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Date August, 1994

On Quantum Commutation Relations for the Coordinate Algebras of Coefficients of Binary Forms.

A Thesis

Presented for the

Master of Science in Mathematics

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Michael Glen Newman

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ABSTRACT

The purpose of this thesis is to take the first step in developing quantum analogs of the classical invariant theory of binary forms. The classical theory begins by considering the linear transformation of a 2×1 vector by a 2×2 matrix. This transformation induces a linear transformation of the coefficients of all the binary forms. The classical invariant theory is concerned with functions of these coefficients that are invariant under the induced linear transformation for all 2×2 matricies.

In the classical theory all elements commute. In this thesis a nonclassical or quantum invariant theory of binary forms will be considered in which elements do not commute. Two types of quantization will be introduced. Each quantization is a deformation into noncommutative or quantum theory and is represented by deformation parameters. By setting the deformation parameters equal to the appropriate values (usually zero or unity) one retrieves the commutative or classical theory. The main goal of this thesis is to find commutation relations for the coefficients of the quantum binary forms. The quantum binary forms considered will be of degree less than or equal to three. It is required that these commutation relations be invariant in form under the transformations induced by the quantum 2×2 matrix groups and it is hoped that these commutation relations will be such that the algebras of the coefficients satisfy PBW property. The determination of these commutation relations is the preliminary step in the development of the invariant theory of quantum binary forms.

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1. Introduction

In this thesis a preliminary step will be taken in developing quantum analogs of the classical invariant theory of binary forms. The classical invariant theory of binary forms was developed in the nineteenth century by Cayley, Sylvester, Clebsch, Gordan and others [Ref 1]. To review the basic notions of this theory consider a field $\mathcal{K}$ of characteristic zero (e.g. complex numbers) and a 2×1 vector

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.1}$$

where $x, y \in \mathcal{K}$. A *binary form* of degree n in the vector $\mathbf{x}$ is a homogeneous polynomial of degree n in the two arguements x, y

$$f^n(\mathbf{x}) = \sum_{k=0}^{n} \binom{n}{k} \alpha'_{k+1} x^{n-k} y^k. \tag{1.2}$$

The quantities $\alpha'_1, \alpha'_2, \ldots, \alpha'_{n+1} \in \mathcal{K}$ are called the *coefficients* of the binary form $f^n(\mathbf{x})$. Now consider a 2×2 matrix

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.3}$$

where $a, b, c, d \in \mathcal{K}$ and the linear transformation of the vector $\mathbf{x}$ by this matrix

$$\mathbf{x}' = \mathbf{M}\mathbf{x} = \begin{pmatrix} x' \\ y' \end{pmatrix} \tag{1.4}$$

where

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy. \end{aligned} \tag{1.5}$$

One can now consider binary forms of degree n in the vector $\mathbf{x}'$

$$f'^n(\mathbf{x}') = \sum_{k=0}^{n} \binom{n}{k} \alpha_{k+1} x'^{n-k} y'^k. \tag{1.6}$$

The requirement $f'^n(\mathbf{M}\mathbf{x}) = f^n(\mathbf{x})$ induces the following linear transformation of the coefficients of an nth degree binary form

$$\alpha_i' = \sum_{j=1}^{n+1} m_{ij}\alpha_j \ , \quad i = 1, 2, \ldots, n+1 \tag{1.7}$$

where m_{ij} are quantities depending on the components of $\mathbf{M}$. The classical invariant theory of binary forms is concerned with functions of the coefficients α_i of an nth degree binary form which are invariant under the action of the induced linear transformation (1.7) for all 2×2 matricies $\mathbf{M}$ [Ref. 1].

In the classical theory the elements $x, y, a, b, c, d, \alpha_1, \alpha_2, \ldots, \alpha_{n+1}, \alpha_1', \alpha_2', \ldots, \alpha_{n+1}'$ commute since they belong to the field $\mathcal{K}$. In this thesis a nonclassical or quantum invariant theory of binary forms will be considered in which these elements will no longer be commuting. Two types of quantization will be considered. In the first quantization $\mathbf{x}$ will be a q quantum vector whose components x, y commute with all other elements while satisfing the commutation relation

$$xy = q^{-1}yx \tag{1.8}$$

and $\mathbf{M}$ will be a q, p quantum matrix whose components a, b, c, d commute with all other elements while satisfying the commutation relations

$$\begin{gathered} ca = qac \ , \ db = qbd \ , \ ba = pab \ , \ dc = pcd \ , \\ cb = p^{-1}qbc \ , \ da = ad + (q - p^{-1})bc. \end{gathered} \tag{1.9}$$

Using the commutation relations (1.8), (1.9) and the transformation equations (1.5) it is easy to show that $x'y' = q^{-1}y'x'$ and thus $\mathbf{x}'$ is a q quantum vector. The quantum binary forms to be considered in this quantization are

$$\begin{gathered} f^1(\mathbf{x}) = \alpha_1'x + \alpha_2'y \\ f'^1(\mathbf{x}') = \alpha_1x' + \alpha_2y' \\ f^2(\mathbf{x}) = \alpha_1'x^2 + (1+pq)\alpha_2'xy + \alpha_3'y^2 \\ f'^2(\mathbf{x}') = \alpha_1x'^2 + (1+pq)\alpha_2x'y' + \alpha_3y'^2 \\ f^3(\mathbf{x}) = \alpha_1'x^3 + (1+pq+p^2q^2)\alpha_2'x^2y + (1+pq+p^2q^2)\alpha_3'xy^2 + \alpha_4'y^3 \\ f'^3(\mathbf{x}') = \alpha_1x'^3 + (1+pq+p^2q^2)\alpha_2x'^2y' + (1+pq+p^2q^2)\alpha_3x'y'^2 + \alpha_4y'^3. \end{gathered} \tag{1.10}$$

The nonzero parameters q, p are called *deformation parameters* because they represent deformation into noncommutative or quantum theory. The case $q = 1$, $p = 1$ represents no deformation, i.e. commutative or classical theory.

In the second type of quantization $\mathbf{x}$ will be an h quantum vector whose components commute with all other elements while satisfying the commutation relation

$$xy = yx + hy^2 \tag{1.11}$$

and $\mathbf{M}$ will be an h quantum matrix whose components commute with all other elements while satisfying the commutation relations

$$\begin{aligned} ba &= ab + h(a^2 + bc - ad) - h^2 ac\ , \ dc = cd + hc^2\ , \\ ca &= ac - hc^2\ , \ db = bd + h(ad - bc - d^2) + h^2 ac\ , \\ cb &= bc - h(ac + cd)\ , \ da = ad + h(ac - cd) - h^2 c^2. \end{aligned} \tag{1.12}$$

Using the commutation relations (1.11), (1.12) and the transformation equations (1.5) it can be shown that $x'y' = y'x' + hy'^2$ and thus $\mathbf{x}'$ is an h quantum vector. The quantum binary forms to be considered in this quantization are

$$\begin{aligned} f^1(\mathbf{x}) &= \alpha_1' x + \alpha_2' y \\ f'^1(\mathbf{x}') &= \alpha_1 x' + \alpha_2 y' \\ f^2(\mathbf{x}) &= \alpha_1' x^2 + 2\alpha_2' yx + \alpha_3' y^2 \\ f'^2(\mathbf{x}') &= \alpha_1 x'^2 + 2\alpha_2 y'x' + \alpha_3 y'^2 \\ f^3(\mathbf{x}) &= \alpha_1' x^3 + 3\alpha_2' yx^2 + 3\alpha_3' y^2 x + \alpha_4' y^3 \\ f'^3(\mathbf{x}') &= \alpha_1 x'^3 + 3\alpha_2 y'x'^2 + 3\alpha_3 y'^2 x' + \alpha_4 y'^3. \end{aligned} \tag{1.13}$$

Classical theory is obtained here by setting the deformation parameter h to zero.

The quantum vectors and quantum matricies introduced above, along with the commutation relations among their components, have arisen from studies of exactly solvable problems in statistical mechanics and quantum field theory. In particular studies of the quantum inverse scattering method and solutions of the quantum Yang-Baxter equations have motivated the development of these ideas.

The main goal of this thesis is to find commutation relations for the coefficients α_i, α'_i of the binary forms $f'^n(\mathbf{x}')$ and $f^n(\mathbf{x})$, $n = 1,2,3$, so that certain conditions are satisfied. One of the desired conditions is that the commutation relations be invariant in form under the induced transformation (1.7), i.e.

(1.14)

$$\alpha'_j\alpha'_k = l_{jk}(\alpha'_1,\alpha'_2,\dots,\alpha'_{n+1}) \implies \alpha_j\alpha_k = l_{jk}(\alpha_1,\alpha_2,\dots,\alpha_{n+1}) \ , \ j,k = 1,2,\dots,n+1 \ , \ n = 1,2,3.$$

In other words, the form of the commutation relations is preserved under the action of the quantum matrix group. It is also hoped that the commutation relations will be such that the algebras of the coefficients will satisfy what is called the Poincaré-Birkhoff-Witt (PBW) property. There exists an important and useful theorem for determining whether or not an algebra satisfies PBW property. This famous theorem is known as the Diamond Lemma. The Diamond Lemma will be explained and applied in this thesis. The development of these commutation relations is the preliminary step in the study of the invariant theory of quantum binary forms.

2. Mathematical Foundations

A basic notion used in this thesis is the concept of an associative algebra. Let $\mathcal{K}$ be a field. Recall that an *associative algebra* $\mathcal{A}$ over $\mathcal{K}$ is a vector space over $\mathcal{K}$ equipped with a multiplication $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, having the following properties:

$$
\begin{gathered}
(a+b)c = ac + bc \\
a(b+c) = ab + ac \\
(\lambda a)b = a(\lambda b) = \lambda(ab) \\
(ab)c = a(bc) \\
\forall\, a, b, c \in \mathcal{A}\,,\ \forall\, \lambda \in \mathcal{K}.
\end{gathered} \tag{2.1}
$$

If there exists an element $1 \in \mathcal{A}$ with the property

$$1a = a1 = a\ ,\ \forall\, a \in \mathcal{A} \tag{2.2}$$

then $\mathcal{A}$ is called a *unital associative algebra*.

Consider the letters x, y. A formal combination of these letters such as x^2, xy, yx, y^2, $x^2y, \ldots$, are called words or *monomials* in the letters x, y. The letters x, y are monomials of degree one while x^2, xy, yx, y^2 are monomials of degree two. Given a field $\mathcal{K}$ consider the set $\mathcal{F}$ of formal symbols or formal finite linear combinations,

$$\mathcal{F} = \{\ \lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 x^2 + \lambda_4 xy + \lambda_5 yx + \lambda_6 y^2 + \cdots \mid \lambda_i \in \mathcal{K}\,,\ \text{finite number of } \lambda_i \neq 0\ \}. \tag{2.3}$$

Given elements $a, b \in \mathcal{F}$,

$$
\begin{gathered}
a = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 yx + a_6 y^2 + \cdots \\
b = b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 yx + b_6 y^2 + \cdots\,,
\end{gathered}
$$

addition and scalar multiplication on $\mathcal{F}$ are defined naturally as

$$a + b = a_0 + b_0 + (a_1 + b_1)x + (a_2 + b_2)y + (a_3 + b_3)x^2 + (a_4 + b_4)xy + (a_5 + b_5)yx + (a_6 + b_6)y^2 + \cdots$$

$$\lambda a = \lambda a_0 + (\lambda a_1)x + (\lambda a_2)y + (\lambda a_3)x^2 + (\lambda a_4)xy + (\lambda a_5)yx + (\lambda a_6)y^2 + \cdots$$

and multiplication on $\mathcal{F}$ is defined by

$$ab = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_2b_0)y + (a_0b_3 + a_1b_1 + a_3b_0)x^2$$
$$+ (a_0b_4 + a_1b_2 + a_4b_0)xy + (a_0b_5 + a_2b_1 + a_5b_0)yx + (a_0b_6 + a_2b_2 + a_6b_0)y^2 + \cdots .$$

With these definitions of addition, scalar multiplication and multiplication it is easily shown that axioms (2.1) are satisfied and $\mathcal{F}$ qualifies as an associative algebra over $\mathcal{K}$ with zero and unity elements given by

$$0 = 0 + 0x + 0y + 0x^2 + 0xy + 0yx + 0y^2 + \cdots$$
$$1 = 1 + 0x + 0y + 0x^2 + 0xy + 0yx + 0y^2 + \cdots .$$

For convenience we write

$$x = 0 + 1x + 0y + 0x^2 + 0xy + 0yx + 0y^2 + \cdots$$
$$y = 0 + 0x + 1y + 0x^2 + 0xy + 0yx + 0y^2 + \cdots$$
$$x^2 = 0 + 0x + 0y + 1x^2 + 0xy + 0yx + 0y^2 + \cdots$$

etc.. In this way the monomials in x, y are identified as elements of $\mathcal{F}$ with the zero and unity elements considered as monomials of degree zero in x, y. Thus it is concluded that the subset

$$\mathcal{B} = \{\, 1 \,, x \,, y \,, x^2 \,, xy \,, yx \,, y^2 \,, \ldots \} \tag{2.4}$$

of monomials in x, y is a basis for $\mathcal{F}$, i.e., the elements of $\mathcal{B}$ generate $\mathcal{F}$ and are linearly independent. Consequently $\mathcal{F}$ is referred to as the *free algebra* generated by the letters x, y. In general a free algebra can be generated by an alphabet consisting of any number of letters.

Recall that a *two-sided ideal* $\mathcal{I}$ in a unital associative algebra $\mathcal{A}$ is a subspace of $\mathcal{A}$ having the property

$$ai \in \mathcal{I} \,,\; ia \in \mathcal{I} \,,\; \forall\, i \in \mathcal{I} \,, \forall\, a \in \mathcal{A}. \tag{2.5}$$

A two-sided ideal in $\mathcal{A}$ can be constructed as follows. Let $g_1, g_2, \ldots, g_n$ be elements of $\mathcal{A}$ and let $\mathcal{V}$ be the subspace of $\mathcal{A}$ generated by these elements,

$$\mathcal{V} = \{ \lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_n g_n \mid \lambda_i \in \mathcal{K} \}.$$

Letting $\mathcal{I}$ be

$$\mathcal{I} = \{ \sum_i a_i v_i b_i \mid v_i \in \mathcal{V} \ , \ a_i, b_i \in \mathcal{A} \}$$

it is easily verified that $\mathcal{I}$ satisfies axiom (2.5) and thus qualifies as a two-sided ideal in $\mathcal{A}$. The elements $g_1, g_2, \ldots, g_n$ are called the *generators* of the ideal $\mathcal{I}$. The *coset* of an element $a \in \mathcal{A}$ induced by $\mathcal{I}$ is defined as

$$\bar{a} = \{ a \} + \mathcal{I} = \{ a + i \mid i \in \mathcal{I} \}$$

and the *factor set* $\mathcal{A} \backslash \mathcal{I}$ is defined as

$$\mathcal{A} \backslash \mathcal{I} = \{ \bar{a} \mid a \in \mathcal{A} \}.$$

Addition, scalar multiplication and multiplication of cosets are defined by

$$\bar{a} + \bar{b} = \{ a' + b' \mid a' \in \bar{a} \ , \ b' \in \bar{b} \}$$

$$\lambda \bar{a} = \{ \lambda a' \mid a' \in \bar{a} \}$$

$$\bar{a}\bar{b} = \{ a'b' \mid a' \in \bar{a} \ , \ b' \in \bar{b} \}.$$

Since $\mathcal{I}$ is a two-sided ideal it follows that

$$\begin{aligned} \bar{a} + \bar{b} &= (a + b)^- \\ \lambda \bar{a} &= (\lambda a)^- \\ \bar{a}\bar{b} &= (ab)^- \end{aligned} \tag{2.6}$$

and $\mathcal{A} \backslash \mathcal{I}$ is closed with respect to addition, scalar multiplication and multiplication of cosets. It follows readily that with these operations axioms (2.1) are satisfied and $\mathcal{A} \backslash \mathcal{I}$ qualifies as an associative algebra over $\mathcal{K}$ with zero element $\bar{0} = \mathcal{I}$ and unity element $\bar{1}$. Hence, $\mathcal{A} \backslash \mathcal{I}$ is called the *factor algebra* induced by the ideal $\mathcal{I}$.

Consider now a nonzero element $q \in \mathcal{K}$ and the two-sided ideal $\mathcal{I}$ in $\mathcal{F}$ generated by the single element $g = xy - q^{-1}yx \in \mathcal{F}$. In the factor algebra $\bar{\mathcal{F}} = \mathcal{F}\backslash\mathcal{I}$ it follows from (2.6) that

$$\bar{x}\bar{y} - q^{-1}\bar{y}\bar{x} = (xy - q^{-1}yx)^{-} = \bar{g} = \mathcal{I} = \bar{0}.$$

Thus the generators $\bar{x}, \bar{y}$ of $\bar{\mathcal{F}}$ satisfy the relation

$$\bar{x}\bar{y} = q^{-1}\bar{y}\bar{x}.$$

From this relation it is concluded that the set

$$\bar{\mathcal{B}} = \{ \bar{1}, \bar{x}, \bar{y}, \bar{x}^2, \bar{x}\bar{y}, \bar{y}\bar{x}, \bar{y}^2, \ldots \} \tag{2.7}$$

is not a basis for $\bar{\mathcal{F}}$. The elements of $\bar{\mathcal{B}}$ generate $\bar{\mathcal{F}}$ but are not linearly independent since the subsets $\{ \bar{x}\bar{y}, \bar{y}\bar{x} \}, \{ \bar{x}^2\bar{y}, \bar{x}\bar{y}\bar{x}, \bar{y}\bar{x}^2 \}, \ldots$, etc. contain linearly dependent monomials. Thus the factor algebra $\bar{\mathcal{F}}$ is not a free algebra but is generated by the elements $\bar{x}, \bar{y}$ subject to the relation $\bar{x}\bar{y} = q^{-1}\bar{y}\bar{x}$.

The specification of an algebra in terms of its generators and relations is called the *presentation* of the algebra. Symbolically the presentation of $\bar{\mathcal{F}}$ is written as

$$\bar{\mathcal{F}} = \langle \bar{x}, \bar{y} \mid \bar{x}\bar{y} = q^{-1}\bar{y}\bar{x} \rangle.$$

Like $\bar{\mathcal{F}}$, an algebra whose relations consist of linear combinations of only second degree monomials is called a *quadratic algebra.* The presentation of the free algebra $\mathcal{F}$ is simply

$$\mathcal{F} = \langle x, y \rangle$$

indicating that $\mathcal{F}$ is generated freely by the letters x, y with no relations.

Since $\bar{\mathcal{F}}$ is not a free algebra the problem of finding a basis for $\bar{\mathcal{F}}$ is of some concern. Any element of $\bar{\mathcal{F}}$ can be expressed as

$$\bar{a} = a_0\bar{1} + a_1\bar{x} + a_2\bar{y} + a_3\bar{x}^2 + a_4\bar{x}\bar{y} + a_5\bar{y}\bar{x} + a_6\bar{y}^2 + \cdots$$

and substitution of the relation $\bar{x}\bar{y} = q^{-1}\bar{y}\bar{x}$ into this expression for $\bar{a}$ leads one to the conclusion that the subset

$$\bar{\mathcal{B}}' = \{\,\bar{1}\,,\,\bar{x}\,,\,\bar{y}\,,\,\bar{x}^2\,,\,\bar{y}\bar{x}\,,\,\bar{y}^2\,,\dots\}\,, \tag{2.8}$$

consisting of all monomials with $\bar{y}$ standing to the left of $\bar{x}$, is a generating set for $\bar{\mathcal{F}}$. However, the linear independence of the elements of $\bar{\mathcal{B}}'$ is unclear. The question of whether or not $\bar{\mathcal{B}}'$ is a basis for $\bar{\mathcal{F}}$ is answered by a famous theorem in ring theory called the *Diamond Lemma* [Ref. 2]. Before stating the Diamond Lemma additional basic concepts must be introduced.

Let $\mathcal{A}$ be a unital associative algebra over $\mathcal{K}$ freely generated by n letters $x_1, x_2, \dots, x_n$ and let $<$ be a *lexicographic ordering* of the generators,

$$x_1 < x_2 < \cdots < x_n. \tag{2.9}$$

This order induces an order on the set of all monomials as follows: $x_{i_1}x_{i_2}\cdots x_{i_k} < x_{j_1}x_{j_2}\cdots x_{j_l}$ if $k < l$ and $x_{i_1}x_{i_2}\cdots x_{i_k} < x_{j_1}x_{j_2}\cdots x_{j_k}$ if $x_{i_1} < x_{j_1}$, or if $x_{i_1} = x_{j_1}$ and $x_{i_2} < x_{j_2}$, etc. The lexicographic ordering process can be described figuratively as follows. The letters $x_1, x_2, \dots, x_n$ are thought of as an alphabet with alphabetical ordering given in (2.9). Consider now a dictionary containing all the words of this alphabet. The dictionary is partitioned into volumes $r = 1, 2, \dots$ with the rth volume containing all words of degree r in alphabetical order. The words in volume k are smaller than the words in volume l if $k < l$ and given two words from the kth volume the first word is smaller than the second word if first word occurs alphabetically before the second word. A linear combination of monomials $l(x_1, x_2, \dots, x_n)$ is said to be smaller than a monomial $x_{i_1}x_{i_2}\cdots x_{i_k}$ and we write

$$l(x_1, x_2, \dots, x_n) < x_{i_1}x_{i_2}\cdots x_{i_k}$$

if every monomial in $l(x_1, x_2, \dots, x_n)$ is smaller than the monomial $x_{i_1}x_{i_2}\cdots x_{i_k}$. By definition the monomials $1, x_1, x_2, \dots, x_n$ are *reduced* with respect to the order $<$ and a monomial of the form $x_{i_1}x_{i_2}\cdots x_{i_k}$ is said to be reduced if $i_1 \le i_2 \le \cdots \le i_k$. From this definition it is easily seen that all words in the set $\bar{\mathcal{B}}'$ given in (2.8) are reduced. An element $a \in \mathcal{A}$ is called reduced if it is a linear combination of reduced monomials. Let us now consider a *reduction system* on $\mathcal{A}$ consisting of the following $\binom{n}{2}$ generators of a two-sided ideal $\mathcal{I}$ in $\mathcal{A}$,

$$g_{ij} = x_i x_j - l_{ij}(x_1, x_2, \ldots, x_n)$$

where $i > j$, $i, j = 1, 2, \ldots, n$ and $l_{ij}(x_1, x_2, \ldots, x_n)$ are linear combinations of quadratic monomials having the property $l_{ij}(x_1, x_2, \ldots, x_n) < x_i x_j$. With this reduction system reduction mappings on $\mathcal{A}$ can now be defined. Let m_1, m_2 be monomials and let the reduction $R_{m_1 x_i x_j m_2}$, $i > j$, be a linear mapping with the following properties:

$$R_{m_1 x_i x_j m_2}(m_1 x_i x_j m_2) = m_1 l_{ij}(x_1, x_2, \ldots, x_n) m_2$$

$$R_{m_1 x_i x_j m_2}(m) = m \ , \ \forall \text{ monomials } m \neq m_1 x_i x_j m_2.$$

Let R be a sequence of composed reductions. The sequence R is said to be *final* on an element $a \in \mathcal{A}$ if $R(a)$ is reduced. The Diamond Lemma for quadratic algebras can now be stated.

DIAMOND LEMMA. *Let $\mathcal{A}$ be a unital associative algebra over $\mathcal{K}$ freely generated by n letters $x_1, x_2, \ldots, x_n$ equipped with a lexicographic ordering of the generators*

$$x_1 < x_2 < \cdots < x_n$$

and a reduction system consisting of the following $\binom{n}{2}$ generators of a two-sided ideal $\mathcal{I}$ in $\mathcal{A}$,

$$g_{ij} = x_i x_j - l_{ij}(x_1, x_2, \ldots, x_n)$$

where $i > j$, $i, j = 1, 2, \ldots, n$ and $l_{ij}(x_1, x_2, \ldots, x_n)$ are linear combinations of quadratic monomials having the property $l_{ij}(x_1, x_2, \ldots, x_n) < x_i x_j$. There exist final sequences of reductions R and R' such that

$$R(l_{ij}(x_1, x_2, \ldots, x_n) x_k) = R'(x_i l_{jk}(x_1, x_2, \ldots, x_n)) \ , \ \forall\, i > j > k \ , \ i, j, k = 1, 2, \ldots, n$$

if and only if the set $\bar{\mathcal{B}}'$ of reduced monomials is a basis for $\bar{\mathcal{A}} = \mathcal{A} \backslash \mathcal{I}$.

If $\mathcal{A}$ is an algebra for which the set $\mathcal{B}'$ of reduced monomials is a basis then $\mathcal{A}$ is said to have or satisfy the *Poincaré-Birkhoff-Witt* (PBW) property. In the algebra $\mathcal{F}$ the lexicographic ordering of the generators is $y < x$ and the reduction system consists of the single element $g = xy - q^{-1}yx$ generating the two-sided ideal $\mathcal{I}$ in $\mathcal{F}$. Thus by applying the Diamond Lemma it is concluded that the set $\bar{\mathcal{B}}'$ given in (2.8) is indeed a basis and $\bar{\mathcal{F}}$ satisfies PBW property. It is noted however that $\bar{\mathcal{F}}$ represents a trivial application of the Diamond Lemma since $\mathcal{F}$ is generated by only two letters and thus there are no third order monomials $x_i x_j x_k$ with $x_i < x_j < x_k$. The Diamond Lemma is an important tool in ring theory because it provides the criteria for determining whether or not an algebra satisfies PBW property.

Let $\mathcal{G}$ be a group (or a semigroup) and let $\mathcal{H}$ be an abelian group. A *grading* is defined to be a homomorphism $|\ \ | : \mathcal{G} \to \mathcal{H}$. Let $\mathbf{Z}$ denote the set of integers and let $\mathcal{X}$ be the set of monomials in the letters $x_1, x_2, \ldots, x_n$. With respect to the operation of multiplication the set $\mathcal{X}$ qualifies as a semigroup and with respect to addition the set $\mathbf{Z} \oplus \mathbf{Z}$ of ordered pairs of integers qualifies as an abelian group. In this thesis we will be interested in gradings of the type $|\ \ | : \mathcal{X} \to \mathbf{Z} \oplus \mathbf{Z}$. Since these gradings are homomorphisms we have

$$|x_{i_1} x_{i_2} \cdots | = |x_{i_1}| + |x_{i_2}| + \cdots .$$

3. Quantum Linear Spaces

Let $\mathcal{K}$ be a field. A Quantum Linear Space over $\mathcal{K}$ is some Quantum Affine Space whose coordinate algebra over $\mathcal{K}$ is a quadratic unital associative algebra.

The space of quantum 2×1 vectors $\mathrm{Vec}_q(2)$ is defined to be the quantum linear space whose coordinate algebra $\mathcal{K}[\mathrm{Vec}_q(2)]$ has the presentation

$$\mathcal{K}[\mathrm{Vec}_q(2)] = \langle\, x, y \mid xy = q^{-1}yx \,\rangle \tag{3.1}$$

where x, y are letters and q is a nonzero element of $\mathcal{K}$. The space of quantum 2×1 vectors $\mathrm{Vec}_h(2)$ is the quantum linear space whose coordinate algebra is given by

$$\mathcal{K}[\mathrm{Vec}_h(2)] = \langle\, x, y \mid xy = yx + hy^2 \,\rangle \tag{3.2}$$

where h is an element of $\mathcal{K}$.

The space of quantum 2×2 matricies $\mathrm{Mat}_{q,p}(2)$ is defined to be the quantum linear space whose coordinate algebra $\mathcal{K}[\mathrm{Mat}_{q,p}(2)]$ has the presentation

$$\mathcal{K}[\mathrm{Mat}_{q,p}(2)] = \left\langle\, a, b, c, d \;\middle|\; \begin{array}{l} ca = qac \;,\; db = qbd \;,\; ba = pab \;,\; dc = pcd \;, \\ cb = p^{-1}qbc \;,\; da = ad + (q - p^{-1})bc \end{array} \right\rangle \tag{3.3}$$

where a, b, c, d are letters and p is a nonzero element of $\mathcal{K}$. The space of quantum 2×2 matricies $\mathrm{Mat}_{h,h'}(2)$ is the quantum linear space whose coordinate algebra is given by

$$\mathcal{K}[\mathrm{Mat}_{h,h'}(2)] = \left\langle\, a, b, c, d \;\middle|\; \begin{array}{l} ba = ab + h'(a^2 + bc - ad) - h'^2ac \;,\; dc = cd + h'c^2 \;, \\ ca = ac - hc^2 \;,\; db = bd + h(ad - bc - d^2) + hh'ac \;, \\ cb = bc - hcd - h'ac \;,\; da = ad - hcd + h'ac - hh'c^2 \end{array} \right\rangle \tag{3.4}$$

where h' is an element of $\mathcal{K}$.

Elements of $\mathrm{Vec}_q(2)$ and $\mathrm{Vec}_h(2)$ are represented as

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

while elements of $\mathrm{Mat}_{q,p}(2)$ and $\mathrm{Mat}_{h,h'}(2)$ are represented as

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In this thesis a special case of the space $\mathrm{Mat}_{h,h'}(2)$ is of interest, namely, the space $\mathrm{Mat}_h(2)$ which is given by

$$\mathrm{Mat}_h(2) = \mathrm{Mat}_{h,h'}(2)\,|_{h'=h}.$$

For reference the presentation of the coordinate algebra of $\mathrm{Mat}_h(2)$ is given.

$$\mathcal{K}[\mathrm{Mat}_h(2)] = \left\langle \; a,b,c,d \;\middle|\; \begin{array}{l} ba = ab + h(a^2 + bc - ad) - h^2 ac \;,\; dc = cd + hc^2 \;, \\[1ex] ca = ac - hc^2 \;,\; db = bd + h(ad - bc - d^2) + h^2 ac \;, \\[1ex] cb = bc - h(ac + cd) \;,\; da = ad + h(ac - cd) - h^2 c^2 \end{array} \right\rangle. \tag{3.5}$$

In order to define the action of the quantum 2×2 matrix $\mathbf{M} \in \mathrm{Mat}_{q,p}(2)$ on the quantum 2×1 vector $\mathbf{x} \in \mathrm{Vec}_q(2)$ consider the tensor product $\mathcal{K}[\mathrm{Vec}_q(2)] \otimes \mathcal{K}[\mathrm{Mat}_{q,p}(2)]$ whose presentation is

$$\mathcal{K}[\mathrm{Vec}_q(2)] \otimes \mathcal{K}[\mathrm{Mat}_{q,p}(2)] = \left\langle \; x,y,a,b,c,d \;\middle|\; \begin{array}{l} xy = q^{-1} yx \;, \\[1ex] ax = xa \;,\; bx = xb \;,\; cx = xc \;,\; dx = xd \;, \\[1ex] ay = ya \;,\; by = yb \;,\; cy = yc \;,\; dy = yd \;, \\[1ex] ca = qac \;,\; db = qbd \;,\; ba = pab \;,\; dc = pcd \;, \\[1ex] cb = p^{-1} q bc \;,\; da = ad + (q - p^{-1}) bc \end{array} \right\rangle. \tag{3.6}$$

The action of $\mathbf{M}$ on $\mathbf{x}$ yields $\mathbf{x}' = \mathbf{M}\mathbf{x}$ whose components $x', y' \in \mathcal{K}[\mathrm{Vec}_q(2)] \otimes \mathcal{K}[\mathrm{Mat}_{q,p}(2)]$ are defined by

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy. \end{aligned} \tag{3.7}$$

It is easily verified that the relation $x'y' = q^{-1} y'x'$ holds and $\mathbf{x}'$ is a quantum vector. Similarly, to define the action of $\mathbf{M} \in \mathrm{Mat}_h(2)$ on $\mathbf{x} \in \mathrm{Vec}_h(2)$ consider the tensor product $\mathcal{K}[\mathrm{Vec}_h(2)] \otimes$

$\mathcal{K}[\mathrm{Mat}_h(2)]$ with the presentation

(3.8)

$$\mathcal{K}[\mathrm{Vec}_h(2)] \otimes \mathcal{K}[\mathrm{Mat}_h(2)] = \left\langle \, x, y, a, b, c, d \; \middle| \; \begin{array}{l} xy = yx + hy^2 \ , \\ ax = xa \ , \ bx = xb \ , \ cx = xc \ , \ dx = xd \ , \\ ay = ya \ , \ by = yb \ , \ cy = yc \ , \ dy = yd \ , \\ ba = ab + h(a^2 + bc - ad) - h^2 ac \ , \ dc = cd + hc^2 \ , \\ ca = ac - hc^2 \ , \ db = bd + h(ad - bc - d^2) + h^2 ac \ , \\ cb = bc - h(ac + cd) \ , \ da = ad + h(ac - cd) - h^2 c^2 \end{array} \right\rangle .$$

The action of $\mathbf{M}$ on $\mathbf{x}$ yields $\mathbf{x}' = \mathbf{M}\mathbf{x}$ as defined in (3.7) and it follows readily that $x'y' = y'x' + hy'^2$. Thus $\mathbf{x}'$ is a quantum vector.

By application of the Diamond Lemma it is noted that each of the coordinate algebras defined thus far satisfies PBW property. For the remainder of this section the coordinate algebras introduced will have unknown relations to be determined.

Consider now the algebra $\mathcal{K}[\mathrm{Poly}^n_{q,p}(2)]$ of coefficients of an nth degree binary form whose presentation is

$$\mathcal{K}[\mathrm{Poly}^n_{q,p}(2)] = \langle \, \alpha_i \mid \alpha_j \alpha_k = l_{jk}(\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) \, \rangle \tag{3.9}$$

where α_i , $i = 1, 2, \ldots, n+1$, are letters and $l_{jk}(\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ are linear combinations of quadratic monomials, $l_{jk}(\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) < \alpha_j \alpha_k$, $j, k = 1, 2, \ldots, n+1$. The presentation of the tensor product $\mathcal{K}[\mathrm{Vec}_q(2)] \otimes \mathcal{K}[\mathrm{Mat}_{q,p}(2)] \otimes \mathcal{K}[\mathrm{Poly}^n_{q,p}(2)]$ is

(3.10)
$$\mathcal{K}[\mathrm{Vec}_q(2)] \otimes \mathcal{K}[\mathrm{Mat}_{q,p}(2)] \otimes \mathcal{K}[\mathrm{Poly}^n_{q,p}(2)] =$$
$$\left\langle x, y, a, b, c, d, \alpha_i \;\middle|\; \begin{array}{l} xy = q^{-1}yx \ , \\ ax = xa \ , \ bx = xb \ , \ cx = xc \ , \ dx = xd \ , \\ ay = ya \ , \ by = yb \ , \ cy = yc \ , \ dy = yd \ , \\ \alpha_i x = x\alpha_i \ , \ \alpha_i y = y\alpha_i \ , \\ ca = qac \ , \ db = qbd \ , \ ba = pab \ , \ dc = pcd \ , \\ cb = p^{-1}qbc \ , \ da = ad + (q - p^{-1})bc \ , \\ \alpha_i a = a\alpha_i \ , \ \alpha_i b = b\alpha_i \ , \ \alpha_i c = c\alpha_i \ , \ \alpha_i d = d\alpha_i \ , \\ \alpha_j\alpha_k = l_{jk}(\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) \end{array} \right\rangle$$
where $i, j, k = 1, 2, \ldots, n+1$.

A first degree binary form in the quantum vector $\mathbf{x} \in \mathrm{Vec}_q(2)$ is

(3.11)
$$f^1(\mathbf{x}) = \alpha'_1 x + \alpha'_2 y$$

where $\alpha'_1, \alpha'_2 \in \mathcal{K}[\mathrm{Vec}_q(2)] \otimes \mathcal{K}[\mathrm{Mat}_{q,p}(2)] \otimes \mathcal{K}[\mathrm{Poly}^1_{q,p}(2)]$. A second degree binary form is defined by

(3.12)
$$f^2(\mathbf{x}) = \alpha'_1 x^2 + (1+pq)\alpha'_2 xy + \alpha'_3 y^2$$

where $\alpha'_1, \alpha'_2, \alpha'_3 \in \mathcal{K}[\mathrm{Vec}_q(2)] \otimes \mathcal{K}[\mathrm{Mat}_{q,p}(2)] \otimes \mathcal{K}[\mathrm{Poly}^2_{q,p}(2)]$ and a third degree binary form is defined as

(3.13)
$$f^3(\mathbf{x}) = \alpha'_1 x^3 + (1+pq+p^2q^2)\alpha'_2 x^2 y + (1+pq+p^2q^2)\alpha'_3 xy^2 + \alpha'_4 y^3$$

where $\alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4 \in \mathcal{K}[\mathrm{Vec}_q(2)] \otimes \mathcal{K}[\mathrm{Mat}_{q,p}(2)] \otimes \mathcal{K}[\mathrm{Poly}^3_{q,p}(2)]$. The factors $1+pq$, $1+pq+p^2q^2$, in (3.12), (3.13) are inserted for future calculational convenience.

In like manner one can consider the algebra $\mathcal{K}[\mathrm{Poly}^n_h(2)]$ of coefficients of an nth degree binary form whose presentation is identical in form to that given in (3.9). The presentation of $\mathcal{K}[\mathrm{Vec}_h(2)] \otimes \mathcal{K}[\mathrm{Mat}_h(2)] \otimes \mathcal{K}[\mathrm{Poly}^n_h(2)]$ is

$$
(3.14) \qquad \mathcal{K}[\mathrm{Vec}_h(2)] \otimes \mathcal{K}[\mathrm{Mat}_h(2)] \otimes \mathcal{K}[\mathrm{Poly}_h^n(2)] =
$$

$$
\left\langle x, y, a, b, c, d, \alpha_i \;\middle|\; \begin{array}{l} xy = yx + hy^2 \ , \\ ax = xa \ , \ bx = xb \ , \ cx = xc \ , \ dx = xd \ , \\ ay = ya \ , \ by = yb \ , \ cy = yc \ , \ dy = yd \ , \\ \alpha_i x = x\alpha_i \ , \ \alpha_i y = y\alpha_i \ , \\ ba = ab + h(a^2 + bc - ad) - h^2 ac \ , \ dc = cd + hc^2 \ , \\ ca = ac - hc^2 \ , \ db = bd + h(ad - bc - d^2) + h^2 ac \ , \\ cb = bc - h(ac + cd) \ , \ da = ad + h(ac - cd) - h^2 c^2 \ , \\ \alpha_i a = a\alpha_i \ , \ \alpha_i b = b\alpha_i \ , \ \alpha_i c = c\alpha_i \ , \ \alpha_i d = d\alpha_i \ , \\ \alpha_j \alpha_k = l_{jk}(\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) \end{array} \right\rangle
$$

where $i, j, k = 1, 2, \ldots, n+1$. A first degree binary form in the quantum vector $\mathbf{x} \in \mathrm{Vec}_h(2)$ is as defined in (3.11) with $\alpha'_1, \alpha'_2 \in \mathcal{K}[\mathrm{Vec}_h(2)] \otimes \mathcal{K}[\mathrm{Mat}_h(2)] \otimes \mathcal{K}[\mathrm{Poly}_h^1(2)]$. A second degree binary form is defined as

$$
(3.15) \qquad f^2(\mathbf{x}) = \alpha'_1 x^2 + 2\alpha'_2 yx + \alpha'_3 y^2
$$

where $\alpha'_1, \alpha'_2, \alpha'_3 \in \mathcal{K}[\mathrm{Vec}_h(2)] \otimes \mathcal{K}[\mathrm{Mat}_h(2)] \otimes \mathcal{K}[\mathrm{Poly}_h^2(2)]$ and a third degree binary form is defined by

$$
(3.16) \qquad f^3(\mathbf{x}) = \alpha'_1 x^3 + 3\alpha'_2 yx^2 + 3\alpha'_3 y^2 x + \alpha'_4 y^3
$$

where $\alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4 \in \mathcal{K}[\mathrm{Vec}_h(2)] \otimes \mathcal{K}[\mathrm{Mat}_h(2)] \otimes \mathcal{K}[\mathrm{Poly}_h^3(2)]$.

Having defined binary forms in the quantum vectors $\mathbf{x} \in \mathrm{Vec}_q(2)$ and $\mathbf{x} \in \mathrm{Vec}_h(2)$, transformed binary forms in the quantum vectors $\mathbf{x}' = \mathbf{M}\mathbf{x}$, $\mathbf{M} \in \mathrm{Mat}_{q,p}(2)$ and $\mathbf{M} \in \mathrm{Mat}_h(2)$, can now be defined. Consider transformed binary forms induced by $\mathbf{M} \in \mathrm{Mat}_{q,p}(2)$. Corresponding to $f^1(\mathbf{x})$ in (3.11) is

$$
(3.17) \qquad f'^1(\mathbf{x}') = \alpha_1 x' + \alpha_2 y',
$$

to $f^2(\mathbf{x})$ in (3.12)

$$f'^2(\mathbf{x}') = \alpha_1 x'^2 + (1+pq)\alpha_2 x'y' + \alpha_3 y'^2, \tag{3.18}$$

and to $f^3(\mathbf{x})$ in (3.13)

$$f'^3(\mathbf{x}') = \alpha_1 x'^3 + (1+pq+p^2q^2)\alpha_2 x'^2 y' + (1+pq+p^2q^2)\alpha_3 x' y'^2 + \alpha_4 y'^3. \tag{3.19}$$

Transformation equations for the coefficients of these binary forms are determined by requiring that $f'^n(\mathbf{Mx}) = f^n(\mathbf{x})$, $n = 1,2,3$. Thus, using the relations in (3.10), substitution of equations (3.7) into (3.17) gives

$$\begin{aligned} \alpha_1' &= m_{11}\alpha_1 + m_{12}\alpha_2 \\ \alpha_2' &= m_{21}\alpha_1 + m_{22}\alpha_2 \end{aligned} \tag{3.20}$$

where

$$m_{11} = a \ , \ m_{12} = c \ , \ m_{21} = b \ , \ m_{22} = d. \tag{3.21}$$

Substituting (3.7) into (3.18) yields

$$\begin{aligned} \alpha_1' &= m_{11}\alpha_1 + m_{12}\alpha_2 + m_{13}\alpha_3 \\ \alpha_2' &= m_{21}\alpha_1 + m_{22}\alpha_2 + m_{23}\alpha_3 \\ \alpha_3' &= m_{31}\alpha_1 + m_{32}\alpha_2 + m_{33}\alpha_3 \end{aligned} \tag{3.22}$$

where

$$\begin{aligned} &m_{11} = a^2 \ , \ m_{12} = (1+pq)ac \ , \ m_{13} = c^2 \\ &m_{21} = ab \ , \ m_{22} = ad + qbc \ , \quad m_{23} = cd \\ &m_{31} = b^2 \ , \ m_{32} = (1+pq)bd \ , \ m_{33} = d^2, \end{aligned} \tag{3.23}$$

and substituting (3.7) into (3.19) yields

$$\begin{aligned} \alpha_1' &= m_{11}\alpha_1 + m_{12}\alpha_2 + m_{13}\alpha_3 + m_{14}\alpha_4 \\ \alpha_2' &= m_{21}\alpha_1 + m_{22}\alpha_2 + m_{23}\alpha_3 + m_{24}\alpha_4 \\ \alpha_3' &= m_{31}\alpha_1 + m_{32}\alpha_2 + m_{33}\alpha_3 + m_{34}\alpha_4 \\ \alpha_4' &= m_{41}\alpha_1 + m_{42}\alpha_2 + m_{43}\alpha_3 + m_{44}\alpha_4 \end{aligned} \tag{3.24}$$

where

$$m_{11} = a^3\ ,\quad m_{12} = (1+pq+p^2q^2)a^2c\ ,\quad m_{13} = (1+pq+p^2q^2)ac^2\ ,\quad m_{14} = c^3$$

$$m_{21} = a^2b\ ,\ m_{22} = a^2d + q(1+pq)abc\ ,\quad m_{23} = (1+pq)acd + q^2bc^2\ ,\ m_{24} = c^2d$$

(3.25)

$$m_{31} = ab^2\ ,\ m_{32} = (1+pq)abd + q^2b^2c\ ,\ m_{33} = ad^2 + q(1+pq)bcd\ ,\quad m_{34} = cd^2$$

$$m_{41} = b^3\ ,\quad m_{42} = (1+pq+p^2q^2)b^2d\ ,\quad m_{43} = (1+pq+p^2q^2)bd^2\ ,\quad m_{44} = d^3.$$

Now consider transformed binary forms induced by $\mathbf{M} \in \mathrm{Mat}_h(2)$. Corresponding to $f^1(\mathbf{x})$ in (3.11) is $f'^1(\mathbf{x}')$ in (3.17), to $f^2(\mathbf{x})$ in (3.15)

(3.26)
$$f'^2(\mathbf{x}') = \alpha_1 x'^2 + 2\alpha_2 y'x' + \alpha_3 y'^2,$$

and to $f^3(\mathbf{x})$ in (3.16)

(3.27)
$$f'^3(\mathbf{x}') = \alpha_1 x'^3 + 3\alpha_2 y'x'^2 + 3\alpha_3 y'^2 x' + \alpha_4 y'^3.$$

Transformation equations for the coefficients of these binary forms are determined again by requiring that $f'^n(\mathbf{M}\mathbf{x}) = f^n(\mathbf{x})$, $n = 1, 2, 3$. Thus, using the relations in (3.14), substitution of equations (3.7) into (3.17) gives once again (3.20) and (3.21). Substitution of (3.7) into (3.26) gives (3.22) where

(3.28)

$$m_{11} = a^2\ ,\qquad m_{12} = 2ac - 2hc^2\ ,\qquad m_{13} = c^2$$

$$m_{21} = ab + \frac{1}{2}h(a^2 - ad + bc) - \frac{1}{2}h^2ac\ ,\ m_{22} = ad + bc - 2hcd - h^2c^2\ ,\qquad m_{23} = cd + \frac{1}{2}hc^2$$

$$m_{31} = b^2 + hab\ ,\qquad m_{32} = 2bd + 2h(ad - d^2) - 2h^2cd\ ,\ m_{33} = d^2 + hcd,$$

and substituting (3.7) into (3.27) yields (3.24) where

$$
\begin{aligned}
m_{11} &= a^3 \\
m_{12} &= 3a^2c - 6hac^2 + 6h^2c^3 \\
m_{13} &= 3ac^2 - 6hc^3 \\
m_{14} &= c^3 \\
m_{21} &= a^2b + h(a^3 - a^2d + abc) - h^2a^2c \\
m_{22} &= a^2d + 2abc + h(a^2c - 5acd - bc^2) + h^2(6c^2d - 4ac^2) + 6h^3c^3 \\
m_{23} &= 2acd + bc^2 + h(ac^2 - 6c^2d) - 6h^2c^3 \\
m_{24} &= c^2d + hc^3 \\
m_{31} &= ab^2 + h(2a^2b - abd + b^2c) + h^2(a^3 - 2a^2d + ad^2 - bcd) \\
&\quad + h^3(2acd - 2a^2c - bc^2) + 2h^4ac^2 \\
m_{32} &= 2abd + b^2c + h(3a^2d - 3ad^2 + abc - 3bcd) + h^2(6cd^2 - 7acd - 2bc^2) \\
&\quad + h^3(12c^2d - 2ac^2) + 6h^4c^3 \\
m_{33} &= ad^2 + 2bcd + h(3acd + bc^2 - 6cd^2) + h^2(ac^2 - 12c^2d) - 6h^3c^3 \\
m_{34} &= cd^2 + 2hc^2d + h^2c^3 \\
m_{41} &= b^3 + 3hab^2 + h^2(3a^2b - abd + b^2c) + h^3(ad^2 - a^2d - abc - bcd) \\
&\quad + h^4(2acd - a^2c - bc^2) + 2h^5ac^2 \\
m_{42} &= 3b^2d + h(9abd - 6bd^2) + h^2(9a^2d - 15ad^2 - 3bcd + 6d^3) \\
&\quad + h^3(18cd^2 - 15acd) + 18h^4c^2d \\
m_{43} &= 3bd^2 + h(6ad^2 + 3bcd - 6d^3) + h^2(6acd - 18cd^2) - 18h^3c^2d \\
m_{44} &= d^3 + 3hcd^2 + 3h^2c^2d.
\end{aligned}
\tag{3.29}
$$

4. Problem Statement

The objective of this thesis is to detemine the relations in (3.9) for the algebras $\mathcal{K}[\mathrm{Poly}^n_{q,p}(2)]$ and $\mathcal{K}[\mathrm{Poly}^n_h(2)]$, $n = 1, 2, 3$. Specifically, it is desired to find the forms of the functions l_{jk} so that

$$\alpha_j\alpha_k = l_{jk}(\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) \implies \alpha'_j\alpha'_k = l_{jk}(\alpha'_1, \alpha'_2, \ldots, \alpha'_{n+1}), \tag{4.1}$$

so that the commutation relations of $\mathcal{K}[\mathrm{Poly}^n_{q,p}(2)]$ are invariant in form under the action of $\mathrm{Mat}_{q,p}(2)$ and likewise for the commutation relations of $\mathcal{K}[\mathrm{Poly}^n_h(2)]$ under the action of $\mathrm{Mat}_h(2)$. If possible it is also desired that the commutation relations be such that $\mathcal{K}[\mathrm{Poly}^n_{q,p}(2)]$ and $\mathcal{K}[\mathrm{Poly}^n_h(2)]$ satisfy PBW property. Each commutation relation must have *homogeneous* grading, i.e., in each relation the gradings of all product terms appearing in that relation must be equal.

We now introduce gradings by assigning to each of the generators of the algebras $\mathcal{K}[\mathrm{Vec}_q(2)] \otimes \mathcal{K}[\mathrm{Mat}_{q,p}(2)] \otimes \mathcal{K}[\mathrm{Poly}^n_{q,p}(2)]$ and $\mathcal{K}[\mathrm{Vec}_h(2)] \otimes \mathcal{K}[\mathrm{Mat}_h(2)] \otimes \mathcal{K}[\mathrm{Poly}^n_h(2)]$ an element of $\mathbf{Z} \oplus \mathbf{Z}$. The reason for introducing gradings is to reduce the number of free parameters in the set of all possible commutation relations. To each of the components of the quantum vectors, $\mathbf{x} \in \mathrm{Vec}_q(2)$ and $\mathbf{x} \in \mathrm{Vec}_h(2)$, we assign the following gradings:

$$|x| = (1,0)\ ,\ |y| = (0,1). \tag{4.2}$$

From the transformation equations (3.7) it follows that

$$\begin{aligned} |x'| &= |ax| = |a| + |x| = |by| = |b| + |y| = (1,0) \\ |y'| &= |cx| = |c| + |x| = |dy| = |d| + |y| = (0,1) \end{aligned} \tag{4.3}$$

and the following gradings are uniquely induced on the components of the quantum matricies $\mathbf{M} \in \mathrm{Mat}_{q,p}(2)$ and $\mathbf{M} \in \mathrm{Mat}_h(2)$:

$$|a| = (0,0)\ ,\ |b| = (1,-1)\ ,\ |c| = (-1,1)\ ,\ |d| = (0,0). \tag{4.4}$$

Thus the action of $\mathbf{M}$ preserves the gradings of the components of $\mathbf{x}$. Since the gradings of the product terms in the binary forms must be homogeneous the gradings of the components of $\mathbf{x}$ and

$\mathbf{x}'$ uniquely induce gradings on the coefficients of the binary forms. From (4.2) and (4.3) it follows immediately that the action of $\mathbf{M}$ preserves the gradings of the coefficients of the binary forms.

Consider first the algebras $\mathcal{K}[\mathrm{Poly}^n_{q,p}(2)]$ under the action of $\mathrm{Mat}_{q,p}(2)$. For $\mathcal{K}[\mathrm{Poly}^1_{q,p}(2)]$ we assign to the generators the ordering

$$\alpha_2 < \alpha_1 \tag{4.5}$$

and from (4.3), (3.17) we assign the gradings

$$|\alpha_1| = (0,1)\ ,\ |\alpha_2| = (1,0). \tag{4.6}$$

Thus

$$\begin{aligned} |\alpha_1^2| &= |\alpha_1| + |\alpha_1| = (0,2) \\ |\alpha_1\alpha_2| = |\alpha_2\alpha_1| &= |\alpha_1| + |\alpha_2| = (1,1) \\ |\alpha_2^2| &= |\alpha_2| + |\alpha_2| = (2,0). \end{aligned} \tag{4.7}$$

From (4.5) and (4.7) it follows that the most general quadratic relation satisfying the requirements stated in (3.9) and having homogeneous grading is

$$\alpha_1\alpha_2 = z\alpha_2\alpha_1 \tag{4.8}$$

where $z \in \mathcal{K}$. From (4.1) the transformed relation is

$$\alpha_1'\alpha_2' = z\alpha_2'\alpha_1'. \tag{4.9}$$

Substituting (3.20) into (4.9), use of (4.8) and the relations in (3.10) yields the following three equations for the scalar z:

$$\begin{aligned} \alpha_1^2 &: m_{11}m_{21} = zm_{21}m_{11} \\ \alpha_2\alpha_1 &: zm_{11}m_{22} + m_{12}m_{21} = z^2m_{21}m_{12} + zm_{22}m_{11} \\ \alpha_2^2 &: m_{12}m_{22} = zm_{22}m_{12}. \end{aligned} \tag{4.10}$$

For $\mathcal{K}[\mathrm{Poly}^2_{q,p}(2)]$ the we order the generators as

$$\alpha_3 < \alpha_2 < \alpha_1 \tag{4.11}$$

and from (4.3), (3.18) we fix the gradings

$$|\alpha_1| = (0,2)\ ,\ |\alpha_2| = (1,1)\ ,\ |\alpha_3| = (2,0). \tag{4.12}$$

Thus

$$\begin{aligned} |\alpha_1^2| &= |\alpha_1| + |\alpha_1| = (0,4) \\ |\alpha_1\alpha_2| &= |\alpha_2\alpha_1| = |\alpha_1| + |\alpha_2| = (1,3) \\ |\alpha_1\alpha_3| &= |\alpha_3\alpha_1| = |\alpha_1| + |\alpha_3| = (2,2) \\ |\alpha_2^2| &= |\alpha_2| + |\alpha_2| = (2,2) \\ |\alpha_2\alpha_3| &= |\alpha_3\alpha_2| = |\alpha_2| + |\alpha_3| = (3,1) \\ |\alpha_3^2| &= |\alpha_3| + |\alpha_3| = (4,0). \end{aligned} \tag{4.13}$$

From (4.11) and (4.13) it follows that the form of the commutation relations satisfying the requirements of (3.9) and having homogeneous grading is

$$\begin{aligned} \alpha_1\alpha_2 &= z_1\alpha_2\alpha_1 \\ \alpha_1\alpha_3 &= z_2\alpha_3\alpha_1 + z_3\alpha_2^2 \\ \alpha_2\alpha_3 &= z_4\alpha_3\alpha_2 \end{aligned} \tag{4.14}$$

where $z_1, z_2, z_3, z_4 \in \mathcal{K}$. From (4.1) the transformed relations are

$$\begin{aligned} \alpha_1'\alpha_2' &= z_1\alpha_2'\alpha_1' \\ \alpha_1'\alpha_3' &= z_2\alpha_3'\alpha_1' + z_3\alpha_2'^2 \\ \alpha_2'\alpha_3' &= z_4\alpha_3'\alpha_2'. \end{aligned} \tag{4.15}$$

Substituting (3.22) into (4.15), use of (4.14) and the relations in (3.10) yields eighteen equations for the scalars z_1, z_2, z_3, z_4. The eighteen equations are listed in (A.1)-(A.3) in the Appendix.

For $\mathcal{K}[\mathrm{Poly}^3_{q,p}(2)]$ we order the generators as

$$\alpha_4 < \alpha_3 < \alpha_2 < \alpha_1 \tag{4.16}$$

and from (4.3), (3.19) we fix the gradings

$$|\alpha_1| = (0,3)\ ,\ |\alpha_2| = (1,2)\ ,\ |\alpha_3| = (2,1)\ ,\ |\alpha_4| = (3,0). \tag{4.17}$$

Thus

$$
(4.18)\qquad
\begin{aligned}
|\alpha_1^2| &= |\alpha_1| + |\alpha_1| = (0,6)\\
|\alpha_1\alpha_2| &= |\alpha_2\alpha_1| = |\alpha_1| + |\alpha_2| = (1,5)\\
|\alpha_1\alpha_3| &= |\alpha_3\alpha_1| = |\alpha_1| + |\alpha_3| = (2,4)\\
|\alpha_1\alpha_4| &= |\alpha_4\alpha_1| = |\alpha_1| + |\alpha_4| = (3,3)\\
|\alpha_2^2| &= |\alpha_2| + |\alpha_2| = (2,4)\\
|\alpha_2\alpha_3| &= |\alpha_3\alpha_2| = |\alpha_2| + |\alpha_3| = (3,3)\\
|\alpha_2\alpha_4| &= |\alpha_4\alpha_2| = |\alpha_2| + |\alpha_4| = (4,2)\\
|\alpha_3^2| &= |\alpha_3| + |\alpha_3| = (4,2)\\
|\alpha_3\alpha_4| &= |\alpha_4\alpha_3| = |\alpha_3| + |\alpha_4| = (5,1)\\
|\alpha_4^2| &= |\alpha_4| + |\alpha_4| = (6,0).
\end{aligned}
$$

From (4.16) and (4.18) it follows that the form of the commutation relations satisfying the requirements of (3.9) and having homogeneous grading is

$$
(4.19)\qquad
\begin{aligned}
\alpha_1\alpha_2 &= z_1\alpha_2\alpha_1\\
\alpha_1\alpha_3 &= z_2\alpha_3\alpha_1 + z_3\alpha_2^2\\
\alpha_1\alpha_4 &= z_4\alpha_4\alpha_1 + z_5\alpha_3\alpha_2\\
\alpha_2\alpha_3 &= z_6\alpha_3\alpha_2 + z_7\alpha_4\alpha_1\\
\alpha_2\alpha_4 &= z_8\alpha_4\alpha_2 + z_9\alpha_3^2\\
\alpha_3\alpha_4 &= z_{10}\alpha_4\alpha_3.
\end{aligned}
$$

where $z_i \in \mathcal{K}$, $i = 1, 2, \ldots, 10$. From (4.1) the transformed relations are

$$
\begin{aligned}
\alpha_1'\alpha_2' &= z_1\alpha_2'\alpha_1' \\
\alpha_1'\alpha_3' &= z_2\alpha_3'\alpha_1' + z_3\alpha_2'^2 \\
\alpha_1'\alpha_4' &= z_4\alpha_4'\alpha_1' + z_5\alpha_3'\alpha_2' \\
\alpha_2'\alpha_3' &= z_6\alpha_3'\alpha_2' + z_7\alpha_4'\alpha_1' \\
\alpha_2'\alpha_4' &= z_8\alpha_4'\alpha_2' + z_9\alpha_3'^2 \\
\alpha_3'\alpha_4' &= z_{10}\alpha_4'\alpha_3'.
\end{aligned} \tag{4.20}
$$

Substitution of (3.24) into (4.20), use of (4.19) and the relations in (3.10) yields sixty equations for the scalars z_i , $i = 1, 2, \ldots, 10$. The sixty equations are listed in (A.4)-(A.9) in the Appendix.

Now consider the algebras $\mathcal{K}[\mathrm{Poly}_h^n(2)]$ under the action of $\mathrm{Mat}_h(2)$. The scalar $h \in \mathcal{K}$ is assigned the grading

$$|h| = (1, -1). \tag{4.21}$$

For $\mathcal{K}[\mathrm{Poly}_h^1(2)]$ we assign to the generators the ordering

$$\alpha_1 < \alpha_2 \tag{4.22}$$

and the gradings given in (4.6). It follows from (3.9), (4.21), (4.7) and (4.22) that the desired form of the commutation relation is

$$\alpha_2\alpha_1 = z_1\alpha_1\alpha_2 + hz_2\alpha_1^2 \tag{4.23}$$

where $z_1, z_2 \in \mathcal{K}$. From (4.1) the transformed relation is then

$$\alpha_2'\alpha_1' = z_1\alpha_1'\alpha_2' + hz_2\alpha_1'^2. \tag{4.24}$$

Substitution of (3.20) into (4.24), use of (4.23) and the relations in (3.14) yields three equations for the scalars z_1, z_2:

$$
\begin{aligned}
\alpha_1^2 &: m_{21}m_{11} + hz_2m_{22}m_{11} = z_1m_{11}m_{21} + hz_1z_2m_{12}m_{21} + hz_2m_{11}^2 + h^2z_2^2m_{12}m_{11} \\
\alpha_1\alpha_2 &: m_{21}m_{12} + z_1m_{22}m_{11} = z_1m_{11}m_{22} + z_1^2m_{12}m_{21} + hz_2m_{11}m_{12} + hz_1z_2m_{12}m_{11} \\
\alpha_2^2 &: m_{22}m_{12} = z_1m_{12}m_{22} + hz_2m_{12}^2.
\end{aligned} \tag{4.25}
$$

For $\mathcal{K}[\mathrm{Poly}_h^2(2)]$ we order the generators as

$$\alpha_1 < \alpha_2 < \alpha_3 \tag{4.26}$$

and by (4.3), (3.26) the gradings are as given in (4.12). It follows from (3.9), (4.21), (4.13) and (4.26) that the desired form of the commutation relations is

$$\begin{aligned}
\alpha_2\alpha_1 &= z_1\alpha_1\alpha_2 + hz_2\alpha_1^2 \\
\alpha_3\alpha_1 &= z_3\alpha_2^2 + z_4\alpha_1\alpha_3 + hz_5\alpha_1\alpha_2 + h^2z_6\alpha_1^2 \\
\alpha_3\alpha_2 &= z_7\alpha_2\alpha_3 + h(z_8\alpha_2^2 + z_9\alpha_1\alpha_3) + h^2z_{10}\alpha_1\alpha_2 + h^3z_{11}\alpha_1^2
\end{aligned} \tag{4.27}$$

where $z_i \in \mathcal{K}$, $i = 1, 2, \ldots, 11$. From (4.1) the transformed relations are

$$\begin{aligned}
\alpha_2'\alpha_1' &= z_1\alpha_1'\alpha_2' + hz_2\alpha_1'^2 \\
\alpha_3'\alpha_1' &= z_3\alpha_2'^2 + z_4\alpha_1'\alpha_3' + hz_5\alpha_1'\alpha_2' + h^2z_6\alpha_1'^2 \\
\alpha_3'\alpha_2' &= z_7\alpha_2'\alpha_3' + h(z_8\alpha_2'^2 + z_9\alpha_1'\alpha_3') + h^2z_{10}\alpha_1'\alpha_2' + h^3z_{11}\alpha_1'^2.
\end{aligned} \tag{4.28}$$

Substituting (3.22) into (4.28), use of (4.27) and the relations in (3.14) yields eighteen equations for the scalars z_i , $i = 1, 2, \ldots, 11$. The eighteen equations are listed in (A.10)-(A.12) in the Appendix.

For $\mathcal{K}[\mathrm{Poly}_h^3(2)]$ we order the generators as

$$\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \tag{4.29}$$

and by (4.3), (3.27) the gradings are as given in (4.17). It follows from (3.9), (4.21), (4.18) and (4.29) that the desired form of the commutation relations is

$$\begin{aligned}
\alpha_2\alpha_1 &= z_1\alpha_1\alpha_2 + hz_2\alpha_1^2 \\
\alpha_3\alpha_1 &= z_3\alpha_2^2 + z_4\alpha_1\alpha_3 + hz_5\alpha_1\alpha_2 + h^2z_6\alpha_1^2 \\
\alpha_4\alpha_1 &= z_7\alpha_2\alpha_3 + z_8\alpha_1\alpha_4 + h(z_9\alpha_2^2 + z_{10}\alpha_1\alpha_3) + h^2z_{11}\alpha_1\alpha_2 + h^3z_{12}\alpha_1^2 \\
\alpha_3\alpha_2 &= z_{13}\alpha_2\alpha_3 + z_{14}\alpha_1\alpha_4 + h(z_{15}\alpha_2^2 + z_{16}\alpha_1\alpha_3) + h^2z_{17}\alpha_1\alpha_2 + h^3z_{18}\alpha_1^2 \\
\alpha_4\alpha_2 &= z_{19}\alpha_3^2 + z_{20}\alpha_2\alpha_4 + h(z_{21}\alpha_2\alpha_3 + z_{22}\alpha_1\alpha_4) + h^2(z_{23}\alpha_2^2 + z_{24}\alpha_1\alpha_3) \\
&\quad + h^3z_{25}\alpha_1\alpha_2 + h^4z_{26}\alpha_1^2 \\
\alpha_4\alpha_3 &= z_{27}\alpha_3\alpha_4 + h(z_{28}\alpha_3^2 + z_{29}\alpha_2\alpha_4) + h^2(z_{30}\alpha_2\alpha_3 + z_{31}\alpha_1\alpha_4) + h^3(z_{32}\alpha_2^2 + z_{33}\alpha_1\alpha_3) \\
&\quad + h^4z_{34}\alpha_1\alpha_2 + h^5z_{35}\alpha_1^2
\end{aligned} \tag{4.30}$$

where $z_i \in \mathcal{K}$, $i = 1, 2, \ldots, 35$. From (4.1) the transformed relations are

$$
(4.31)\quad
\begin{aligned}
\alpha_2'\alpha_1' &= z_1\alpha_1'\alpha_2' + hz_2\alpha_1'^2 \\
\alpha_3'\alpha_1' &= z_3\alpha_2'^2 + z_4\alpha_1'\alpha_3' + hz_5\alpha_1'\alpha_2' + h^2z_6\alpha_1'^2 \\
\alpha_4'\alpha_1' &= z_7\alpha_2'\alpha_3' + z_8\alpha_1'\alpha_4' + h(z_9\alpha_2'^2 + z_{10}\alpha_1'\alpha_3') + h^2z_{11}\alpha_1'\alpha_2' + h^3z_{12}\alpha_1'^2 \\
\alpha_3'\alpha_2' &= z_{13}\alpha_2'\alpha_3' + z_{14}\alpha_1'\alpha_4' + h(z_{15}\alpha_2'^2 + z_{16}\alpha_1'\alpha_3') + h^2z_{17}\alpha_1'\alpha_2' + h^3z_{18}\alpha_1'^2 \\
\alpha_4'\alpha_2' &= z_{19}\alpha_3'^2 + z_{20}\alpha_2'\alpha_4' + h(z_{21}\alpha_2'\alpha_3' + z_{22}\alpha_1'\alpha_4') + h^2(z_{23}\alpha_2'^2 + z_{24}\alpha_1'\alpha_3') \\
&\quad + h^3z_{25}\alpha_1'\alpha_2' + h^4z_{26}\alpha_1'^2 \\
\alpha_4'\alpha_3' &= z_{27}\alpha_3'\alpha_4' + h(z_{28}\alpha_3'^2 + z_{29}\alpha_2'\alpha_4') + h^2(z_{30}\alpha_2'\alpha_3' + z_{31}\alpha_1'\alpha_4') + h^3(z_{32}\alpha_2'^2 + z_{33}\alpha_1'\alpha_3') \\
&\quad + h^4z_{34}\alpha_1'\alpha_2' + h^5z_{35}\alpha_1'^2.
\end{aligned}
$$

Substituting (3.24) into (4.31), using (4.30) and the relations in (3.14) yields sixty equations for the scalars z_i , $i = 1, 2, \ldots, 35$. The sixty equations are listed in (A.13)-(A.18) in the Appendix.

The equations in (4.10), (A.1)-(A.3), (A.4)-(A.9), (4.25), (A.10)-(A.12), (A.13)-(A.18) can now be used to solve for the unknown scalar coefficients and thus determine the relations for $\mathcal{K}[\mathrm{Poly}^n_{q,p}(2)]$ and $\mathcal{K}[\mathrm{Poly}^n_h(2)]$, $n = 1, 2, 3$.

Substitution of (3.21) into equations (4.10), use of the relations in (3.10) and solving gives

$$z = p^{-1}. \tag{5.1}$$

Therefore the presentation of $\mathcal{K}[\mathrm{Poly}^1_{q,p}(2)]$ is

$$\mathcal{K}[\mathrm{Poly}^1_{q,p}(2)] = \langle\, \alpha_1, \alpha_2 \mid \alpha_1\alpha_2 = p^{-1}\alpha_2\alpha_1 \,\rangle. \tag{5.2}$$

Substitution of (3.23) into equations (A.1)-(A.3), use of the relations in (3.10) and solving gives

$$z_1 = p^{-2}\ ,\ z_2 = p^{-2}q^2\ ,\ z_3 = p^{-1}(1-p^2q^2)\ ,\ z_4 = p^{-2}. \tag{5.3}$$

Therefore the presentation of $\mathcal{K}[\mathrm{Poly}^2_{q,p}(2)]$ is

$$\mathcal{K}[\mathrm{Poly}^2_{q,p}(2)] = \left\langle\ \alpha_1, \alpha_2, \alpha_3 \ \middle|\ \begin{aligned} \alpha_1\alpha_2 =& p^{-2}\alpha_2\alpha_1\ , \\ \alpha_1\alpha_3 =& p^{-2}q^2\alpha_3\alpha_1 + p^{-1}(1-p^2q^2)\alpha_2^2\ , \\ \alpha_2\alpha_3 =& p^{-2}\alpha_3\alpha_2 \end{aligned} \right\rangle. \tag{5.4}$$

Substitution of (3.25) into equations (A.4)-(A.9) , use of the relations in (3.10) and solving gives

$$\begin{aligned}
&z_1 = p^{-3}\ , \qquad z_2 = p^{-3}q^3\ , \qquad z_3 = p^{-2}(1-p^3q^3) \\
&z_4 = \frac{p^{-6}q^3(1+pq+p^2q^2) - p^{-7}q^2}{1+pq}\ ,\ z_5 = \frac{p^{-4}(1-p^3q^3)(1+pq+p^2q^2)}{1+pq} \\
&z_6 = \frac{p^{-3}(1+pq+p^2q^2) - q^3}{1+pq}\ , \qquad z_7 = \frac{p^{-6}q^2(pq-1)}{1+pq} \\
&z_8 = p^{-3}q^3\ ,\ \ z_9 = p^{-2}(1-p^3q^3)\ ,\ \ z_{10} = p^{-3}.
\end{aligned} \tag{5.5}$$

Thus the presentation of $\mathcal{K}[\mathrm{Poly}^3_{q,p}(2)]$ is

(5.6)

$$\mathcal{K}[\mathrm{Poly}^3_{q,p}(2)] = \left\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \;\middle|\; \begin{array}{l} \alpha_1\alpha_2 = p^{-3}\alpha_2\alpha_1 \;,\; \alpha_1\alpha_3 = p^{-3}q^3\alpha_3\alpha_1 + p^{-2}(1-p^3q^3)\alpha_2^2 \;, \\ \\ \alpha_1\alpha_4 = \left(\dfrac{p^{-6}q^3(1+pq+p^2q^2) - p^{-7}q^2}{1+pq}\right)\alpha_4\alpha_1 \\ \qquad + \left(\dfrac{p^{-4}(1-p^3q^3)(1+pq+p^2q^2)}{1+pq}\right)\alpha_3\alpha_2, \\ \\ \alpha_2\alpha_3 = \left(\dfrac{p^{-3}(1+pq+p^2q^2) - q^3}{1+pq}\right)\alpha_3\alpha_2 \\ \qquad + \left(\dfrac{p^{-6}q^2(pq-1)}{1+pq}\right)\alpha_4\alpha_1, \\ \\ \alpha_2\alpha_4 = p^{-3}q^3\alpha_4\alpha_2 + p^{-2}(1-p^3q^3)\alpha_3^2 \;,\; \alpha_3\alpha_4 = p^{-3}\alpha_4\alpha_3 \end{array} \right\rangle .$$

Substitution of (3.21) into equations (4.25), use of the relations in (3.14) and solving gives

(5.7) $$z_1 = 1 \;,\; z_2 = 1.$$

Therefore the presentation of $\mathcal{K}[\mathrm{Poly}^1_h(2)]$ is

(5.8) $$\mathcal{K}[\mathrm{Poly}^1_h(2)] = \langle\, \alpha_1, \alpha_2 \mid \alpha_2\alpha_1 = \alpha_1\alpha_2 + h\alpha_1^2 \,\rangle .$$

Substitution of (3.28) into equations (A.10)-(A.12), use of the relations in (3.14) and solving gives

(5.9)

$$z_1 = 1 \;,\; z_2 = 2 \;,\; z_3 = 0 \;,\; z_4 = 1 \;,\; z_5 = 4 \;,\; z_6 = 6 \;,\; z_7 = 1 \;,\; z_8 = 0 \;,\; z_9 = 2 \;,\; z_{10} = 2 \;,\; z_{11} = 3.$$

Thus the presetation of $\mathcal{K}[\mathrm{Poly}^2_h(2)]$ is

(5.10) $$\mathcal{K}[\mathrm{Poly}^2_h(2)] = \left\langle \alpha_1, \alpha_2, \alpha_3 \;\middle|\; \begin{array}{l} \alpha_2\alpha_1 = \alpha_1\alpha_2 + 2h\alpha_1^2 \;, \\ \\ \alpha_3\alpha_1 = \alpha_1\alpha_3 + 4h\alpha_1\alpha_2 + 6h^2\alpha_1^2 \;, \\ \\ \alpha_3\alpha_2 = \alpha_2\alpha_3 + 2h\alpha_1\alpha_3 + 2h^2\alpha_1\alpha_2 + 3h^3\alpha_1^2 \end{array} \right\rangle .$$

Substitution of (3.29) into equations (A.13)-(A.18), use of the relations in (3.14) and solving gives

(5.11)

$z_1 = 1$, $z_2 = 3$, $z_3 = 0$, $z_4 = 1$, $z_5 = 6$, $z_6 = 12$, $z_7 = 0$, $z_8 = 1$, $z_9 = 0$, $z_{10} = 9$,

$z_{11} = 36$, $z_{12} = 60$, $z_{13} = 1$, $z_{14} = 0$, $z_{15} = 2$, $z_{16} = 1$, $z_{17} = 2$, $z_{18} = 4$, $z_{19} = 0$, $z_{20} = 1$,

$z_{21} = 3$, $z_{22} = 3$, $z_{23} = 6$, $z_{24} = 9$, $z_{25} = 24$, $z_{26} = 36$, $z_{27} = 1$, $z_{28} = -3$, $z_{29} = 6$, $z_{30} = 0$,

$z_{31} = 12$, $z_{32} = 18$, $z_{33} = 6$, $z_{34} = 36$, $z_{35} = 48$.

Thus the presentation of $\mathcal{K}[\mathrm{Poly}^3_h(2)]$ is

$$(5.12)\quad \mathcal{K}[\mathrm{Poly}^3_h(2)] = \left\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \;\middle|\; \begin{array}{l} \alpha_2\alpha_1 = \alpha_1\alpha_2 + 3h\alpha_1^2 , \\ \alpha_3\alpha_1 = \alpha_1\alpha_3 + 6h\alpha_1\alpha_2 + 12h^2\alpha_1^2 , \\ \alpha_4\alpha_1 = \alpha_1\alpha_4 + 9h\alpha_1\alpha_3 + 36h^2\alpha_1\alpha_2 + 60h^3\alpha_1^2 , \\ \alpha_3\alpha_2 = \alpha_2\alpha_3 + h(2\alpha_2^2 + \alpha_1\alpha_3) + 2h^2\alpha_1\alpha_2 + 4h^3\alpha_1^2 , \\ \alpha_4\alpha_2 = \alpha_2\alpha_4 + h(3\alpha_2\alpha_3 + 3\alpha_1\alpha_4) + h^2(6\alpha_2^2 + 9\alpha_1\alpha_3) \\ \qquad + 24h^3\alpha_1\alpha_2 + 36h^4\alpha_1^2 , \\ \alpha_4\alpha_3 = \alpha_3\alpha_4 + h(6\alpha_2\alpha_4 - 3\alpha_3^2) + 12h^2\alpha_1\alpha_4 \\ \qquad + h^3(18\alpha_2^2 + 6\alpha_1\alpha_3) + 36h^4\alpha_1\alpha_2 + 48h^5\alpha_1^2 \end{array} \right\rangle .$$

Many of the equations listed in the Appendix are very complicated. These equations can be simplified as follows. By setting the component c in the quantum matrix $\mathbf{M}$ to zero the relations in (3.10), (3.14) and the transformation components in (3.23), (3.25), (3.28), (3.29) are simplified, thus, simplifying the equations in the Appendix. The commutation relations in (5.4), (5.6), (5.10), (5.12) were obtained by setting $c = 0$ and solving the simplifed equations in (A.1)-(A.3), (A.4)-(A.9), (A.10)-(A.12), (A.13)-(A.18), respectively, for the unknown scalar coefficients. The solutions for the scalar coefficients were then substituted back into the origional equations with $c \neq 0$ to check that they hold true for all quantum matricies $\mathbf{M}$. These equations were checked on a personal computer using the symbolic computation program *Mathematica* version 2.2. Unfortunately due to technical problems involving computer memory it was not possible to check all of the equations in (A.15)-(A.18).

6. Conclusions

Applying the Diamond Lemma to (5.2) it is concluded that $\mathcal{K}[\mathrm{Poly}^1_{q,p}(2)]$ satisfies PBW property. Thus, the commutation relation of $\mathcal{K}[\mathrm{Poly}^1_{q,p}(2)]$ is invariant in form under the action of $\mathbf{M} \in \mathrm{Mat}_{q,p}(2)$ and has homogeneous grading, and $\mathcal{K}[\mathrm{Poly}^1_{q,p}(2)]$ satisfies PBW property. Using the relations in (5.4) it follows that there exist final reduction sequences R, R' such that

$$R(l_{12}(\alpha_1,\alpha_2,\alpha_3)\alpha_3) = R'(\alpha_1 l_{23}(\alpha_1,\alpha_2,\alpha_3)). \tag{6.1}$$

Therefore, by the Diamond Lemma $\mathcal{K}[\mathrm{Poly}^2_{q,p}(2)]$ satisfies PBW property. Thus, the commutation relations of $\mathcal{K}[\mathrm{Poly}^2_{q,p}(2)]$ are invariant in form under the action of $\mathbf{M} \in \mathrm{Mat}_{q,p}(2)$ and have homogeneous grading, and $\mathcal{K}[\mathrm{Poly}^2_{q,p}(2)]$ satisfies PBW property. Using the relations in (5.6) it follows that there exist final reduction sequences R_i, R'_i, $i = 1,2,3,4$, such that

$$\begin{aligned}
&R_1(l_{12}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\alpha_3) = R'_1(\alpha_1 l_{23}(\alpha_1,\alpha_2,\alpha_3,\alpha_4))\\
&R_2(l_{12}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\alpha_4) = R'_2(\alpha_1 l_{24}(\alpha_1,\alpha_2,\alpha_3,\alpha_4))\\
&R_3(l_{13}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\alpha_4) = R'_3(\alpha_1 l_{34}(\alpha_1,\alpha_2,\alpha_3,\alpha_4))\\
&R_4(l_{23}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\alpha_4) = R'_4(\alpha_2 l_{34}(\alpha_1,\alpha_2,\alpha_3,\alpha_4))
\end{aligned} \tag{6.2}$$

if and only if the following condition holds:

$$z_5 z_7 = \frac{p^{-4}(1-p^3q^3)(1+pq+p^2q^2)}{1+pq}\frac{p^{-6}q^2(pq-1)}{1+pq} = 0. \tag{6.3}$$

This condition implies that $p = q^{-1}$. Therefore, by the Diamond Lemma $\mathcal{K}[\mathrm{Poly}^3_{q,p}(2)]$ satisfies PBW property if and only if $p = q^{-1}$. Thus, the commutation relations of $\mathcal{K}[\mathrm{Poly}^3_{q,p}(2)]$ are invariant in form under the action of $\mathbf{M} \in \mathrm{Mat}_{q,p}(2)$ and have homogeneous grading, and $\mathcal{K}[\mathrm{Poly}^3_{q,p}(2)]$ satisfies PBW property if and only if $p = q^{-1}$.

Applying the Diamond Lemma to (5.8) it is concluded that $\mathcal{K}[\mathrm{Poly}^1_h(2)]$ satisfies PBW property. Thus, the commutation relation of $\mathcal{K}[\mathrm{Poly}^1_h(2)]$ is invariant in form under the action of $\mathbf{M} \in \mathrm{Mat}_h(2)$ and has homogeneous grading, and $\mathcal{K}[\mathrm{Poly}^1_h(2)]$ satisfies PBW property. Using the relations in (5.10) it follows that there exist final reduction sequences R, R' such that

$$R(l_{32}(\alpha_1, \alpha_2, \alpha_3)\alpha_1) = R'(\alpha_3 l_{21}(\alpha_1, \alpha_2, \alpha_3)). \tag{6.4}$$

Therefore, by the Diamond Lemma $\mathcal{K}[\mathrm{Poly}_h^2(2)]$ satisfies PBW property. Thus, the commutation relations of $\mathcal{K}[\mathrm{Poly}_h^2(2)]$ are invariant in form under the action of $\mathbf{M} \in \mathrm{Mat}_h(2)$ and have homogeneous grading, and $\mathcal{K}[\mathrm{Poly}_h^2(2)]$ satisfies PBW property. Using the relations in (5.12) it follows that there exist final reduction sequences R_i, R_i', $i = 1, 2, 3, 4$, such that

$$\begin{aligned}
R_1(l_{32}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)\alpha_1) &= R_1'(\alpha_3 l_{21}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)) \\
R_2(l_{42}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)\alpha_1) &= R_2'(\alpha_4 l_{21}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)) \\
R_3(l_{43}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)\alpha_1) &= R_3'(\alpha_4 l_{31}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)) \\
R_4(l_{43}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)\alpha_2) &= R_4'(\alpha_4 l_{32}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)).
\end{aligned} \tag{6.5}$$

Therefore, by the Diamond Lemma $\mathcal{K}[\mathrm{Poly}_h^3(2)]$ satisfies PBW property. Thus, the commutation relations of $\mathcal{K}[\mathrm{Poly}_h^3(2)]$ are invariant in form under the action of $\mathbf{M} \in \mathrm{Mat}_h(2)$ and have homogeneous grading, and $\mathcal{K}[\mathrm{Poly}_h^3(2)]$ satisfies PBW property.

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APPENDIX

APPENDIX

$$\alpha_1'\alpha_2' = z_1\alpha_2'\alpha_1'$$

(A.1)

$$\alpha_1^2 : m_{11}m_{21} = z_1 m_{21}m_{11}$$

$$\alpha_2\alpha_1 : z_1 m_{11}m_{22} + m_{12}m_{21} = z_1^2 m_{21}m_{12} + z_1 m_{22}m_{11}$$

$$\alpha_3\alpha_1 : z_2 m_{11}m_{23} + m_{13}m_{21} = z_1 z_2 m_{21}m_{13} + z_1 m_{23}m_{11}$$

$$\alpha_2^2 : z_3 m_{11}m_{23} + m_{12}m_{22} = z_1 z_3 m_{21}m_{13} + z_1 m_{22}m_{12}$$

$$\alpha_3\alpha_2 : z_4 m_{12}m_{23} + m_{13}m_{22} = z_1 z_4 m_{22}m_{13} + z_1 m_{23}m_{12}$$

$$\alpha_3^2 : m_{13}m_{23} = z_1 m_{23}m_{13}$$

$$\alpha_1'\alpha_3' = z_2\alpha_3'\alpha_1' + z_3\alpha_2'^2$$

(A.2)

$\alpha_1^2 \ : m_{11}m_{31} = z_2 m_{31}m_{11} + z_3 m_{21}^2$

$\alpha_2\alpha_1 \ : z_1 m_{11}m_{32} + m_{12}m_{31} = z_1 z_2 m_{31}m_{12} + z_2 m_{32}m_{11} + z_1 z_3 m_{21}m_{22} + z_3 m_{22}m_{21}$

$\alpha_3\alpha_1 \ : z_2 m_{11}m_{33} + m_{13}m_{31} = z_2^2 m_{31}m_{13} + z_2 m_{33}m_{11} + z_2 z_3 m_{21}m_{23} + z_3 m_{23}m_{21}$

$\alpha_2^2 \ : z_3 m_{11}m_{33} + m_{12}m_{32} = z_2 z_3 m_{31}m_{13} + z_2 m_{32}m_{12} + z_3^2 m_{21}m_{23} + z_3 m_{22}^2$

$\alpha_3\alpha_2 \ : z_4 m_{12}m_{33} + m_{13}m_{32} = z_2 z_4 m_{32}m_{13} + z_2 m_{33}m_{12} + z_3 z_4 m_{22}m_{23} + z_3 m_{23}m_{22}$

$\alpha_3^2 \ : m_{13}m_{33} = z_2 m_{33}m_{13} + z_3 m_{23}^2$

$$\alpha_2'\alpha_3' = z_4\alpha_3'\alpha_2'$$

(A.3)

$$\begin{aligned}
\alpha_1^2 &: m_{21}m_{31} = z_4 m_{31}m_{21} \\
\alpha_2\alpha_1 &: z_1 m_{21}m_{32} + m_{22}m_{31} = z_1 z_4 m_{31}m_{22} + z_4 m_{32}m_{21} \\
\alpha_3\alpha_1 &: z_2 m_{21}m_{33} + m_{23}m_{31} = z_2 z_4 m_{31}m_{23} + z_4 m_{33}m_{21} \\
\alpha_2^2 &: z_3 m_{21}m_{33} + m_{22}m_{32} = z_3 z_4 m_{31}m_{23} + z_4 m_{32}m_{22} \\
\alpha_3\alpha_2 &: z_4 m_{22}m_{33} + m_{23}m_{32} = z_4^2 m_{32}m_{23} + z_4 m_{33}m_{22} \\
\alpha_3^2 &: m_{23}m_{33} = z_4 m_{33}m_{23}
\end{aligned}$$

$$\alpha_1'\alpha_2' = z_1\alpha_2'\alpha_1'$$

(A.4)

$$\alpha_1^2 : m_{11}m_{21} = z_1m_{21}m_{11}$$

$$\alpha_2\alpha_1 : z_1m_{11}m_{22} + m_{12}m_{21} = z_1^2m_{21}m_{12} + z_1m_{22}m_{11}$$

$$\alpha_3\alpha_1 : z_2m_{11}m_{23} + m_{13}m_{21} = z_1z_2m_{21}m_{13} + z_1m_{23}m_{11}$$

$$\alpha_4\alpha_1 : z_4m_{11}m_{24} + z_7m_{12}m_{23} + m_{14}m_{21} = z_1z_4m_{21}m_{14} + z_1z_7m_{22}m_{13} + z_1m_{24}m_{11}$$

$$\alpha_2^2 : z_3m_{11}m_{23} + m_{12}m_{22} = z_1z_3m_{21}m_{13} + z_1m_{22}m_{12}$$

$$\alpha_3\alpha_2 : z_5m_{11}m_{24} + z_6m_{12}m_{23} + m_{13}m_{22} = z_1z_5m_{21}m_{14} + z_1z_6m_{22}m_{13} + z_1m_{23}m_{12}$$

$$\alpha_4\alpha_2 : z_8m_{12}m_{24} + m_{14}m_{22} = z_1z_8m_{22}m_{14} + z_1m_{24}m_{12}$$

$$\alpha_3^2 : z_9m_{12}m_{24} + m_{13}m_{23} = z_1z_9m_{22}m_{14} + z_1m_{23}m_{13}$$

$$\alpha_4\alpha_3 : z_{10}m_{13}m_{24} + m_{14}m_{23} = z_1z_{10}m_{23}m_{14} + z_1m_{24}m_{13}$$

$$\alpha_4^2 : m_{14}m_{24} = z_1m_{24}m_{14}$$

$$\alpha_1'\alpha_3' = z_2\alpha_3'\alpha_1' + z_3\alpha_2'^2$$

(A.5)

$$\alpha_1^2 : m_{11}m_{31} = z_2 m_{31}m_{11} + z_3 m_{21}^2$$

$$\alpha_2\alpha_1 : z_1 m_{11}m_{32} + m_{12}m_{31} = z_1 z_2 m_{31}m_{12} + z_2 m_{32}m_{11} + z_1 z_3 m_{21}m_{22} + z_3 m_{22}m_{21}$$

$$\alpha_3\alpha_1 : z_2 m_{11}m_{33} + m_{13}m_{31} = z_2^2 m_{31}m_{13} + z_2 m_{33}m_{11} + z_2 z_3 m_{21}m_{23} + z_3 m_{23}m_{21}$$

$$\alpha_4\alpha_1 : z_4 m_{11}m_{34} + z_7 m_{12}m_{33} + m_{14}m_{31}$$

$$= z_2 z_4 m_{31}m_{14} + z_2 z_7 m_{32}m_{13} + z_2 m_{34}m_{11} + z_3 z_4 m_{21}m_{24} + z_3 z_7 m_{22}m_{23} + z_3 m_{24}m_{21}$$

$$\alpha_2^2 : z_3 m_{11}m_{33} + m_{12}m_{32} = z_2 z_3 m_{31}m_{13} + z_2 m_{32}m_{12} + z_3^2 m_{21}m_{23} + z_3 m_{22}^2$$

$$\alpha_3\alpha_2 : z_5 m_{11}m_{34} + z_6 m_{12}m_{33} + m_{13}m_{32}$$

$$= z_2 z_5 m_{31}m_{14} + z_2 z_6 m_{32}m_{13} + z_2 m_{33}m_{12} + z_3 z_5 m_{21}m_{24} + z_3 z_6 m_{22}m_{23} + z_3 m_{23}m_{22}$$

$$\alpha_4\alpha_2 : z_8 m_{12}m_{34} + m_{14}m_{32} = z_2 z_8 m_{32}m_{14} + z_2 m_{34}m_{12} + z_3 z_8 m_{22}m_{24} + z_3 m_{24}m_{22}$$

$$\alpha_3^2 : z_9 m_{12}m_{34} + m_{13}m_{33} = z_2 z_9 m_{32}m_{14} + z_2 m_{33}m_{13} + z_3 z_9 m_{22}m_{24} + z_3 m_{23}^2$$

$$\alpha_4\alpha_3 : z_{10} m_{13}m_{34} + m_{14}m_{33} = z_2 z_{10} m_{33}m_{14} + z_2 m_{34}m_{13} + z_3 z_{10} m_{23}m_{24} + z_3 m_{24}m_{23}$$

$$\alpha_4^2 : m_{14}m_{34} = z_2 m_{34}m_{14} + z_3 m_{24}^2$$

$$\alpha_1'\alpha_4' = z_4\alpha_4'\alpha_1' + z_5\alpha_3'\alpha_2'$$

(A.6)

$$\alpha_1^2\ : m_{11}m_{41} = z_4m_{41}m_{11} + z_5m_{31}m_{21}$$

$$\alpha_2\alpha_1\ : z_1m_{11}m_{42} + m_{12}m_{41} = z_1z_4m_{41}m_{12} + z_4m_{42}m_{11} + z_1z_5m_{31}m_{22} + z_5m_{32}m_{21}$$

$$\alpha_3\alpha_1\ : z_2m_{11}m_{43} + m_{13}m_{41} = z_2z_4m_{41}m_{13} + z_4m_{43}m_{11} + z_2z_5m_{31}m_{23} + z_5m_{33}m_{21}$$

$$\alpha_4\alpha_1\ : z_4m_{11}m_{44} + z_7m_{12}m_{43} + m_{14}m_{41}$$

$$= z_4^2m_{41}m_{14} + z_4z_7m_{42}m_{13} + z_4m_{44}m_{11} + z_4z_5m_{31}m_{24} + z_5z_7m_{32}m_{23} + z_5m_{34}m_{21}$$

$$\alpha_2^2\ : z_3m_{11}m_{43} + m_{12}m_{42} = z_3z_4m_{41}m_{13} + z_4m_{42}m_{12} + z_3z_5m_{31}m_{23} + z_5m_{32}m_{22}$$

$$\alpha_3\alpha_2\ : z_5m_{11}m_{44} + z_6m_{12}m_{43} + m_{13}m_{42}$$

$$= z_4z_5m_{41}m_{14} + z_4z_6m_{42}m_{13} + z_4m_{43}m_{12} + z_5^2m_{31}m_{24} + z_5z_6m_{32}m_{23} + z_5m_{33}m_{22}$$

$$\alpha_4\alpha_2\ : z_8m_{12}m_{44} + m_{14}m_{42} = z_4z_8m_{42}m_{14} + z_4m_{44}m_{12} + z_5z_8m_{32}m_{24} + z_5m_{34}m_{22}$$

$$\alpha_3^2\ : z_9m_{12}m_{44} + m_{13}m_{43} = z_4z_9m_{42}m_{14} + z_4m_{43}m_{13} + z_5z_9m_{32}m_{24} + z_5m_{33}m_{23}$$

$$\alpha_4\alpha_3\ : z_{10}m_{13}m_{44} + m_{14}m_{43} = z_4z_{10}m_{43}m_{14} + z_4m_{44}m_{13} + z_5z_{10}m_{33}m_{24} + z_5m_{34}m_{23}$$

$$\alpha_4^2\ : m_{14}m_{44} = z_4m_{44}m_{14} + z_5m_{34}m_{24}$$

$$\alpha_2'\alpha_3' = z_6\alpha_3'\alpha_2' + z_7\alpha_4'\alpha_1'$$

(A.7)

$$\alpha_1^2 : m_{21}m_{31} = z_6m_{31}m_{21} + z_7m_{41}m_{11}$$

$$\alpha_2\alpha_1 : z_1m_{21}m_{32} + m_{22}m_{31} = z_1z_6m_{31}m_{22} + z_6m_{32}m_{21} + z_1z_7m_{41}m_{12} + z_7m_{42}m_{11}$$

$$\alpha_3\alpha_1 : z_2m_{21}m_{33} + m_{23}m_{31} = z_2z_6m_{31}m_{23} + z_6m_{33}m_{21} + z_2z_7m_{41}m_{13} + z_7m_{43}m_{11}$$

$$\alpha_4\alpha_1 : z_4m_{21}m_{34} + z_7m_{22}m_{33} + m_{24}m_{31}$$

$$= z_4z_6m_{31}m_{24} + z_6z_7m_{32}m_{23} + z_6m_{34}m_{21} + z_4z_7m_{41}m_{14} + z_7^2m_{42}m_{13} + z_7m_{44}m_{11}$$

$$\alpha_2^2 : z_3m_{21}m_{33} + m_{22}m_{32} = z_3z_6m_{31}m_{23} + z_6m_{32}m_{22} + z_3z_7m_{41}m_{13} + z_7m_{42}m_{12}$$

$$\alpha_3\alpha_2 : z_5m_{21}m_{34} + z_6m_{22}m_{33} + m_{23}m_{32}$$

$$= z_5z_6m_{31}m_{24} + z_6^2m_{32}m_{23} + z_6m_{33}m_{22} + z_5z_7m_{41}m_{14} + z_6z_7m_{42}m_{13} + z_7m_{43}m_{12}$$

$$\alpha_4\alpha_2 : z_8m_{22}m_{34} + m_{24}m_{32} = z_6z_8m_{32}m_{24} + z_6m_{34}m_{22} + z_7z_8m_{42}m_{14} + z_7m_{44}m_{12}$$

$$\alpha_3^2 : z_9m_{22}m_{34} + m_{23}m_{33} = z_6z_9m_{32}m_{24} + z_6m_{33}m_{23} + z_7z_9m_{42}m_{14} + z_7m_{43}m_{13}$$

$$\alpha_4\alpha_3 : z_{10}m_{23}m_{34} + m_{24}m_{33} = z_6z_{10}m_{33}m_{24} + z_6m_{34}m_{23} + z_7z_{10}m_{43}m_{14} + z_7m_{44}m_{13}$$

$$\alpha_4^2 : m_{24}m_{34} = z_6m_{34}m_{24} + z_7m_{44}m_{14}$$

$$\alpha_2'\alpha_4' = z_8\alpha_4'\alpha_2' + z_9\alpha_3'^2$$

(A.8)

$$\alpha_1^2 : m_{21}m_{41} = z_8m_{41}m_{21} + z_9m_{31}^2$$

$$\alpha_2\alpha_1 : z_1m_{21}m_{42} + m_{22}m_{41} = z_1z_8m_{41}m_{22} + z_8m_{42}m_{21} + z_1z_9m_{31}m_{32} + z_9m_{32}m_{31}$$

$$\alpha_3\alpha_1 : z_2m_{21}m_{43} + m_{23}m_{41} = z_2z_8m_{41}m_{23} + z_8m_{43}m_{21} + z_2z_9m_{31}m_{33} + z_9m_{33}m_{31}$$

$$\alpha_4\alpha_1 : z_4m_{21}m_{44} + z_7m_{22}m_{43} + m_{24}m_{41}$$

$$= z_4z_8m_{41}m_{24} + z_7z_8m_{42}m_{23} + z_8m_{44}m_{21} + z_4z_9m_{31}m_{34} + z_7z_9m_{32}m_{33} + z_9m_{34}m_{31}$$

$$\alpha_2^2 : z_3m_{21}m_{43} + m_{22}m_{42} = z_3z_8m_{41}m_{23} + z_8m_{42}m_{22} + z_3z_9m_{31}m_{33} + z_9m_{32}^2$$

$$\alpha_3\alpha_2 : z_5m_{21}m_{44} + z_6m_{22}m_{43} + m_{23}m_{42}$$

$$= z_5z_8m_{41}m_{24} + z_6z_8m_{42}m_{23} + z_8m_{43}m_{22} + z_5z_9m_{31}m_{34} + z_6z_9m_{32}m_{33} + z_9m_{33}m_{32}$$

$$\alpha_4\alpha_2 : z_8m_{22}m_{44} + m_{24}m_{42} = z_8^2m_{42}m_{24} + z_8m_{44}m_{22} + z_8z_9m_{32}m_{34} + z_9m_{34}m_{32}$$

$$\alpha_3^2 : z_9m_{22}m_{44} + m_{23}m_{43} = z_8z_9m_{42}m_{24} + z_8m_{43}m_{23} + z_9^2m_{32}m_{34} + z_9m_{33}^2$$

$$\alpha_4\alpha_3 : z_{10}m_{23}m_{44} + m_{24}m_{43} = z_8z_{10}m_{43}m_{24} + z_8m_{44}m_{23} + z_9z_{10}m_{33}m_{34} + z_9m_{34}m_{33}$$

$$\alpha_4^2 : m_{24}m_{44} = z_8m_{44}m_{24} + z_9m_{34}^2$$

$$\alpha_3'\alpha_4' = z_{10}\alpha_4'\alpha_3'$$

(A.9)

$$\alpha_1^2 : m_{31}m_{41} = z_{10}m_{41}m_{31}$$

$$\alpha_2\alpha_1 : z_1m_{31}m_{42} + m_{32}m_{41} = z_1z_{10}m_{41}m_{32} + z_{10}m_{42}m_{31}$$

$$\alpha_3\alpha_1 : z_2m_{31}m_{43} + m_{33}m_{41} = z_2z_{10}m_{41}m_{33} + z_{10}m_{43}m_{31}$$

$$\alpha_4\alpha_1 : z_4m_{31}m_{44} + z_7m_{32}m_{43} + m_{34}m_{41} = z_4z_{10}m_{41}m_{34} + z_7z_{10}m_{42}m_{33} + z_{10}m_{44}m_{31}$$

$$\alpha_2^2 : z_3m_{31}m_{43} + m_{32}m_{42} = z_3z_{10}m_{41}m_{33} + z_{10}m_{42}m_{32}$$

$$\alpha_3\alpha_2 : z_5m_{31}m_{44} + z_6m_{32}m_{43} + m_{33}m_{42} = z_5z_{10}m_{41}m_{34} + z_6z_{10}m_{42}m_{33} + z_{10}m_{43}m_{32}$$

$$\alpha_4\alpha_2 : z_8m_{32}m_{44} + m_{34}m_{42} = z_8z_{10}m_{42}m_{34} + z_{10}m_{44}m_{32}$$

$$\alpha_3^2 : z_9m_{32}m_{44} + m_{33}m_{43} = z_9z_{10}m_{42}m_{34} + z_{10}m_{43}m_{33}$$

$$\alpha_4\alpha_3 : z_{10}m_{33}m_{44} + m_{34}m_{43} = z_{10}^2m_{43}m_{34} + z_{10}m_{44}m_{33}$$

$$\alpha_4^2 : m_{34}m_{44} = z_{10}m_{44}m_{34}$$

$$\alpha'_2\alpha'_1 = z_1\alpha'_1\alpha'_2 + hz_2\alpha'^2_1$$

(A.10)

$$\alpha_1^2 : m_{21}m_{11} + hz_2m_{22}m_{11} + h^2z_6m_{23}m_{11} + h^3z_{11}m_{23}m_{12}$$

$$= z_1m_{11}m_{21} + hz_1z_2m_{12}m_{21} + h^2z_1z_6m_{13}m_{21} + h^3z_{11}m_{13}m_{22} + hz_2m_{11}^2 + h^2z_2^2m_{12}m_{11}$$

$$+ h^3z_2z_6m_{13}m_{11} + h^4z_2z_{11}m_{13}m_{12}$$

$$\alpha_1\alpha_2 : m_{21}m_{12} + z_1m_{22}m_{11} + hz_5m_{23}m_{11} + h^2z_{10}m_{23}m_{12}$$

$$= z_1m_{11}m_{22} + z_1^2m_{12}m_{21} + hz_1z_5m_{13}m_{21} + h^2z_1z_{10}m_{13}m_{22} + hz_2m_{11}m_{12} + hz_1z_2m_{12}m_{11}$$

$$+ h^2z_2z_5m_{13}m_{11} + h^3z_2z_{10}m_{13}m_{12}$$

$$\alpha_1\alpha_3 : m_{21}m_{13} + z_4m_{23}m_{11} + hz_9m_{23}m_{12}$$

$$= z_1m_{11}m_{23} + z_1z_4m_{13}m_{21} + hz_1z_9m_{13}m_{22} + hz_2m_{11}m_{13} + hz_2z_4m_{13}m_{11} + h^2z_2z_9m_{13}m_{12}$$

$$\alpha_2^2 : m_{22}m_{12} + z_3m_{23}m_{11} + hz_8m_{23}m_{12}$$

$$= z_1m_{12}m_{22} + z_1z_3m_{13}m_{21} + hz_1z_8m_{13}m_{22} + hz_2m_{12}^2 + hz_2z_3m_{13}m_{11} + h^2z_2z_8m_{13}m_{12}$$

$$\alpha_2\alpha_3 : m_{22}m_{13} + z_7m_{23}m_{12} = z_1m_{12}m_{23} + z_1z_7m_{13}m_{22} + hz_2m_{12}m_{13} + hz_2z_7m_{13}m_{12}$$

$$\alpha_3^2 : m_{23}m_{13} = z_1m_{13}m_{23} + hz_2m_{13}^2$$

$$\alpha_3'\alpha_1' = z_3\alpha_2'^2 + z_4\alpha_1'\alpha_3' + hz_5\alpha_1'\alpha_2' + h^2z_6\alpha_1'^2$$

(A.11)

$\alpha_1^2 : m_{31}m_{11} + hz_2m_{32}m_{11} + h^2z_6m_{33}m_{11} + h^3z_{11}m_{33}m_{12}$

$= z_3m_{21}^2 + hz_2z_3m_{22}m_{21} + h^2z_3z_6m_{23}m_{21} + h^3z_3z_{11}m_{23}m_{22} + z_4m_{11}m_{31} + hz_2z_4m_{12}m_{31}$

$+ h^2z_4z_6m_{13}m_{31} + h^3z_4z_{11}m_{13}m_{32} + hz_5m_{11}m_{21} + h^2z_2z_5m_{12}m_{21} + h^3z_5z_6m_{13}m_{21} + h^4z_5z_{11}m_{13}m_{22}$

$+ h^2z_6m_{11}^2 + h^3z_2z_6m_{12}m_{11} + h^4z_6^2m_{13}m_{11} + h^5z_6z_{11}m_{13}m_{12}$

$\alpha_1\alpha_2 : m_{31}m_{12} + z_1m_{32}m_{11} + hz_5m_{33}m_{11} + h^2z_{10}m_{33}m_{12}$

$= z_3m_{21}m_{22} + z_1z_3m_{22}m_{21} + hz_3z_5m_{23}m_{21} + h^2z_3z_{10}m_{23}m_{22} + z_4m_{11}m_{32} + z_1z_4m_{12}m_{31}$

$+ hz_4z_5m_{13}m_{31} + h^2z_4z_{10}m_{13}m_{32} + hz_5m_{11}m_{22} + hz_1z_5m_{12}m_{21} + h^2z_5^2m_{13}m_{21} + h^3z_5z_{10}m_{13}m_{22}$

$+ h^2z_6m_{11}m_{12} + h^2z_1z_6m_{12}m_{11} + h^3z_5z_6m_{13}m_{11} + h^4z_6z_{10}m_{13}m_{12}$

$\alpha_1\alpha_3 : m_{31}m_{13} + z_4m_{33}m_{11} + hz_9m_{33}m_{12}$

$= z_3m_{21}m_{23} + z_3z_4m_{23}m_{21} + hz_3z_9m_{23}m_{22} + z_4m_{11}m_{33} + z_4^2m_{13}m_{31} + hz_4z_9m_{13}m_{32}$

$+ hz_5m_{11}m_{23} + hz_4z_5m_{13}m_{21} + h^2z_5z_9m_{13}m_{22} + h^2z_6m_{11}m_{13} + h^2z_4z_6m_{13}m_{11} + h^3z_6z_9m_{13}m_{12}$

$\alpha_2^2 : m_{32}m_{12} + z_3m_{33}m_{11} + hz_8m_{33}m_{12}$

$= z_3m_{22}^2 + z_3^2m_{23}m_{21} + hz_3z_8m_{23}m_{22} + z_4m_{12}m_{32} + z_3z_4m_{13}m_{31} + hz_4z_8m_{13}m_{32}$

$+ hz_5m_{12}m_{22} + hz_3z_5m_{13}m_{21} + h^2z_5z_8m_{13}m_{22} + h^2z_6m_{12}^2 + h^2z_3z_6m_{13}m_{11} + h^3z_6z_8m_{13}m_{12}$

$\alpha_2\alpha_3 : m_{32}m_{13} + z_7m_{33}m_{12}$

$= z_3m_{22}m_{23} + z_3z_7m_{23}m_{22} + z_4m_{12}m_{33} + z_4z_7m_{13}m_{32} + hz_5m_{12}m_{23} + hz_5z_7m_{13}m_{22}$

$+ h^2z_6m_{12}m_{13} + h^2z_6z_7m_{13}m_{12}$

$\alpha_3^2 : m_{33}m_{13} = z_3m_{23}^2 + z_4m_{13}m_{33} + hz_5m_{13}m_{23} + h^2z_6m_{13}^2$

$$\alpha_3'\alpha_2' = z_7\alpha_2'\alpha_3' + h(z_8\alpha_2'^2 + z_9\alpha_1'\alpha_3') + h^2 z_{10}\alpha_1'\alpha_2' + h^3 z_{11}\alpha_1'^2$$

(A.12)

$$\begin{aligned}
\alpha_1^2\ :\ & m_{31}m_{21} + hz_2m_{32}m_{21} + h^2z_6m_{33}m_{21} + h^3z_{11}m_{33}m_{22}\\
&= z_7m_{21}m_{31} + hz_2z_7m_{22}m_{31} + h^2z_6z_7m_{23}m_{31} + h^3z_7z_{11}m_{23}m_{32} + hz_8m_{21}^2 + h^2z_2z_8m_{22}m_{21}\\
&+h^3z_6z_8m_{23}m_{21} + h^4z_8z_{11}m_{23}m_{22} + hz_9m_{11}m_{31} + h^2z_2z_9m_{12}m_{31} + h^3z_6z_9m_{13}m_{31} + h^4z_9z_{11}m_{13}m_{32}\\
&+h^2z_{10}m_{11}m_{21} + h^3z_2z_{10}m_{12}m_{21} + h^4z_6z_{10}m_{13}m_{21} + h^5z_{10}z_{11}m_{13}m_{22} + h^3z_{11}m_{11}^2 + h^4z_2z_{11}m_{12}m_{11}\\
&+h^5z_6z_{11}m_{13}m_{11} + h^6z_{11}^2m_{13}m_{12}
\end{aligned}$$

$$\begin{aligned}
\alpha_1\alpha_2\ :\ & m_{31}m_{22} + z_1m_{32}m_{21} + hz_5m_{33}m_{21} + h^2z_{10}m_{33}m_{22}\\
&= z_7m_{21}m_{32} + z_1z_7m_{22}m_{31} + hz_5z_7m_{23}m_{31} + h^2z_7z_{10}m_{23}m_{32} + hz_8m_{21}m_{22} + hz_1z_8m_{22}m_{21}\\
&+h^2z_5z_8m_{23}m_{21} + h^3z_8z_{10}m_{23}m_{22} + hz_9m_{11}m_{32} + hz_1z_9m_{12}m_{31} + h^2z_5z_9m_{13}m_{31} + h^3z_9z_{10}m_{13}m_{32}\\
&+h^2z_{10}m_{11}m_{22} + h^2z_1z_{10}m_{12}m_{21} + h^3z_5z_{10}m_{13}m_{21} + h^4z_{10}^2m_{13}m_{22} + h^3z_{11}m_{11}m_{12} + h^3z_1z_{11}m_{12}m_{11}\\
&+h^4z_5z_{11}m_{13}m_{11} + h^5z_{10}z_{11}m_{13}m_{12}
\end{aligned}$$

$$\begin{aligned}
\alpha_1\alpha_3\ :\ & m_{31}m_{23} + z_4m_{33}m_{21} + hz_9m_{33}m_{22}\\
&= z_7m_{21}m_{33} + z_4z_7m_{23}m_{31} + hz_7z_9m_{23}m_{32} + hz_8m_{21}m_{23} + hz_4z_8m_{23}m_{21} + h^2z_8z_9m_{23}m_{22}\\
&+hz_9m_{11}m_{33} + hz_4z_9m_{13}m_{31} + h^2z_9^2m_{13}m_{32} + h^2z_{10}m_{11}m_{23} + h^2z_4z_{10}m_{13}m_{21} + h^3z_9z_{10}m_{13}m_{22}\\
&+h^3z_{11}m_{11}m_{13} + h^3z_4z_{11}m_{13}m_{11} + h^4z_9z_{11}m_{13}m_{12}
\end{aligned}$$

$$\begin{aligned}
\alpha_2^2\ :\ & m_{32}m_{22} + z_3m_{33}m_{21} + hz_8m_{33}m_{22}\\
&= z_7m_{22}m_{32} + z_3z_7m_{23}m_{31} + hz_7z_8m_{23}m_{32} + hz_8m_{22}^2 + hz_3z_8m_{23}m_{21} + h^2z_8^2m_{23}m_{22}\\
&+hz_9m_{12}m_{32} + hz_3z_9m_{13}m_{31} + h^2z_8z_9m_{13}m_{32} + h^2z_{10}m_{12}m_{22} + h^2z_3z_{10}m_{13}m_{21} + h^3z_8z_{10}m_{13}m_{22}\\
&+h^3z_{11}m_{12}^2 + h^3z_3z_{11}m_{13}m_{11} + h^4z_8z_{11}m_{13}m_{12}
\end{aligned}$$

$$\begin{aligned}
\alpha_2\alpha_3\ :\ & m_{32}m_{23} + z_7m_{33}m_{22}\\
&= z_7m_{22}m_{33} + z_7^2m_{23}m_{32} + hz_8m_{22}m_{23} + hz_7z_8m_{23}m_{22} + hz_9m_{12}m_{33} + hz_7z_9m_{13}m_{32}\\
&+h^2z_{10}m_{12}m_{23} + h^2z_7z_{10}m_{13}m_{22} + h^3z_{11}m_{12}m_{13} + h^3z_7z_{11}m_{13}m_{12}
\end{aligned}$$

$$\alpha_3^2\ :\ m_{33}m_{23} = z_7m_{23}m_{33} + hz_8m_{23}^2 + hz_9m_{13}m_{33} + h^2z_{10}m_{13}m_{23} + h^3z_{11}m_{13}^2$$

$$\alpha_2'\alpha_1' = z_1\alpha_1'\alpha_2' + hz_2\alpha_1'^2$$

(A.13)

$$\alpha_1^2 : m_{21}m_{11} + hz_2m_{22}m_{11} + h^2z_6m_{23}m_{11} + h^3z_{18}m_{23}m_{12} + h^3z_{12}m_{24}m_{11} + h^4z_{26}m_{24}m_{12} + h^5z_{35}m_{24}m_{13}$$

$$= z_1m_{11}m_{21} + hz_1z_2m_{12}m_{21} + h^2z_1z_6m_{13}m_{21} + h^3z_1z_{18}m_{13}m_{22} + h^3z_1z_{12}m_{14}m_{21} + h^4z_1z_{26}m_{14}m_{22}$$

$$+ h^5z_1z_{35}m_{14}m_{23} + hz_2m_{11}^2 + h^2z_2^2m_{12}m_{11} + h^3z_2z_6m_{13}m_{11} + h^4z_2z_{18}m_{13}m_{12} + h^4z_2z_{12}m_{14}m_{11}$$

$$+ h^5z_2z_{26}m_{14}m_{12} + h^6z_2z_{35}m_{14}m_{13}$$

$$\alpha_1\alpha_2 : m_{21}m_{12} + z_1m_{22}m_{11} + hz_5m_{23}m_{11} + h^2z_{17}m_{23}m_{12} + h^2z_{11}m_{24}m_{11} + h^3z_{25}m_{24}m_{12} + h^4z_{34}m_{24}m_{13}$$

$$= z_1m_{11}m_{22} + z_1^2m_{12}m_{21} + hz_1z_5m_{13}m_{21} + h^2z_1z_{17}m_{13}m_{22} + h^2z_1z_{11}m_{14}m_{21} + h^3z_1z_{25}m_{14}m_{22}$$

$$+ h^4z_1z_{34}m_{14}m_{23} + hz_2m_{11}m_{12} + hz_1z_2m_{12}m_{11} + h^2z_2z_5m_{13}m_{11} + h^3z_2z_{17}m_{13}m_{12} + h^3z_2z_{11}m_{14}m_{11}$$

$$+ h^4z_2z_{25}m_{14}m_{12} + h^5z_2z_{34}m_{14}m_{13}$$

$$\alpha_1\alpha_3 : m_{21}m_{13} + z_4m_{23}m_{11} + hz_{16}m_{23}m_{12} + hz_{10}m_{24}m_{11} + h^2z_{24}m_{24}m_{12} + h^3z_{33}m_{24}m_{13}$$

$$= z_1m_{11}m_{23} + z_1z_4m_{13}m_{21} + hz_1z_{16}m_{13}m_{22} + hz_1z_{10}m_{14}m_{21} + h^2z_1z_{24}m_{14}m_{22} + h^3z_1z_{33}m_{14}m_{23}$$

$$+ hz_2m_{11}m_{13} + hz_2z_4m_{13}m_{11} + h^2z_2z_{16}m_{13}m_{12} + h^2z_2z_{10}m_{14}m_{11} + h^3z_2z_{24}m_{14}m_{12} + h^4z_2z_{33}m_{14}m_{13}$$

$$\alpha_1\alpha_4 : m_{21}m_{14} + z_{14}m_{23}m_{12} + z_8m_{24}m_{11} + hz_{22}m_{24}m_{12} + h^2z_{31}m_{24}m_{13}$$

$$= z_1m_{11}m_{24} + z_1z_{14}m_{13}m_{22} + z_1z_8m_{14}m_{21} + hz_1z_{22}m_{14}m_{22} + h^2z_1z_{31}m_{14}m_{23} + hz_2m_{11}m_{14}$$

$$+ hz_2z_{14}m_{13}m_{12} + hz_2z_8m_{14}m_{11} + h^2z_2z_{22}m_{14}m_{12} + h^3z_2z_{31}m_{14}m_{13}$$

$$\alpha_2^2 : m_{22}m_{12} + z_3m_{23}m_{11} + hz_{15}m_{23}m_{12} + hz_9m_{24}m_{11} + h^2z_{23}m_{24}m_{12} + h^3z_{32}m_{24}m_{13}$$

$$= z_1m_{12}m_{22} + z_1z_3m_{13}m_{21} + hz_1z_{15}m_{13}m_{22} + hz_1z_9m_{14}m_{21} + h^2z_1z_{23}m_{14}m_{22} + h^3z_1z_{32}m_{14}m_{23}$$

$$+ hz_2m_{12}^2 + hz_2z_3m_{13}m_{11} + h^2z_2z_{15}m_{13}m_{12} + h^2z_2z_9m_{14}m_{11} + h^3z_2z_{23}m_{14}m_{12} + h^4z_2z_{32}m_{14}m_{13}$$

$$\alpha_2\alpha_3 : m_{22}m_{13} + z_{13}m_{23}m_{12} + z_7m_{24}m_{11} + hz_{21}m_{24}m_{12} + h^2z_{30}m_{24}m_{13}$$

$$= z_1m_{12}m_{23} + z_1z_{13}m_{13}m_{22} + z_1z_7m_{14}m_{21} + hz_1z_{21}m_{14}m_{22} + h^2z_1z_{30}m_{14}m_{23} + hz_2m_{12}m_{13}$$

$$+ hz_2z_{13}m_{13}m_{12} + hz_2z_7m_{14}m_{11} + h^2z_2z_{21}m_{14}m_{12} + h^3z_2z_{30}m_{14}m_{13}$$

$$\alpha_2\alpha_4 : m_{22}m_{14} + z_{20}m_{24}m_{12} + hz_{29}m_{24}m_{13}$$

$$= z_1m_{12}m_{24} + z_1z_{20}m_{14}m_{22} + hz_1z_{29}m_{14}m_{23} + hz_2m_{12}m_{14} + hz_2z_{20}m_{14}m_{12} + h^2z_2z_{29}m_{14}m_{13}$$

$$\alpha_3^2 : m_{23}m_{13} + z_{19}m_{24}m_{12} + hz_{28}m_{24}m_{13}$$

$$= z_1m_{13}m_{23} + z_1z_{19}m_{14}m_{22} + hz_1z_{28}m_{14}m_{23} + hz_2m_{13}^2 + hz_2z_{19}m_{14}m_{12} + h^2z_2z_{28}m_{14}m_{13}$$

$$\alpha_3\alpha_4 : m_{23}m_{14} + z_{27}m_{24}m_{13} = z_1m_{13}m_{24} + z_1z_{27}m_{14}m_{23} + hz_2m_{13}m_{14} + hz_2z_{27}m_{14}m_{13}$$

$$\alpha_4^2 : m_{24}m_{14} = z_1m_{14}m_{24} + hz_2m_{14}^2$$

$$\alpha_3'\alpha_1' = z_3\alpha_2'^2 + z_4\alpha_1'\alpha_3' + hz_5\alpha_1'\alpha_2' + h^2z_6\alpha_1'^2$$

(A.14)

$$\begin{aligned}
\alpha_1^2\ :\ & m_{31}m_{11} + hz_2m_{32}m_{11} + h^2z_6m_{33}m_{11} + h^3z_{18}m_{33}m_{12} + h^3z_{12}m_{34}m_{11} + h^4z_{26}m_{34}m_{12} + h^5z_{35}m_{34}m_{13} \\
&= z_3m_{21}^2 + hz_2z_3m_{22}m_{21} + h^2z_3z_6m_{23}m_{21} + h^3z_3z_{18}m_{23}m_{22} + h^3z_3z_{12}m_{24}m_{21} + h^4z_3z_{26}m_{24}m_{22} \\
&+ h^5z_3z_{35}m_{24}m_{23} + z_4m_{11}m_{31} + hz_2z_4m_{12}m_{31} + h^2z_4z_6m_{13}m_{31} + h^3z_4z_{18}m_{13}m_{32} + h^3z_4z_{12}m_{14}m_{31} \\
&+ h^4z_4z_{26}m_{14}m_{32} + h^5z_4z_{35}m_{14}m_{33} + hz_5m_{11}m_{21} + h^2z_2z_5m_{12}m_{21} + h^3z_5z_6m_{13}m_{21} + h^4z_5z_{18}m_{13}m_{22} \\
&+ h^4z_5z_{12}m_{14}m_{21} + h^5z_5z_{26}m_{14}m_{22} + h^6z_5z_{35}m_{14}m_{23} + h^2z_6m_{11}^2 + h^3z_2z_6m_{12}m_{11} + h^4z_6^2m_{13}m_{11} \\
&+ h^5z_6z_{18}m_{13}m_{12} + h^5z_6z_{12}m_{14}m_{11} + h^6z_6z_{26}m_{14}m_{12} + h^7z_6z_{35}m_{14}m_{13} \\
\alpha_1\alpha_2\ :\ & m_{31}m_{12} + z_1m_{32}m_{11} + hz_5m_{33}m_{11} + h^2z_{17}m_{33}m_{12} + h^2z_{11}m_{34}m_{11} + h^3z_{25}m_{34}m_{12} + h^4z_{34}m_{34}m_{13} \\
&= z_3m_{21}m_{22} + z_1z_3m_{22}m_{21} + hz_3z_5m_{23}m_{21} + h^2z_3z_{17}m_{23}m_{22} + h^2z_3z_{11}m_{24}m_{21} + h^3z_3z_{25}m_{24}m_{22} \\
&+ h^4z_3z_{34}m_{24}m_{23} + z_4m_{11}m_{32} + z_1z_4m_{12}m_{31} + hz_4z_5m_{13}m_{31} + h^2z_4z_{17}m_{13}m_{32} + h^2z_4z_{11}m_{14}m_{31} \\
&+ h^3z_4z_{25}m_{14}m_{32} + h^4z_4z_{34}m_{14}m_{33} + hz_5m_{11}m_{22} + hz_1z_5m_{12}m_{21} + h^2z_5^2m_{13}m_{21} + h^3z_5z_{17}m_{13}m_{22} \\
&+ h^3z_5z_{11}m_{14}m_{21} + h^4z_5z_{25}m_{14}m_{22} + h^5z_5z_{34}m_{14}m_{23} + h^2z_6m_{11}m_{12} + h^2z_1z_6m_{12}m_{11} + h^3z_5z_6m_{13}m_{11} \\
&+ h^4z_6z_{17}m_{13}m_{12} + h^4z_6z_{11}m_{14}m_{11} + h^5z_6z_{25}m_{14}m_{12} + h^6z_6z_{34}m_{14}m_{13} \\
\alpha_1\alpha_3\ :\ & m_{31}m_{13} + z_4m_{33}m_{11} + hz_{16}m_{33}m_{12} + hz_{10}m_{34}m_{11} + h^2z_{24}m_{34}m_{12} + h^3z_{33}m_{34}m_{13} \\
&= z_3m_{21}m_{23} + z_3z_4m_{23}m_{21} + hz_3z_{16}m_{23}m_{22} + hz_3z_{10}m_{24}m_{21} + h^2z_3z_{24}m_{24}m_{22} + h^3z_3z_{33}m_{24}m_{23} \\
&+ z_4m_{11}m_{33} + z_4^2m_{13}m_{31} + hz_4z_{16}m_{13}m_{32} + hz_4z_{10}m_{14}m_{31} + h^2z_4z_{24}m_{14}m_{32} + h^3z_4z_{33}m_{14}m_{33} \\
&+ hz_5m_{11}m_{23} + hz_4z_5m_{13}m_{21} + h^2z_5z_{16}m_{13}m_{22} + h^2z_5z_{10}m_{14}m_{21} + h^3z_5z_{24}m_{14}m_{22} + h^4z_5z_{33}m_{14}m_{23} \\
&+ h^2z_6m_{11}m_{13} + h^2z_4z_6m_{13}m_{11} + h^3z_6z_{16}m_{13}m_{12} + h^3z_6z_{10}m_{14}m_{11} + h^4z_6z_{24}m_{14}m_{12} + h^5z_6z_{33}m_{14}m_{13} \\
\alpha_1\alpha_4\ :\ & m_{31}m_{14} + z_{14}m_{33}m_{12} + z_8m_{34}m_{11} + hz_{22}m_{34}m_{12} + h^2z_{31}m_{34}m_{13} \\
&= z_3m_{21}m_{24} + z_3z_{14}m_{23}m_{22} + z_3z_8m_{24}m_{21} + hz_3z_{22}m_{24}m_{22} + h^2z_3z_{31}m_{24}m_{23} + z_4m_{11}m_{34} \\
&+ z_4z_{14}m_{13}m_{32} + z_4z_8m_{14}m_{31} + hz_4z_{22}m_{14}m_{32} + h^2z_4z_{31}m_{14}m_{33} + hz_5m_{11}m_{24} + hz_5z_{14}m_{13}m_{22} \\
&+ hz_5z_8m_{14}m_{21} + h^2z_5z_{22}m_{14}m_{22} + h^3z_5z_{31}m_{14}m_{23} + h^2z_6m_{11}m_{14} + h^2z_6z_{14}m_{13}m_{12} + h^2z_6z_8m_{14}m_{11} \\
&+ h^3z_6z_{22}m_{14}m_{12} + h^4z_6z_{31}m_{14}m_{13}
\end{aligned}$$

$$\alpha_2^2 : m_{32}m_{12} + z_3 m_{33}m_{11} + hz_{15}m_{33}m_{12} + hz_9 m_{34}m_{11} + h^2 z_{23}m_{34}m_{12} + h^3 z_{32}m_{34}m_{13}$$

$$= z_3 m_{22}^2 + z_3^2 m_{23}m_{21} + hz_3 z_{15}m_{23}m_{22} + hz_3 z_9 m_{24}m_{21} + h^2 z_3 z_{23}m_{24}m_{22} + h^3 z_3 z_{32}m_{24}m_{23}$$

$$+ z_4 m_{12}m_{32} + z_3 z_4 m_{13}m_{31} + hz_4 z_{15}m_{13}m_{32} + hz_4 z_9 m_{14}m_{31} + h^2 z_4 z_{23}m_{14}m_{32} + h^3 z_4 z_{32}m_{14}m_{33}$$

$$+ hz_5 m_{12}m_{22} + hz_3 z_5 m_{13}m_{21} + h^2 z_5 z_{15}m_{13}m_{22} + h^2 z_5 z_9 m_{14}m_{21} + h^3 z_5 z_{23}m_{14}m_{22} + h^4 z_5 z_{32}m_{14}m_{23}$$

$$+ h^2 z_6 m_{12}^2 + h^2 z_3 z_6 m_{13}m_{11} + h^3 z_6 z_{15}m_{13}m_{12} + h^3 z_6 z_9 m_{14}m_{11} + h^4 z_6 z_{23}m_{14}m_{12} + h^5 z_6 z_{32}m_{14}m_{13}$$

$$\alpha_2\alpha_3 : m_{32}m_{13} + z_{13}m_{33}m_{12} + z_7 m_{34}m_{11} + hz_{21}m_{34}m_{12} + h^2 z_{30}m_{34}m_{13}$$

$$= z_3 m_{22}m_{23} + z_3 z_{13}m_{23}m_{22} + z_3 z_7 m_{24}m_{21} + hz_3 z_{21}m_{24}m_{22} + h^2 z_3 z_{30}m_{24}m_{23} + z_4 m_{12}m_{33}$$

$$+ z_4 z_{13}m_{13}m_{32} + z_4 z_7 m_{14}m_{31} + hz_4 z_{21}m_{14}m_{32} + h^2 z_4 z_{30}m_{14}m_{33} + hz_5 m_{12}m_{23} + hz_5 z_{13}m_{13}m_{22}$$

$$+ hz_5 z_7 m_{14}m_{21} + h^2 z_5 z_{21}m_{14}m_{22} + h^3 z_5 z_{30}m_{14}m_{23} + h^2 z_6 m_{12}m_{13} + h^2 z_6 z_{13}m_{13}m_{12} + h^2 z_6 z_7 m_{14}m_{11}$$

$$+ h^3 z_6 z_{21}m_{14}m_{12} + h^4 z_6 z_{30}m_{14}m_{13}$$

$$\alpha_2\alpha_4 : m_{32}m_{14} + z_{20}m_{34}m_{12} + hz_{29}m_{34}m_{13}$$

$$= z_3 m_{22}m_{24} + z_3 z_{20}m_{24}m_{22} + hz_3 z_{29}m_{24}m_{23} + z_4 m_{12}m_{34} + z_4 z_{20}m_{14}m_{32} + hz_4 z_{29}m_{14}m_{33}$$

$$+ hz_5 m_{12}m_{24} + hz_5 z_{20}m_{14}m_{22} + h^2 z_5 z_{29}m_{14}m_{23} + h^2 z_6 m_{12}m_{14} + h^2 z_6 z_{20}m_{14}m_{12} + h^3 z_6 z_{29}m_{14}m_{13}$$

$$\alpha_3^2 : m_{33}m_{13} + z_{19}m_{34}m_{12} + hz_{28}m_{34}m_{13}$$

$$= z_3 m_{23}^2 + z_3 z_{19}m_{24}m_{22} + hz_3 z_{28}m_{24}m_{23} + z_4 m_{13}m_{33} + z_4 z_{19}m_{14}m_{32} + hz_4 z_{28}m_{14}m_{33}$$

$$+ hz_5 m_{13}m_{23} + hz_5 z_{19}m_{14}m_{22} + h^2 z_5 z_{28}m_{14}m_{23} + h^2 z_6 m_{13}^2 + h^2 z_6 z_{19}m_{14}m_{12} + h^3 z_6 z_{28}m_{14}m_{13}$$

$$\alpha_3\alpha_4 : m_{33}m_{14} + z_{27}m_{34}m_{13}$$

$$= z_3 m_{23}m_{24} + z_3 z_{27}m_{24}m_{23} + z_4 m_{13}m_{34} + z_4 z_{27}m_{14}m_{33} + hz_5 m_{13}m_{24} + hz_5 z_{27}m_{14}m_{23}$$

$$+ h^2 z_6 m_{13}m_{14} + h^2 z_6 z_{27}m_{14}m_{13}$$

$$\alpha_4^2 : m_{34}m_{14} = z_3 m_{24}^2 + z_4 m_{14}m_{34} + hz_5 m_{14}m_{24} + h^2 z_6 m_{14}^2$$

$$\alpha_4'\alpha_1' = z_7\alpha_2'\alpha_3' + z_8\alpha_1'\alpha_4' + h(z_9\alpha_2'^2 + z_{10}\alpha_1'\alpha_3') + h^2z_{11}\alpha_1'\alpha_2' + h^3z_{12}\alpha_1'^2$$

(A.15)

$$\begin{aligned}
\alpha_1^2 \ :\ & m_{41}m_{11} + hz_2m_{42}m_{11} + h^2z_6m_{43}m_{11} + h^3z_{18}m_{43}m_{12} + h^3z_{12}m_{44}m_{11} + h^4z_{26}m_{44}m_{12} + h^5z_{35}m_{44}m_{13} \\
&= z_7m_{21}m_{31} + hz_2z_7m_{22}m_{31} + h^2z_6z_7m_{23}m_{31} + h^3z_7z_{18}m_{23}m_{32} + h^3z_7z_{12}m_{24}m_{31} + h^4z_7z_{26}m_{24}m_{32} \\
&+h^5z_7z_{35}m_{24}m_{33} + z_8m_{11}m_{41} + hz_2z_8m_{12}m_{41} + h^2z_6z_8m_{13}m_{41} + h^3z_8z_{18}m_{13}m_{42} + h^3z_8z_{12}m_{14}m_{41} \\
&+h^4z_8z_{26}m_{14}m_{42} + h^5z_8z_{35}m_{14}m_{43} + hz_9m_{21}^2 + h^2z_2z_9m_{22}m_{21} + h^3z_6z_9m_{23}m_{21} + h^4z_9z_{18}m_{23}m_{22} \\
&+h^4z_9z_{12}m_{24}m_{21} + h^5z_9z_{26}m_{24}m_{22} + h^6z_9z_{35}m_{24}m_{23} + hz_{10}m_{11}m_{31} + h^2z_2z_{10}m_{12}m_{31} + h^3z_6z_{10}m_{13}m_{31} \\
&+h^4z_{10}z_{18}m_{13}m_{32} + h^4z_{10}z_{12}m_{14}m_{31} + h^5z_{10}z_{26}m_{14}m_{32} + h^6z_{10}z_{35}m_{14}m_{33} + h^2z_{11}m_{11}m_{21} + h^3z_2z_{11}m_{12}m_{21} \\
&+h^4z_6z_{11}m_{13}m_{21} + h^5z_{11}z_{18}m_{13}m_{22} + h^5z_{11}z_{12}m_{14}m_{21} + h^6z_{11}z_{26}m_{14}m_{22} + h^7z_{11}z_{35}m_{14}m_{23} + h^3z_{12}m_{11}^2 \\
&+h^4z_2z_{12}m_{12}m_{11} + h^5z_6z_{12}m_{13}m_{11} + h^6z_{12}z_{18}m_{13}m_{12} + h^6z_{12}^2m_{14}m_{11} + h^7z_{12}z_{26}m_{14}m_{12} + h^8z_{12}z_{35}m_{14}m_{13} \\
\alpha_1\alpha_2 \ :\ & m_{41}m_{12} + z_1m_{42}m_{11} + hz_5m_{43}m_{11} + h^2z_{17}m_{43}m_{12} + h^2z_{11}m_{44}m_{11} + h^3z_{25}m_{44}m_{12} + h^4z_{34}m_{44}m_{13} \\
&= z_7m_{21}m_{32} + z_1z_7m_{22}m_{31} + hz_5z_7m_{23}m_{31} + h^2z_7z_{17}m_{23}m_{32} + h^2z_7z_{11}m_{24}m_{31} + h^3z_7z_{25}m_{24}m_{32} \\
&+h^4z_7z_{34}m_{24}m_{33} + z_8m_{11}m_{42} + z_1z_8m_{12}m_{41} + hz_5z_8m_{13}m_{41} + h^2z_8z_{17}m_{13}m_{42} + h^2z_8z_{11}m_{14}m_{41} \\
&+h^3z_8z_{25}m_{14}m_{42} + h^4z_8z_{34}m_{14}m_{43} + hz_9m_{21}m_{22} + hz_1z_9m_{22}m_{21} + h^2z_5z_9m_{23}m_{21} + h^3z_9z_{17}m_{23}m_{22} \\
&+h^3z_9z_{11}m_{24}m_{21} + h^4z_9z_{25}m_{24}m_{22} + h^5z_9z_{34}m_{24}m_{23} + hz_{10}m_{11}m_{32} + hz_1z_{10}m_{12}m_{31} + h^2z_5z_{10}m_{13}m_{31} \\
&+h^3z_{10}z_{17}m_{13}m_{32} + h^3z_{10}z_{11}m_{14}m_{31} + h^4z_{10}z_{25}m_{14}m_{32} + h^5z_{10}z_{34}m_{14}m_{33} + h^2z_{11}m_{11}m_{22} + h^2z_1z_{11}m_{12}m_{21} \\
&+h^3z_5z_{11}m_{13}m_{21} + h^4z_{11}z_{17}m_{13}m_{22} + h^4z_{11}^2m_{14}m_{21} + h^5z_{11}z_{25}m_{14}m_{22} + h^6z_{11}z_{34}m_{14}m_{23} + h^3z_{12}m_{11}m_{12} \\
&+h^3z_1z_{12}m_{12}m_{11} + h^4z_5z_{12}m_{13}m_{11} + h^5z_{12}z_{17}m_{13}m_{12} + h^5z_{11}z_{12}m_{14}m_{11} + h^6z_{12}z_{25}m_{14}m_{12} + h^7z_{12}z_{34}m_{14}m_{13} \\
\alpha_1\alpha_3 \ :\ & m_{41}m_{13} + z_4m_{43}m_{11} + hz_{16}m_{43}m_{12} + hz_{10}m_{44}m_{11} + h^2z_{24}m_{44}m_{12} + h^3z_{33}m_{44}m_{13} \\
&= z_7m_{21}m_{33} + z_4z_7m_{23}m_{31} + hz_7z_{16}m_{23}m_{32} + hz_7z_{10}m_{24}m_{31} + h^2z_7z_{24}m_{24}m_{32} + h^3z_7z_{33}m_{24}m_{33} \\
&+z_8m_{11}m_{43} + z_4z_8m_{13}m_{41} + hz_8z_{16}m_{13}m_{42} + hz_8z_{10}m_{14}m_{41} + h^2z_8z_{24}m_{14}m_{42} + h^3z_8z_{33}m_{14}m_{43} \\
&+hz_9m_{21}m_{23} + hz_4z_9m_{23}m_{21} + h^2z_9z_{16}m_{23}m_{22} + h^2z_9z_{10}m_{24}m_{21} + h^3z_9z_{24}m_{24}m_{22} + h^4z_9z_{33}m_{24}m_{23} \\
&+hz_{10}m_{11}m_{33} + hz_4z_{10}m_{13}m_{31} + h^2z_{10}z_{16}m_{13}m_{32} + h^2z_{10}^2m_{14}m_{31} + h^3z_{10}z_{24}m_{14}m_{32} + h^4z_{10}z_{33}m_{14}m_{33} \\
&+h^2z_{11}m_{11}m_{23} + h^2z_4z_{11}m_{13}m_{21} + h^3z_{11}z_{16}m_{13}m_{22} + h^3z_{10}z_{11}m_{14}m_{21} + h^4z_{11}z_{24}m_{14}m_{22} + h^5z_{11}z_{33}m_{14}m_{23} \\
&+h^3z_{12}m_{11}m_{13} + h^3z_4z_{12}m_{13}m_{11} + h^4z_{12}z_{16}m_{13}m_{12} + h^4z_{10}z_{12}m_{14}m_{11} + h^5z_{12}z_{24}m_{14}m_{12} + h^6z_{12}z_{33}m_{14}m_{13}
\end{aligned}$$

$$
\begin{aligned}
\alpha_1\alpha_4\ :\ & m_{41}m_{14} + z_{14}m_{43}m_{12} + z_8m_{44}m_{11} + hz_{22}m_{44}m_{12} + h^2z_{31}m_{44}m_{13} \\
&= z_7m_{21}m_{34} + z_7z_{14}m_{23}m_{32} + z_7z_8m_{24}m_{31} + hz_7z_{22}m_{24}m_{32} + h^2z_7z_{31}m_{24}m_{33} + z_8m_{11}m_{44} \\
&+ z_8z_{14}m_{13}m_{42} + z_8^2m_{14}m_{41} + hz_8z_{22}m_{14}m_{42} + h^2z_8z_{31}m_{14}m_{43} + hz_9m_{21}m_{24} + hz_9z_{14}m_{23}m_{22} \\
&+ hz_8z_9m_{24}m_{21} + h^2z_9z_{22}m_{24}m_{22} + h^3z_9z_{31}m_{24}m_{23} + hz_{10}m_{11}m_{34} + hz_{10}z_{14}m_{13}m_{32} + hz_8z_{10}m_{14}m_{31} \\
&+ h^2z_{10}z_{22}m_{14}m_{32} + h^3z_{10}z_{31}m_{14}m_{33} + h^2z_{11}m_{11}m_{24} + h^2z_{11}z_{14}m_{13}m_{22} + h^2z_8z_{11}m_{14}m_{21} + h^3z_{11}z_{22}m_{14}m_{22} \\
&+ h^4z_{11}z_{31}m_{14}m_{23} + h^3z_{12}m_{11}m_{14} + h^3z_{12}z_{14}m_{13}m_{12} + h^3z_8z_{12}m_{14}m_{11} + h^4z_{12}z_{22}m_{14}m_{12} + h^5z_{12}z_{31}m_{14}m_{13} \\
\alpha_2^2\ :\ & m_{42}m_{12} + z_3m_{43}m_{11} + hz_{15}m_{43}m_{12} + hz_9m_{44}m_{11} + h^2z_{23}m_{44}m_{12} + h^3z_{32}m_{44}m_{13} \\
&= z_7m_{22}m_{32} + z_3z_7m_{23}m_{31} + hz_7z_{15}m_{23}m_{32} + hz_7z_9m_{24}m_{31} + h^2z_7z_{23}m_{24}m_{32} + h^3z_7z_{32}m_{24}m_{33} \\
&+ z_8m_{12}m_{42} + z_3z_8m_{13}m_{41} + hz_8z_{15}m_{13}m_{42} + hz_8z_9m_{14}m_{41} + h^2z_8z_{23}m_{14}m_{42} + h^3z_8z_{32}m_{14}m_{43} \\
&+ hz_9m_{22}^2 + hz_3z_9m_{23}m_{21} + h^2z_9z_{15}m_{23}m_{22} + h^2z_9^2m_{24}m_{21} + h^3z_9z_{23}m_{24}m_{22} + h^4z_9z_{32}m_{24}m_{23} \\
&+ hz_{10}m_{12}m_{32} + hz_3z_{10}m_{13}m_{31} + h^2z_{10}z_{15}m_{13}m_{32} + h^2z_9z_{10}m_{14}m_{31} + h^3z_{10}z_{23}m_{14}m_{32} + h^4z_{10}z_{32}m_{14}m_{33} \\
&+ h^2z_{11}m_{12}m_{22} + h^2z_3z_{11}m_{13}m_{21} + h^3z_{11}z_{15}m_{13}m_{22} + h^3z_9z_{11}m_{14}m_{21} + h^4z_{11}z_{23}m_{14}m_{22} + h^5z_{11}z_{32}m_{14}m_{23} \\
&+ h^3z_{12}m_{12}^2 + h^3z_3z_{12}m_{13}m_{11} + h^4z_{12}z_{15}m_{13}m_{12} + h^4z_9z_{12}m_{14}m_{11} + h^5z_{12}z_{23}m_{14}m_{12} + h^6z_{12}z_{32}m_{14}m_{13} \\
\alpha_2\alpha_3\ :\ & m_{42}m_{13} + z_{13}m_{43}m_{12} + z_7m_{44}m_{11} + hz_{21}m_{44}m_{12} + h^2z_{30}m_{44}m_{13} \\
&= z_7m_{22}m_{33} + z_7z_{13}m_{23}m_{32} + z_7^2m_{24}m_{31} + hz_7z_{21}m_{24}m_{32} + h^2z_7z_{30}m_{24}m_{33} + z_8m_{12}m_{43} \\
&+ z_8z_{13}m_{13}m_{42} + z_7z_8m_{14}m_{41} + hz_8z_{21}m_{14}m_{42} + h^2z_8z_{30}m_{14}m_{43} + hz_9m_{22}m_{23} + hz_9z_{13}m_{23}m_{22} \\
&+ hz_7z_9m_{24}m_{21} + h^2z_9z_{21}m_{24}m_{22} + h^3z_9z_{30}m_{24}m_{23} + hz_{10}m_{12}m_{33} + hz_{10}z_{13}m_{13}m_{32} + hz_7z_{10}m_{14}m_{31} \\
&+ h^2z_{10}z_{21}m_{14}m_{32} + h^3z_{10}z_{30}m_{14}m_{33} + h^2z_{11}m_{12}m_{23} + h^2z_{11}z_{13}m_{13}m_{22} + h^2z_7z_{11}m_{14}m_{21} + h^3z_{11}z_{21}m_{14}m_{22} \\
&+ h^4z_{11}z_{30}m_{14}m_{23} + h^3z_{12}m_{12}m_{13} + h^3z_{12}z_{13}m_{13}m_{12} + h^3z_7z_{12}m_{14}m_{11} + h^4z_{12}z_{21}m_{14}m_{12} + h^5z_{12}z_{30}m_{14}m_{13} \\
\alpha_2\alpha_4\ :\ & m_{42}m_{14} + z_{20}m_{44}m_{12} + hz_{29}m_{44}m_{13} \\
&= z_7m_{22}m_{34} + z_7z_{20}m_{24}m_{32} + hz_7z_{29}m_{24}m_{33} + z_8m_{12}m_{44} + z_8z_{20}m_{14}m_{42} + hz_8z_{29}m_{14}m_{43} \\
&+ hz_9m_{22}m_{24} + hz_9z_{20}m_{24}m_{22} + h^2z_9z_{29}m_{24}m_{23} + hz_{10}m_{12}m_{34} + hz_{10}z_{20}m_{14}m_{32} + h^2z_{10}z_{29}m_{14}m_{33} \\
&+ h^2z_{11}m_{12}m_{24} + h^2z_{11}z_{20}m_{14}m_{22} + h^3z_{11}z_{29}m_{14}m_{23} + h^3z_{12}m_{12}m_{14} + h^3z_{12}z_{20}m_{14}m_{12} + h^4z_{12}z_{29}m_{14}m_{13} \\
\alpha_3^2\ :\ & m_{43}m_{13} + z_{19}m_{44}m_{12} + hz_{28}m_{44}m_{13} \\
&= z_7m_{23}m_{33} + z_7z_{19}m_{24}m_{32} + hz_7z_{28}m_{24}m_{33} + z_8m_{13}m_{43} + z_8z_{19}m_{14}m_{42} + hz_8z_{28}m_{14}m_{43} \\
&+ hz_9m_{23}^2 + hz_9z_{19}m_{24}m_{22} + h^2z_9z_{28}m_{24}m_{23} + hz_{10}m_{13}m_{33} + hz_{10}z_{19}m_{14}m_{32} + h^2z_{10}z_{28}m_{14}m_{33} \\
&+ h^2z_{11}m_{13}m_{23} + h^2z_{11}z_{19}m_{14}m_{22} + h^3z_{11}z_{28}m_{14}m_{23} + h^3z_{12}m_{13}^2 + h^3z_{12}z_{19}m_{14}m_{12} + h^4z_{12}z_{28}m_{14}m_{13}
\end{aligned}
$$

$\alpha_3\alpha_4 \; : m_{43}m_{14} + z_{27}m_{44}m_{13}$

$$= z_7 m_{23}m_{34} + z_7 z_{27} m_{24} m_{33} + z_8 m_{13} m_{44} + z_8 z_{27} m_{14} m_{43} + h z_9 m_{23} m_{24} + h z_9 z_{27} m_{24} m_{23}$$

$$+ h z_{10} m_{13} m_{34} + h z_{10} z_{27} m_{14} m_{33} + h^2 z_{11} m_{13} m_{24} + h^2 z_{11} z_{27} m_{14} m_{23} + h^3 z_{12} m_{13} m_{14} + h^3 z_{12} z_{27} m_{14} m_{13}$$

$\alpha_4^2 \; : m_{44}m_{14} = z_7 m_{24} m_{34} + z_8 m_{14} m_{44} + h z_9 m_{24}^2 + h z_{10} m_{14} m_{34} + h^2 z_{11} m_{14} m_{24} + h^3 z_{12} m_{14}^2$

$$\alpha_3'\alpha_2' = z_{13}\alpha_2'\alpha_3' + z_{14}\alpha_1'\alpha_4' + h(z_{15}\alpha_2'^2 + z_{16}\alpha_1'\alpha_3') + h^2 z_{17}\alpha_1'\alpha_2' + h^3 z_{18}\alpha_1'^2$$

(A.16)

$$\begin{aligned}
\alpha_1^2 \ : \ & m_{31}m_{21} + hz_2m_{32}m_{21} + h^2z_6m_{33}m_{21} + h^3z_{18}m_{33}m_{22} + h^3z_{12}m_{34}m_{21} + h^4z_{26}m_{34}m_{22} + h^5z_{35}m_{34}m_{23} \\
&= z_{13}m_{21}m_{31} + hz_2z_{13}m_{22}m_{31} + h^2z_6z_{13}m_{23}m_{31} + h^3z_{13}z_{18}m_{23}m_{32} + h^3z_{12}z_{13}m_{24}m_{31} + h^4z_{13}z_{26}m_{24}m_{32} \\
&+ h^5z_{13}z_{35}m_{24}m_{33} + z_{14}m_{11}m_{41} + hz_2z_{14}m_{12}m_{41} + h^2z_6z_{14}m_{13}m_{41} + h^3z_{14}z_{18}m_{13}m_{42} + h^3z_{12}z_{14}m_{14}m_{41} \\
&+ h^4z_{14}z_{26}m_{14}m_{42} + h^5z_{14}z_{35}m_{14}m_{43} + hz_{15}m_{21}^2 + h^2z_2z_{15}m_{22}m_{21} + h^3z_6z_{15}m_{23}m_{21} + h^4z_{15}z_{18}m_{23}m_{22} \\
&+ h^4z_{12}z_{15}m_{24}m_{21} + h^5z_{15}z_{26}m_{24}m_{22} + h^6z_{15}z_{35}m_{24}m_{23} + hz_{16}m_{11}m_{31} + h^2z_2z_{16}m_{12}m_{31} + h^3z_6z_{16}m_{13}m_{31} \\
&+ h^4z_{16}z_{18}m_{13}m_{32} + h^4z_{12}z_{16}m_{14}m_{31} + h^5z_{16}z_{26}m_{14}m_{32} + h^6z_{16}z_{35}m_{14}m_{33} + h^2z_{17}m_{11}m_{21} + h^3z_2z_{17}m_{12}m_{21} \\
&+ h^4z_6z_{17}m_{13}m_{21} + h^5z_{17}z_{18}m_{13}m_{22} + h^5z_{12}z_{17}m_{14}m_{21} + h^6z_{17}z_{26}m_{14}m_{22} + h^7z_{17}z_{35}m_{14}m_{23} + h^3z_{18}m_{11}^2 \\
&+ h^4z_2z_{18}m_{12}m_{11} + h^5z_6z_{18}m_{13}m_{11} + h^6z_{18}^2m_{13}m_{12} + h^6z_{12}z_{18}m_{14}m_{11} + h^7z_{18}z_{26}m_{14}m_{12} + h^8z_{18}z_{35}m_{14}m_{13}
\end{aligned}$$

$$\begin{aligned}
\alpha_1\alpha_2 \ : \ & m_{31}m_{22} + z_1m_{32}m_{21} + hz_5m_{33}m_{21} + h^2z_{17}m_{33}m_{22} + h^2z_{11}m_{34}m_{21} + h^3z_{25}m_{34}m_{22} + h^4z_{34}m_{34}m_{23} \\
&= z_{13}m_{21}m_{32} + z_1z_{13}m_{22}m_{31} + hz_5z_{13}m_{23}m_{31} + h^2z_{13}z_{17}m_{23}m_{32} + h^2z_{11}z_{13}m_{24}m_{31} + h^3z_{13}z_{25}m_{24}m_{32} \\
&+ h^4z_{13}z_{34}m_{24}m_{33} + z_{14}m_{11}m_{42} + z_1z_{14}m_{12}m_{41} + hz_5z_{14}m_{13}m_{41} + h^2z_{14}z_{17}m_{13}m_{42} + h^2z_{11}z_{14}m_{14}m_{41} \\
&+ h^3z_{14}z_{25}m_{14}m_{42} + h^4z_{14}z_{34}m_{14}m_{43} + hz_{15}m_{21}m_{22} + hz_1z_{15}m_{22}m_{21} + h^2z_5z_{15}m_{23}m_{21} + h^3z_{15}z_{17}m_{23}m_{22} \\
&+ h^3z_{11}z_{15}m_{24}m_{21} + h^4z_{15}z_{25}m_{24}m_{22} + h^5z_{15}z_{34}m_{24}m_{23} + hz_{16}m_{11}m_{32} + hz_1z_{16}m_{12}m_{31} + h^2z_5z_{16}m_{13}m_{31} \\
&+ h^3z_{16}z_{17}m_{13}m_{32} + h^3z_{11}z_{16}m_{14}m_{31} + h^4z_{16}z_{25}m_{14}m_{32} + h^5z_{16}z_{34}m_{14}m_{33} + h^2z_{17}m_{11}m_{22} + h^2z_1z_{17}m_{12}m_{21} \\
&+ h^3z_5z_{17}m_{13}m_{21} + h^4z_{17}^2m_{13}m_{22} + h^4z_{11}z_{17}m_{14}m_{21} + h^5z_{17}z_{25}m_{14}m_{22} + h^6z_{17}z_{34}m_{14}m_{23} + h^3z_{18}m_{11}m_{12} \\
&+ h^3z_1z_{18}m_{12}m_{11} + h^4z_5z_{18}m_{13}m_{11} + h^5z_{17}z_{18}m_{13}m_{12} + h^5z_{11}z_{18}m_{14}m_{11} + h^6z_{18}z_{25}m_{14}m_{12} + h^7z_{18}z_{34}m_{14}m_{13}
\end{aligned}$$

$$\begin{aligned}
\alpha_1\alpha_3 \ : \ & m_{31}m_{23} + z_4m_{33}m_{21} + hz_{16}m_{33}m_{22} + hz_{10}m_{34}m_{21} + h^2z_{24}m_{34}m_{22} + h^3z_{33}m_{34}m_{23} \\
&= z_{13}m_{21}m_{33} + z_4z_{13}m_{23}m_{31} + hz_{13}z_{16}m_{23}m_{32} + hz_{10}z_{13}m_{24}m_{31} + h^2z_{13}z_{24}m_{24}m_{32} + h^3z_{13}z_{33}m_{24}m_{33} \\
&+ z_{14}m_{11}m_{43} + z_4z_{14}m_{13}m_{41} + hz_{14}z_{16}m_{13}m_{42} + hz_{10}z_{14}m_{14}m_{41} + h^2z_{14}z_{24}m_{14}m_{42} + h^3z_{14}z_{33}m_{14}m_{43} \\
&+ hz_{15}m_{21}m_{23} + hz_4z_{15}m_{23}m_{21} + h^2z_{15}z_{16}m_{23}m_{22} + h^2z_{10}z_{15}m_{24}m_{21} + h^3z_{15}z_{24}m_{24}m_{22} + h^4z_{15}z_{33}m_{24}m_{23} \\
&+ hz_{16}m_{11}m_{33} + hz_4z_{16}m_{13}m_{31} + h^2z_{16}^2m_{13}m_{32} + h^2z_{10}z_{16}m_{14}m_{31} + h^3z_{16}z_{24}m_{14}m_{32} + h^4z_{16}z_{33}m_{14}m_{33} \\
&+ h^2z_{17}m_{11}m_{23} + h^2z_4z_{17}m_{13}m_{21} + h^3z_{16}z_{17}m_{13}m_{22} + h^3z_{10}z_{17}m_{14}m_{21} + h^4z_{17}z_{24}m_{14}m_{22} + h^5z_{17}z_{33}m_{14}m_{23} \\
&+ h^3z_{18}m_{11}m_{13} + h^3z_4z_{18}m_{13}m_{11} + h^4z_{16}z_{18}m_{13}m_{12} + h^4z_{10}z_{18}m_{14}m_{11} + h^5z_{18}z_{24}m_{14}m_{12} + h^6z_{18}z_{33}m_{14}m_{13}
\end{aligned}$$

$$\alpha_1\alpha_4 : m_{31}m_{24} + z_{14}m_{33}m_{22} + z_8m_{34}m_{21} + hz_{22}m_{34}m_{22} + h^2z_{31}m_{34}m_{23}$$

$$= z_{13}m_{21}m_{34} + z_{13}z_{14}m_{23}m_{32} + z_8z_{13}m_{24}m_{31} + hz_{13}z_{22}m_{24}m_{32} + h^2z_{13}z_{31}m_{24}m_{33} + z_{14}m_{11}m_{44}$$

$$+ z_{14}^2m_{13}m_{42} + z_8z_{14}m_{14}m_{41} + hz_{14}z_{22}m_{14}m_{42} + h^2z_{14}z_{31}m_{14}m_{43} + hz_{15}m_{21}m_{24} + hz_{14}z_{15}m_{23}m_{22}$$

$$+ hz_8z_{15}m_{24}m_{21} + h^2z_{15}z_{22}m_{24}m_{22} + h^3z_{15}z_{31}m_{24}m_{23} + hz_{16}m_{11}m_{34} + hz_{14}z_{16}m_{13}m_{32} + hz_8z_{16}m_{14}m_{31}$$

$$+ h^2z_{16}z_{22}m_{14}m_{32} + h^3z_{16}z_{31}m_{14}m_{33} + h^2z_{17}m_{11}m_{24} + h^2z_{14}z_{17}m_{13}m_{22} + h^2z_8z_{17}m_{14}m_{21} + h^3z_{17}z_{22}m_{14}m_{22}$$

$$+ h^4z_{17}z_{31}m_{14}m_{23} + h^3z_{18}m_{11}m_{14} + h^3z_{14}z_{18}m_{13}m_{12} + h^3z_8z_{18}m_{14}m_{11} + h^4z_{18}z_{22}m_{14}m_{12} + h^5z_{18}z_{31}m_{14}m_{13}$$

$$\alpha_2^2 : m_{32}m_{22} + z_3m_{33}m_{21} + hz_{15}m_{33}m_{22} + hz_9m_{34}m_{21} + h^2z_{23}m_{34}m_{22} + h^3z_{32}m_{34}m_{23}$$

$$= z_{13}m_{22}m_{32} + z_3z_{13}m_{23}m_{31} + hz_{13}z_{15}m_{23}m_{32} + hz_9z_{13}m_{24}m_{31} + h^2z_{13}z_{23}m_{24}m_{32} + h^3z_{13}z_{32}m_{24}m_{33}$$

$$+ z_{14}m_{12}m_{42} + z_3z_{14}m_{13}m_{41} + hz_{14}z_{15}m_{13}m_{42} + hz_9z_{14}m_{14}m_{41} + h^2z_{14}z_{23}m_{14}m_{42} + h^3z_{14}z_{32}m_{14}m_{43}$$

$$+ hz_{15}m_{22}^2 + hz_3z_{15}m_{23}m_{21} + h^2z_{15}^2m_{23}m_{22} + h^2z_9z_{15}m_{24}m_{21} + h^3z_{15}z_{23}m_{24}m_{22} + h^4z_{15}z_{32}m_{24}m_{23}$$

$$+ hz_{16}m_{12}m_{32} + hz_3z_{16}m_{13}m_{31} + h^2z_{15}z_{16}m_{13}m_{32} + h^2z_9z_{16}m_{14}m_{31} + h^3z_{16}z_{23}m_{14}m_{32} + h^4z_{16}z_{32}m_{14}m_{33}$$

$$+ h^2z_{17}m_{12}m_{22} + h^2z_3z_{17}m_{13}m_{21} + h^3z_{15}z_{17}m_{13}m_{22} + h^3z_9z_{17}m_{14}m_{21} + h^4z_{17}z_{23}m_{14}m_{22} + h^5z_{17}z_{32}m_{14}m_{23}$$

$$+ h^3z_{18}m_{12}^2 + h^3z_3z_{18}m_{13}m_{11} + h^4z_{15}z_{18}m_{13}m_{12} + h^4z_9z_{18}m_{14}m_{11} + h^5z_{18}z_{23}m_{14}m_{12} + h^6z_{18}z_{32}m_{14}m_{13}$$

$$\alpha_2\alpha_3 : m_{32}m_{23} + z_{13}m_{33}m_{22} + z_7m_{34}m_{21} + hz_{21}m_{34}m_{22} + h^2z_{30}m_{34}m_{23}$$

$$= z_{13}m_{22}m_{33} + z_{13}^2m_{23}m_{32} + z_7z_{13}m_{24}m_{31} + hz_{13}z_{21}m_{24}m_{32} + h^2z_{13}z_{30}m_{24}m_{33} + z_{14}m_{12}m_{43}$$

$$+ z_{13}z_{14}m_{13}m_{42} + z_7z_{14}m_{14}m_{41} + hz_{14}z_{21}m_{14}m_{42} + h^2z_{14}z_{30}m_{14}m_{43} + hz_{15}m_{22}m_{23} + hz_{13}z_{15}m_{23}m_{22}$$

$$+ hz_7z_{15}m_{24}m_{21} + h^2z_{15}z_{21}m_{24}m_{22} + h^3z_{15}z_{30}m_{24}m_{23} + hz_{16}m_{12}m_{33} + hz_{13}z_{16}m_{13}m_{32} + hz_7z_{16}m_{14}m_{31}$$

$$+ h^2z_{16}z_{21}m_{14}m_{32} + h^3z_{16}z_{30}m_{14}m_{33} + h^2z_{17}m_{12}m_{23} + h^2z_{13}z_{17}m_{13}m_{22} + h^2z_7z_{17}m_{14}m_{21} + h^3z_{17}z_{21}m_{14}m_{22}$$

$$+ h^4z_{17}z_{30}m_{14}m_{23} + h^3z_{18}m_{12}m_{13} + h^3z_{13}z_{18}m_{13}m_{12} + h^3z_7z_{18}m_{14}m_{11} + h^4z_{18}z_{21}m_{14}m_{12} + h^5z_{18}z_{30}m_{14}m_{13}$$

$$\alpha_2\alpha_4 : m_{32}m_{24} + z_{20}m_{34}m_{22} + hz_{29}m_{34}m_{23}$$

$$= z_{13}m_{22}m_{34} + z_{13}z_{20}m_{24}m_{32} + hz_{13}z_{29}m_{24}m_{33} + z_{14}m_{12}m_{44} + z_{14}z_{20}m_{14}m_{42} + hz_{14}z_{29}m_{14}m_{43}$$

$$+ hz_{15}m_{22}m_{24} + hz_{15}z_{20}m_{24}m_{22} + h^2z_{15}z_{29}m_{24}m_{23} + hz_{16}m_{12}m_{34} + hz_{16}z_{20}m_{14}m_{32} + h^2z_{16}z_{29}m_{14}m_{33}$$

$$+ h^2z_{17}m_{12}m_{24} + h^2z_{17}z_{20}m_{14}m_{22} + h^3z_{17}z_{29}m_{14}m_{23} + h^3z_{17}m_{12}m_{14} + h^3z_{17}z_{20}m_{14}m_{12} + h^4z_{17}z_{29}m_{14}m_{13}$$

$$\alpha_3^2 : m_{33}m_{23} + z_{19}m_{34}m_{22} + hz_{28}m_{34}m_{23}$$

$$= z_{13}m_{23}m_{33} + z_{13}z_{19}m_{24}m_{32} + hz_{13}z_{28}m_{24}m_{33} + z_{14}m_{13}m_{43} + z_{14}z_{19}m_{14}m_{42} + hz_{14}z_{28}m_{14}m_{43}$$

$$+ hz_{15}m_{23}^2 + hz_{15}z_{19}m_{24}m_{22} + h^2z_{15}z_{28}m_{24}m_{23} + hz_{16}m_{13}m_{33} + hz_{16}z_{19}m_{14}m_{32} + h^2z_{16}z_{28}m_{14}m_{33}$$

$$+ h^2z_{17}m_{13}m_{23} + h^2z_{17}z_{19}m_{14}m_{22} + h^3z_{17}z_{28}m_{14}m_{23} + h^3z_{18}m_{13}^2 + h^3z_{18}z_{19}m_{14}m_{12} + h^4z_{18}z_{28}m_{14}m_{13}$$

$\alpha_3\alpha_4\ : m_{33}m_{24} + z_{27}m_{34}m_{23}$

$= z_{13}m_{23}m_{34} + z_{13}z_{27}m_{24}m_{33} + z_{14}m_{13}m_{44} + z_{14}z_{27}m_{14}m_{43} + hz_{15}m_{23}m_{24} + hz_{15}z_{27}m_{24}m_{23}$

$+hz_{16}m_{13}m_{34} + hz_{16}z_{27}m_{14}m_{33} + h^2z_{17}m_{13}m_{24} + h^2z_{17}z_{27}m_{14}m_{23} + h^3z_{18}m_{13}m_{14} + h^3z_{18}z_{27}m_{14}m_{13}$

$\alpha_4^2\ : m_{34}m_{24} = z_{13}m_{24}m_{34} + z_{14}m_{14}m_{44} + hz_{15}m_{24}^2 + hz_{16}m_{14}m_{34} + h^2z_{17}m_{14}m_{24} + h^3z_{18}m_{14}^2$

$$\alpha_4'\alpha_2' = z_{19}\alpha_3'^2 + z_{20}\alpha_2'\alpha_4' + h(z_{21}\alpha_2'\alpha_3' + z_{22}\alpha_1'\alpha_4') + h^2(z_{23}\alpha_2'^2 + z_{24}\alpha_1'\alpha_3') + h^3 z_{25}\alpha_1'\alpha_2' + h^4 z_{26}\alpha_1'^2$$

(A.17)

$$\begin{aligned}
\alpha_1^2 \ : \ & m_{41}m_{21} + hz_2m_{42}m_{21} + h^2z_6m_{43}m_{21} + h^3z_{18}m_{43}m_{22} + h^3z_{12}m_{44}m_{21} + h^4z_{26}m_{44}m_{22} + h^5z_{35}m_{44}m_{23} \\
& = z_{19}m_{31}^2 + hz_2z_{19}m_{32}m_{31} + h^2z_6z_{19}m_{33}m_{31} + h^3z_{18}z_{19}m_{33}m_{32} + h^3z_{12}z_{19}m_{34}m_{31} + h^4z_{19}z_{26}m_{34}m_{32} \\
& + h^5z_{19}z_{35}m_{34}m_{33} + z_{20}m_{21}m_{41} + hz_2z_{20}m_{22}m_{41} + h^2z_6z_{20}m_{23}m_{41} + h^3z_{18}z_{20}m_{23}m_{42} + h^3z_{12}z_{20}m_{24}m_{41} \\
& + h^4z_{20}z_{26}m_{24}m_{42} + h^5z_{20}z_{35}m_{24}m_{43} + hz_{21}m_{21}m_{31} + h^2z_2z_{21}m_{22}m_{31} + h^3z_6z_{21}m_{23}m_{31} + h^4z_{18}z_{21}m_{23}m_{32} \\
& + h^4z_{12}z_{21}m_{24}m_{31} + h^5z_{21}z_{26}m_{24}m_{32} + h^6z_{21}z_{35}m_{24}m_{33} + hz_{22}m_{11}m_{41} + h^2z_2z_{22}m_{12}m_{41} + h^3z_6z_{22}m_{13}m_{41} \\
& + h^4z_{18}z_{22}m_{13}m_{42} + h^4z_{12}z_{22}m_{14}m_{41} + h^5z_{22}z_{26}m_{14}m_{42} + h^6z_{22}z_{35}m_{14}m_{43} + h^2z_{23}m_{21}^2 + h^3z_2z_{23}m_{22}m_{21} \\
& + h^4z_6z_{23}m_{23}m_{21} + h^5z_{18}z_{23}m_{23}m_{22} + h^5z_{12}z_{23}m_{24}m_{21} + h^6z_{23}z_{26}m_{24}m_{22} + h^7z_{23}z_{35}m_{24}m_{23} + h^2z_{24}m_{11}m_{31} \\
& + h^3z_2z_{24}m_{12}m_{31} + h^4z_6z_{24}m_{13}m_{31} + h^5z_{18}z_{24}m_{13}m_{32} + h^5z_{12}z_{24}m_{14}m_{31} + h^6z_{24}z_{26}m_{14}m_{32} + h^7z_{24}z_{35}m_{14}m_{33} \\
& + h^3z_{25}m_{11}m_{21} + h^4z_2z_{25}m_{12}m_{21} + h^5z_6z_{25}m_{13}m_{21} + h^6z_{18}z_{25}m_{13}m_{22} + h^6z_{12}z_{25}m_{14}m_{21} + h^7z_{25}z_{26}m_{14}m_{22} \\
& + h^8z_{25}z_{35}m_{14}m_{23} + h^4z_{26}m_{11}^2 + h^5z_2z_{26}m_{12}m_{11} + h^6z_6z_{26}m_{13}m_{11} + h^7z_{18}z_{26}m_{13}m_{12} + h^7z_{12}z_{26}m_{14}m_{11} \\
& + h^8z_{26}^2m_{14}m_{12} + h^9z_{26}z_{35}m_{14}m_{13}
\end{aligned}$$

$$\begin{aligned}
\alpha_1\alpha_2 \ : \ & m_{41}m_{22} + z_1m_{42}m_{21} + hz_5m_{43}m_{21} + h^2z_{17}m_{43}m_{22} + h^2z_{11}m_{44}m_{21} + h^3z_{25}m_{44}m_{22} + h^4z_{34}m_{44}m_{23} \\
& = z_{19}m_{31}m_{32} + z_1z_{19}m_{32}m_{31} + hz_5z_{19}m_{33}m_{31} + h^2z_{17}z_{19}m_{33}m_{32} + h^2z_{11}z_{19}m_{34}m_{31} + h^3z_{19}z_{25}m_{34}m_{32} \\
& + h^4z_{19}z_{34}m_{34}m_{33} + z_{20}m_{21}m_{42} + z_1z_{20}m_{22}m_{41} + hz_5z_{20}m_{23}m_{41} + h^2z_{17}z_{20}m_{23}m_{42} + h^2z_{11}z_{20}m_{24}m_{41} \\
& + h^3z_{20}z_{25}m_{24}m_{42} + h^4z_{20}z_{34}m_{24}m_{43} + hz_{21}m_{21}m_{32} + hz_1z_{21}m_{22}m_{31} + h^2z_5z_{21}m_{23}m_{31} + h^3z_{17}z_{21}m_{23}m_{32} \\
& + h^3z_{11}z_{21}m_{24}m_{31} + h^4z_{21}z_{25}m_{24}m_{32} + h^5z_{21}z_{34}m_{24}m_{33} + hz_{22}m_{11}m_{42} + hz_1z_{22}m_{12}m_{41} + h^2z_5z_{22}m_{13}m_{41} \\
& + h^3z_{17}z_{22}m_{13}m_{42} + h^3z_{11}z_{22}m_{14}m_{41} + h^4z_{22}z_{25}m_{14}m_{42} + h^5z_{22}z_{34}m_{14}m_{43} + h^2z_{23}m_{21}m_{22} + h^2z_1z_{23}m_{22}m_{21} \\
& + h^3z_5z_{23}m_{23}m_{21} + h^4z_{17}z_{23}m_{23}m_{22} + h^4z_{11}z_{23}m_{24}m_{21} + h^5z_{23}z_{25}m_{24}m_{22} + h^6z_{23}z_{34}m_{24}m_{23} + h^2z_{24}m_{11}m_{32} \\
& + h^2z_1z_{24}m_{12}m_{31} + h^3z_5z_{24}m_{13}m_{31} + h^4z_{17}z_{24}m_{13}m_{32} + h^4z_{11}z_{24}m_{14}m_{31} + h^5z_{24}z_{25}m_{14}m_{32} + h^6z_{24}z_{34}m_{14}m_{33} \\
& + h^3z_{25}m_{11}m_{22} + h^3z_1z_{25}m_{12}m_{21} + h^4z_5z_{25}m_{13}m_{21} + h^5z_{17}z_{25}m_{13}m_{22} + h^5z_{11}z_{25}m_{14}m_{21} + h^6z_{25}^2m_{14}m_{22} \\
& + h^7z_{25}z_{34}m_{14}m_{23} + h^4z_{26}m_{11}m_{12} + h^4z_1z_{26}m_{12}m_{11} + h^5z_5z_{26}m_{13}m_{11} + h^6z_{17}z_{26}m_{13}m_{12} + h^6z_{11}z_{26}m_{14}m_{11} \\
& + h^7z_{25}z_{26}m_{14}m_{12} + h^8z_{26}z_{34}m_{14}m_{13}
\end{aligned}$$

$$\alpha_1\alpha_3 : m_{41}m_{23} + z_4m_{43}m_{21} + hz_{16}m_{43}m_{22} + hz_{10}m_{44}m_{21} + h^2z_{24}m_{44}m_{22} + h^3z_{33}m_{44}m_{23}$$

$$= z_{19}m_{31}m_{33} + z_4z_{19}m_{33}m_{31} + hz_{16}z_{19}m_{33}m_{32} + hz_{10}z_{19}m_{34}m_{31} + h^2z_{19}z_{24}m_{34}m_{32} + h^3z_{19}z_{33}m_{34}m_{33}$$

$$+ z_{20}m_{21}m_{43} + z_4z_{20}m_{23}m_{41} + hz_{16}z_{20}m_{23}m_{42} + hz_{10}z_{20}m_{24}m_{41} + h^2z_{20}z_{24}m_{24}m_{42} + h^3z_{20}z_{33}m_{24}m_{43}$$

$$+ hz_{21}m_{21}m_{33} + hz_4z_{21}m_{23}m_{31} + h^2z_{16}z_{21}m_{23}m_{32} + h^2z_{10}z_{21}m_{24}m_{31} + h^3z_{21}z_{24}m_{24}m_{32} + h^4z_{21}z_{33}m_{24}m_{33}$$

$$+ hz_{22}m_{11}m_{43} + hz_4z_{22}m_{13}m_{41} + h^2z_{16}z_{22}m_{13}m_{42} + h^2z_{10}z_{22}m_{14}m_{41} + h^3z_{22}z_{24}m_{14}m_{42} + h^4z_{22}z_{33}m_{14}m_{43}$$

$$+ h^2z_{23}m_{21}m_{23} + h^2z_4z_{23}m_{23}m_{21} + h^3z_{16}z_{23}m_{23}m_{22} + h^3z_{10}z_{23}m_{24}m_{21} + h^4z_{23}z_{24}m_{24}m_{22} + h^5z_{23}z_{33}m_{24}m_{23}$$

$$+ h^2z_{24}m_{11}m_{33} + h^2z_4z_{24}m_{13}m_{31} + h^3z_{16}z_{24}m_{13}m_{32} + h^3z_{10}z_{24}m_{14}m_{31} + h^4z_{24}^2m_{14}m_{32} + h^5z_{24}z_{33}m_{14}m_{33}$$

$$+ h^3z_{25}m_{11}m_{23} + h^3z_4z_{25}m_{13}m_{21} + h^4z_{16}z_{25}m_{13}m_{22} + h^4z_{10}z_{25}m_{14}m_{21} + h^5z_{24}z_{25}m_{14}m_{22} + h^6z_{25}z_{33}m_{14}m_{23}$$

$$+ h^4z_{26}m_{11}m_{13} + h^4z_4z_{26}m_{13}m_{11} + h^5z_{16}z_{26}m_{13}m_{12} + h^5z_{10}z_{26}m_{14}m_{11} + h^6z_{24}z_{26}m_{14}m_{12} + h^7z_{26}z_{33}m_{14}m_{13}$$

$$\alpha_1\alpha_4 : m_{41}m_{24} + z_{14}m_{43}m_{22} + z_8m_{44}m_{21} + hz_{22}m_{44}m_{22} + h^2z_{31}m_{44}m_{23}$$

$$= z_{19}m_{31}m_{34} + z_{14}z_{19}m_{33}m_{32} + z_8z_{19}m_{34}m_{31} + hz_{19}z_{22}m_{34}m_{32} + h^2z_{19}z_{31}m_{34}m_{33} + z_{20}m_{21}m_{44}$$

$$+ z_{14}z_{20}m_{23}m_{42} + z_8z_{20}m_{24}m_{41} + hz_{20}z_{22}m_{24}m_{42} + h^2z_{20}z_{31}m_{24}m_{43} + hz_{21}m_{21}m_{34} + hz_{14}z_{21}m_{23}m_{32}$$

$$+ hz_8z_{21}m_{24}m_{31} + h^2z_{21}z_{22}m_{24}m_{32} + h^3z_{21}z_{31}m_{24}m_{33} + hz_{22}m_{11}m_{44} + hz_{14}z_{22}m_{13}m_{42} + hz_8z_{22}m_{14}m_{41}$$

$$+ h^2z_{22}^2m_{14}m_{42} + h^3z_{22}z_{31}m_{14}m_{43} + h^2z_{23}m_{21}m_{24} + h^2z_{14}z_{23}m_{23}m_{22} + h^2z_8z_{23}m_{24}m_{21} + h^3z_{22}z_{23}m_{24}m_{22}$$

$$+ h^4z_{23}z_{31}m_{24}m_{23} + h^2z_{24}m_{11}m_{34} + h^2z_{14}z_{24}m_{13}m_{32} + h^2z_8z_{24}m_{14}m_{31} + h^3z_{22}z_{24}m_{14}m_{32} + h^3z_{24}z_{31}m_{14}m_{33}$$

$$+ h^3z_{25}m_{11}m_{24} + h^3z_{14}z_{25}m_{13}m_{22} + h^3z_8z_{25}m_{14}m_{21} + h^4z_{22}z_{25}m_{14}m_{22} + h^5z_{25}z_{31}m_{14}m_{23} + h^4z_{26}m_{11}m_{14}$$

$$+ h^4z_{14}z_{26}m_{13}m_{12} + h^4z_8z_{26}m_{14}m_{11} + h^5z_{22}z_{26}m_{14}m_{12} + h^6z_{26}z_{31}m_{14}m_{13}$$

$$\alpha_2^2 : m_{42}m_{22} + z_3m_{43}m_{21} + hz_{15}m_{43}m_{22} + hz_9m_{44}m_{21} + h^2z_{23}m_{44}m_{22} + h^3z_{32}m_{44}m_{23}$$

$$= z_{19}m_{32}^2 + z_3z_{19}m_{33}m_{31} + hz_{15}z_{19}m_{33}m_{32} + hz_9z_{19}m_{34}m_{31} + h^2z_{19}z_{23}m_{34}m_{32} + h^3z_{19}z_{32}m_{34}m_{33}$$

$$+ z_{20}m_{22}m_{42} + z_3z_{20}m_{23}m_{41} + hz_{15}z_{20}m_{23}m_{42} + hz_9z_{20}m_{24}m_{41} + h^2z_{20}z_{23}m_{24}m_{42} + h^3z_{20}z_{32}m_{24}m_{43}$$

$$+ hz_{21}m_{22}m_{32} + hz_3z_{21}m_{23}m_{31} + h^2z_{15}z_{21}m_{23}m_{32} + h^2z_9z_{21}m_{24}m_{31} + h^3z_{21}z_{23}m_{24}m_{32} + h^4z_{21}z_{32}m_{24}m_{33}$$

$$+ hz_{22}m_{12}m_{42} + hz_3z_{22}m_{13}m_{41} + h^2z_{15}z_{22}m_{13}m_{42} + h^2z_9z_{22}m_{14}m_{41} + h^3z_{22}z_{23}m_{14}m_{42} + h^4z_{22}z_{32}m_{14}m_{43}$$

$$+ h^2z_{23}m_{22}^2 + h^2z_3z_{23}m_{23}m_{21} + h^3z_{15}z_{23}m_{23}m_{22} + h^3z_9z_{23}m_{24}m_{21} + h^4z_{23}^2m_{24}m_{22} + h^5z_{23}z_{32}m_{24}m_{23}$$

$$+ h^2z_{24}m_{12}m_{32} + h^2z_3z_{24}m_{13}m_{31} + h^3z_{15}z_{24}m_{13}m_{32} + h^3z_9z_{24}m_{14}m_{31} + h^5z_{23}z_{24}m_{14}m_{32} + h^6z_{24}z_{32}m_{14}m_{33}$$

$$+ h^3z_{25}m_{12}m_{22} + h^3z_3z_{25}m_{13}m_{21} + h^4z_{15}z_{25}m_{13}m_{22} + h^4z_9z_{25}m_{14}m_{21} + h^5z_{23}z_{25}m_{14}m_{22} + h^6z_{25}z_{32}m_{14}m_{23}$$

$$+ h^4z_{26}m_{12}^2 + h^4z_3z_{26}m_{13}m_{11} + h^5z_{15}z_{26}m_{13}m_{12} + h^5z_9z_{26}m_{14}m_{11} + h^6z_{23}z_{26}m_{14}m_{12} + h^7z_{26}z_{32}m_{14}m_{13}$$

$\alpha_2\alpha_3$: $m_{42}m_{23} + z_{13}m_{43}m_{22} + z_7m_{44}m_{21} + hz_{21}m_{44}m_{22} + h^2z_{30}m_{44}m_{23}$

$= z_{19}m_{32}m_{33} + z_{13}z_{19}m_{33}m_{32} + z_7z_{19}m_{34}m_{31} + hz_{19}z_{21}m_{34}m_{32} + h^2z_{19}z_{30}m_{34}m_{33} + z_{20}m_{22}m_{43}$

$+ z_{13}z_{20}m_{23}m_{42} + z_7z_{20}m_{24}m_{41} + hz_{20}z_{21}m_{24}m_{42} + h^2z_{20}z_{30}m_{24}m_{43} + hz_{21}m_{22}m_{33} + hz_{13}z_{21}m_{23}m_{32}$

$+ hz_7z_{21}m_{24}m_{31} + h^2z_{21}^2m_{24}m_{32} + h^3z_{21}z_{30}m_{24}m_{33} + hz_{22}m_{12}m_{43} + hz_{13}z_{22}m_{13}m_{42} + hz_7z_{22}m_{14}m_{41}$

$+ h^2z_{21}z_{22}m_{14}m_{42} + h^3z_{22}z_{30}m_{14}m_{43} + h^2z_{23}m_{22}m_{23} + h^2z_{13}z_{23}m_{23}m_{22} + h^2z_7z_{23}m_{24}m_{21} + h^3z_{21}z_{23}m_{24}m_{22}$

$+ h^4z_{23}z_{30}m_{24}m_{23} + h^2z_{24}m_{12}m_{33} + h^2z_{13}z_{24}m_{13}m_{32} + h^2z_7z_{24}m_{14}m_{31} + h^3z_{21}z_{24}m_{14}m_{32} + h^4z_{24}z_{30}m_{14}m_{33}$

$+ h^3z_{25}m_{12}m_{23} + h^3z_{13}z_{25}m_{13}m_{22} + h^3z_7z_{25}m_{14}m_{21} + h^4z_{21}z_{25}m_{14}m_{22} + h^5z_{25}z_{30}m_{14}m_{23} + h^4z_{26}m_{12}m_{13}$

$+ h^4z_{13}z_{26}m_{13}m_{12} + h^4z_7z_{26}m_{14}m_{11} + h^5z_{21}z_{26}m_{14}m_{12} + h^6z_{26}z_{30}m_{14}m_{13}$

$\alpha_2\alpha_4$: $m_{42}m_{24} + z_{20}m_{44}m_{22} + hz_{29}m_{44}m_{23}$

$= z_{19}m_{32}m_{34} + z_{19}z_{20}m_{34}m_{32} + hz_{19}z_{29}m_{34}m_{33} + z_{20}m_{22}m_{44} + z_{20}^2m_{24}m_{42} + hz_{20}z_{29}m_{24}m_{43}$

$+ hz_{21}m_{22}m_{34} + hz_{20}z_{21}m_{24}m_{32} + h^2z_{21}z_{29}m_{24}m_{33} + hz_{22}m_{12}m_{44} + hz_{20}z_{22}m_{14}m_{42} + h^2z_{22}z_{29}m_{14}m_{43}$

$+ h^2z_{23}m_{22}m_{24} + h^2z_{20}z_{23}m_{24}m_{22} + h^3z_{23}z_{29}m_{24}m_{23} + h^2z_{24}m_{12}m_{34} + h^2z_{20}z_{24}m_{14}m_{32} + h^3z_{24}z_{29}m_{14}m_{33}$

$+ h^3z_{25}m_{12}m_{24} + h^3z_{20}z_{25}m_{14}m_{22} + h^4z_{25}z_{29}m_{14}m_{23} + h^4z_{26}m_{12}m_{14} + h^4z_{20}z_{26}m_{14}m_{12} + h^5z_{26}z_{29}m_{14}m_{13}$

α_3^2 : $m_{43}m_{23} + z_{19}m_{44}m_{22} + hz_{28}m_{44}m_{23}$

$= z_{19}m_{33}^2 + z_{19}^2m_{34}m_{32} + hz_{19}z_{28}m_{34}m_{33} + z_{20}m_{23}m_{43} + z_{19}z_{20}m_{24}m_{42} + hz_{20}z_{28}m_{24}m_{43}$

$+ hz_{21}m_{23}m_{33} + hz_{19}z_{21}m_{24}m_{32} + h^2z_{21}z_{28}m_{24}m_{33} + hz_{22}m_{13}m_{43} + hz_{19}z_{22}m_{14}m_{42} + h^2z_{22}z_{28}m_{14}m_{43}$

$+ h^2z_{23}m_{23}^2 + h^2z_{19}z_{23}m_{24}m_{22} + h^3z_{23}z_{28}m_{24}m_{23} + h^2z_{24}m_{13}m_{33} + h^2z_{19}z_{24}m_{14}m_{32} + h^3z_{24}z_{28}m_{14}m_{33}$

$+ h^3z_{25}m_{13}m_{23} + h^3z_{19}z_{25}m_{14}m_{22} + h^4z_{25}z_{28}m_{14}m_{23} + h^4z_{26}m_{13}^2 + h^4z_{19}z_{26}m_{14}m_{12} + h^5z_{26}z_{28}m_{14}m_{13}$

$\alpha_3\alpha_4$: $m_{43}m_{24} + z_{27}m_{44}m_{23}$

$= z_{19}m_{33}m_{34} + z_{19}z_{27}m_{34}m_{33} + z_{20}m_{23}m_{44} + z_{20}z_{27}m_{24}m_{43} + hz_{21}m_{23}m_{34} + hz_{21}z_{27}m_{24}m_{33}$

$+ hz_{22}m_{13}m_{44} + hz_{22}z_{27}m_{14}m_{43} + h^2z_{23}m_{23}m_{24} + h^2z_{23}z_{27}m_{24}m_{23} + h^2z_{24}m_{13}m_{34} + h^2z_{24}z_{27}m_{14}m_{33}$

$+ h^3z_{25}m_{13}m_{24} + h^3z_{25}z_{27}m_{14}m_{23} + h^4z_{26}m_{13}m_{14} + h^4z_{26}z_{27}m_{14}m_{13}$

α_4^2 : $m_{44}m_{24}$

$= z_{19}m_{34}^2 + z_{20}m_{24}m_{44} + hz_{21}m_{24}m_{34} + hz_{22}m_{14}m_{44} + h^2z_{23}m_{24}^2 + h^2z_{24}m_{14}m_{34}$

$+ h^3z_{25}m_{14}m_{24} + h^4z_{26}m_{14}^2$

$$\alpha_4'\alpha_3' = z_{27}\alpha_3'\alpha_4' + h(z_{28}\alpha_3'^2 + z_{29}\alpha_2'\alpha_4') + h^2(z_{30}\alpha_2'\alpha_3' + z_{31}\alpha_1'\alpha_4') + h^3(z_{32}\alpha_2'^2 + z_{33}\alpha_1'\alpha_3') + h^4 z_{34}\alpha_1'\alpha_2' + h^5 z_{35}\alpha_1'^2$$

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$$\begin{aligned}
\alpha_1^2 \ :\ & m_{41}m_{31} + hz_2m_{42}m_{31} + h^2z_6m_{43}m_{31} + h^3z_{18}m_{43}m_{32} + h^3z_{12}m_{44}m_{31} + h^4z_{26}m_{44}m_{32} + h^5z_{35}m_{44}m_{33} \\
&= z_{27}m_{31}m_{41} + hz_2z_{27}m_{32}m_{41} + h^2z_6z_{27}m_{33}m_{41} + h^3z_{18}z_{27}m_{33}m_{42} + h^3z_{12}z_{27}m_{34}m_{41} + h^4z_{26}z_{27}m_{34}m_{42} \\
&+h^5z_{27}z_{35}m_{34}m_{43} + hz_{28}m_{31}^2 + h^2z_2z_{28}m_{32}m_{31} + h^3z_6z_{28}m_{33}m_{31} + h^4z_{18}z_{28}m_{33}m_{32} + h^4z_{12}z_{28}m_{34}m_{31} \\
&+h^5z_{26}z_{28}m_{34}m_{32} + h^6z_{28}z_{35}m_{34}m_{33} + hz_{29}m_{21}m_{41} + h^2z_2z_{29}m_{22}m_{41} + h^3z_6z_{29}m_{23}m_{41} + h^4z_{18}z_{29}m_{23}m_{42} \\
&+h^4z_{12}z_{29}m_{24}m_{41} + h^5z_{26}z_{29}m_{24}m_{42} + h^6z_{29}z_{35}m_{24}m_{43} + h^2z_{30}m_{21}m_{31} + h^3z_2z_{30}m_{22}m_{31} + h^4z_6z_{30}m_{23}m_{31} \\
&+h^5z_{18}z_{30}m_{23}m_{32} + h^5z_{12}z_{30}m_{24}m_{31} + h^6z_{26}z_{30}m_{24}m_{32} + h^7z_{30}z_{35}m_{24}m_{33} + h^2z_{31}m_{11}m_{41} + h^3z_2z_{31}m_{12}m_{41} \\
&+h^4z_6z_{31}m_{13}m_{41} + h^5z_{18}z_{31}m_{13}m_{42} + h^5z_{12}z_{31}m_{14}m_{41} + h^6z_{26}z_{31}m_{14}m_{42} + h^7z_{31}z_{35}m_{14}m_{43} + h^3z_{32}m_{21}^2 \\
&+h^4z_2z_{32}m_{22}m_{21} + h^5z_6z_{32}m_{23}m_{21} + h^6z_{18}z_{32}m_{23}m_{22} + h^6z_{12}z_{32}m_{24}m_{21} + h^7z_{26}z_{32}m_{24}m_{22} + h^8z_{32}z_{35}m_{24}m_{23} \\
&+h^3z_{33}m_{11}m_{31} + h^4z_2z_{33}m_{12}m_{31} + h^5z_6z_{33}m_{13}m_{31} + h^6z_{18}z_{33}m_{13}m_{32} + h^6z_{12}z_{33}m_{14}m_{31} + h^7z_{26}z_{33}m_{14}m_{32} \\
&+h^8z_{33}z_{35}m_{14}m_{33} + h^4z_{34}m_{11}m_{21} + h^5z_2z_{34}m_{12}m_{21} + h^6z_6z_{34}m_{13}m_{21} + h^7z_{18}z_{34}m_{13}m_{22} + h^7z_{12}z_{34}m_{14}m_{21} \\
&+h^8z_{26}z_{34}m_{14}m_{22} + h^9z_{34}z_{35}m_{14}m_{23} + h^5z_{35}m_{11}^2 + h^6z_2z_{35}m_{12}m_{11} + h^7z_6z_{35}m_{13}m_{11} + h^8z_{18}z_{35}m_{13}m_{12} \\
&+h^8z_{12}z_{35}m_{14}m_{11} + h^9z_{26}z_{35}m_{14}m_{12} + h^{10}z_{35}^2m_{14}m_{13}
\end{aligned}$$

$$\begin{aligned}
\alpha_1\alpha_2 \ :\ & m_{41}m_{32} + z_1m_{42}m_{31} + hz_5m_{43}m_{31} + h^2z_{17}m_{43}m_{32} + h^2z_{11}m_{44}m_{31} + h^3z_{25}m_{44}m_{32} + h^4z_{34}m_{44}m_{33} \\
&= z_{27}m_{31}m_{42} + z_1z_{27}m_{32}m_{41} + hz_5z_{27}m_{33}m_{41} + h^2z_{17}z_{27}m_{33}m_{42} + h^2z_{11}z_{27}m_{34}m_{41} + h^3z_{25}z_{27}m_{34}m_{42} \\
&+h^4z_{27}z_{34}m_{34}m_{43} + hz_{28}m_{31}m_{32} + hz_1z_{28}m_{32}m_{31} + h^2z_5z_{28}m_{33}m_{31} + h^3z_{17}z_{28}m_{33}m_{32} + h^3z_{11}z_{28}m_{34}m_{31} \\
&+h^4z_{25}z_{28}m_{34}m_{32} + h^5z_{28}z_{34}m_{34}m_{33} + hz_{29}m_{21}m_{42} + hz_1z_{29}m_{22}m_{41} + h^2z_5z_{29}m_{23}m_{41} + h^3z_{17}z_{29}m_{23}m_{42} \\
&+h^3z_{11}z_{29}m_{24}m_{41} + h^4z_{25}z_{29}m_{24}m_{42} + h^5z_{29}z_{34}m_{24}m_{43} + h^2z_{30}m_{21}m_{32} + h^2z_1z_{30}m_{22}m_{31} + h^3z_5z_{30}m_{23}m_{31} \\
&+h^4z_{17}z_{30}m_{23}m_{32} + h^4z_{11}z_{30}m_{24}m_{31} + h^5z_{25}z_{30}m_{24}m_{32} + h^6z_{30}z_{34}m_{24}m_{33} + h^2z_{31}m_{11}m_{42} + h^2z_1z_{31}m_{12}m_{41} \\
&+h^3z_5z_{31}m_{13}m_{41} + h^4z_{17}z_{31}m_{13}m_{42} + h^4z_{11}z_{31}m_{14}m_{41} + h^5z_{25}z_{31}m_{14}m_{42} + h^6z_{31}z_{34}m_{14}m_{43} + h^3z_{32}m_{21}m_{22} \\
&+h^3z_1z_{32}m_{22}m_{21} + h^4z_5z_{32}m_{23}m_{21} + h^5z_{17}z_{32}m_{23}m_{22} + h^5z_{11}z_{32}m_{24}m_{21} + h^6z_{25}z_{32}m_{24}m_{22} + h^7z_{32}z_{34}m_{24}m_{23} \\
&+h^3z_{33}m_{11}m_{32} + h^3z_1z_{33}m_{12}m_{31} + h^4z_5z_{33}m_{13}m_{31} + h^5z_{17}z_{33}m_{13}m_{32} + h^5z_{11}z_{33}m_{14}m_{31} + h^6z_{25}z_{33}m_{14}m_{32} \\
&+h^7z_{33}z_{34}m_{14}m_{33} + h^4z_{34}m_{11}m_{22} + h^4z_1z_{34}m_{12}m_{21} + h^5z_5z_{34}m_{13}m_{21} + h^6z_{17}z_{34}m_{13}m_{22} + h^6z_{11}z_{34}m_{14}m_{21} \\
&+h^7z_{25}z_{34}m_{14}m_{22} + h^8z_{34}^2m_{14}m_{23} + h^5z_{35}m_{11}m_{12} + h^5z_1z_{35}m_{12}m_{11} + h^6z_5z_{35}m_{13}m_{11} + h^7z_{17}z_{35}m_{13}m_{12} \\
&+h^7z_{11}z_{35}m_{14}m_{11} + h^8z_{25}z_{35}m_{14}m_{12} + h^9z_{34}z_{35}m_{14}m_{13}
\end{aligned}$$

$$\alpha_1\alpha_3 \ : m_{41}m_{33} + z_4m_{43}m_{31} + hz_{16}m_{43}m_{32} + hz_{10}m_{44}m_{31} + h^2z_{24}m_{44}m_{32} + h^3z_{33}m_{44}m_{33}$$

$$= z_{27}m_{31}m_{43} + z_4z_{27}m_{33}m_{41} + hz_{16}z_{27}m_{33}m_{42} + hz_{10}z_{27}m_{34}m_{41} + h^2z_{24}z_{27}m_{34}m_{42} + h^3z_{27}z_{33}m_{34}m_{43}$$

$$+ hz_{28}m_{31}m_{33} + hz_4z_{28}m_{33}m_{31} + h^2z_{16}z_{28}m_{33}m_{32} + h^2z_{10}z_{28}m_{34}m_{31} + h^3z_{24}z_{28}m_{34}m_{32} + h^4z_{28}z_{33}m_{34}m_{33}$$

$$+ hz_{29}m_{21}m_{43} + hz_4z_{29}m_{23}m_{41} + h^2z_{16}z_{29}m_{23}m_{42} + h^2z_{10}z_{29}m_{24}m_{41} + h^3z_{24}z_{29}m_{24}m_{42} + h^4z_{29}z_{33}m_{24}m_{43}$$

$$+ h^2z_{30}m_{21}m_{33} + h^2z_4z_{30}m_{23}m_{31} + h^3z_{16}z_{30}m_{23}m_{32} + h^3z_{10}z_{30}m_{24}m_{31} + h^4z_{24}z_{30}m_{24}m_{32} + h^5z_{30}z_{33}m_{24}m_{33}$$

$$+ h^2z_{31}m_{11}m_{43} + h^2z_4z_{31}m_{13}m_{41} + h^3z_{16}z_{31}m_{13}m_{42} + h^3z_{10}z_{31}m_{14}m_{41} + h^4z_{24}z_{31}m_{14}m_{42} + h^5z_{31}z_{33}m_{14}m_{43}$$

$$+ h^3z_{32}m_{21}m_{23} + h^3z_4z_{32}m_{23}m_{21} + h^4z_{16}z_{32}m_{23}m_{22} + h^4z_{10}z_{32}m_{24}m_{21} + h^5z_{24}z_{32}m_{24}m_{22} + h^6z_{32}z_{33}m_{24}m_{23}$$

$$+ h^3z_{33}m_{11}m_{33} + h^3z_4z_{33}m_{13}m_{31} + h^4z_{16}z_{33}m_{13}m_{32} + h^4z_{10}z_{33}m_{14}m_{31} + h^5z_{24}z_{33}m_{14}m_{32} + h^6z_{33}^2m_{14}m_{33}$$

$$+ h^4z_{34}m_{11}m_{23} + h^4z_4z_{34}m_{13}m_{21} + h^5z_{16}z_{34}m_{13}m_{22} + h^5z_{10}z_{34}m_{14}m_{21} + h^6z_{24}z_{34}m_{14}m_{22} + h^7z_{33}z_{34}m_{14}m_{23}$$

$$+ h^5z_{35}m_{11}m_{13} + h^5z_4z_{35}m_{13}m_{11} + h^6z_{16}z_{35}m_{13}m_{12} + h^6z_{10}z_{35}m_{14}m_{11} + h^7z_{24}z_{35}m_{14}m_{12} + h^8z_{33}z_{35}m_{14}m_{13}$$

$$\alpha_1\alpha_4 \ : m_{41}m_{34} + z_{14}m_{43}m_{32} + z_8m_{44}m_{31} + hz_{22}m_{44}m_{32} + h^2z_{31}m_{44}m_{33}$$

$$= z_{27}m_{31}m_{44} + z_{14}z_{27}m_{33}m_{42} + z_8z_{27}m_{34}m_{41} + hz_{22}z_{27}m_{34}m_{42} + h^2z_{27}z_{31}m_{34}m_{43} + hz_{28}m_{31}m_{34}$$

$$+ hz_{14}z_{28}m_{33}m_{32} + hz_8z_{28}m_{34}m_{31} + h^2z_{22}z_{28}m_{34}m_{32} + h^3z_{28}z_{31}m_{34}m_{33} + hz_{29}m_{21}m_{44} + hz_{14}z_{29}m_{23}m_{42}$$

$$+ hz_8z_{29}m_{24}m_{41} + h^2z_{22}z_{29}m_{24}m_{42} + h^3z_{29}z_{31}m_{24}m_{43} + h^2z_{30}m_{21}m_{34} + h^2z_{14}z_{30}m_{23}m_{32} + h^2z_8z_{30}m_{24}m_{31}$$

$$+ h^3z_{22}z_{30}m_{24}m_{32} + h^3z_{30}z_{31}m_{24}m_{33} + h^2z_{31}m_{11}m_{44} + h^2z_{14}z_{31}m_{13}m_{42} + h^2z_8z_{31}m_{14}m_{41} + h^3z_{22}z_{31}m_{14}m_{42}$$

$$+ h^4z_{31}^2m_{14}m_{43} + h^3z_{32}m_{21}m_{24} + h^3z_{14}z_{32}m_{23}m_{22} + h^3z_8z_{32}m_{24}m_{21} + h^4z_{22}z_{32}m_{24}m_{22} + h^5z_{31}z_{32}m_{24}m_{23}$$

$$+ h^3z_{33}m_{11}m_{34} + h^3z_{14}z_{33}m_{13}m_{32} + h^3z_8z_{33}m_{14}m_{31} + h^4z_{22}z_{33}m_{14}m_{32} + h^5z_{31}z_{33}m_{14}m_{33} + h^4z_{34}m_{11}m_{24}$$

$$+ h^4z_{14}z_{34}m_{13}m_{22} + h^4z_8z_{34}m_{14}m_{21} + h^5z_{22}z_{34}m_{14}m_{22} + h^6z_{31}z_{34}m_{14}m_{23} + h^5z_{35}m_{11}m_{14} + h^5z_{14}z_{35}m_{13}m_{12}$$

$$+ h^5z_8z_{35}m_{14}m_{11} + h^6z_{22}z_{35}m_{14}m_{12} + h^7z_{31}z_{35}m_{14}m_{13}$$

$$\alpha_2^2 \;: m_{42}m_{32} + z_3m_{43}m_{31} + hz_{15}m_{43}m_{32} + hz_9m_{44}m_{31} + h^2z_{23}m_{44}m_{32} + h^3z_{32}m_{44}m_{33}$$

$$= z_{27}m_{32}m_{42} + z_3z_{27}m_{33}m_{41} + hz_{15}z_{27}m_{33}m_{42} + hz_9z_{27}m_{34}m_{41} + h^2z_{23}z_{27}m_{34}m_{42} + h^3z_{27}z_{32}m_{34}m_{43}$$

$$+ hz_{28}m_{32}^2 + hz_3z_{28}m_{33}m_{31} + h^2z_{15}z_{28}m_{33}m_{32} + h^2z_9z_{28}m_{34}m_{31} + h^3z_{23}z_{28}m_{34}m_{32} + h^4z_{28}z_{32}m_{34}m_{33}$$

$$+ hz_{29}m_{22}m_{42} + hz_3z_{29}m_{23}m_{41} + h^2z_{15}z_{29}m_{23}m_{42} + h^2z_9z_{29}m_{24}m_{41} + h^3z_{23}z_{29}m_{24}m_{42} + h^4z_{29}z_{32}m_{24}m_{43}$$

$$+ h^2z_{30}m_{22}m_{32} + h^2z_3z_{30}m_{23}m_{31} + h^3z_{15}z_{30}m_{23}m_{32} + h^3z_9z_{30}m_{24}m_{31} + h^4z_{23}z_{30}m_{24}m_{32} + h^5z_{30}z_{32}m_{24}m_{33}$$

$$+ h^2z_{31}m_{12}m_{42} + h^2z_3z_{31}m_{13}m_{41} + h^3z_{15}z_{31}m_{13}m_{42} + h^3z_9z_{31}m_{14}m_{41} + h^4z_{23}z_{31}m_{14}m_{42} + h^5z_{31}z_{32}m_{14}m_{43}$$

$$+ h^3z_{32}m_{22}^2 + h^3z_3z_{32}m_{23}m_{21} + h^4z_{15}z_{32}m_{23}m_{22} + h^4z_9z_{32}m_{24}m_{21} + h^5z_{23}z_{32}m_{24}m_{22} + h^6z_{32}^2m_{24}m_{23}$$

$$+ h^3z_{33}m_{12}m_{32} + h^3z_3z_{33}m_{13}m_{31} + h^4z_{15}z_{33}m_{13}m_{32} + h^4z_9z_{33}m_{14}m_{31} + h^5z_{23}z_{33}m_{14}m_{32} + h^6z_{32}z_{33}m_{14}m_{33}$$

$$+ h^4z_{34}m_{12}m_{22} + h^4z_3z_{34}m_{13}m_{21} + h^5z_{15}z_{34}m_{13}m_{22} + h^5z_9z_{34}m_{14}m_{21} + h^6z_{23}z_{34}m_{14}m_{22} + h^7z_{32}z_{34}m_{14}m_{23}$$

$$+ h^5z_{35}m_{12}^2 + h^5z_3z_{35}m_{13}m_{11} + h^6z_{15}z_{35}m_{13}m_{12} + h^6z_9z_{35}m_{14}m_{11} + h^7z_{23}z_{35}m_{14}m_{12} + h^8z_{32}z_{35}m_{14}m_{13}$$

$$\alpha_2\alpha_3 \;: m_{42}m_{33} + z_{13}m_{43}m_{32} + z_7m_{44}m_{31} + hz_{21}m_{44}m_{32} + h^2z_{30}m_{44}m_{33}$$

$$= z_{27}m_{32}m_{43} + z_{13}z_{27}m_{33}m_{42} + z_7z_{27}m_{34}m_{41} + hz_{21}z_{27}m_{34}m_{42} + h^2z_{27}z_{30}m_{34}m_{43} + hz_{28}m_{32}m_{33}$$

$$+ hz_{13}z_{28}m_{33}m_{32} + hz_7z_{28}m_{34}m_{31} + h^2z_{21}z_{28}m_{34}m_{32} + h^3z_{28}z_{30}m_{34}m_{33} + hz_{29}m_{22}m_{43} + hz_{13}z_{29}m_{23}m_{42}$$

$$+ hz_7z_{29}m_{24}m_{41} + h^2z_{21}z_{29}m_{24}m_{42} + h^3z_{29}z_{30}m_{24}m_{43} + h^2z_{30}m_{22}m_{33} + h^2z_{13}z_{30}m_{23}m_{32} + h^2z_7z_{30}m_{24}m_{31}$$

$$+ h^3z_{21}z_{30}m_{24}m_{32} + h^4z_{30}^2m_{24}m_{33} + h^2z_{31}m_{12}m_{43} + h^2z_{13}z_{31}m_{13}m_{42} + h^2z_7z_{31}m_{14}m_{41} + h^3z_{21}z_{31}m_{14}m_{42}$$

$$+ h^4z_{30}z_{31}m_{14}m_{43} + h^3z_{32}m_{22}m_{23} + h^3z_{13}z_{32}m_{23}m_{22} + h^3z_7z_{32}m_{24}m_{21} + h^4z_{21}z_{32}m_{24}m_{22} + h^5z_{30}z_{32}m_{24}m_{23}$$

$$+ h^3z_{33}m_{12}m_{33} + h^3z_{13}z_{33}m_{13}m_{32} + h^3z_7z_{33}m_{14}m_{31} + h^4z_{21}z_{33}m_{14}m_{32} + h^5z_{30}z_{33}m_{14}m_{33} + h^4z_{34}m_{12}m_{23}$$

$$+ h^4z_{13}z_{34}m_{13}m_{22} + h^4z_7z_{34}m_{14}m_{21} + h^5z_{21}z_{34}m_{14}m_{22} + h^6z_{30}z_{34}m_{14}m_{23} + h^5z_{35}m_{12}m_{13} + h^5z_{13}z_{35}m_{13}m_{12}$$

$$+ h^5z_7z_{35}m_{14}m_{11} + h^6z_{21}z_{35}m_{14}m_{12} + h^7z_{30}z_{35}m_{14}m_{13}$$

$\alpha_2\alpha_4 \; : m_{42}m_{34} + z_{20}m_{44}m_{32} + hz_{29}m_{44}m_{33}$

$$=z_{27}m_{32}m_{44} + z_{20}z_{27}m_{34}m_{42} + hz_{27}z_{29}m_{34}m_{43} + hz_{28}m_{32}m_{34} + hz_{20}z_{28}m_{34}m_{32} + h^2z_{28}z_{29}m_{34}m_{33}$$

$$+hz_{29}m_{22}m_{44} + hz_{20}z_{29}m_{24}m_{42} + h^2z_{29}^2m_{24}m_{43} + h^2z_{30}m_{22}m_{34} + h^2z_{20}z_{30}m_{24}m_{32} + h^3z_{29}z_{30}m_{24}m_{33}$$

$$+h^2z_{31}m_{12}m_{44} + h^2z_{20}z_{31}m_{14}m_{42} + h^3z_{29}z_{31}m_{14}m_{43} + h^3z_{32}m_{22}m_{24} + h^3z_{20}z_{32}m_{24}m_{22} + h^4z_{29}z_{32}m_{24}m_{23}$$

$$+h^3z_{33}m_{12}m_{34} + h^3z_{20}z_{33}m_{14}m_{32} + h^4z_{29}z_{33}m_{14}m_{33} + h^4z_{34}m_{12}m_{24} + h^4z_{20}z_{34}m_{14}m_{22} + h^5z_{29}z_{34}m_{14}m_{23}$$

$$+h^5z_{35}m_{12}m_{14} + h^5z_{20}z_{35}m_{14}m_{12} + h^6z_{29}z_{35}m_{14}m_{13}$$

$\alpha_3^2 \; : m_{43}m_{33} + z_{19}m_{44}m_{32} + hz_{28}m_{44}m_{33}$

$$=z_{27}m_{33}m_{43} + z_{19}z_{27}m_{34}m_{42} + hz_{27}z_{28}m_{34}m_{43} + hz_{28}m_{33}^2 + hz_{19}z_{28}m_{34}m_{32} + h^2z_{28}^2m_{34}m_{33}$$

$$+hz_{29}m_{23}m_{43} + hz_{19}z_{29}m_{24}m_{42} + h^2z_{28}z_{29}m_{24}m_{43} + h^2z_{30}m_{23}m_{33} + h^2z_{19}z_{30}m_{24}m_{32} + h^3z_{28}z_{30}m_{24}m_{33}$$

$$+h^2z_{31}m_{13}m_{43} + h^2z_{19}z_{31}m_{14}m_{42} + h^3z_{28}z_{31}m_{14}m_{43} + h^3z_{32}m_{23}^2 + h^3z_{19}z_{32}m_{24}m_{22} + h^4z_{28}z_{32}m_{24}m_{23}$$

$$+h^3z_{33}m_{13}m_{33} + h^3z_{19}z_{33}m_{14}m_{32} + h^4z_{28}z_{33}m_{14}m_{33} + h^4z_{34}m_{13}m_{23} + h^4z_{19}z_{34}m_{14}m_{22} + h^5z_{28}z_{34}m_{14}m_{23}$$

$$+h^5z_{35}m_{13}^2 + h^5z_{19}z_{35}m_{14}m_{12} + h^6z_{28}z_{35}m_{14}m_{13}$$

$\alpha_3\alpha_4 \; : m_{43}m_{34} + z_{27}m_{44}m_{33}$

$$=z_{27}m_{33}m_{44} + z_{27}^2m_{34}m_{43} + hz_{28}m_{33}m_{34} + hz_{27}z_{28}m_{34}m_{33} + hz_{29}m_{23}m_{44} + hz_{27}z_{29}m_{24}m_{43}$$

$$+h^2z_{30}m_{23}m_{34} + h^2z_{27}z_{30}m_{24}m_{33} + h^2z_{31}m_{13}m_{44} + h^2z_{27}z_{31}m_{14}m_{43} + h^3z_{32}m_{23}m_{24} + h^3z_{27}z_{32}m_{24}m_{23}$$

$$+h^3z_{33}m_{13}m_{34} + h^3z_{27}z_{33}m_{14}m_{33} + h^4z_{34}m_{13}m_{24} + h^4z_{27}z_{34}m_{14}m_{23} + h^5z_{35}m_{13}m_{14} + h^5z_{27}z_{35}m_{14}m_{13}$$

$\alpha_4^2 \; : m_{44}m_{34}$

$$=z_{27}m_{34}m_{44} + hz_{28}m_{34}^2 + hz_{29}m_{24}m_{44} + h^2z_{30}m_{24}m_{34} + h^2z_{31}m_{14}m_{44} + h^3z_{32}m_{24}^2$$

$$+h^3z_{33}m_{14}m_{34} + h^4z_{34}m_{14}m_{24} + h^5z_{35}m_{14}^2$$

VITA

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