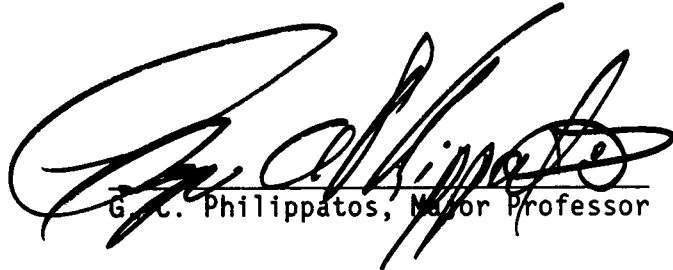


To the Graduate Council:

I am submitting herewith a dissertation written by Michael A. S. Guth entitled "Theories of Speculation and the Economics of Information." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Economics.


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THEORIES OF SPECULATION AND THE ECONOMICS OF INFORMATION

A Dissertation

Presented for the

Doctor of Philosophy

Degree

The University of Tennessee, Knoxville

Michael Anthony Stephen Guth

June 1988

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The best reward an economic theorist can get is praise from a fellow theorist, and thus I especially enjoyed discussing the dissertation with Errol Glustoff. It was also reassuring to get his views on technical points such as which fixed-point theorem to use, how many random variables should be subsumed into the "state of the world," and why welfare effects of speculative bubbles should wash in a general equilibrium model. Glustoff's suggestions for laying out the material greatly improved the way the text now reads. Of course, I take responsibility for any errors that remain.

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ABSTRACT

This dissertation studies informational asymmetries and explores some new areas of uncertainty in economic decision-making. The first chapter contains a statement of the main contributions of each chapter and a literature review on speculation theory, informational content of prices, and speculative bubbles. Chapter 2 discusses intrinsic uncertainty and higher-order beliefs. The chapter's main theme is that individuals can agree without knowing they agree and that agreed common knowledge priors can eliminate this intrinsic uncertainty. Chapter 2 introduces a new framework for common knowledge beliefs and two propositions that clarify the very fundamental assumptions that guarantee equilibrium consensus.

Chapter 3 examines the impact of intrinsic uncertainty on the major asset pricing paradigms. Since intrinsic uncertainty has been a neglected research topic in mathematical economics and financial economics, it is not surprising to find the Modigliani-Miller paradigm, Arbitrage Pricing Theory, and Option Pricing Theory depend on common knowledge beliefs assumptions. The Capital Asset Pricing Model subsumes common knowledge through assumptions about perfect information about the returns to assets.

Whereas Chapter 3 illustrates speculation based on intrinsic uncertainty, Chapter 4 illustrates speculation with agreed common knowledge priors but with a known dichotomy between informed and uninformed traders. Rather than attempting to iterate to consensus as in Chapter 2, the traders in Chapter 4 buy and sell assets and draw

inferences from the behavior of others. Chapter 4 presents a Bayesian Nash equilibrium of speculative bubbles and emphasizes the feedback effect stemming from informed trader purchases. This chapter presents the first game theoretic analysis of speculative bubbles as well as the first fully rational, payoff-maximizing approach to bubble formation.

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CHAPTER 1

TOPICS IN SPECULATION THEORY AND THE ECONOMICS OF INFORMATION

1. INTRODUCTION

This dissertation studies informational asymmetries that can lead to speculation and strategic play. Chapter 2 is a methodological essay pointing out how intrinsic uncertainty can enter in an environment in which individuals have common priors. The contribution of the chapter is to elucidate the foundations under which a consensus equilibrium is guaranteed. A second contribution is the introduction of a framework to analyze and work with higher-order beliefs and common knowledge beliefs.

Chapter 3 is primarily an applications piece. The intrinsic uncertainty in Chapter 2 is shown to have important consequences for the major paradigms in the theory of finance. Models with three or more periods and intrinsic uncertainty can exhibit capital gains motivated speculation. Chapter 3 contains several significant contributions. First, we show the validity of the famous Modigliani-Miller (MM) paradigm in a multi-period environment actually requires agreed common knowledge beliefs. This is truly a monumental achievement - worthy of a dissertation by itself. After revolutionizing the field of finance, the paradigm has been so extensively scrutinized over the last thirty years that the chances of making an original contribution to MM theory are miniscule. Yet that is what is accomplished in Chapter 3, and in the process the foundations of finance theory have been shaken.

A second contribution of Chapter 3 is to illustrate how intrinsic uncertainty leads to a nonuniqueness of possible factor models in the Arbitrage Pricing Theory and of option pricing formulae in Option Pricing Theory.

A third and equally significant contribution of Chapter 3 is the demonstration that only some and not all models that employ homogeneous beliefs actually require common knowledge beliefs. After reading Chapter 2 some theorists may perceive that "agreed common knowledge priors" (ACKP) should be substituted for "common priors" throughout all economic and financial theory. Our findings that some models require only homogeneous beliefs for unique, closed-form solutions is truly startling and shows common knowledge is not implicit in many of these older models.

Chapter 4 presents a new theoretical analysis of information inference through prices. Whereas the source of speculation in Chapter 3 is intrinsic uncertainty over the priors used by others, the source of speculation in Chapter 4 is a common knowledge dichotomy between informed and uninformed traders. The contributions of Chapter 4 include the first game theoretic analysis of speculative bubbles, and the first fully rational, payoff-maximizing derivation of bubbles.

Several themes unify the different chapters of the dissertation. First, all of the chapters deal with choice under uncertainty and sophisticated use of probabilities. The next two chapters deal with common knowledge probabilities and higher-order beliefs, which are in fact probability distributions over probability distributions. Aside

from ACKP, the fourth chapter deals with Bayesian updates of prior probabilities and equilibrium posterior probabilities.

Second, each of the chapters isolates a form of uncertainty and examines what inferences can be drawn in this environment. The second and third chapters focus on intrinsic uncertainty over the unknown beliefs of other market participants; the fourth chapter focuses on uncertainty over the relation between private signals and publicly revealed prices in the market.

Finally, each of the chapters focuses in part on game theory. Chapter 4 is specifically a Bayesian game of incomplete information. Chapter 2 examines one of the fundamental assumptions of game theory, use of ACKP and what can happen without ACKP. The third chapter shows market transactions based on strategic play that provide wealth to asset holders.

The text of this chapter proceeds as follows. In Section 2 we present a taxonomy of information settings and trading rules/circumstances used in the economics of information literature. Section 3 discusses the definition of common knowledge and why economic theorists have come to rely heavily upon the concept itself. This section will set up the theoretical work on common knowledge in Chapter 2 as well as the application of "agreed common knowledge priors" in Chapter 3. Section 4 provides background material on the Kihlstrom-Mirman and Grossman-Stiglitz work that forms the basis for the Bayesian game contained in Chapter 4. Moreover, since a primary result of the work in Chapter 4 shows how to construct a fully rational speculative bubble, we describe in Section 5 of this chapter

some key observations from the speculative bubble literature.

Section 6 contains a short dissertation overview.

2. TAXONOMY OF ECONOMIC ENVIRONMENTS

Speculation requires (1) intertemporal investment and (2) decision-making under uncertainty. Economic models with only one period contain no uncertainty over external factors, e.g., the arrival of information events or the revelation of the state of the world. All information is known at the time of trading, and economic agents optimize without intertemporal considerations.

Yet even in models with only one period, "intrinsic uncertainty" can exist over the unknown attributes of other economic agents in the model. For example, the intrinsic uncertainty can relate to the beliefs, preferences, income constraints, and in short any standard object of comparative statics. This form of uncertainty is intrinsic to the agents in the standard economic model and can be distinguished from extrinsic uncertainty about, e.g., sunspots.

Many single period markets involve a tatonnement process, whereby a hypothetical auctioneer adjusts prices to clear markets for given aggregate demand and supply schedules. No trading can take place outside equilibrium, i.e., at non-market-clearing prices. None of the models we formulate in Chapters 2, 3, and 4 incorporate non-equilibrium trading.

The common knowledge literature is predominantly written in a two-period environment with iterations occurring in the second period. In the first period, the agents receive a private signal

about the true state of the world. In the second period, the agents call out their probability beliefs back and forth and sometimes arrive at a consensus. Alternatively, the common knowledge literature can be said to be grounded on a single period model, where agents come into the model endowed with a private signal about the true state of the world.

The reader may benefit from contrasting the common knowledge literature environment with that of the classic Hirshleifer (1975, 1976) model of speculation incorporating three periods. In Hirshleifer's first period, agents trade to an intermediate position knowing they will have a chance to revise their holdings before the state of the world is revealed. In period 2, the agents receive a signal about the state of the world and the market again opens for trading. In period 3, the state of the world is revealed and the state-contingent claims pay off accordingly.

Tirole (1982) criticized the main determinants of speculation in the Hirshleifer model and attempted to argue that no trading, and hence no speculation, could take place in a Rational Expectations equilibrium. However, Tirole employed a single period, static equilibrium model. In such a static model there can be neither intertemporal investment nor uncertainty over exogenous events, the traditional source of speculative behavior. Thus, as Hirshleifer (1984) pointed out in his rebuttal to Tirole, the latter adopted an inadequate framework to analyze such a complex phenomenon as speculation.

The main point of Chapter 2 will be that a limited form of

uncertainty exists even in a static economy, but that this source of uncertainty comes from within the model rather than outside it. Even in a static economy agents can face uncertainty over the unknown attributes of other agents in the model. In circumstances where one agent's payoff depends on the motives, beliefs, or preferences of either his trading partner or the market collectively, then this intrinsic uncertainty can be a source of misinference, in the static economy, and speculation, in a multiperiod economy.

In either the single period or two period view of the common knowledge literature, potential misinferences can be drawn in the absence of common knowledge: intrinsic uncertainty still exists in the conventional common knowledge literature model. This intrinsic uncertainty can give rise to speculation, in an intertemporal environment, as we show in Chapter 3.

Some of our original models, most notably the Bayesian Nash equilibrium model of Chapter 4, do present a sequence of temporary equilibria, where markets clear and a publicly revealed price can be observed by all. As we show in Chapter 4, the prevailing price in turn draws other economic forces into play that can lead the market-clearing price to rise, even above fundamental values. We emphasize that in Chapter 4's sequence of economies, markets clear each trading period.

Perhaps the dominant view, though by no means the only view, of the Kihlstrom and Mirman (1974) and Grossman and Stiglitz (1976) models envisions a single period Rational Expectations equilibrium. For a single period, feedback effects will be irrelevant. In Section

4 we quote Kihlstrom and Mirman to show these two authors explicitly meant to consider an intertemporal environment in which feedback effects can be relevant. Furthermore, concerning their underlying intertemporal environment, Grossman and Stiglitz (1976) state:

there are constantly new shocks to the economy; although each of these shocks may have certain individual characteristics . . . we are interested . . . in how these shocks affect market returns; and we postulate that we can describe the occurrence of these different shocks, in terms of their effects on returns, by a stationary stochastic process. The capital market must continually adjust to these shocks. We have formulated an equilibrium notion which explicitly takes account of the economy's response to these various shocks. Others have described this as a disequilibrium situation, but have been unable to say much about it. In the structure we have developed, the market never fully adjusts. (p.248)

Thus it becomes evident that Grossman and Stiglitz envision a (potentially infinite) sequence of temporary equilibria each market trading period, in which the exchange of information through prices will be a long-run equilibrium. They do not attempt to model a single period tatonnement, market-clearing process. However, theorists who thought the Grossman and Stiglitz (1976) concept applies to a single (long-run) period model are partially vindicated by the tendency of short-term equilibria to move towards the long-run equilibrium. It would also be reasonable to point out that Grossman and Stiglitz left the statement of underlying trading periods and information events somewhat vague.

3. COMMON KNOWLEDGE

Common knowledge has been defined by Aumann (1976) as the ultimate equilibrium emerging from a DeGroot (1974)-type convergence of agents' posterior beliefs. In words, an event is common knowledge between agents i and j when both agents know that event has occurred, i knows that j knows the event has occurred, j knows that i knows that j knows the event has occurred, ad infinitum.

Following Aumann's definition, a series of no-trading theorems were proposed by Milgrom and Stokey (1982), Tirole (1982), and Geanakoplis and Polemarchakis (1983). These papers allege that with common prior beliefs, the mere willingness of others to trade should convince at least one trading partner that his own position must be disadvantageous. Hence all the required trading partners would allegedly never agree to the trade.

However, these papers did not address the subtle distinction between homogeneous priors and priors that are common knowledge. In short, individuals in these models have no way of knowing a priori that their priors are the same. Therefore, while the agents may objectively agree, they would not typically know they agree. This distinction forms the basis for the theoretical work in Chapter 2.

We are interested in two separate definitions of common knowledge. The first borrows some concepts from Lattice Theory, namely the meet and join of partitions, and was introduced by Aumann (1976). This definition can adequately describe common knowledge events or other objects in the state space. However, to describe

higher-order beliefs and common knowledge beliefs, we formulate our own definitions and notation.

For three of the asset pricing paradigms of Chapter 3, we prove that "homogeneous beliefs" is too weak an assumption to permit aggregation across individuals; the models actually need the more restrictive ACKP assumption. However, "homogeneous beliefs" assumptions should not be blindly replaced with "common knowledge beliefs," since Chapter 3 explains why some valuation theories only require homogeneity of beliefs.

The common knowledge definitions can be most easily understood in a state-preference approach to financial theory. Investors are assumed to buy and sell financial assets based in part on their beliefs about what state of the world will be revealed. In game-theoretic terms, the state of the world will determine the payoff from alternative strategies adopted by the players.

The partitions of the state spaces correspond to education and/or information individuals have that enables them to infer what a signal means. The smaller the classes of these partitions, the more precisely the individual can determine what state of the world will be revealed. Thus, information that the true state is a member of a large class of a partition is less helpful than information that the true state is in a smaller class, *ceteris paribus*.

Exchanging information on probability beliefs also raises interesting behavioral questions about whether a person places more confidence in his own priors or those of someone else. Based on his degree of self-confidence, a person may resist altering his priors.

4. RATIONAL EXPECTATIONS AND MARKET PRICES' INFORMATIONAL CONTENT

This section provides a critique of two works that led to our development of the Bayes game in Chapter 4. An extensive rational expectations literature has expanded around the general theme that traders use prices to infer additional information about the profitability of their trades. Surprisingly, the literature lacks any accurate portrayal of speculative impacts on such informational inferences. In particular, the often-cited work by Grossman and Stiglitz (1976) completely assumes away complex problems posed by capital gains motivations.

We shall begin with a work published two years earlier. Kihlstrom and Mirman (1974, hereafter KM) considered an economy where insiders witness a random variable correlated with a firm's future dividend policy. The authors then attempted to characterize sufficient conditions ensuring that the rest of the market will glean the private information simply by observing share price changes. Among other things, these conditions require a one-to-one correspondence between the information, I , and the market price vector $P(I)$, so that the uninformed traders can invert the function $P(I)$ and learn $I = f^{-1}(P)$.

KM traders buy and sell assets solely for their real earnings prospects. Without attempting to incorporate capital gains motives, KM nevertheless applied their model to the stock market and "price indicators of product quality" transactions. Consequently, the KM price invertibility propositions depend crucially on this no capital

gains assumption. In fact, their sufficient conditions do not even apply to multiperiod transactions - where speculation can occur.

KM seemed to recognize their model's simplifications when their informed traders become smart enough to realize their purchases can trigger "feedback effects."

There is one important difficulty that arises when individual economic units use prices to make inferences about the economy. In order to make correct inferences, the individual must understand how the economy operates and how the economic structure is related to equilibrium prices. An integral part of his understanding must be a realization that if he and others use information inferred from prices to adapt their behavior, a feedback is introduced which in turn affects the determination of prices. This difficulty is avoided in the analysis . . . because of the simplifying assumptions (employed). These assumptions imply that individual behavior is independent of expectations. In general, however, this difficulty cannot be avoided, and a complete analysis of the conditions under which Bayesian expectations converge must eventually take it into consideration. (KM (1974), pp.359-368, emphasis added)

Grossman and Stiglitz (1976, hereafter GS) expanded KM's basic theme to include the costs of purchasing information. The GS model assumes, analogous to KM, that an "informed" group observes a random variable correlated to an asset's expected return. GS attempted to resolve the old finance theory conundrum that no one has an incentive to purchase information if capital markets are efficient, i.e., prices reflect all available information. But if no one gather information, then market prices can reflect no information. Hence everyone will have an incentive to gather information.

Their principal result states that uninformed traders in a noisy environment will gather information until the last trader to purchase information is indifferent between buying the information or

remaining uninformed, i.e., the marginal cost equals the marginal benefit from gathering information. Individuals in a decentralized economy will typically have differential information, and rational expectations equilibria will not fully reveal (or fully aggregate a la Hayek) information. The difference between fully revealing prices and market clearing prices suffices to compensate those purchasing the information. Grossman (1981) reiterated this point and provided references to the related literature.

Clearly the KM and GS price inference models accurately portray a wide range of socio-economic phenomena. However, we have reason to question the robustness of their models. Several caveats crop up in both the KM and GS models when these models are applied to capital markets. Whereas these authors have asserted that, under their simplifying assumptions, market participants can judge the quality of goods and services by their price; in fact, the purveyors of these consumer goods have no incentive to accurately signal their quality through prices.

For example, a poor quality restaurant might set high prices when it knows the market associates high prices with good food. Within the KM and GS models any poor quality establishment can lure imperfectly informed customers by advertising high prices. The price indicator of quality model necessarily requires price taking behavior by all purveyors of consumer goods. Apart from perfect competition, KM and GS market traders must attempt to decipher good from bad price signals, and needless to say, the KM and GS price-information relationships become infused with white noise.

The second problem with the KM and GS models concerns the feedback effect, previously mentioned by KM. In general, "informed" traders may have incentives to purchase some asset after receiving a negative reading on its future earning potential. If the informed traders realize other investors are observing their transactions in the hopes of gleaning the private information, they might try to conceal their knowledge or fool the market. When stock purchases trigger subsequent buying sprees by uninformed traders, the informed group may speculate regardless of an asset's real earnings prospects. The rather obvious KM and GS feedback mechanisms create potential for speculative profits for the informed group to exploit.

Third, the KM and GS models assume that an informed trader, who receives a favorable signal on an asset's earnings, always proceeds to buy into the stock. In fact, these "informed" traders must know they have foreknowledge to exploit their position. The "informed" traders in the KM and GS models may well suspect that some other group of traders has already obtained the same information and bought into the stock. The prevailing market price may already impound enough so-called "private" information to render the "informed" trader's position valueless. In addition, if the other market traders plan to sell the stock in the near future, the security's price may plummet regardless of the signal the informed KM and GS traders received. Thus all the KM and GS informed traders face uncertainty from the rest of the market's intended portfolio revisions. Essentially, receiving a favorable signal fails to guarantee these "informed" traders respond by immediately buying into the asset.

Finally, the KM and GS models avoid any portrayal of the capital gains motive; in fact, their models have virtually nothing to say about prices' informational content in real financial markets. Once their models incorporate speculative purchases, we find KM and GS traders then have incentives to gather information about the market's purchasing behavior. Moreover, rising prices may reflect only the perception about increased market valuations in the future, independent of the real earnings of assets: rising prices may reflect market euphoria, not inside information on real earnings.

In a single period market, feedback signals are irrelevant. The state of the world is revealed at the same time that trades occur, and no trader has an opportunity to engage in speculation, even if he so desired. The relevance of feedback effects is not affected by the tatonnement market clearing process; feedback signals become important in a multi-period context, whether a hypothetical auctioneer clears markets or some other institution.

As explained in the quotations from Section 2 and again in the present section, KM and GS might have viewed their models as defining a series of temporary equilibria. If too many traders became informed, then market forces would react such that traders would have incentives to pay no cost for information and would obtain it from public prices. Similarly, if too little information was purchased in any given equilibrium, market forces would act to correct the situation in subsequent equilibria.

The model in Chapter 4 specifically examines what market forces come into play if uninformed traders continually draw inferences from

informed traders' behavior. We posit that these forces will actually induce the informed traders occasionally to take advantage of this feedback through a manipulation. Like KM and GS, our model contains forces to keep the equilibrium in steady state where prices are shown to deviate from fundamentals with positive probability. If too much manipulation occurs, uninformed traders react by ignoring purchasing signals by the informed. Too little manipulation will uninformed traders to mimic the behavior identified in the KM and GS models, which in turn gives the informed traders incentives to attempt to manipulate.

Chapter 4 does not present a nonequilibrium model. Markets clear each period, and we solve for the equilibrium strategies adopted by the informed and uninformed traders.

5. ALTERNATIVE MODELS OF SPECULATIVE BUBBLES

The model presented in Chapter 4 contains a number of firsts. That model presents the first game theoretic analysis of speculative bubble behavior, and the first derivation of bubbles from fully rational, payoff-maximizing behavior.

To assist the reader in determining the novelty of our approach to speculative bubble behavior, we present in this section a brief overview of the bubble literature. Many of these models are not directly relevant to our own work in Chapters 2, 3, and especially Chapter 4. However, these alternative models may give insights into considerations other theorists found important in modeling bubbles.

In the Rational Expectations literature, one stream employs a

finite number of infinitely-lived individuals. Within these models market traders would always buy an asset if its price falls below its fundamental.¹ The buyers would then want to hold the asset and receive its dividend stream forever. This possibility excludes so-called "negative bubbles." When the price exceeds the market fundamental, the infinitely-lived individuals would want to sell the asset short forever. However, even without short selling, no one would want to purchase an asset whose price exceeds its fundamental unless they expected to realize a capital gain from reselling it.

Tirole (1982) proves that if all agents intend to sell in finite time, then at some point no one will be holding the asset - a condition violating the market-clearing equilibrium notion at a positive price. Tirole's exclusion of price bubbles extends to economies with differential information as well.

For the infinitely-lived individuals economy, Tirole states that "unless traders have different priors about the value of a given asset or are able to use the corresponding market for insurance purposes, this market does not give rise to gains from trade."

(Tirole (1982), p.1163). However Tirole failed to recognize that agents may have the same prior beliefs about a given asset's value without knowing they agree. This topic is expanded in Chapter 2. Thus traders may still speculate when they have concordant, but not

¹Flood and Garber (1980) introduced this term into the economics literature. In financial theory, the fundamental is the present discounted value of the stream of dividends. However, individuals generally condition their expectation for the stream of dividends on differential information.

common knowledge beliefs. Indeed most speculators determine financial asset's resale profitability based on how they believe the rest of the market values the assets.

Tirole would argue that the very willingness of others to trade conveys information, and consequently no agent in Tirole's model would allegedly accept a trade. However, individuals in his economy could not distinguish between traders with better information, and hence differing posteriors, versus traders with different priors. Unless the traders know that they share the same prior beliefs, and Tirole's model provides no mechanism to communicate this information, then the agents might well trade - even under Tirole's assumptions. See also Hirshleifer's (1984) critique of Tirole's model.

In overlapping generations economies, Tirole (1985) summarized the literature expanding Samuelson's (1958) pure consumption loan model; the main contribution of that literature was to show money can have positive value even though its market fundamental is zero.

The traditional approach to speculative bubbles, as typified in the work by Kindleberger (1978), explains bubble formation as irrational behavior. Economists have difficulty attributing observed phenomena to irrationality, because it violates one of the fundamental principles of economic theory. Any problem or perplexing event can be explained away by appealing to sufficient irrational behavior. This is one reason why the Bayes game model in Chapter 4, which employs payoff-maximizing (rational) strategies, offers such a significant contribution to the theory of speculative bubbles.

Other works have emphasized mathematical treatments of price

trajectories and examined conditions that could lead to explosive price paths. Articles on price paths over time include Blanchard (1979), Blanchard and Weston (1981), and Obstfeld and Rogoff (1983).

Finally, a stream of the literature has examined how "extrinsic" factors, such as sunspots or the price of gold, can influence market participants in such a way that their expectations become self-fulfilling. Canzoneri (1984) has shown how gold prices can affect an unpegged foreign exchange regime, and his work illustrates exchange rate deviations that approximate bubbles. Cass and Shell (1983) have written the definitive work on sunspot equilibria.

6. DISSERTATION OVERVIEW

Using the material in Chapter 1 for background, this dissertation will study speculative markets and explore some new areas of uncertainty in economic decision-making. The next chapter discusses probabilities and higher-order beliefs, which can lead to speculation based on the unknown information others possess. Chapter 2 provides a framework for common knowledge, several numerical examples, and two propositions that clarify the conditions that guarantee a consensus equilibrium.

Chapter 3 applies common knowledge beliefs to the major asset pricing paradigms. The common knowledge literature itself is esoteric and concentrated among a few writers in the Journal of Economic Theory. This chapter will hopefully broaden the interest to financial economists and others who employ models of market valuation. For three of these paradigms, homogeneous beliefs alone will not

yield the conventional formulation of these models; hence we impose agreed common knowledge beliefs to derive the familiar forms.

Chapter 4 presents a Bayesian Nash equilibrium of speculative bubbles and reexamines the "feedback effect" induced by informed trader purchases. This chapter presents the first game theoretic analysis of speculative bubbles as well as the first fully rational, payoff-maximizing approach to bubble formation.

7. REFERENCES

- Aumann, Robert J., (1976), "Agreeing to Disagree," Annals of Statistics, Vol. 4, pp.1236-1239.
- Blanchard, Olivier Jean, (1979), "Speculative Bubbles, Crashes and Rational Expectations," Economic Letters, Vol. 3, pp.387-389.
- and Mark W. Watson, (1982), "Bubbles, Rational Expectations, and Financial Markets," in Crises in the Economic and Financial Structure, ed. by P. Wachtel, Lexington, Mass.: Lexington Books, 1982.
- Canzoneri, Matthew B., (1983), "Rational Destabilizing Speculation and Exchange Intervention Policy," Journal of Macroeconomics, Vol. 36, pp.59-89.
- Cass, David and Karl Shell, (1983), "Do Sunspots Matter?," Journal of Political Economy, Vol. 91, pp.193-227.
- DeGroot, Morris H., (1974), "Reaching a Consensus," Journal of the American Statistical Association, Vol. 60, pp.118-121.
- Flood, Robert P., and Peter M. Garber, (1980), "Market Fundamentals Versus Price-Level Bubbles: The First Tests," Journal of Political Economy, Vol. 88, No. 4, pp.745-770.
- Geanakoplos, John D., and Heraklis M. Polemarchakis, (1982), "We Can't Disagree Forever," Journal of Economic Theory, Vol. 28, pp.192-200.
- Grossman, Sanford J., (1981), "An Introduction to the Theory of Rational Expectations Under Asymmetric Information," Review of Economic Studies, Vol. 48, pp.541-559.
- and Joseph E. Stiglitz, (1976), "Information and Competitive Price Systems," American Economic Review, Vol. 66, pp.246-253.
- Hirshleifer, Jack, (1984), "Two Models of Speculation and Information," UCLA Department of Economics Working Paper #329, June 1984.
- Kihlstrom, Richard E., and Leonard J. Mirman, (1974), "Information and Market Equilibrium," Bell Journal of Economics, Vol. 5, pp.357-376.
- Kindleberger, Charles P., (1978), Manias, Panics, and Crashes, (New York: Basic Books, 1978).

- Mertens, J.-F., and S. Zamir, (1985), "Formulation of Bayesian Analysis for Games with Incomplete Information," International Journal of Game Theory, Vol. 14, pp.1-29.
- Milgrom, Paul, and Nancy Stokey, (1982), "Information, Trade, and Common Knowledge," Journal of Economic Theory, Vol. 26, pp.17-27.
- Obstfeld, Maurice, and Kenneth Rogoff, (1983), "Speculative Hyperinflation in Maximizing Models: Can We Rule Them Out?," Journal of Political Economy, Vol. 91, pp.675-687.
- Samuelson, Paul A., (1958), "An Exact Consumption-Loan Model of Interest With or Without the Social Contrivance of Money," Journal of Political Economy, Vol. 66, pp.467-482.
- Tirole, Jean, (1982), "On the Possibility of Speculation under Rational Expectations," Econometrica, Vol. 50, pp.1163-1181.
- (1985), "Asset Bubbles and Overlapping Generations: A Synthesis," Econometrica, Vol. 53, No. 5, pp.1071-1100.

CHAPTER 2

COMMON PRIORS AND COMMON KNOWLEDGE POSTERiors

1. INTRODUCTION

In economic models that incorporate uncertainty, individuals typically face intrinsic uncertainty over the unknown beliefs, preferences, income constraints, and other attributes of others in the model. This chapter shows how intrinsic uncertainty over the unknown probability beliefs of others can lead to a lack of consensus about the probabilities for various states of the world. The main contribution of this chapter is to prove that the popular common knowledge consensus equilibrium is guaranteed only under a highly restrictive condition not previously recognized in the literature.

To eliminate disagreement associated with heterogeneous probability beliefs, theorists in economics and finance typically assume individuals have "homogeneous beliefs." This chapter will show that a "homogeneous beliefs" assumption is not sufficient to achieve consensus, and that a stronger "agreed common knowledge beliefs" assumption is actually required. This chapter is primarily a methodological essay, and it is intended to lay a strong foundation for the speculation based on intrinsic uncertainty applications in Chapter 3. Thus the focus of this chapter will be on the intrinsic uncertainty itself; we will show the impact of this uncertainty in the form of capital gains speculation in Chapter 3.

Intrinsic uncertainty can be eliminated by imposing common

knowledge. When the value of some parameter θ is common knowledge among agents, it means that each agent knows the value of θ , knows other agents know the value, knows other agents know he knows the value, on through an infinite regression.

This chapter clarifies the distinction between "same priors" and "common knowledge priors." In words, people with the same priors agree on the likelihood of some outcome. However, they can agree without knowing they agree; this phrase is the recurring theme of Chapter 2. With agreed common knowledge priors individuals not only have the same priors but know they have the same priors.

In the following sections we prove that Aumann's (1976) proposition on the relation of common knowledge events to common knowledge beliefs is invalid under the "same prior" assumption with which it is stated. The first proposition proves if two people have the same priors and their posteriors for an event are common knowledge, then - contrary to the widely held view - these posteriors may be unequal. The two people can agree to disagree.

This result is truly significant, because we have obtained the precise opposite conclusion of the common knowledge literature,¹ even under their existing rigid assumptions. Many of the best economic

¹The common knowledge concept has been employed to show how consensus among subjective probabilities will terminate market trading (Tirole 1982, Milgrom and Stokey 1982, Geanakoplos and Polemarchakis 1982), betting or contingent agreements (Sebenius and Geanakoplos 1983), and further revision of posterior probabilities (Aumann 1976). These works do not distinguish between common priors and common knowledge priors. These same works go to great lengths, by contrast, to point out the distinction between common posteriors and common knowledge posteriors.

theorists in the profession have contributed to or written about the common knowledge consensus result. All of these theorists agree on its validity as stated. By showing the possibility of multiple equilibria under a "common priors" regime, we have clarified and extended the very fundamental assumptions that underpin the existence and uniqueness of an equilibrium in economic theory.

The intuitive logic for this multiplicity of equilibria can be stated as follows. If two people merely have the same prior, they could not infer whether differences in their posteriors are due to different priors or due to differences in information. If they could somehow find out that the differences in their posteriors are solely due to informational asymmetries, then they might arrive at consensus.

To guarantee equilibrium consensus we impose common knowledge on the existing "common prior" assumption. The chapter then contains a second proposition stating if both the priors and posteriors are common knowledge and the priors are equal, then the posteriors must be equal. This proposition correctly restates the popular consensus result currently circulating under just a common priors assumption.²

In this chapter we also introduce a new rigorous formulation of higher-order probability beliefs. We show how it becomes fairly easy

²The present paper is an excerpt from a longer unpublished manuscript Guth (1984). In a subsequent work, Aumann (1987) conceded the technical point made in this paper: his theorem actually requires common knowledge priors. However, Aumann would seek to minimize this omission by arguing that this is what theorists had intended all along. In contrast, we have found many financial economics models can achieve equilibrium consensus without assuming common knowledge priors. Moreover, we argue here that this form of intrinsic uncertainty was not generally recognized - and certainly was not published - in the various literatures that use common beliefs assumptions.

to represent a statement such as "i believes j believes i assigns probability .5 to state A" in the new framework.

Thus this chapter is intended to refine the common knowledge concept and discuss common knowledge first-order, second-order and higher-order beliefs, and not merely common knowledge events.

Aumann's definition is somewhat circular, in that to define a common knowledge event he must first postulate common knowledge prior beliefs.

Intrinsic uncertainty and higher-order beliefs are important in financial economics, game theory, and indeed any field where the payoffs to people are determined in part by the beliefs and actions of others. For example, intrinsic uncertainty may be the single most important determinant of volatility on the New York stock exchange! This significance for financial economics explains why we devote the entire next chapter to applications of intrinsic uncertainty and common knowledge beliefs in the financial theory paradigms.

2. FRAMEWORK FOR COMMON KNOWLEDGE EVENTS

Consider a model with n people and operating under uncertainty with the value of θ denoting the state of the world. The random variable θ is defined on the probability space (Ω, F, ρ) . We can interpret (Ω, F) as the space of states of the world, ρ as the probability measure defined on the σ -field F , and Φ_i as the information partition of individual i which separates the elements of Ω into disjoint subsets and completely spans the space. In game theoretic terms, the information partitions correspond to information

sets comprising indistinguishable nodes for a given player.

Let $\omega \in \Omega$ denote a state of the world. Each individual i has a probability distribution, p_i^1 , over states in Ω with $p_i^1(\omega)$ denoting the probability i assigns to ω . For any set $\psi \subseteq \Omega$, $p_i^1(\psi)$ is the probability i assigns to the event $\theta \in \psi$. We refer to the distribution p_i^1 as the first-order beliefs of i , hence the probability distribution is written with "1" as a superscript.

Definition 1: The individuals have common (same) first-order beliefs when $p_i^1(\omega) = p_j^1(\omega)$, for all individuals i, j , over events ω .

Following Aumann (1976) when the actual state of nature is ω , let individual i receive information that the state is an element of a class of his partition, $\Phi_i(\omega)$, where $\Phi_i(\omega) \in \Phi_i$. Thus i knows the event E occurs with probability 1 only if $\Phi_i(\omega) \subset E$. Depending on his prior probability distribution and how he updates his prior beliefs, i can potentially assign any probability in the interval $[0,1]$ to E when $E \subset \Phi_i(\omega)$.

Let J denote the join of the partitions $\Phi_1, \Phi_2, \dots, \Phi_n$, where the join of the partitions, as distinguished from the join of elements of a set,³ is defined in Lattice Theory as the "finest common coarsening" of the partitions. Let $J(\omega)$ be that element of the join containing ω .

³The meet of two elements, a and b , of a lattice can be written " $a \wedge b$ " or " $a \cdot b$ " and derives from the intersection operator of set theory. The join of the two elements of the lattice is written " $a \vee b$ " or " $a + b$," and is based on the union operator of set theory. The intersection operator of set theory is homeomorphic to the AND operator of Boolean algebra. Indeed, a Boolean algebra is a special class of lattices. Similarly, there exists a homeomorphism between the union operator of set theory and the OR operator of Boolean algebra.

The basis for meets and joins of partitions in Lattice Theory becomes coarsenings and refinements of the partitions. The following illustration is taken from Donnellan (1968, p.18-19). If a set $(abcd)$ is partitioned into $P_1 = (abc/d)$ and $P_2 = (ab/cd)$, then the meet of P_1 and P_2 is $(ab/c/d)$. The meet of the partitions is the coarsest common refinement of the partitions. The partition P_1 contains two classes: (abc) and (d) .

If $P_3 = (a/b/cd)$ and $P_4 = (ab/c/d)$, then the join of P_3 and P_4 , written $P_3 + P_4$ is (ab/cd) . The join of the partitions gives a coarsening of either partition. $P_3 + P_4$ is characterized as the most refined partition having the property that overlapping classes of P_3 and P_4 are wholly contained in just one of its classes: the finest common coarsening.⁴

Definition 2: An event E is common knowledge at ω if $J(\omega) \subset E$.⁵

The terms "same prior" and "homogeneous beliefs" as they have been used in economic theory and financial economics refer to Definition 1. Note that Definition 2 is stated only for events, not for beliefs. We must now turn to a rigorous definition of probability assignments or beliefs that are common knowledge.

⁴The common knowledge literature of Aumann (1976), Milgrom and Stokey (1982), and Geanakoplos and Polemarchakis (1982), has used the term "meet of the partitions" to stand for their finest common coarsening. For clarification see Donnellan (1968), Birkhoff (1961), or Szasz (1963).

⁵Aumann (1976) proposed this definition for a common knowledge event substituting the expression for "meet" to "join."

3. FRAMEWORK FOR INTRINSIC UNCERTAINTY

The first-order beliefs of the other $n-1$ agents are typically unknown to an agent, but standard Bayesian reasoning encourages the treatment of any unknown by assigning a subjective probability distribution to it. Therefore, in addition to his own distribution p_i^1 for θ , each agent i must also have a subjective probability distribution for the unknown first-order beliefs $(p_1^1, \dots, p_{i-1}^1, p_{i+1}^1, \dots, p_n^1)$ of the other $n-1$ agents in the model. Let p_i^2 denote this subjective probability distribution over the first-order beliefs of the other $n-1$ agents.⁶

Definition 3: The second-order belief of agent i , p_i^2 , is a probability distribution for agent i over the possible values of $(p_1^1, \dots, p_{i-1}^1, p_{i+1}^1, \dots, p_n^1)$.

Note that agent i 's second-order belief is a probability distribution over the possible probability distributions the other agents might have for the values of θ . We could now proceed to define third-order beliefs, which would be a probability distribution over second-order beliefs, and so on for an infinite regress.⁷

⁶There exists no unique homeomorphism between the first-order, second-order, or any higher-order beliefs of an individual. Theoretically, the second-order beliefs can be defined as a separate random variable with a separate probability space (Γ, G, p^2) , where $\gamma \in \Gamma$ denotes an $(n-1)$ -tuple of possible probability distributions for the other $n-1$ agents, and the probability measure p^2 is defined on the σ -field G .

⁷To discuss the complexity of the infinite hierarchy of higher-order beliefs, it is necessary to consider a mathematical framework consisting of a Cartesian product of the set of all possible Borel-

Given a definition for second-order beliefs, let us now turn to the issue of how intrinsic uncertainty crops up in the prevailing model of common knowledge. Let E be an event, and let $p_i^1[E | \Phi_i(\omega)]$ denote agent i 's posterior probability for E given the true state is a member of $\Phi_i(\omega)$. Then for all $\omega \in \Omega$ and for any agent i

$$p_i^1[E | \Phi_i(\omega)] = p_i^1[E, \Phi_i(\omega)] / p_i^1[\Phi_i(\omega)]. \quad (4)$$

Although Aumann's (1976) definition of common knowledge is stated in terms of an event, E , Aumann's proposition actually relies on the concept of "common knowledge posteriors." To fill the gap in Aumann's proof, Geanakoplos and Polemarchakis (1982) proposed a variation of Aumann's common knowledge event definition.

Let $\Psi_i = \{\omega' \in \Omega : p_i^1[A | \Phi_i(\omega')] = c\}$, and let $J_{ij}(\omega)$ be the element of the join of Φ_i and Φ_j that contains ω . Then according to Geanakoplos and Polemarchakis, if $J_{ij}(\omega) \subset \Psi_i$, the posterior $p_i^1[A | \Phi_i(\omega)] = c$ will be common knowledge between i and j .

If the Aumann-Geanakoplos-Polemarchakis (AGP) condition, $J_{ij}(\omega) \subset \Psi_i$, is imposed on Aumann's model, it severely restricts the events, A , which can be common knowledge and also restricts the nature of the partitions for each of the agents. For example, if $J_{ij}(\omega) = \{1, 2, 3, 4, 5, 6\}$, and $\Phi_i = \{1, 2\}, \{3\}, \{4, 5, 6\}$, then the only posterior that can be common knowledge under the AGP condition is

$p_i^1[(1, 2, 3, 4, 5, 6) | \Phi_i(\omega)] = 1$. But surely the profession is interested

measurable beliefs of all orders for all the agents in the economy. Mertens and Zamir (1985) have developed such a framework, but they define higher-order beliefs by induction based on the first-order beliefs.

in a proposition that tells us more than that the probability of the whole state space is one. And even Geanakoplos and Polemarchakis (1982) found this underlying assumption rather unmanageable:

the posteriors of the two agents on an event A can be common knowledge only if it is the case that the probability of A conditional on any element of (ϕ_j) contained in $(J_{ij}(\omega))$ is the same. But this is a rather unlikely situation - for most events A it will not be the case. As a result, Aumann's proposition demonstrates the identity of posteriors for a very limited class of events. (p.194)

In general, a posterior can be common knowledge without applying the AGP condition. Aumann (1976) stated, "In our case, if 1 and 2 tell each other their posteriors and trust each other, then the posteriors are common knowledge." No restrictions are necessary on the individual partitions of the state space or the events for these posteriors to become common knowledge; consequently, since Aumann's proposition actually requires the hidden AGP assumption for its proof, we would prefer to formulate an alternative expression for common knowledge posteriors.⁸

⁸DeGroot (1974) described a revision process where agents, with different information about the unknown value of some parameter θ , call out their probability beliefs back and forth and then successively update the measures based on the posteriors they hear. In DeGroot's model, the agents eventually converge to equal posterior beliefs, i.e., they assign the same probability to particular values of the parameter. DeGroot shows how the agents might reach a consensus and form a common subjective probability distribution for θ by pooling their individual opinions. While the consensus result may seem appealing, it typically only holds for the limiting case where the agents' priors for the possible values of θ are equal and common knowledge. The intuition rests on individuals with common priors for the possible values of θ not knowing that others assign the same probability to θ . The absence of common knowledge can give rise to intrinsic uncertainty, where individuals face another unknown in the form of the probability distributions others employ.

In words, a posterior, $p_i^1[E|\Phi_i(\omega)] = c$, is common knowledge between i and j when both i and j know it, i knows j knows it, j knows i knows j knows, and so on. We can construct a rigorous definition for common knowledge probability measures using our K -order belief, intrinsic uncertainty framework. To simplify the notation, let us consider a two-person model with agents i and j . Thus the second-order belief for agent i has been reduced from a probability distribution over the $(n-1)$ -tuple, $(p_i^1, \dots, p_{i-1}^1, p_{i+1}^1, \dots, p_n^1)$, to simply a probability distribution over p_j^1 . Similarly p_j^2 is now a probability distribution over p_i^1 .

In a two-person model, the third-order belief of agent i , p_i^3 , is a probability distribution over p_j^2 . Thus p_i^3 is a probability distribution for agent i over the possible probability distributions p_j^2 agent j might employ which in turn is a probability distribution over p_i^1 .

We have argued that $p_i^2(\omega)$ should typically be a genuine, nondegenerate probability distribution as the actual probability $p_j^1(\omega)$ that agent j assigns to the event ω is unknown to agent i . But common knowledge is the opposite extreme of complete intrinsic uncertainty, and for the mathematics of higher-order beliefs, common knowledge first-order beliefs will often imply mass points on the higher-order beliefs: all probability will be assigned to the known value of the first-order belief.

If agent i assigned probability to more than one probability distribution in his second-order belief for agent j , then he must be

uncertain as to the first-order probability distribution employed by agent j . Hence agents i and j cannot "know" they agree if their respective second-order beliefs for each other assign positive probability to more than one first-order belief of the other agent. The following definitions apply to a two-person model with agents i and j , and $E \subset \Omega$.

Definition 3: j is said to know $\rho_i^1(E) = c$ if and only if $\rho_j^2(E) = c$.

Definition 4: i is said to know j knows $\rho_i^1(E) = c$ if and only if $\rho_i^3(E) = \rho_j^2(E) = \rho_i^1(E) = c$.

Definition 5: j is said to know i knows j knows $\rho_i^1(E) = c$ if and only if $\rho_j^4(E) = \rho_i^3(E) = \rho_j^2(E) = \rho_i^1(E) = c$.

Definition 6: The first-order posterior $\rho_i^1[E|\Phi_i(\omega)] = c$ is common knowledge between i and j at ω if and only if for any even integer K , and any odd integer L , $\rho_j^K[E|\Phi_i(\omega)] = \rho_i^L[E|\Phi_i(\omega)] = \rho_i^1[E|\Phi_i(\omega)] = c$.

Note that in Definition 6 the conditioning event $\Phi_i(\omega)$ is typically known only to agent i . The posterior for agent j conditioned on agent i 's informed signal corresponds to the posterior agent j would assign to the event given the informed class of i 's partition.

When the true state is ω , agent i is informed that the state lies within the class $\Phi_i(\omega) \subset \Phi_i$, and agent j is informed the true state lies within the class $\Phi_j(\omega) \subset \Phi_j$. Following Aumann (1976), it is then common knowledge between i and j that the true state lies within $J_{ij}(\omega)$. When the agents are given further information about a

posterior for some specific event E , they can refine the set $J_{ij}(\omega)$ to those states where the posterior is possible as well.

Proposition 1: Let $\omega, \omega' \in \Omega$, and let $c \in [0,1]$ and $d \in [0,1]$. Define the set $C = \{\omega': \omega' \in J_{ij}(\omega) \text{ and } \rho_j^2[E|\Phi_i(\omega')] = c\}$, and similarly define the set $D = \{\omega': \omega' \in J_{ij}(\omega) \text{ and } \rho_i^2[E|\Phi_j(\omega')] = d\}$. Let the posteriors $\rho_i^1[E|\Phi_i(\omega)] = c$ and $\rho_j^1[E|\Phi_j(\omega)] = d$ be common knowledge between i and j at ω . Then $c \neq d$, if

$$\rho_i^2[E \cap D] / \rho_i^2[D] \neq \rho_j^2[E \cap C] / \rho_j^2[C].$$

Proof: Since $\rho_i^1[E|\Phi_i(\omega)] = c$ is common knowledge between agents i and j at ω , then in general

$$c = \rho_i^1[E|\Phi_i(\omega)] = \rho_j^K[E|\Phi_i(\omega)], \quad (5)$$

where $K = \text{even interger}$, and in particular

$$c = \rho_i^1[E|\Phi_i(\omega)] = \rho_j^2[E|\Phi_i(\omega)]. \quad (6)$$

By definition $C = \{\omega': \omega' \in J_{ij}(\omega) \text{ and } \rho_j^2[E|\Phi_i(\omega')] = c\}$ and $D = \{\omega': \omega' \in J_{ij}(\omega) \text{ and } \rho_i^2[E|\Phi_j(\omega')] = d\}$. Let $C = \bigcup_y \Phi_i(\omega)^y$. Since the classes $\Phi_i(\omega)^y$ are disjoint and $\rho_j^2[E|\Phi_j(\omega)] = c$ throughout C , then

$$\begin{aligned} c &= \rho_j^2[E|\Phi_i(\omega)^y] \\ &= \rho_j^2[E \cap \Phi_i(\omega)^y] / \rho_j^2[\Phi_i(\omega)^y] \\ &= \rho_j^2[E \cap C] / \rho_j^2[C]. \end{aligned} \quad (7)$$

Similarly, it follows that $\rho_i^2[E \cap D] / \rho_i^2[D] = d$. Therefore $c \neq d$, as

$$p_i^2[E \cap D] / p_i^2[D] \neq p_j^2[E \cap C] / p_j^2[C],$$

and $c = d$ if

$$p_i^2[E \cap D] / p_i^2[D] = p_j^2[E \cap C] / p_j^2[C].$$

Proposition 1 shows that even when two people have the same priors and their posteriors for an event E are common knowledge, as when they both call out their posteriors, then their posteriors may remain unequal.

This is precisely the opposite result obtained by the common knowledge literature, and so we must clarify the reason for the difference. The common knowledge literature has assumed that if two people have the same prior over the events in the state space, i.e., $p_i^1(\omega) = p_j^1(\omega)$ for events ω , then the two also have the same second and higher-order beliefs. Yet the two can have the same first-order beliefs without knowing they are the same. This subtle distinction between "same priors" and common knowledge priors is explained in more detail in Aumann (1987).

When an individual hears someone call out a posterior probability for an event which is different from his own posterior for the event, he can attribute the difference between the posteriors to either different information, different priors, or both. In short, the individual could not distinguish between someone with the same prior and different information versus someone with a different prior.

If the first-order priors for all the agents in the model were the same and were common knowledge, then the agents could attribute any difference in posteriors to informational asymmetries. Aumann's

argument should be stated "people with the same and common knowledge priors cannot agree to disagree."

4. ILLUSTRATION OF INTRINSIC UNCERTAINTY WITH SAME PRIORS

Perhaps the easiest way to demonstrate the gist of Section II is in a series of counter-examples which point out the distinction between same priors, agreed common knowledge priors, and common knowledge priors that are not equal.

Example 1: Same Priors, Different Posteriors

Suppose $\Omega = \{\alpha, \beta, \gamma, \delta\}$, $\Phi_i = \{\alpha\beta, \gamma\delta\}$, $\Phi_j = \{\alpha\gamma, \beta\delta\}$, $E = \alpha\delta$, and $\omega = \alpha$. When Ω is finite, the σ -field F is taken to be the power set 2^Ω . Suppose both i and j have the same prior probability distribution over the four events in Ω : (.4, .1, .2, .3) for $(\alpha, \beta, \gamma, \delta)$. We can think of these priors as based on a history of ten trials, where α came up four times, β came up once, and so on. Then

$$p_i^1[E|\Phi_i(\omega)] \equiv p_i^1(\alpha\delta|\alpha\beta) = .4/.5 = 4/5, \text{ and } p_j^1(\alpha\delta|\alpha\gamma) = 4/6.$$

At this point it might be erroneously inferred that by hearing agent i call out the posterior $4/5$ for the event E , agent j can somehow tell that agent i received the informed signal $\alpha\beta$.

In fact, as far as agent j is concerned, agent i might well have received the informed signal $\gamma\delta$. Although the agents have the same prior probability distribution for the events in Ω , they do not know that they employ the same prior.

One of the fundamental principles of Bayesian subjectivist probability theory is that an individual will assign positive probability

to every possible value of an unknown, unless he has information to exclude some of the possibilities from consideration. When i and j hear that their posteriors are not equal, they can attribute the difference to informational asymmetries, different priors, or both. At the very least, agent j could not exclude the possibility of different priors.

Therefore, if agent j assigns positive probability to i employing any of the following priors: $(.2, .3, .1, .4)$; $(.1, .4, .1, .4)$; $(.4, .1, .1, .4)$; $(.35, .6, .01, .04)$; then the posterior $\rho_i^1[E|\Phi(\omega)] = 4/5$ would mean i received the informed signal $\Phi_i(\omega) = \gamma\delta$, rather than $\Phi_i(\omega) = \alpha\beta$.

The same is true for agent i . Upon hearing the posterior $\rho_j^1[E|\Phi_j(\omega)] = 4/6$, agent i might conclude that agent j received the signal $\Phi_j(\omega) = \alpha\gamma$. But since he does not know how agent i assigns prior probability to the elements of Ω , agent j would assign some positive probability in his second-order distribution ρ_j^2 to the possible priors $(.3, .2, .1, .4)$; $(.2, .2, .2, .4)$. For both of these possible priors, agent j 's posterior for E of $4/6$ would indicate he received the signal $\Phi_j(\omega) = \beta\delta$.

Finally, we must consider that the agents could draw the correct inference about informed signals from erroneous assumptions about the priors. For example, if agent j assigns positive probability to agent i using the prior $(.4, .1, .3, .2)$ instead of agent i 's actual prior $(.4, .1, .2, .3)$, agent j might still correctly infer that i 's posterior for E of $4/5$ means agent i received the signal $\Phi_i(\omega) = \alpha\beta$.

Example 2: Same Priors, Same Posteriors

Now let us change the numerical values of the common prior for i and j . We again take $\Omega = \{\alpha, \beta, \gamma, \delta\}$, $\Phi_i = \{\alpha\beta, \gamma\delta\}$, $\Phi_j = \{\alpha\gamma, \beta\delta\}$, $E = \alpha\delta$, and $\omega = \alpha$. But now let both i and j assign equal prior probability to the four events in Ω : $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ for $(\alpha, \beta, \gamma, \delta)$. Then $p_i^1[E|\Phi_i(\omega)] \equiv p_i^1(\alpha\delta|\alpha\beta) = \frac{1}{4} / \frac{1}{2} = 1/2$, and $p_j^1(\alpha\delta|\alpha\gamma) = 1/2$. So we find equal posteriors for the two agents without any further revising. Yet even for this case the agents cannot know whether the equality in their posteriors derives from their having the same priors or having different priors and different information.

Example 3: Same Prior, Different Posteriors

Suppose i and j have the same prior for a random variable that has two possible outcomes $X1$ and $X2$; let the prior probability for $X1$ be $\frac{1}{2}$ and the prior for $X2$ be $\frac{1}{2}$. Let E be the event that the random variable comes up $X1$ on the next trial. Suppose that each person is permitted to observe one previous outcome, and that these trials came up $X1$ for agent i and $X2$ for agent j . If each one's information consists solely of his endowed prior distribution and the outcome of his one observation, then the posteriors for E will be $2/3$ and $1/3$ respectively.

If each one then informs the other of his posterior, what will they conclude about the previous outcomes? The common knowledge literature suggests that agent j would immediately be able to infer that i 's observation came up $X1$ and that i would infer j 's trial came up $X2$, so that both people would revise their posteriors to $1/2$.

But how could agent i infer anything from agent j 's posterior unless he knew the prior upon which the posterior is based? It is clear that neither agent would know on how many observations the other's posterior is based, let alone the outcome of these observations. Therefore, it is not clear that the $2/3$ and $1/3$ posteriors must be revised to equality.

This counter-example is based on the example provided by Aumann (1976) at page 1238. Aumann recognized that if the agents did not know how many trials are allowed the other one, then their posteriors might well remain divergent. However, even if each knew the number of trials the other one observed, their posteriors still might diverge since each is uncertain about the other's prior.

Example 4: Same Prior, Different Posterior, Further Refinement

When the agents' posteriors are announced and common knowledge, then each agent has three pieces of information. He knows his own prior, his own private signal (in the form of a class of his partition), and he knows both his and the other agent's posterior is common knowledge. For many problems this amount of information may suffice to lead to posterior revision and equality, even when the priors are only the same but not common knowledge.

Consider $\Phi_i = \{1, 2, 3\}, \{4, 5\}, \{6, 7, 8, 9\}$; $\Phi_j = \{8, 1, 4\}, \{2, 5, 7, 9\}, \{3, 6\}$; $\omega = 6$; $\Phi_i(\omega) = \{6, 7, 8, 9\}$; $\Phi_j(\omega) = \{3, 6\}$; $M_{ij}(\omega) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$; $E = (3,7)$; and both i and j assign equal $(1/9)$ probability to the nine events in Ω .

Both agent i 's and agent j 's posterior for E , $p_i^1(E|6,7,8,9) = \frac{1}{2}$

and $p_j^1(E|3,6) = \frac{1}{2}$, are common knowledge. In the Aumann framework, the agents partitions are common knowledge. Thus when agent j calls out positive probability for the event $(3,7)$ given his private signal, it becomes common knowledge that $\Phi_j(\omega) \neq \{8, 1, 4\}$ since $E \cap \{8, 1, 4\} = \emptyset$. Similarly, it becomes common knowledge that $\Phi_i(\omega) \neq \{4, 5\}$.

Using the notation in the proof of Proposition 1, we could define the set $C = \{\omega': \omega' \in J_{ij}(\omega) \text{ and } p_j^2[E|\Phi_i(\omega')] = 1/4\}$, and analogously define the set $D = \{\omega': \omega' \in J_{ij}(\omega) \text{ and } p_i^2[E|\Phi_j(\omega')] = 1/2\}$. We could then note that $\{8, 1, 4\} \not\subset D$ and $\{4, 5\} \not\subset C$, so these states would be removed from consideration when the posteriors are common knowledge. From the formal definition of common knowledge posteriors, we can state that j knows i assigns zero probability to the set $\{4,5\}$ implies $p_j^2[4,5|\Phi_i(\omega)] = 0$, i knows j knows i assigns zero posterior probability to $\{4,5\}$ means $p_i^3[4,5|\Phi_i(\omega)] = 0$, so that

$$p_j^K[4,5|\Phi_i(\omega)] = p_i^L[4,5|\Phi_i(\omega)] = 0,$$

where K is an even integer, and L is an odd integer.

Whereas previously with only their private signals the set of states which were common knowledge between i and j at ω was given by $J_{ij}(\omega)$, now that the posteriors for E are common knowledge as well, the set of possible states which are common knowledge has been refined to $\{2, 3, 6, 7, 9\}$. Now i knows that the element "8" of his informed class $\{6, 7, 8, 9\}$ is not possible. Agent i thus revises his posterior to $p_i^1(E|6,7,9) = 1/3$.

The refinement of the set of common knowledge states can evolve

from common knowledge posteriors. We impose no restrictions on what events can have common knowledge posteriors, nor do we restrict the nature of the partitions as suggested by the AGP condition.

We are left at this stage with a posterior for i of $1/3$ conditioned on both his original informed signal and the refinement of the state space, and a posterior for j of $1/2$. The two have moved closer to consensus but have not reached it. In some examples the revision process that evolves from common knowledge posteriors may be sufficient to enable the agents to reach consensus without imposing further restrictions.

To illustrate how consensus is obtained in the present example, suppose we impose a new condition, namely, that the agents not only have the same priors but that these priors are common knowledge as well: the agents have agreed common knowledge priors.

If agent i 's prior was common knowledge, then by definition

$$\rho_j^K(\omega) = \rho_i^L(\omega) = \rho_i^1(\omega), \quad (8)$$

where K is an even integer and L is an odd integer, and condition (8) holds for all $\omega \in \Omega$. Similarly, if agent j 's prior is common knowledge, then

$$\rho_i^K(\omega) = \rho_j^L(\omega) = \rho_j^1(\omega). \quad (9)$$

The agents have "agreed" priors if their priors are equal:

$$\rho_j^1(\omega) = \rho_i^1(\omega), \quad (10)$$

again for all $\omega \in \Omega$. Combining (8), (9), and (10) means that every odd and even order belief of agent i equals every odd and even order belief of agent j for a particular event under consideration. The

notation in the following definition applies to a two-person model.

Definition 7: i and j have agreed common knowledge priors (ACKP) when for any pair of natural numbers, N and M , and $\omega \in \Omega$,

$$\rho_j^N(\omega) = \rho_i^M(\omega). \quad (11)$$

In the preceding example, the posterior $\rho_i^1[(3,7)|\Phi_i(\omega)] = 1/4$ immediately informs agent j that i received the information signal $\Phi_i(\omega) = \{6, 7, 8, 9\}$. Similarly, i knows j received the signal $\{3, 6\}$ when he hears the posterior $\rho_j^1[(3,7)|\Phi_j(\omega)] = 1/2$. Then each individual can determine that $\omega = 6$, in this particular example, by taking the intersection of $\{3, 6\}$ and $\{6, 7, 8, 9\}$. Thus both agent i and j would then assign zero posterior probability to $(3,7)$ conditioned on ACKP, their respective informed classes of their partitions, and the common knowledge posteriors.

A formal definition could also be written for ACKP for n agents, although the definition would be notationally dense. Agent i 's second-order beliefs, $\rho_i^2(\omega)$, would be a probability distribution over the first-order beliefs of the other $n-1$ agents for ω . Let $\rho_i^2(j(\omega))$ denote that element of $\rho_i^2(\omega)$ that corresponds to i 's belief about j . Then with ACKP, $\rho_i^2(j(\omega)) = \rho_j^1(\omega) = \rho_i^1(\omega)$. But then if all n agents have ACKP, then $\rho_i^2(k(\omega)) = \rho_j^2(j(\omega)) = \rho_j^1(\omega) = \rho_i^1(\omega)$, for any two agents j and k , and $\omega \in \Omega$.

5. CONSENSUS WITH AGREED COMMON KNOWLEDGE PRIORS (ACKP)

The consensus result of the common knowledge literature will typically apply to the limiting case where the agents have ACKP. Our

analysis has shown that when the agents merely have the same prior, they face uncertainty over the prior distribution employed by others. We view this intrinsic uncertainty as separate and distinct from the probability each assigns to elements of the state space Ω .

A corollary to Proposition 1 can now be derived where ACKP are imposed on the model by assumption. Suppose the two agents i and j have ACKP and their posteriors $\rho_i^1[E|\Phi_i(\omega)] = c$ and $\rho_j^1[E|\Phi_j(\omega)] = d$ are common knowledge at ω .

Whereas in general, $\rho_i^2[E|\Phi_j(\omega)]$ and $\rho_j^2[E|\Phi_i(\omega)]$ would be entire probability distributions, because the agents first-order posteriors are common knowledge it follows that $\rho_i^2[E|\Phi_j(\omega)] = d$ and $\rho_j^2[E|\Phi_i(\omega)] = c$, and indeed $\rho_i^K[E|\Phi_j(\omega)] = d$ and $\rho_j^K[E|\Phi_i(\omega)] = c$, for any even integer K .

When the second-order beliefs are equal to a number, rather than an entire distribution, then each agent "knows" with certainty the others' first-order belief for the event in question.

Define the set $C = \{\omega' : \omega' \in J_{ij}(\omega), \rho_i^1[E|\Phi_i(\omega)] = c\}$. Since the agents have the same first order priors by assumption, it follows that $\rho_i^1[E|\Phi_i(\omega)] = \rho_j^1[E|\Phi_i(\omega)] = c$. The partitions for i and j are common knowledge in this framework, so that j knows precisely what elements are in the set C . Agent j knows what elements are contained in each of i 's partitions; consequently, j knows which of i 's partitions are contained in C since they must satisfy the condition $\rho_j^1[E|\Phi_i(\omega)] = c$.

Similarly, we can define the set $D = \{\omega' : \omega' \in J_{ij}(\omega), \rho_j^1[E|\Phi_j(\omega)] = d\}$, and both i and j will know all the elements of the

set D by virtue of the ACKP assumption. The elements of C and D must be in $J_{ij}(\omega)$, since the posteriors are common knowledge at ω by assumption and it is common knowledge between i and j that the true event lies within $J_{ij}(\omega)$.

Write $C = \bigcup_y \Phi_i(\omega)^y$. Since the partitions $\Phi_i(\omega)^y$ are disjoint and $\rho_i^1[E|\Phi_i(\omega)] = c$ throughout C, then $c = \rho_i^1[E|C]$. Similarly, $\rho_j^1[E|\Phi_j(\omega)] = d$ throughout D, so that $d = \rho_j^1[E|D]$. Observe that the intersection of C and D is nonempty, since it must include ω , the true state of the world. Since $\{C \cap D\} \subset C$ and $\{C \cap D\} \subset D$, then

$$c = \rho_i^1[E|C \cap D], \quad (12)$$

and

$$d = \rho_j^1[E|C \cap D]. \quad (13)$$

The right-hand sides of (12) and (13) are equal from the "agreed" part of the ACKP assumption, namely, that agents i and j have the same prior. Therefore, $c = d$. This result establishes the following proposition.

Proposition 2: If i and j have ACKP, and their posteriors

$\rho_i^1[E|\Phi_i(\omega)] = c$ and $\rho_j^1[E|\Phi_j(\omega)] = d$ are common knowledge between i and j at ω , then $c = d$.

We have emphasized agreed common knowledge beliefs. It is certainly conceivable that the agents could have common knowledge priors that were not equal; it could be common knowledge between i and j that $\rho_i^1(\omega) \neq \rho_j^1(\omega)$, for $\omega \in \Omega$. Different common knowledge beliefs are a form of heterogeneous beliefs. Assuming "same prior beliefs" is more restrictive than "heterogeneous beliefs," but ACKP is even more restrictive than "same priors."

Does it make (economic) sense to talk about common knowledge but heterogeneous beliefs? In the real world we often witness two equally intelligent and knowledgeable mutual fund managers who have different and common knowledge beliefs about the market's future performance. Both mutual fund managers seem to have access to the same information, yet one may be bullish and one may be bearish on the market. When these conflicting prognoses for the market are publicly announced, we have a situation where heterogeneous beliefs are common knowledge.⁹

Proposition 2 follows directly from Aumann (1976), but it settles a lingering question on the fundamental assumptions necessary to derive consensus. Our goal in clarifying this result was not to herald the use of "common knowledge beliefs" assumptions in place of the standard "homogeneous beliefs," or "same prior beliefs" assumptions.

This work was intended, first, to refine the common knowledge concept in a way not previously defined rigorously in the literature: common knowledge beliefs not just events. Second, our results have demonstrated the importance of intrinsic uncertainty - the antithesis of common knowledge - and will hopefully encourage theorists to incorporate this phenomenon in future modeling of decision-making and consensus under uncertainty.

⁹To have heterogeneous, common knowledge beliefs, each mutual fund manager must consider himself either a better analyst or the recipient of better information than the party with whom he disagrees. Each must believe that in the final analysis he will be proven correct. This willingness to openly disagree also raises some interesting behavioral questions about aggressiveness and self-confidence (in one's own priors) that have not been addressed to date in the common knowledge literature.

6. SHOULD BELIEFS BE SUBSUMED IN THE STATE OF THE WORLD?

To answer the question posed in the title in the section title, we must split it into two separate questions: (1) is it technically justifiable to include other forms of uncertainty in the "state of the world?" (2) did theorists consider intrinsic uncertainty in much of the rational expectations, common knowledge, and financial economics literatures?

While it is technically feasible to redefine the state space to include other random variables besides the actual state of the world, we must question the rationale for doing so.¹⁰ In the conventional Arrow-Debreu contingent claims economy, the state variable refers to a "state of nature," and thus there were numerous extension using "rain or shine" choices or states "A and B." If these state variables were to include possible first-order beliefs as well, then they should also include possible second-order beliefs, on up to countable numbers of higher-order beliefs. Furthermore, this redefined state space would have to include, e.g., a unique "sunspot" value for each of the possible first-order beliefs, since some segment of the market may face uncertainty over sunspots. We would end up with a state space Ω ,

¹⁰When I first began studying intrinsic uncertainty in 1984, I too at first thought about redefining the state space so that events in the Aumann framework not only represented true states of the world, but also an event where, e.g., everyone knew everyone else's beliefs. As soon as that event became common knowledge, then the beliefs would be ipso facto common knowledge as well. However, I declined to proceed further with this redefinition, because it opened the possibility of including every (whimsical) attribute in the state of the world variable that some noninfinitesimal segment of the market believed to be relevant. The list of variables became endless.

where each $\omega \in \Omega$ is defined as a particular element on a Cartesian product space of true states, higher-order beliefs, sunspots, and any other relevant random variable.

Even if one wanted to state ω as a complete description of uncertain variables in the model, it would always be possible to split ω into several states each time a new random variable introduced uncertainty. Those who would argue that conditional on a given ω , "everybody knows everything" are confusing the distinction between ex ante beliefs and ex post realizations. Furthermore, when the state of the world is revealed to be, e.g., either rain or shine, it in no way limits the multiple a priori beliefs in the economy that could have achieved the price vectors and other endogenous variables in the revealed state.

The standard model for incomplete information in a multi-person context would assume individuals are given endowments, preferences, budget constraints, beliefs, and other fixed parameters. We could then pose comparative statics questions, e.g., what happens if beliefs are changed? What happens if preferences are changed?

By contrast, if all these exogenous specifications - which are potential sources of uncertainty - were included in the state space, then the states space itself would have to be redefined so that no meaningful comparative statics could take place. So we see that the great pitfall of redefining the state space to include any uncertain variable is that it creates extreme complexity for different permutations of all the uncertain variables and it mixes exogenous and endogenous variables, e.g., prevailing future price, because these

endogenous variables are legitimate objects of uncertainty as well.

Another important argument against redefining the state space to include any and all objects of uncertainty concerns the very nature of abstraction. In formulating a model, a theorist must abstract from reality and choose what parameters to look at and what to exclude. Implicit in any abstraction is the idea that certain variables or parameters that are important have intentionally been excluded for the sake of simplification. The mark of a good theorist then is choosing which variables to put into his model. Under a redefined state space, however, the theorist would find all the excluded abstractions creeping back into the model via the "state of the world."

Aumann (1987) seems to confuse the complexity of the redefined state space when he states at page 10, "neither a partition nor a prior is an event." (emphasis added). Yet $\omega \in \Omega$ is a sample point, while $A \subset \Omega$ is an event. Thus A would be a subset of true states and prior beliefs in Aumann's proposed redefinition of the state space, so that priors in fact can now distinguish between separate events in Ω .

Furthermore, a theorist can develop a fully consistent model of differential information in which the partitions of the state space are not common knowledge. Such a model will not lead to the kind of inference possibilities examined here and elsewhere in the common knowledge literature, but that does not mean that such models may not be useful in other contexts. Finally, we would argue that the structure of the information partitions Φ_i are more than mere "dictionary classifications." The distinction between $\{1,2,3\}$ and $\{1\},\{2\},\{3\}$ would clearly serve to the latter's advantage in any game of

incomplete information; the latter partition enables a much more precise inference of the same signal.

The second part of the question concerned whether the various literatures recognized the need for common knowledge beliefs and not just homogeneous beliefs assumptions. The common knowledge literature has explicitly written out the assumption of common knowledge posteriors, which indicates recognition of the distinction between (posterior) beliefs that may be equal versus common knowledge. Yet leading economic uncertainty theorists have now recognized the literature's reliance on the unstated more restrictive ACKP assumption:

the (common knowledge literature's) conclusion, that this process leads inevitably to agreed beliefs and thus to non-trading, is a special result with very little robustness. For one thing, and Tirole (1982) concedes this, it fails if traders' priors differ. And indeed, for the validity of the non-trading theorem something even stronger than agreed priors is needed: the parties' priors must not only be agreed but must be 'common knowledge' in the sense of Aumann (1976) -- i.e., each must know the other shares his beliefs, must know that the other knows he knows, etc. This point is due to Guth (1984), who has also analyzed the implications for speculation theory of shared priors that are not common knowledge. Hirshleifer (1984), p.5.

In fact, the common knowledge literature's "no-trading theorem" is essentially an attempted restatement of necessary conditions for Akerlof's (1970) non-trading of "lemons" theorem.

Critics of the macroeconomics literature on Rational Expectations have also found that the literature routinely assumes that individuals "know the model," but that kind of assumption fails to eliminate intrinsic uncertainty over the unknown beliefs of others. Essentially the individuals could know the model and agree without knowing they

agree. The critics of Rational Expectations paradigm have explained that Rational Expectations models actually require both "perceived and actual unanimity of beliefs." [Frydman and Phelps (1983)]. This is tantamount to agreed common knowledge priors. To the best of our knowledge, no one has proposed redefining the state space in the Rational Expectations models to include beliefs as well.¹¹

7. CONCLUSIONS

This chapter has focused on the "common priors" assumption that permeates the statistical literature on consensus, financial economics, and Rational Expectations. We have shown the common priors assumption still leaves open the possibility for intrinsic uncertainty, in which individuals do not know what priors are used by others in the model. The consensus equilibrium results for the common knowledge literature requires the more restrictive agreed common knowledge priors (ACKP) assumption.

This chapter has proven two propositions. The first states if (any) two individuals have common priors and their posteriors for an event E are common knowledge, then these posteriors may be unequal

¹¹The experimental economics literature has found evidence which partially supports our hypothesis. In repeated experiments, economists have found more rapid convergence to equilibrium when the participants knew the beliefs and strategies of others in the market. See, e.g., Plott (1982) for the experimental economics literature, and Gould (1986) for a psychological perspective on multiple equilibria (potentially from intrinsic uncertainty) in the Rational Expectations literature. To fully test our hypothesis, economists would have to conduct an experiment, in which the subjects were not told about the beliefs of other market participants, and then asked to trade contingent claims that payoff in part based on the beliefs/strategies of others.

(depending on a condition stated in Proposition 1). This result is precisely the opposite conclusion reached by the common knowledge literature, which had argued the conditions in Proposition 1 were sufficient to guarantee consensus (equal posteriors).

The intuitive logic behind Proposition 1 is that individuals cannot distinguish, based on hearing someone's posterior is different from their own, whether that person has different priors or whether he has different information. With merely common priors, nothing guarantees the individuals will reach consensus, and thus they may choose to disagree by assigning unequal posteriors.

The second proposition states that with ACKP and common knowledge posteriors, the posteriors must be equal. This proposition clarifies and corrects the principal result of the common knowledge literature to include ACKP. The intuitive logic for this result rests with excluding the possibility of different priors to account for differences in posterior informed beliefs. The individuals continue to draw inferences about the true state until they have reached agreement.

The iterations to consensus in the common knowledge literature raise interesting behavioral issues. First, it must be assumed that the individuals will honestly reveal their true posteriors. Such honest revelation may not be the case for the financial settings discussed in Chapter 3. Furthermore, the individuals must earnestly be trying to reach consensus. If one person becomes entrenched or feels he is superior, he may not revise his posteriors. A person listening to the entrenched person repeatedly call out his same posterior may draw misinferences that the entrenched person actually

has superior information. Therefore, the consensus literature implicitly assumes cooperative behavior.

Although the consensus literature requires ACKP, Chapter 3 explains why some financial economics paradigms, such as the Capital Asset Pricing Model (CAPM) or the Modigliani-Miller Propositions in a two-period economy, actually need only the less restrictive assumption of homogeneous beliefs or equal priors. Therefore, Aumann's (1987) main contention that "common knowledge beliefs" should replace "homogeneous beliefs" throughout the various literatures is invalid. First, it was not implicitly assumed by these theorists all along. Second, as we show in Chapter 3, some of the financial economics models can achieve equilibrium consensus with the less restrictive "homogeneous beliefs/same priors" assumption.

The difference between "same priors" and "agreed common knowledge priors," is that in the former case the agents agree without knowing they agree. Each agent may treat the prior distribution used by others as an unknown and may, therefore, assign a nondegenerate probability distribution to the possible priors of the other $n-1$ agents. With common knowledge priors, the higher-order beliefs collapse to a certainty case.

Harsanyi (1968) argued eloquently that there is no rational basis for people who have been fed precisely the same information to maintain different subjective probabilities. Of course individuals must have a prior probability distribution to update given the information they receive. As long as the individual priors over the state space Ω are not common knowledge, there can be intrinsic uncertainty

where individuals are uncertain as to the probabilities assigned by others to events in Ω .

Intrinsic uncertainty has nothing to do with "innate" differences in priors. There will be intrinsic uncertainty whether the priors are equal or unequal. Intrinsic uncertainty is based on the amount of a priori information available to each individual.

If we stand back from the Bayesian understanding of probability, then it is clear that your probability calculations are worth a lot more if ground rules were established in advance. If no such rules were established, then you really do not know what is going on, and the assertion that you have probabilities for what rules I might have been following does not help you much. Indeed, the assertion that I was following some unknown set of rules may not have much content. I did what I did and said what I said, and in the absence of any prior agreement or deliberation, the question of what I might have said lies more in the realm of imagination than in the realm of fact. Shafer (1986), p.13.

Uncertainty over the ground rules of play is akin to uncertainty over the prior probability distributions of others. In the absence of any a priori deliberation or information, each individual will generally face uncertainty over the unknown priors of others.

This chapter has shown that the widely cited proposition that "if two agents have the same prior and their posteriors are common knowledge, then their posteriors must be equal" is not generally valid. Several numerical counter-examples were developed that employ all the assumptions of the original model, but specifically focus on how intrinsic uncertainty over priors can still arise in a model where individuals agree but have no way of knowing a priori they agree. Imposing an ACKP assumption allowed us to rederive the standard consensus result and eliminate intrinsic uncertainty over priors.

8. REFERENCES

- Akerlof, George, (1970), "The Market for 'Lemons': Qualitative Uncertainty and the Market Mechanism," Vol. 89, pp.488-500.
- Aumann, Robert J., (1976), "Agreeing to Disagree," Annals of Statistics, Vol. 4, pp.1236-1239.
- (1987), "Correlated Equilibrium as an Expression of Bayesian Rationality," Econometrica, Vol. 55, No. 1, pp.1-18.
- Billingsly, P., (1968), Convergence of Probability Measures, (New York: John Wiley and Sons, Publishers, 1968).
- Birkhoff, Garret, (1961), Lattice Theory, (New York: American Mathematics Society, 1961).
- DeGroot, Morris H., (1974), "Reaching a Consensus," Journal of the American Statistical Association, Vol. 60, pp.118-121.
- Donnellan, Thomas, (1968), Lattice Theory, (London: Pergamon Press, 1968)
- Frydman, Roman and Edmund S. Phelps, (1983), Individual Forecasting and Aggregate Outcomes: "Rational Expectations" Examined, (Cambridge, England: Cambridge University Press, 1983).
- Geanakoplos, John D., and Heraklis M. Polemarchakis, (1982), "We Can't Disagree Forever," Journal of Economic Theory, Vol. 28, pp.192-200.
- Gould, John P., (1986), "Is the Rational Expectations Hypothesis Enough?" Journal of Business, Vol. 59, No. 4, pp.S371-S377.
- Guth, Michael A. S., (1984), "Intrinsic Uncertainty in the Contingent Claims Economy," unpublished manuscript, California Institute of Technology, Pasadena, California.
- Harsanyi, John, (1967-1968), "Games of Incomplete Information Played by Bayesian Players, Parts I - III," Management Science, Vol. 14, pp.159-182, 320-334, 486-502.
- Hirshleifer, Jack, (1984), "Two Models of Speculation and Information," UCLA Department of Economics Working Paper #329, June 1984.
- Mertens, J.-F., and S. Zamir, (1985), "Formulation of Bayesian Analysis for Games with Incomplete Information," International Journal of Game Theory, Vol. 14, pp.1-29.
- Milgrom, Paul, and Nancy Stokey, (1982), "Information, Trade, and Common Knowledge," Journal of Economic Theory, Vol. 26, pp.17-27.

- Myerson, R., (1985), "Bayesian Equilibrium and Incentive Compatibility: An Introduction," in L. Hurwicz, et al., editors, Social Goals and Social Organization.
- Plott, Charles R., (1982), "Industrial Organization Theory and Experimental Economics," Journal of Economic Literature, Vol. 20, pp.1485-1527.
- Sebenius, James K., and John Geanakoplos, (1983), "Don't Bet on It: Contingent Agreements with Asymmetric Information," Journal of the American Statistical Association, Vol. 78, pp.424-426.
- Shafer, Glenn, (1986), "The Construction of Probability Arguments," University of Kansas, Lawrence, Business School Working Paper No. 183, January 1986.
- Szasz, Gabor, (1963), Introduction to Lattice Theory, (New York: Academic Press, 1963).
- Tan, T. C. C. and S. R. C. Werlang, (1986), "On Aumann's Notion of Common Knowledge -- An Alternative Approach," published in Theoretical Aspects of Reasoning About Knowledge, Proceedings of 1986 Conference, Edited by Joseph Y. Halpern.
- Tirole, Jean, (1982), "On the Possibility of Speculation under Rational Expectations," Econometrica, Vol. 50, pp.1163-1181.

CHAPTER 3

COMMON KNOWLEDGE FOUNDATIONS OF MODERN ASSET PRICING MODELS:

RESTATEMENT AND RECONCILIATION OF THE MM, CAPM,

APT, AND OPT PARADIGMS

This chapter examines the pivotal role of common knowledge in the major asset pricing paradigms of the theory of financial economics. A common thread of arbitrage pricing runs throughout most asset valuation theories. The need for common knowledge manifests itself whenever investors consider capital gains potential. Then for whatever financial instrument is being valued - security, option, bond - the market valuation, in general, and the resale value, in particular, will be determined by beliefs and expectations for the majority of market participants.

Most of these asset pricing paradigms were developed under a "homogeneous beliefs" assumption, which is analogous to the "individuals know the model" assumption in the Rational Expectations literature. In reexamining the different asset pricing models for their reliance on common knowledge, we first examine how intrinsic uncertainty can distort the conventional formulation of these models. For the CAPM and a two-period version of the MM paradigm, common knowledge is not required; the weaker homogeneous beliefs assumption can achieve equilibrium. For the paradigms that require common knowledge beliefs, we impose the necessary assumption to derive the unique formulation of the pricing theories originally sought by their proponents.

1. THE MODIGLIANI-MILLER PARADIGM

Perhaps no series of papers has had a greater role in shaping the way financial theorists view market valuations of corporate debt and equity than the classic works by Modigliani and Miller (MM). Beginning with their seminal work in 1958 and continuing through the presidential addresses of both Merton Miller (1977) and Franco Modigliani (1982) to the American Finance Association, the MM valuation theorems have evolved into a major paradigm of the theory of finance.

Although they may disagree over the MM valuation theorems' practical importance, financial theorists have often employed MM-style arbitrage. For example, the APT and OPT paradigms discussed in the next sections rely on MM-style arbitrage. The standing of the MM propositions among financial theorists is reflected in the following typical quotation: "It has been shown in practice that there is an optimal capital structure for the firm, and that its existence really follows from the logic of the MM analysis supplemented only by a few simple institutional observations."¹ Indeed even after the long series of exchanges between MM and their critics,² as well as additional commentary by other financial theorists,³ the basic arguments and conclusions were left intact.

Despite the extensive scrutiny of the MM paradigm and its fundamental assumptions over the last thirty years, we have found a new

¹Baumol and Malkiel (1967).

²Modigliani and Miller (1958), (1959), (1963), (1965), Durand (1959), Rose (1959).

³Stiglitz (1969), (1974), Masulis (1980), Merton (1977).

and unpublished discrepancy in the treatment of capital gains-motivated investors who are concerned with wealth in addition to income. In the existing MM literature, the fundamental valuation of corporate debt and equity is premised on investors having homogeneous beliefs. However, as we will show in this chapter the logic of the MM analysis in a multi-period economy needs common knowledge. Without common knowledge in a multi-period economy, the typical MM result may still be a valid possibility but will no longer be guaranteed.

The MM form of riskless "arbitrage" actually relies on an absent assumption that investor beliefs are common knowledge. Even under the simplifying assumptions identified and scrutinized extant, including perfect and complete contingent claims markets, the MM paradigm will still require a common knowledge assumption.

A. Logical Intuition

Under the "homogeneous beliefs" rubric, the MM paradigm assumed that individuals completely agree on each firm's state-contingent earnings and the probability distribution over possible states of the world. Thus all investors agree on which firms belong to a given risk class; however, the investors do not know a priori that they completely agree. Indeed, the MM paradigm at present contains no mechanism to transmit this information (about the beliefs of others) to individual market participants.

MM-style arbitrage opportunities relate to balancing corporate debt with private borrowing and are defined in terms of differences in market valuation for firms in the same risk class. When stocks are

undervalued or overvalued - as measured by MM's risk classes - an investor should be able to buy and sell securities in such a way that the income the investor receives more than offsets his debt obligations. The investor thus obtains a riskless profit.

It turns out that in a multi-period environment, the income arbitrage opportunity fails to guarantee that investors will actually shift their portfolio holdings from shares in overvalued firms to shares in undervalued firms. This result breaks tradition not only with the MM paradigm but also with most of orthodox finance theory. The reason an investor might continue holding an overvalued firm's shares or even buy additional overvalued shares would be in the expectation of realizing a capital gain.

With a "homogeneous beliefs" or "common priors" assumption, an investor knows that although he personally feels some firm is overvalued, the rest of the market (or some fraction of it) may regard that same firm as undervalued. He may believe the market believes the firm has a higher state-contingent earnings stream. If that is the case, the market may then increase demand for the same "overvalued" shares.

When investors are concerned with both income and capital gains, the rationale for the MM arbitrage strategy is not complete. Investors may personally disagree with the market's valuation yet fear that its future valuation will exceed his personal estimate by an even greater amount. An investor motivated by capital gains may buy into an "overvalued" stock at present if he believes the future overvaluations will continue to rise. Alternatively, an investor may

choose to sell "undervalued" stock at present with the speculative intent of repurchasing the stock at some future date at an even lower price. From a wealth standpoint, this investment strategy makes perfect sense.

Almost all investors in the stock market are concerned with wealth. Thus even under the simplifying assumptions identified in the MM paradigm, investors may continue to hold "overvalued" shares or even buy additional "overvalued" shares, when they believe market demand for those assets will increase.

Investors who desire capital gains must consider the beliefs and preferences of other market participants. Therefore, the preceding portfolio revision, which reversed the logic of MM, can apply to all capital gains-motivated transactions, not just speculative bubble purchases.

There are two methods to salvage the MM arbitrage maneuver given this limitation. First, the model could explicitly assume away the existence of speculators - investors whose objective is capital gains. But this assumption would also eliminate any approximation to real financial markets. Alternatively, financial theorists must assume that investors have not only "homogeneous" beliefs about firm risk classes, but also that these beliefs are common knowledge. Otherwise, investors risk losing wealth from purchasing stocks the market may intend to devalue.

A common knowledge assumption would leave unanswered how the investors arrived at that particular environment. Speculators have incentives to conceal their true beliefs lest other traders scoop

their potential investments. And aside from the incentive compatibility issue, the standard finance framework has no mechanism to communicate information about investors' beliefs a priori to all market transactors.

MM theory has used homogeneous beliefs assumptions to derive expressions for the market's collective behavior. Homogeneous beliefs, as commonly defined in the literature, require no more than that individual investors have uniform (prior) beliefs about some unknown parameter. We believe that the MM paradigm in a multi-period environment actually requires a stronger foundation: agreed common knowledge beliefs.

In the formal analysis of this chapter, we prove three results. First, when investors have ACKP about the assignment of firms to risk classes, then the market values of two firms in the same risk class must be equal. This proposition holds in either a multi-period or two-period model of the MM paradigm.

Second, in a multi-period economy where investors have only common priors, we prove that a risk neutral investor may speculate by buying shares in an "overvalued" firm. The decision to purchase "overvalued" shares rather than sell them off cuts squarely against the prediction of orthodox finance theory. This result relies on the ability to buy and resell the shares before the state of the world is revealed; thus, three or more periods are required.

Third, we prove that the market values of two firms in the same risk class will be equal in a two-period economy without ACKP and even without homogeneous beliefs. The key assumption is that a majority of

the purchasing power of the market agrees the firms belong to the same risk class. Then the majority can follow the MM-style income arbitrage to bring the market values into equality.

Proposition 3 partially supports Stiglitz's (1969) position that the validity of the MM debt/equity proposition does not depend on homogeneous beliefs. That statement is correct for a two-period market, analyzed by Stiglitz (1969), where even heterogeneous beliefs could be incorporated into the model.

Speculation combines intertemporal investment and choice under uncertainty. To speculate over the intrinsic uncertainty discussed in this chapter, investors must be able to buy (sell) and sell (repurchase) before the state of the world is revealed. A two-period paradigm would only allow one trading period before the state of the world is revealed and intrinsic uncertainty over firms' risk class earnings would thus be eliminated.

B. A Numerical Example

Since the insight we offer for an intertemporal perspective of MM-style arbitrage is truly profound and likely to shake the existing foundations of the paradigm, we propose to develop first a numerical example and defer a theoretical generalization to the next section. Consider now two investors i and j in a contingent claims economy with five possible states of the world: states 1, 2, 3, 4, and 5.

We begin with period 0. Let individual i 's partition for the possible states be $\{1,2,3\}$, $\{4,5\}$, and let j 's partition for the possible states $\{1\}$, $\{2,3\}$, $\{4,5\}$. Just as in the MM paradigm, assume

i and j have "homogeneous beliefs" in the form of a common prior over the five states. Let this common prior assign probabilities $(.7, .1, .1, 0, .1)$ over the states $(1, 2, 3, 4, 5)$, respectively.

Let there exist a homeomorphism between states 1 through 5 and the value of some firm's equity when the state is revealed such that in state 1 the equity has value 1, in state 2 the equity has value of 2. Given the prior beliefs $(.7, .1, .1, 0, .1)$, both investor i and j would believe the fundamental value of this firm's equity is 1.7 in period 0. If all investors have "homogeneous beliefs" or the same prior, then the prevailing market price for the firm's equity in period 0 will be 1.7.

In period 1, both i and j receive information on what will be the true state of the world. Suppose the true state of the world is 2. To conform to the investors' state partitions, let investor i be informed the true state lies within his class $\{1,2,3\}$. Investor j is informed the true state lies within j 's class $\{2,3\}$. Investor i now redistributes the .1 probability he previously assigned to state 5 to the three remaining states known to him. Now investor i 's posterior probability distribution is $(.7333, .1333, .1333)$ for the states $\{1,2,3\}$, which yields a fundamental value of 1.4. Investor j also redistributes the probability on his two remaining states; investor j 's posteriors of $(.5, .5)$ for states $\{2,3\}$ yields a fundamental of 2.5. Investors i and j now disagree on the fundamental value of the firm's equity.

After the investors receive their private signals, markets again reopen in period 1. If the market participants are distributed such

that half the market shares investor i 's partitioning and half the market shares investor j 's partitioning, then the prevailing market price will be 1.95, which is the average of 1.4 and 2.5.

Investor i 's fundamental value has dropped from 1.7 to 1.4 in period 1, so that investor i now regards the firm as overvalued at market clearing prices of 1.95. In contrast, investor j 's fundamental value has risen from 1.7 to 2.5 in period 1. Investor j now finds the firm's equity undervalued at a market clearing price of 1.95.

The traditional arbitrage argument suggests investor i would want to sell any shares of the firm he owned, which he now regards as overvalued. Similarly investor j would want to buy shares at the 1.95 market price. At this point we will demonstrate a fundamental departure of the traditional "arbitrage" argument, for we will now explain why investor i may want to continue holding his "overvalued" shares or even buy more shares at the "overvalued" price of 1.95.

Suppose both investors anticipate that additional information will be revealed in period 2; this information may eliminate some state from consideration. If both investors i and j learn that the true state is not state 1, then i will revise his posteriors to assign (.5, .5) probability to the two remaining states $\{2,3\}$. Thus investor i 's posterior fundamental would shift from 1.4 to 2.5. However, investor j would learn nothing new from the fact that state 1 is not the true state; consequently, his posterior would remain unchanged at (.5, .5) for $\{2,3\}$. Under these conditions, the prevailing market price in period 2 would be 2.5.

Looking at this possible scenario in period 1, the key point to

recognize is the capital gains potential. If investor i sells shares, which he personally values at 1.4, at a price of 1.95, he will realize a pure profit of .55. However, if he waits until period 2, he may be able to sell the same shares at a price of 2.5. Thus he might realize a pure profit of 1.1 based on his fundamental in period 1. This potential profit could induce investor i to continue holding his shares to period 2 and not sell in period 1. It could even induce investor i to buy more shares, which he personally regards overvalued at a price of 1.95, just to resell them in period 2 at an even higher price.

If it turns out the true state is 2, then a market price trajectory of 1.7 in period 0, 1.95 in period 1, 2.5 in period 2, and 2.0 in period 3 when the state is revealed, would exhibit a deviation in period 2 away from the fully-informed equilibrium. If the true state turns out to be 3, then the price trajectory of 1.7, 1.95, 2.5, and 3.0, across periods 0 through 3, respectively, would simply reveal a growing consensus towards the fully-informed equilibrium.

When the investors merely have the same priors (common priors), then each can attribute differences in posteriors either to different information or to differences in their priors. With common priors the investors could agree without knowing they agree. With common knowledge priors the investors will always reach a consensus about the expected fundamental value of the shares.

The main result of this numerical example is that capital gains potential in the future can induce an investor to pass up a "riskless profit" in the current period in the hopes of realizing even greater

profits in the future. The potential for capital gains depends on both intertemporal investment - the market horizon extending beyond two periods - and choice under uncertainty.

Another way of seeing this same result would be to consider an investor currently holding, in period 1, some asset that he personally values at 1.4. Suppose in period 1 someone offers this owner a price of 1.5 for his asset. According to orthodox financial economics theory, the owner should jump at the chance to sell his asset for 1.5. Orthodox theory maintains the owner should be indifferent between holding the asset or holding 1.4 in cash, and the .1 differential amounts to a pure profit for the owner. In the case of the MM paradigm, it would be typical at this point for a theorist to argue the owner should sell any "overvalued" asset.

However most arbitrage pricing theories neglect the incentives the owner has to pass up this "sure profit" in the hopes of getting even more in the future. Suppose the owner believes in period 2 that his fundamental value will remain at 1.4, but that he might be offered a price of 2.1 in period 2. Clearly $2.1 - 1.4 > 1.5 - 1.4$. The owner might also be offered a price of 1.2 in period 2, in which case he would have a paper loss of $-.2$. If the owner believes the prevailing price is split with equal probability between 2.1 and 1.2, then his expected market fundamental is 1.65.

A risk preferring owner clearly would prefer to speculate, take the gamble, and wait for period 2. A risk neutral owner finds $1.65 > 1.5$ and anticipates he could do better by waiting until period 2. A risk averse owner may grab the sure profit in period 1 or may take a

chance on period 2, depending on his degree of risk aversion.

It may turn out that the owner was selling at an overvalued price, or it may turn out that he was incorrectly valuing the asset and sold it for less than its true worth. The main point here is uncertainty that can lead to capital gains. As we will show in the next section, common priors or homogeneous beliefs is not sufficient to eliminate the intrinsic uncertainty that can be a source of speculative activity.

C. The Modern MM "Arbitrage" Strategy

Define a firm's "risk class" according to its earnings over different states of the world. A crucial assumption, as shown in the subsequent literature, is that complete and perfect contingent claims markets exist.⁴ Let X_{is} denote firm i 's earnings in state of the world s . A given investor believes firms i and j belong to the same risk class if their earnings correlate perfectly over states of the world:

$$X_{is} = c X_{js}, \text{ for all } s, \quad (1)$$

where c is a constant.

The MM-style "arbitrage" typically employs the following kind of

⁴We must emphasize the importance of complete and perfect contingent claims markets in the proof of the MM Theorem. Hirshleifer has stated that "perfect" contingent claims here implies the absence of monopoly power, i.e., that no firm could issue a unique security providing a "sandwich" of state-claims not already present in the market. Coupling the complete and perfect contingent claims assumptions with no "external drains," e.g., bankruptcy or taxes, permits a simple proof of the MM Theorem for a two-period economy. See, e.g., Hirshleifer (1971), pp.263-265. For assumptions in the subsequent literature see Stiglitz (1969) and (1974), and Merton (1977).

argument. Suppose an investor believes firms 1 and 2 belong to the same risk class, and for simplicity assume both firms have the same earnings stream, i.e., $c = 1$, and hence the same expected return. Suppose firm 1 has only equity in its capital structure, while firm 2 has both debt and equity.

Let E_1 represent the quantity of outstanding shares in firm 1, with price per share of P_1 . Let E_2 and P_2 represent the corresponding equity and share price variables for firm 2, and let D_2 denote firm 2's outstanding debt. Firms 1 and 2 then have respective market values

$$M_1 = P_1 E_1,$$

and

$$M_2 = P_2 E_2 + D_2.$$

Suppose $M_2 > M_1$, and the investor holds S_2 shares of firm 2 totaling $\alpha\%$ of firm 2's outstanding stock: $S_2 = \alpha E_2$. Let Y_{2s} denote the investor's income in state s . Then

$$Y_{2s} = \alpha (X_{2s} - rD_2),$$

where r is the rate of interest. By assumption $X_{1s} = X_{2s}$ for all s , hence let $X_s \equiv X_{1s} = X_{2s}$ and thus

$$Y_{2s} = \alpha (X_s - rD_2).$$

Suppose the investor sells his shares in firm 2 and borrows αD_2 dollars using these proceeds to invest in firm 1 equity. Let S_1 denote his new holdings in firm 1, thus

$$\begin{aligned} P_1 S_1 &= \alpha (P_2 E_2 + D_2) \\ &= \alpha M_2. \end{aligned}$$

Define β as the percentage of firm 1 shares he acquires. The investor

then receives income from his firm 1 holdings equal to

$$\begin{aligned}
 Y_{1s} &= \beta X_s - \alpha rD_2 \\
 &= (S_1/E_1) X_s - \alpha rD_2 \\
 &= (\alpha M_2/P_1E_1) X_s - \alpha rD_2 \\
 &= \alpha [(M_2/M_1) X_s - rD_2].
 \end{aligned}$$

Since $M_2 > M_1$ by hypothesis, and where $X_s > 0$, then $Y_{1s} > Y_{2s}$ for every state of the world s . Thus, according to the financial literature, all investors find it profitable, whatever their preference towards risk, to sell their "overvalued" shares in firm 2 and acquire shares in the "undervalued" firm 1. Every investor can allegedly earn a risk free profit.

In fact, for long-lived assets such portfolio revisions involve risk unless investors know with certainty that the market agrees with their own beliefs. Apart from a common knowledge assumption, investors in a multi-period economy can have uncertainty that their wealth will change with anticipated or unanticipated changes in market valuations.

The future market value of a firm is an unknown random variable to each investor in the market. Bayesian subjectivist theory encourages the treatment of any unknown by assigning a probability distribution over its possible values. The future market value of any firm f , M_f , at some point in time, T , is a random variable defined on a probability space (Ω, F, P) , where the measurable space (Ω, F) contains all the possible market values for firm f . Therefore at time T , the expectation of M_f is defined by

$$E(M_f) = \int_{\Omega} M_f dP, \quad (2)$$

which exists since $\int_{\Omega} M_f dP < \infty$ by assumption.

Assume for the moment that changes in firms' market valuations are borne by changes in the price of their equity. If any investor believes at some future period T

$$\alpha \left[\int_{\Omega} M_2 dP - M_2 \right] - \beta \left[\int_{\Omega} M_1 dP - M_1 \right] > Y_{1s} - Y_{2s}, \quad (3)$$

then he will likely hold the overvalued firm 2's stock in the expectation of realizing some additional future capital gain. Moreover, in direct contradiction to the logic of the MM "arbitrage" maneuver, the investor may actually buy more shares in firm 2 and sell off his holdings in firm 1. Since his expectation of realizing further capital gains, shown in the left-hand side of equation (3), is greater than his income, shown in the right-hand side of (3), the investor may decide to purchase additional shares in firm 2.

D. Common Knowledge of Firm Risk Classes

For the MM paradigm, the applicable common knowledge beliefs assumption categorizes firms into risk classes. If all the market participants (coincidentally) agree on each firm's state-contingent earnings and the probability distribution over states of the world, can the market participants, individually and collectively, assign a firm to a given risk class?⁵ Not without common knowledge!

It has long been recognized in finance that investors typically buy assets which they believe the market has undervalued. For

⁵The practical significance of firm risk classes was shown by expression (1).

example, if an investor planned on buying shares in a firm, he would have to purchase those shares at a price which the market collectively has determined to be its best estimate of the shares' value. If the investor had access to the same information as the rest of the market, he would find the shares selling at a price, ceteris paribus,⁶ that would leave him indifferent between purchasing those shares or not.

In practice, then, an investor who purchases stock must find the prevailing market price for the stock to be less than his estimated value of the shares based on the information available to him. This concept is among the most important in all the theory of finance. It implies that the purchaser of financial assets believes not only that the asset will perform well in the future, but that the asset will perform better than the market's best estimate as reflected in the prevailing price for the asset.

The distinction between individual valuations and beliefs about other investors' valuations introduces "intrinsic uncertainty" into the model. This intrinsic uncertainty over the unknown beliefs, and therefore portfolio revisions, of other traders in the market is the heart of the subtle distinction between common prior beliefs and priors that are common knowledge. With common priors, the investors (objectively) agree. With ACKP, investors not only agree, but know they agree, know everyone else knows they agree, and so on.

The foundation of the MM paradigm robust enough to incorporate

⁶It is necessary to add a ceteris paribus qualifier here, because risk aversion liquidity, institutional, and other constraints may allow some share transactions at prices below the market's best estimate of their value.

multi-period analysis rests with the assumption that firm "risk class" assignments are common knowledge. This assumption requires in turn that investors in the MM model have ACKP about the state-world probability distribution and the corresponding state-contingent earnings for each firm. Appendix A examines the robustness of the MM debt-equity propositions under differing scenarios involving common knowledge assumptions.

E. Conclusions and Issues for Further Research

Rather than merely summarizing the preceding analysis of the MM paradigm, this section poses several questions about the model and answers them in such a way as to stress the main points of the analysis. We have raised a very general capital gains objection to arbitrage arguments and illustrated it in the particular context of capital structure. The MM capital structure paradigm was chosen to illustrate these points because of its path-breaking significance to the theory of finance. The remaining sections of Chapter 3 show that the same general objection applies to the APT and OPT but not the CAPM.

We have shown that in the absence of common knowledge, investors in an MM environment could agree without knowing they agree. This intrinsic uncertainty among the investors may lead to portfolio revisions that directly contradict the logic of the MM analysis even under the restrictive assumptions identified to date in the MM literature. Even more than showing that investors will not necessarily want to buy/sell in the way recommended by MM, we have taken the argument

one step further. In Section C, we have shown that if the MM strategy is followed, investors face risk. This risk is defined in terms of potential lost capital gains given by equation (3) and leads some capital gains-motivated investors to adopt strategies identified in Proposition 2 of Appendix A.

In a model with three or more periods, we have illustrated that even though an investor may personally feel some firm is overvalued, he may also feel that "the market" regards this same firm as under-valued. Under these circumstances, we find investors buying shares in firms which they subjectively find overvalued, simply to realize further capital gains.

A second form of risk faced by the investor executing the MM strategy is potential capital losses from being forced to liquidate his holdings before the state is revealed. Proposition 3 in Appendix A shows if the intertemporal model is converted effectively to a two-period model, by restricting the investor to one transaction period before the state is revealed, then a "riskless" profit can be earned. But the hidden risk is the ability of the investor to commit to a long enough investment horizon to witness the state revelation.

How do these arguments relate to speculative bubbles? Does the strategy in Proposition 2 require bubbles which last forever? What kind of linkage between underlying fundamentals and market prices is being suggested? The strategy we have described requires capital gains-motivated investors. Speculators receive bubble profits from buying in relatively early stages of the bubble and selling before the price reverts back to its fundamental. The strategy identified in

Proposition 2 does not require bubbles which last forever. In fact, it does not even require the objective existence of a bubble. The strategy only requires an individual to perceive the subjective overvaluation or undervaluation will last long enough for him to earn a capital gain.

The link between underlying fundamentals and market prices is blurred for the individual investor. Finance theory commonly assumes market price equals the "market" fundamental by definition. Yet the individual faces risk and uncertainty as to how the rest of the market values a firm's assets as opposed to the way he subjectively values those same assets. The individual's fundamental may differ from the market's fundamental. Our analysis has explained how subjectively determined "overvalued" assets can nevertheless be attractive investment opportunities for the individual.

What is the motivation for the importance of the MM paradigm potentially failing when beliefs are homogeneous but not common knowledge? Do the finance or rational expectations literatures contain arguments to justify the MM assumptions about beliefs? In discovering a critical assumption for the validity of the MM paradigm, i.e., agreed common knowledge prior beliefs in a multi-period environment, we have advanced the profession's knowledge of the fundamental determinants of asset valuations. As to the justification for ACKP beliefs, a theorist might borrow from the literature's own justification for homogeneous beliefs: the dominance of institutional traders in securities markets. With homogeneous beliefs, the investors agree just by chance! An environment where professional

investment managers realize other managers have access to the same information sources can easily give rise to common knowledge.

In a model with a finite number of periods, does the arbitrage strategy of MM still work, because even without common knowledge about distributions, the strategy locks in a sure profit? This question cannot be answered without clarifying that an MM strategy typically entails some initial transaction and then waiting for the state to be revealed. If the investment horizon extends to the state revelation period, then based on income arbitrage an investor could lock in a sure profit.

However, the existence of an income arbitrage profit may be dominated by a superior strategy of attempting to obtain capital gains, by a strategy of leaving money in a safe account and collecting the prevailing rate of interest, or placing funds in assets that do not require long-term investment until the state of the world is revealed.

How interesting is the point that the MM paradigm (or other arbitrage-based theories) can fail? What superior theory can replace it? First, no theory can be accepted by any discipline which claims to be a science without sufficient rigor of proof. On mathematical principles, any deficiency in the proof must be highlighted and explored or the profession could not advance from contributions to theory. Second, the superior theory that replaces the old MM formulation is a refortified MM theory equipped with ACKP.

ACKP eliminates the intrinsic uncertainty which forms the basis for capital gains-motivated transactions in a multi-period version of the paradigm. We end up with a multi-period competitive model, with

no external drains like taxes or bankruptcy and individual borrowing at the same rate as corporate borrowing, where the financial structure of the firm is irrelevant to its market valuation.

2. MEAN-VARIANCE THEORY AND THE CAPM

Unlike the MM paradigm, ACKP are not required for the standard formulation of the Capital Asset Pricing Model (CAPM). The contribution of this section is to explain why only assuming common priors or "homogeneous beliefs" fails to engender intrinsic uncertainty and capital gains speculation in the CAPM. The significance of this contribution is that it negates the common knowledge literature's claim that theorists need to substitute "common knowledge beliefs" for "homogeneous beliefs" throughout economic and finance theory. This section shows that common priors or "homogeneous beliefs" are sufficient for the CAPM.

The "homogeneous beliefs" assumption has crucially simplified mean-variance asset theory and related empirical works. By homogeneous beliefs financial theorists assume

every investor attaches the same values for the mean m_x and standard-deviation s_x for the distribution of returns of every security and for the correlations r_{xy} among the returns. It follows from this that the opportunity sets of all individuals have the same proportionate shape, differing from one another by only the scale factor of individual endowed capital K_0 . (Hirshleifer (1970), p.289)

Although Sharpe (1964) acknowledged the homogeneous beliefs assumption was "highly restrictive and undoubtedly unrealistic," he nevertheless defended its use. "Since these assumptions imply

equilibrium conditions which form a major part of classical financial doctrine, it is far from clear that this formulation should be rejected - especially in view of the dearth of alternative models leading to similar results." (Sharpe 1964, p.434).

The two-period CAPM assumes investors know the return on assets, and this return includes both dividends and capital gains potential. Since the purchases and sales of other investors in the market determine capital gains and losses, investors implicitly must know the purchasing tendencies of the market in order to calculate any asset's capital gains return. A common knowledge beliefs assumption is generally required to eliminate speculation based on how changing beliefs will affect capital gains potential; the two-period CAPM has substituted perfect knowledge (and indeed common knowledge) about capital gains in place of an agreed common knowledge beliefs assumption. Actually, the two assumptions are logically equivalent.

The multi-period CAPM, derived by Merton (1973a) and extended by Breeden (1979), assumes price trajectories into the future are known. It can be shown that intermediate period prices are common knowledge in period 0 only when individuals have ACKP about the state space. Without ACKP individuals might not reach consensus agreement about what future prices will prevail. Therefore, the two-period CAPM eliminates intrinsic uncertainty through individuals knowing the capital gain on assets. The multi-period CAPM subsumes ACKP through assumptions on the price trajectories across intermediate periods, from which investors can compute potential capital gains.

In his classic beauty contest analogy, Keynes (1936) reveals why

estimates about higher-order beliefs are crucial to speculation based on intrinsic uncertainty. Since stock market investors realize other investors revise their portfolios from the same capital gains-motivated viewpoint, investors need to consider more than which stocks average opinion believes will perform well. Investors must actually choose stock they think average opinion thinks average opinion thinks will perform well. This thought-process is known as "metathinking." In general, agents second and third-order beliefs could differ, and consequently one would expect investors to hold differing quantities of a given asset depending on which higher-order beliefs guides their portfolio revision.

Both the two-period and multi-period versions of the CAPM cleverly avoids financial metathinking behavior by simply assuming that investors know the assets' returns. Once they know these returns, which may depend on any level of metathinking or a lack thereof, they have all the information they need to minimize the variance of their portfolios for a given expected return.

Chapter 2 showed that when investors agree without knowing they agree, intrinsic uncertainty may still arise. If CAPM investors agreed on assets' returns without knowing they agree, one might expect speculative trades to arise in this model as well. However, the CAPM's definition of a return incorporates both dividends and capital gains; consequently, knowing the capital gains potential implies knowledge about the beliefs and preferences of others. Keynes would argue that stock market investors cannot know assets' returns unless they know the higher-order beliefs of other market traders. The CAPM

simply assumes they do know the returns. Appendix B contains a derivation of the standard CAPM and highlights the importance of knowing the assets' returns.

3. ARBITRAGE PRICING THEORY (APT)

The Arbitrage Pricing Theory (APT), introduced by Ross (1976) and (1977), has now received considerable attention as an alternative to the CAPM. Like the MM paradigm of Section 1, the APT is currently written under a common prior beliefs assumption; the APT actually requires the more restrictive agreed common knowledge beliefs assumption. The terms "agreed common knowledge beliefs" (ACKB) and "agreed common knowledge priors" (ACKP) can be used interchangeably.

The contribution of this section is to show that the APT in its present form yields non-unique solutions to the aggregate, market-level asset pricing model. This non-uniqueness of a relevant solution can be established by imposing common knowledge beliefs on the model, thereby eliminating intrinsic uncertainty over the beliefs of other investors and creating true arbitrage opportunities.

The significance of this contribution, aside from shaking the foundations of a well-established paradigm in the theory of finance, is that it shows how intrinsic uncertainty can lead to multiple pricing relationships presently not accounted for in the APT. Another contribution shows theorists how to reconstruct the standard APT formulation under common knowledge.

In the APT, investors are assumed to have "homogeneous" beliefs about the factor model generating asset returns. As Roll and Ross

(1980) have suggested, "The key point in aggregation is to make strong enough assumptions on the homogeneity of individual anticipations to produce a testable theory. To do so with the APT we need to assume that individuals agree on both the factor coefficients, β_{jk} , and the expected returns, $E(R_j)$." However, merely assuming individuals have homogenous beliefs would still permit uncertainty over the market's (other investors') investment behavior. This uncertainty can lead to alternative asset pricing formulations revealing the potential for multiple equilibria within the standard APT framework.

Since its introduction in 1976, the APT has been refined and extended by Huberman (1982), Chamberlain and Rothchild (1983), and Ingersoll (1983), among others.⁷ Subsequently, Dybvig (1983) and Grinblatt and Titman (1983) independently derived upper bounds on the deviations from APT-predicted prices.

All of these papers begin with the same underlying assumption that asset returns are generated by a factor model.⁸ Consider a market with Z risky assets and a riskless asset with return R_f . There exists a subset of the assets numbered 1 through T whose returns are governed by the factor model

⁷However, these analyses were complex and their derivations of the APT formula lacked economic intuition, as articulated in the criticism by Shanken (1982).

⁸Ross (1976) and Huberman (1982) went to some length to specify the necessary conditions that ensure arbitrage opportunities would apply to any asset deviating from the formula of the APT. These conditions include von Neumann-Morgenstern utility functions for investors, as well as assumptions on the factor and error distributions. Yet, as we show in this section, all these restrictive conditions failed to produce true arbitrage opportunities in the absence of common knowledge.

$$R_j = E(R_j|\Phi) + \beta_{j1} F_1 + \dots + \beta_{jM} F_M + \epsilon_j, \quad (4)$$

where R_j = the return of asset j ; Φ = the information set upon which the expected value is conditioned; β_{jk} = sensitivity of asset j 's return to movements in factor k ; ϵ_j = mean zero idiosyncratic risk component of the j th asset, $j = 1, \dots, T$. Assume that the error terms, ϵ_j , are independent of the factors, the returns and of each other. Therefore, $\text{cov}(\epsilon_i, \epsilon_j) = 0$ for all $i \neq j$. Further assume that $T > M$ in order to ensure that the system of equations is determined.

Our model depicts the standard form of the APT with the addition of the information set Φ .⁹ Including the information set will enable us to show one example of some effects from the absence of common knowledge beliefs.

A. Logical Intuition and Example

The theoretical intuition behind the APT is that for some subset of the assets under consideration, the returns will be a linear function of the factors in equation (4). In practice, these factors are "grand aggregates" such as the level of employment or the interest rate. Conversely stated, the APT implies that for a large enough subset of assets, the only possible common determinants of their returns are the various grand aggregates, although empirical studies reveal little agreement on what these factors should be.

Our interest in the significance of common knowledge beliefs for

⁹Note that we have assumed nothing about investor preferences or the objective function they seek to maximize.

the APT rests with theoretical and not empirical considerations. When reformulated to incorporate common knowledge beliefs, the APT will yield a global solution, i.e., a solution which will identify a unique market investment formula from aggregating individual investor beliefs, rather than the non-unique solutions provided in the APT's present formulations. Ross (1976) himself has indicated this topic is "one of the most difficult and important areas for future research."¹⁰

The market valuation of returns to assets is a subjective concept applying to each individual. Under the homogeneous beliefs assumption investors in the APT paradigm might agree without knowing they agree. The intrinsic uncertainty inherent in the APT can lead investors to question what factors or what functional relationship or what information the rest of the market employs in deciding market values for assets.

An investor in the APT paradigm knows what factors and relationships he personally considers the determinants of asset returns, but he faces risk over the factors and relationships others feel are important. Consider now the following three factor equations. All three are possible candidates for the unknown factor model employed by others, and all three are perfectly consistent with the existing assumptions of the APT paradigm. First, we have

$$R_j = E(R_j|\Phi) + \beta_{j1} F_1 + \dots + \beta_{jM-K} F_{M-K} + \epsilon_j ,$$

¹⁰See Ross (1976) at page 356. Ross concluded that relaxing the homogeneous expectations assumption would require study of disequilibrium dynamics and movement from ex ante expectations to ex post observations.

where the number of factors M-K is less than the M factors in (4).

Second,

$$R_j = E(R_j|\Phi) + \gamma_{j1} F_1 + \dots + \gamma_{jM} F_M + \epsilon_j ,$$

where $\gamma_{jk} \neq \beta_{jk}$, and third

$$R_j = E(R_j|\Psi) + \beta_{j1} F_1 + \dots + \beta_{jM} F_M + \epsilon_j .$$

The intrinsic uncertainty present in the APT as it is now formulated would not allow any investor from excluding the possibility that one of these three factor models is employed by other investors in the market.

In the first example, we have altered the number of factors believed to determine the return on the assets. Shanken (1982) has pointed out that financial economists have yet to determine empirically the precise number of factors in the APT. The statistical data on asset returns often reveal the great majority of variance to be explained by a single factor, and that the other factors though containing significant coefficients nevertheless account for a negligible amount of the variance.¹¹ If both the APT and the CAPM explain the variance in asset returns with primarily one factor, then these empirical observations raise questions about the utility of switching from the CAPM to the APT.

¹¹See Trzcinka (1987). It has been pointed out that the empirical tests of the APT often find seven or eight significant factors, but one factor accounts for nearly 99% of the variance with the remaining 1% split between the other six or seven factors. Using this criteria a researcher could find twenty "significant" factors.

The second example has replaced the factor coefficients, because although investors may objectively agree on the values of these coefficients, they do not know the values other investors assign to these coefficients. Again the empirical literature has arrived at no consensus on the factor coefficients. This lack of empirical agreement might well be explained by theoretical intrinsic uncertainty that investors do not know they can exclude models with different factor coefficients.

Finally, the third factor model illustrates this same point for the expected returns. When $\Psi < \Phi$ in the third model, an investor believes he is better informed than at least some subset of the market; conversely, if $\Phi < \Psi$, an investor feels the market has additional information upon which to condition their expectation for the asset's return.

In the next section we establish three propositions, with proofs contained in Appendix C. The first proposition reverses the main theorem of Ross's (1976) original model. This proposition states the sum of squared deviations from (4) is not bounded above as the number of assets increases. The intuition for the proof rests with understanding that trading optimality and "homogeneous beliefs" are consistent with one investor believing he can profit off another. So long as the number of assets approaches infinity and presumably the economy grows to allow new sources of revenue, the profits from transactions based outside the APT may diverge.

The second proposition uses the counter-examples in this section to show Ross's paradigm at present contains multiple expressions for

the market's model of asset returns that are perfectly consistent with the homogeneous beliefs assumption. In fact, further counter-examples could be constructed using combinations of these same examples. By introducing a new functional form,

$$R_j = E(R_j|\Phi) + \beta_{j1} (F_1)^3 + \dots + \beta_{jM} (F_M)^3 + \epsilon_j ,$$

we can highlight the importance of linearity to the APT.

The standard APT return expression was written in linear form, and the APT cannot account for nonlinear relationships through interpreting the factors in the APT as nonlinear functions of real variables.¹² Here we find another twist where the homogeneous beliefs assumption might lead to an inconsistency. For if it is assumed that the individuals know the factors that determine the return on assets, but these theorists fail to assume that these beliefs are common knowledge, then it is possible for any individual to believe personally these factors affect returns in a linear form but feel the rest of the market employs a nonlinear relationship.

B. Reexamining the APT With and Without Common Knowledge Beliefs

Consider an economy with N individuals. We shall restate the following assumptions utilized in part by Ross (1976), Grinblatt and Titman (1983), and Dybvig (1983) among others.

Assumption 3.1: (Riskless return opportunity) There exists a

¹²Chamberlain and Rothschild (1983) prove that a necessary and sufficient condition for returns to be generated by a K-factor linear model is that K eigenvalues of the covariance matrix of returns grow large as the number of securities increases.

riskfree asset with rate of return, ρ .

Assumption 3.2: (Nonnegligibility of individual 1) There exists at least one individual, identified with subscript 1, who constitutes a noninfinitesimal portion of the market.

For notational simplicity, let equation (15) be denoted in equivalent form as follows:

$$\begin{aligned} R_j &= E(R_j|\Phi) + \beta_{j1} F_1 + \dots + \beta_{jM} F_M + \epsilon_j \\ &\equiv E_j + \beta_j F + \epsilon_j. \end{aligned} \quad (16)$$

Let α_{jk} denote the fraction of individual k 's wealth, ω_k , held in asset j .

Assumption 3.3: (Boundedness of subjective expectations) For all assets j , the sequence of subjective expected returns is uniformly bounded:

$$\| E \| \equiv \sup_j |E_j| < \infty.$$

Assumption 3.4: (Homogenous beliefs) All individuals believe the asset returns are governed by equation (15), and in particular all individuals objectively agree on the M factors, F_i , the β coefficients, and all objectively have the same information set Φ .

Assumption 3.5: The number of assets is infinite. Summation terms over j should be interpreted as summing from $j = 1$ to ∞ .

Proposition 3.1. Given Assumptions 3.1 through 3.5, there exists a value for ρ and a matrix of factors F such that

$$\sum_j (E_j - \rho - \beta_j F)^2 < \infty$$

depending on whether or not there exists at least one individual, k , for whom $\sum w_k \alpha_{jk} E_j \rightarrow -\infty$.

Proof. See Appendix C.

Define a probability space (Ω, F, p) over the possible form of asset return models, with p_i^1 representing individual i 's first-order subjective probability distribution and equation (4) representing an element of Ω .

Assumption 3.6: The individuals "know the model generating factor returns" so that if $(\bullet)$ denotes equation (4), then $p_i^1(\bullet) = 1$, for all individuals i , $i = 1, \dots, N$.

If $p_i^1(\bullet) < 1$, this inequality would imply individual i was uncertain about the factor model generating returns to assets. By assigning probability 1 to the particular model given by equation (15), each individual is said to "know with certainty" that model. Equation (15) therefore represents an individual factor model for the returns to assets.

Definition 3.1: The market valuation for the returns to assets is a regression equation with R_j the dependent variable and independent variables chosen by the market.

Since the market valuation of returns to assets is a subjective concept applying to each individual, intrinsic uncertainty can lead each individual to question what factor model is chosen by others. Proposition 3.2 formalizes this result.

Proposition 3.2. Under Assumptions 3.1 through 3.6, equation (15) yields a non-unique expression for the market valuation of returns to assets.

Proof: See Appendix C.

To surmount this intrinsic uncertainty over the beliefs of others, we must impose common knowledge beliefs on the APT.

Assumption 3.7: Investors have agreed common knowledge priors (ACKP) about the individual asset return equation (4),

$$R_j = E(R_j|\Phi) + \beta_{j1} F_1 + \dots + \beta_{jM} F_M + \epsilon_j ,$$

and the independent distributions of the variables in (4).

Proposition 3.2 has shown that common knowledge assumptions are necessary to aggregate the individual investment equations into a market evaluation. We must now show that ACKP is sufficient. Assumption 3.7 eliminates intrinsic uncertainty between the investors; thus, the investors cannot "agree without knowing they agree." Consequently, each investor's individual return model (4) becomes common knowledge and coincident with the market valuation. This logic leads naturally to the following result.

Proposition 3.3: Given Assumption 3.7, equation (4) will yield unique solution for each individual's expectation for the market valuation of the returns to assets.

C. Conclusions and Some Empirical Observations

This section has shown that under a homogeneous beliefs assumption, individual investors in the APT paradigm agree without knowing they agree. Investors thus face intrinsic uncertainty about the beliefs of other market participants about which factors govern the returns to assets. Due to this uncertainty, investors cannot exclude the possibility that others use a different model. Under these conditions the APT yields multiple pricing equilibria all consistent with the stated assumptions of the paradigm.

With ACKP about the factor model generating returns to assets, the individual investment model yields a unique expression for the market level factor model. From a theoretical standpoint, we have opted for the necessary tradeoff in loss of generality by imposing a more restrictive assumption on the standard APT framework. The alternative would be to abandon the APT due to logical inconsistencies in its present formulation with only homogeneous beliefs. However, the APT has become such an important and widely referenced paradigm of financial theory that we felt it was simply too good a model to abandon.

Empirically, testing for common knowledge beliefs should prove no more difficult than the existing asset pricing tests under homogeneous beliefs. We would advance the argument that in a market consisting of 90% institutional investors, such as the New York Stock Exchange, it is entirely reasonable to assume that large mutual fund investors not only know and have paid researchers studying the factors that

influence returns to assets, but that these institutional investors are smart enough to realize that other institutional investors are employing researchers as well. Thus common knowledge may turn out to be a more plausible assumption empirically than the conventional "homogeneous" or "common prior beliefs" assumption, which would likely occur only by chance.

Various extensions or reformulations of the APT require different assumptions regarding common knowledge beliefs. Sharpe (1982) requires common knowledge beliefs, not simply homogeneous beliefs, about the covariance of F_j and F_k and the $\text{var}(\epsilon)$. Similarly, Shanken (1982) actually relies on common knowledge about "equivalent sets:" common knowledge about which sets of securities are merely different packagings of the same underlying returns. But for Ross's formulation of the APT, and the upper bound models of Grinblatt and Titman (1983) and Dybvig (1983), Assumption 3.7 is both necessary and sufficient to derive the familiar market expression for the APT.

4. OPTION PRICING THEORY

Few recent developments in the modern theory and practice of finance have had a greater impact than the introduction of Option Pricing Theory (OPT) by Black and Scholes (1973) and Merton (1973). The OPT was quickly seen to be a suitable framework to describe the underlying value of all financial assets, and it has since become a universally accepted and widely applied paradigm of finance.

The two main contributions of this section are the illustration that the Black-Scholes formula requires a common knowledge beliefs

assumption, and a new hypothesis that options can be used to hedge against changes in market valuations caused by intrinsic uncertainty. The significance of these contributions is that it clarifies the fundamental assumptions upon which finance paradigms are based. In addition, intrinsic uncertainty as a basis for option trading may turn out to be a fruitful research area.

Proponents of the OPT have heralded the fact that the only nonobservable variable in the OPT is the volatility of the stock price. The reasons for the economy of inputs to the model include a reliance upon arbitrage and valuing options in terms of the price of the underlying security, an observable phenomenon. In practice, financial theorists will assume that the price of the underlying security equals its value; economic theorists will often be interested in price deviations from fundamentals.

Homogeneous beliefs assumptions are inherent in the OPT to derive a common valuation among market participants for both the option and the underlying asset. To illustrate the importance of homogenous beliefs in the OPT, the Black-Scholes formula represents a convenient starting point. Their formula defines the price of a call to be

$$C = S_0 N(d_1) - E e^{-rt} N(d_2), \quad (5)$$

where

$$d_1 = [\ln(S_0/E) + (r + \frac{1}{2} \sigma^2) t] / \sigma \sqrt{t}, \quad (6)$$

$$d_2 = [\ln(S_0/E) + (r - \frac{1}{2} \sigma^2) t] / \sigma \sqrt{t}, \quad (7)$$

and S_0 = the current price of the stock; r = the continuously compounded riskless interest rate; E = the exercise price of the option;

t = the remaining time before expiration (as a fraction of a year);
 σ = the standard deviation of the continuously compounded rate of return, i.e., σ^2 is the volatility of the underlying stock price; and
 $N(d)$ = the value of the cumulative normal distribution at d .

To derive a price given by equation (5), an investor must have an estimate for σ to substitute into (6) and (7). There are several ways to estimate σ including use of historical data on returns. Indeed, Black and Scholes derived their model under the assumption of independent, identically distributed (i.i.d.) rates of return over time. They assume that the stock price follows a "random walk in continuous time with a variance rate proportional to the square of the stock price."¹³ Thus the variance of the rate of return is constant.

A second way to estimate σ is to assume that the Black-Scholes model, or some variant of it, holds on average. The price estimation problem then reduces to an equation with one unknown for which the formula yields a solution. A third method of estimating σ would be to average the estimates obtained from each outstanding call on the stock. Thus the OPT in its present form can only give a subjective value of an option based on each individual's subjective estimate for σ .

A. Logical Intuition

To aggregate across individuals, option pricing theorists frequently employ "homogeneous beliefs" assumptions. When market

¹²Black and Scholes (1973), p.640.

participants have homogeneous beliefs, they are presumed, inter alia, to use the same method to derive an estimate for σ .

Subjective estimates of σ and other variables in OPT may well differ from their objective values. When option pricing theorists assume that stock prices follow a random walk or that asset returns are governed by a linear K-factor model, they make no additional assumptions about the information available to investors in this environment a priori.

Like the MM and APT asset pricing paradigms already discussed in this chapter, the OPT actually requires the "agreed common knowledge beliefs" assumption. Therefore, to determine a market value for an option (or a combination of options) requires knowledge not only of the option value perceived by an individual investor, but also knowledge of how other investors in the market value those same options.

However, if all investors agreed on the value or, more precisely, the expected value of financial assets, then these assets would never be traded, apart from risk aversion or liquidity considerations. The main contribution of this section is to show how intrinsic uncertainty underpins much of option trading.

To illustrate, consider a straddle, which is a combination of a put and a call on a security with the same exercise price and expiration date on each option. An investor who believes the price of the underlying shares will move substantially either up or down, without knowing which direction for sure, would be a likely candidate to purchase a straddle. Yet if all investors in the market had identical beliefs, preferences for risk, and liquidity constraints, then the

prevailing market price for the straddle option would leave these investors indifferent between purchasing the straddle or not.

If an investor believes that the market has underestimated the magnitude of future price changes, then he might purchase a straddle at the prevailing market price. Clearly, the purchaser of the straddle option holds differing beliefs from the market, or whoever is selling him the option package.

The writer of a straddle is an investor who believes that the underlying share price will remain close to the exercise price of the straddle. He gains money from small changes in stock prices and loses when the price changes are large. Yet the market prices for put and call options already incorporate the market's best estimate of σ^2 , the instantaneous variance of the price of the underlying security.

Buyers and sellers of options are not merely betting that the stock price variance will be high or low, respectively. Rather, these investors are betting that the variance will be higher or lower than the market's best estimate, as reflected in the prevailing market prices for the security itself and for options on the security.

What is the nature of the differing beliefs between financial investors? Are options traded because of uncertainty about exogenous returns to underlying assets, because of intrinsic uncertainty as to how the rest of the market perceives the exogenous risk, or both? We will answer these questions by focusing on two sources of uncertainty in such models: exogenous uncertainty about the actual state of the world, and intrinsic uncertainty over the beliefs and potential portfolio revisions of other investors in the market. It has already

been shown that purchasing options can reduce the risk of exogenous changes in asset returns.¹⁴ Options offer a form of insurance against exogenous changes in the real return to assets. The next section offers some heuristic examples to illustrate how intrinsic uncertainty can form a basis for option trading under differing investor belief regimes.

B. Examples of Intrinsic Uncertainty as a Basis for Option Trading

In this section we analyze intrinsic uncertainty and its complement, common knowledge, as a foundation for option trading. We illustrate how OPT and option trading can evolve under three different regimes of investor beliefs. The first regime posits investors have common (homogeneous) beliefs, but not common knowledge beliefs. The second regime considers the most restrictive case for investor beliefs: agreed common knowledge beliefs. The third regime discusses differing common knowledge beliefs.

Consider three periods in a stock and option market economy, where investors buy and sell financial instruments in period 1. At the end of period 1, the markets receive an imperfect signal about the exogenously determined returns on the assets. To simplify the analysis of this section, suppose there are only two possible information events: A and B. Let $E(\sigma^2|A) = a^2$ and $E(\sigma^2|B) = b^2$, where E

¹⁴It is well known that option trading allows diversification of risk from holding stocks. See, e.g., Malkiel and Quandt (1969), Merton, Scholes, and Gladstein (1978), Gastineau (1979), and Gladstein, Merton, and Scholes (1982), Boyle and Emanuel (1980), Cox and Rubinstein (1984), Galai and Masulis (1976), Ingersoll (1977), Smith (1976), Merton (1976a) and (1976b), among others.

represents the expected value operator and $a^2 < b^2$. In period 2, the investors revise their portfolios. In period 3, the exogenous factors generate new returns to the assets.

We can now consider financial trades under differing investor belief regimes. Specifically, we focus on option trading as a method of eliminating risk due to intrinsic uncertainty over the unknown beliefs of other market participants.

Regime 1: Common (Homogeneous) Beliefs

Assume each of the investors in period 1 subjectively believes the information events will be A with probability 1 and B with probability 0. Thus the investors have common prior beliefs, or homogeneous beliefs, about the exogenous returns on the assets. Yet we have assumed nothing about the information the investors have about the market, the information available to the market, and the beliefs which the market hold. Option trading in this regime can be a source of capital gains based not only on the exogenous real return of the assets, but also on purchasing tendencies of the market.

It is certainly possible that an individual investor might believe that everyone else in the market shares his own beliefs. However, it is just as possible that the individual may believe, even in a market where investors objectively have common prior beliefs, that others in the market hold different views from his own. At least the individual would be uncertain as to how others assign probability to the A and B states of the world. Since the latter case of intrinsic uncertainty is more likely to be the norm than the exception, it

is of some interest to both financial theorists and practitioners.

Suppose investor i assigns $\Pr(A) = 1$ and $\Pr(B) = 0$, but he believes the majority of other investors assign $\Pr(A) = .4$ and $\Pr(B) = .6$. Based on his first-order distribution, investor i perceives $E(\sigma^2|A) = a^2$. Based on his second-order beliefs, i feels the market's estimate for σ^2 is given by

$$E(\sigma^2) = .4 a^2 + .6 b^2 \equiv c^2.$$

From the Black-Scholes formula (5), (6), and (7) we see that the derivative of the value of the call option (5) with respect to σ^2 is positive. Since $c^2 > a^2$, the call option takes on greater value to investor i when he considers capital gains potential in addition to real earnings. Essentially, every investor in the stock market places himself at risk not only to fluctuations in the real returns to assets but also to capital gains and losses from the purchasing tendencies of other market participants. Thus purchasing a call option can reduce the risk for an individual from purchasing a share at a time when other investors are driving up the price of shares with their purchases.

Suppose in period 1 investor i owns shares in a firm and would like to buy more shares in period 2. If he believes the majority of the market currently evaluating $E(\sigma^2) = c^2$ will revise their estimate to $\sigma^2 = a^2$ by period 2, then he may expect the majority to buy shares in period 2. In period 2 the majority of the market will learn the stock has less exogenous risk, which will induce them to buy more of the stock at a given price, ceteris paribus.

Investor i can eliminate the intrinsic risk he faces from

fluctuations in market buying behavior by purchasing a call option in period 1 entitling investor i to purchase a share in period 2 at a price which investor i believes is less than the price that will prevail in period 2 with the majority's bidding up of the asset.

A change in the market's information about risk of fluctuating market values will not necessarily in itself induce trading in options on the asset. To back up any assertion that an option trade actually takes place, we would have to invoke a theory which explains how options are allocated in equilibrium portfolios. Thus depending on each investor's equilibrium portfolio constraints, our conjectures must be qualified to indicate possible trades that might take place but are not guaranteed.

The investor i we describe above might be paired with another investor j , who (correctly) believes the majority of the market in fact shares the same probability assignments of $\Pr(A) = 1$ and $\Pr(B) = 0$. Alternatively i might be able to trade with some investor k , who believes the majority view is given by $\Pr(A) = .5$ and $\Pr(B) = .5$. Since the second-order beliefs of k , $(.5, .5)$, are closer to his own second-order beliefs, investor i might be able to obtain better terms of trade from investor j . If $.5 a^2 + .5 b^2 = d^2$, then we note $a^2 < d^2 < c^2$, and $c^2 - a^2 < c^2 - b^2$. Therefore, investors i and j have larger terms of trade than investors i and k .¹⁵

¹⁵We assume for convenience that investors in this illustration weight heavily their own second-order beliefs in determining the value of the option. In theory, the investors could place more emphasis on their third or higher-order beliefs, as Keynes has already noted with respect to underlying security valuations.

Analogously, if investor i wanted to sell his stock in period 2, but was concerned the majority of the market would be selling in period 2 rather than buying, then investor i would face intrinsic uncertainty over whether the sales of other investors might depress prices in period 2. Investor i could realize a capital loss if he took part in the majority's selling action. Consequently, investor i could eliminate this intrinsic risk by purchasing a put option at a price higher than the price he expected to prevail in period 2 with the majority's sales.

In a regime where investors only have common priors, options can be traded between investors who hold differing views of how the "rest of the market" assigns probability to the information events A and B. Thus intrinsic uncertainty over the unknown beliefs and portfolio revisions of others in the market can lead investors to buy and sell options as a method of securing their wealth.

Regime 2: Agreed Common Knowledge Beliefs (ACKB)

If intrinsic uncertainty provides a basis for option trading, what can be said about option trading in an economy where investors have agreed common knowledge priors (ACKB)? In such a regime, investors would find the market's best estimate for the value of the options as well as the value of the underlying stocks to be identical with their own. Therefore, each investor would find no undervalued or overvalued financial instruments in the economy.

In general, options could provide no means of reducing risk from exogenous changes in the real returns to assets, because the

equilibrium market prices should, ceteris paribus, leave each investor indifferent between purchasing the financial instruments or not. Furthermore, the option cannot be traded over intrinsic uncertainty, because in an ACKB regime no intrinsic uncertainty exists.

We have employed a ceteris paribus qualifier elsewhere in this paper, but this ACKB regime highlights its importance. When investors share ACKB, the only reason for trading financial assets must derive from differences in risk aversion, liquidity constraints, or some other external feature of the model such as institutional requirements.

Again, by way of illustration, consider any investor i who believes $\Pr(A) = 1$ and $\Pr(B) = 0$. Thus $E(\sigma^2) = a^2$. Each investor i not only personally believes the volatility is a^2 but also knows everyone else knows, and so on. No investor i can be paired with another investor to trade options based on differing higher-order probability assignments, because each investor knows he assigns the same value to the volatility as everyone else.

Regime 3: Different Common Knowledge Beliefs (DCKB)

In this regime the Black-Scholes assumption of homogeneous beliefs is not met. Therefore, we cannot refer to the call value emerging from their formula. However, we can offer some comparisons based on the common sense rule that the greater the (perceived) volatility of the underlying stock, the greater the (perceived) value of an option. If a share price has very little volatility, then the option would likely be left unexercised leaving the purchaser with the

purchase cost of the call or put option.

One special case of heterogeneous beliefs is when investors hold different common knowledge beliefs (DCKB). Suppose it was common knowledge between i and j that $p_i^1(\omega) \neq p_j^1(\omega)$, for all i, j , over events $\omega \in \Omega$ in the probability space (Ω, F, ρ) . Therefore,

$$p_i^3(\omega) \equiv p_j^2(\omega) \equiv p_i^1(\omega) \neq p_j^1(\omega) \equiv p_i^2(\omega) \equiv p_j^3(\omega), \quad (8)$$

so that in general

$$p_i^N(\omega) = p_j^{N+1}(\omega) \quad (9)$$

where N is a natural number. Equation (8) states that i and j disagree over the probability assigned to the generic event ω . (In our continuing illustration ω represents either the information event A or B). The left-hand side of (8) further states that i knows j knows i 's assignment is different from his own; while the right-hand side states j knows i knows j 's assignment is different from his own. Equation (9) carries this knowledge of disagreement up to the N th level.

Whereas homogeneous beliefs imply the individuals agree without knowing they agree, DCKB imply the individuals disagree and know they disagree. Options would not be traded in this regime due to intrinsic uncertainty; however, options could be traded over the exogenous real return potential for assets.

Suppose i believed $E(\sigma^2) = a^2$ and j believed $E(\sigma^2) = b^2$. It is conceivable that i and j might be able to bargain between themselves for a call option at a price that i would be willing to write and j willing to buy since $b^2 > a^2$. Thus with a higher expectations for the

volatility, any option should be more valuable to j than i . The uncertainty leading to different valuations for the same option is based solely on differing beliefs about who has better information about the state of the world. The investors face no uncertainty in a DCKB regime over how the other traders assign probability between possible events A and B.

To summarize all these scenarios, in regime 1 investors agree about exogenous randomness, yet options might reduce or eliminate risk over intrinsic uncertainty in the market. In regime 2, options would be traded only for liquidity constraints, risk aversion, and the like. In regime 3, no intrinsic uncertainty exists, but options can be traded over the exogenous real returns to assets.

C. Indeterminism in the OPT Without Common Knowledge

As shown in the OPT literature, the Black-Scholes formula can be derived under a set of ideal assumptions.

Assumption 4.1: The short-term interest rate is a known constant through time.

Assumption 4.2: There are frictionless markets.

Assumption 4.3: Financial instruments can be continuously traded, i.e., there exists divisibility of capital which makes it possible to buy or sell any fraction of an option or underlying security.

Assumption 4.4: The stock price follows a Brownian motion process over time.

Assumption 4.5: The stock pays no dividends or other distributions.

Assumption 4.6: The formula gives the value of a European call, i.e., the call only exercised at maturity.

Assumption 4.7: There are no penalties for short-selling.

Under these simplifying assumptions, a risk-free hedged portfolio - one is reminded here of the risk-free arbitrage maneuver of the MM paradigm - can allegedly be formed from a combination of a long position in the stock and a short position in the European call written on the stock. As the stock price changes over time, an investor could maintain the hedge by adjusting the proportion of stock and calls in his portfolio. The value of this hedged portfolio, V_h , would be given by the product of the prices and quantities of the assets it comprises:

$$V_h = S Q_S + C Q_C , \quad (10)$$

where S is the price per share of the stock, Q_S is the number of shares, C is the price of the European call, and Q_C is the quantity of calls. If we take the total derivative of (10), then the change in the value of the hedged position is:

$$dV_h = Q_S dS + Q_C dC . \quad (11)$$

The price of the call option can be written, $C(S,t)$, as a function of the stock price, S , and time, t . Under assumption (A3), Black and Scholes use stochastic calculus and Ito's Lemma to expand dC as follows:

$$dC = C(S + dS, t + dt) - C(S,t) \quad (12)$$

$$= C_1 dS + \frac{1}{2} C_{11} \sigma^2 S^2 dt + C_2 dt . \quad (13)$$

The subscripts on C in equation (13) denote partial derivatives with

respect to the first or second argument in $C(S,t)$.

At this point the Black-Scholes derivation of the value of a call option reveals the fundamental indeterminism of OPT. The estimate for σ^2 is being treated as a known constant. In fact, σ^2 is still an unknown parameter depending not on individuals' own estimates for σ^2 , since no single individual has enough market power to control the stock price variance, but upon each individual's estimate for the market's estimation of σ^2 .

Bayesian subjectivist theory encourages the treatment of any unknown by assigning a probability distribution over the possible values of the unknown. In general, each investor in the market should have an entire probability distribution over possible values for the market's estimate of σ^2 . However, the literature in finance has treated σ^2 as a constant, which implies every investor has a degenerate probability distribution assigning all probability to a single possible value for σ^2 .

By assumption 4.4 or some variant of it, option pricing theorists have sought to superimpose agreed or homogeneous beliefs on the market participants. Yet this assumption only implies that each individual knows his own estimate for σ^2 and that objectively these values are the same. Neither the Black and Scholes (1973) paper nor any other extension of the OPT contains any mechanism that would transmit information about the beliefs of others to the market participants.

The next step in the derivation of the pricing formula would be to substitute equation (13) for dC into equation (31), which yields

$$dV_h = Q_S dS + Q_C [C_1 dS + \frac{1}{2} C_{11} \sigma^2 S^2 dt + C_2 dt]. \quad (14)$$

Black and Scholes observed that the riskless hedge could be maintained by purchasing one share of stock and selling $1/C_1$ calls:

$$Q_S = 1 \text{ and } Q_C = -1/C_1. \quad (15)$$

After incorporating (32) into equation (31), it now reads

$$dV_h = dS + -1/C_1 [C_1 dS + \frac{1}{2} C_{11} \sigma^2 S^2 dt + C_2 dt]. \quad (16)$$

Moreover, since the return on the hedged portfolio is known and risk-free, it must be given by the prevailing interest rate, r , on riskless assets, so that

$$dV_h = rV_h dt. \quad (17)$$

Substituting (17) for the left-hand side of (16) and simplifying we have

$$C_2 = r V_h (-C_1) - \frac{1}{2} C_{11} \sigma^2 S^2;$$

and, finally, substituting (30) and (35) yields

$$C_2 = rC - rSC_1 - \frac{1}{2} C_{11} \sigma^2 S^2. \quad (18)$$

Equation (18) is an ordinary differential equation yielding each individual's subjective value of the call, based on each individual's subjective estimate for σ^2 .

The call will only be exercised if the price of a share at the maturity date is greater than or equal to the exercise price, E , of the call. Let " T " be the maturity date of the call then

$$C(S,T) = S - E \quad \text{for } S \geq E \quad (19a)$$

$$= 0 \quad \text{for } S < E \quad (19b)$$

We should note that the terminal conditions (19a) and (19b) apply

to a European call, i.e., a call which cannot be exercised before time T .

Equation (18) has a unique solution subject to the boundary conditions (19a) and (19b), given by

$$C = S N(d_1) - E e^{-rt} N(d_2) \quad (20)$$

where

$$d_1 = [\ln(S/E) + (r + \frac{1}{2} \sigma^2) t] / \sigma \sqrt{t}, \quad (21)$$

$$d_2 = [\ln(S/E) + (r - \frac{1}{2} \sigma^2) t] / \sigma \sqrt{t}. \quad (22)$$

From each individual's perspective, the value of the call option emerging from equation (20) gives only his own subjective value, not the market's collective valuation.

As outside observers, we may note that Assumption 4.4 implies that the stock price is distributed normally with a fixed variance. But individuals in this economy, even under Assumptions 4.1 - 4.7, have no way of knowing what values others in the economy assign to σ^2 . No mechanism in the Black and Scholes framework or any other variant of the OPT conveys this information. Indeed this type of information about the unknown beliefs of others is not provided in the market or anywhere else. Yet the functioning of financial markets depends crucially on individual estimates of the market's beliefs, since individual investors confront asset prices which reflect the market's best estimate for the return or value of those assets.

To restore the OPT formula as a market-level expression for the value of an option, we need to impose common knowledge on the model. The necessary assumption can take either of two forms:

Assumption 4.8a: Individuals have agreed common knowledge estimates for σ^2 .

Assumption 4.8b: The random walk process that stock prices follow is common knowledge.

Both of these assumptions will lead to the same result, namely, that the individual investor will know that other investors share his own estimate for the variance of the underlying stock price.

To summarize, the OPT requires a "common knowledge" assumption to transform the individual investment expression into a market formula for the pricing of options. Without common knowledge, individual investors in the market will have intrinsic uncertainty over the unknown beliefs and portfolio decisions of others. By imposing common knowledge on the paradigm, we eliminate this intrinsic uncertainty, which is certainly a part of real world fluctuations in both the stock and option exchanges.

D. Conclusions

Section 4 has shown how intrinsic uncertainty - in the absence of common knowledge - can enter into option pricing and yield differing subjective valuations for an option. When individuals face intrinsic uncertainty over market valuations and the beliefs of others, options can be used to hedge against this risk. An individual who expected the market to be selling off some asset at the same time he wanted to sell his own stock may seek to purchase a put option. Those who believe the market demand for a stock may increase at a time when they

wish to purchase the same stock may seek a call option on the stock. subjected to changes in market valuations of securities.

Section 4 has also reconciled the Black-Scholes OPT with common knowledge definitions about the random process generating returns to assets, or more precisely, the variance of the underlying stock. Without a common knowledge beliefs assumption, the Black-Scholes formula yields the subjective value of an option to one particular individual. By assuming common knowledge about the volatility term σ^2 , the OPT formula becomes not only an expression for each individual's value of the option but also an expression for the market's agreed valuation as well.

Since common knowledge belief assumptions will yield a unique expression for the market valuation of an option, it is of some interest to both financial theorists and practitioners that generally the purchaser and seller of an option hold differing views of the value of that option. We have shown that intrinsic uncertainty can form a second foundation for option trading.¹⁶ It has already been

¹⁶The form of intrinsic uncertainty we consider for option trading was previously considered by Keynes (1936) for securities: "[P]rofessional investment may be likened to those newspaper competitions in which the competitors have to pick out the six prettiest faces from a hundred photographs, the prize being awarded to the competitor whose choice most nearly corresponds to the average preferences of the competitors as a whole; so that each competitor has to pick, not those faces which he himself finds prettiest, but those which he thinks likeliest to catch the fancy of other competitors, all of whom are looking at the problem from the same point of view. It is not a case of choosing those which, to the best of one's judgment, are really the prettiest, nor even those which average opinion genuinely thinks the prettiest. We have reached the third degree where we devote our intelligences to anticipating what average opinion expects the average opinion to be." (Keynes 1936, p.156).

widely recognized that options allow diversification of risk from exogenous factors, such as droughts, wars, or even business cycles, that influence the real returns to assets. This section has stressed that options can also protect investors from unanticipated or anticipated portfolio revisions by other investors in the market.

5. CONCLUSIONS FOR THE FOUR ASSET PRICING PARADIGMS

Intrinsic uncertainty is a significant part of the risk which financial investors face; hence further theoretical work is needed to incorporate this form of uncertainty into the standard finance valuation models. In pursuing this research we have been led naturally to questioning the wisdom of the conventional perfect-market model used for financial analysis, in which investors are assumed to have access to costless information. Common knowledge imposes a stringent requirement about the kind of information (about the beliefs of other traders in the market) that real world markets may have no mechanism to provide. As Merton (1987) has reminded the profession, recognition of informational costs and the absence of such mechanisms, (particularly where common knowledge is concerned), "can go a long way toward explaining financial behavior that is otherwise seen as anomalous to the standard (perfect-market) model."¹⁷

The MM paradigm was shown to be valid in a two-period economy whether or not common knowledge is assumed. With multiperiods an

¹⁷Merton (1987, p.486-487). Merton is careful to use the term "common knowledge" rather than merely "common priors" or "homogeneous beliefs" in discussing the informational assumptions of financial asset pricing paradigms.

investor following the MM-style arbitrage strategy faces risk in the form of price fluctuations that leave him with a capital loss or a foregone capital gain. Furthermore, in a multiperiod model an MM-style strategy entails a commitment to remain invested until the state of the world is revealed.

The CAPM avoids the explicit need for common knowledge assumptions by implicitly giving the investor perfect and correct information about the real returns and capital gains from holding assets. In order to have perfect knowledge about capital gains across assets, each investor must know the beliefs and preferences of the other market participants.

The APT requires common knowledge about the K-factor model governing returns to assets. Without common knowledge, investors can question the market's beliefs about the factor coefficients, the number of factors, the functional form, and the information upon which the market conditions its expected return. Consequently, it is little wonder that the empirical literature on testing the APT has found no agreement as to what the factors in the APT should be.

Finally, for the OPT we have shown that intrinsic uncertainty over the volatility term σ^2 can lead to different calculations for the value of put and call options. It is not sufficient in the Black-Scholes formula to assume investors "know" σ^2 , for they can "know" and agree on σ^2 without knowing they agree. To rectify this problem over potential multiple valuations, we showed the need to impose common knowledge beliefs on the model. However, the absence of common knowledge raises several interesting questions about how

options might be traded, not only as insurance against exogenous shocks to the economy, but also as insurance against endogenous fluctuations in price due to intrinsic uncertainty over the beliefs of others. When investors face intrinsic uncertainty, options might be used as a hedge against changes in market valuations of a security independent of the exogenous determinants of the security's real earnings.

APPENDIX A. THE DEBT/EQUITY PROPOSITIONS

The classic work of MM stated that the cost of capital for a firm was independent of its debt-equity ratio. This appendix delineates what impact common knowledge assumptions have on the proof of the irrelevancy proposition. We employ five assumptions.

Assumption 1.1: The markets for assets are frictionless and competitive.

Assumption 1.2: Individuals can borrow at the same market rate of interest as firms.

Assumption 1.3: It is common knowledge that firms cannot go bankrupt.

Assumption 1.4: Investors have agreed common knowledge prior (ACKP) beliefs about the assignment of each firm to a given risk class.

Let M_1 and M_2 denote the market values of firms 1 and 2, respectively, and let firms 1 and 2 belong to the same risk class. Then the following proposition can be obtained. Let X_{is} denote the earnings of firm i in state s .

Assumption 1.5: $X_{1s} = X_{2s} \equiv X_s$.

Proposition 1: Given Assumptions 1.1 - 1.5, $M_1 = M_2$.

Proof: Suppose $M_1 \neq M_2$ and $M_1 > M_2$. Under Assumption 1.4 an arbitrage opportunity now exists, namely, buy α % of firm 1's bonds and α % of firm 2's equity at a price of $\alpha(P_2 E_2 + D_2) = \alpha M_2$. The investor can then short sell α % of firm 1's bonds and equity at price $\alpha(P_1 E_1 + D_1) = \alpha M_1$. The immediate profit from these two transactions is $\alpha M_1 - \alpha M_2 > 0$, by supposition. When state of the world "s" is

revealed, the investor will receive income of $Y_{2s} = \alpha (X_{2s} - rD_2) + \alpha rD_2$ from his purchase of firm 2 bonds and equity; the investor will also have to payout income $Y_{1s} = \alpha (X_{1s} - rD_1) + \alpha rD_1$, based on his short sales. By Assumption 1.5, $X_{1s} = X_{2s}$ and the investor's return and expenditure obligations exactly offset. Therefore, the investor has achieved a riskless profit. If $M_1 < M_2$, the preceding strategy can be reversed. Either inequality condition is inconsistent with equilibrium, hence the only remaining possibility is $M_1 = M_2$.

Assumption 1.6: The investors have common priors or homogeneous beliefs about the assignment of firms to risk classes.

For intertemporal models, the market value of firm i , M_i , can be perceived by each individual investor as a random variable defined on some probability space (Ω, F, p) .

Proposition 2: Under Assumptions 1.1 - 1.3, 1.5 and 1.6, in an economy with three or more periods, let $X_{1s} = X_{2s} = X_s$ for all states s , and let $M_2 > M_1$. It is possible for a risk neutral investor to anticipate and receive capital gains from purchasing shares and bonds in firm 2, the "overvalued" firm.

Proof. Let the investor have endowed wealth W . If the investor purchases α % of firm 2's equity and α % of firm 2's debt, where $W = \alpha(P_2 E_2 + D_2) = \alpha M_2$, he will receive income of $Y_{2s} = \alpha X_s$ when state s is revealed. Alternatively, the investor could sell his shares before the state is revealed and realize a capital gain or loss. Let investor i expect the market value of firm 2 at some period T to equal

$\int M_2 dp_i$. Then investor i expects to realize a capital gain of $\alpha (\int M_2 dp_i - M_2)$ if he sells his firm 2 shares and bonds at time T . Similarly, if $\gamma M_1 = W$, then the investor could receive income of $Y_{1s} = \gamma X_s$ in state s from purchasing firm 1 bonds and equity. The same investor could also sell his firm 1 shares in some intermediate period T and realize a capital gain of $\gamma (\int M_1 dp_i - M_1)$. If $\alpha (\int M_2 dp_i - M_2) - \gamma (\int M_1 dp_i - M_1) > Y_{1s} - Y_{2s}$, a risk neutral investor will purchase firm 2 debt and equity.

Several comments are required about the assumptions that underpin Proposition 2. First, $Y_{1s} > Y_{2s}$ because $\gamma > \alpha$ when $M_2 > M_1$. Second, if the investor is risk averse rather than risk neutral, he may wish to execute an MM-style portfolio revision and wait for the state of the world to be revealed. A risk averse investor may also be willing to take a chance on the perceived capital gain, depending on his degree of risk aversion. Finally, capital gains-motivated speculation requires three or more periods: the speculator must be able to buy (sell) and resell (repurchase) before the state of the world is revealed.

Assumption 1.7: A majority of the purchasing power of the market believes $X_{1s} = X_{2s} \equiv X_s$.

Proposition 3: Given Assumptions 1.1 - 1.3, and 1.7, if the economy comprises two periods, then $M_1 = M_2$.

Proof. In a two-period regime, the revelation of the state-of-the-world eliminates any intrinsic uncertainty over the beliefs of other

market participants. In Proposition 1 this intrinsic uncertainty was eliminated via common knowledge beliefs among the market. Thus the conditions of Proposition 1 are essentially met, and the proof follows from the proof of Proposition 1. Assumption 1.7 assures that even with some heterogeneous beliefs, the market can effect purchasing strategies that force $M_1 = M_2$.

Remark 1. An alternative proof of the MM debt-equity proposition in a two-period market is contained in Hirshleifer (1971).

Remark 2. Common knowledge beliefs are not required for a proof of the MM proposition in a two-period economy.

APPENDIX B. DERIVATION OF THE CAPM WITH KNOWN ASSET RETURNS

The standard CAPM equation derives from the minimization problem,

$$\begin{aligned} \min_{X_{ik}} \quad & \sum_i \sum_j X_{ik} X_{jk} \text{COV}(\tilde{R}_i, \tilde{R}_j) \\ & + \lambda_k \left[\sum_i X_{ik} E(\tilde{R}_i) + (1 - \sum_i X_{ik}) R_f - E(\tilde{R}_p) \right], \end{aligned} \quad (\text{B.1})$$

where X_{ik} represents the proportion of individual k 's portfolio held in asset i ; $\tilde{R}_i$ and $\tilde{R}_j$ are the returns on assets i and j , respectively, where the return denotes the summed capital gain, dividends, and interest payments divided by the asset's price when the period commenced; λ_k is the Lagrange multiplier associated with individual k 's wealth, R_f is the riskless rate of interest, and $E(\tilde{R}_p)$ is a constant denoting individual k 's expected return on his portfolio.

The first-order interior condition stated for individual k is

$$\sum_j X_{jk} \text{COV}(\tilde{R}_i, \tilde{R}_j) + \lambda_k [E(\tilde{R}_i) - R_f] = 0 \quad (\text{B.2})$$

Let W_k denote the fraction of total market wealth individual k holds.

Then

$$\sum_j \sum_k W_k X_{jk} \equiv \sum_j X_{jm}, \quad (\text{B.3})$$

and

$$\sum_j X_{jm} \tilde{R}_j \equiv \tilde{R}_m, \quad (\text{B.4})$$

where X_{jm} denotes asset j 's fraction in the total market portfolio, and R_m is the market's return on all assets taken together. Multiplying each term in (B.2) by W_k , summing over k , and substituting (B.3) into the first term after switching the order of summation yields

$$\sum_j X_{jm} \text{COV}(\tilde{R}_i, \tilde{R}_j) + \sum_k W_k \lambda_k [E(\tilde{R}_i) - R_f] = 0. \quad (\text{B.5})$$

Next, we substitute (B.4) and let the variable Z denote $\sum_k W_k \lambda_k$, then the first-order condition becomes

$$\text{COV}(\tilde{R}_i, \tilde{R}_m) + Z[E(\tilde{R}_i) - R_f] = 0. \quad (\text{B.6})$$

Let $\tilde{R}_i = \tilde{R}_m$ and observe that (B.6) is now

$$\text{VAR}(\tilde{R}_m) + Z[E(\tilde{R}_m) - R_f] = 0, \quad (\text{B.7})$$

and solving for Z yields

$$Z = - \text{VAR}(\tilde{R}_m) / [E(\tilde{R}_m) - R_f]. \quad (\text{B.8})$$

Again substituting this value of Z into (B.6) yields the familiar CAPM equation:

$$E(\tilde{R}_i) - R_f = \frac{\text{COV}(\tilde{R}_i, \tilde{R}_m)}{\text{VAR}(\tilde{R}_m)} [E(\tilde{R}_m) - R_f], \quad (\text{B.9})$$

$$= \beta_i [E(\tilde{R}_m) - R_f]. \quad (\text{B.10})$$

This result only requires agents to use, e.g., the same estimate $\tilde{R}_j$ for asset j 's return. In equilibrium, when all agents revise their portfolios according to their beliefs about the returns, they will have mean-varient efficient holdings.

APPENDIX C. PROOFS OF APT PROPOSITIONS

Proof of Proposition 3.1: (Derived in part from Ross (1976) at page 352) If the condition of Proposition 3.1 approaches infinity, then by Assumption 3.2 for some individual 1

$$\sum_j \alpha_{j1} E_j \rightarrow \infty. \quad (C.1)$$

Denote the total fraction of wealth held by individual 1 be denoted by ω_1 and for the rest of the market by ω_{m-1} . We can write out the aggregate demand, D_j , for asset j as

$$D_j \equiv \omega_1 \alpha_{j1} + \omega_{m-1} \alpha_{j,m-1} \geq 0.$$

Since in equilibrium the sum of the aggregate demand across assets must comprise the total assets in the economy,

$$\sum_j D_j = 1.$$

Therefore

$$\begin{aligned} \sup_j |E_j| &\geq \sum_j D_j E_j \\ &= \sum_j (\omega_1 \alpha_{j1} + \omega_{m-1} \alpha_{j,m-1}) E_j \\ &= \omega_1 \sum_j \alpha_{j1} E_j + \omega_{m-1} \sum_j \alpha_{j,m-1} E_j \quad (C.2) \end{aligned}$$

Together equation (C.1) and the noninfinitesimal size of individual 1 imply the first summation term in (C.2) approaches infinity. In order for $\sup |E_j|$ to be bounded,

$$\omega_{m-1} \sum_j \alpha_{j,m-1} E_j \rightarrow -\infty. \quad (C.3)$$

In order for the rest of the market's summation in (C.3) to hold, it must be the case that for some individual k , $k \neq 1$,

$$\sum_j \omega_k \alpha_{jk} E_j \rightarrow -\infty \quad (C.4)$$

on a subsequence. Equation (C.4) states a condition that for individual 1 to expect to earn divergent positive profits, he must expect to earn these profits at the expense of some other individual k . If the expectation held true, individual i must expect individual k to earn divergent negative profits when summed across an infinite stream of assets in the economy. Equations (C.1) and (C.4) are consistent with optimality and Assumption 3.3, since the latter assumption refers only to each individual's own expectation for the return to assets and not his expectation for other individual's expected return. Conversely, if $\sum (E_j - \rho - \beta_j F)^2 < \infty$, then the same steps of the proof can be repeated to show that there can exist no individual k for whom $\sum \omega_k \alpha_{jk} E_j \rightarrow -\infty$, as shown in Ross (1976).

Proof of Proposition 3.2: (By counter-example) From Assumptions 3.4 and 3.6, each individual believes the returns on the assets is governed by

$$R_j = E(R_j|\Phi) + \beta_{j1} F_1 + \dots + \beta_{jM} F_M + \epsilon_j. \quad (4)$$

The particular form of the factor models used by the other $N-1$ investors are still unknown random variables to each individual. The following equations are well-defined factor models for the returns to assets:

$$R_j = E(R_j|\Phi) + \beta_{j1} F_1 + \dots + \beta_{jM-K} F_{M-K} + \epsilon_j, \quad (C.5)$$

where $M-K < M$, or

$$R_j = E(R_j|\Phi) + \gamma_{j1} F_1 + \dots + \gamma_{jM} F_M + \epsilon_j, \quad (C.6)$$

where $\gamma_{jk} \neq \beta_{jk}$, or

$$R_j = E(R_j|\Psi) + \beta_{j1} F_1 + \dots + \beta_{jM} F_M + \epsilon_j . \quad (C.7)$$

Each individual i could assign positive probability that either (C.5), (C.6), or (C.7) might be used by others in the market. Any individual who believes a noninfinitesimal fraction of the other investors will use (C.5) or (C.6) or (C.7), must modify his own portfolio transactions accordingly. By modifying his investment behavior to account for alternative equations such as (C.5) or (C.6) or (C.7), each investor i will no longer believe investments are governed solely by (4). As long as the fraction of investors, who face uncertainty over the investment equations chosen by others, is noninfinitesimal, equation (4) will not predict the market valuation for returns.

6. REFERENCES

- Baumol, William J., and Burton Malkiel, (1967), "The Firm's Optimal Debt-Equity Combination and the Cost of Capital," Quarterly Journal of Economics, Vol. 81, pp. 548-578.
- Bhattacharya, M., (1980), "Empirical Properties of the Black-Scholes Formula Under Ideal Conditions," Journal of Financial and Quantitative Analysis, Vol. 15, pp.1081-1105.
- Black, Fischer and Myron Scholes, (1973), "The Pricing of Options and Corporate Liabilities," Journal of Political Economy, Vol. 81, pp.637-659.
- Boyle, P. and D. Emanuel, (1980), "Discretely Adjusted Option Hedges," Journal of Financial Economics, Vol. 8, No. 3, pp. 259-282.
- Breeden, Douglas, (1979), "An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities," Journal of Financial Economics, Vol. 7, pp.265-296.
- Chamberlain, G. and Michael Rothschild, (1983), "Arbitrage, Factor Structure, and Mean-Variance Analysis on Large Asset Markets," Econometrica, Vol. 51, pp.1281-1304.
- Chen, Nai-fu and Jonathan Ingersoll, Jr., (1983), "Exact Pricing in Linear Factor Models with Finitely Many Assets," Journal of Finance, Vol. 38, No. 3, pp.983-988.
- Cox, J. C., and M. Rubinstein, (1984), Option Markets, (Englewood Cliffs, N.J.: Prentice-Hall, 1984).
- Cragg, John and Burton Malkiel, (1982), Expectations and the Structure of Share Prices, (Chicago: University of Chicago Press, 1982).
- Durand, David, (1959), "Comment," American Economic Review, Vol. 49, pp.639-654.
- Dybvig, P., (1983), "An Explicit Bound on Individual Assets' Deviations from APT Pricing in a Finite Economy," Journal of Financial Economics, Vol. 12, pp.483-496.
- Galai, D. and R. W. Masulis, (1976), "The Option Pricing Model and the Risk Factor of Stock," Journal of Financial Economics, Vol. 3, pp.53-81.

- Gastineau, G., (1979), The Stock Options Manual, (New York: McGraw-Hill, 1979).
- Geske, R., (1977), "The Valuation of Corporate Liabilities as Compound Options," Journal of Financial and Quantitative Analysis, Vol. 4, pp.541-552.
- Gladstein, M., Robert Merton, and Myron Scholes, (1982), "The Returns and Risks of Alternative Put Option Portfolio Investment Strategies," Journal of Business, January 1982, pp.1-55.
- Grinblatt, Mark and Sheridan Titman, (1983), "Factor Pricing in a Finite Economy," Journal of Financial Economics, Vol. 12, pp.497-507.
- Harrison, J. Michael and David M. Kreps, (1979), "Martingales and Arbitrage in Multiperiod Securities Markets," Journal of Economic Theory, Vol. 20, pp.381-408.
- Hirshleifer, Jack, (1970), Investment, Interest, and Capital, (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1970).
- Huberman, Gur, (1982), "A Simple Approach to Arbitrage Pricing Theory," Journal of Economic Theory, Vol. 28, pp. 183-191.
- Ingersoll, Jonathan E., Jr., (1977), "An Examination of Corporate Call Policies on Convertible Securities," Journal of Finance, Vol. 32, pp.463-478.
- (1981), "Some Results in the Theory of Arbitrage Pricing," Working Paper, School of Management, Yale University.
- Keynes, John Maynard, (1936), The General Theory of Employment, Interest, and Money, (New York: McGraw Hill).
- Malkiel, Burton M., and Richard Quandt, (1969), Strategies and Rational Decisions in Securities Options Market, (Cambridge, Mass.: MIT Press, 1969).
- Masulis, R. W., (1980), "The Effects of Capital Structure Change on Security Prices: A Study of Exchange Offers," Journal of Financial Economics, Vol. 8, pp. 107-126.
- Merton, Robert, (1973a), "Theory of Rational Option Pricing," Bell Journal of Economics and Management Science, Vol. 11, pp. 141-183.
- (1973b), "An Intertemporal Capital Asset Pricing Model," Econometrica, Vol. 41, No. 5, pp.867-887.

- (1976a), "Option Pricing When Underlying Stock Returns are Discontinuous," Journal of Financial Economics, Vol. 3, pp. 125-144.
- (1976b), "The Impact on Option Pricing of Specification Error in the Underlying Stock Price Returns," Journal of Finance, Vol. 31, No. 2, pp.333-350.
- (1977), "On the Pricing of Contingent Claims and the Modigliani-Miller Theorem," Journal of Financial Economics, Vol. 15, No. 2, pp.241-250.
- (1987), "A Simplified Model of Capital Market Equilibrium with Incomplete Information," Journal of Finance, Vol. 42, No. 3, pp.483-501.
- Myron Scholes, and M. Gladstein, (1978), "A Simulation of Returns and Risk of Alternative Option Portfolio Investment Strategies," Journal of Business, April 1978, pp.183-242.
- Miller, Merton H., (1977), "Debt and Taxes," Journal of Finance, Vol. 32, pp. 261-276.
- Modigliani, Franco, (1982), "Debt, Dividend Policy, Taxes, Inflation, and Market Valuation," Journal of Finance, Vol. 37, pp. 255-274.
- Modigliani, Franco and Merton Miller, (1958), "The Cost of Capital, Corporation Finance, and the Theory of Investment," American Economic Review, Vol. 48, pp.261-297.
- (1959), "Reply," American Economic Review, Vol. 49, pp. 655-668.
- (1961), "Dividend Policy, Growth and the Valuation of Shares," Journal of Business, Vol. 34, pp. 411-433.
- (1963), "Corporate Income Taxes and the Cost of Capital: A Correction," American Economic Review, Vol. 53, pp. 433-443.
- (1965), "Reply," American Economic Review, Vol. 55, pp. 524-526.
- Roll, Richard and Stephen A. Ross, (1980), "An Empirical Investigation of the Arbitrage Pricing Theory," Journal of Finance, Vol. 35, No. 5, pp.1073-1103.
- Rose, J., (1959), "Comment," American Economic Review, Vol. 49, pp. 638-639.
- Ross, Stephen A., (1976), "The Arbitrage Theory of Capital Asset Pricing," Journal of Economic Theory, Vol. 13, pp. 341-360.

(1977), "Return, Risk, and Arbitrage," in Risk and Return in Finance, edited by I. Friend and J. Bicksler, (Cambridge, MA: Ballinger).

Shanken, Jay, (1982), "The Arbitrage Pricing Theory: Is It Testable?," Journal of Finance, Vol. 37, pp.1129-1140.

Sharpe, William F., (1964), "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk," Journal of Finance, Vol. 19, pp.425-442.

(1982), "Factor Models, the APT and CAPMS," Working Paper, Stanford University.

Stambaugh, Robert F., (1983), "Arbitrage Pricing with Information," Journal of Financial Economics, Vol. 12, pp.357-369.

Stiglitz, Joseph, (1969), "A Reexamination of the Modigliani-Miller Theorem," American Economic Review, Vol. 59, pp. 784-793.

(1974), "On the Irrelevance of Corporate Financial Policy," American Economic Review, Vol. 64, pp. 851-866.

Smith, C. W., (1976), "Applications of Option Pricing Analysis," in Handbook of Financial Economics, ed. J. L. Bicksler, (Amsterdam: North Holland Publishing Co., 1979), pp.80-121.

Trzcinka, Charles, (1986), "On the number of Factors in the Arbitrage Pricing Model," Journal of Finance, Vol. 46, No. 2, June 1986, pp.347-368.

CHAPTER 4

THEORIES OF INFORMATION AND PRICES: FULLY RATIONAL SPECULATIVE BUBBLES

In Chapter 2 we showed the impact of intrinsic uncertainty on the standard consensus equilibrium and thereby reversed a major conclusion of the common knowledge branch of the economics of information literature. In Chapter 3 we applied this intrinsic uncertainty concept to four paradigms in the theory of finance. We found common knowledge beliefs are required for the validity of three of the paradigms. We now turn attention to another major branch of the economics of information literature: the information inference through prices literature.

Whereas in Chapter 3 we illustrated speculation based on intrinsic uncertainty, in Chapter 4 we explicitly assume the common knowledge structure of game theory but nevertheless derive speculation based on feedback effects. Chapter 4 presents a game of incomplete information, where the players' probability beliefs, possible payoffs from alternative decisions, and game tree structure are all common knowledge.

Speculation requires (1) intertemporal investment and (2) choice under uncertainty. In Chapter 3 intrinsic uncertainty over beliefs in the state space formed the basis for speculation. In the present chapter, the uncertainty leading to speculation centers on the type of signal informed traders have received.

Chapter 4 presents a game played between two homogeneous groups

acting as a single player: one group is informed, one group is uninformed. The informed group moves first and may sometimes try to fool the uninformed traders into believing that asset returns are better than they actually are. Sometimes this deception succeeds. A bubble results, and after the crash is over and the dust has cleared, the informed group turns out to have gained at the expense of the uninformed.

The significance of our model is that it raises fundamental questions about the robustness of the traditional "information inference through prices" theory for securities markets. This chapter derives speculative behavior based on the feedback effect generated from purchases by informed traders. In the process of analyzing these feedback effects, we have arrived at an original theory for the formation of price bubbles. This chapter presents the first game theoretic analysis of speculative bubbles and also the first derivation of bubbles from fully rational, payoff-maximizing behavior on the part of all market participants.

A major branch of the economics of information theory suggests market prices convey information in models with uncertainty. If a seller is offering a lower than expected price for some good, the potential buyer should beware that the good may have something wrong with it. The key result here is that prices are doing more than just clearing markets; prices serve a dual purpose in conveying information.

The leading approach to the study of information inference through prices follows from the work of Kihlstrom and Mirman (1974)

and Grossman and Stiglitz (1976). This approach, which has been widely heralded as the Rational Expectations equilibrium for the economics of information, contends that when informed traders receive private information and buy some asset or good, their purchases will convey information to uninformed traders through the prevailing market prices. Uninformed individuals can observe market prices and frequently infer the content of the informed individuals private signals. For example, if the stock of some resource is fixed, uninformed individuals might infer that a higher price is associated with high demand by the informed individuals based on favorable information. The higher the price the more favorable the private information is perceived.

When further randomness is introduced to these models, e.g., shifts in the consumer demand or in the stock of some asset, then the price still conveys some information to the uninformed. However, prices have now become noisy signals. A price may be high because informed traders have received a favorable signal (which leads them to buy the asset or good), because the supply of the asset is low, or because consumer demand has shifted upwards. Nevertheless, even in models with additional randomness, prices will on average be high when informed individuals have received favorable signals and low when they have received unfavorable signals.

The central feature of the Kihlstrom-Mirman and Grossman-Stiglitz analyses is that prices convey at least some information, and uninformed traders can glean information from observing prevailing prices. Indeed, where information is costly, Grossman and Stiglitz argue that it may be more cost-efficient for some individuals to obtain

information from others, e.g., through observing market prices, than obtain it directly.

However, there exists an important caveat in the Kihlstrom-Mirman and Grossman-Stiglitz framework. This caveat focuses on the treatment of informed traders and their lack of understanding of the feedback effect generated by uninformed individuals buying some asset because they have inferred informed traders are buying it. The object of this paper is to draw attention to some of the more fundamental, yet unpublished implications of the Kihlstrom-Mirman and Grossman-Stiglitz approach, and then use these new findings to address several issues on the efficient market hypothesis and the economics of information.

We will illustrate a Bayesian Nash equilibrium for a game with incomplete information and show that the concept of "uninformed traders buying when they infer that informed traders are buying" can lead to price deviations from fundamentals. Alternatively stated, such purchases by uninformed individuals can lead to speculative bubbles.

The Grossman-Stiglitz analysis implicitly assumes uninformed individuals are smart enough to deduce that high prices are a (noisy) signal that informed traders received a favorable signal, e.g., about the real return to some asset. Yet Grossman and Stiglitz assume the informed individuals are not smart enough to realize their purchases will trigger subsequent purchases by the uninformed group. Consequently, the informed individuals fail to perceive that their purchases will induce a feedback effect among uninformed individuals.

If informed individuals decide to take advantage of this feedback, then they may purchase an asset just to exploit capital gains profits off the uninformed individuals. The informed individuals can take advantage of the feedback even when they have received an unfavorable signal about some asset's return.

Just as Kihlstrom and Mirman (1974) and Grossman and Stiglitz (1976) introduced a new approach to the study of the economics of information, so do we now propose adopting a new, richer framework for considering the dichotomy between informed and uninformed individuals. This paper formulates the Grossman-Stiglitz model within a Bayesian game of incomplete information, and proposes a new rational, utility-maximizing approach to price deviations from market fundamentals. Although our discussion focuses on capital markets, the analysis developed here can be applied to any competitive market with a dichotomy of informed and uninformed individuals.

1. PRINCIPALS AND EXTENSIVE FORM GAME

Consider a market containing two assets, one safe and one risky. Let the return, r , to the risky asset depend on a random variable, η , which can be observed at some cost, and another random variable, ϵ , that cannot be observed:

$$r = \eta + \epsilon, \quad (1)$$

where η and ϵ are independent, normally distributed random variables. Suppose the informed individuals learn the actual value of η . Some exogenous risk remains from ϵ , but the informed traders have purchased private information that reduces the risk on the asset's return.

The per capita demand for the risky asset by those individuals who are informed of η , X_i , will vary proportionately with η and inversely with the price, P , of the risky asset: $\delta X_i / \delta \eta > 0$ and $\delta X_i / \delta P < 0$. At this point, Grossman and Stiglitz (1976) have written the per capita demand by the informed individuals as a function solely of η and P . However, there is another important component of informed individuals' demand for the risky asset: capital gains potential.

In our Bayesian game framework, uninformed individuals will buy with some probability, β , conditioned on observing a high market price. Informed individuals will attempt to realize a capital gain off the feedback effect, with probability α , by buying the risky asset even when they have learned the value of η is low.

Demand must equal supply each period in equilibrium:

$$\lambda X_i(P, \eta, \beta) + (1-\lambda) X_u(P, \alpha) = X^S, \quad (2)$$

where λ is the fraction of the individuals who are informed, X_u is the per capita demand of the uninformed, and X^S is the per capita supply of the asset. Uninformed individuals observe only price, but they also know with some probability, α , that the informed individuals try to manipulate the game. From observing the price, uninformed individuals can no longer infer η , because now a higher P can be associated either with a higher η or a manipulation with probability α of the informed individuals who have really learned the value of η will be low.

When the informed individuals recognize the feedback effect and act in their own best interests, there will no longer be a one-to-one mapping of precisely one η for any P . The homeomorphism between P and

η , upon which Grossman and Stiglitz rely, does not exist with the addition of capital gains-motivated purchases by the informed individuals. Hence, the conditional distribution of r given P will differ from the conditional distribution of r given η .

Before proceeding to derive the Bayesian Nash equilibrium of this game with incomplete information, we can intuitively conclude that if β were sufficiently large, it might always pay for the informed individuals to try to manipulate the uninformed individuals. If β were sufficiently small, so that uninformed individuals almost never bought the risky asset solely from observing its price was high, then there would be little opportunity for the uninformed traders to be manipulated. Consequently, if β were known to be small, then α would be small.

We will focus on the interior solution where both $0 < \alpha < 1$ and $0 < \beta < 1$. Depending on how the exogenous parameters in the model are specified, we will demonstrate an equilibrium where it pays in terms of expected profit for the uninformed to "buy when they observe a high price."

One surprising fact that emerges from this analysis is that even though all the actors are fully rational and maximize expected utility, their Bayesian Nash Equilibrium behavior induces occasional deviations of price above the market fundamental. This speculative bubble results from the case where informed individuals buy the risky asset when they know η is low, uninformed individuals mimic their purchases, and the price of the risky asset is driven above its fundamental (conditioned on η).

The first move in the extensive form game is assigned to chance and given by the private signal the informed individuals receive on the actual value of η . To simplify the construction of the game tree, we will consider two possible values for η : high or low.

The second move of the tree is assigned to the informed individuals, who know the actual value of η . The informed individual must select a purchasing strategy for the risky asset. If they receive a signal that η will be high, we assume the informed traders will have high demand for the risky asset. However, it does not follow that they will necessarily have low demand for the asset when η is known to be low; rather, the informed traders are now given a choice of whether to try to manipulate the market or not. Their expected profit from trying to manipulate the market will depend on the purchasing proclivity of uninformed individuals, who follow the Grossman and Stiglitz model and demand the risky asset when they observe a high price.

The third move is by the uninformed individuals, who observe the level of prices bid up by the informed traders in the previous move but not what happened by chance in the first move. The uninformed traders draw Bayesian inferences about the real (correct) versus purely speculative (incorrect) prospects for the asset's future valuation - and hence their own expected payoffs - from observing the informed traders behavior.

Following these moves the game ends and payoffs are realized according to the strategies adopted by the parties. The real return of the asset will determine the payoffs to the players when η is high.

When η is low the payoffs will depend on the capital gain/loss realized by the informed investor in trying to resell his risky asset shares. We assume the informed individuals can resell their holdings on the market, if they so desire, as the uninformed individuals bid up the price of the risky asset. The uninformed segment can see market clearing prices, but they cannot observe the price dynamics outside equilibrium. The extensive form game tree is shown in Figure 1.

Note that the uninformed individuals can always infer η is low when informed demand is low. When the uninformed individuals observe a high price, they must decide whether η is high or low based on Bayesian inference. In this game, the informed have no basis to manipulate the uninformed when η is high. Consequently, the informed have no incentive but to select high demand when η is known to be high.

2. PAYOFFS FROM ALTERNATIVE STRATEGIES

For the reader's convenience, all notation is defined in the glossary at the end of the chapter. Informed traders who receive a high signal (η_H) and thus have high demand for the risky asset (X_{iH}) will receive their profits from holding the risky asset and realizing its real return. In this case the profits to the informed traders, π_{i1} , are independent of whether the uninformed traders have high demand for the risky asset (X_{uH}) or low demand (X_{uL}). π_{i1} is the highest payoff the informed traders can receive, and this payoff results from collecting high returns from the asset. Since collecting the real returns when the informed traders receive signal η_H is a

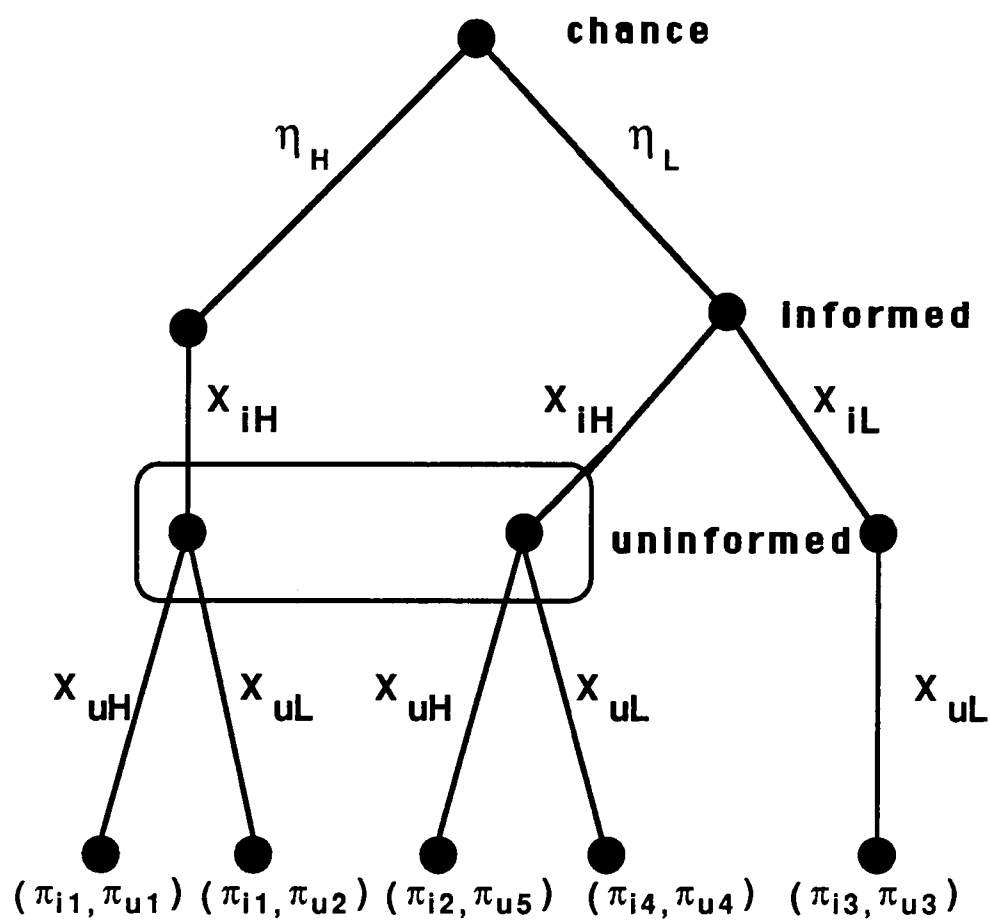


Figure 1. Bayes Game of Information Inference Through Prices.

dominant strategy, the tree only shows the informed traders offering X_{iH} when the signal is η_H .

The second highest payoff for the informed traders is when they receive a low signal (η_L), exhibit high demand for the risky asset (X_{iH}), and succeed in fooling the uninformed to purchase the risky asset with high demand (X_{uH}). In this case the informed trades will receive profits of π_{i2} . The rationale for this payoff follows from the ability of informed traders to sell off (a portion of) their risky asset holdings while the uninformed traders are driving up the price with their high demand.

Obviously, for the informed traders to realize a capital gain the aggregate demand for the risky assets - comprising both informed and uninformed traders' demand - must move outward. A shift in aggregate demand outward at the same time the informed traders are selling off their risky asset holdings implies that the informed group constitute a smaller fraction of the market than the uninformed. Either the informed must be outnumbered by the uninformed or they must constitute a smaller percentage of the purchasing power in the market.

The third highest payoff for the informed (π_{i3}) in our game comes when the informed traders receive a low signal and have low demand. Then the risky asset has a relatively lower return, but the informed have only demand a low quantity of it as well.

The informed traders receive the lowest payoff (π_{i4}) in the game when they receive a low signal (η_L) and exhibit high demand for the risky asset (X_{iH}) but fail to induce the uninformed traders to buy: the informed traders have low demand (X_{uL}) anyways. In this worst

case scenario for the informed, they get stuck holding the risky asset in large quantities when it only has a low return.

The uninformed traders will receive their two highest payoffs when the return on the risky asset is high (η_H). Their highest payoff (π_{U1}) will be when they have had high demand (X_{UH}), and their second highest payoff (π_{U2}) will come when they failed to bid up demand for the asset with low demand (X_{UL}).

Turning now to the η_L branch of the tree, the uninformed receive the next best payoff (π_{U3}) when they correctly infer η_L from X_{iL} and consequently do not bid up demand for the risky asset. The fourth highest payoff (π_{U4}) comes when they decide to buy low (X_{UL}) and the informed traders have attempted to fool them with high demand X_{iH} . Although the uninformed have adopted the same strategy of X_{UL} , $\pi_{U3} > \pi_{U4}$ because the informed traders' high demand has driven up the price that the uninformed must pay for the low quantity of the risky asset they demand (X_{UL}).

The worst case and fifth highest payoff (π_{U5}) for the uninformed comes when they are successfully fooled by the informed into buying high (X_{UH}) even though the informed traders received signal η_L . Here the informed traders exploit capital gains profits off the uninformed, whose purchases are induced as a feedback effect having inferred from prices that informed traders have high demand.

The following assumptions restate the relative magnitudes of the payoffs.

Assumption 1: $\pi_{ij} > 0$ and $\pi_{uk} > 0$ for all $j \in \{1,2,3,4\}$, $k \in \{1,2,3,4,5\}$.

Assumption 2: $\pi_{i1} > \pi_{i2} > \pi_{i3} > \pi_{i4}$.

Assumption 3: $\pi_{u1} > \pi_{u2} > \pi_{u3} > \pi_{u4} > \pi_{u5}$.

Assumption 1 states that in any event, both informed and uninformed traders earn positive profits and thereby explains why uninformed traders are willing to play the game. The key part of Assumption 2 for our game is that the payoff to the informed traders when they have received a low signal is greater if they can successfully fool the uninformed than if they honestly reveal η_L with their low demand purchases: π_{i2} must be greater than π_{i3} . If this one payoff relationship did not hold, the informed traders would have no incentive to try to fool the uninformed.

The key part of Assumption 3 for our Bayes game is that the uninformed receive a higher payoff when they buy high and the returns are high than would be their payoff from buying low and the returns are low: π_{u1} must be greater than π_{u3} . If this payoff relationship did not strictly hold, the uninformed traders would always be better off adopting a strategy of low demand. The fact that they can occasionally get higher payoffs from bidding high demand and collecting real returns induces them to play this speculative feedback game.

Definition 1: The payoff to the informed traders when they receive a low signal, η_L , is

$$V^i(\alpha; \beta) = \alpha [\beta \pi_{i2} + (1 - \beta) \pi_{i4}] + (1 - \alpha) \pi_{i3}. \quad (3)$$

The payoff to uninformed traders who infer from prices that informed traders have high demand is

$$V^U(\beta; \alpha, X_{iH}) = \beta [\Pr(\eta_L | X_{iH}) \pi_{u5} + \Pr(\eta_H | X_{iH}) \pi_{u1}] \\ + (1 - \beta) [\Pr(\eta_L | X_{iH}) \pi_{u4} + \Pr(\eta_H | X_{iH}) \pi_{u2}]. \quad (4)$$

Definition 1 defines the expected value to the informed traders who receive low signals based on the probability, α , the informed traders try to manipulate the uninformed. When the informed do manipulate, their payoff will depend on the probability, β , that the uninformed subsequently purchase. Similarly, the payoff function for the uninformed individuals gives the expected value to them of deciding to purchase with probability β , upon witnessing the informed traders purchase. The expected payoff to the uninformed traders will depend on the perceived probability of high and low signals conditioned on witnessing the informed traders' high demand.

Notice that $V^I(\alpha; \beta)$ is continuous in its choice argument α over the interval $[0,1]$ and likewise $V^U(\beta; \alpha, X_{iH})$ is continuous in its choice argument β over the interval $[0,1]$. We can therefore invoke a fixed point theorem to prove the following preliminary lemma on the existence of an equilibrium.

Lemma 1: There exist unique $\alpha' \in [0,1]$ and $\beta' \in [0,1]$ which maximize $V^1(\alpha; \beta)$ and $V^2(\beta; \alpha, X_{iH})$, respectively.

Proof. By the Brouwer Fixed Point Theorem and the continuity of $V^1(\alpha; \beta)$ and $V^2(\beta; \alpha, X_{iH})$, there exists unique $\alpha' \in [0,1]$ which maximizes $V^1(\alpha; \beta)$ for any given β and a unique $\beta' \in [0,1]$ which maximizes $V^2(\beta; \alpha, X_{iH})$ for any given α and X_{iH} . Furthermore, α' is defined by

$$\alpha' = 0 \quad \text{and} \quad \beta \pi_{i2} + (1 - \beta) \pi_{i4} - \pi_{i3} < 0, \quad (5)$$

$$\text{or } \alpha' = 1 \text{ and } \beta \pi_{i2} + (1 - \beta) \pi_{i4} - \pi_{i3} > 0, \quad (6)$$

$$\text{or } \alpha' \in [0,1] \text{ and } \beta \pi_{i2} + (1 - \beta) \pi_{i4} - \pi_{i3} = 0. \quad (7)$$

The optimal β' is defined by

$$\beta' = 0 \text{ and } \Pr(\eta_L|X_{ih})(\pi_{u5} - \pi_{u4}) + \Pr(\eta_H|X_{ih})(\pi_{u1} - \pi_{u2}) < 0, \quad (8)$$

$$\beta' = 1 \text{ and } \Pr(\eta_L|X_{ih})(\pi_{u5} - \pi_{u4}) + \Pr(\eta_H|X_{ih})(\pi_{u1} - \pi_{u2}) > 0, \quad (9)$$

or $\beta' \in [0,1]$ and

$$\Pr(\eta_L|X_{ih})(\pi_{u5} - \pi_{u4}) + \Pr(\eta_H|X_{ih})(\pi_{u1} - \pi_{u2}) = 0. \quad (10)$$

Conditions (5) through (7) state that informed traders who receive low signals will always bid low demand for the risky asset ($\alpha' = 0$), always try to manipulate the uninformed traders with a high demand bluff ($\alpha' = 1$), or be indifferent between these strategies depending on whether the profit from bidding low, π_{i3} , exceeds the expected profit from attempting to manipulate the uninformed, $\beta \pi_{i2} + (1 - \beta) \pi_{i4}$.

Conditions (8) through (10) state that uninformed traders witnessing high demand by the informed traders will always follow suit and demand high ($\beta' = 1$), always bid low demand ($\beta' = 0$), or be indifferent between these strategies depending on the magnitude of the difference between their profits with high versus low demand and the conditional probabilities of a high or low signal given informed traders have high demand.

3. BAYESIAN NASH EQUILIBRIUM IN PURE STRATEGIES

The problem set out in Section 1 (that of determining an optimal investment strategy in light of the feedback effect generated by

informed traders' purchases), with payoffs defined in Section 2, can be modeled and solved in a game theoretic framework. The solution for games with incomplete information is the Bayesian Nash Equilibrium introduced by Harsanyi (1967).

Definition 2: The strategy space for trading group j is $S_j = [0,1]$. A pure strategy for the informed (i) traders is a scalar $\alpha \in S_i$, and for the uninformed (u) traders is a scalar $\beta \in S_u$.

Definition 3: The set of best responses for the informed traders to β is $\phi_i(\beta) = \{\alpha \in S_i \mid V^i(\alpha; \beta) \geq V^i(\alpha'; \beta) \text{ for all } \alpha' \in S_i\}$. The set of best responses for the uninformed traders to α is $\phi_u(\alpha) = \{\beta \in S_u \mid V^u(\beta; \alpha, X_{iH}) \geq V^u(\beta'; \alpha, X_{iH}) \text{ for all } \beta' \in S_u\}$.

The functions $\phi_i: S_u \rightarrow S_i$ and $\phi_u: S_i \rightarrow S_u$ are the best response correspondence for the informed and uninformed traders, respectively.

Assumption 5: The uninformed traders use Bayes' Rule to compute $\Pr(\eta_L | X_{iH})$ and $\Pr(\eta_H | X_{iH})$.

Definition 4: A strategy pair (α^*, β^*) is a Nash Equilibrium for the game $G = (V^i, V^u, S_i, S_u)$ if

- (a) $\alpha^* \in S_i$ and $\beta^* \in S_u$;
- (b) $V^i(\alpha^*; \beta^*) \geq V^i(\alpha; \beta^*)$ for all $\alpha \in S_i$; and
- (c) $V^u(\beta^*; \alpha^*, X_{iH}) \geq V^u(\beta; \alpha^*, X_{iH})$ for all $\beta \in S_u$.

Alternatively, the pair (α^*, β^*) is a Nash Equilibrium if $\alpha^* \in \phi_i(\beta^*)$ and $\beta^* \in \phi_u(\alpha^*)$: each strategy is a best response to the other.

Let p_L denote the prior $\Pr(\eta_L)$.

THEOREM 1: Under Assumption 5 there exist a Bayesian Nash Equilibrium in pure strategies given by

$$(\alpha^*, \beta^*) = \left(\left(\frac{\pi_{u2} - \pi_{u1}}{\pi_{u5} - \pi_{u4}} \right) \left(\frac{1 - \rho_L}{\rho_L} \right), \frac{\pi_{i3} - \pi_{i4}}{\pi_{i2} - \pi_{i4}} \right).$$

Proof: From condition (7) we know that for an interior solution for α it must be the case that

$$\beta \pi_{i2} + (1 - \beta) \pi_{i4} - \pi_{i3} = 0. \quad (11)$$

Define β^* as that probability of a high demand by the uninformed traders that leaves the informed traders who receive a low signal indifferent between attempting to manipulate the uninformed traders or not. Then solving (11) for β^* yields

$$\beta^* = \frac{\pi_{i3} - \pi_{i4}}{\pi_{i2} - \pi_{i4}}. \quad (12)$$

To obtain an interior solution for the probability β , it is necessary but not sufficient from condition (10) that $\beta \in [0,1]$ and

$$\Pr(n_L | X_{iH})(\pi_{u5} - \pi_{u4}) + \Pr(n_H | X_{iH})(\pi_{u1} - \pi_{u2}) = 0. \quad (13)$$

Given $\rho_L + \rho_H = 1$ and Assumption 5, the uninformed traders update their prior beliefs conditioned on inferring that X_{iH} using Bayes' Rule:

$$\Pr(n_L | X_{iH}) = \frac{\alpha \rho_L}{\alpha \rho_L + 1 \rho_H} = \frac{\alpha \rho_L}{\alpha \rho_L + (1 - \rho_L)}. \quad (14)$$

The "1" in the denominator of the second term in (14) comes from the fact that informed traders who receive a high signal have high demand

for the risky asset with probability 1. Substituting (14) into (13) coupled with the fact that $\Pr(\eta_L | X_{iH}) + \Pr(\eta_H | X_{iH}) = 1$, yields

$$\alpha p_L (\pi_{u5} - \pi_{u4} - \pi_{u1} + \pi_{u2}) - (\pi_{u2} - \pi_{u1})[\alpha p_L + (1 - p_L)] = 0. \quad (15)$$

Solving (15) for that value of α that leaves uninformed traders facing a high demand by the informed traders indifferent between following suit or not, we obtain

$$\alpha^* = \frac{\pi_{u2} - \pi_{u1}}{\pi_{u5} - \pi_{u4}} \frac{1 - p_L}{p_L}. \quad (16)$$

By definition of the Kuhn-Tucker conditions (7) and (10), $V^i(\alpha^*; \beta^*) = V^i(\alpha; \beta^*)$ for all $\alpha \in [0, 1]$, and $V^u(\beta^*; \alpha^*, X_{iH}) = V^u(\beta; \alpha^*, X_{iH})$ for all $\beta \in [0, 1]$. Therefore, $V^i(\alpha^*; \beta^*) \geq V^i(\alpha; \beta^*)$ and $V^u(\beta^*; \alpha^*, X_{iH}) \geq V^u(\beta; \alpha^*, X_{iH})$ so that (α^*, β^*) is a Nash Equilibrium where α^* is computed using Bayesian inference. $\parallel$

Equation (12) is an informed trader's reaction function when he has received a low signal, η_L . If $\beta > \beta^*$, an informed traders with a low signal would always try to manipulate the uninformed traders. If the uninformed traders followed suit with probability $\beta < \beta^*$, the informed traders would never expect to gain from manipulating the uninformed; consequently, in these circumstances $\alpha = 0$.

Note that under Assumption 2, $0 < \beta^* < 1$. The reaction function for the uninformed traders given that the informed traders have high demand for the risky asset is equation (16). The right-hand side of (16) ≥ 0 , depending on the value of p_L ; however, it may also exceed

unity. In this case $\alpha \leq 1 < \alpha^*$, and optimality condition (9) would apply to the uninformed traders' decision problems so that $\beta = 1$.

Nash equilibria are best responses by one player taking the strategies of all other players as given, and Harsanyi's (1967) original equilibrium concept relied on players' optimal reactions. Figure 2 illustrates the informed and uninformed traders reaction functions for $\alpha^* < 1$, with the Bayesian Nash Equilibrium given by the intersection of the reaction functions.

In Figure 2 when $\alpha < \alpha^*$, the uninformed traders will always want to buy ($\beta = 1$) the risky asset after observing the informed traders buy: the probability, α , of the informed traders attempting to manipulate the uninformed is sufficiently small. When $\alpha = \alpha^*$, all of the uninformed traders' strategies, or mixtures of them, result in the same expected payoff. Thus the optimal uninformed traders' reaction function at $\alpha = \alpha^*$ is the entire interval $\beta \in [0,1]$. Finally, when $\alpha > \alpha^*$ the uninformed traders never want to purchase ($\beta = 0$) because the probability that they are being manipulated is too high. The informed traders' reaction function has the inverse interpretation varying with the probability, β , that the uninformed will subsequently purchase.

We must emphasize the significance of the equilibrium obtained in Theorem 1 and shown in Figure 2. First, we have shown that attempting to infer information from prices, i.e., a high price associated with favorable news, can lead to strategic manipulation of those attempting the inference.

Second, and more importantly, we have just presented the first

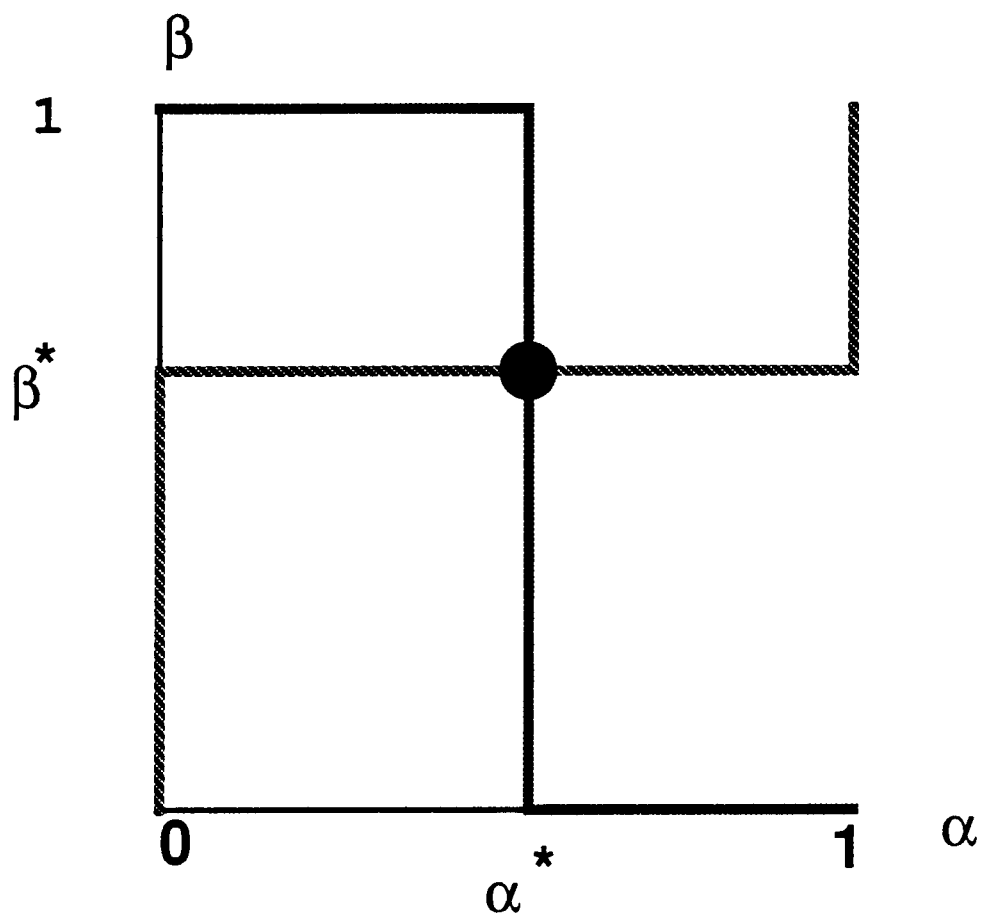


Figure 2. Bayesian Nash Equilibrium when
 $\alpha^* < 0$ ($z < \rho_L < 1$)

fully rational, payoff-maximizing derivation of a speculative bubble! Theorem 1 and Figure 2 show that with positive probability informed traders will purchase when they have a low signal and the uninformed traders will subsequently purchase with positive probability as well. Their combined behavior drives the risky asset's price above its fundamental value known to the informed (and to an objective observer).

Since the right-hand side of (16) depends on the exogenously specified prior probability for a low signal, the following set of propositions describe the Bayesian Nash equilibria of the game under differing p_L assignments. First observe that for some value of p_L the equilibrium value of $\alpha^* = 1$. Let $p_L = z$ denote this value that sets $\alpha^* = 1$, then

$$z = \frac{\pi_{u2} - \pi_{u1}}{\pi_{u5} - \pi_{u4} + \pi_{u2} - \pi_{u1}} . \quad (17)$$

Proposition 1: If $p_L = 1$, the Bayesian Nash equilibrium of the game is $(\alpha^*, \beta^*) = (0, 0)$.

Proof: If $p_L = 1$, then from (16) $\alpha^* = 0$ and

$$\Pr(\eta_L | X_{iH}) = \frac{\alpha(1)}{\alpha(1) + 1(0)} = 1 .$$

Accordingly, $\Pr(\eta_L | X_{iH})(\pi_{u5} - \pi_{u4}) + \Pr(\eta_H | X_{iH})(\pi_{u1} - \pi_{u2}) = \pi_{u5} - \pi_{u4} < 0$; therefore $\beta = 0$ for all $\alpha \in [0, 1]$, which implies $\beta \pi_{i2} + (1 - \beta) \pi_{i4} - \pi_{i3} < 0$ and $\alpha = 0$. Therefore $0 \in \phi_i(0)$ and $0 \in \phi_u(0)$ so that each strategy is a best response to the other.

Proposition 2: If $z < \rho_L < 1$, then the Bayesian Nash equilibrium of the game is (α^*, β^*) defined in Theorem 1.

Proof: The condition $z < \rho_L < 1$ is necessary and sufficient to ensure $0 < \alpha^* < 1$ from (16). It has already been established that $0 < \beta^* < 1$ by definition (12) and Assumption (2). $V^i(\alpha^*; \beta^*) \geq V^i(\alpha; \beta^*)$ and $V^u(\beta^*; \alpha^*, X_{iH}) \geq V^u(\beta; \alpha^*, X_{iH})$ so that (α^*, β^*) is a Bayesian Nash Equilibrium.

Proposition 3: If $\rho_L = z$, the Bayesian Nash equilibria are given by $\alpha = 1$ and $\beta \in [\beta^*, 1]$.

Proof: By definition when $\rho_L = z$, $\alpha^* = 1$. This value for α^* implies $V^u(1; \alpha < 1, X_{iH}) \geq V^u(\beta; \alpha < 1, X_{iH})$ and $V^u(\beta; 1, X_{iH}) = V^u(\beta'; 1, X_{iH})$ for all $\beta, \beta' \in S_u$, from the uninformed traders' perspective. For the informed traders taking β as given and for all $\beta' \geq \beta^*$, $V^i(1; \beta') \geq V^i(\alpha; \beta')$ for all $\alpha \in S_i$, and in particular $V^i(1; \beta^*) = V^i(\alpha; \beta^*)$ for $\alpha \neq 1$. For all $\beta < \beta^*$,

$$\beta \pi_{i2} + (1 - \beta) \pi_{i4} - \pi_{i3} < 0 ,$$

and $\alpha = 0$. Therefore, the Bayesian Nash equilibria will be given by $\alpha = 1$ together with any value of β in the interval $[\beta^*, 1]$.

Proposition 4: If $0 \leq \rho_L < z$, then the Bayesian Nash equilibrium of the game is $(\alpha, \beta) = (1, 1)$.

Proof: The condition $0 \leq \rho_L < z$ implies $\alpha^* > 1$ from (16). This fact implies $V^u(1; \alpha', X_{iH}) \geq V^u(\beta; \alpha', X_{iH})$ for all $\beta, \alpha' \in [0, 1]$. From the informed traders perspective taking $\beta = 1$ as given, $V^i(1; 1) \geq V^i(\alpha; 1)$

for all $\alpha \in S_i$. Therefore, the strategy pair $(\alpha, \beta) = (1, 1)$ each is a best response to the other strategy; consequently, $(1, 1)$ is a Bayesian Nash equilibrium.

Proposition 1 states that when the (prior) probability of a low signal is true with probability 1, then the uninformed traders know with certainty that any high demand by the informed traders is an attempt to take advantage of the feedback effect. Consequently, the uninformed traders would never have high demand upon witnessing high demand by the informed: $\beta = 0$. The informed traders realize $\beta = 0$, which implies they could never hope to exploit the uninformed traders through the feedback. Therefore, in equilibrium $\alpha = 0$. The traders' reaction correspondences and $(0, 0)$ equilibrium are shown in Figure 3.

The informed traders' reaction function in Figure 3 has remained the same shape as in Figure 2. The informed traders are still indifferent for values of $\beta = \beta^*$, will manipulate with probability $\alpha = 1$ for $\beta > \beta^*$, and will manipulate with probability $\alpha = 0$ for the remaining values of $\beta < \beta^*$. However, the uninformed will never buy after they have seen the informed traders buy, i.e., $\beta = 0$ for all values of α , because they are certain the private signal is low. Given these optimal reactions, the equilibrium is at the origin.

Proposition 2 merely states the condition on the exogenous prior that guarantees $\alpha^* < 1$ previously shown in Figure 2. Proposition 3 shows that when $\alpha^* = 1$, the uninformed traders will be indifferent between any of their strategies at the value of $\alpha = 1$. For any $\alpha < 1$, $\beta = 1$ will be the dominant strategy. Thus the uninformed traders'

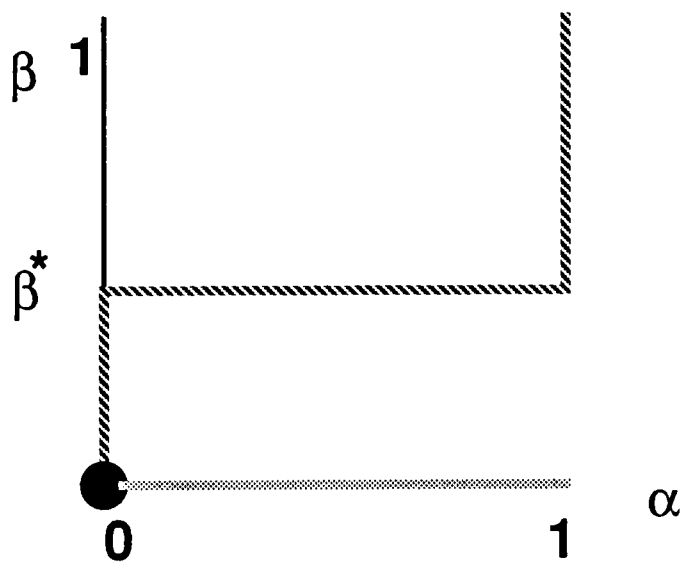


Figure 3. $\rho_L = 1$.

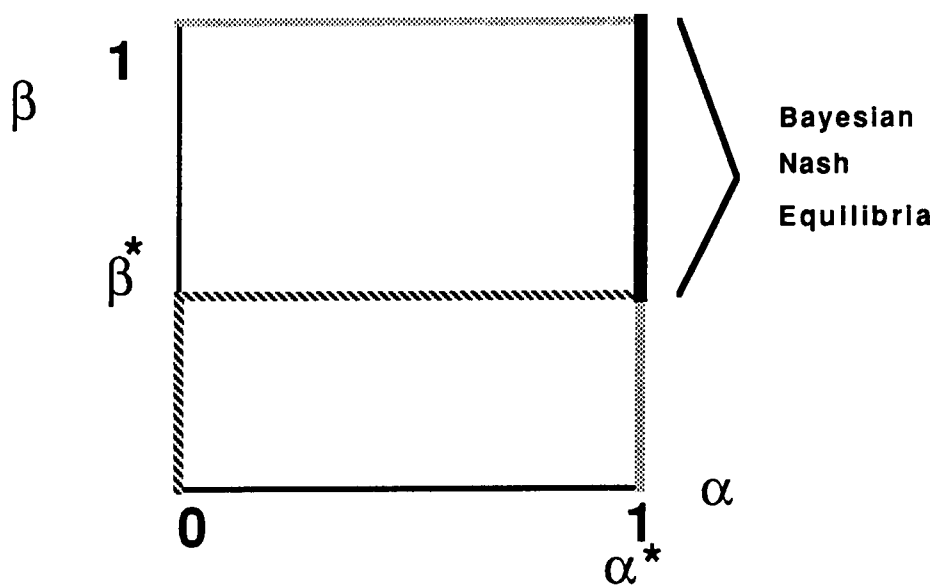


Figure 4. $\rho_L = z$.

optimal reaction function comprises the upper border and right border of the unit square. The informed traders reaction function remains the same shape as in the conditions for Propositions 1 and 2. As noted earlier, the informed trades find $\alpha = 1$ to be a best reaction for values of β on the interval $[\beta^*, 1]$. The multiple equilibria are shown in Figure 4 by a darkened line connecting endpoints $(1, 1)$ and $(1, \beta^*)$.

Proposition 4 states that when the probability of a low signal is so low ($p_L < z$), the uninformed traders believe X_{iH} indicates the informed traders received a high signal with sufficiently high probability to induce them to always bid high demand as well: $\beta = 1$. The informed traders, taking $\beta = 1$ as given, will always profit from exploiting the feedback effect when they have low signals since they realize the uninformed traders will mimic their purchases with probability 1. Consequently, the optimal strategies will be the pair $(\alpha, \beta) = (1, 1)$.

Except when $p_L = 1$, in equilibrium the informational asymmetry and feedback effect permits the informed to exploit speculative bubble profits off uninformed traders.

Propositions 1 through 4 consider comparative statics for changes in the prior probability of a low signal. The payoffs to uninformed traders' affect the informed traders' strategy via inferences about the most profitable reaction, and vice versa for the uninformed traders' strategies. These comparative statics results for the other exogenous parameters affecting equilibrium strategies - the payoffs - are summarized in the following corollary to Theorem 1.

Corollary 1.

$$(a) \partial \alpha^* / \partial (\pi_{u2} - \pi_{u1}) > 0,$$

$$(b) \partial \alpha^* / \partial (\pi_{u5} - \pi_{u4}) < 0,$$

$$(c) \partial \beta^* / \partial (\pi_{i3} - \pi_{i4}) > 0,$$

$$(d) \partial \beta^* / \partial (\pi_{i2} - \pi_{i4}) < 0.$$

Corollary 1 each trading group's decision to purchase is actually affected by the profits to the other group and thus their willingness to play the game. For example, part (a) states that as the profit margin widens for uninformed traders who mimic informed traders' purchases when the informed traders have received high signals versus those who demand low, more uninformed traders will be induced to have high demand conditioned on informed traders having high demand. This higher β^* in turn induces by reaction a higher α^* .

Since the information set surrounding the informed traders' high demand nodes occur with positive probability, the Nash equilibrium of this Bayes game will also be Trembling-Hand Perfect. [See Selten (1975) and Simon (1987)]. Moreover, because the information sets connect all the tree branches, this Bayes game has no subgame short of the entire tree itself. Therefore, the concept of a Subgame Perfect Equilibrium has no relevance for the present model. [See Hellwig and Leininger (1987)].

4. THE WELFARE ASPECTS OF PURELY SPECULATIVE PURCHASES

Economists have traditionally been interested in the welfare aspects of speculative phenomena. From a general equilibrium theory

perspective, speculative bubbles have no welfare effects. The gains and losses from bubble profits represent redistributions within an economy, but not a net loss to the economy.

Using partial equilibrium analysis, we can identify at least three sources for potential losses in society's welfare. First, the speculation may replace productive activity. Such was the case with the Seventeenth Century tulip bulb mania, in which some Dutch farmers quit plowing their fields and merchants ceased their enterprises all to take part in the speculative bidding up of tulip bulbs. At the height of the bubble some bulbs were selling for the price of entire farms.

Second, if the speculative activity interferes with the accumulation of productive capital, society's welfare can suffer. In this case bidding up the speculative asset's price diverts funds away from necessary capital formation to sustain business production. It is possible to construct a partial equilibrium scenario in which the resulting underinvestment in productive capital will lead to real losses in some society's output. From a general equilibrium perspective, the flow of capital into the bubble asset's sector should offset the partial equilibrium welfare effect.

A third source of potential welfare losses, in a partial equilibrium setting, would arise from external drains on the economy. If foreign investors finance the initial stages of the bubble and are able to realize capital gains before the bubble collapses, then the foreign investors receive gains at the expense of the domestic investors. One nation or region's welfare loss is measured by the

value of the gains to the foreign investors. From a general equilibrium perspective, we must look to the "global" economy and view the bubble profits as just transfers within the economy. The reader interested in the welfare aspects of historical bubbles and "swindles" will find excellent reading in Kindleberger's (1978) book.

Our Bayes game model, like the Grossman and Stiglitz model on which it was inspired, represents a general equilibrium analysis with both the demand and supply side represented in the informed and uninformed traders. Thus we are able to show demand equals supply in equation (2). Consequently, welfare remains unchanged in our analysis. All gains to the informed traders from bubble purchases are strictly redistributions from the uninformed traders, and these transfers represent no real loss to the economy modeled in Section 3.

Partial equilibrium welfare effects could be added to the model by proposing an alternative sector competes for funds with the risky asset sector in which the price bubble now grows.

Suppose we reformulate the model in Section 3 in such a way that the attempted manipulation by the informed traders cannot occur, e.g., if all traders are equally informed. By assuming economic growth and real profits from holding assets, the model would depict an infusion of income into the economy with the gains distributed among individuals according to real dividend payoffs from their asset holdings. Without the speculative capital gains motive, assets would be held strictly for their fundamental value. Under Assumption 1, the economy experiences growth in all scenarios. In both this fully informed scenario and the scenario model in Section 3 with informed and uninformed

traders, all profits from asset holdings represent redistributions between individuals rather than net gains or losses to society.

The existence of the bubble presents real opportunities for the informed traders to sell off some of their risky asset holdings to capture the capital gains. The redistribution comes from traders joining the bubble in its latter stages and is paid out to traders purchasing at the initial stages of the bubble. But again, in any general equilibrium analysis, speculative bubbles will have no welfare effects. Blanchard and Weston (1982) have examined partial equilibrium welfare costs.

5. FINANCIAL ECONOMICS SIGNALING

In our model informed traders' purchases constitute a mixed signal about the information they have received. Like other signaling models in the financial economics literature, the extent to which private information becomes reflected in the prevailing market prices will depend on the extent of strategic play. The "cat and mouse style" game we illustrate between informed and uninformed traders can also evolve between other economic players.

Myers and Majluf (1984) present a signaling equilibrium for a firm trying to decide whether to raise cash for investment by issuing new shares. The informed group (managers) know more about the firm's value and investment opportunities than the uninformed group (potential investors). In some cases, managers may decline to invest because the cost to existing shareholders of issuing new shares at a bargain price may outweigh the investment's net present value. The

(uninformed) new investors can take advantage of the (informed) managers' signal in the Myers and Majluf model analogous to the way uninformed traders take advantage of the informed traders' purchasing signal in the Grossman and Stiglitz model.

Investors must decide upon an equilibrium share price given the issue-invest signal. In the Myers and Majluf model, the decision to issue new stock always reduces the stock price. Given an environment in which rational investors know both the firm's decision rule as well as the actual decision to issue new shares, can strategic play lead to a speculative bubble analogous to the model in Section 3?

Managers who know a firm's shares are overvalued could induce a greater deviation from the fundamental by publicly deciding to issue no new shares. In some circumstances, investors would incorrectly interpret this signal to mean the true value of slack and assets in place exceeds the price of old shares with a new issue. Thus our Bayesian Nash equilibrium model of bubble creation can be applied to the firm managers' stock issue decision presented in the Myers and Majluf model. The equilibrium will depend on the objective probability that the firm has limited cash available in much the same way that the equilibrium of the model in Section 3 depends on the objective probability of a low or high signal.

John and Kalay (1985) have examined the signal emerging from a firm's decision to cut dividend payments. Some investors may interpret the reduction in dividends as a signal of lower future expected returns from the firm's investments; the lower returns would tend to reduce the price of shares in the firm. Alternatively, the

decreased dividend could signal higher expected return based on increased expected investment, which would tend to raise the asset's price.

When the firm has outstanding risky debt, shareholders have incentives to underinvest, which leads bondholders to reduce the value of the firm's bonds. John and Kalay then illustrate an equilibrium in which the informed group (shareholders) precommit the firm to some minimum investment level through constraints on dividend payouts. The uninformed group (bondholders) use the dividend signal as a means to infer the private information known to shareholders: the firm's investment opportunities.

Firms with poor investment opportunities who try to fool the market with a large investment precommitment must pay the penalty of having to overinvest. The costs of overinvestment thus serve to deter too many false dividend signals.

The John and Kalay model is similar in one respect to our own Bayesian Nash equilibrium work: using the dividend/investment precommitment signal, a firm with poor investment opportunities can induce a speculative bubble on its bond prices by trying to manipulate the uninformed group's feedback effect.

However, their work can be distinguished from our own in at least two respects. First, the informed group in our model actually contribute to the speculative bidding up of prices, whereas in the John and Kalay model the informed group send out a signal that induces the uninformed group to buy bonds (at a price deviating from the fundamental). Second, the informed group in our model profit at the

expense of the uninformed group, whereas in the John and Kalay model shareholders who falsely try to manipulate the market are penalized by having to overinvest. This certain penalty may well eliminate any incentive to try to manipulate the market, unless the firm's bond prices rose so dramatically as to fully compensate the shareholders for the firm's overinvestment.

Finally, the unexpected crash of the New York Stock Exchange and other world stock markets circa October 19, 1987, represents an irresistible topic for a bubble theorist. The model presented in this paper depicts how purchases by uninformed investors, based not so much on knowledge of the underlying fundamental as on a signal of seeing others buy, can lead to speculative bubbles. We believe an element of this purchasing mentality exists in the real stock market, and our model explains why fully rational "uninformed" investors might on average receive gains from purchasing this way.

Reports that the Dow Jones industrial average was rising above the market's fundamental value went unheaded. The historical movement of the Dow Jones industrial average from 1984 - 1987, characterized by periodic reports of new record highs, led many in the market to project the historical pattern into the future. Recall from our model that when the probability of a low signal, p_L , becomes sufficiently low, then both informed and uninformed investors will always buy the risky asset. The uninformed investors are willing to accept a few capital gains to the benefit of the informed group, because the probability of a high signal will cover their losses over time.

Informed investors can judge better than uninformed investors the

fundamental value to the market of these external factors: the national deficit, the American trade deficit, the value of the American dollar, and fears of rising interest rates. Yet just as in our model, if β is sufficiently high the informed investors will buy irrespective of the signal they received (and with hindsight should have heeded). The professional mutual fund institutional investors who suffered substantial losses in 1987 are like informed investors in our model who believe if they keep purchasing the risky asset, they can induce the uninformed traders to keep the flow of money going. When the cash flow from the uninformed is reduced, the ensuing bubble collapses.

6. CONCLUSIONS AND ISSUES FOR FUTURE RESEARCH

In conclusion, the Bayesian Nash equilibrium model of speculative behavior takes into account the feedback effect generated by uninformed traders purchasing when they have inferred informed traders bought. In the process of analyzing these feedback effects, we have introduced the first game theoretic analysis of speculative bubbles and the first derivation of speculative bubbles from fully rational, payoff-maximizing behavior. Further study of feedback effects, particularly in speculative markets, should provide rich new models that can explain factors ranging from stock market overshooting and volatility to dividend signals sent by firm managers.

The model we have presented here could be generalized to allow the traders a continuum of strategies and to allow for a continuum of possible signals received by the informed. While this generalization

may add a more realistic portrayal of information transmission in financial markets, we believe the speculative bubbles can be more easily seen in a simpler high-low signal and strategy space. We have opted for the simpler model to introduce our new Bayes game model of information inference through prices so as to highlight the surprising result: creation of a fully rational bubble.

Another issue for future research would be to look at the incentives individuals in each of the two groups have to depart from their group's behavior. We have solved the mathematics in this game employing two players with the groups acting as a homogeneous body. Nevertheless, an informed individual may have an incentive to sell information directly to the uninformed when his own group is attempting to fool the uninformed. Peering into this issue, we would find that no sooner do individuals have incentives to depart from group behavior, than the groups themselves have incentives to form binding commitments or institutions that would prevent individual members from breaking the group's strategy.

Individual departures from group behavior is also an unaddressed issue in the Kihlstrom-Mirman and Grossman-Stiglitz paradigm as well. Informed individuals in those economies could essentially go on the black market and sell news directly to the uninformed. The stability of the Grossman-Stiglitz equilibrium depends on achieving the optimal number of informed traders so that the marginal informed trader is just compensated for the costs of the information. If others convey this same information to the uninformed on the black market at a lower cost, then the entire information gathering mechanism will unravel or

continually be in a state of disequilibrium.

Furthermore, with many players of each type, the resulting game theory equilibrium will no longer be a Nash equilibrium but rather a correlated equilibrium taking into account intrinsic uncertainty and higher-order beliefs identified in Chapters 2 and 3. In fact, the modified game will no doubt contain multiple correlated equilibria and no unique equilibrium. Whether the correlated equilibria would be a form of separating equilibria or pooling equilibria is unclear. The separating equilibria would be in the Grossman-Stiglitz tradition and the approach adopted in our formulation as well. The pooling equilibria would eliminate the possibilities for speculative bubbles. In the correlated equilibria, the probabilities α and β of our Bayes game might be understood as a certain frequency of the population who take part in the game.

This Chapter has derived speculation based on uncertainty over the private signal received by informed traders. The game tree, prior probabilities for a high or low signal, and payoffs at the bottom of the tree are all common knowledge, which is a requirement of game theory. If these features were not common knowledge, the players would not know what game they were playing. Analyzing uninformed traders' purchases as a feedback effect, we have derived a speculative bubble and shown scenarios under varying probabilities of a high signal. Except for the case where the probability of a high signal is zero, we have shown Nash equilibrium behavior leading to the formation of bubbles with positive probability in each case.

7. GLOSSARY

p_L - (objective) prior probability of a low signal, η_L .

p_H - (objective) prior probability of a high signal, η_H .

X_{iH} - high demand for the risky asset by the informed traders.

X_{uH} - high demand for the risky asset by the uninformed traders.

X_{uL} - low demand for the risky asset by the uninformed traders.

α - the probability that the informed traders will buy when they have received a low signal, $\Pr(X_{iH}|\eta_L)$.

α^* - the value of α that leaves the uninformed traders indifferent between their strategies of purchasing the risky asset upon witnessing the informed traders have purchased (and have high demand).

β - the probability that the uninformed traders will purchase when they have inferred the informed traders buy the risky asset, $\Pr(X_{uH}|X_{iH})$.

β^* - the value of β that leaves the informed individuals indifferent between their strategies (of various probabilities α) of trying to manipulate the uninformed traders.

z - the value of p_L which sets $\alpha^* = 1$.

8. REFERENCES

- Altman, Edward I., and Marti G. Subrahmanyam, Editors, (1985), Recent Advances in Corporate Finance, (Homewood, Illinois: Richard D. Irwin Publishers, 1985).
- Blanchard, Olivier J., and Mark W. Watson, (1982), "Bubbles, Rational Expectations, and Financial Markets," in Crises in the Economic and Financial Structure, ed. by P. Wachtel, (Lexington, Mass.: Lexington Books, 1982).
- Grossman, Sanford and Joseph Stiglitz, (1976), "Information and Competitive Price Systems," American Economic Review, Vol. 66, pp.246-254.
- Harsanyi, John C., (1967), "Games with Incomplete Information Played by Bayesian Players, Parts I-III," Management Science, Vol. 14, pp.159-189, pp.320-334, pp.486-502.
- Hellwig, Martin, and Wolfgang Leininger, (1987), "On the Existence of Subgame-Perfect Equilibrium in Infinite-Action Games of Perfect Information," Journal of Economic Theory, Vol. 43, pp.55-75.
- John, Kose and Avner Kalay, (1985), "Informational Content of Optimal Debt Contracts," in Altman and Subrahmanyam, Editors, Recent Advances in Corporate Finance, (Homewood, Illinois: Richard D. Irwin Publishers, 1985).
- Kihlstrom, Richard E., and Leonard J. Mirman, (1974), "Information and Market Equilibrium," Bell Journal of Economics, Vol. 5, pp.357-376.
- Kindleberger, Charles P., (1978), Manias, Panics, and Crashes, (New York: Basic Books, 1978).
- Myers, Stewart, and N. Majluf, (1984), "Corporate Financing and Investment Decisions When Firms Have Information Investors Do Not Have," Journal of Financial Economics, Vol. 13, pp.187-221.
- Selten, Robert, (1975), "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games," International Journal of Game Theory, Vol. 4, pp.25-55.
- Simon, Leo K., (1987), "Local Perfection," Journal of Economic Theory, Vol. 43, pp.134-156.
- Tirole, Jean, (1982), "On the Possibility of Speculation under Rational Expectations," Econometrica, Vol. 50, pp.1163-1181.
- (1985), "Asset Bubbles in Overlapping Generations," Econometrica, Vol. 53, No. 5, pp.1071-1100.

CHAPTER 5

SUMMARY OF THE DISSERTATION'S CONTRIBUTIONS

This dissertation has advanced several original theories in the economics of information and finance areas. In Chapter 2, we present a methodological essay defining higher-order beliefs and common knowledge beliefs. A major contribution of that chapter is to show that the standard consensus equilibrium result, in both the statical literature on consensus and the common knowledge literature, actually requires a missing "agreed common knowledge priors" (ACKP) assumption.

Without common knowledge beliefs, individuals typically face uncertainty over the unknown beliefs of others in a model. These individuals are said to face "intrinsic uncertainty." If theorists assume individuals have "common priors" over some state space, they have assumed nothing more than that the individuals (coincidentally) agree. In Chapter 2 we take up the question of what happens if individuals agree without knowing they agree. We demonstrate that such individuals, upon hearing someone call out posterior probability beliefs different from his own, cannot distinguish whether that person simply started out with different priors or actually has different information.

To arrive at a consensus equilibrium in which individuals can attribute differences in probability assignments solely to differences in information, we have shown that the restrictive ACKP assumption must be imposed on the model. With ACKP, individuals not only agree but know they agree. Thus each individual knows that everyone else

knows that everyone else assigns the same probability assignments to the state space. Under an ACKP regime, individuals can call out their posterior probability beliefs and they will iterate to agreement in the form of equal posteriors.

A second major contribution of Chapter 2 is to show theorists how they can define and thereby work with "common knowledge beliefs" and "higher-order beliefs" in economic models. Outside a common knowledge beliefs environment, the beliefs of others represent an unknown random variable to each individual. Therefore, these beliefs about beliefs can have a genuine probability distribution assigned by each individual. It follows by induction that K -order beliefs can be defined as probability distributions over the (unknown) $(K-1)$ -order probability distributions employed by others.

In Chapter 3, the intrinsic uncertainty identified in Chapter 2 is examined within the major paradigms in the theory of finance. For models with three or more periods, intrinsic uncertainty in these models can lead to speculation over how the market values assets. In the case of the Modigliani-Miller paradigm, the main propositions of that literature are still valid in a two period economy in which capital gains-motivated speculation cannot occur by definition. Individuals in a two-period model of the MM paradigm may still face intrinsic uncertainty, but the state of the world is revealed before they can speculate.

For a multi-period economy, we show investors may have incentives to follow investment strategies that directly contradict the logic of the MM analysis. This result is truly startling, because the

overwhelming majority of the finance profession now accepts that the MM propositions are valid under the restrictive conditions of the MM paradigm identified extant. We show in Chapter 3 that an ACKP assumption is required for the validity of the MM paradigm in a multi-period economy.

Chapter 3 also explains why the validity of the CAPM does not hinge on common knowledge beliefs assumptions. Both the two-period and multi-period versions of the CAPM assume knowledge about the capital gains potential of assets. The only way investors could calculate future capital gains is if they know something about the rest of the market's beliefs and how these beliefs might change with information events. Thus the CAPM indirectly has common knowledge beliefs. Nevertheless, having assumed knowledge about future prices, the general validity of the CAPM will still hold in a homogeneous beliefs environment: CAPM theorists do not need to impose common knowledge beliefs about the state space as in the MM paradigm.

The significance of this contribution on the CAPM is that it throws caution to the common knowledge literature's claims that economic and financial theorists need to substitute "common knowledge beliefs" for "homogeneous beliefs" across most of economic theory. In contrast, we would argue that these various models need to be reviewed on an individual basis rather than assuming automatically that theorists really meant to imply common knowledge beliefs with their stated "homogeneous beliefs" assumptions.

The APT and OPT paradigms are shown to depend on common knowledge beliefs to guarantee a globally unique solution. Without common

knowledge, investors can buy and sell securities or options based on uncertainty over the market's valuation. For the APT, an individual may personally feel some factor does not affect asset returns. With intrinsic uncertainty present under the APT's homogeneous beliefs assumption, the individual may nevertheless take this same factor into account in his portfolio decisions out of concern the market may employ the factor.

For the OPT, we find investors deriving a subjective estimate for the volatility of an underlying security and then questioning what estimate is used by others in the market value of the option. If the market intends to sharply increase or decrease demand for the asset, an investor attempting to either purchase or sell the asset finds himself at risk over the changing market valuations. Under these conditions, options can become a useful hedge against unfavorable changes in (uncertain) market valuations.

The fourth chapter presents the first game theoretic analysis of speculative bubble behavior as well as the first derivation of speculative bubbles from fully rational, payoff-maximizing behavior. This model was formulated in response to the standard theory on "information inference through prices." This literature examines how uninformed traders can glean information simply by observing prices after informed traders have transacted. A high price is associated with favorable news and a low (or lower) price is associated with unfavorable news.

Yet these information inference models contain an air of unreality when applied to financial markets. In these environments we

show that informed individuals should be smart enough to realize how they might profit from the feedback effect generated by their purchases. If the uninformed traders buy on average after witnessing the informed traders have bought, then the informed traders may seek to induce subsequent purchases even when they have received a low signal. The informed traders are induced into this behavior by the uninformed traders' purchasing proclivities and the potential to realize capital gains.

VITA

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