentially Subnormal Operators

In this section we want to introduce the idea of an *essentially subnormal operator*. We also prove some basic properties of such operators, including the equivalence of various “reasonable” definitions. The question of unitary equivalence modulo the compact operators is addressed and some nice results and examples arise by considering essential isometries; this was done for instance in Section 5. The class of essentially subnormal operators has been considered in Lange [1983] and Williams [1990]. One of our main results in this section (Theorem 7.3) answers a question in Williams [1990].

It’s well known that subnormality is a C^* -algebra concept; see Theorem I.1.1 and Corollary I.1.4. Thus for an operator T it makes sense to ask if $\pi(T)$ is a subnormal element in the Calkin algebra $\mathcal{B}/\mathcal{B}_0$. If $\pi(T)$ is subnormal in $\mathcal{B}/\mathcal{B}_0$, then we say that T is an *essentially subnormal operator*.

Consider the *Fundamental Diagram*.

$$\begin{array}{ccc} C^*(S) & & \\ \downarrow \pi & & \\ C^*(\pi(S)) & \xrightarrow[\rho]{} & C^*(T) \end{array}$$

We know that if S is a subnormal operator, then T is also subnormal. Now consider the question, when is T subnormal? Since subnormality is a C^* -property, it is preserved by $*$ -homomorphisms. Thus since ρ is a $*$ -isomorphism, T is subnormal if and only if $\pi(S)$ is subnormal in $\mathcal{B}/\mathcal{B}_0$. That is, T is subnormal if and only if S is essentially subnormal.

If $p(z, \bar{z}) = z^n \bar{z}^k$, then let's agree that $p(S) = S^{*k} S^n$. So, "the adjoints" are all grouped together on the left. Thus if p is any polynomial in z and $\bar{z}$, then we may define $p(S)$ by first writing p as a linear combination of terms like $z^n \bar{z}^k$ and then applying the above convention. This is called the *hereditary functional calculus* for the operator S developed by Agler; see Chapter I. It is easy to check that if S is a subnormal operator and p is any polynomial in z and $\bar{z}$, then $p(S) = S_p$, the Toeplitz operator with symbol p . Further, recall from Theorem I.1.1, that an operator S is subnormal if and only if there exists a disk Δ containing $\sigma(S)$, such that $p(S) \geq 0$ for every polynomial (in z and $\bar{z}$) $p \geq 0$ on Δ .

Proposition 7.1 Let S be an operator and Δ any closed disk containing the essential spectrum of S . The operator S is essentially subnormal if and only if for every polynomial $p(z, \bar{z})$ that is non-negative on Δ we have $p(S) = A + K$, with A positive and K compact.

Proof. The operator S is essentially subnormal if and only if $\pi(S)$ is subnormal in $\mathcal{B}/\mathcal{B}_0$. This means by definition, that for any polynomial $p(z, \bar{z}) \geq 0$ on $\sigma(\pi(S))$ must satisfy that $p(\pi(S)) \geq 0$, where the above convention is used to define $p(\pi(S))$. Now, since π is a $*$ -homomorphism, we see that $\pi(p(S)) = p(\pi(S)) \geq 0$. Also it is easy to see that positive elements in the Calkin algebra can be "lifted" to compact perturbations of positive elements in $\mathcal{B}(\mathcal{H})$. That is, there exists a positive operator A (depending on p) so that $\pi(p(S)) = \pi(A)$. Thus, $p(S) = A + K$, for some compact operator K . Conversely, this condition clearly implies that $p(\pi(S)) \geq 0$ for each polynomial $p(z, \bar{z}) \geq 0$ on Δ and this is sufficient to guarantee that $\pi(S)$ is subnormal in $\mathcal{B}/\mathcal{B}_0$; see Section I.1. $\square$

Theorem 7.3 below is the main theorem in this section and it allows one to understand more fully the concept of essential subnormality and produce several examples. First, observe that if T is an operator and $\mathcal{M}$ is a closed subspace, then $\mathcal{M}$ is an invariant subspace for T if and only if $P^\perp TP = 0$, where P is the orthogonal projection onto $\mathcal{M}$. Thus, we say that $\mathcal{M}$ is an *essentially invariant subspace* for T if $P^\perp TP$ is compact.

Lemma 7.2 If T is an operator and $\mathcal{M}$ is a closed subspace, then $\mathcal{M}$ is essentially invariant for T if and only if $\mathcal{M}$ is invariant for some compact perturbation of T .

Proof. Let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with respect to $\mathcal{M} \oplus \mathcal{M}^\perp$. If P is the projection on $\mathcal{M}$, then $C =$

$P^\perp TP$. Thus, $T = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}$. If $\mathcal{M}$ is essentially invariant, then we have T written

as an operator that leaves $\mathcal{M}$ invariant plus a compact operator. The converse is similar. $\square$

We now answer affirmatively the question raised in Williams[1990] whether (a) and (e) in Theorem 7.3 are equivalent.

Theorem 7.3 If T is any operator on $\mathcal{H}$, then the following are equivalent:

- (a) T is essentially subnormal.
- (b) There exists a faithful representation ρ of $C^*(\pi(T))$ such that $\rho(\pi(T))$ is a subnormal operator.
- (c) For every faithful representation ρ of $C^*(\pi(T))$, the operator $\rho(\pi(T))$ is a subnormal operator.
- (d) T is the restriction to an invariant subspace of an operator of the form $N + K$ where N is normal, K is compact and $\sigma_e(N) \subseteq \sigma_e(T)$.
- (e) T is the restriction to an invariant subspace of an essentially normal operator.
- (f) T is the compression of an essentially normal operator to an essentially invariant subspace.

Proof. Since subnormality is preserved by $*$ -homomorphisms and every C^* -algebra has a faithful representation, we see that (a), (b) and (c) are all equivalent. It's also clear that (d) implies (e) implies (f). It thus suffices to show that (b) implies (d) and (f) implies (a).

We first show that (b) implies (d). If T is an operator on $\mathcal{H}$ such that there exists a faithful representation ρ of $C^*(\pi(T))$ with $S = \rho(\pi(T))$ a subnormal operator, then $\sigma(S) = \sigma_e(T)$. Thus we may define a map $\Phi: C(\sigma_e(T)) \rightarrow C^*(\pi(T))$ as follows. For $f \in C(\sigma_e(T))$, let $\Phi(f) = \rho^{-1}(S_f)$. It is clear that Φ is a positive linear map. As such, by Theorem I.1.6 we may lift Φ to $\mathcal{B}(\mathcal{H})$; that is, there exists a positive linear map $\Phi_1: C(\sigma_e(T)) \rightarrow \mathcal{B}(\mathcal{H})$ such that $\Phi = \pi \circ \Phi_1$. Now, since Φ_1 is positive we may dilate Φ_1 ; see Theorem I.1.2 and I.1.3. That is, there exists a larger Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a $*$ -homomorphism $\varphi: C(\sigma_e(T)) \rightarrow \mathcal{B}(\mathcal{K})$ such that for each $f \in C(\sigma_e(T))$, $\Phi_1(f) = P\varphi(f)|_{\mathcal{H}}$ where P is the projection of $\mathcal{K}$ onto $\mathcal{H}$. If $N = \varphi(z)$, then N is a normal operator on $\mathcal{K}$ and $\sigma(N) \subseteq \sigma_e(T)$. Furthermore, it follows that $\varphi(f) = f(N)$ for any $f \in C(\sigma_e(T))$. Also, if $T_1 = \Phi_1(z)$, then T_1 is a compact perturbation of T since $\pi(T_1) = \pi(\Phi_1(z)) = \Phi(z) = \rho^{-1}(S) = \pi(T)$. Also, $T_1 = \Phi_1(z) = P\varphi(z)|_{\mathcal{H}} = PN|_{\mathcal{H}}$. We now show that $\mathcal{H}$ is essentially invariant for N . Let $N = \begin{bmatrix} T_1 & B \\ A & C \end{bmatrix}$

with respect to the decomposition $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$. We need to show that A is compact.

Since $S_{z^2} = S^*S$ and positive linear maps are self-adjoint, we see that $\Phi(|z|^2) = \Phi(z)^*\Phi(z)$. Thus $\Phi_1(|z|^2) = \Phi_1(z)^*\Phi_1(z) \pmod{\mathcal{B}_0}$. It follows that $PN^*N|_{\mathcal{H}} = (PN^*|_{\mathcal{H}})(PN|_{\mathcal{H}}) = T_1^*T_1 \pmod{\mathcal{B}_0}$. Now a matrix computation gives that $N^*N =$

$\begin{bmatrix} T_1^* T_1 + A^* A & * \\ * & * \end{bmatrix}$. So, $T_1^* T_1 + A^* A = T_1^* T_1 \pmod{\mathcal{B}_0}$. Thus A is compact. If $C = T -$

T_1 , then C is also compact. Now, if we define $K = \begin{bmatrix} C & 0 \\ -A & 0 \end{bmatrix}$, then K is compact and the operator $N + K$ has $\mathcal{H}$ as an invariant subspace. Further, $N+K$ is an extension of T and $\sigma_e(N+K) = \sigma_e(N) \subseteq \sigma(N) \subseteq \sigma_e(T)$. Thus (d) follows.

We now show that (f) implies (a). Suppose that R is an essentially normal operator with $\mathcal{M}$ an essentially invariant subspace and $T = PR|_{\mathcal{M}}$, where P is the projection on $\mathcal{M}$. By Lemma 7.2, there is an operator S that is a compact perturbation of R such that $\mathcal{M} \in \text{Lat} S$. In fact the proof of Lemma 7.2 shows that S may be chosen so that $T = S|_{\mathcal{M}}$. Now let Δ be any disk containing both $\sigma_e(T)$ and $\sigma_e(S)$. If p is any polynomial in z and $\bar{z}$ that is non-negative on Δ , then we must show that $p(T)$ equals a positive operator plus a compact operator. Now, since S is essentially normal, $\pi(S)$ is normal, hence $p(\pi(S)) \geq 0$. Thus, $p(S) = A + K$ where A is positive and K is compact. Since $S = \begin{bmatrix} T & * \\ 0 & * \end{bmatrix}$, an easy

computation gives that $p(S) = \begin{bmatrix} p(T) & * \\ * & * \end{bmatrix}$ if the convention from the hereditary functional

calculus is used. If $K = \begin{bmatrix} C & * \\ * & * \end{bmatrix}$ where C is necessarily compact, then $A = p(S) - K =$

$\begin{bmatrix} p(T) - C & * \\ * & * \end{bmatrix}$. Thus, since compressions of positive operators are positive, we see that

$p(T) - C \geq 0$. Thus $p(T) = (p(T) - C) + C$ is the sum of a positive operator and a compact operator. So Proposition 7.1 implies that T is essentially subnormal. $\square$

Corollary 7.4 If S is an essentially subnormal operator and $\mathcal{M}$ is an invariant subspace for S , then $S|_{\mathcal{M}}$ is also essentially subnormal.

Proof. This follows immediately from Theorem 7.3(e). $\square$

If K is a compact set in $\mathbb{C}$, then we denote by $R(K)$ the uniform closure of the rational functions with poles off K .

Theorem 7.5 If S is an essentially subnormal operator, then the following hold.

- (a) $\|[S^*, S]\|_e \leq \text{dist}[\bar{\mathbb{Z}}, R(\sigma_e(S))]^2$.
- (b) If $R(\sigma_e(S)) = C(\sigma_e(S))$, then S is essentially normal.
- (c) If $C^*(S) \cap \mathcal{B}_0 = (0)$, then S is subnormal.
- (d) $S = S_0 \oplus \bigoplus_{n=1}^{\infty} S_n$ where S_0 is a subnormal operator and each S_n , $n \geq 1$, is an essentially subnormal operator with the property that $\mathcal{B}_0 \subseteq C^*(S_n)$; in particular, S_n is irreducible.

Proof. (a) Consider the fundamental diagram for S .

$$\begin{array}{ccc} C^*(S) & & \\ \downarrow \pi & & \\ C^*(\pi(S)) & \xrightarrow[\rho]{} & C^*(T) \end{array}$$

Since, S is essentially subnormal we know that T is subnormal. Further since ρ is an isometry, we have that $\|[S^*, S]\|_e = \|[T^*, T]\| \leq \text{dist}[\bar{\mathbb{Z}}, R(\sigma(T))]^2 = \text{dist}[\bar{\mathbb{Z}}, R(\sigma_e(S))]^2$. The

inequality above for T is well known and due to Axler and Shapiro (see Conway [1991], p. 177 or Corollary III.1.3 below). So (a) follows and thus (b) does also by the Stone-Weierstrass Theorem.

(c) If $C^*(S) \cap \mathcal{B}_0 = (0)$, then the $*$ -homomorphism $\pi: C^*(S) \rightarrow C^*(\pi(S))$ is actually a $*$ -isomorphism. Thus, $\pi^{-1}: C^*(\pi(S)) \rightarrow C^*(S)$ is a well-defined $*$ -homomorphism. As $\pi(S)$ is a subnormal element, we see that S is also (see Corollary I.1.4).

(d) By Theorem 1.8 we see that there are reducing subspaces $\mathcal{H}_n$, $n \geq 0$, for S such that if

$S_n = S|_{\mathcal{H}_n}$, then $S = S_0 \oplus \bigoplus_{n=1}^{\infty} S_n$; where each S_n , $n \geq 1$, is an irreducible operator with the

property that $\mathcal{B}_0 \subseteq C^*(S_n)$ and $C^*(S_0) \cap \mathcal{B}_0 = (0)$. Since S is essentially subnormal, Corollary 7.4 says that each S_n is also. Furthermore, property (c) says that S_0 is subnormal. $\square$

Observe that in Theorem 7.5(d) the subnormal operator S_0 need not be a direct sum of irreducible operators. See Example 1.12 and Proposition 1.14.

Example 7.6

- (a) $S + K$ is essentially subnormal whenever S is subnormal and K is compact.
- (b) For any subnormal operator S , the Toeplitz operator with symbol $f \in L^\infty(\mu)$, S_f , is essentially subnormal whenever the Hankel operator H_f is compact.
- (c) $S^*|_{\mathcal{M}^\perp}$ is essentially subnormal whenever S is a subnormal operator with compact self-commutator and $\mathcal{M} \in \text{Lat} S$.
- (d) $S^{(\infty)}$ is essentially subnormal if and only if S is subnormal.

In particular in (c) one may choose, for example, S to be any cyclic subnormal operator or M_z on the Bergman space or Hardy space of a bounded region; see Theorem 7.5(b).

Proof. (a) and (c) are clear from Theorem 7.3 and Example (d) follows from Theorem 7.5(c). For (b) let $N = mneS$ and $\mu = svsmN$. If $f \in L^\infty(\mu)$ and S acts on $\mathcal{H}$, then $f(N) =$

$\begin{bmatrix} S_f & * \\ H_f & * \end{bmatrix}$ with respect to $\mathcal{H} \oplus \mathcal{H}^\perp$. So if P is the projection onto $\mathcal{H}$, then $H_f = P^\perp f(N)|_{\mathcal{H}}$ is

the usual Hankel operator with symbol f mapping $\mathcal{H}$ into $\mathcal{H}^\perp$. So if H_f is compact, then Lemma 7.1 says that $\mathcal{H}$ is an essentially invariant subspace for $f(N)$. Thus by Theorem 7.3(f), S_f is essentially subnormal since it is the compression of a normal operator to an essentially invariant subspace. $\square$

Regarding Example 7.6(b) it is known that when S is the unilateral shift and $f \in L^\infty(\partial\mathbf{D})$, then the Hankel operator H_f constructed from S is compact if and only if $f \in H^\infty + C$, where C denotes the continuous functions on $\partial\mathbf{D}$. It is also known which functions on $\mathbf{D}$ give compact Hankel operators for the Bergman shift; see Luecking [1992] or Stroethoff [1990]. These results give interesting examples of essentially subnormal operators and more motivation for understanding which Hankel operators are compact.

We now consider some basic questions about essentially subnormal operators. Some of which may be answered for essentially normal operators using Brown-Douglas-Fillmore

Theory. Recall that B-D-F implies that two essentially normal operators S, T are unitarily equivalent mod $\mathcal{B}_0$ if and only if $\sigma_e(S) = \sigma_e(T)$ and $\text{ind}(S-\lambda) = \text{ind}(T-\lambda)$ (see Davidson[1996], p. 275). Since $\pi(S)$ and $\pi(T)$ are normal elements in the Calkin algebra, this is equivalent to requiring that $C^*(\pi(S)) \approx C^*(\pi(T))$ and $\text{ind}(S-\lambda) = \text{ind}(T-\lambda)$. Thus it is natural to ask the following.

Question 7.7 If S and T are essentially subnormal, $C^*(\pi(S)) \approx C^*(\pi(T))$, and S and T have the same index function, then is S unitarily equivalent (mod $\mathcal{B}_0$) to T ?

As mentioned above, Question 7.7 has an affirmative answer when S and T are essentially normal. Also, one can show (as in Theorem 5.3) that Question 7.7 is also true for essential isometries. Recall that before the full force of B-D-F Theory, that applies to essentially normal operators, was established, the special case for normal operators was solved. This is known as the Weyl-von Neumann-Berg Theorem (see Theorem I.1.12 or Davidson [1996], p.59). Likewise, before the general problem in Question 7.7 is attacked, it would be nice to consider the case when S and T are subnormal.

Example 7.8 If λ_n is a sequence of positive numbers increasing to one, then let $S = \bigoplus_{n=1}^{\infty} (\lambda_n U)$, where U is the unilateral shift. One easily checks that S and $S \oplus S$ satisfy the hypotheses of Question 7.7, see Example 8.21 below. Is S unitarily equivalent to $S \oplus S$ mod $\mathcal{B}_0$?

To illustrate a possible line of questioning, recall that in B-D-F Theory one considers extensions of the Calkin Algebra. An extension of $\mathcal{B}/\mathcal{B}_0$ can simply be viewed as a $*$ -homomorphism $\varphi:C(K) \rightarrow \mathcal{B}/\mathcal{B}_0$. In fact there is a natural correspondence between essentially normal operators and extensions of the Calkin algebra. The following observation shows a similar result for essentially subnormal operators. Although, instead of considering $*$ -homomorphisms into $\mathcal{B}/\mathcal{B}_0$, one considers positive linear maps. Observe that if S is an essentially subnormal operator, then the *essential normal spectrum* of S is well defined, exactly as in Section 4. We shall denote it, as before, by $\sigma_n(\pi(S))$.

Proposition 7.9 An operator S is essentially subnormal if and only if there is a positive linear map $\rho:C(K) \rightarrow \mathcal{B}/\mathcal{B}_0$ such that $\rho(z) = \pi(S)$ and $\rho(\bar{z}z) = \pi(S)^*\pi(S)$.

Suppose S is an essentially subnormal operator and φ is a faithful representation of $C^*(\pi(S))$. If $T = \varphi(\pi(S))$, then T is subnormal. One can define a positive linear map $\rho_S:C(\sigma_n(\pi(S))) \rightarrow \mathcal{B}/\mathcal{B}_0$ as follows. If $f \in C(\sigma_n(\pi(S))) = C(\sigma_n(T))$, then $\rho_S(f) = \varphi^{-1}(T_f)$. Notice that by Lemma 3.8, ρ_S is one-to-one. Observe that ρ_S is independent of the faithful representation φ . In fact, ρ_S is uniquely determined by its defining properties. We shall call ρ_S the positive linear map associated to S .

It is natural to ask if one can consider equivalence classes of positive linear maps $\rho:C(K) \rightarrow \mathcal{B}/\mathcal{B}_0$ such that $\rho(\bar{z}z) = \rho(z)^*\rho(z)$ and make any progress on Question 7.7.

Another natural question is to characterize the operators of the form “subnormal plus compact” among all essentially subnormal operators. Notice that if $T = S + K$ where S is subnormal and K is compact, then $\text{ind}(T - \lambda) \leq 0$ for all $\lambda \notin \sigma_e(T)$. Thus, the following is a reasonable conjecture.

Question 7.10 If T is an essentially subnormal operator such that $\text{ind}(T - \lambda) \leq 0$ for all $\lambda \notin \sigma_e(T)$, then is T unitarily equivalent mod $\mathcal{B}_0$ to a subnormal operator ?

We shall give two results regarding Question 7.10. The first result is well known. It follows from B-D-F Theory and the fact that we can construct essentially normal subnormal operators with any spectral picture, provided the index function is non-positive.

Proposition 7.11 If T is essentially normal and $\text{ind}(T - \lambda) \leq 0$ for all $\lambda \notin \sigma_e(T)$, then T is unitarily equivalent mod $\mathcal{B}_0$ to a subnormal operator.

The following result is similar to one in B-D-F Theory.

Proposition 7.12 If T is an essentially subnormal operator on $\mathcal{H}$, then T is unitarily equivalent mod $\mathcal{B}_0$ to a subnormal operator if and only if the associated positive linear map ρ_T has a lift; that is, there exists a compact set $K \supseteq \sigma_n(\pi(S))$ and a positive linear map $\rho: C(K) \rightarrow B(\mathcal{H})$ such that $\rho(\bar{z}z) = \rho(z)^* \rho(z)$ and $\rho_T(f) = \pi \circ \rho(f)$ for each $f \in C(K)$.

Proof. If ρ exists, then, by Corollary I.1.4, $\rho(z)$ is a subnormal operator. Also, $\pi \circ \rho(z) = \rho_T(z) = \pi(T)$. So, $\rho(z)$ is a compact perturbation of T . Conversely, if there exists a subnormal operator S such that $S - T$ is compact, then $\sigma_n(\pi(S)) = \sigma_n(\pi(T))$. If $K = \sigma_n(S)$, then, by Proposition 4.2(a), $K \supseteq \sigma_n(\pi(S))$. Define $\rho: C(K) \rightarrow B(H)$ by $\rho(f) = S_f$, the

Toeplitz operator with symbol f . Clearly, ρ is a positive linear map such that $\rho(\bar{z}z) = \rho(z)^*\rho(z)$. Thus, ρ_T and $\pi \circ \rho$ are two positive linear maps on $C(K)$ that agree on z and $\bar{z}z$. Hence $\rho_T = \pi \circ \rho$ on $C(K)$. $\square$

Notice that it is not clear that Question 7.7 implies Question 7.10. If such an implication is to hold, then we must be able to construct some subnormal operators.

Question 7.13 If S is any subnormal operator and n_U is a non-positive integer for each bounded component U of $\mathbb{C} - \sigma_e(S)$, then can one find a subnormal operator T such that $C^*(\pi(T)) \approx C^*(S)$ and $\text{ind}(T - \lambda) = n_U$ for $\lambda \in U$.

Suppose S is essentially subnormal. Let ϕ be a faithful representation of $C^*(\pi(S))$ and let $T = \phi(\pi(S))$. So T is a subnormal operator. Also S is essentially normal if and only if T is normal. Let's say that S is *essentially pure* if ϕ may be chosen such that T is pure. For example, a quasinormal operator $U \otimes A$ is essentially pure if and only if $\sigma(A) = \sigma_e(A)$; see Corollary 8.22. Also notice that Theorem 1.4 will be useful for these matters. See Theorem 9.11 and Example 9.12 below for a non-trivial example of an irreducible subnormal operator that is not essentially pure.

Concerning the invariant subspace problem for essentially subnormal operators one should see Lange [1983] where some partial results are given. His approach is via spectral sets. The following question is an "essential version" of the Berger-Shaw Theorem. If true, then it would reduce the invariant subspace problem for essentially subnormal operators back to that for essentially normal operators.

Question 7.14 If S is a cyclic essentially subnormal operator, then is $[S^*, S]$ compact?

8. Tensor Products

In this section we shall use C^* -algebra techniques to investigate tensor products of subnormal operators more closely. If S and T are subnormal operators, then $S \otimes T$ is a subnormal operator. If N and M are normal extensions of S and T , then $N \otimes M$ is a normal extension of $S \otimes T$. However $N \otimes M$ is seldom the minimal normal extension of $S \otimes T$. Even if S is the Bergman shift, then the minimal normal extension of $S \otimes S$ is complicated to describe (see Conway [1991]). Thus we shall concentrate on a more manageable class of tensor products, namely $S \otimes N$ where S is a subnormal and N is a normal. These are natural generalizations of quasinormal operators, yet offer much greater diversity. A pure quasinormal operator has the form $U \otimes A$, where U is the unilateral shift and A is a positive operator with dense range. It is easy to check that the self-commutator of $S \otimes N$ is $[S^*, S] \otimes N^*N$. Hence $S \otimes N$ has compact self-commutator if and only if both $[S^*, S]$ and N are compact. Thus, these operators are nice computable examples of subnormal operators that generally do not have compact self-commutator. However $S \otimes N$ is generally reducible. In fact, every reducing subspace for N gives rise to a reducing subspace of $S \otimes N$. Hence they have a rich lattice of reducing subspaces and as such they generate non-trivial von Neumann algebras.

Recall that a normal operator N is one-to-one if and only if it has dense range. This follows because $\|N^*x\| = \|Nx\|$ for all x .

Proposition 8.1 If S is a subnormal operator and N is a normal operator, then $S \otimes N$ is a pure subnormal operator if and only if S is pure and N has dense range.

Proof. If S acts on $\mathcal{H}$ and N acts on $\mathcal{K}$, then $T = S \otimes N$ acts on $\mathcal{H} \otimes \mathcal{K}$. As mentioned above T is subnormal. Assume that S is pure and N has dense range. To show T is pure it suffices to show that the smallest invariant subspace of T which contains the range of $[T^*, T]$ is all of $\mathcal{H} \otimes \mathcal{K}$. So, if $\mathcal{M}$ is an invariant subspace of T which contains the range of $[T^*, T] = [S^*, S] \otimes N^*N$, then we have that $\mathcal{M}$ contains all vectors of the form $[T^*, T](h \otimes k) = [S^*, S]h \otimes N^*Nk$. As N has dense range, so does N^*N . Hence as k varies we see that $\mathcal{M}$ contains all vectors of the form $[S^*, S]h \otimes k$ for any h and k . Now let $\mathcal{L} = \{x \in \mathcal{H} : \text{for every } k \in \mathcal{K}, x \otimes k \in \mathcal{M}\}$. As noted above the range of $[S^*, S]$ is contained in $\mathcal{L}$. Furthermore as $\mathcal{M}$ is invariant for T , one easily checks that $\mathcal{L}$ is closed and invariant for S . Since S is pure we have that $\mathcal{L} = \mathcal{H}$. Thus we see that $\mathcal{M}$ contains every vector of the form $h \otimes k$ for any $h \in \mathcal{H}$ and $k \in \mathcal{K}$. Since these vectors have dense linear span in $\mathcal{H} \otimes \mathcal{K}$, we have $\mathcal{M} = \mathcal{H} \otimes \mathcal{K}$ and thus T is pure.

Conversely, if N is not one-to-one, then $S \otimes N$ has a non-trivial kernel, hence cannot be pure. If $S = S_1 \oplus M$ where M is normal, then $S \otimes N = (S_1 \otimes N) \oplus (M \otimes N)$ is not pure. $\square$

Corollary 8.2 If S is a pure subnormal operator and T any subnormal operator such that $mneT$ is one-to-one, then $S \otimes T$ is a pure subnormal operator.

Proof. That $S \otimes T$ is subnormal is clear. Let $N = mneT$. Since $S \otimes T$ is the restriction of the pure subnormal operator $S \otimes N$ to an invariant subspace, $S \otimes T$ must also be pure. $\square$

Proposition 8.3 If S is a pure subnormal operator and N is a normal operator with dense range, then $\text{mne}(S \otimes N) = \text{mne}(S) \otimes N$.

Proof. If $M = \text{mne}S$, then $M \otimes N$ is a normal extension of $S \otimes N$. If S acts on $\mathcal{H}$ and N acts on $\mathcal{K}$, then $S \otimes N$ acts on $\mathcal{H} \otimes \mathcal{K}$. Also, if M acts on $\mathcal{K}_0$, then $M \otimes N$ acts on $\mathcal{K}_0 \otimes \mathcal{K}$. If $\mathcal{M}$ is a reducing subspace for $M \otimes N$ that contains $\mathcal{H} \otimes \mathcal{K}$, then we must show that $\mathcal{M} = \mathcal{K}_0 \otimes \mathcal{K}$. Let $\mathcal{L} = \{x \in \mathcal{K}_0 : x \otimes k \in \mathcal{M} \text{ for every } k \in \mathcal{K}\}$. Clearly, $\mathcal{H} \subseteq \mathcal{L}$ and $\mathcal{L}$ is a closed linear subspace of $\mathcal{K}_0$. In fact, since N has dense range and $\mathcal{M}$ is a reducing subspace for $M \otimes N$, it follows that $\mathcal{L}$ is a reducing subspace for M . Since $\mathcal{H} \subseteq \mathcal{L}$ and $M = \text{mne}S$, it follows that $\mathcal{H} = \mathcal{L}$. Thus $h \otimes k \in \mathcal{M}$ for every $h \in \mathcal{K}_0$ and $k \in \mathcal{K}$. So, $\mathcal{M} = \mathcal{K}_0 \otimes \mathcal{K}$ and thus $M \otimes N = \text{mne}(S \otimes N)$. $\square$

Corollary 8.4 If S is a pure subnormal operator and N is a normal operator with dense range, then $\text{dual}(S \otimes N) = \text{dual}(S) \otimes N^*$.

Proof. Keeping the notation from Proposition 8.3, we have that $M \otimes N = \text{mne}(S \otimes N)$. So, $\text{dual}(S \otimes N)$ is the restriction of $M^* \otimes N^*$ to $(\mathcal{H} \otimes \mathcal{K})^\perp$. Since $(\mathcal{H} \otimes \mathcal{K})^\perp = \mathcal{H}^\perp \otimes \mathcal{K}$, we see that $(M^* \otimes N^*)|_{(\mathcal{H}^\perp \otimes \mathcal{K})} = M^*|_{\mathcal{H}^\perp} \otimes N^* = \text{dual}(S) \otimes N^*$. $\square$

We now give a nice application of C^* -algebra techniques to determine the spectral properties of the operators $S \otimes N$. If K and L are two compact subsets of $\mathbb{C}$, then KL denotes the set of all possible products from K and L ; that is, the range of the multiplication map $m: K \times L \rightarrow \mathbb{C}$ given by $m(z, w) = zw$. Since $K \times L$ is compact, we see

that KL is also compact. Part (a) of Theorem 8.5 is well known and holds for any two operators, see Brown and Pearcy [1966].

Theorem 8.5 If S is a pure subnormal operator and N is a normal operator with dense range, then the following hold:

- (a) $\sigma(S \otimes N) = \sigma(S)\sigma(N)$,
- (b) $\sigma_{ap}(S \otimes N) = \sigma_{ap}(S)\sigma(N)$,
- (c) $\sigma_n(S \otimes N) = \sigma_n(S)\sigma(N)$.

Proof. A version of the Weyl-von Neumann-Berg Theorem says that there exists a diagonal operator D such that $D \sim_{sa} N$, see Theorem 1.1.12. In fact D may be any diagonal operator such that $\sigma_e(D) = \sigma_e(N)$ and D has the same isolated eigenvalues as N with the same multiplicities. In particular it follows that if D has eigenvalues $\{\lambda_n\}$, then $cl\{\lambda_n\} = \sigma(N)$. Now, since $\sigma(D) = \sigma(N)$, Corollary 2.8 says that $C^*(S \otimes N) \approx C^*(S \otimes D)$. It follows that $S \otimes N$ and $S \otimes D$ have the same spectrum, approximate spectrum and normal spectrum. That the operators have the same spectrum is clear. It follows from a result of Bunce, that they have the same approximate point spectrum and from Corollary 3.3 that they have the same normal spectrum. Thus it suffices to determine the spectral properties of $S \otimes D$. However, since D is diagonal, $D = \bigoplus_{n=1}^{\infty} \lambda_n$, we see that $S \otimes D \cong$

$\bigoplus_{n=1}^{\infty} \lambda_n S$. That is, $S \otimes D$ is unitarily equivalent to a direct sum of multiplies of S . Since,

$$\sigma\left(\bigoplus_{n=1}^{\infty} \lambda_n S\right) = cl\left[\bigcup_{n=1}^{\infty} \lambda_n \sigma(S)\right] = cl[\{\lambda_n\}\sigma(S)] = \sigma(D)\sigma(S) = \sigma(N)\sigma(S), \quad (a) \text{ follows.}$$

Similarly, (b) and (c) follow. $\square$

Observe that property (a) above holds for any operator S and (b) holds whenever S is hyponormal. Also, Theorem 8.5 allows us to compute the essential normal spectrum in some cases.

Corollary 8.6 If S is a pure subnormal operator such that $\sigma_{\text{ap}}(S) = \sigma_n(S)$ and N is a normal operator with dense range, then $\sigma_n(\pi(S \otimes N)) = \sigma_{\text{ap}}(S \otimes N) = \sigma_n(S \otimes N)$.

Proof. Since $\sigma_{\text{ap}}(S) = \sigma_n(S)$, we see from Theorem 8.5 that $\sigma_{\text{ap}}(S \otimes N) = \sigma_n(S \otimes N)$. Now Proposition 4.2(a) says that $\sigma_n(\pi(S \otimes N))$ always lies in between $\sigma_{\text{ap}}(S \otimes N)$ and $\sigma_n(S \otimes N)$. Thus all three coincide. $\square$

Corollary 8.7 If S is a pure quasinormal operator, then $\sigma_n(\pi(S)) = \sigma_{\text{ap}}(S) = \sigma_n(S)$.

Here are some natural questions to consider for general subnormal operators, although they are often easier to answer for the operators of the form $S \otimes N$ when S is well behaved. In particular, if N and M are normal operators with dense range, when do $S \otimes N$ and $S \otimes M$ have isomorphic C^* -algebras? When are they unitarily equivalent? What does $C^*(\pi(S \otimes N))$ look like? When does $C^*(S \otimes N)$ contain a compact operator?

For the first question see Corollary 2.8. We now consider the question of when $S \otimes N$ is unitarily equivalent to $S \otimes M$. Notice that if $N \cong M$, then of course $S \otimes N \cong S \otimes M$. But generally the converse is not true. If $S = U$, the unilateral shift, and N is a normal operator with dense range, then we always have that $U \otimes N \cong U \otimes |N|$. In general $S \otimes N$ is not unitarily equivalent to $S \otimes A$ for any self-adjoint operator A . In fact there are three different classes of operators here: $S \otimes A$ where A is positive; $S \otimes B$ where B is self-

adjoint; and $S \otimes N$ where N is normal. Furthermore these classes get progressively more complex and are not equal. If S is the unilateral shift, however, then the three classes all coincide.

We want to see when the following is true: If A and B are self-adjoint with dense range and $S \otimes A \cong S \otimes B$, then is $|A| \cong |B|$? Notice that the self-commutator of $S \otimes A$ is $[S^*, S] \otimes A^2$. So, given that $S \otimes A \cong S \otimes B$ we see that $[S^*, S] \otimes A^2 \cong [S^*, S] \otimes B^2$ must be true. Hence we would like to “cancel” the term $[S^*, S]$ and conclude that $A^2 \cong B^2$. To do this requires a little work.

First observe that if $A \otimes B \cong A \otimes C$ for positive operators A , B and C , then B and C need not be unitarily equivalent. Simply choose $A = I_\infty$, $B = I_n$ and $C = I_k$ where $n \neq k$ and where we denote the identity operator on a space of dimension α by I_α . It follows that both $A \otimes B$ and $A \otimes C$ are unitarily equivalent to the identity operator on an infinite dimensional space, but clearly B is not equivalent to C . So, some additional hypothesis is necessary. For any self-adjoint operator A , let n_A denote the multiplicity function for A . Also, any reference to the a.e. behavior of n_A naturally means with respect to a scalar valued spectral measure (svsm) for A . We say that A has a finite multiplicity function if n_A is finite a.e.

Theorem 8.8 Suppose A , B and C are positive operators such that A is compact and either B or C has a finite multiplicity function. If $A \otimes B \cong A \otimes C$, then $B \cong C$.

Corollary 8.9 Suppose S is a hyponormal operator with compact self-commutator.

- (a) If N and M are normal operators with dense ranges such that either $|N|$ or $|M|$ has a finite multiplicity function and $S \otimes N \cong S \otimes M$, then $|N| \cong |M|$.
- (b) If A and B are self-adjoint operators with dense ranges such that either A or B has a finite multiplicity function and $S \otimes A \cong S \otimes B$, then $|A| \cong |B|$.
- (c) If A and B are positive operators with dense ranges such that either A or B has a finite multiplicity function, then $S \otimes A \cong S \otimes B$ if and only if $A \cong B$.

Proof. (a) As $S \otimes N \cong S \otimes M$, the self-commutators are also unitarily equivalent. So $[S^*, S] \otimes N^*N \cong [S^*, S] \otimes M^*M$. Since the multiplicity functions of $|N|$ and $|M|$ are finite a.e., so are the multiplicity functions of $N^*N = |N|^2$ and $M^*M = |M|^2$. Hence Theorem 8.8 implies that $N^*N \cong M^*M$, and this is equivalent to $|N| \cong |M|$.

(b) and (c) These follow exactly as above except notice that the self-commutators are $[S^*, S] \otimes A^2$ and $[S^*, S] \otimes B^2$. Since A and B have finite multiplicity functions, so do A^2 and B^2 . Thus Theorem 8.8 applies to give that $A^2 \cong B^2$. That is $|A| \cong |B|$. $\square$

Now let's begin with some preliminary results that will help us restate our hypothesis in Theorem 8.8 involving the multiplicity functions.

Lemma 8.10 If S is any irreducible subnormal operator with compact self-commutator acting on $\mathcal{H}_1$ and N is any normal operator acting on $\mathcal{H}_2$ with $\sigma(N) \subseteq \sigma_{\text{ap}}(S)$, then the projection P of $\mathcal{H}_1 \oplus \mathcal{H}_2$ onto $\mathcal{H}_1$ is in $W^*(S \oplus N)$.

Proof. By Lemma 1.5, we know that the map which sends $S_f + K \rightarrow (S_f + K) \oplus f(N)$ is a *-isomorphism of $C^*(S)$ onto $C^*(S \oplus N)$. In particular $K \oplus 0 \in C^*(S \oplus N)$ for any compact operator K on $\mathcal{H}_1$. Hence if we choose compact operators $\{K_n\}$ on $\mathcal{H}_1$ which converge strongly to the identity on $\mathcal{H}_1$, then clearly $\{K_n \oplus 0\}$ converges strongly to P . Hence $P \in W^*(S \oplus N)$. $\square$

Lemma 8.11 If $W = U \oplus V$ is an isometry on $\mathcal{H}_1 \oplus \mathcal{H}_2$ where U is a unilateral shift on $\mathcal{H}_1$ and V is a unitary on $\mathcal{H}_2$, then the projection P onto $\mathcal{H}_1$ is in $W^*(W)$.

Proof. Notice that U may not have compact self-commutator. However, if U_1 is a unilateral shift of multiplicity 1, then the projection onto the first summand of $U_1 \oplus V$ belongs to $W^*(U_1 \oplus V)$ by Lemma 8.10. Now, observe that the map from $W^*(U_1 \oplus V)$ to $W^*(W)$ given by $A \oplus B$ goes to $A^{(\infty)} \oplus B$ is a *-isomorphism. Thus, since $I \oplus 0$ belongs to $W^*(U_1 \oplus V)$, we see that P belongs to $W^*(W)$. $\square$

Lemma 8.12 Let W be a unilateral shift of arbitrary multiplicity on $\mathcal{H}$. If a self-adjoint operator A on $\mathcal{H}$ commutes with W , then A has uniform infinite multiplicity.

Proof. Observe that W is quasinormal. In fact $W \cong U \otimes I_k$, where I_k is the identity on a k -dimensional space $\mathcal{L}$ and U is the unilateral shift (with multiplicity one), and $U \otimes I_k$ acts on the infinite direct sum $\mathcal{L}^{(\infty)}$. So, if A commutes with $U \otimes I_k$ then the matrix representation of A with respect to $\mathcal{L}^{(\infty)}$ must be lower triangular and constant along the diagonals. But as A is self-adjoint this means that the matrix for A must actually be

diagonal. Since the diagonal entries must all be equal to some operator B we have that $A \cong B^{(\infty)}$ for some self-adjoint operator B acting on $\mathcal{L}$. Hence A has uniform infinite multiplicity. $\square$

Now if $\mathcal{A}$ is a von Neumann algebra, then two projections P and Q in $\mathcal{A}$ are said to be equivalent (in $\mathcal{A}$) if there is a partial isometry W in $\mathcal{A}$ so that $P = W^*W$ and $Q = WW^*$. A projection is said to be infinite (in $\mathcal{A}$) if it is equivalent to a proper subprojection of itself, which is also in $\mathcal{A}$. Otherwise it is finite. A von Neumann algebra $\mathcal{A}$ is finite if the identity operator I is finite in $\mathcal{A}$. So, for any partial isometry W in $\mathcal{A}$ with $W^*W = I$, we must also have $WW^* = I$. But $W^*W = I$ if and only if W is an isometry, and $W^*W = WW^* = I$ if and only if W is a unitary. So, $\mathcal{A}$ is finite if and only if every isometry in $\mathcal{A}$ is actually a unitary. Notice that if $\mathcal{A}$ is abelian, then this is trivially true. So every abelian von Neumann algebra is finite. We now need to investigate when the commutant of a self-adjoint operator is finite. So, let $\{A\}'$ denote the commutant of the operator A . Although, the following result is classical we include a proof for completeness.

Lemma 8.13 If A is self-adjoint, then $\{A\}'$ is a finite von Neumann algebra if and only if the multiplicity function n_A is finite a.e..

Proof. Suppose the multiplicity function n_A is finite a.e.. Let W be an isometry in $\{A\}'$. We must show that W is a unitary. So, let $W = U \oplus V$ be the Wold decomposition where U is a unilateral shift and V is a unitary. Let P be the projection onto the first coordinate

space in the decomposition for W . So $PW = U \oplus 0$. By Lemma 8.11, $P \in W^*(W) = \{W, W^*\}''$. Now as A is self-adjoint and commutes with W , A also commutes with W^* . Hence $A \in \{W, W^*\}'$. So P also commutes with A . Now if $\mathcal{M}$ is the range of P , then $\mathcal{M}$ is a reducing subspace for A and, on $\mathcal{M}$, A and U commute. By Lemma 8.12, A must have uniform infinite multiplicity on $\mathcal{M}$. Thus as n_A is finite a.e. we have that $\mathcal{M} = 0$. So $W = V$ is a unitary. Thus $\{A\}'$ is a finite von Neumann algebra.

Conversely, suppose there is a Borel set Δ with $n_A = \infty$ on Δ . If E is the spectral measure for A , then let $\mathcal{M}$ be the range of $E(\Delta)$. Thus $\mathcal{M}$ reduces A and $A|_{\mathcal{M}}$ has uniform infinite multiplicity. If $B = M_t$ on $L^2(\mu|\Delta)$ where $\mu = \text{svsm}A$, then there are reducing subspaces $\mathcal{M}_n$ for A such that $\mathcal{M} = \oplus \mathcal{M}_n$ and such that $A|_{\mathcal{M}_n} \cong B$. If $U_n: \mathcal{M}_n \rightarrow L^2(\mu|\Delta)$ is a unitary

such that $U_n^* B U_n = A|_{\mathcal{M}_n}$, then define U on $\mathcal{M}$ by $U = \begin{bmatrix} 0 & 0 & 0 & 0 \\ U_2^* U_1 & 0 & 0 & 0 \\ 0 & U_3^* U_2 & 0 & 0 \\ 0 & 0 & U_4^* U_3 & \dots \end{bmatrix}$ with

respect to the decomposition of $\mathcal{M}$ as $\oplus \mathcal{M}_n$. An easy computation shows that $A|_{\mathcal{M}}$ commutes with U . So if $W = U \oplus I$ on $\mathcal{M} \oplus \mathcal{M}^\perp$, then it's easy to see that W is an isometry that is not a unitary and $W \in \{A\}'$. So $\{A\}'$ is not a finite von Neumann algebra, and the converse follows. $\square$

Finally let us state, for completeness, the following result which will be used.

Theorem 8.14 (Kadison & Singer [1957])

Suppose A , B , and C are operators such that $W^*(A \oplus B)$ and $W^*(A \oplus B)'$ are finite von Neumann algebras. If $A \oplus B \cong A \oplus C$, then $B \cong C$.

Proof of Theorem 8.8. Suppose $A \otimes B \cong A \otimes C$ where A , B , and C are positive. Suppose also that A is compact and B has a finite multiplicity function. We may suppose that $\|A\| = \|B\| = \|C\| = 1$ and that A is one-to-one. Let the distinct eigenvalues of A be $\{1, \lambda_1, \lambda_2, \dots\}$, where $1 = \lambda_0 > \lambda_1 > \lambda_2 > \dots$ and where λ_k has multiplicity n_k . So we have the following,

$$(*) \quad A \otimes B \cong \begin{bmatrix} B^{(n_0)} & 0 & 0 & 0 \\ 0 & \lambda_1 B^{(n_1)} & 0 & 0 \\ 0 & 0 & \lambda_2 B^{(n_2)} & 0 \\ 0 & 0 & 0 & \dots \end{bmatrix} = B^{(n_0)} \oplus \left\{ \bigoplus_{k=1}^{\infty} \lambda_k B^{(n_k)} \right\}$$

Likewise for the operator C . As $A \otimes B \cong A \otimes C$, they have equivalent scalar valued spectral measures. Let $\mu = \text{svsm}B$ and $\nu = \text{svsm}C$. Now the only summand of $A \otimes B$ which puts positive spectral measure on the interval $(\lambda_1, 1]$ is $B^{(n_0)}$. Likewise $C^{(n_0)}$ is the only summand from $A \otimes C$ which contributes to the spectral measure on $(\lambda_1, 1]$. Thus we have that $\mu|_{(\lambda_1, 1]} \approx \nu|_{(\lambda_1, 1]}$. Also, for $t \in (\lambda_1, 1]$, $n_0 n_B(t) = n_{A \otimes B}(t) = n_{A \otimes C}(t) = n_0 n_C(t)$, so $n_B(t) = n_C(t)$ for a.e. $t \in (\lambda_1, 1]$. By considering the spectral projection for B corresponding to the interval $(\lambda_1, 1]$ we can decompose B as $B = B_1 \oplus B_2$ where $\text{svsm}B_1 = \mu|_{(\lambda_1, 1]}$ and $\text{svsm}B_2$ is supported on $[0, \lambda_1]$. Also write $C = C_1 \oplus C_2$ in a similar way. Since for B_1 and C_1 the spectral measures are equivalent and multiplicity functions agree, we have that $B_1 \cong C_1$. Thus we also have $A \otimes B_1 \cong A \otimes C_1$. So we have the following,

$$(A \otimes B_1) \oplus (A \otimes B_2) \cong (A \otimes C_1) \oplus (A \otimes C_2).$$

Hence by the Kadison/Singer Theorem we can conclude that $(A \otimes B_2) \cong (A \otimes C_2)$ because by using the matrix representation as in (*) above it's easy to see that $A \otimes B$ has a finite multiplicity function, since B does. Thus Lemma 8.13 says that the hypothesis of the Kadison/Singer Theorem is satisfied.

As $(A \otimes B_2) \cong (A \otimes C_2)$, we have $\|B_2\| = \|C_2\| \leq \lambda_1$. So arguing as above we again see that on the interval $J_2 = (\lambda_1 \|B_2\|, \|B_2\|] = (\lambda_1 \|C_2\|, \|C_2\|]$, B_2 and C_2 have equivalent spectral measures and the same multiplicity function. Hence we can write $B_2 = B_3 \oplus B_4$ and $C_2 = C_3 \oplus C_4$ where B_3, C_3 have spectral measures carried by J_2 and B_4, C_4 have spectral measures supported on $[0, \lambda_1 \|B_2\|] = [0, \lambda_1 \|C_2\|]$. So, again we get $(A \otimes B_3) \oplus (A \otimes B_4) \cong (A \otimes C_3) \oplus (A \otimes C_4)$ and, because $B_3 \cong C_3$ and B has a finite multiplicity function, the Kadison/Singer Theorem gives that $A \otimes B_4 \cong A \otimes C_4$. Also, $\|B_4\| = \|C_4\| \leq \lambda_1 \|B_2\| \leq \lambda_1^2$.

Continuing in the manner and since $\lambda_1^n \rightarrow 0$, we see that $B \cong C$. $\square$

We now give an example to show that the finiteness assumption in Theorem 8.8 and Corollary 8.9 is really necessary. Recall an example was mentioned above involving the identity operator acting on spaces of different dimensions. This example does not work if the operator A is required to be compact.

Example 8.15 Let A be the diagonal operator whose entries are $\{1, \frac{1}{2}, \frac{1}{4}, \dots, (\frac{1}{2})^n, \dots\}$ each with multiplicity one. For $0 \leq \delta < 1$, let B_δ be M_t on $L^2(\delta, 1)$. If $0 \leq \delta, \varepsilon \leq \frac{1}{2}$, then $A \otimes (B_\delta)^{(\infty)} \cong A \otimes (B_\varepsilon)^{(\infty)}$. However, if $\delta \neq \varepsilon$, then $B_\delta^{(\infty)}$ is not equivalent to $B_\varepsilon^{(\infty)}$.

Proof. If $\delta \leq \frac{1}{2}$, then $\text{svsm}(A \otimes B_\delta)$ is Lebesgue measure on $(0,1)$. So, if $0 \leq \delta, \varepsilon \leq \frac{1}{2}$ then $A \otimes (B_\delta)^{(\infty)} \cong (A \otimes B_\delta)^{(\infty)} \cong (A \otimes B_\varepsilon)^{(\infty)} \cong A \otimes (B_\varepsilon)^{(\infty)}$; but clearly if $\delta \neq \varepsilon$, then $B_\delta^{(\infty)}$ is not equivalent to $B_\varepsilon^{(\infty)}$. $\square$

We now use Example 8.15 to construct another example illustrating how far things can go wrong in Corollary 8.9 if A or B does not have finite multiplicity.

Example 8.16 There exists a pure subnormal operator S with compact self-commutator and two positive operators B, C such that $S \otimes B \cong S \otimes C$, yet $\sigma(B) \neq \sigma(C)$.

Proof. Keeping the notation from Example 8.15, if $S = U \otimes A$, where U is the unilateral shift, then S is a pure subnormal operator with compact self-commutator; namely $A^2 \oplus 0_\infty$. So, if $\varepsilon \neq \delta$ and $0 \leq \delta, \varepsilon \leq \frac{1}{2}$ and we set $B = B_\delta^{(\infty)}$ and $C = B_\varepsilon^{(\infty)}$, then we have $A \otimes B \cong A \otimes C$. Hence $U \otimes (A \otimes B) \cong U \otimes (A \otimes C)$. Thus $S \otimes B \cong S \otimes C$ for positive operators B, C but B and C do not even have the same spectrum. $\square$

Question 8.17 What are necessary and sufficient conditions for $C^*(S \otimes N) \approx C^*(S \otimes M)$ where S is a pure subnormal operator and N, M are normal operators?

Observe that Corollary 2.8 says that if $\sigma(N) = \sigma(M)$, then $C^*(S \otimes N) \approx C^*(S \otimes M)$. In view of Example 8.16 the converse is certainly not true. Also note that the example given in 8.16 has another interesting property. If $\varepsilon \neq \delta$ and $0 \leq \delta, \varepsilon \leq \frac{1}{2}$, then define $S_1 = S \otimes B_\varepsilon$ and $S_2 = S \otimes B_\delta$. Now, Example 8.16 shows that $S_1^{(\infty)} \cong S_2^{(\infty)}$ and yet Corollary 8.9

says that S_1 is not unitarily equivalent to S_2 . This should be contrasted with the following.

If T_1 and T_2 are any two operators such that $T_1^{(2)} \cong T_2^{(2)}$, then $T_1 \cong T_2$ (see Kadison and Singer [1957]).

We now consider the question of identifying $C^*(\pi(S \otimes N))$. We start with a general result and then specialize to quasinormal operators.

Theorem 8.18 If $\sigma(N) = \sigma_e(N)$, then $C^*(S \otimes N)$ contains no compact operators.

Proof. If $\sigma(N) = \sigma_e(N)$, then $N \sim_a N^{(2)}$. Hence it follows that $S \otimes N \sim_a S \otimes N^{(2)}$ (see the proof of Proposition 2.7). Thus, (the proof of) Theorem 2.6 gives that $C^*(S \otimes N)$ contains no compact operators. $\square$

It follows from Theorem 8.18 that if $\sigma(N) = \sigma_e(N)$, then $C^*(\pi(S \otimes N)) \approx C^*(S \otimes N)$.

Note also that the converse of Theorem 8.18 is not true. See Example 8.24.

Corollary 8.19 If $\sigma(N) = \sigma_e(N)$, then $\sigma_n(\pi(S \otimes N)) = \sigma_n(S \otimes N)$.

We now consider the quasinormal case. In what follows U is the unilateral shift.

Theorem 8.20 Let $S = U \otimes A$ where A is a positive operator with dense range, and let κ be a faithful representation of $C^*(\pi(S))$ on a separable Hilbert space. If we set $T = \kappa(\pi(S))$ and let N and T_p be the normal and pure parts of T , respectively, then the following hold:

- (a) T is quasinormal;
- (b) $\text{cl}\{z : |z| \in \sigma(A) - \sigma_e(A)\} \subseteq \sigma(N) \subseteq \{z : |z| \in \sigma(A)\}$;
- (c) $T_p \cong U \otimes B$ for some positive operator B with dense range and $\sigma(B) = \sigma_e(A)$.

Conversely, if T is any operator which satisfies the above three conditions, then there is a faithful representation κ of $C^*(\pi(S))$ so that $T = \kappa(\pi(S))$.

Proof. Fix a faithful representation κ of $C^*(\pi(S))$ on a separable Hilbert space, set $\varphi = \kappa \circ \pi$ and let $T = \varphi(S)$. So $\varphi: C^*(S) \rightarrow C^*(T)$ is a $*$ -homomorphism and $\varphi(S) = T$. Clearly T is quasinormal. Let $T = N \oplus T_p$ with N normal and T_p pure. Now, $\sigma_{ap}(T) = \sigma_{ap}(S)$ because $\sigma_{ap}(T) = \sigma_l(T) = \sigma_{lc}(S) = \sigma_{ap}(S)$. By Theorem 8.5(b), we see that $\sigma_{ap}(S) = \{z : |z| \in \sigma(A)\}$ and clearly $\sigma_{ap}(T) = \sigma(N) \cup \sigma_{ap}(T_p)$. Thus, $\sigma(N) \subseteq \{z : |z| \in \sigma(A)\}$. Furthermore, since T is quasinormal, T_p is also. Hence $T_p \cong U \otimes B$ for some positive operator B with dense range (see Theorem 1.1.16). Now, $\sigma_e([S^*, S]) = \sigma([T^*, T]) = \sigma([T_p^*, T_p]) \cup \{0\}$. But $[S^*, S] = A^2 \oplus 0_\infty$ and $[T_p^*, T_p] = B^2 \oplus 0_\infty$. Hence we see that $\sigma_e(A) \cup \{0\} = \sigma(B) \cup \{0\}$. Now we want to conclude that $\sigma_e(A) = \sigma(B)$.

If $0 \in \sigma_e(A)$, then, as A is one-to-one, 0 cannot be an eigenvalue. Hence it must be a limit point of the non-zero elements in $\sigma_e(A)$. Hence also $0 \in \sigma(B)$, since these sets have the same non-zero elements. If $0 \in \sigma(B)$, then $0 \in \sigma_{ap}(T_p) \subseteq \sigma_{ap}(S) = \{z : |z| \in \sigma(A)\}$.

Thus, $0 \in \sigma(A)$. Further as A is one-to-one we must have $0 \in \sigma_e(A)$. So we have that $0 \in \sigma_e(A)$ if and only if $0 \in \sigma(B)$. Thus we do have that $\sigma_e(A) = \sigma(B)$.

It remains to show that $\text{cl}\{z : |z| \in \sigma(A) - \sigma_e(A)\} \subseteq \sigma(N)$. But as noted above $\{z : |z| \in \sigma(A)\} = \sigma_{\text{ap}}(S) = \sigma_{\text{ap}}(T) = \sigma(N) \cup \sigma_{\text{ap}}(T_p) = \sigma(N) \cup \{z : |z| \in \sigma(B) = \sigma_e(A)\}$. So we must have that $\text{cl}\{z : |z| \in \sigma(A) - \sigma_e(A)\} \subseteq \sigma(N)$. Thus all three properties hold.

For the converse, we will show that any two operators satisfying the three conditions have isomorphic C^* -algebras. One can easily check that this will prove the theorem.

So suppose $T = N \oplus T_p$ is given and satisfies the conditions (a), (b) and (c). If B is the bilateral shift, D is a diagonal operator whose entries are exactly the points in $\sigma(A) - \sigma_e(A)$ and C is any positive operator with dense range satisfying $\sigma(C) = \sigma_e(A)$, then $S_1 = (B \otimes D) \oplus (U \otimes C)$ is one operator that satisfies the three conditions above. We now show that $C^*(T) \approx C^*(S_1)$.

Let $\mu = \text{svsm}N$. If $\mu(\sigma_{\text{ap}}(T_p)) > 0$, then we may write $N = N_1 \oplus N_2$ where $\sigma(N_2) \subseteq \sigma_{\text{ap}}(T_p) = \{z : |z| \in \sigma(B) = \sigma_e(A)\}$ and $\text{svsm}N_1$ is carried by $\{z : |z| \in \sigma(A) - \sigma_e(A)\}$. Thus we have $T = N_1 \oplus N_2 \oplus T_p$ and, by (b), $\sigma(N_1) = \text{cl}\{z : |z| \in \sigma(A) - \sigma_e(A)\}$. Furthermore by Theorem 1.4 there is a $*$ -isomorphism $\psi_1 : C^*(N_2 \oplus T_p) \rightarrow C^*(T_p)$ with $\psi_1(N_2 \oplus T_p) = T_p$. Also as $T_p \cong U \otimes B$ there is, by Corollary 2.8, a $*$ -isomorphism $\psi_2 : C^*(T_p) \rightarrow C^*(U \otimes C)$

with $\psi_2(T_p) = U \otimes C$. Finally as $\sigma(N_1) = \sigma(B \otimes D)$ and these are normal operators, there is a $*$ -isomorphism $\psi_3: C^*(N_1) \rightarrow C^*(B \otimes D)$ with $\psi_3(N_1) = B \otimes D$. Thus $\psi = \psi_3 \oplus \psi_2 \circ \psi_1$ gives a $*$ -isomorphism from $C^*(T) = C^*(N_1 \oplus (N_2 \oplus T_p))$ onto $C^*(S_1)$ with $\psi(T) = S_1$. $\square$

Example 8.21 Let A be a diagonal operator whose entries are positive and increase to one. If $S = U \otimes A$, then there is a faithful representation κ of $C^*(\pi(S))$ such that $\kappa(\pi(S)) = (B \otimes A) \oplus U$, where B is the bilateral shift and U is the unilateral shift.

Notice that by Theorem 8.20 we can answer several natural questions about faithful representations of $C^*(\pi(S))$ for quasinormal operators S .

Corollary 8.22 If U is the unilateral shift and A is a positive operator with dense range, then there is a faithful representation κ of $C^*(\pi(S))$ such that:

- (a) $\kappa(\pi(S))$ is essentially normal if and only if $\sigma_e(A)$ is either finite or is a sequence converging to zero;
- (b) $\kappa(\pi(S))$ is pure if and only if $\sigma(A) = \sigma_e(A)$.
- (c) $\kappa(\pi(S))$ is irreducible (acting on an infinite dimensional space) if and only if $\sigma(A) = \sigma_e(A) = \{\lambda\}$, in which case $S = \lambda U^{(\infty)}$ and $\kappa(\pi(S))$ may be chosen to be λU .

Observe that all these various conditions on A are really conditions on the self-commutator of S , because $[S^*, S] = A^2 \oplus 0$.

Corollary 8.23 If $S = U \otimes A$ is a pure quasinormal operator, then $C^*(S)$ contains a compact operator if and only if $\sigma(A) = \sigma_e(A)$.

Proof. One direction was proved in Theorem 8.18. If $C^*(S)$ contains no compact operators, then $\pi: C^*(S) \rightarrow C^*(\pi(S))$ is a *-isomorphism. Hence $\sigma([S^*, S]) = \sigma_e([S^*, S])$. Since $[S^*, S] = A^2 \oplus 0_\infty$, we have $\sigma(A) \cup \{0\} = \sigma_e(A) \cup \{0\}$. Now since S is pure, A has dense range and thus $0 \in \sigma(A)$ if and only if $0 \in \sigma_e(A)$. So $\sigma(A) = \sigma_e(A)$. $\square$

We now show that Corollary 8.23 is not true for general operators of the form $S \otimes N$. That is, the converse of Theorem 8.18 is not true.

Example 8.24 There exists a pure subnormal operator S with compact self-commutator and a positive operator A with dense range such that $C^*(S \otimes A)$ contains no compact operators, yet $\sigma(A) \neq \sigma_e(A)$.

Proof. Let B be the diagonal operator whose entries are $\{1, \frac{1}{2}, \frac{1}{4}, \dots, (\frac{1}{2})^n, \dots\}$, each with multiplicity one, and set $S = U \otimes B$. Let $A = [M_t \text{ on } L^2(\frac{1}{2}, 1)] \oplus C$ where C is any positive one-to-one compact operator with norm one. Since $B \otimes A \cong \bigoplus_{n=0}^{\infty} (2^{-n} A)$, it is easy to check that $\sigma(B \otimes A) = [0, 1] = \sigma_e(B \otimes A)$. Thus, Corollary 8.23 says that $C^*(S \otimes A) = C^*(U \otimes (B \otimes A))$ contains no compact operators. But clearly, $\sigma(A) \neq \sigma_e(A)$. $\square$

Corollary 8.25 If S is a pure quasinormal operator and A is a positive operator with dense range, then $C^*(S \otimes A)$ contains a compact operator if and only if $\sigma([T^*, T]) = \sigma_e([T^*, T])$ where $T = S \otimes A$.

9. Applications and Examples

Recall the *Fundamental Diagram* for the operator S .

$$\begin{array}{ccc} C^*(S) & & \\ \downarrow \pi & & \\ C^*(\pi(S)) & \xrightarrow[\rho]{} & C^*(T) \end{array}$$

In this section we give some more examples of irreducible subnormal operators S that do not have compact self-commutator and identify $C^*(\pi(S))$ by determining the operator T . We also consider some natural questions on constructing $*$ -homomorphisms.

We shall begin with a nice result illustrating the difference between completely positive maps and $*$ -homomorphisms.

Proposition 9.1 If S and T are operators and there is a completely positive map $\varphi: C^*(S) \rightarrow C^*(T)$ with $\varphi(S) = T$, $\varphi(S^*S) = T^*T$ and $\varphi(SS^*) = TT^*$, then φ is a $*$ -homomorphism.

Proof. Suppose T acts on the Hilbert space $\mathcal{H}$. As φ is completely positive we may dilate φ . So there is an operator S_1 on a Hilbert space $\mathcal{K}$ that contains $\mathcal{H}$ and a $*$ -homomorphism $\psi: C^*(S) \rightarrow C^*(S_1)$ such that $\psi(S) = S_1$ and $\varphi(A) = P\psi(A)|_{\mathcal{H}}$ for all A in $C^*(S)$, where P is the projection of $\mathcal{K}$ onto $\mathcal{H}$. If $S_1 = \begin{bmatrix} T & B \\ A & * \end{bmatrix}$ with respect to $\mathcal{H} \oplus \mathcal{H}^\perp$, then the (1,1) entry of S_1 is T because the compression of S_1 to $\mathcal{H}$ is T . Also, the compression of $S_1^*S_1$ is T^*T . But notice that

$$S_1^* S_1 = \begin{bmatrix} T^* & A^* \\ * & * \end{bmatrix} \begin{bmatrix} T & * \\ A & * \end{bmatrix} = \begin{bmatrix} T^* T + A^* A & * \\ * & * \end{bmatrix},$$

so we see that $A = 0$. Thus $\mathcal{H}$ is invariant for S_1 . Likewise the compression of $S_1 S_1^*$ is TT^* , so we have

$$S_1 S_1^* = \begin{bmatrix} T & B \\ 0 & * \end{bmatrix} \begin{bmatrix} T^* & 0 \\ B^* & * \end{bmatrix} = \begin{bmatrix} TT^* + BB^* & * \\ * & * \end{bmatrix},$$

again we must have $B = 0$. So, in fact $\mathcal{H}$ is a reducing subspace for S_1 . Thus the map φ is obtained from the $*$ -homomorphism ψ followed by the restriction to a reducing subspace. So, φ is a $*$ -homomorphism. $\square$

If S is irreducible with compact self-commutator, then by Theorem 1.1 we have that $C^*(S) = \{ S_f + K : f \in C(\sigma_n(S)) \text{ and } K \text{ is compact} \}$. If T is another such operator, then we know from Theorem 2.2 that $C^*(S) \approx C^*(T)$ if and only if $S \cong T$. So, if $\varphi: C^*(S) \rightarrow C^*(T)$ is a $*$ -isomorphism with $\varphi(S) = T$, then $\varphi(S_f + K) = U^*(S_f + K)U = T_f + U^*KU$ for some unitary operator U . However for any unitary U , the map $(S_f + K) \rightarrow (T_f + U^*KU)$ may be defined as a linear map.

Lemma 9.2 If S and T are irreducible subnormal operators with compact self-commutator acting on $\mathcal{H}$ and $\mathcal{K}$, respectively, $U: \mathcal{K} \rightarrow \mathcal{H}$ is any unitary and $\sigma_n(T) \subseteq \sigma_n(S) = \sigma_e(S)$, then the map $\varphi_U: C^*(S) \rightarrow C^*(T)$ given by $\varphi_U(S_f + K) = T_f + U^*KU$ is a well defined linear map.

Proof. Although φ_U is defined “piecewise” on the Toeplitz operators and separately on the compact operators, the fact that $\sigma_n(S) = \sigma_e(S)$ guarantees that there are no non-zero

compact Toeplitz operators with continuous symbols in $C^*(S)$. Also, since $\sigma_n(T) \subseteq \sigma_n(S)$, we see that T_f is well defined whenever S_f is defined. Thus in view of Theorem 1.1 and the fact that $\text{Comm}(S) = \mathcal{B}_0$, we have that φ_U is well-defined and clearly linear. $\square$

Corollary 9.3 If S and T are irreducible subnormal operators with compact self-commutator satisfying

$$(a) \sigma_e(S) = \sigma_n(S) = \sigma_n(T) = \sigma_e(T),$$

$$(b) [S^*, S] \cong [T^*, T]$$

and U is a unitary so that $U^*[S^*, S]U = [T^*, T]$, then $S \cong T$ if and only if the natural map $\varphi_U: C^*(S) \rightarrow C^*(T)$ given by $\varphi_U(S_f + K) = T_f + U^*KU$ is completely positive.

Proof. By property (a) and Lemma 9.2 we see that φ_U is well defined and linear. Also since $[S^*, S] = U[T^*, T]U^*$, we may consider φ_{U^*} and this will be an inverse to φ_U . Hence φ_U is a one-to-one onto linear map such that $\varphi_U(S) = T$. Since S^*S is the Toeplitz operator with symbol $|z|^2$ we see that $\varphi_U(S^*S) = T^*T$. Also since $U^*[S^*, S]U = [T^*, T]$ and $\varphi_U(S^*S) = T^*T$, we must have that $\varphi_U(SS^*) = TT^*$. Hence Proposition 9.1 says that φ_U is completely positive if and only if φ_U is a $*$ -isomorphism and this happens if and only if $S \cong T$, (see Theorem 2.2). $\square$

We now show that in fact something stronger than Corollary 9.3 holds. Namely the natural map φ_U only needs to be positive. For the next lemma suppose S and T are irreducible subnormal operators with compact self-commutator acting on $\mathcal{H}$ and $\mathcal{K}$,

respectively, $U: \mathcal{K} \rightarrow \mathcal{H}$ is any unitary and $\sigma_n(T) \subseteq \sigma_n(S) = \sigma_e(S)$. Let φ_U be the natural map given by $\varphi_U(S_f + K) = T_f + U^*KU$.

Lemma 9.4 The linear map φ_U is positive if and only if $U^*S_fU \leq T_f$ for all positive continuous functions f on $\sigma_n(S)$.

Proof. If $U^*S_fU \leq T_f$ for all positive continuous functions f on $\sigma_n(S)$, then we must show φ_U is positive. So, suppose $S_f + K \geq 0$. Then $0 \leq U^*(S_f + K)U = U^*S_fU + U^*KU \leq T_f + U^*KU = \varphi_U(S_f + K)$. So we have that φ_U is positive. Conversely, suppose φ_U is positive and let f be a positive continuous function on $\sigma_n(S)$. If $\{P_n\}$ is a sequence of finite rank projections increasing strongly to the identity, then set $K_n = P_n S_f$. So $\{K_n\}$ is a sequence of compact operators converging strongly to S_f . Furthermore, $\|K_n x\| \leq \|S_f x\|$ for each x . Hence squaring both sides and changing to inner products we get $\langle K_n^* K_n x, x \rangle \leq \langle S_f^2 x, x \rangle$, because S_f is self-adjoint. Hence $0 \leq K_n^* K_n \leq S_f^2$ from which we also get, $0 \leq |K_n| \leq S_f$. Also, as $K_n \rightarrow S_f$ (SOT), we get $K_n^* K_n \rightarrow S_f^2$ (SOT). Hence using Lemma I.7.2 of Davidson[1995] we also get that taking square roots preserves (SOT) convergence. So $|K_n| \rightarrow S_f$ (SOT). Thus, we have that $0 \leq |K_n| \leq S_f$ and $|K_n| \rightarrow S_f$ (SOT). As φ_U is positive, we also get that $0 \leq U^*|K_n|U \leq T_f$ for all n . So,

$$\langle T_f x, x \rangle \geq \langle U^*|K_n|U x, x \rangle = \langle |K_n|U x, U x \rangle \rightarrow \langle S_f U x, U x \rangle = \langle U^*S_f U x, x \rangle.$$

Hence we have that $U^*S_fU \leq T_f$. $\square$

Theorem 9.5 If the same assumptions hold for S and T as in Lemma 9.2, then the natural map $\varphi_U: C^*(S) \rightarrow C^*(T)$ is positive if and only if $U^*SU = T$.

Proof. If φ_U is positive, then Lemma 9.4 gives that $U^*S_fU \leq T_f$ for all positive continuous functions f on $\sigma_n(S)$. In particular, by also considering $(\|f\|_\infty - f)$ we see that in fact $U^*S_fU = T_f$ for all positive continuous functions on $\sigma_n(S)$. Since every function is a linear combination of four positive functions, we see that $U^*S_fU = T_f$ for every continuous function on $\sigma_n(S)$. In particular, if $f(z) = z$, then we have $U^*SU = T$. $\square$

We now turn to examining some more irreducible subnormal operators S and determining the associated operator T in the Fundamental Diagram for S . Once T is known, one is often able to check whether $C^*(S)$ contains the compact operators or not.

Lemma 9.6 If S is an irreducible operator and there exists a faithful representation ρ of $C^*(\pi(S))$ such that $T = \rho(\pi(S))$ is irreducible with compact self-commutator, then $C^*(S)$ contains the compact operators.

Proof. If $\varphi = \rho \circ \pi: C^*(S) \rightarrow C^*(T)$, then $\ker(\varphi) = C^*(S) \cap \mathcal{B}_0$. If $C^*(S)$ contains no compact operators, then φ is a $*$ -isomorphism. Since $\mathcal{B}_0 \subseteq C^*(T)$, Theorem 2.2 gives that $S \cong T$. Thus S has compact self-commutator and $C^*(S)$ does contain a compact operator, a contradiction. So, in fact $C^*(S)$ does contain a non-zero compact operator. As S is irreducible, $C^*(S)$ contains all the compact operators. $\square$

Lemma 9.6 will be useful. One should also see Proposition 5.9 where it is also useful to determine when ρ may be chosen so that $T = \rho(\pi(S))$ has compact self-commutator.

We now consider a class of operators where we can completely understand the faithful representations of $C^*(\pi(S))$, as was done in Theorem 8.20 for quasinormal operators.

If S is an operator, then let $\mathcal{C}(S)$ denote the collection of all operators T on separable Hilbert spaces such that $C^*(T) \approx C^*(\pi(S))$. That is, $\mathcal{C}(S)$ is the collection of all possible T 's that may appear in the *Fundamental Diagram* for S . Recall, that if G is a bounded open set, then S_G denotes M_z on $H^2(G)$.

Theorem 9.10 Let S be a pure subnormal operator. Suppose $P^\infty(S) \approx H^\infty(G)$, where G is connected and $\sigma_e(S) = \partial G$. Also let $\mathcal{M} \in \text{Lat} S$ and set $T = S|_{\mathcal{M}}$.

- (a) If $\text{ind}(T - \lambda) \neq -\infty$ for $\lambda \in G$, then $\mathcal{C}(T)$ is exactly the class of normal operators with spectrum equal to ∂G .
- (b) If $\text{ind}(T - \lambda) = -\infty$ for $\lambda \in G$, then $\mathcal{C}(T)$ is exactly those operators unitarily equivalent to one of the form $S_G^{(n)} \oplus N$, where $n \geq 1$ and N is a normal operator with $\sigma(N) \subseteq \partial G$.

In particular, in part (b) if $G = \mathbf{D}$, then there is a faithful representation ρ of $C^*(\pi(T))$ so that $\rho(\pi(T)) = U$, the unilateral shift.

Proof. (a) If $\text{ind}(T - \lambda) \neq -\infty$ for $\lambda \in G$, then $\sigma_e(T) \subseteq \sigma_e(S) = \partial G$. Thus since G is connected we have that $R(\partial G) = C(\partial G)$. Hence by Theorem 7.5(b) T is essentially normal. Thus, $C^*(\pi(T))$ is abelian with maximal ideal space $\sigma_e(T) = \partial G$.

(b) If $\text{ind}(T-\lambda) = -\infty$ for $\lambda \in G$, then $\sigma_e(T) \supseteq \text{cl}G$ and since $R(\text{cl}G) \subseteq P^\infty(\text{svsm}S)$, we have that $\sigma(T) \subseteq \text{cl}G$. Hence $\sigma_e(T) = \text{cl}G$. Since $T = S|_{\mathcal{M}}$, Theorem 4.11 says that $\sigma_n(\pi(T)) \subseteq \sigma_n(\pi(S)) = \sigma_e(S) = \partial G$. Since $\partial G \subseteq \sigma_{\text{ap}}(T) \subseteq \sigma_n(\pi(T))$, we have $\sigma_n(\pi(T)) = \partial G$. Now, in order to apply Theorem 5.8 we need to show that $P^\infty(T) \approx H^\infty(G)$. Since T is the restriction of S to an invariant subspace, it follows that the Sarason hull for T is contained in that of S ($= G$). But we also have $\sigma(T) - \sigma_{\text{ap}}(T)$ contained in the Sarason Hull of T (this is true for any subnormal operator). But $G = \sigma(T) - \sigma_{\text{ap}}(T)$. Thus $P^\infty(T) \approx H^\infty(G)$. So, Theorem 5.8 applies to say that there exists a faithful representation ρ of $C^*(\pi(T))$ such that $\rho(\pi(T)) = S_G$. Now since G is connected, S_G is irreducible and essentially normal (see Theorem 7.5(b)). Since the collection $\mathcal{C}(T)$ is exactly those operators R such that $C^*(R) \approx C^*(S_G)$, Theorem 6.14 applies to finish the proof. $\square$

We now consider the duals of such operators, although we must specialize to the disk.

Theorem 9.11 Let S_1 be a pure subnormal operator. Suppose $P^\infty(S_1) \approx H^\infty(\mathbf{D})$ and $\sigma_e(S_1) = \partial\mathbf{D}$. Also let $\mathcal{M} \in \text{Lat}S_1$ and set $S = S_1|_{\mathcal{M}}$ and $T = \text{dual}S$. Assume that $\text{mne}S = \text{mne}S_1$.

- (a) If $\text{ind}(S) \neq -\infty$, then $\mathcal{C}(T)$ consists of those normal operators with $\sigma(N) = \sigma_e(\text{mne}T)$.
- (b) If $\text{ind}(S) = -\infty$, then $\mathcal{C}(T)$ consists of those operators unitarily equivalent to one of the form $S_{\mathbf{D}}^{(n)} \oplus N$, where $n \geq 1$ and N is a normal operator with $\sigma(N) = \sigma_e(\text{mne}T)$.

Proof. (a) If $\text{ind}(S) \neq -\infty$, then $\sigma_e(S) = \partial\mathbf{D}$, so Theorem 7.5(b) implies that S has compact self-commutator. Thus T does also. It follows from Theorem 4.13 and Corollary 4.3(a)

that $\sigma_e(\text{mne}S) = \sigma_e(S) \cup \sigma_e(T)^*$. So one has that $\sigma_e(T)^* = \sigma_e(\text{mne}S)$. Hence $\sigma_e(T) = \sigma_e(\text{mne}T)$. Since $C^*(\pi(T))$ is abelian with maximal ideal space $\sigma_e(T)$, (a) follows.

(b) If $\text{ind}S = -\infty$, then by Corollary 5.10 we see that $\sigma_e([S^*, S]) = \{0, 1\}$. Now, it is not hard to check that $\sigma_e([S^*, S]) = \sigma_e([T^*, T])$. So $\sigma_e([T^*, T]) = \{0, 1\}$. If ρ is a faithful representation of $C^*(\pi(T))$, then let $\rho(\pi(T)) = T_p \oplus N$ where T_p is pure and N is normal. Since the operator $\rho(\pi(T))$ is a subnormal operator whose self-commutator is a projection and T_p has norm at most one, T_p is a unilateral shift (see Theorem I.2.2). Now as S_1 is an extension of S we easily see that T is an extension of $\text{dual}S_1$. So, $\sigma_{ap}(\text{dual}S_1) \subseteq \sigma_{ap}(T)$. But it follows from Theorem 4.13 that $\sigma_{ap}(\text{dual}S_1) = \sigma_e(\text{mne}S_1)^*$. Since S is pure we have that $\text{mne}T = (\text{mne}S_1)^*$. Hence $\sigma_e(\text{mne}T) \subseteq \sigma_{ap}(T)$. Since Theorem 4.13 gives that $\sigma_{ap}(T) \subseteq \sigma_e(\text{mne}T)$, we have $\sigma_{ap}(T) = \sigma_e(\text{mne}T)$. Thus, $\sigma_e(\text{mne}T) = \sigma_{ap}(T) = \sigma_{ap}(\kappa(\pi(T))) = \sigma_{ap}(T_p \oplus N) = \partial\mathbf{D} \cup \sigma(N)$. So, $\sigma(N) = \sigma_e(\text{mne}T)$. Thus $\kappa(\pi(T))$ has the required form. $\square$

Example 9.12 Let B be the Bergman shift, $\mathcal{M} \in \text{Lat}B$ and set $S = B|_{\mathcal{M}}$ and $T = \text{dual}S$. If $\text{ind}(S) = -\infty$, then $\mathcal{C}(S)$ consists of all isometries that are not unitary and $\mathcal{C}(T)$ consists of those operators of the form $U^{(n)} \oplus N$ where U is the unilateral shift, $n \geq 1$ and N is any normal operator with $\sigma(N) = \text{cl}\mathbf{D}$.

In particular, Example 9.12 shows that T is not essentially pure. That is, there are no pure operators in $\mathcal{C}(T)$.

The following Corollary is a nice application of Theorem 9.10 and Theorem 9.11 and does hold in more generality.

Corollary 9.13 If B is the Bergman shift, $\mathcal{M} \in \text{Lat} B$, $S = B|_{\mathcal{M}}$ and $T = \text{dual} S$, then S and T are irreducible subnormal operators, $\mathcal{B}_0 \subseteq C^*(S)$ and $\mathcal{B}_0 \subseteq C^*(T)$.

Proof. It is clear that $\text{mne} S = \text{mne} B$. Since any projection that commutes with S lifts to the commutant of $\text{mne} S$, we see that S is irreducible. Thus it follows from Conway [1981] that T is also irreducible. So, Lemma 9.6 together with Theorem 9.10 shows that $\mathcal{B}_0 \subseteq C^*(S)$. Now consider $C^*(T)$. If T does not have compact self-commutator, then by Theorem 9.11 there is a faithful representation ρ of $C^*(\pi(T))$ with $\rho(\pi(T)) = U \oplus N$ where U is the unilateral shift and N is some normal operator with $\sigma(N) = \text{cl} \mathbf{D}$. Set $\varphi = \rho \circ \pi$. So $\varphi: C^*(T) \rightarrow C^*(U \oplus N)$ is a $*$ -homomorphism with $\ker(\varphi) = C^*(T) \cap \mathcal{B}_0$. If φ is an isomorphism, then since T is pure Theorem 1.4 says that we must have $\sigma(N) \subseteq \sigma_{\text{ap}}(U) = \partial \mathbf{D}$. This is clearly not true. So we must have $\ker(\varphi) \neq (0)$. Hence $C^*(T)$ contains a compact operator. Thus as T is irreducible, $\mathcal{B}_0 \subseteq C^*(T)$. $\square$

Another nice class of operators consists of bundle shifts (see Theorem 3.14). It follows from Corollary 3.2 that if S is a bundle shift over a region R and φ is a representation of $C^*(S)$, then $\varphi(S)$ is also a bundle shift over R . Thus for a bundle shift S , the collection $\mathcal{A}(S)$ consists of bundle shifts. Can we identify this collection?

CHAPTER III

Norm Estimates on the Self-Commutator

In this Chapter we shall discuss the size of the self-commutator in three different norms: the usual norm, the essential norm and the trace norm. In Section 1 we give a more general version and an essential version of a classical inequality due to Axler and Shapiro on the norm of the self-commutator. Also a necessary and sufficient condition for compactness of the self-commutator is given. Furthermore, it is shown that compactness of the self-commutator is a similarity invariant for subnormal operators. This section uses some of the work from Chapter II.

In section 2 we present a sharp form of the Berger-Shaw Theorem for cyclic subnormal operators. In particular we use Thomson's Theorem on the existence of analytic bounded point evaluations to prove that every pure cyclic subnormal operator S has trace class self-commutator and show that $\text{tr}[S^*, S] = \frac{1}{\pi} \text{Area}[\sigma(S) - \sigma_e(S)]$. We also determine exactly which operators in the commutant of S have trace class self-commutator. Some partial results on trace estimates for the self-commutators of rationally cyclic subnormal operators are presented as well as some examples.

1. The Compactness of the Self-Commutator

In 1985 Axler and Shapiro proved that for any subnormal operator S , $\|[S^*, S]\| \leq \text{dist}[\bar{z}, R(\sigma(S))]^2$. We now prove a similar estimate (in a similar way) for functions of S , although we shall replace $R(\sigma(S))$ by the restriction algebra $\mathcal{R}(S)$ for S . Not only is this a larger algebra, it is more closely connected with S .

If S is a subnormal operator on $\mathcal{H}$, $\mu = \text{svsm} S$ and $N = \text{mne} S$, then the restriction algebra, $\mathcal{R}(S)$, for S consists of those functions $f \in L^\infty(\mu)$ such that $f(N)$ leaves $\mathcal{H}$ invariant. In this case we define $f(S)$ to be $f(N)|_{\mathcal{H}}$. We first present three examples. The first two are standard, but the third is quite surprising.

Example 1.1 Let G be any bounded region and μ a compactly supported measure in $\mathbb{C}$.

- (a) If $S = M_z$ on $P^2(\mu)$, then $\mathcal{R}(S) = P^2(\mu) \cap L^\infty(\mu)$.
- (b) If $S = M_z$ on $L_a^2(G)$, then $\mathcal{R}(S) = H^\infty(G)$.
- (c) If $S = M_z$ on $H^2(G)$, then $\mathcal{R}(S) = \{ f \in H^\infty(G) : f \text{ has "boundary values"} \}$.

Let's clarify what we mean by "boundary values" in (c). Let $\tau: \mathbf{D} \rightarrow G$ be a universal analytic covering map. If $f \in H^\infty(G)$, then $f \circ \tau \in H^\infty(\mathbf{D})$, and hence has a non-tangential boundary function $(f \circ \tau)^*$ defined a.e. on $\partial \mathbf{D}$. We shall say that f has "boundary values" if there exists a function $f^* \in L^\infty(\partial G, \omega_G)$ such that $f^* \circ \tau = (f \circ \tau)^*$ a.e. on $\partial \mathbf{D}$. For example, if G is the slit disk, then $f \in H^\infty(G)$ has "boundary values" if and only if the non-tangential limits of f from above and below agree a.e. along the slit.

Theorem 1.2 If S is a subnormal operator, $\mu = \text{svsm} S$ and $f \in L^\infty(\mu)$ is such that the Toeplitz operator S_f is hyponormal, then $\|[S_f^*, S_f]\| \leq \text{dist}_{L^\infty(\mu)}[\bar{f}, \mathcal{R}(S)]^2$.

Proof. Let P be the projection onto $\mathcal{H}$. We have, for $x \in \mathcal{H}$,

$$\begin{aligned} \|[S_f^*, S_f]\| &= \sup_{\|x\|=1} \langle [S_f^*, S_f]x, x \rangle = \sup_{\|x\|=1} [\|S_f x\|^2 - \|S_f^* x\|^2] \leq \sup_{\|x\|=1} [\|f(N)x\|^2 - \|P\bar{f}(N)x\|^2] = \\ &= \sup_{\|x\|=1} [\|\bar{f}(N)x\|^2 - \|P\bar{f}(N)x\|^2] = \sup_{\|x\|=1} \|\bar{f}(N)x - P\bar{f}(N)x\|^2 = \sup_{\|x\|=1} \text{dist}[\bar{f}(N)x, \mathcal{H}]^2 \leq \\ &\leq \sup_{\|x\|=1} \inf_{g \in \mathcal{R}(S)} \|\bar{f}(N)x - g(N)x\|^2 \leq \inf_{g \in \mathcal{R}(S)} \|\bar{f}(N) - g(N)\|^2 = \text{dist}_{L^\infty(\mu)}[\bar{f}, \mathcal{R}(S)]^2. \end{aligned}$$

Observe that since S_f is hyponormal, we do not need absolute values on the first supremum. Thus the second equality holds true, and hence also the first inequality. $\square$

Corollary 1.3 If S is subnormal and $f \in \mathcal{R}(S)$, then $\|[f(S)^*, f(S)]\| \leq \text{dist}_{L^\infty(\mu)}[\bar{f}, \mathcal{R}(S)]^2$.

Proof. If $f \in \mathcal{R}(S)$, then the Toeplitz operator $S_f = f(S)$ is subnormal. $\square$

We now present a version of Corollary 1.3 involving essential norms. Recall the Fundamental Diagram for S , first introduced in II.4.

Let π be the natural map into the Calkin algebra and ρ any faithful representation of $C^*(\pi(S))$ on a separable Hilbert space.

We have,

$$\begin{array}{ccc} C^*(S) & & \\ \downarrow \pi & & \\ C^*(\pi(S)) & \xrightarrow[\rho]{} & C^*(T) \end{array}$$

If $\varphi = \rho \circ \pi$, then φ is a *-homomorphism from $C^*(S)$ onto $C^*(T)$ with $\varphi(S) = T$. It follows from the remarks after Corollary II.3.11 that the continuous elements in the restriction algebra for T do not depend on ρ . So, we define the *essential restriction algebra* for S , $\mathcal{R}_e(S)$, to be $\mathcal{R}(T) \cap C(\sigma_n(T))$. Recall that $\sigma_n(T) = \sigma_n(\pi(S)) \subseteq \sigma_n(S)$.

Lemma 1.4 Let S be a subnormal operator. If $f \in C(\sigma_n(\pi(S)))$ and $\tilde{f}$ is any continuous extension of f to $\sigma_n(S)$, then $f \in \mathcal{R}_e(S)$ if and only if the Hankel operator $H_{\tilde{f}}$ is compact.

Proof. As noted in Corollary II.3.11, $f \in \mathcal{R}(T)$ if and only if $T_{\tilde{f}}T_f - T_{|f|^2} = 0$. Thus, since $\varphi(S_{\tilde{f}}S_{\tilde{f}} - S_{|\tilde{f}|^2}) = T_{\tilde{f}}T_f - T_{|f|^2}$, we have that $f \in \mathcal{R}(T)$ if and only if $S_{\tilde{f}}S_{\tilde{f}} - S_{|\tilde{f}|^2}$ is compact. However the following identity holds: $S_{|\tilde{f}|^2} - S_{\tilde{f}}S_{\tilde{f}} = H_{\tilde{f}}^*H_{\tilde{f}}$. So, $f \in \mathcal{R}_e(S)$ if and only if the Hankel operator $H_{\tilde{f}}$ is compact. $\square$

Theorem 1.5 If S is a subnormal operator and $f \in \mathcal{R}(S) \cap C(\sigma_n(S))$, then

$$\left\| [f(S)^*, f(S)] \right\|_e \leq \text{dist}_{C(\sigma_n(\pi(S)))}[\tilde{f}, \mathcal{R}_e(S)]^2.$$

Proof. If $f \in \mathcal{R}(S) \cap C(\sigma_n(S))$, then using the Fundamental Diagram for S and Corollary II.3.4, we see that $\varphi([f(S)^*, f(S)]) = [f(T)^*, f(T)]$. Now, applying Corollary 1.3 to T and using the fact that ρ is an isometry, we have,

$$\left\| [f(S)^*, f(S)] \right\|_e = \left\| [f(T)^*, f(T)] \right\| \leq \text{dist}[\tilde{f}, \mathcal{R}(T)]^2 \leq \text{dist}[\tilde{f}, \mathcal{R}_e(S)]^2. \quad \square$$

Corollary 1.6 If S is a subnormal operator and $f \in \mathcal{R}(S) \cap C(\sigma_n(S))$, then

$$\left\| [f(S)^*, f(S)] \right\|_e \leq \text{dist}[\bar{f}, R(\sigma_e(S))]^2.$$

Proof. This follows immediately because $R(\sigma_e(S)) = R(\sigma(T)) \subseteq \mathcal{R}(T) = \mathcal{R}_e(S)$. $\square$

The following result was first proven in Aleman [1996] with a different technique.

Corollary 1.7 If S is a subnormal operator, then $\left\| [S^*, S] \right\|_e \leq \text{dist}[\bar{z}, R(\sigma_e(S))]^2$.

Corollary 1.8 If S is a subnormal operator and $R(\sigma_e(S)) = C(\sigma_e(S))$, then S has compact self-commutator.

Corollary 1.9 Let S be a subnormal operator. If one of the following hold,

- (a) $\text{Area}[\sigma_e(S)] = 0$ or
- (b) $\sigma(S) - \sigma_e(S)$ is connected and dense in $\sigma(S)$,

then S has compact self-commutator.

Proof. If (a) holds, then by the Hartogs-Rosenthal Theorem (see Conway [1991], p. 175), we know that $R(\sigma_e(S)) = C(\sigma_e(S))$, hence Corollary 1.8 applies. For (b), if $G = \sigma(S) - \sigma_e(S)$ and G is dense in $\sigma(S)$, then $\sigma_e(S) = \partial G$. Furthermore, as G is connected, we know that $R(\partial G) = C(\partial G)$. Thus again Corollary 1.8 applies. $\square$

Some immediate applications of Corollary 1.9(b) are to subnormal operators of the form M_z on $\mathcal{H}$, where $\mathcal{H}$ is a Hilbert space of analytic functions on a region G . In Corollary 1.9(b) one may replace connected by “has at most a finite number of components”. But as we will show below it is not sufficient to just assume that $\sigma(S) - \sigma_e(S)$ is dense in $\sigma(S)$.

We now present a necessary and sufficient condition for a subnormal operator to have compact self-commutator. Recall, that if μ is a measure, then its Cauchy transform is given by $\hat{\mu}(w) = \int \frac{1}{z-w} d\mu(z)$. If g is a compactly supported locally integrable function on $\mathbb{C}$, then $\hat{g}$ denotes the Cauchy transform of the measure $g dA$. If g is bounded, then $\hat{g}$ is a continuous function on $\mathbb{C}$ and is analytic off of the support of g . We will denote the characteristic function of a set E by χ_E .

Theorem 1.10 If S is a subnormal operator and $f = \hat{\chi}_{\sigma_e(S)}$, then S has compact self-commutator if and only if $[S_f, S]$ is compact.

Proof. Consider the Fundamental Diagram for S . Let $\varphi: C^*(S) \rightarrow C^*(T)$ be the *-homomorphism there such that $\varphi(S) = T$. If S has compact self-commutator, then T is normal and by Corollary II.3.4 we have that $\varphi([S_f, S]) = [f(T), T] = 0$. Thus, since $\ker(\varphi) = C^*(S) \cap \mathcal{B}_0$, we have that $[S_f, S]$ is compact.

Now, suppose that $[S, S_f]$ is compact. Choose a C^1 function $u(z)$ with compact support such that $u(z) = \bar{z}$ in a neighborhood of $\sigma(S)$. Since u has compact support we may

represent it as, $u(w) = \frac{-1}{\pi} \int \frac{\bar{\partial} u(z)}{z-w} dA(z)$. Now this integral may be written as,

$$\frac{-1}{\pi} \left[\int_{\mathbb{C}-\sigma(S)} \frac{\bar{\partial} u(z)}{z-w} dA(z) + \int_{\sigma(S)-\sigma_e(S)} \frac{1}{z-w} dA(z) + \int_{\sigma_e(S)} \frac{1}{z-w} dA(z) \right]$$

Let $u_1(w)$, $u_2(w)$ and $u_3(w)$ denote the three corresponding integrals above. So, we have that $u(w) = (-1/\pi)[u_1(w) + u_2(w) + u_3(w)]$. Thus, since $u(w) = \bar{w}$ near $\sigma(S)$, we also have that $[S^*, S] = [S_u, S] = (-1/\pi)\{ [S_{u_1}, S] + [S_{u_2}, S] + [S_{u_3}, S] \}$. Thus we need to show that each commutator is compact. Observe that our assumption is that $[S_{u_3}, S]$ is compact. Now, since u_1 is the Cauchy transform of a bounded function that vanishes on $\sigma(S)$, we have that $u_1 \in R(\sigma(S))$. Hence $[S_{u_1}, S] = 0$. Likewise, u_2 is a Cauchy transform of a bounded function that vanishes on $\sigma_e(S)$, so $u_2 \in R(\sigma_e(S))$. Using the Fundamental Diagram for S and Corollary II.3.4 (and that $u_2 \in R(\sigma(T))$), we have that $\varphi([S_{u_2}, S]) = [u_2(T), T] = 0$. Thus $[S_{u_2}, S] \in \ker(\varphi) = C^*(S) \cap \mathcal{B}_0$. So, $[S_{u_2}, S]$ is compact. $\square$

The nice thing about Theorem 1.10 is that very often one can verify that S_f actually commutes with S ; thus $[S_f, S] = 0$. For instance, if $S = M_z$ on $L_a^2(G)$ where G is *any* bounded open set, then $[S_f, S] = 0$ and thus S has compact self-commutator. For the next result, if G is a bounded open set, then $A(G)$ denotes those functions analytic on G and continuous on $\text{cl}G$. Notice that $f = \hat{\chi}_{\sigma_e(S)} \in A(\sigma(S) - \sigma_e(S))$.

Corollary 1.11 Let S be a subnormal operator and $G = \sigma(S) - \sigma_e(S)$. If G is dense in $\sigma(S)$ and $\{ S_f : f \in A(G) \} \subseteq \{ S \}'$, then S has compact self-commutator.

In particular, if G is dense in $\sigma(S)$ and $R(\sigma(S)) = A(G)$, then S has compact self-commutator.

We now give an example of a pure subnormal operator S so that $G = \sigma(S) - \sigma_e(S)$ is dense in $\sigma(S)$, yet S does not have compact self-commutator. Thus in general something extra is needed, such as in Corollary 1.9(b) and Corollary 1.11.

Example 1.12 There exists a pure subnormal operator such that $G = \sigma(S) - \sigma_e(S)$ is dense in $\sigma(S)$, yet S does not have compact self-commutator.

Proof. Suppose L is a swiss cheese; that is, there are open disks Δ_n inside the unit disk $\mathbf{D}$

whose union is dense in $\mathbf{D}$ and with summable radii such that $L = \text{cl}\mathbf{D} - \bigcup_{n=1}^{\infty} \Delta_n$. Let ν_1 be

arc length measure on $\bigcup_{n=1}^{\infty} \partial\Delta_n$. Now, let B_n be an open disk with the same center as Δ_n

but with half the radius. Also, let ν_2 be arc length measure on $\bigcup_{n=1}^{\infty} \partial B_n$. If we set $S_1 = M_z$

on $R^2(K, \nu_1)$, then each S_i is a pure rationally cyclic subnormal operator. Let $S = S_1^{(\infty)} \oplus$

S_2 . It follows that $\sigma(S) = L \cup \bigcup_{n=1}^{\infty} \text{cl}B_n$, $\sigma_e(S) = L \cup \bigcup_{n=1}^{\infty} \partial B_n$ and $G = \bigcup_{n=1}^{\infty} B_n$. So, G is

dense in $\sigma(S)$, yet S does not have compact self-commutator. $\square$

There is a technique developed by John B. Conway such that under some reasonable conditions one may construct an irreducible subnormal operator similar to a given direct sum of operators. Hence it is desirable to know what properties of subnormal operators remain invariant under similarity. We will show that compactness of the self-commutator is one of these properties, but first an example.

Example 1.13 There exists an operator T similar to a normal operator, yet T does not have compact self-commutator.

Proof. If R is any operator that is similar to a normal operator and is not normal, then $T = R^{(\infty)}$ will do the job. To find such an R , try 2×2 matrices. $\square$

Theorem 1.14 If S and T are similar subnormal operators, then S has compact self-commutator if and only if T has compact self-commutator.

Proof. We may suppose that both S and T act on the same space $\mathcal{H}$. Let $X: \mathcal{H} \rightarrow \mathcal{H}$ be an invertible map, so that $X^{-1}SX = T$. Consider $C^*(S, X, T)$. Let ρ be a faithful representation of $C^*(\pi(S), \pi(X), \pi(T))$, where π is the natural map into the Calkin algebra. If $S_1 = \rho(S)$, $Y = \rho(X)$ and $T_1 = \rho(T)$, then we have that Y is invertible and $Y^{-1}S_1Y = T_1$. Thus, S_1 and T_1 are similar subnormal operators. If say, S has compact self-commutator, then S_1 is normal, hence T_1 is also normal (see Conway[1991], p. 82). Thus T also has compact self-commutator. $\square$

Question 1.15 Suppose S and T are pure subnormal operators and S has compact self-commutator. If S and T are quasisimilar, then must T have compact self-commutator?

Notice that if $R(\sigma_e(S)) = C(\sigma_e(S))$, then Corollary 1.8 says that any subnormal operator with the same essential spectrum as S will have compact self-commutator. In particular, since quasisimilarity preserves the essential spectrum (see Yang [1990]) we see that Question 1.15 has an affirmative answer in this case.

2. The Berger Shaw Theorem for Cyclic Subnormal Operators

The Berger-Shaw Theorem (see Conway[1991], p. 152) implies that every rationally cyclic hyponormal operator T has trace class self-commutator and $\text{tr}[T^*, T] \leq \frac{1}{\pi} \text{Area}[\sigma(T)]$. In this section we consider the case of a pure cyclic subnormal operator

S and give another proof that S has trace class self-commutator and compute the trace as

$$\text{tr}[S^*, S] = \frac{1}{\pi} \text{Area}[\sigma(S) - \sigma_e(S)].$$

It follows that the principal function for a cyclic

subnormal operator vanishes a.e. on the essential spectrum. We also characterize the operators in the commutant of S that have trace class self-commutators. The main tools used are Thomson's Theorem on the existence of bounded point evaluations for cyclic subnormal operators and the Intertwining Lemma of Berger and Shaw allowing one to compare the trace of the self-commutators of two hyponormal operators.

If μ is a positive compactly supported regular Borel measure in $\mathbb{C}$, let $P^2(\mu)$ denote the closure of the (analytic) polynomials in $L^2(\mu)$. If $S_\mu = M_z$ on $P^2(\mu)$, then S_μ is a cyclic subnormal operator. Furthermore, every cyclic subnormal operator can be represented in this form for some measure μ . If $f \in P^2(\mu) \cap L^\infty(\mu)$, then $f(S_\mu)$ denotes the multiplication operator on $P^2(\mu)$ with symbol f .

For a given measure μ let $G = \text{abpe}(\mu)$ be the set of analytic bounded point evaluations for $P^2(\mu)$. If S_μ is pure, then $G = \sigma(S_\mu) - \sigma_e(S_\mu)$. For each λ in G , the reproducing kernel for

the point λ is a function k_λ in $P^2(\mu)$ such that $p(\lambda) = \langle p, k_\lambda \rangle$ for every polynomial p . Also for f in $P^2(\mu)$, let $\hat{f}(\lambda) = \langle f, k_\lambda \rangle$ be the analytic extension of f to G .

We shall use Thomson's Theorem (see Thomson[1991] or Conway [1991], p.375) on the plentiful supply of analytic bounded point evaluations to prove that $f(S_\mu)$ has trace class self-commutator if and only if $\hat{f}$ is in the Dirichlet space on G ; that is, $\int_G |\hat{f}'|^2 dA < \infty$.

Furthermore, we shall show that $\text{tr}[f(S_\mu)^*, f(S_\mu)] = \frac{1}{\pi} \int_G |\hat{f}'|^2 dA$. See Arazy, Fisher, Janson and Peetre [1990] for some nice work on computing traces, as above, for multiplication operators on weighted Bergman spaces on arbitrary regions.

Theorem 2.1 If $S_\mu = M_z$ on $P^2(\mu)$ is a pure cyclic subnormal operator, $G = \text{abpe}(\mu)$ and $f \in P^2(\mu) \cap L^\infty(\mu)$, then $[f(S_\mu)^*, f(S_\mu)]$ is trace class if and only if $\hat{f}$ is in the Dirichlet space on G . In that case, $\text{tr}[f(S_\mu)^*, f(S_\mu)] = \frac{1}{\pi} \int_G |\hat{f}'|^2 dA$.

We shall state the Intertwining Lemma for completeness. See Clancey[1979], p. 73, for a nice proof.

The Intertwining Lemma Let S and T be hyponormal operators on $\mathcal{H}$ and $\mathcal{K}$ respectively. If there exists a one-to-one Hilbert-Schmidt operator $W: \mathcal{H} \rightarrow \mathcal{K}$ with dense range satisfying $WS = TW$, then $\text{tr}[T^*, T] \leq \text{tr}[S^*, S]$.

In what follows, $H^\infty(G)$ is the algebra of bounded analytic functions on the region G and $H^2(G)$ is the Hardy space. Also, let $S_G = M_z$ on $H^2(G)$ and $f(S_G) = M_f$ on $H^2(G)$.

Lemma 2.2 If G is a bounded simply connected region and $f \in H^\infty(G)$, then the multiplication operator $f(S_G)$ has trace class self-commutator if and only if f is in the Dirichlet space on G . Also, $\text{tr}[f(S_G)^*, f(S_G)] = \frac{1}{\pi} \int_G |f'|^2 dA$.

Proof. If $G = \mathbf{D}$, then this is an easy computation involving the Fourier coefficients of f . In general, let $\varphi: \mathbf{D} \rightarrow G$ be a Riemann map. If $f \in H^\infty(G)$, then the operator $f(S_G)$ is unitarily equivalent to $(f \circ \varphi)(S_{\mathbf{D}})$. Thus with a simple change of variables we see that the general case can be reduced to the unit disk, $\mathbf{D}$. $\square$

Let G be any bounded region in $\mathbf{C}$. We say that a Hilbert space $\mathcal{H}$ is a Hilbert space of analytic functions on G if $\mathcal{H}$ is a vector subspace of $\text{Hol}(G)$, the Frechet space of holomorphic functions on G , such that the inclusion of $\mathcal{H}$ into $\text{Hol}(G)$ is continuous. The following Lemma shall be proved after Theorem 2.4.

Lemma 2.3 If G is any bounded region, then there exists a sequence of regions $\{G_n\}$ such that for each n , G_n is bounded by an analytic Jordan curve, $\text{cl}G_n \subseteq G_{n+1} \subseteq G$, $\bigcup_{n=1}^{\infty} G_n$ is dense in G and $\text{Area}(G_n) \rightarrow \text{Area}(G)$ as $n \rightarrow \infty$.

Theorem 2.4 If $\mathcal{H}$ is any Hilbert space of analytic functions on a bounded region G that contains the polynomials and $f \in H^\infty(G)$ is such that multiplication by f , M_f , on $\mathcal{H}$ is a hyponormal operator, then $\text{tr}[M_f^*, M_f] \geq \frac{1}{\pi} \int_G |f'|^2 dA$.

Proof. Let $\{G_n\}$ be as in Lemma 2.3. For a fixed n , define $W_n: \mathcal{H} \rightarrow H^2(G_n)$ as the restriction operator. Since G_n is compactly contained in G , this is a well-defined bounded linear operator. Clearly W_n is one-to-one, and, since $\mathcal{H}$ contains the polynomials and ∂G_n is a Jordan curve, W_n has dense range. Clearly W_n intertwines $S = (M_f, \mathcal{H})$ and $T = (M_f, H^2(G_n))$ and both are hyponormal operators. We now show that W_n is Hilbert-Schmidt.

So, let $\{f_k\}$ be any orthonormal basis for $\mathcal{H}$. Also, let K_z be the reproducing kernels in $\mathcal{H}$ for $z \in G$. Observe that

$$K_z = \sum_{k=1}^{\infty} \langle K_z, f_k \rangle f_k = \sum_{k=1}^{\infty} \overline{\hat{f}_k(z)} f_k ,$$

$$\|K_z\|_{\mu}^2 = \sum_{k=1}^{\infty} |\hat{f}_k(z)|^2 .$$

Thus we have

$$\sum_{k=1}^{\infty} \|W_n f_k\|_{H^2(G_n)}^2 = \sum_{k=1}^{\infty} \int_{\partial G_n} |\hat{f}_k(z)|^2 d\omega_n = \int_{\partial G_n} \sum_{k=1}^{\infty} |\hat{f}_k(z)|^2 d\omega_n = \int_{\partial G_n} \|K_z\|_{\mathcal{H}}^2 d\omega_n < \infty .$$

The last integral is finite because the reproducing kernels $\{ K_z : z \in \text{cl} G_n \}$ form a bounded set in $\mathcal{H}$. Thus W_n is Hilbert-Schmidt. The Intertwining Lemma now applies to give that $\text{tr}[S^*, S] \geq \text{tr}[T^*, T]$. But since G_n is simply connected, Lemma 2.2 implies that

$$\text{tr}[T^*, T] = \frac{1}{\pi} \int_{G_n} |f'|^2 dA . \text{ Thus, } \text{tr}[S^*, S] \geq \frac{1}{\pi} \int_{G_n} |f'|^2 dA \text{ for all } n . \text{ Now, since } \text{Area}(G_n) \rightarrow$$

$\text{Area}(G)$, as $n \rightarrow \infty$, the Monotone Convergence Theorem gives the desired conclusion. $\square$

Proof of Lemma 2.3. If G is a bounded region, let $\tau: \mathbf{D} \rightarrow G$ be a universal covering map. A fundamental region for τ is a region F in $\mathbf{D}$ so that $\tau(\text{cl}F \cap \mathbf{D}) = G$ and τ is one-to-one on F . Let F be a “Dirichlet region” for τ ; that is, a particular type of fundamental region that is an intersection of hyperbolic half-planes in $\mathbf{D}$. See Katok [1992], p. 52. Since the region F is simply connected, in fact hyperbolically convex, and τ is one-to-one on F , it follows that $\tau(F)$ is a simply connected region contained in G . Also, $\tau(F)$ is dense in G , since τ maps $\text{cl}F \cap \mathbf{D}$ onto G . Furthermore, since ∂F is contained in a countable number of hyperbolic geodesics, ∂F has area zero. Now, as analytic functions map sets of area zero onto sets of area zero, it follows that $\tau(\partial F)$ has area zero. Since $G - \tau(F) \subseteq \tau(\partial F)$, and this latter set has area zero, it follows that $\text{Area}(G) = \text{Area}(\tau(F))$. Now, let $\varphi: \mathbf{D} \rightarrow \tau(F)$ be a Riemann map. If $G_n = \varphi\left(\left\{z \in \mathbf{D}: |z| < 1 - \frac{1}{n}\right\}\right)$, then G_n has the required properties. $\square$

Notice that if G is a simply connected region, then Lemma 2.3 follows immediately from the Riemann mapping theorem.

Corollary 2.5 If S_μ is an irreducible cyclic subnormal operator, $G = \text{abpe}(\mu)$ and $f \in P^2(\mu) \cap L^\infty(\mu)$, then $\text{tr}[f(S_\mu)^*, f(S_\mu)] \geq \frac{1}{\pi} \int_G |\hat{f}'|^2 dA$, where $\hat{f}$ is the analytic extension of f to G .

Proof. Thomson's Theorem allows us to use the natural map $\wedge: P^2(\mu) \rightarrow \text{Hol}(G)$ to identify $P^2(\mu)$ with a space of analytic functions on G . Also, if $f \in P^2(\mu) \cap L^\infty(\mu)$, then $f(S_\mu)$ is a subnormal operator. Hence Theorem 2.4 applies. $\square$

Lemma 2.6 If S_μ is a pure cyclic subnormal operator, $\text{abpe}(\mu) = \mathbf{D}$ and f is analytic on a neighborhood of the closed unit disk, $\text{cl } \mathbf{D}$, then $\text{tr}[f(S_\mu)^*, f(S_\mu)] \leq \frac{1}{\pi} \int_{\mathbf{D}} |f'|^2 dA$.

Proof. Suppose f is analytic on $\{z : |z| < r\}$ for some $r > 1$. Let $G = \{z : |z| < r_0\}$ for some $1 < r_0 < r$. So, $f \in H^\infty(G)$. Now let $S = M_f$ on $H^2(G)$ and set $T = f(S_\mu)$. Also, define $W: H^2(G) \rightarrow P^2(\mu)$, by $Wh = h|_{\text{cl } \mathbf{D}}$. Clearly, $WS = TW$ and W is one-to-one with dense range. Also, an argument as in Theorem 2.4 shows that W is Hilbert-Schmidt. So the Intertwining Lemma gives us that $\text{tr}[T^*, T] \leq \text{tr}[S^*, S] = \frac{1}{\pi} \int_G |f'|^2 dA$, the last equality holding by Lemma 2.2. Letting r_0 decrease to 1 gives the result. $\square$

Proof Theorem 2.1.

Special Case: $\text{abpe}(\mu) = \mathbf{D}$.

In this case, let $f \in P^2(\mu) \cap L^\infty(\mu)$. By Corollary 2.5, it suffices to show that

$$\text{tr}[f(S_\mu)^*, f(S_\mu)] \leq \frac{1}{\pi} \int_{\mathbf{D}} |\hat{f}'|^2 dA. \quad \text{Suppose that } \frac{1}{\pi} \int_{\mathbf{D}} |\hat{f}'|^2 dA < \infty; \text{ so } \hat{f} \in D_1, \text{ the Dirichlet}$$

space on $\mathbf{D}$. Define $f_j(z) = \hat{f}(r_j z)$ for some sequence $\{r_j\}$ increasing to 1. It is easy to

check that f_j converges to $\hat{f}$ in the Dirichlet norm; that is, $\frac{1}{\pi} \int_{\mathbf{D}} |f'_j|^2 dA \rightarrow \frac{1}{\pi} \int_{\mathbf{D}} |\hat{f}'|^2 dA$.

Now, as each f_j is analytic on a neighborhood of $\text{cl } \mathbf{D}$, Corollary 2.5 and Lemma 2.6 give

that $\text{tr}[f_j(S_\mu)^*, f_j(S_\mu)] = \frac{1}{\pi} \int_{\mathbf{D}} |f'_j|^2 dA$. Since $f_j(S_\mu)$ is subnormal it follows that $[f_j(S_\mu)^*, f_j(S_\mu)]$

is positive. Hence its trace equals its trace norm. Thus the sequence $\{[f_j(S_\mu)^*, f_j(S_\mu)] : n \geq 1\}$ is norm bounded in the space of trace class operators, $\mathcal{B}_1$. Since $\mathcal{B}_1$ is the dual of the separable space of compact operators $\mathcal{B}_0$, there is a weak* convergent subsequence. Let's assume that $[f_j(S_\mu)^*, f_j(S_\mu)] \rightarrow A$ weak* in $\mathcal{B}_1$. That is, if K is any compact operator, then $\text{tr}(K[f_j(S_\mu)^*, f_j(S_\mu)]) \rightarrow \text{tr}(KA)$. Now, by choosing K to be the rank one operator $h \otimes h$ for some $h \in P^2(\mu)$, one sees that $[f_j(S_\mu)^*, f_j(S_\mu)]$ is positive for all n , and so A must be a positive operator in $\mathcal{B}_1$.

$$\text{Claim 1: } \text{tr} A \leq \frac{1}{\pi} \int_{\mathbf{D}} |\hat{f}'|^2 dA.$$

Since A is a positive trace class operator, its trace equals its trace norm. Thus since the norm is lower semi-continuous in the weak* topology, we have

$$\text{tr} A = \|A\|_1 \leq \liminf_{j \rightarrow \infty} \|[f_j(S_\mu)^*, f_j(S_\mu)]\|_1 = \liminf_{j \rightarrow \infty} \frac{1}{\pi} \int_{\mathbf{D}} |f'_j|^2 dA = \frac{1}{\pi} \int_{\mathbf{D}} |\hat{f}'|^2 dA.$$

Observe that the last limit exists because f_j converges to $\hat{f}$ in the Dirichlet norm.

$$\text{Claim 2: } A = [f(S_\mu)^*, f(S_\mu)].$$

If $h \in P^2(\mu)$, then by considering rank one operators, we have $\langle [f_j(S_\mu)^*, f_j(S_\mu)]h, h \rangle \rightarrow \langle Ah, h \rangle$. Also, as f_j converges pointwise boundedly to f , we get that $f_j h \rightarrow fh$ in $L^2(\mu)$ norm and $\bar{f}_j h \rightarrow \bar{f}h$ in $L^2(\mu)$ norm. Hence if P is the projection of $L^2(\mu)$ onto $P^2(\mu)$, then

$$\langle [f_j(S_\mu)^*, f_j(S_\mu)]h, h \rangle = \|f_j(S_\mu)h\|^2 - \|f_j(S_\mu)^*h\|^2 = \|f_j h\|^2 - \|P(\bar{f}_j h)\|^2 \rightarrow$$

$$\|fh\|^2 - \|P(\bar{f}h)\|^2 = \langle [f(S_\mu)^*, f(S_\mu)]h, h \rangle.$$

Hence we have that for each $h \in P^2(\mu)$, $\langle Ah, h \rangle = \langle [f(S_\mu)^*, f(S_\mu)]h, h \rangle$. As both operators are self-adjoint, they must be equal.

Thus, since $[f(S_\mu)^*, f(S_\mu)] = A$, $[f(S_\mu)^*, f(S_\mu)]$ is trace class and $\text{tr}[f(S_\mu)^*, f(S_\mu)] = \text{tr}A \leq$

$$\frac{1}{\pi} \int_D |\hat{f}'|^2 dA. \text{ Combining this with Corollary 2.5, gives } \text{tr}[f(S_\mu)^*, f(S_\mu)] = \frac{1}{\pi} \int_D |\hat{f}'|^2 dA.$$

General Case: S_μ is pure and $G = \text{abpe}(\mu)$.

It follows from Thomson's Theorem that each component of G corresponds to an irreducible summand of S_μ . Hence S_μ is a direct sum of irreducible cyclic operators and the self-commutator is therefore also a direct sum of self-commutators. Thus it suffices to consider the case where S_μ is irreducible. In particular G is connected.

If τ is a Riemann map of $\mathbf{D}$ onto G , then there is a measure ν on $\text{cl}\mathbf{D}$ such that S_ν is pure, $\text{abpe}(\nu) = \mathbf{D}$, $\mu = \nu \circ \tau^{-1}$ and S_μ is unitarily equivalent to $\tau(S_\nu)$. (This strongly uses Thomson's Theorem, see Olin and Yang [1995] or Theorem I.2.3.) Further, the unitary is given by composition with τ .

If $f \in P^2(\mu) \cap L^\infty(\mu)$, then $g = f \circ \tau \in P^2(\nu) \cap L^\infty(\nu)$ and so it follows that $f(S_\mu) \cong g(S_\nu)$.

Thus, $\text{tr}[f(S_\mu)^*, f(S_\mu)] = \text{tr}[g(S_\nu)^*, g(S_\nu)] = \frac{1}{\pi} \int_{\mathbf{D}} |\hat{g}'|^2 dA$. Now, as $\hat{g} = \hat{f} \circ \tau$, a simple change

of variables gives that this last integral equals $\frac{1}{\pi} \int_G |\hat{f}'|^2 dA$. So the proof of the Theorem is

complete. $\square$

If S is any cyclic subnormal operator that is not pure, then, since S is cyclic, the spectral measure of the normal part cannot put any mass on $G = \text{abpe}(S)$.

Corollary 2.7 If S is a cyclic subnormal operator, then $[S^*, S]$ is trace class and

$$\text{tr}[S^*, S] = \frac{1}{\pi} \text{Area}[\sigma(S) - \sigma_e(S)].$$

If T is any pure hyponormal operator with trace class self-commutator, then the principal function for T , g_T , is a non-negative function defined on $\sigma(T)$ such that $g_T(\lambda) = -\text{ind}(T - \lambda)$

for all $\lambda \notin \sigma_e(T)$ and $\text{tr}[T^*, T] = \frac{1}{\pi} \int_{\sigma(T)} g_T dA$ (see Clancey [1979] or Martin and Putinar

[1989]).

Corollary 2.8 If S is a pure cyclic subnormal operator, then the principal function for S vanishes a.e. on $\sigma_e(S)$.

Proof. If g_S is the principal function for S and $G = \sigma(S) - \sigma_e(S)$, then we have

$$\frac{1}{\pi} \text{Area}(G) = \text{tr}[S^*, S] = \frac{1}{\pi} \int_{\sigma(S)} g_S dA = \frac{1}{\pi} \int_G 1 dA + \frac{1}{\pi} \int_{\sigma_e(S)} g_S dA$$

where we have used that $g_S(\lambda) = -\text{ind}(S-\lambda) = 1$ on G . Thus, since g_S is non-negative, we see that $g_S = 0$ a.e. on $\sigma_e(S)$. $\square$

We now give some examples. The following examples show that it is necessary to leave out the essential spectrum in computing the trace of $[S_\mu^*, S_\mu]$ and that one does not always have equality in the Berger-Shaw Theorem.

Example 2.9 It's known that there exist Jordan curves with positive area (see Osgood [1903] or Sagan [1994]). If G is the simply connected region bounded by such a curve, then there exists a measure μ on G so that S_μ is an irreducible subnormal operator and $\text{abpe}(\mu) = G$ (see Conway [1991], p. 405). Hence $\sigma_e(S_\mu) = \partial G$ has positive area.

Lemma 2.10 If B is any disk and $K \subseteq B$ is a nowhere dense compact set, then there exists a sequence $\{\Delta_k\}$ of open disks contained in $B - K$ such that Δ_n and Δ_k have disjoint closures if $n \neq k$, the radii of the disks is summable and the disks cluster precisely on K .

By this last statement we mean that if $\{z_k\}$ is a sequence with $z_k \in \Delta_k$, then the limit points of the sequence $\{z_k\}$ are contained in K and every point in K is a limit point of such a sequence.

Proof. Since K is nowhere dense, $B - K$ is dense in B . Now, choose a sequence $\{a_n\}$ of points in $B - K$ that clusters precisely on K . One may do this as follows, for each integer n , cover K with a finite number of open disks with radius $1/n$. Since $B - K$ is dense in B , one may choose one point in each of these disks. In this way the constructed sequence clusters precisely on K . So for each n , $a_n \in B - K$ and is isolated from the other points of the sequence, hence we may choose a disk $\Delta_n \subseteq B - K$ centered at a_n so that the disks $\{\Delta_n\}$ have disjoint closures. If we now shrink the disks as necessary to guarantee that their radii are summable, then they will still have disjoint closures. Also, since the radii tend to zero, the disks will cluster at precisely the same points as the sequence $\{a_n\}$. $\square$

We now use Lemma 2.10 to give two examples. The first example illustrates that even for pure cyclic subnormal operators the essential spectrum can be a very thick subset of the spectrum. In particular it shows how far the inequality in the Berger-Shaw Theorem can be from an equality.

Example 2.11 If $\varepsilon > 0$, then there exists a pure cyclic subnormal operator S_μ such that $\frac{1}{\pi} \text{Area}[\sigma(S_\mu)] \geq 1 - \varepsilon$ and yet $\text{tr}[S_\mu^*, S_\mu] < \varepsilon$.

Proof. Let $0 < \varepsilon < 1/2$. Choose a compact totally disconnected set $K \subseteq \mathbf{D}$, such that

$\frac{1}{\pi} \text{Area}(\mathbf{D} - K) < \varepsilon$. Now, let Δ_n be the disks guaranteed by Lemma 2.10. Since the disks

cluster exactly on the set K , we have $\text{cl}\left(\bigcup_{n=1}^{\infty} \Delta_n\right) = K \cup \left(\bigcup_{n=1}^{\infty} \text{cl}\Delta_n\right)$. Also since the sum of

the radii of Δ_n is finite, if we let μ be arc length measure on $\bigcup_{n=1}^{\infty} \partial\Delta_n$, then S_μ is pure. Let

$L = K \cup \left(\bigcup_{n=1}^{\infty} \text{cl}\Delta_n \right)$. Since K is totally disconnected, $\mathbb{C} - K$ is connected, hence so is $\mathbb{C} - L$.

As such, we know from Mergelyan's Theorem (see Conway [1991], p. 253) that $P(L) = A(L) := C(L) \cap \text{Hol}(\text{int}L)$, where $P(L)$ is the uniform closure of the polynomials on L . It

follows that $\sigma(S_\mu) = K \cup \left(\bigcup_{n=1}^{\infty} \text{cl}\Delta_n \right)$ and $G = \text{abpe}(\mu) = \bigcup_{n=1}^{\infty} \Delta_n$. Thus,

$$\frac{1}{\pi} \text{Area}[\sigma(S_\mu)] \geq \frac{1}{\pi} \text{Area}(K) \geq 1 - \varepsilon \text{ and } \text{tr}[S_\mu^*, S_\mu] = \frac{1}{\pi} \text{Area}(G) < \varepsilon. \quad \square$$

In trying to generalize Theorem 2.1, one naturally asks if certain spectral properties guarantee that an operator has trace class self-commutator. We know that spectral information can guarantee compactness of the self-commutator, see Corollary 1.8. It is natural to ask what else is needed to get a trace class self-commutator. The next example shows that a reasonable conjecture fails. Namely, even if the essential spectrum of S is small (as in Corollary 1.8) and the index function of S is integrable, then S need not have trace class self-commutator. First a simple lemma is needed.

Lemma 2.12 Suppose $\{K_n : n \in \mathbb{Z}^+ \cup \{\infty\}\}$ is a sequence of disjoint compact sets such that the sets $\{K_n : n \geq 1\}$ cluster only on K_∞ . If $R(K_n) = C(K_n)$ for $n \in \mathbb{Z}^+ \cup \{\infty\}$ and $K = \text{cl}\left(\bigcup_{n=1}^{\infty} K_n\right)$, then $R(K) = C(K)$.

Proof. If μ is a measure on K and $\mu \perp R(K)$, then since each K_n , $n < \infty$, is isolated from the others, including K_∞ , Runge's Theorem gives that the characteristic function of K_n belongs to $R(K)$. Thus, if $f \in R(K_n)$ and f is defined to be zero off K_n , then $f \in R(K)$. So, $\mu|_{K_n} \perp R(K_n)$, thus $\mu|_{K_n} = 0$ for all $n < \infty$. Hence μ is supported on K_∞ and annihilates $R(K)$ (although it does not necessarily annihilate $R(K_\infty)$). Now, if U is any bounded component of $C - K_\infty$, then $U \cap K \neq \emptyset$, else $U \subseteq \bigcup_{n=1}^{\infty} K_n$ and this contradicts the Baire Category Theorem. Hence Runge's Theorem says that $R(K)|_{K_\infty} = R(K_\infty)$. Thus since μ is supported on K_∞ and $R(K_\infty) = C(K_\infty)$ we see that $\mu = 0$. Thus $R(K) = C(K)$. $\square$

Example 2.13 There exists a pure subnormal operator with compact self-commutator such that $R(\sigma_e(S)) = C(\sigma_e(S))$, the index function, $\text{ind}(S - \lambda)$, is integrable on $\sigma(S) - \sigma_e(S)$ and yet $[S^*, S]$ is not trace class.

Proof. To start, let $B_0 = \mathbf{D}$ the unit disk and choose a totally disconnected compact set K_0 in B_0 with $\frac{1}{\pi} \text{Area}(B_0 - K_0) < \frac{1}{2}$. Now, let B_n , $n \geq 1$, be the open disks guaranteed by Lemma 2.10. Also, let r_n be the radius of B_n . Next choose positive integers m_n such that $\sum_{n=0}^{\infty} m_n r_n^2 = \infty$. For each $n \geq 1$, choose a totally disconnected compact set $K_n \subseteq B_n$ such that $(m_n + 2) \text{Area}(B_n - K_n) \leq \frac{1}{n^2}$. Next for each $n \geq 1$, apply Lemma 2.10 to get a sequence $\{\Delta_{nk} : k \geq 1\}$ of open disks in $B_n - K_n$ clustering exactly on K_n . For $n \geq 0$ define, $S_n = M_z$ on $H^2(B_n)$ and for $n, k \geq 1$ define $S_{nk} = M_z$ on $H^2(\Delta_{nk})$. Let $S =$

$\bigoplus_{n=0}^{\infty} S_n^{(m_n)} \oplus \bigoplus_{n,k=1}^{\infty} S_{nk}$. We claim that S is the desired operator. Clearly S is a pure

subnormal operator. Also, if K is compact and totally disconnected, then $R(K) = C(K)$.

Hence Lemma 2.12 implies that $R(\sigma_e(S)) = C(\sigma_e(S))$. Now, this last property guarantees

that S has compact self-commutator (see Corollary 1.8). Also, if we set $T = \bigoplus_{n=0}^{\infty} S_n^{(m_n)}$,

then $\text{tr}[T^*, T] = \sum_{n=0}^{\infty} m_n r_n^2 = \infty$. It follows that T does not have trace class self-

commutator, hence S cannot either. Finally, to show that the index function is integrable

on $G = \sigma(S) - \sigma_e(S)$, it suffices to check this on the union U of the disks B_n , because

$|\text{ind}(S - \lambda)| \leq 1$ for λ not in U . Since $|\text{ind}(S - \lambda)| \leq (m_n + 2)$ for $\lambda \in B_n$ and because $G \cap$

$B_n \subseteq B_n - K_n$, we have

$$\int_{G \cap U} |\text{ind}(S - \lambda)| dA = \sum_{n=1}^{\infty} \int_{G \cap B_n} |\text{ind}(S - \lambda)| dA \leq \sum_{n=1}^{\infty} (m_n + 2) \text{Area}(B_n - K_n) < \infty.$$

This last sum is finite because $(m_n + 2) \text{Area}(B_n - K_n) \leq \frac{1}{n^2}$. $\square$

Since the previous construction strongly hinges on the fact that $\sigma_e(S)$ has positive area, the

next question is a natural one; namely a “trace class version” of Theorem 1.10.

Question 2.14 If S is a pure subnormal operator, $f = \hat{\chi}_{\sigma_e(S)}$, $[S, S]$ is trace class and the index function of S is integrable over $\sigma(S) - \sigma_e(S)$, then must $[S^*, S]$ be trace class?

It follows from Hadwin and Nordgren [1988], that Question 2.14 has an affirmative answer when $\text{Area}[\sigma_e(S)] = 0$.

Theorem 2.4 applies to some rationally cyclic subnormal operators. Notice though that the region G in Theorem 2.4 needs to be connected or have at most a finite number of components. Let $R^2(K, \mu)$ denote the $L^2(\mu)$ closure of the rational functions with poles off K .

Corollary 2.15 If $S = M_z$ on $R^2(K, \mu)$ is pure, $G = \text{abpe}(R^2(K, \mu))$ is connected and the reproducing kernels are dense in $R^2(K, \mu)$, then for $f \in R^2(K, \mu) \cap L^\infty(\mu)$ we have $\text{tr}[f(S)^*, f(S)] \geq \frac{1}{\pi} \int_G |\hat{f}'|^2 dA$.

Proof. Since the reproducing kernels are dense, the natural map $\wedge: R^2(K, \mu) \rightarrow \text{Hol}(G)$ allows us to identify $R^2(K, \mu)$ with a Hilbert space of analytic functions on G . Also, since G is connected and $f(S)$ is a subnormal operator Theorem 2.4 applies. $\square$

Corollary 2.16 If $S = M_z$ on $R^2(K, \mu)$ is pure, $G = \text{abpe}(R^2(K, \mu))$ is connected and $\text{cl } G = \sigma(S)$, then for $f \in R^2(K, \mu) \cap L^\infty(\mu)$ we have $\text{tr}[f(S)^*, f(S)] \geq \frac{1}{\pi} \int_G |\hat{f}'|^2 dA$.

Proof. Since G is connected and $\text{cl } G = \sigma(S)$, we have $\sigma_e(S) = \partial G$ and thus $R(\sigma_e(S)) = C(\sigma_e(S))$. Now, if $\mathcal{M}$ is the collection of all functions in $R^2(K, \mu)$ that are orthogonal to the reproducing kernels, then $\mathcal{M}$ is a closed invariant subspace for S and $\sigma(S|_{\mathcal{M}}) \subseteq \sigma_e(S)$. Thus, since $R(\sigma_e(S)) = C(\sigma_e(S))$, this implies that $S|_{\mathcal{M}}$ is normal. Since S is pure, $\mathcal{M} = (0)$ and the reproducing kernels are dense. Hence Theorem 2.15 applies. $\square$

The Berger-Shaw Theorem together with Corollary 2.16 gives the following result.

Corollary 2.17 If $S = M_z$ on $R^2(K, \mu)$ is pure, $G = \text{abpe}(R^2(K, \mu))$ is connected and $\text{cl}G = \sigma(S)$ then $\frac{1}{\pi} \text{Area}(G) \leq \text{tr}[S^*, S] \leq \frac{1}{\pi} \text{Area}[\sigma(S)]$.

Corollary 2.18 If $S = M_z$ on $R^2(K, \mu)$ is pure, $G = \text{abpe}(R^2(K, \mu))$ is connected, $\text{cl}G = \sigma(S)$ and $\text{Area}[\partial G] = 0$, then $\text{tr}[S^*, S] = \frac{1}{\pi} \text{Area}(G)$.

If $S = M_z$ on $R^2(K, \mu)$, then it is known that $\text{abpe}(R^2(K, \mu)) = \sigma(S) - \sigma_e(S)$. Exactly when $\sigma_e(S)$ contributes to the trace of $[S^*, S]$ is unknown. For *cyclic* subnormal operators we have shown above that it does not. We now construct a pure rationally cyclic subnormal operator with $G = \text{abpe}(R^2(K, \mu))$ dense in $\sigma(S)$, yet $\sigma_e(S)$ has positive area and is part of the trace of $[S^*, S]$. That is, the principal function g_S for S , is *not* zero a.e. on $\sigma_e(S)$.

Example 2.19 There exists a pure rationally cyclic subnormal operator S such that $G = \sigma(S) - \sigma_e(S)$ is non-empty and dense in $\sigma(S)$, yet $\text{tr}[S^*, S] > \frac{1}{\pi} \text{Area}(G)$.

Proof. Suppose L is a swiss cheese; that is, there are disjoint open disks Δ_n inside the unit

disk $\mathbf{D}$ whose union is dense in $\mathbf{D}$ and with summable radii such that $L = \text{cl}\mathbf{D} - \bigcup_{n=1}^{\infty} \Delta_n$. Let

ν_1 be arc length measure on $\bigcup_{n=1}^{\infty} \partial\Delta_n$. Now, let B_n be an open disk with the same center as

Δ_n but with half the radius. Let ν_2 be arc length measure on $\bigcup_{n=1}^{\infty} \partial B_n$. If we set $K = L \cup$

$\bigcup_{n=1}^{\infty} \text{cl} B_n$ and $\mu = \nu_1 + \nu_2$, then $S = M_z$ on $R^2(K, \mu)$ is a pure rationally cyclic subnormal

operator. Further, $\sigma(S) = K$, $\sigma_e(S) = L \cup \bigcup_{n=1}^{\infty} \partial B_n$ and $G = \text{abpe}(R^2(K, \mu)) = \bigcup_{n=1}^{\infty} B_n$. Now,

if $S_i = M_z$ on $R^2(K, \nu_i)$, then $S = S_1 \oplus S_2$. Since S_2 is a direct sum of functions of the

unilateral shift, Lemma 2.2 applies to give that $\text{tr}[S_2^*, S_2] = \frac{1}{\pi} \text{Area}(G)$. Also since S_1 is

pure, $\text{tr}[S_1^*, S_1] > 0$. Thus $\text{tr}[S^*, S] = \text{tr}[S_1^*, S_1] + \text{tr}[S_2^*, S_2] = \text{tr}[S_1^*, S_1] + \frac{1}{\pi} \text{Area}(G) >$

$\frac{1}{\pi} \text{Area}(G)$. $\square$

The previous example, is one of the standard examples of an $R^2(K, \mu)$ space where the reproducing kernels are not dense. We shall close with a natural question.

Question 2.20 If $S = M_z$ on $R^2(K, \mu)$ is a pure rationally cyclic subnormal operator, $G = \text{abpe}(R^2(K, \mu))$ and $R(\sigma_e(S)) = C(\sigma_e(S))$, then is $\text{tr}[S^*, S] = \frac{1}{\pi} \text{Area}(G)$?

Notice that this hypothesis $R(\sigma_e(S)) = C(\sigma_e(S))$ is satisfied by every irreducible cyclic subnormal operator, so this is a natural generalization to the rationally cyclic case. Also this hypothesis guarantees that the reproducing kernels are dense in $R^2(K, \mu)$. Further, the example mentioned above with $\text{tr}[S^*, S] > \frac{1}{\pi} \text{Area}(G)$ does not satisfy $R(\sigma_e(S)) = C(\sigma_e(S))$.

CHAPTER IV

The Injectivity of the Self-Commutator

In this chapter we shall consider the question of when zero is an eigenvalue for the self-commutator of a subnormal operator. Since zero is always in the essential spectrum of the self-commutator this is a natural question. Also, it is related to a general approximation problem and this will be explored more in Section 1. In Section 2 we will consider the above question for the Hardy operators; that is, $S_G = (M_z, H^2(G))$ where G is a bounded simply connected region. It will be shown that zero is an eigenvalue for $[S_G^*, S_G]$ if and only if G is a generalized quadrature domain. In Section 3 we shall consider cyclic subnormal operators. Here the results are not complete, but they *suggest* that if $[S_\mu^*, S_\mu]$ has zero as an eigenvalue, then $G = \text{abpe}(\mu)$ must be a generalized quadrature domain and μ must be (weighted) harmonic measure plus a discrete sum of point masses in G that form an $H^\infty(G)$ zero set. Precise statements will be given below.

1. Basic Properties and Some Approximation Questions

The following result will be fundamental in what follows.

Proposition 1.1 If S is a pure subnormal operator on $\mathcal{H}$ and $N = mneS$ on $\mathcal{K}$, then $\ker[S^*, S] = \{ h \in \mathcal{H} : N^*h \in \mathcal{H} \}$. Furthermore, $\ker[S^*, S]$ is invariant for any operator on $\mathcal{H}$ that lifts to the commutant of N .

Proof. If $N = \begin{bmatrix} S & X \\ 0 & T^* \end{bmatrix}$ with respect to $\mathcal{H} \oplus \mathcal{H}^\perp$, then a matrix computation gives that $[S^*, S] = XX^*$. So, $\ker[S^*, S] = \ker X^*$, but $X^* = P^\perp N^*|_{\mathcal{H}}$ where P is the projection of $\mathcal{K}$ onto $\mathcal{H}$. Thus $\ker X^* = \{h \in \mathcal{H} : N^*h \in \mathcal{H}\}$. Now, if A is an operator on $\mathcal{K}$ that commutes with N and leaves $\mathcal{H}$ invariant, then A also commutes with N^* . So if $h \in \ker[S^*, S]$, then $N^*Ah = AN^*h \in A\mathcal{H} \subseteq \mathcal{H}$. So, $Ah \in \ker[S^*, S]$. Hence A leaves $\ker[S^*, S]$ invariant. $\square$

Corollary 1.2 If S is a pure subnormal operator, $\mathcal{M} \in \text{Lat} S$ and $T = S|_{\mathcal{M}}$, then $\ker[T^*, T] \subseteq \ker[S^*, S]$.

Proof. Simply note that the minimal normal extension of T is N restricted to a reducing subspace. Thus, $\ker[T^*, T] = \{h \in \mathcal{M} : N^*h \in \mathcal{M}\} \subseteq \{h \in \mathcal{H} : N^*h \in \mathcal{H}\} = \ker[S^*, S]$. $\square$

It follows that if S is an *extension* of a subnormal operator T and $0 \in \sigma_p([T^*, T])$, then also $0 \in \sigma_p([S^*, S])$.

Example 1.3 The following operators have zero as an eigenvalue for their self-commutators.

- (a) Operators with finite rank self-commutator.
- (b) Quasinormal operators.

- (c) $S_\mu = M_z$ on $P^2(\mu)$ where $\mu = m|\partial\mathbf{D}| + \sum_{n=1}^{\infty} c_n \delta_{a_n}$, where $\{a_n\}$ form a Blaschke sequence in $\mathbf{D}$ and $\sum_{n=1}^{\infty} c_n < \infty$.

Proof. (a) This is clear. (b) Every quasinormal operator S has the form $S = U \otimes A$ where U is the unilateral shift and A is a positive one-to-one operator. Thus, one easily checks that $[S^*, S] = A^2 \oplus 0$. Hence $\ker[S^*, S] \neq (0)$.

(c) It is easy to check that $\text{abpe}(\mu) = \mathbf{D}$. So, if $\mathcal{M}$ consists of those functions that vanish at the points $\{a_n\}$, then $\mathcal{M}$ is an invariant subspace for S_μ and $S_\mu|_{\mathcal{M}}$ is the unilateral shift. Hence (a) implies that $S_\mu|_{\mathcal{M}}$ has zero as an eigenvalue for its self-commutator. Thus Corollary 1.2 gives that S_μ does also. $\square$

We now present a result to give some motivation for why one may consider the question of when $[S^*, S]$ has zero as an eigenvalue. There are several fundamental questions involving self-commutators, one is how well does the self-commutator determine the operator. That is, if two operators have unitarily equivalent self-commutators how much can the operators differ? In particular, how can one construct different operators with unitarily equivalent self-commutators. The following result shows that for an operator and its dual, there is a very simple obstruction to them having unitarily equivalent self-commutators.

Proposition 1.4 If S is any pure subnormal operator and $T = \text{dual}S$, then $[S^*, S]|_{\mathcal{M}_1} \cong [T^*, T]|_{\mathcal{M}_2}$ where $\mathcal{M}_1 = \ker[S^*, S]^\perp$ and $\mathcal{M}_2 = \ker[T^*, T]^\perp$.

Proof. Suppose S acts on $\mathcal{H}$ and $N = mneS$. If $N = \begin{bmatrix} S & X \\ 0 & T^* \end{bmatrix}$ with respect to $\mathcal{H} \oplus \mathcal{H}^\perp$, then a matrix computation gives that $[S^*, S] = XX^*$ and $[T^*, T] = X^*X$. So, clearly $X^*[S^*, S] = [T^*, T]X^*$. Since the self-commutators are self-adjoint and $X^*: \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is one-to-one with dense range, the result follows from Theorem 1.2.1. $\square$

So, if $T = \text{dual}S$, then the only obstruction to $[S^*, S] \cong [T^*, T]$ is that one self-commutator has zero as an eigenvalue and the other does not.

Corollary 1.5 If S is any pure subnormal operator with finite rank self-commutator and $T = \text{dual}S$, then $[S^*, S] \cong [T^*, T]$.

We now show that the question of whether zero is an eigenvalue for $[S^*, S]$ is equivalent to an approximation problem.

Proposition 1.6 If S is a pure subnormal operator on $\mathcal{H}$ and $N = mneS$ on $\mathcal{K}$, then $[S^*, S]$ is one-to-one if and only if $N(\mathcal{H}^\perp) + \mathcal{H}^\perp$ is dense in $\mathcal{K}$.

Proof. Observe that $\{N(\mathcal{H}^\perp) + \mathcal{H}^\perp\}^\perp = \{h \in \mathcal{H} : N^*h \in \mathcal{H}\}$ and apply Proposition 1.1. $\square$

Corollary 1.7 If $S = M_z$ on $R^2(K, \mu)$ is a pure rationally cyclic subnormal operator and $T = \text{dual}S$, then the following hold:

- (a) $0 \in \sigma_p([S^*, S])$ if and only if $zR^2(K, \mu)^\perp + R^2(K, \mu)^\perp$ is not dense in $L^2(\mu)$,
- (b) $0 \in \sigma_p([T^*, T])$ if and only if $\bar{z}R^2(K, \mu) + R^2(K, \mu)$ is not dense in $L^2(\mu)$.

The previous result is a function theoretic characterization of when $[S^*, S]$ and $[T^*, T]$ are one-to-one and it leads to some interesting L^2 approximation problems. In particular, from (b) one may suspect that zero is always an eigenvalue of $[T^*, T]$. But as we shall see later this is not true.

Proposition 1.8 Let S be a pure subnormal operator on $\mathcal{H}$ and μ a svsm for S . If $0 \in \sigma_p([S^*, S])$, then $\overline{R}^\infty(\sigma(S), \mu) + R^\infty(\sigma(S), \mu)$ is not weak* dense in $L^\infty(\mu)$.

Proof. Let $N = mneS$. By Proposition 1.1 there is a vector $h \in \mathcal{H}$ with $N^*h \in \mathcal{H}$. Now, if $f \in \overline{R}^\infty(\sigma(S), \mu) + R^\infty(\sigma(S), \mu)$, then $f(N)h \in \mathcal{H}$. So if $\overline{R}^\infty(\sigma(S), \mu) + R^\infty(\sigma(S), \mu)$ is weak* dense in $L^\infty(\mu)$, then $\text{cl}\{f(N)h : f \in \overline{R}^\infty(\sigma(S), \mu) + R^\infty(\sigma(S), \mu)\}$ would be a reducing subspace for N that is contained in $\mathcal{H}$, contradicting the purity of S . $\square$

We shall see that there are some quite natural measures for which the above set is weak* dense in L^∞ . Although the above result is quite simple in nature it has the following surprising consequence.

Corollary 1.9 If S is a pure subnormal operator and $0 \in \sigma_p([S^*, S])$, then $\sigma(S)$ has non-empty interior.

Proof. By a result of Trent and Wang [1981], if K is any compact set with no interior, then $\overline{R}(K) + R(K)$ is uniformly dense in $C(K)$. $\square$

For example, if the spectrum of S is a swiss cheese, then $[S^*, S]$ is one-to-one. Notice that there are quasinormal operators S such that $0 \in \sigma_p([S^*, S])$ and yet $\sigma(S) = \sigma_{ap}(S)$.

We now characterize the scalar-valued spectral measures of pure subnormal operators whose self-commutators are not one-to-one.

Theorem 1.10 If μ is a positive compactly supported Borel measure in $\mathbb{C}$, then there is a pure subnormal operator S such that $0 \in \sigma_p([S^*, S])$ and $\mu = \text{svsm} S$ if and only if $P^\infty(\mu)$ has no L^∞ summand and $\overline{z}P^\infty(\mu) + P^\infty(\mu)$ is not weak* dense in $L^\infty(\mu)$.

Proof. One direction follows from Proposition 1.8. So, assume that $P^\infty(\mu)$ has no L^∞ summand and $\overline{z}P^\infty(\mu) + P^\infty(\mu)$ is not weak* dense in $L^\infty(\mu)$.

First suppose $\overline{z}P^\infty(\mu) + P^\infty(\mu)$ has no L^∞ summand.

In this case, Chaumat's Lemma implies that there is a function $f \in L^1(\mu)$ so that $|f| > 0$ μ -a.e. and $f \perp \overline{z}P^\infty(\mu) + P^\infty(\mu)$ (see Conway[1991], p.246). Write $f = g\overline{h}$ where $g, h \in L^2(\mu)$. Now let $S = M_z$ on $\mathcal{H}$ where $\mathcal{H} = \text{cl}[\{\overline{z}P^\infty(\mu) + P^\infty(\mu)\}g]$ (closure in $L^2(\mu)$). It follows that S is a subnormal operator and clearly the self-commutator is not one-to-one (g is in the kernel, see Proposition 1.1). Furthermore as g is never zero, $\mu = \text{svsm} S$. Also as h is never zero and $h \perp \mathcal{H}$, S is pure.

Now in the general case, as $P^\infty(\mu)$ has no L^∞ summand there is an equivalent measure ν so that $S_\nu = M_z$ on $P^2(\nu)$ is pure; see Conway[1991], p. 87. Also, by Chaumat's Lemma,

there exists an $f \in L^1(\mu)$ that annihilates $\bar{z}P^\infty(\mu) + P^\infty(\mu)$ and is maximal with respect to absolute continuity, among all functions that annihilate $\bar{z}P^\infty(\mu) + P^\infty(\mu)$. Let $\Delta = \{f \neq 0\}$ and let $\mu_1 = \mu|_\Delta$. Then $\bar{z}P^\infty(\mu_1) + P^\infty(\mu_1)$ has no L^∞ summand, so by the first case there is a pure subnormal operator S_1 with $\text{svsm} S_1 = \mu_1$ and $0 \in \sigma_p([S_1^*, S_1])$. Now let $S = S_1 \oplus S_\nu$. Clearly S is a pure subnormal operator and $\text{svsm} S = \mu$. Furthermore, it easily follows from Proposition 1.1 that $\ker[S_1^*, S_1] \oplus (0) \subseteq \ker[S^*, S]$, hence $0 \in \sigma_p([S^*, S])$. $\square$

Example 1.11 If $\mu = m|_{\partial D} + \nu$ where ν is any positive finite Borel measure carried by D , then there is a pure subnormal operator S such that $\mu = \text{svsm} S$ and $0 \in \sigma_p([S^*, S])$.

Proof. We shall simply show that Theorem 1.10 applies. The fact that $P^\infty(\mu) = H^\infty(D)$ is clear. Also, $\bar{z}P^\infty(\mu) + P^\infty(\mu)$ is not weak* dense in $L^\infty(\mu)$, because if it is weak* dense in $L^\infty(\mu)$, then also $\{ \bar{z}p + q : p, q \text{ are polynomials} \}$ is weak* dense in $L^\infty(\mu|_\Delta)$ for any measurable set Δ . But for the measure μ , $\{ \bar{z}p + q : p, q \text{ are polynomials} \}$ is clearly not weak* dense in $L^\infty(\partial D)$. $\square$

It is natural to ask if $\bar{z}P^\infty(\mu) + P^\infty(\mu)$ is weak* dense in $L^\infty(\mu)$, then does this place any restrictions on the Sarason hull of $P^\infty(\mu)$? The answer is yes, when μ is harmonic measure (see Corollary 2.2). So, the following is a natural question.

Question 1.12 Is there a measure μ on $\text{cl} D$ such that $P^\infty(\mu) = H^\infty(D)$ and $\bar{z}P^\infty(\mu) + P^\infty(\mu)$ is weak* dense in $L^\infty(\mu)$?

We now give some examples showing how $\overline{z}P^\infty(\mu) + P^\infty(\mu)$ can be weak* dense in $L^\infty(\mu)$. In what follows, $P(K)$ will be the uniform closure of the polynomials on K .

Proposition 1.13 If G is the inside of a Jordan curve and an arc I on ∂G is also an arc on a circle, then $\overline{z}P(\partial G) + P(\partial G)$ is uniformly dense in $C(\partial G)$ if and only if G is not a disk.

Proof. If $\overline{z}P(\partial G) + P(\partial G)$ is dense in $C(\partial G)$, then it is clear that G is not a disk. Now suppose that G is not a disk. Let μ be any measure on ∂G with $\mu \perp \overline{z}P(\partial G) + P(\partial G)$. If I is an arc on the circle $\{z : |z - a| = r\}$ and an arc on ∂G , then let $R(z) = r^2 - |z - a|^2$. So, $R(z) = 0$ for $z \in I$, but R is not zero on all of ∂G . Also, $R \in \overline{z}P(\partial G) + P(\partial G)$, so $\int p R d\mu = 0$ for all polynomials p . Observe that the measure $Rd\mu$ is supported on $K = \text{cl}(\partial G - I)$ and that this compact set is polynomially convex with no interior. It follows that $P(K) = C(K)$, thus $Rd\mu$ is the zero measure. That is μ is zero on $\partial G \cap \{R \neq 0\}$. Since G is not a disk, this latter set is an open subset of ∂G hence contains an open arc J . Thus μ is supported on the compact set $\partial G - J$ and annihilates the polynomials, so as above $\mu = 0$ on $\partial G - J$. Hence $\mu = 0$ and $\overline{z}P(\partial G) + P(\partial G)$ is uniformly dense in $C(\partial G)$. $\square$

The above technique can be used to handle some regions G where ∂G is not a Jordan curve. For example, if G is the upper half of the unit disk with some slits. More examples will be given in Section 2.

We now consider a property stronger than having zero as an eigenvalue. Namely, every *part* of S has zero as an eigenvalue for its self-commutator. If S is an operator, then a *part* of S is $S|_{\mathcal{M}}$ for some invariant subspace $\mathcal{M}$ of S (see Halmos [1982], problem 152). Thus the parts of an operator are all those operators obtained by restricting S to its invariant subspaces. So, let's consider those pure subnormal operators such that every part of S has zero as an eigenvalue for its self-commutator. Observe that by Corollary 1.2 it suffices to check the cyclic parts. If U is the unilateral shift and B is the Bergman shift, then $S = U \oplus B$ is a pure subnormal operator such that $0 \in \sigma_p([S^*, S])$ (see Corollary 1.2) yet not every part of S has this property (because $[B^*, B]$ is one-to-one). See Example 3.17 for an irreducible operator with this property.

Proposition 1.14 Let S be a pure subnormal operator on $\mathcal{H}$ with $mneS = N$. If every part of S has zero as an eigenvalue for its self-commutator, then for each positive integer n there exists a vector $h \in \mathcal{H}$ such that $N^{*k}h \in \mathcal{H}$ for all k satisfying $0 \leq k \leq n$.

Proof. Proceeding by induction, for $n = 1$ simply apply Proposition 1.1. Now suppose the result is true for $n = n_0$; we must show it to be true for $n = n_0 + 1$. So, suppose there is an $h \in \mathcal{H}$ with $N^{*k}h \in \mathcal{H}$ for all k with $0 \leq k \leq n_0$. Let $\mathcal{M}$ be the invariant subspace for S generated by h . Set $T = S|_{\mathcal{M}}$. As T is a part of S , $0 \in \sigma_p([T^*, T])$. So, Proposition 1.1 gives that there exists a (non-zero) vector $g \in \mathcal{M}$ with $N^*g \in \mathcal{M}$. Also notice that for every vector x in $\mathcal{M}$, it follows that $N^{*k}x \in \mathcal{H}$ for all k with $0 \leq k \leq n_0$, because $\{p(N)h : p \text{ a polynomial}\}$ is a dense set in $\mathcal{M}$. So as $N^*g \in \mathcal{M}$ we have that $N^{*k}g \in \mathcal{H}$ for all k with $0 \leq k \leq n_0 + 1$. Thus the result follows. $\square$

This leads to an interesting function theoretic property connected with pure subnormal operators all of whose parts have zero as an eigenvalue for its self-commutator. This should be compared to Proposition 1.8, whose proof is essentially the same.

Corollary 1.15 Let S be a pure subnormal operator and $\mu = \text{svsm}S$. If every part of S has zero as an eigenvalue for its self-commutator, then for each positive integer n $R^\infty(\sigma(S), \mu) + \bar{z}R^\infty(\sigma(S), \mu) + \cdots + \bar{z}^n R^\infty(\sigma(S), \mu)$ is not weak* dense in $L^\infty(\mu)$.

2. Hardy Spaces and Generalized Quadrature Domains

In this section we shall examine more closely the operators $S_G = M_z$ on $H^2(G)$ for a bounded simply connected region G . If $\varphi: \mathbf{D} \rightarrow G$ is a Riemann map, then S_G is unitarily equivalent to $T_\varphi = M_\varphi$ on $H^2(\mathbf{D})$. If φ is any Nevanlinna function on $\mathbf{D}$, then we say that φ has a *pseudocontinuation* if there exists a Nevanlinna class function Φ on the exterior of $\mathbf{D}$ such that φ and Φ has the same radial limits a.e. on $\partial\mathbf{D}$. If g is any function defined in either the unit disk or its exterior, then g^* is the function given by $g^*(z) = \overline{g\left(\frac{1}{\bar{z}}\right)}$. Notice that g is in the Nevanlinna class on $\mathbf{D}$ if and only if g^* is in the Nevanlinna class of $\mathbf{C} - \text{cl}\mathbf{D}$. Further if g has boundary values then $g^*(e^{i\theta}) = \overline{g(e^{i\theta})}$ for m-a.e. θ . In particular, if φ is a Nevanlinna function on $\mathbf{D}$ and Φ is a function defined on the exterior of the disk, then Φ is a pseudocontinuation of φ if and only if Φ^* is a Nevanlinna function on $\mathbf{D}$ and $\Phi^*(e^{i\theta}) = \overline{\varphi(e^{i\theta})}$ for a.e. θ . Thus it is clear that rational functions have pseudocontinuations; however, it is well known (see Proposition 2.4) that these are the only functions with pseudocontinuations that are analytic across the unit circle.

We now make use of pseudocontinuations to classify those bounded analytic functions φ such that the Toeplitz operator T_φ on $H^2(\mathbf{D})$ has zero as an eigenvalue for its self-commutator.

Theorem 2.1 If $\varphi \in H^\infty(\mathbf{D})$, then the following are equivalent:

- (a) $0 \in \sigma_p([T_\varphi^*, T_\varphi])$;
- (b) $0 \in \sigma_p([S^*, S])$ where $S = \text{dual}(T_\varphi)$;
- (c) φ has a pseudo-continuation;
- (d) $\varphi \perp \mathcal{M}$ for some invariant subspace $\mathcal{M}$ of the unilateral shift;
- (e) $H^2 + \overline{\varphi}H^2$ is not dense in $L^2(\partial\mathbf{D})$;
- (f) $H^\infty + \overline{\varphi}H^\infty$ is not weak*-dense in $L^\infty(\partial\mathbf{D})$;

If φ is continuous on $\partial\mathbf{D}$, then the above are equivalent to:

- (g) $P(\partial\mathbf{D}) + \overline{\varphi}P(\partial\mathbf{D})$ is not uniformly dense in $C(\partial\mathbf{D})$.

Proof. We first show (a) $\Leftrightarrow$ (c). If $g \in \ker[T_\varphi^*, T_\varphi]$ and $g \neq 0$, then by Proposition 1.1 we have that there is a function $h \in H^2$ so that $\overline{\varphi}g = h$, m-a.e. on $\partial\mathbf{D}$. So h/g is a Nevanlinna class function on $\mathbf{D}$ whose boundary values agree with those of $\overline{\varphi}$. Hence h^*/g^* is a Nevanlinna class function on $\mathbf{C}_\infty - \text{cl}\mathbf{D}$ whose boundary values agree with those of φ . So h^*/g^* is a pseudocontinuation of φ .

Now suppose φ has a pseudo-continuation Φ . If $\Phi = g/h$, where g and h are bounded analytic function on $\mathbf{C}_\infty - \text{cl}\mathbf{D}$, then g^*/h^* is a Nevanlinna function on $\mathbf{D}$ whose boundary values agree with those of $\overline{\varphi}$. Hence $\overline{\varphi}h^* = g^*$ a.e. on $\partial\mathbf{D}$ and thus $\overline{\varphi}h^* \in H^2$. So by Proposition 1.1 we see that $\ker[T_\varphi^*, T_\varphi] \neq (0)$.

(a) $\Leftrightarrow$ (b) It is known that $\text{dual}(T_\varphi) \cong T_{\varphi^\#}$ where $\varphi^\#(z) = \overline{\varphi(\bar{z})}$ (see Conway [1981]).

Now simply observe that φ has a pseudo-continuation if and only if $\varphi^\#$ does also.

(b) $\Leftrightarrow$ (e) This follows immediately from Proposition 1.6.

(e) $\Rightarrow$ (f) $\Rightarrow$ (g) These are obvious.

(g) $\Rightarrow$ (c) Here we will assume that φ is continuous on $\text{cl}\mathbf{D}$. If this is not the case, then a similar proof shows that (f) $\Rightarrow$ (c). If $P(\partial\mathbf{D}) + \overline{\varphi} P(\partial\mathbf{D})$ is not uniformly dense in $C(\partial\mathbf{D})$, then there is a non-zero measure ν on $\partial\mathbf{D}$ so that $\nu \perp \{ P(\partial\mathbf{D}) + \overline{\varphi} P(\partial\mathbf{D}) \}$. Since $\nu \perp P(\partial\mathbf{D})$ the F. & M Riesz Theorem says that $\nu \ll m|_{\partial\mathbf{D}}$ and $d\nu/dm \in H^1$. So there exists an $h \in H^1$ with $\nu = h \, dm$. Now also, $\nu \perp \overline{\varphi} P(\partial\mathbf{D})$. So $0 = \int z^n \overline{\varphi} \, d\nu = \int z^n \overline{\varphi} h \, dm$, for all $n \geq 0$. Thus $\overline{\varphi} h \in H^1$. So, let $g \in H^1$ be such that $\overline{\varphi} h = g$ m-a.e. on $\partial\mathbf{D}$. Thus g/h is a Nevanlinna function on $\mathbf{D}$ that agrees with $\overline{\varphi}$ a.e. on $\partial\mathbf{D}$. Hence g^*/h^* is the required pseudo-continuation of φ .

Thus (a) through (c) and (e) through (g) are all equivalent. The fact that (c) and (d) are equivalent first appeared in R.G. Douglas, H.S. Shapiro and A.L. Shields [1970].

(d) $\Leftrightarrow$ (a) Simply observe that if $\varphi \perp \mathcal{M}$ for some shift invariant subspace $\mathcal{M}$, then $\mathcal{M} \subseteq \ker[T_\varphi^*, T_\varphi]$. Also, a simple computation gives that $\varphi \perp \ker[T_\varphi^*, T_\varphi]$ and this latter subspace is invariant for the unilateral shift by Proposition 1.1. $\square$

It is an unsolved problem to characterize those compact sets K such that $\bar{z}P(K) + P(K)$ is uniformly dense in $C(K)$. The following result solves this problem for Jordan curves.

Corollary 2.2 If Γ is a Jordan curve, $G = \text{ins}\Gamma$ and ω is harmonic measure for some point of G , then the following are equivalent:

- (a) $\bar{z}P^2(\omega) + P^2(\omega)$ is dense in $L^2(\omega)$,
- (b) $\bar{z}P^\infty(\omega) + P^\infty(\omega)$ is weak*-dense in $L^\infty(\omega)$,
- (c) $\bar{z}P(\Gamma) + P(\Gamma)$ is uniformly dense in $C(\Gamma)$,
- (d) The Riemann map of $\mathbf{D}$ onto G does not have a pseudocontinuation.

Proof. Let $\phi: \mathbf{D} \rightarrow G$ be a Riemann map. Since $\Gamma = \partial G$ is a Jordan curve, ϕ extends to a homeomorphism of $\text{cl}\mathbf{D}$ onto $\text{cl}G$. Thus composition with ϕ is an isometry from the appropriate function space on ∂G onto the corresponding space on $\partial\mathbf{D}$. Thus we may apply Theorem 2.1. $\square$

Recall Proposition 1.13. It together with Corollary 2.2 gives several examples of Riemann maps onto the inside of a Jordan curve that do not have pseudocontinuations.

Example 2.3 If ϕ is an inner function, then $\ker[T_\phi^*, T_\phi] = \phi H^2$. Also if $f = \phi g$, where ϕ is inner and g is outer, then $\phi \ker[T_g^*, T_g] \subseteq \ker[T_f^*, T_f] \subseteq \ker[T_g^*, T_g]$.

The next proposition is classical, although we include its proof. It shows that having a pseudocontinuation is a strong condition. However, as noted in Example 2.3, every inner function φ has a pseudocontinuation; namely $(1/\varphi)^*$.

Proposition 2.4 If φ is analytic in a neighborhood of the closed unit disk, then φ has a pseudocontinuation if and only if φ is a rational function.

Proof. If φ has a pseudocontinuation, then there is a Nevanlinna function Φ meromorphic outside the unit disk and $\varphi = \Phi$ a.e. on $\partial\mathbf{D}$. Now if $r > 1$ and φ is analytic on the disk with radius r , then φ and Φ are Nevanlinna functions on $1 < |z| < r$ with the same boundary values on $\partial\mathbf{D}$. Hence $\varphi(z) = \Phi(z)$ for all z satisfying $1 < |z| < r$. Thus Φ is a true analytic continuation of φ across $\partial\mathbf{D}$. Further as Φ is analytic on $1 < |z| < r$ and meromorphic at infinity, Φ can have at most a finite number of poles. So if $F(z)$ is defined to be φ inside the disk and Φ outside the disk, then F is meromorphic on $\mathbf{C}_\infty$ with at most a finite number of poles. Hence Liouville's Theorem gives that F is a rational function. So φ is also. Conversely if φ is a rational function, then it is its own pseudocontinuation. $\square$

Corollary 2.5 If G is a region bounded by an analytic Jordan curve, $\varphi: \mathbf{D} \rightarrow G$ is a Riemann map and $S_G = M_z$ on $H^2(G)$, then the following are equivalent:

- (a) $0 \in \sigma_p([S_G^*, S_G])$;
- (b) $[S_G^*, S_G]$ is finite rank;
- (c) φ is a rational function.

If G is any bounded simply connected region, then by a *Nevanlinna function* on G we mean a quotient of bounded analytic functions on G . Let $\varphi: \mathbf{D} \rightarrow G$ be a Riemann map. If f is a Nevanlinna function on G , then we say that f has *boundary values* when there exists a function f^* defined a.e. on ∂G with respect to harmonic measure such that the boundary values of $f \circ \varphi$ agree a.e. on $\partial \mathbf{D}$ with $f^* \circ \varphi$. Thus f has boundary values exactly when the radial limit function of $f \circ \varphi$ is constant on $\varphi^{-1}(p)$ for a.e. $[\omega]$ $p \in \partial G$, where we are considering φ as a map of $\partial \mathbf{D}$ onto ∂G via radial limits. For a bounded simply connected region G , we say that G is a *generalized quadrature domain* if there exists a Nevanlinna class function R on G , called the *generalized Schwarz function*, such that R has boundary values in the sense above and $R(z) = \bar{z}$ for a.e. $z \in \partial G$. Recall that for a quadrature domain the Schwarz function is required to be continuous up to ∂G and can have only a finite number of poles in G (see Aharonov and Shapiro [1976]).

Corollary 2.6 If G is a bounded simply connected region and $\varphi: \mathbf{D} \rightarrow G$ is a Riemann map, then the following are equivalent:

- (a) $0 \in \sigma_p([S_G^*, S_G])$;
- (b) φ has a pseudocontinuation;
- (c) G is a generalized quadrature domain.

For comparison, we state and prove the following well known result (see Aharonov and Shapiro [1976], p. 49 for the equivalence of (b) and (c)).

Corollary 2.7 If G is a bounded simply connected region and $\varphi: \mathbf{D} \rightarrow G$ is a Riemann map, then the following are equivalent:

- (a) $[S_G^*, S_G]$ is finite rank;
- (b) φ is a rational function;
- (c) G is a quadrature domain.

Proof of Corollary 2.6. Since S_G is unitarily equivalent to T_φ on $H^2(\mathbf{D})$, Theorem 2.1 implies that (a) and (b) are equivalent. We now show that (b) and (c) are equivalent. If φ has a pseudocontinuation Φ , then let $R = \Phi^* \circ \varphi^{-1}$. Since Φ is a Nevanlinna function so is R . To see that R has boundary values that agree with $\bar{z}$ a.e. $[\omega_G]$ on ∂G , we need to check that $\bar{z} \circ \varphi$ agrees a.e. on $\partial \mathbf{D}$ with the boundary function of $R \circ \varphi$. However, $R \circ \varphi = \Phi^*$ and the radial limits of Φ^* agree a.e. on $\partial \mathbf{D}$ with those of $\bar{\varphi}$. Thus R has boundary values and $R(z) = \bar{z}$ for a.e. $[\omega_G]$ $z \in \partial G$. Hence G is a generalized quadrature domain. Conversely, if G is a generalized quadrature domain with generalized Schwarz function R , then $R \circ \varphi$ is a Nevanlinna function on $\mathbf{D}$ whose boundary values agree a.e. with those of $\bar{\varphi}$. Hence $\Phi = (R \circ \varphi)^*$ is the required pseudocontinuation. $\square$

Proof of Corollary 2.7. Again, S_G is unitarily equivalent to T_φ on $H^2(\mathbf{D})$. It is easy to check that $\varphi \perp \text{zker}[T_\varphi^*, T_\varphi]$. If S_G has a finite rank self-commutator, then $\text{zker}[T_\varphi^*, T_\varphi]$ has finite codimension in $H^2(\mathbf{D})$. Hence φ is a linear combination of reproducing kernels and thus is a rational function. To show that (b) implies (c) one uses the same argument as in Corollary 2.6. Finally, if G is a quadrature domain with Schwarz function R , then $R \circ \varphi$ is a meromorphic function on $\mathbf{D}$ with only a finite number of poles whose boundary values agree a.e. with those of $\bar{\varphi}$. If p is a polynomial such that the product $p(z) R \circ \varphi(z)$ is analytic, then $p R \circ \varphi \in H^\infty(\mathbf{D})$; since R is bounded near ∂G . Thus if $g \in pH^2(\mathbf{D})$, then $g = pf$ for some $f \in H^2(\mathbf{D})$; and so $\bar{\varphi} g = \bar{\varphi} pf$ and this last function agrees a.e. on $\partial \mathbf{D}$ with

$(R \circ \varphi)pf = (pR \circ \varphi)f \in H^2(\mathbf{D})$. Hence $\overline{\varphi}g \in H^2(\mathbf{D})$ for every $g \in pH^2(\mathbf{D})$. Thus by

Proposition 1.1, $pH^2(\mathbf{D}) \subseteq \ker[T_\varphi^*, T_\varphi]$. So, $[S_G^*, S_G]$ is finite rank. $\square$

It is now shown that there are regions G such that $0 \in \sigma_p([S_G^*, S_G])$ and yet $[S_G^*, S_G]$ has infinite rank. Thus there are regions that are generalized quadrature domains but not quadrature domains. This does not seem too surprising, although in view of Corollary 2.5 it is not obvious.

Lemma 2.8 Suppose $\{f_n\}$, f are analytic on $\{|z| < r_0\}$, f is univalent and $f_n \rightarrow f$ uniformly on compact subsets of $\{|z| < r_0\}$, then for each $r < r_0$ there exists an integer N_r such that f_n is univalent on $\{|z| < r\}$ for all $n \geq N_r$.

Now let $\{a_k\}$ be an infinite Blaschke sequence in $\mathbf{D}$ and $B(z)$ the corresponding Blaschke product with zeros at precisely the points $\{a_k\}$. Also let $B_n(z)$ be the Blaschke product that corresponds to the first n terms of the sequence $\{a_k\}$ ($B_0 = 1$). Consider the shift invariant subspace BH^2 . We want to construct a bounded univalent function orthogonal

to this subspace that is not a rational function. Let $h_n(z) = \frac{B_n(z)}{(1 - \overline{a_{n+1}}z)}$, $n \geq 0$. So h_n is the

Blaschke product $B_n(z)$ times the reproducing kernel for the point a_{n+1} . Using this it is easy to see that $\{h_n/\|h_n\|_{H^2} : n \geq 0\}$ is a basis for $(BH^2)^\perp$. For more on the orthogonal complement of shift invariant subspaces, see Ahern and Clark [1970].

Theorem 2.9 If $\{a_k\}$ any infinite Blaschke sequence in $\mathbf{D}$ and B is the associated Blaschke product, then there is a univalent function φ that is continuous on $\text{cl}\mathbf{D}$, orthogonal to BH^2 and is not a rational function.

Proof. Keeping the above notation, we are going to define inductively a sequence of

scalars $\{c_n\}$ and let $\varphi(z) = \sum_{n=0}^{\infty} c_n h_n(z)$. To begin let $c_0 = 1/\|h_0\|_{\infty}$. For convenience we

may suppose that $a_1 \neq 0$. Now let $f(z) = c_0 h_0$. Notice that f is univalent on the disk

centered at the origin with radius $1/|a_1|$. If we let $f_n = f + (1/n) h_1$, then f and $\{f_n\}$ satisfy

the hypothesis of Lemma 2.8. Hence for all large n , f_n is univalent on some neighborhood

of the closed unit disk. If n is an integer large enough so that f_n is univalent in a

neighborhood of the closed unit disk and $(1/n) < 1/2\|h_1\|_{\infty}$, then set $c_1 = 1/n$ for this

value of n . For the next step, set $f = c_0 h_0 + c_1 h_1$ and $f_n = f + (1/n) h_2$. By construction, f is

univalent on a neighborhood of $\text{cl}\mathbf{D}$ and again Lemma 2.8 implies that for all large n , f_n is

univalent on a neighborhood of the closed unit disk. So choose an n large enough such

that f_n is univalent on a neighborhood of the closed unit disk and $(1/n) < 1/2^2\|h_2\|_{\infty}$. Set

$c_2 = 1/n$.

Continue in this manner to construct a sequence $\{c_k\}$ so that for each n , if we set $S_n(z) =$

$\sum_{k=0}^n c_k h_k(z)$, then $S_n(z)$ is univalent on a neighborhood of $\text{cl}\mathbf{D}$ and $\|S_n\|_{\infty} \leq 1$ because for

each n we have $c_n\|h_n\|_{\infty} < 1/2^n$. Now define $\varphi(z) = \sum_{n=0}^{\infty} c_n h_n(z)$. By construction the

series converges uniformly on $\text{cl}\mathbf{D}$. Thus φ is an analytic function that is continuous on $\text{cl}\mathbf{D}$. Also as the partial sums S_n are univalent in $\mathbf{D}$, φ is also univalent. Furthermore by the comments preceding the Theorem, $\varphi \perp \text{BH}^2$. Finally φ is not a rational function or else φ would be analytic on an open set containing $\text{cl}\mathbf{D}$. But, in fact, φ does not continue analytically past any point on $\partial\mathbf{D}$ where the sequence $\{a_k\}$ clusters. Recall that the Blaschke product $B(z)$ does converge uniformly on compact subsets of $\Omega = \mathbf{C} - \text{cl}\{1/\bar{a}_k\}$. Thus the partial products $B_n(z)$ are uniformly bounded on compact subsets of Ω . So, it follows that the series defining φ actually converges uniformly on compact subsets of Ω . Hence the series defining φ also defines an analytic extension of φ to Ω . Now it is clear that φ has poles at the points $\{1/\bar{a}_k\}$, hence φ cannot be analytic at any point where these poles cluster. $\square$

Example 2.10. There exists a bounded simply connected region G such that the Riemann map $\varphi:\mathbf{D} \rightarrow G$ extends continuously up to $\partial\mathbf{D}$ and $0 \in \sigma_p([S_G^*, S_G])$ and yet $[S_G^*, S_G]$ has infinite rank.

Proof. If B be any infinite Blaschke product, then let φ be the bounded univalent function given in Theorem 2.9. Since φ is not a rational function, Corollary 2.7 implies that $[S_G^*, S_G]$ has infinite rank. However, since $\varphi \perp \text{BH}^2(\mathbf{D})$, Theorem 2.1 implies that zero is an eigenvalue for $[T_\varphi^*, T_\varphi]$. However this is unitarily equivalent to $[S_G^*, S_G]$. $\square$

In fact, notice that in Example 2.10 we may choose φ such that it does not continue analytically past any point on $\partial\mathbf{D}$. Or at the other extreme, we may choose φ such that it continues past every point of $\partial\mathbf{D}$ except one.

Question 2.11 Does there exist a bounded univalent function φ and a singular inner function S , such that $\varphi \perp SH^2(\mathbf{D})$?

Notice that this is equivalent to finding a generalized quadrature domain G such that its generalized Schwarz function has no poles.

3. Cyclic Subnormal Operators

The general cyclic subnormal operator has the form $S_\mu = M_z$ on $P^2(\mu)$, the closure of the analytic polynomials in $L^2(\mu)$. In this section we investigate irreducible cyclic subnormal operators S_μ with $0 \in \sigma_p([S_\mu^*, S_\mu])$. The cyclic subnormal operators with finite rank self-commutator were characterized by Olin, Thomson and Trent [preprint] without the use of Thomson's Theorem. Since Thomson's Theorem appeared, another proof of the characterization of these operators was given by McCarthy and Yang [1995]. We shall state the result so it may be compared with the results obtained for cyclic subnormal operators with $0 \in \sigma_p([S_\mu^*, S_\mu])$.

Theorem 3.1 If S_μ is an irreducible cyclic subnormal operator, then S_μ has finite rank self-commutator if and only if $G = \text{abpe}(\mu)$ is a quadrature domain, the Riemann map of $\mathbf{D}$ onto G is a weak* generator of $H^\infty(\mathbf{D})$, and $\mu \approx \omega_G + \sum_{k=1}^n w_k \delta_{a_k}$, where ω_G is harmonic measure for G , $w_k > 0$ and $a_k \in G$.

An equivalent way to state Theorem 3.1 is to pull the operator back to the disk as follows.

Theorem 3.2 If S is an irreducible cyclic subnormal operator, then S has finite rank self-commutator if and only if S is unitarily equivalent to $r(S_v)$ where r is a rational function with poles off $\partial\mathbf{D}$ that is a weak*-generator of $H^\infty(\mathbf{D})$ and $v = m|\partial\mathbf{D} + \sum_{k=1}^n w_k \delta_{a_k}$, where $w_k > 0$ and $a_k \in \mathbf{D}$.

The idea for cyclic operators with $0 \in \sigma_p([S_\mu^*, S_\mu])$ is that they should have the form

$\phi(S_v)$ where ϕ has a pseudocontinuation and $v \approx m|\partial\mathbf{D} + \sum_{k=1}^\infty w_k \delta_{a_k}$ where the $\{a_k\}$ form

a Blaschke sequence in $\mathbf{D}$. We now show that in fact every such operator has a slightly stronger property than $0 \in \sigma_p([S_\mu^*, S_\mu])$.

Theorem 3.3 If S_v is a pure cyclic subnormal operator such that $v \approx m|\partial\mathbf{D} + \sum_{k=1}^\infty w_k \delta_{a_k}$ where $\{a_k\}$ is a Blaschke sequence in $\mathbf{D}$ and ϕ is any bounded analytic function that has a pseudocontinuation, then there is a non-zero bounded function in $\ker[\phi(S_v)^*, \phi(S_v)]$.

Proof. Since ϕ has a pseudocontinuation, Theorem 2.1 implies that T_ϕ on $H^2(\mathbf{D})$ has zero as an eigenvalue for its self-commutator. Hence by Proposition 1.1 we see that $\ker[T_\phi^*, T_\phi]$ is a non-zero shift invariant subspace. Hence by Beurling's Theorem there exists a non-zero bounded function $f \in \ker[T_\phi^*, T_\phi]$. So, by Proposition 1.1, we have that $\bar{\phi} f \in H^2$. Since ϕ and f are both bounded, there exists a bounded analytic function g on $\mathbf{D}$ such that $\bar{\phi} f = g$ a.e. on $\partial\mathbf{D}$. Now, let B be a Blaschke product with zeros at the points

$\{a_k\}$. It follows that $\overline{\varphi} fB = gB$ ν -a.e.. Thus since $\overline{\varphi} fB = gB \in P^2(\nu)$ we have by

Proposition 1.1 that fB is a non-zero bounded function in $\ker[\varphi(S_\nu)^*, \varphi(S_\nu)]$. $\square$

Corollary 3.4 If S_μ is an irreducible cyclic subnormal operator such that $G = \text{abpe}(\mu)$ is a generalized quadrature domain and $\mu \approx \omega_G + \sum_{k=1}^{\infty} w_k \delta_{a_k}$, where $\{a_k\}$ is an $H^\infty(G)$ zero set, then there exists a non-zero bounded function in $\ker[S_\mu^*, S_\mu]$.

Proof. It follows from Theorem 1.2.3 that $S_\mu \cong \varphi(S_\nu)$ where $\varphi: \mathbf{D} \rightarrow G$ is a Riemann map and ν is a measure such that S_ν is pure, $\text{abpe}(\nu) = \mathbf{D}$ and $\mu = \nu \circ \varphi^{-1}$. Furthermore, the unitary is given by composition with φ . It follows from Corollary 2.6 that φ has a pseudocontinuation and from $\mu = \nu \circ \varphi^{-1}$ that $\nu \approx m|\partial\mathbf{D}| + \sum_{k=1}^{\infty} w_k \delta_{a_k}$. Hence Theorem 3.3 applies to give a non-zero bounded function in $\ker[\varphi(S_\nu)^*, \varphi(S_\nu)]$. Since the unitary conjugating $\varphi(S_\nu)$ to S_μ is given by composition with φ , it preserves bounded functions. Thus, there exists a non-zero bounded function in $\ker[S_\mu^*, S_\mu]$. $\square$

The author believes that the converse to Corollary 3.4 is also true. We shall present some partial results, although we do not have the full converse at this time. We can show that if there exists a non-zero bounded function in $\ker[S_\mu^*, S_\mu]$, then G is a generalized quadrature domain. It is unclear though if the measure must be of the form above. This question involves a curious question about univalent functions with pseudocontinuations.

Theorem 3.5 If S_v is a pure cyclic subnormal operator, $\text{abpe}(v) = \mathbf{D}$ and $\varphi \in P^2(v) \cap L^\infty(v)$ is such that there exists a non-zero bounded function in $\ker[\varphi(S_v)^*, \varphi(S_v)]$, then φ has a pseudocontinuation.

Note that since $\text{abpe}(v) = \mathbf{D}$, we shall consider φ as both a function in $P^2(v)$ and in $H^\infty(\mathbf{D})$.

Proof. If g is a non-zero bounded function in $\ker[\varphi(S_v)^*, \varphi(S_v)]$, then by Proposition 1.1 we have that $\overline{\varphi} g \in P^2(v)$. So, there exists a function $h \in P^2(v) \cap L^\infty(v)$ such that $\overline{\varphi} g = h$ v a.e.. In particular, $\overline{\varphi} g = h$ v a.e. on $\partial\mathbf{D}$. It is known that $v|_{\partial\mathbf{D}}$ is absolutely continuous with respect to arc length measure (although, v may put little or no mass on $\partial\mathbf{D}$). However, since S_v is pure and $P^\infty(S_v) \approx H^\infty(\mathbf{D})$ it follows that for every measurable set $E \subseteq \partial\mathbf{D}$ with positive length, either $v(E) > 0$ or there exists a subsequence of $\text{supp}(v) \cap \mathbf{D}$ that clusters *non-tangentially* at almost every point of E . Since $\overline{\varphi} g = h$ a.e. v and φ, g and h are all bounded and analytic on $\mathbf{D}$, we actually have that $\overline{\varphi} g = h$ everywhere on $\text{supp}(v) \cap \mathbf{D}$. By taking non-tangential limits, if necessary, we have that $\overline{\varphi} g = h$ a.e. on $\partial\mathbf{D}$ with respect to arc length measure. Thus by Proposition 1.1, we see that $0 \in \sigma_p([T_\varphi^*, T_\varphi])$. So Theorem 2.1 implies that φ has a pseudocontinuation. $\square$

Corollary 3.6 If S_μ is an irreducible cyclic subnormal operator and there exists a non-zero bounded function in $\ker[S_\mu^*, S_\mu]$, then $G = \text{abpe}(\mu)$ is a generalized quadrature domain.

We now present a converse to Corollary 3.4 when G is a quadrature domain. It is not known whether this converse holds for all generalized quadrature domains. Notice, in particular that the unit disk $\mathbf{D}$ is a quadrature domain with Schwarz function $R(z) = 1/\bar{z}$. Hence, Theorem 3.7 applies in that case.

Theorem 3.7 If S_μ is a pure cyclic subnormal operator such that $G = \text{abpe}(\mu)$ is a quadrature domain, then there exists a non-zero bounded function in $\ker[S_\mu^*, S_\mu]$ if and only if $\mu \approx \omega_G + \sum_{k=1}^{\infty} w_k \delta_{a_k}$, where $\{a_k\}$ is an $H^\infty(G)$ zero set.

We shall use the following Lemma from Olin, Thomson and Trent [Preprint]. It was crucial in characterizing cyclic operators with finite rank self-commutator. A similar result was used in McCarthy and Yang[1995]. Its proof uses facts about algebraic curves.

Lemma 3.8 If r is a rational function with poles off of $\text{cl}\mathbf{D}$ and is univalent on $\mathbf{D}$, then the cardinality of $\{z \in \mathbf{D} : r(z) = r\left(\frac{1}{\bar{z}}\right)\}$ is finite.

Proof of Theorem 3.7. If μ has the required form, then Corollary 3.4 applies. Now suppose that there exists a non-zero bounded function in $\ker[S_\mu^*, S_\mu]$. Since G is a quadrature domain, Corollary 2.7 implies that the Riemann map of $\mathbf{D}$ onto G is a rational function $r(z)$. Now Theorem 1.2.3 implies that there exists a measure ν on $\text{cl}\mathbf{D}$ such that μ

$= v \circ r^{-1}$, S_v is pure, $\text{abpe}(v) = \mathbf{D}$ and S_μ is unitarily equivalent to $r(S_v)$ via composition with $r(z)$. Since there is a non-zero bounded function in $\ker[S_\mu^*, S_\mu]$, we also get a non-zero bounded function in $\ker[r(S_v)^*, r(S_v)]$. So, by Proposition 1.1, there exists a non-zero bounded function $f \in P^2(v)$ such that $\bar{r}f \in P^2(v)$. Thus, there exists a bounded function $g \in P^2(v)$ such that $\bar{r}f = g$ a.e. $[v]$. Now it is easy to see that $\overline{r(z)}\hat{f}(z) = \hat{g}(z)$ everywhere

on $\text{supp}(v) \cap \mathbf{D}$. Hence it follows that $\Phi(z) = \left(\frac{\hat{g}(z)}{\hat{f}(z)} \right)$ satisfies $\Phi(z) = \overline{r(z)}$ a.e. $[v]$.

In particular, since r and Φ are Nevanlinna functions, it follows that $\Phi(z) = \overline{r(z)}$ a.e. on $\partial\mathbf{D}$. Thus Φ^* gives a pseudocontinuation of $r(z)$. However, $r(z)$ is its own

pseudocontinuation. Hence $r(z) = \Phi^*(z)$ for all z with $|z| > 1$. Thus, $\overline{r\left(\frac{1}{\bar{z}}\right)} = r^*(z) = \Phi(z)$

$= \left(\frac{\hat{g}(z)}{\hat{f}(z)} \right)$ everywhere on $\mathbf{D}$. Hence $\overline{r\left(\frac{1}{\bar{z}}\right)}\hat{f}(z) = \hat{g}(z)$ everywhere on $\mathbf{D}$. Thus,

$\overline{r\left(\frac{1}{\bar{z}}\right)}\hat{f}(z) = \overline{r(z)}\hat{f}(z)$ everywhere on $\text{supp}(v) \cap \mathbf{D}$. Thus, $\text{supp}(v) \cap \mathbf{D} \subseteq \{ z \in \mathbf{D} :$

$\hat{f}(z) = 0 \} \cup \{ z \in \mathbf{D} : r(z) = r\left(\frac{1}{\bar{z}}\right) \}$. Using Lemma 3.8 and the fact that $\hat{f}$ is a bounded

analytic function, we see that $\text{supp}(v) \cap \mathbf{D}$ is a Blaschke sequence; call it $\{b_j\}$. Now, if $\mathcal{M}$

$= \{ h \in P^2(v) : \hat{h}(b_j) = 0 \text{ for all } j \}$, then $\mathcal{M}$ is a closed invariant subspace for S_v and

$\text{svsm}(S_v|\mathcal{M})$ is supported on $\partial\mathbf{D}$. Since S_v is pure, we see that $v|\partial\mathbf{D} \approx m|\partial\mathbf{D}$. Hence $v \approx$

$m|\partial\mathbf{D} + \sum_{k=1}^{\infty} w_k \delta_{b_k}$. Since $\mu = v \circ r^{-1}$, μ also has the required form. $\square$

The only place we used that G was a quadrature domain in the proof of Theorem 3.7 was to get the Riemann map to be a rational function so that Lemma 3.8 would apply. If we could affirmatively answer the following natural analogue of Lemma 3.8, then we would be able to have a full converse to Corollary 3.4

Question 3.9 If φ is a bounded univalent function on $\mathbf{D}$ that has a pseudocontinuation Φ , is the set $\{z \in \mathbf{D} : \varphi(z) = \Phi\left(\frac{1}{\bar{z}}\right)\}$ a Blaschke sequence?

However large the above set can be is exactly how “large” the measure $\mu|_G$ can be. An equivalent formulation of Question 3.9 is as follows.

Question 3.10 If φ is a bounded univalent function on $\mathbf{D}$ and Ψ is a Nevanlinna function on $\mathbf{D}$ such that $\Psi = \overline{\varphi}$ a.e. on $\partial\mathbf{D}$, then is $\{z \in \mathbf{D} : \Psi(z) = \overline{\varphi(z)}\}$ a Blaschke sequence?

Thus far we have considered cyclic subnormal operators S_μ such that there exists a non-zero bounded function in $\ker[S_\mu^*, S_\mu]$. We now consider the case that $\ker[S_\mu^*, S_\mu] \neq (0)$ and we do not necessarily have a bounded function as an eigenvector. The following basic question remains open.

Question 3.11 If S_μ is an irreducible cyclic subnormal operator and $\ker[S_\mu^*, S_\mu] \neq (0)$, then does there exist a non-zero bounded function in $\ker[S_\mu^*, S_\mu]$?

The following simple observation can be helpful.

If $f \in P^2(\mu)$, then we say that f is a Nevanlinna function if $\hat{f}$ is a Nevanlinna function on G
 $= \text{abpe}(\mu)$. That is, $\hat{f}$ is a quotient of bounded analytic functions on G .

Proposition 3.12 If S_μ is an irreducible cyclic subnormal operator and there exists a non-zero Nevanlinna function in $\ker[S_\mu^*, S_\mu]$, then there exists a non-zero bounded function in $\ker[S_\mu^*, S_\mu]$.

Proof. Let $G = \text{abpe}(\mu)$. If f is a non-zero Nevanlinna function in $\ker[S_\mu^*, S_\mu]$, then $\hat{f}$ is a quotient of bounded analytic functions on G . Since Thomson's Theorem says that $\wedge: P^2(\mu) \cap L^\infty(\mu) \rightarrow H^\infty(G)$ is onto, there exists $g, h \in P^2(\mu) \cap L^\infty(\mu)$, such that $\hat{f} = \frac{\hat{g}}{\hat{h}}$. Now, Proposition 1.1 implies that $\ker[S_\mu^*, S_\mu]$ is a hyperinvariant subspace. Thus it is invariant under multiplication by h . So, $hf \in \ker[S_\mu^*, S_\mu]$. Now, since $\widehat{(hf)} = \hat{h}\hat{f} = \hat{g}$ and the $\wedge$ map is one-to-one, it follows that $hf = g$. Thus g is a non-zero bounded function in $\ker[S_\mu^*, S_\mu]$. $\square$

In particular, if $\mu|_{\partial G} \approx \omega_G = \text{harmonic measure for } G$, then every function in $P^2(\mu)$ is a Nevanlinna function. Hence Question 3.11 is related to the following.

Question 3.13 If S_μ is an irreducible cyclic subnormal operator and $\ker[S_\mu^*, S_\mu] \neq (0)$, then is $\mu|_{\partial G} \approx \omega_G$, where $G = \text{abpe}(\mu)$ and ω_G is harmonic measure for G ?

This last question is not known even when $G = \mathbf{D}$. We shall give some examples shortly.

For the next result, if $G = \text{abpe}(\mu)$ and $a \in G$, then let K_a denote the reproducing kernel for $P^2(\mu)$ at the point a .

Theorem 3.14 If S_μ is irreducible and $P^\infty(\mu|\partial G) = L^\infty(\mu|\partial G)$, then $[S_\mu^*, S_\mu]$ is one-to-one if and only if $\{ (z-a)K_a : a \in \text{supp}(\mu) \cap G \}$ has dense linear span in $P^2(\mu)$.

Notice that since $P^\infty(\mu|\partial G) = L^\infty(\mu|\partial G)$, $\text{supp}(\mu) \cap G \neq \emptyset$.

Proof. Suppose that $\{ (z-a)K_a : a \in \text{supp}(\mu) \cap G \}$ has dense linear span in $P^2(\mu)$. If $f \in \ker[S_\mu^*, S_\mu]$, then $\bar{z}f \in P^2(\mu)$ (by Proposition 1.1) and $(\bar{z}f)^\wedge(a) = \bar{a}\hat{f}(a)$ for each $a \in \text{supp}(\mu) \cap G$. Thus we see that $\langle \bar{z}f, K_a \rangle = \bar{a}\langle f, K_a \rangle$ holds for each $a \in \text{supp}(\mu) \cap G$. It then follows easily that $\langle f, (z-a)K_a \rangle = 0$ for each $a \in \text{supp}(\mu) \cap G$. So, we must have $f = 0$. Thus $[S_\mu^*, S_\mu]$ is one-to-one. (Note that this direction of the proof did not use the assumption on $\mu|\partial G$.)

Now, suppose that $[S_\mu^*, S_\mu]$ is one-to-one and let $f \in P^2(\mu)$ be such that $f \perp \{ (z-a)K_a : a \in \text{supp}(\mu) \cap G \}$. We must show that $f = 0$. As above we see that $\langle S_\mu^* f, K_a \rangle = \langle \bar{z}f, K_a \rangle = \bar{a}\langle f, K_a \rangle$ holds for each $a \in \text{supp}(\mu) \cap G$. Thus $(S_\mu^* f)^\wedge(a) = (\bar{z}f)^\wedge(a) = \bar{a}\hat{f}(a) = \bar{a}\hat{f}(a)$ μ -a.e. on G . Now, let $g = (\bar{z}f - S_\mu^* f) = (I - P)(\bar{z}f)$ where P is the orthogonal projection of $L^2(\mu)$ onto $P^2(\mu)$. So as $I-P$ projects onto $P^2(\mu)^\perp$, we have $g \in P^2(\mu)^\perp$ and $g = 0$ μ a.e. on G . If $g \neq 0$, then let $T = \text{dual}S_\mu$ and consider T restricted to

the invariant subspace generated by g . This is a pure cyclic subnormal operator with svsm supported on ∂G^* . Hence $P^\infty(\mu^*|\partial G^*) \neq L^\infty(\mu^*|\partial G^*)$, a contradiction. So, $g = 0$. Thus, $\bar{z}f = S_\mu^*f \in P^2(\mu)$. Hence $f \in \ker[S_\mu^*, S_\mu]$, so $f = 0$ as required. $\square$

Theorem 3.15 If S_μ is an irreducible cyclic subnormal operator, $G = \text{abpe}(\mu)$ and $0 \in \sigma_p([S_\mu^*, S_\mu])$, then:

- (a) Each compact set $K \subseteq \text{supp}(\mu) \cap G$ satisfies $P(K) = C(K)$;
- (b) For each $r > 0$, $\text{supp}(\mu) \cap G \cap \{z : |z| = r\}$ is discrete in G .

First recall that $G = \text{abpe}(\mu)$ is a simply connected open set. Furthermore an open set U is simply connected if and only if the polynomials are dense in the space of all analytic functions on U , with the topology of uniform convergence on compact subsets of U .

Proof of Theorem 3.15. (a) If $f \in \ker[S_\mu^*, S_\mu]$, then by Proposition 1.1 we have $\bar{z}f \in P^2(\mu)$. Let $K \subseteq \text{supp}(\mu) \cap G$ be compact and choose a simply connected region G_1 that contains K and whose closure is contained in G . This can always be done by going back to the unit disk via a Riemann map. So we have that $\widehat{(\bar{z}f)} = \bar{z}f = \widehat{\bar{z}}\hat{f} \mu$ - a.e. on G . Since $\hat{f}$ is analytic on G and $\text{cl}(G_1)$ is compactly contained in G , $\hat{f}$ has only a finite number of zeros on $\text{cl}(G_1)$ and each of these is a zero of $\widehat{(\bar{z}f)}$. Hence $g = \widehat{(\bar{z}f)} / \hat{f}$ is analytic on G_1 . Thus $g \in P(K)$ by the remarks made preceding the proof (and since G_1 is simply connected). But $\bar{z} = g$ on K , so $\bar{z} \in P(K)$, hence $P(K) = C(K)$.

(b) Keeping the same notation, notice that $zg(z)$ is meromorphic on G and constant on $\text{supp}(\mu) \cap G \cap \{z : |z| = r\}$ for each $r > 0$. Hence this set must be discrete in G ; that is, every limit point must lie in ∂G . $\square$

So the previous theorem says that if $0 \in \sigma_p([S_\mu^*, S_\mu])$, then $\text{supp}(\mu) \cap G$ cannot be too large. For example it cannot have any interior points. In fact (b) implies that the Hausdorff dimension of $\text{supp}(\mu) \cap G$ is at most one; in particular it has area zero.

We now present an interesting application of Theorem 3.15.

Proposition 3.16 If S is a pure subnormal operator and every part of S has zero as an eigenvalue for its self-commutator, then $\sigma_n(S)$ has no interior.

Proof. Suppose that S acts on $\mathcal{H}$ and $N = mneS$. Let $h \in \mathcal{H}$ be a separating vector for N (see Conway [1991], page 249). Thus the cyclic part of S determined by h has zero as an eigenvalue for its self-commutator. That is, if $\mathcal{M}$ is the invariant subspace generated by h , then $S|_{\mathcal{M}}$ is a pure cyclic subnormal operator, and, by assumption, it has zero as an eigenvalue for its self-commutator. Hence Theorem 3.15 implies that $\sigma_n(S|_{\mathcal{M}})$ has no interior. However, since h is a separating vector, $\text{svsm}(S|_{\mathcal{M}}) \approx \text{svsm}(S)$. Thus $\sigma_n(S) = \sigma_n(S|_{\mathcal{M}})$. So $\sigma_n(S)$ has no interior. $\square$

In particular, Proposition 3.16 applies to some quasinormal operators.

Example 3.17 If B is the Bergman shift and $S = \text{dual}B$, then S is irreducible, $0 \in \sigma_p([S^*, S])$ but not every part of S has zero as an eigenvalue for its self-commutator.

Proof. It is known that S is irreducible, see Conway [1981]. It is easy to check that $\overline{z}L_a^2(\mathbf{D}) + L_a^2(\mathbf{D})$ is not dense in $L^2(\mathbf{D})$, since z^2 is orthogonal to it. Thus, Corollary 1.7(b) implies that $0 \in \sigma_p([S^*, S])$. Since $\sigma_n(S) = \text{cl}\mathbf{D}$, Proposition 3.16 says that not every part of S has zero as an eigenvalue for its self-commutator. $\square$

Corollary 3.18 If S is a pure subnormal operator and every part of S has zero as an eigenvalue for its self-commutator, then $\sigma(S) \neq \sigma_{ap}(S)$.

Proof. Since $0 \in \sigma_p([S^*, S])$, Corollary 1.9 implies that $\sigma(S)$ has non-empty interior. However, $\sigma_{ap}(S) \subseteq \sigma_n(S)$ and $\sigma_n(S)$ has no interior by Proposition 3.16. Hence $\sigma(S) \neq \sigma_{ap}(S)$. $\square$

We now want to consider some examples of cyclic subnormal operators on $\mathbf{D}$ whose measures put no mass on the boundary and the mass inside $\mathbf{D}$ is very sparse, namely a sum of point masses. Such operators actually exist in abundance; see Theorem I.2.4.

We say that $\{a_n\}$ is a *sampling sequence* for $L_a^2(\mathbf{D}, (1 - |z|^2)^{2\alpha-1} dA)$, $\alpha > 0$, if the identity map on polynomials extends to give a similarity between the weighted Bergman operator

and S_μ , where $\mu = \sum_{n=1}^{\infty} (1-|a_n|^2)^{2\alpha+1} \delta_{a_n}$. Sampling sequences always exist for these

weighted Bergman spaces and have been characterized by Seip[1993].

If $\alpha = 1/2$, then S_μ is similar to the Bergman shift. In general, if $\alpha > 0$, then is $[S_\mu^*, S_\mu]$ one-to-one? Even though the self-commutator of the weighted Bergman operator is one-to-one (Theorem 3.15) it could still happen that $[S_\mu^*, S_\mu]$ is not one-to-one.

Observe that every sampling sequence is an $H^\infty(\mathbf{D})$ dominating sequence. This holds because S_μ is similar to a weighted Bergman operator, hence must satisfy $P^\infty(\mu) \approx H^\infty(\mathbf{D})$.

But for such discrete measures, it is known that $P^\infty(\mu) \approx H^\infty(\mathbf{D})$ if and only if the atoms of μ form a dominating set (see Conway [1991], p. 306).

Example 3.19 If $\{a_n\}$ is a sampling sequence for $L_a^2(\mathbf{D})$ and $\mu = \sum_{n=1}^{\infty} (1-|a_n|^2)^2 \delta_{a_n}$, then $[S_\mu^*, S_\mu]$ is one-to-one.

Proof. If $f \in P^2(\mu)$ and $f \in \ker[S_\mu^*, S_\mu]$, then Proposition 1.1 implies that $\bar{z}f \in P^2(\mu)$.

Hence also $(1-|z|^2)f \in P^2(\mu)$. But since $\hat{f} \in L_a^2(\mathbf{D})$ we know that $(1-|z|^2)f(z) \rightarrow 0$ as

$|z| \rightarrow 1$ (because the normalized reproducing kernels go to zero weakly as $|z| \rightarrow 1$). So,

$(1-|z|^2)f \in P^2(\mu) \cap L^\infty(\mu)$. That is, there is a function $g \in H^\infty(\mathbf{D})$ with

$(1-|a_n|^2)f(a_n) = g(a_n)$. Thus $g(a_n) \rightarrow 0$ as $n \rightarrow \infty$. Since $\{a_n\}$ is an H^∞ dominating set, we must have that $g = 0$ identically. Hence $f = 0$ μ -a.e. Thus $[S_\mu^*, S_\mu]$ is one-to-one. $\square$

The above proof works because when $\alpha \leq 1/2$, the functions in the weighted Bergman space satisfy the growth condition $(1-|z|^2)|f(z)| \rightarrow 0$ as $|z| \rightarrow 1$. We now prove that $[S_\mu^*, S_\mu]$ is one-to-one whenever $\alpha \leq 1$. When $\alpha > 1/2$, the functions in the weighted Bergman spaces do not need to satisfy the growth condition above, hence a different proof is needed.

Example 3.20 If $0 < \alpha \leq 1$, $\{a_n\}$ is a sampling sequence for $L_a^2(\mathbf{D}, (1-|z|^2)^{2\alpha-1} dA)$ and $\mu = \sum_{n=1}^{\infty} (1-|a_n|^2)^{2\alpha+1} \delta_{a_n}$, then $[S_\mu^*, S_\mu]$ is one-to-one.

Proof. If $f \in \ker[S_\mu^*, S_\mu]$, then $(1-|z|^2)f \in P^2(\mu)$. Suppose $g \in P^2(\mu)$ and $g(z) =$

$(1-|z|^2)f(z)$ μ -a.e.. It follows that $f(z) = \frac{g(z)}{(1-|z|^2)}$ on the sequence $\{a_n\}$. Since $\alpha \leq 1$,

we have $2\alpha-1 \leq 1$, so $\sum_{n=1}^{\infty} (1-|a_n|^2)|g(a_n)|^2 \leq \sum_{n=1}^{\infty} (1-|a_n|^2)^{2\alpha-1}|g(a_n)|^2 = \|f\|^2 < \infty$.

Now for $\varepsilon > 0$, let $I_\varepsilon = \{a_k : |g(a_k)| > \varepsilon\}$. By the above inequalities, I_ε is a Blaschke sequence for each $\varepsilon > 0$. So if $\varepsilon = 1$, then g is bounded by 1 on $\{a_n\} - I_1$. If $B(z)$ is the Blaschke product with zeros at precisely the points in I_1 , then Bg is bounded on the sampling sequence $\{a_n\}$. That is, $Bg \in P^2(\mu) \cap L^\infty(\mu)$. So, there exists an $h \in H^\infty(\mathbf{D})$

such that $h = Bg$ on $\{a_n\}$. Thus we have $B(z)f(z) = \frac{B(z)g(z)}{(1-|z|^2)} = \frac{h(z)}{(1-|z|^2)}$ on the sequence

$\{a_n\}$ and h is bounded. Now, repeating the argument above, we see that for each $\varepsilon > 0$,

the set $J_\varepsilon = \{a_k : |h(a_k)| > \varepsilon\}$ is a Blaschke sequence. However if one removes a

Blaschke sequence from a dominating sequence, then the new sequence is still dominating.

Thus, $\{a_n\} - J_\varepsilon$ is a dominating sequence, $h \in H^\infty(\mathbf{D})$ and $|h| < \varepsilon$ on $\{a_n\} - J_\varepsilon$. Hence

$\|h\|_\infty \leq \varepsilon$. Since this holds for each $\varepsilon > 0$, we have $h = 0$. Thus, $Bf = 0$ μ -a.e.. It follows

easily that $f = 0$ μ -a.e. and so $[S_\mu^*, S_\mu]$ is one-to-one. $\square$

Question 3.21 If $\alpha > 1$, $\{a_n\}$ is a sampling sequence for $L_a^2(\mathbf{D}, (1-|z|^2)^{2\alpha-1} dA)$ and $\mu = \sum_{n=1}^{\infty} (1-|a_n|^2)^{2\alpha+1} \delta_{a_n}$, then is $[S_\mu^*, S_\mu]$ one-to-one?

We now prove that if $\alpha > 0$ and μ is as above, then not every part of S_μ has zero as an eigenvalue for its self-commutator.

Theorem 3.22 If $\alpha > 0$, $\{a_n\}$ is a sampling sequence for $L_a^2(\mathbf{D}, (1-|z|^2)^{2\alpha-1} dA)$ and $\mu = \sum_{n=1}^{\infty} (1-|a_n|^2)^{2\alpha+1} \delta_{a_n}$, then not every part of S_μ has zero as an eigenvalue for its self-commutator.

Proof. If every part of S_μ has zero as an eigenvalue for its self-commutator, then by

Proposition 1.14 we have that for each integer $m > 0$, there exists a non-zero function $f \in$

$P^2(\mu)$ such that $\bar{z}^k f \in P^2(\mu)$ for each positive integer $k \leq m$. In particular, $(1 - |z|^2)^m f \in P^2(\mu)$. However, one easily checks that $(1 - |a_n|^2)^{2\alpha+1} |f(a_n)|^2 \leq \|f\|^2$. So, fix an $m > 2\alpha+1$. If $f, g \in P^2(\mu)$ are such that $(1 - |z|^2)^m f = g$ on $\{a_n\}$, then $g \in P^2(\mu) \cap L^\infty(\mu)$ and $g(a_n) \rightarrow 0$ as $n \rightarrow \infty$. Since, $\{a_n\}$ is a dominating sequence, we must have $g = 0$ μ a.e.. Hence $f = 0$, a contradiction. So not every part of S_μ can have zero as an eigenvalue for its self-commutator. $\square$

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Vita

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