To the Graduate Council:

I am submitting herewith a dissertation written by Manuel Jackson Sanders III entitled "Existence of Certain Compact Contractible Manifolds Containing Disjoint Spines." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree Doctor of Philosophy, with a major in Mathematics.

Robert J. Daverman, Major Professor

We have read this dissertation
and recommend its acceptance:

Accepted for the Council:

Associate Vice Chancellor and
Dean of The Graduate School

EXISTENCE OF CERTAIN COMPACT CONTRACTIBLE MANIFOLDS CONTAINING DISJOINT SPINES

A Dissertation
Presented for the
Doctor of Philosophy
Degree
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Manuel Jackson Sanders III

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For Helen and Gretchen

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ABSTRACT

A polyhedron K in the interior of a compact PL manifold M (with boundary) is said to be a (PL) spine of M provided M collapses to K. The manifold M has disjoint spines provided it collapses (independently) to two disjoint such polyhedra in its interior. A long-standing conjecture asserts that a certain class of compact, contractible 4-manifolds constructed by Mazur do not have disjoint spines. More recently, the question as to which compact, contractible manifolds have a pair of disjoint spines has been expanded by C. Guilbault so as to include other compact, contractible manifolds, including those in higher dimensions.

A technique of M.H.A. Newman provides compact, contractible manifolds which are not balls. A Newman manifold is constructed as the closure of the complement of a regular neighborhood of a non-simply connected, finite, acyclic, simplicial complex, K, in S^n for sufficiently large n. When well-defined, it is denoted $New(K, n)$. Guilbault has shown that if K is a non-simply connected, finite, acyclic, simplicial k-complex, then, $New(K, n)$ has disjoint spines provided $n > 4k$. His techniques, thereby, provide interesting examples of disjoint spine phenomena in dimensions $n \geq 9$ and are the only known examples of such occurrences.

In this work, it is shown that the result holds in dimension $n = 4k$ as well. Moreover, if K is a non-simply connected, finite, acyclic 2-complex, it is shown that

$New(K, n)$ has disjoint spines if $n \geq 5$, thereby producing relatively low-dimensional examples of the phenomenon. Since it is known that no interesting examples can occur in dimensions $n \leq 3$, these results show that the question regarding existence of interesting examples is, now, undecided only in dimension 4.

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CHAPTER I

PRELIMINARIES

I call our world Flatland, not because we call it so, but to make its nature clearer to you, my happy readers, who are privileged to live in Space.

And so begins an enchanting story by Edwin Abbott. First published in 1884, the story titled *Flatland: A Romance of Many Dimensions* [Abb] has captured the imaginations of generations of readers, young and old alike. Its modern popularity can be seen from its availability in electronic form [www]. *Flatland* is the story a 2-dimensional planar world. The story is centered on a character named A Square who is imprisoned in Flatland for attempting to teach his fellow Flatlanders about the third dimension. Through its satirical commentary on Victorian society as well as its spiritual implications, Abbott is able to make us appreciate the profound differences that exist between phenomena occurring in two dimensions and those which occur in dimension three. The character A Square escapes his Flatland prison by rising out of the plane to break his bonds.

A Square calls our world Spaceland. Our physical beings seem confined to this 3-dimensional world in which we believe we live. Our minds, however, are free to roam; they allow us to consider higher-dimensional spaces. We owe A Square (and E. Abbott) some credit for this freedom. Just as Abbott freed A Square from his planar bonds, so he empowers us to envision phenomena that are out of our physical reach. In much the same way as A Square does, we encounter ideas that seem strange and, perhaps, improbable to us. Nevertheless, modern physics would be very different if not for the mathematical notion of infinite-dimensional Hilbert space. Quantum mechanics absorbs high-dimensional mathematical notions rather nicely. It is well known that Spaceland is a relatively tame place compared with

the higher dimensional spaces. Mathematically speaking, many interesting things which occur in high dimensions are not allowed in Spaceland.

While we shall confine ourselves to considerations involving finite-dimensional spaces in this work, we will encounter phenomena that seem strange and even improbable at some levels. To put a mathematical perspective on *Flatland*, one might say that a mathematical moral of the story is: *While a simple closed curve separates the plane, it does not separate* 3*-space.* This idea, though simple to us, is quite formidable to the inhabitants of Flatland. Likewise, it is important for us to reach beyond our natural intuition so that we may understand occurrences which seem remote to us.

In the pages ahead, we shall study phenomena which do not exist in any 3-dimensional *world.* In fact, the unsolved 3-dimensional Poincaré Conjecture provides doubt as to the existence of any 3-dimensional space in which the phenomena we study might or might not exist. Regardless of this existence issue, the phenomena we study do not arise in dimension 3 settings. If this makes these notions seem strange or desolate, we remind ourselves of A Square's struggle to comprehend the self-description of a visiting sphere whose cross-sectional image in Flatland is a circle. According to A Square [www, pp. 34-8],

> *Every reader in Spaceland will easily understand that my mysterious Guest was speaking the language of truth and even of simplicity. But to me, proficient though I was in Flatland Mathematics, it was by no means a simple matter...There stood before me, visibly incorporate, all that I had before inferred, conjectured, dreamed, of perfect Circular beauty.*

Understanding 3-dimensional space is a mystical experience for A Square; it is a journey into a higher reality. For us, our excursion into higher dimensional phenomena is simply a matter of trying to understand mathematical truth. The beauty of

such truth is all that we seek. It is our single desire.

Like so many other modern results in topology, dimension plays a key role in our investigations. We will be devoted to examining questions regarding contractible PL (= piecewise linear) manifolds. More specifically, the work of Guilbault [Gui] shows that certain contractible manifolds have the feature that they *collapse* independently to two disjoint subsets. His results seem improbable for reasons that we will outline in some detail later. Moreover, there is a dimensional requirement introduced by his techniques; Guilbault is able to produce manifolds in dimensions nine and above that collapse to distinct interior subsets.

While it is true that dimensional considerations have played key roles in many recent topological results, in these results, one typically expects common techniques to produce output in dimensions five or six. That is, the emergence of new phenomena in dimension nine is sufficiently surprising so as to warrant suspicion. The expectation behind our work is that there *should* be examples of these types of contractible manifolds in lower dimensions as well. Furthermore, if they do indeed exist, we expect to be able to provide a constructive method for producing them (much as is done in [Gui] for the higher dimensions). Ultimately, we shall improve upon Guilbault's techniques so as to provide these more expected examples.

We shall begin by recalling some of the necessary and useful elements regarding the nature of the objects under investigation.

1.1 Introduction

There are several notions that are used throughout the work. It seems proper then, that these matters should be addressed at the outset. As mentioned above, we will be working with PL manifolds. Most of these manifolds will have boundaries.

Therefore, it will be convenient to use the term *manifold* so as to include the possibility of a boundary rather than specifying the distinction of a ∂-manifold as is sometimes done. Also, any manifold encountered should be assumed to be PL unless otherwise specified. We shall, at times, use the word *complex* when referring to a topological space. In these instances, we are referring either to a simplicial complex or to a PL CW complex. We shall make the distinction only if it seems necessary or appropriate. The symbol $\approx$ will be used from this point forward to denote a PL homeomorphism. In the way of other conventions, the word *map* will (of course) always mean a continuous function as is usual in topological matters. Moreover, our maps will always be PL unless otherwise noted. Similarly, the word *embedding* in use with a polyhedron will always mean a PL embedding, homeomorphism means PL homeomorphism. In some instances, we may emphasize characteristics by saying that a manifold is PL or we may say that e is a PL embedding of a polyhedron or we may use a term like PL manifold. This distinction is for emphasis only and the absence of such distinction should not be misconstrued; the conventions above will always apply. Also, S^n will denote the n-sphere and $I = [0,1]$.

We will assume the fundamentals of PL topology, [R-S] being the standard reference. Therefore, the notion of a collapse of a polyhedron Y to a polyhedron X is inherent from this assumption. This notion has already been encountered in the preceding commentary. When X and Y are polyhedra, we shall use the common notation $Y \searrow X$ to express symbolically that Y collapses to X (or equivalently, $X \nearrow Y$ to denote that X expands to Y). The notions of collapsing and expanding are special cases of more general mathematical ideas: Those of simple-homotopy theory. We shall utilize tools from this beautiful area of mathematics because they will be powerful, efficient, and convenient (as well as required!). [Coh] is a standard source regarding simple-homotopy theory and provides a wealth of information on

the subject. The proofs of almost all facts regarding simple-homotopy equivalences which will be important to our work can be found (or at least suggested) there.

We shall, at times when it appears conducive to providing a crisper argument, supply proofs of some standard arguments. However, the arrangement is, admittedly, sporadic. We will begin with some background information though.

1.2 Definitions and Relevant Data

Definition 1 *Let M be a compact manifold. Let K be a subpolyhedron in the interior of M. Then, if M collapses to K, we say that K is a PL spine of M (or simply a spine of M).*

The simplest example occurs when M is a ball, i.e., $M \approx I^n$. Then, since a manifold is a ball if and only if it collapses to a point of its interior, balls have point spines.

It should be noted that regular neighborhood theory asserts that if K is a PL spine of M, then $M - K \approx \partial M \times [0,1)$. This property is the defining element in a generalization of the notion of PL spine that we present next. The following objects have previously played key roles in what is known about the manifold phenomena we study. Likewise, they will continue to play a role in some of the new techniques developed here.

Definition 2 *Let M be a manifold. Let X be a compactum in the interior of M. Then, X is a PL pseudospine of M if $M - X \approx \partial M \times [0,1)$.*

For emphasis, note that the homeomorphism in the previous definition is required to be PL. We should also point out that it is occasionally convenient to use an alternate equivalent definition for a PL spine of a manifold which involves the following notion.

Definition 3 *If X and Y are spaces and $f : X \to Y$ is a topological map, the* mapping cylinder *of f, denoted M_f, is defined to be the space obtained from $X \times I$ together with the disjoint union of Y with identifications given by $x \times \{1\} \sim f(x)$ for each $x \in X$. Briefly,*

$$M_f = (X \times I \oplus Y)/\sim (f(x) = x \times \{1\}).$$

The equivalence of the following characterization of PL spine with that previously given follows again from regular neighborhood theory.

Theorem 1.1 *Let M be a compact manifold. Let K be a subpolyhedron in the interior of M. Then, K is said to be a PL spine of M provided that M is PL homeomorphic to a mapping cylinder M_α of some map $\alpha : \partial M \to K'$ where K' is homeomorphic to K and the homeomorphism sends ∂M and K' to the natural copies of each in the mapping cylinder.*

Remark: In general, given a topological manifold X, a spine of X is a compactum C in $intX$ so that X is topologically homeomorphic to a mapping cylinder X_f where $f : \partial X \to C'$ is a topological map with C' (topolologically) homeomorphic to C so that the homeomorphism sends ∂X and C' to the natural copies of each in M_f. Furthermore, more generally, a pseudospine of X is a compactum C in $intX$ so that $X - C$ is topologically homeomorphic to $\partial X \times [0, 1)$. Because our work will be done solely in the PL setting, we adopt the following convention:

Convention: *The word spine will mean PL spine and the word pseudospine will mean PL pseudospine throughout the remainder of the work.*

Homology groups encountered will use integer coefficients. In dealing with homology, the following definitions are useful:

Definition 4 *A homology n-cell, X, is a compact, n-dimensional, topological manifold with the homology of a point. That is, $\widetilde{H}_i(X) \cong 0$ for each i.*

Equivalently said, a homology n-cell is a compact, acyclic topological n-manifold.

Definition 5 *A homology n-sphere, Y, is a compact, n-dimensional, topological manifold with the homology of S^n. That is, $\widetilde{H}_i(Y) \cong Z$ if $i = n$ and $\widetilde{H}_i(Y) \cong 0$ otherwise.*

Standard algebraic arguments show that if X is a homology cell, then ∂X is a homology sphere. Moreover, it follows from the same type of arguments that gluing two homology cells together along a common boundary results in a homology sphere. If the factors are PL manifolds, then, of course, the result is as well.

Now, for clarity, we'll attempt to describe the question fully which is at the core of our work. A formal definition will leave no room for doubt as to the issue.

Definition 6 *Let M be a manifold. Let K_1 and K_2 be polyhedra in the interior of M with $K_1 \cap K_2 = \emptyset$. Suppose that K_1 and K_2 are both spines of M. Then, we say that M has disjoint spines.*

Question 1 *Do there exist interesting contractible manifolds with disjoint spines?*

The answer to this question is yes. We shall describe what is known about the subject momentarily. The work presented in this document provides new examples of the phenomenon. The word "interesting" above simply means contractible manifolds which are not balls.

To begin, we will recall that the *trivial range* of an n-dimensional manifold is comprised of all dimensions k so that $2k + 2 \leq n$. We shall make use of the fact that homotopic embeddings of complexes in the trivial range are ambiently isotopic [B-K]. In order to provide a precise characterization of Guilbault's result, we continue by describing a particularly nice class of interesting, compact, contractible manifolds.

1.3 Newman Contractible Manifolds

A technique of M.H.A. Newman produces compact contractible manifolds which are not balls. The technique is as follows: Take a finite, acyclic, simplicial k-complex, K, with nontrivial fundamental group and embed it in S^n for a sufficiently large n. Let N denote a regular neighborhood of K. Then, N is a PL homology n-cell. Moreover, it follows from duality that $M = cl(S^n - N)$ is also a PL homology cell. Furthermore, general position shows that M is simply connected (provided $k + 2 < n$) and therefore, contractible. However, M is not a ball when $k + 2 < n$ as $\partial M = \partial N \hookrightarrow N$ is a π_1-isomorphism and yet $\pi_1(N) \neq \{1\}$. Hence, M is a contractible manifold with non-simply connected boundary. (In fact, by the High Dimensional Poincaré Theorems, any interesting, compact, contractible topological manifold of dimension at least 5, must have non-simply connected boundary.)

Contractible manifolds constructed in the above fashion are called Newman Contractible Manifolds. When the manifold constructed in this manner is unique, it will be denoted $New(K, n)$. Well-definedness is typically not an issue. Routine arguments quickly show that in the trivial range, $New(K, n)$ is well-defined. While this comment suffices for now, the following statement derived from an argument of [A-G] (applied in the PL setting) will prove to be valuable to us later. The essence of the proof involves applying the h-cobordism theorem (boundary version) to a

cobordism between contractible manifolds with identical boundaries.

Theorem 1.2 *Every compact, contractible (PL) n-manifold ($n \geq 5$) is uniquely determined up to homeomorphism by its boundary.*

It is also true that a contractible 4-manifold is uniquely determined by its boundary up to topological homeomorphism. However, this classification will not be important for our work.

1.4 Handlebody Techniques for Later Use

The following observation will play an important role for us. We will freely use this idea without further comment in what follows. Suppose that $New(K, n)$ is well defined where K is a simplicial k-complex. Given an embedding $e : K \to S^n$, we can form a preferred regular neighborhood of $e(K) \subset S^n$ as a handle decomposition with 1 handle for each simplex of K by thickening up the simplices of $e(K)$. However, we can eliminate (combine) all the 0-handles by choosing a maximal tree in K and performing the usual handle-theoretic operations to combine all the 0-handles of the regular neighborhood together with the 1-handles corresponding to this maximal tree into a single 0-handle.

The PL CW-complex obtained by collapsing all the handles back onto their cores consists of a single 0-cell along with some higher dimensional cells (cells having dimension no more than k though). Call this PL CW complex J. The point of this idea is that we can regard $New(K, n)$ as the closure of the complement of a regular neighborhood of an embedding of J into S^n. Hence, any Newman Contractible Manifold $New(K, n)$ can be regarded as obtained from a PL CW-complex with a single 0-cell using the techniques described above. Our construction efforts in Chapter III will make use of this phenomenon.

1.5 Known Results

To date, the only known examples of interesting contractible manifolds with disjoint spines occur within the class of Newman Contractible Manifolds. We provide the following statement of a theorem of Guilbault.

Theorem 1.3 [Gui, Theorem 4.1] *Let K be a k-dimensional finite acyclic complex. For any $n > 4k$, the compact contractible n-manifold $M = New(K, n)$ contains a disjoint pair of PL spines.*

Of course 1-dimensional acyclic complexes are contractible (in fact, collapsible). Therefore, they are of no value in the production of interesting contractible manifolds using Newman's method. However, non-simply connected acyclic 2-complexes exist in abundance. The above theorem, therefore, provides examples of interesting contractible manifolds with disjoint spines in dimensions $n \geq 9$.

A main focus in examining manifolds with disjoint spines in this work is to try to determine if the dimensional requirement invoked by the proof of the preceding theorem is best possible. That is ...

Question 2 *Are there interesting contractible n-dimensional manifolds with disjoint spines with $n < 9$?*

Attempting to find an answer to this question is at the core of what follows. As alluded to in the introductory remarks, it is known that in dimensions $n \leq 3$, nothing interesting can happen (regardless of the 3-dimensional Poincaré Conjecture). The argument used to see this revolves around decidedly 3-dimensional characteristics. See [Gui, Closing Remarks] for some 3-dimensional considerations.

The conclusion of Theorem 1.3 seems improbable for various reasons. Consider the following proposition whose proof follows readily from general position arguments.

Proposition 1.1 *Let M be an n-manifold. Let K be a k-dimensional spine of M. Then, if $i : \partial M \to M$ denotes inclusion, $i_j : \pi_j(\partial M) \to \pi_j(M)$ is an isomorphism when $1 + j + k < n$ and an epimorphism when $j + k < n$.*

From this, observe that if M is a contractible n-manifold with non-simply connected boundary, then any spine of M must have dimension at least $n - 2$. Said more efficiently, any spine of M must have codimension at most 2.

Now, if a manifold has disjoint spines, then [Gui, Lemma 2.3] shows that any spine can be isotoped off itself. Intuitively, one can understand why this seems plausible since a second spine lies in the interior of a collar and can be *pushed* (isotoped) along collar lines until it has been completely moved off itself. The merit in the argument lies in the fact that this can be done for *any* spine in the manifold. Hence, where general position arguments fail miserably to *push a spine off itself* via ambient isotopy, Guilbault's result implies that, nevertheless, an isotopy does indeed exist. It should be pointed out that the isotopy provided by existence of disjoint spines has no motion control like that of an ϵ-isotopy produced via general position. Still, this fact seems surprising.

It is also interesting to note that Guilbault's examples occur in dimensions $n \geq 9$. If a 9-manifold has a (best-case scenario) 7-dimensional spine, one would expect that attempting to isotope this spine off itself would result in unmanageable singularities. That is, one expects general position type arguments to fail rather badly in attempting to isotope a spine off itself.

Notice that if a contractible 4-manifold has a 2-dimensional spine (e.g., Mazur's 4-manifold [Maz]), general position *almost* produces disjoint spines with a small isotopy. The singularities which remain (after a general position isotopy) consist of a finite set of points. It seems conceivable that one might be able to provide techniques to remove these singularities. However, we shall see that in this

one dimension where general position seems likely to be of use, namely dimension 4, there remains, upon the completion of this work, nothing that is known about the existence of disjoint spines in contractible 4-manifolds.

This fact strikes an ironic chord as it was a 4-dimensional issue which led to the work of Guilbault and, hence, to this work. More specifically in terms of historical progression, while examining contractible manifolds in an attempt to understand which compact, contractible manifolds had interiors which could non-trivially cover some compact manifold, D. Wright [Wri] began to consider the following question: Does the interior of the Mazur 4-manifold nontrivial cover some compact manifold? From earlier results of his along with a suggestion by R. D. Edwards, he was led to the realization that if the Mazur 4-manifold did not have disjoint spines, then it could not cover in this fashion. Hence, the question regarding disjoint spine phenomena for this particular manifold was born.

Interestingly enough, the issue regarding disjoint spines in the Mazur 4-manifold was never resolved in Wright's work. However, Wright did go on to show that the interior of the Mazur 4-manifold could not nontrivial cover using a completely different line of argument. The question regarding existence of disjoint spines in contractible 4-manifolds remains open.

Guilbault expanded the disjoint spines question to include other contractible manifolds, including those in higher dimensions. He then went on to provide the only known examples of interesting, compact, contractible manifolds containing disjoint spines. It is at this point that our work adds to what is known about the issue. Chapter II will provide slight improvements to Guilbault's work. In Chapter III, we will produce techniques which provide interesting contractible manifolds with disjoint spines in all dimensions $n \geq 5$. The techniques will be constructive in nature, much in the manner that Guilbault's examples were constructed.

CHAPTER II

SLIGHT IMPROVEMENTS

In this chapter, we show that Guilbault's result actually holds in dimension $n = 4k$. To do this, we will need some data regarding regular neighborhoods in the so-called *stable* range of a k-dimensional complex; that is, all dimensions n so that $2k + 1 \leq n$.

2.1 A Stable Range Result

A common practice in working with complexes is to let $K^{(j)}$ denote the j-skeleton of K. Additionally, the notation $A \cup_C B$ is used to describe the adjunction space formed from A and B by gluing along a subset C (presumably, A and B each contain a natural copy of C). Also, if $f : A \to B$ is a (topological) map, one writes $A \cup_f B$ to denote the space obtained from A and B by identifying $x \in A$ with $f(x) \in B$. We will make use of these conventions when they prove to be convenient. The idea for the following proof is due to J. Bryant.

Proposition 2.1 *Let K be a finite k-dimensional complex and Q an n-manifold. Then, if e_1 and e_2 are homotopic embeddings of K into Q with $2k + 1 \leq n$ and N_1 and N_2 are regular neighborhoods of $e_1(K)$ and $e_2(K)$ respectively, $N_1 \approx N_2$.*

Proof: If $n \geq 2k+2$, trivial range results coupled with regular neighborhood theory apply to provide the result. It is the case $n = 2k + 1$ that is of interest here.

Notice that the $(k - 1)$-skeleton is trivial range in the ambient dimension. Hence, without loss of generality, we may assume that e_1 and e_2 agree on the $(k-1)$-skeleton via an ambient isotopy. So, we have two embeddings of K which agree on the $(k - 1)$-skeleton. For each embedding, thicken so as to obtain a handle decomposition with each decomposition possessing one handle for each cell of $e_i(K)$ for

$i = 1, 2$. Since $e_1(K^{(k-1)}) \approx e_2(K^{(k-1)})$, we can assume that all handles of dimension at most $k-1$ are identical. Let N_1 and N_2 denote the regular neighborhoods of $e_1(K)$ and $e_2(K)$ respectively formed by these handle decompositions.

Now, in order to prove that N_1 and N_2 are homeomorphic, let M denote the manifold (with boundary) consisting of all handles of dimension at most $k-1$. To form N_1 and N_2 we attach k-handles to M along ∂M. Fix a k-handle H_1^k of N_1. Then, there exists a k-handle H_2^k of N_2 which corresponds naturally to H_1^k via the homeomorphism between $e_1(K)$ and $e_2(K)$. The claim is that the attaching maps of these two handles are homotopic in ∂M. To see this, note that the map which collapses M back onto the embedded $(k-1)$-complex provides a homotopy in M between the attaching maps of H_1^k and H_2^k. General position allows us to push this k-dimensional homotopy off of the $(k-1)$-skeleton spine of M and out to ∂M. Thus, the claim holds: The attaching maps of corresponding k-handles are homotopic *in the boundary of* M.

Now, continue by considering the dimensions. We have embedded $(k-1)$-spheres (the attaching spheres of the k-handles) in ∂M and a homotopy between these (in ∂M as well). But, ∂M is $2k$-dimensional. Hence, this is trivial range for an embedded $(k-1)$-sphere. Thus, trivial range results apply to provide an isotopy between attaching maps of corresponding handles in ∂M. Compactness of K insures that there are only finitely many k-handles to consider. Hence, it follows from the usual handlebody techniques that $N_1 \approx N_2$. □

The following lemma and corollary will be required. They are each interesting in their own right.

Lemma 2.1 *Every finite k-dimensional complex embeds in a $2k$-dimensional manifold.*

Proof: Let K be a finite k-dimensional complex. Let $f : K \to R^{2k}$ be a general position map of K into R^{2k}. Without loss of generality, we assume that K is subdivided so that the map f is an embedding on each simplex of K. Then f is an embedding on the $(k-1)$-skeleton of K as any singularities lie in top-dimensional simplices. In a simplicial neighborhood of $f(K)$, begin forming a handle decomposition as usual. In particular, identify all handles up to and including handles of dimension $k-1$. Now, because the singular points of f lie in top-dimensional (i.e., $2k$-dimensional) simplexes, the $2k$-dimensional "handles" (call these singular handles) have possible intersection problems. But, since we know how we would like to attach these $2k$-dimensional handles as their attaching regions are non-problematic, we simply abstractly attach $2k$-dimensional handles which do not intersect. Attach these handles so they are in 1-1 correspondence with the singular handles and so that they respect the attaching regions of the singular handles.

The new handle decomposition gives rise to a $2k$-dimensional manifold M and this manifold collapses onto a copy of K by collapsing each handle back onto its core. Hence, K is embedded in a $2k$-dimensional manifold as desired. □

While certainly interesting, the real value of the preceding lemma lies in connection with what is discovered when the complex in question is acyclic. Under this scenario, we obtain a fundamentally useful result.

Corollary 2.1 *Every finite, acyclic, k-dimensional complex ($k \geq 3$) embeds in R^{2k}.*

Proof: Construct a $2k$-dimensional manifold M as in the lemma so that M collapses to K. Then, M is a PL homology $2k$-cell and ∂M is a PL homology sphere of dimension $2k-1$. Now, ∂M bounds a contractible manifold when $k \geq 3$ [Ker]. Form a manifold by identification of M with this contractible manifold along their homeomorphic boundaries in the natural way.

The result is a PL homology sphere of dimension $2k$. General position shows it to be simply connected and, hence, by the High Dimensional PL Poincaré Theorem, homeomorphic with S^{2k}. Hence, M is embedded in R^{2k}. By construction, $K \subset M$. Thus, K embeds in R^{2k} as well. □

As a footnote to this section, we point out that the dimension requirement on K in the previous corollary is superfluous if we allow topological embeddings. When $k = 1$, K is a tree which, in fact, does embed in R^2. However, when $k = 2$, it is only known that K topologically embeds in R^4. Of related interest, for any balanced presentation P of the fundamental group of a finite, acyclic 2-complex (i.e., one with the same number of generators as relators), [Huc] provides a (PL) embedding of the standard 2-complex, K, based on this presentation into S^4. He does so in such a way that the complement of K is, even, contractible.

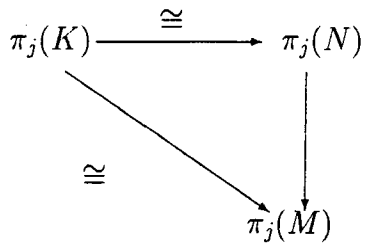
Finally, acyclicity is required in Corollary 2.1 as it is well known that there is no embedding of the k-skeleton of a $(2k+2)$-simplex into R^{2k}.

2.2 Useful Techniques and Data

We now proceed to develop an argument providing information about certain vanishing relative homotopy groups. We shall employ this argument in several places in the material which follows.

Proposition 2.2 *If M is a compact manifold with boundary and $K \subset intM$ a complex with $K \hookrightarrow M$ a homotopy equivalence so that codimension $K \geq 3$, then for any regular neighborhood N of K in $intM$, $\pi_j(M - intN, \partial N) \cong 0$ for each j.*

Proof: Note that for each j, the inclusions $K \hookrightarrow N$, $K \hookrightarrow M$ give rise to isomorphisms of homotopy groups: $\pi_j(K) \xrightarrow{\cong} \pi_j(N)$, $\pi_j(K) \xrightarrow{\cong} \pi_j(M)$ respectively. Hence, by commutativity of the inclusion induced diagram,



$N \hookrightarrow M$ is a homotopy equivalence as well. Therefore, $H_j(M,N) \cong 0$ for all j. Let $W = M - intN$.

Using the strong excision property (Excision in Simplicial Theory), we can observe that $H_j(W, \partial N) \cong 0$ for all j as well. We mention this because a similar argument applies in the universal covering space.

Let $\widetilde{N}, \widetilde{M}, \widetilde{W}$, and $\widetilde{\partial N}$ denote the universal covers of these manifolds. Then, $\widetilde{W} \subset \widetilde{M}$ and $\widetilde{\partial N} \subset \widetilde{N} \subset \widetilde{M}$. Let $p : \widetilde{M} \to M$ be the covering map.

Since $N \hookrightarrow M$ is a homotopy equivalence, so is $\widetilde{N} \hookrightarrow \widetilde{M}$. In particular, $H_j(\widetilde{M}, \widetilde{N}) = 0$ for each $j \geq 0$. Note that $p^{-1}(\partial N) = \widetilde{\partial N} = \partial \widetilde{N}$.

Moreover, $p^{-1}(W) = \widetilde{W}$ as well. Then, $\widetilde{W} = p^{-1}(M - intN)$. But, $p^{-1}(M - intN) = p^{-1}(M) - p^{-1}(intN)$. Similarly, $p^{-1}(intN) = p^{-1}(N) - p^{-1}(\partial N)$. So, $p^{-1}(intN) = \widetilde{N} - \partial \widetilde{N}$. It transpires from this and the last paragraph that $\widetilde{W} = \widetilde{M} - int\widetilde{N}$.

Applying strong excision to the pair $(\widetilde{M}, \widetilde{N})$, we see that homology is preserved: $H_j(\widetilde{M}, \widetilde{N}) \cong H_j(\widetilde{M} - int\widetilde{N}, \widetilde{N} - int\widetilde{N})$ for each j. But, $H_j(\widetilde{M} - int\widetilde{N}, \widetilde{N} - int\widetilde{N}) = H_j(\widetilde{W}, \partial\widetilde{N})$. As mentioned though, $\partial\widetilde{N} = \widetilde{\partial N}$. Hence, we conclude that $H_j(\widetilde{W}, \widetilde{\partial N}) \cong 0$ for all j. From the Relative Hurewicz Theorem, it follows that $\pi_j(\widetilde{W}, \widetilde{\partial N}) \cong 0$ for each j and so $\pi_j(W, \partial N) \cong 0$ for each j as well. □

The previous proposition turns out to be very useful. Its utility will be displayed by the number of times it is invoked in proving the following theorem.

Theorem 2.1 *Let Y be an n-dimensional regular neighborhood of a finite j-complex J. Suppose that there is a k-dimensional complex $K \subset intY$ so that $K \hookrightarrow Y$ is a homotopy equivalence. If J and K both have codimension at least 3 and if $j+k < n$, then, K is a pseudospine of Y.*

Proof: By general position, assume that $J \cap K = \emptyset$.

Remark: If N is a regular neighborhood of K in $intY$, then $Y - intN \approx Y - K$. Hence, it suffices to check that N is a pseudospine of Y. Furthermore, by isotopy uniqueness of regular neighborhoods of K in $intY$, it suffices to show that *some* regular neighborhood of K is a pseudospine of Y. Moreover, Proposition 2.2 shows that $\pi_j(Y - intN, \partial N) \cong 0$ for each j. If one knew that $\pi_j(Y - intN, \partial Y) \cong 0$ for $j = 0, 1, 2$, engulfing techniques would provide the result. Alternately, we will show that $Y - intN$ strongly deforms to ∂N.

Let N be a regular neighborhood of K in $intY$ so that $N \cap J = \emptyset$. Let M_0 be a collar on ∂N in $int(Y - intN)$ so that $M_0 \cap J = \emptyset$ as well. By Proposition 2.2, $\pi_j(Y - intN, \partial N) \cong 0$ for each j. Since $J \subset Y - intN$ is codimension 3, J can be engulfed from M_0. Let M_0' be the new collar on ∂N in $int(Y - intN)$ which contains J.

Let V be a regular neighborhood of J in $intM_0'$ and let V_0 be a collar on ∂V in $int(Y - intN) - intV$ such that $V_0 \subset intM_0'$ too.

Then, $V \subset N \cup M_0' \subset Y$ and $N \cup M_0'$ is a regular neighborhood of K in $intY$. See Figure 2.1.

The inclusion induced diagram (Figure 2.2) shows that $V \hookrightarrow N \cup M_0'$ is a homotopy equivalence. Note that $V \subset int(N \cup M_0')$.

The invocation of Proposition 2.2 now shows that $\pi_j((N \cup M_0') - intV, \partial V) \cong 0$ for each j. Because K is codimension 3, we can engulf K from V_0 in $int((N \cup M_0') - intV)$. Let V_0' be the new collar on ∂V in $int((N \cup M_0') - intV)$ which contains K.

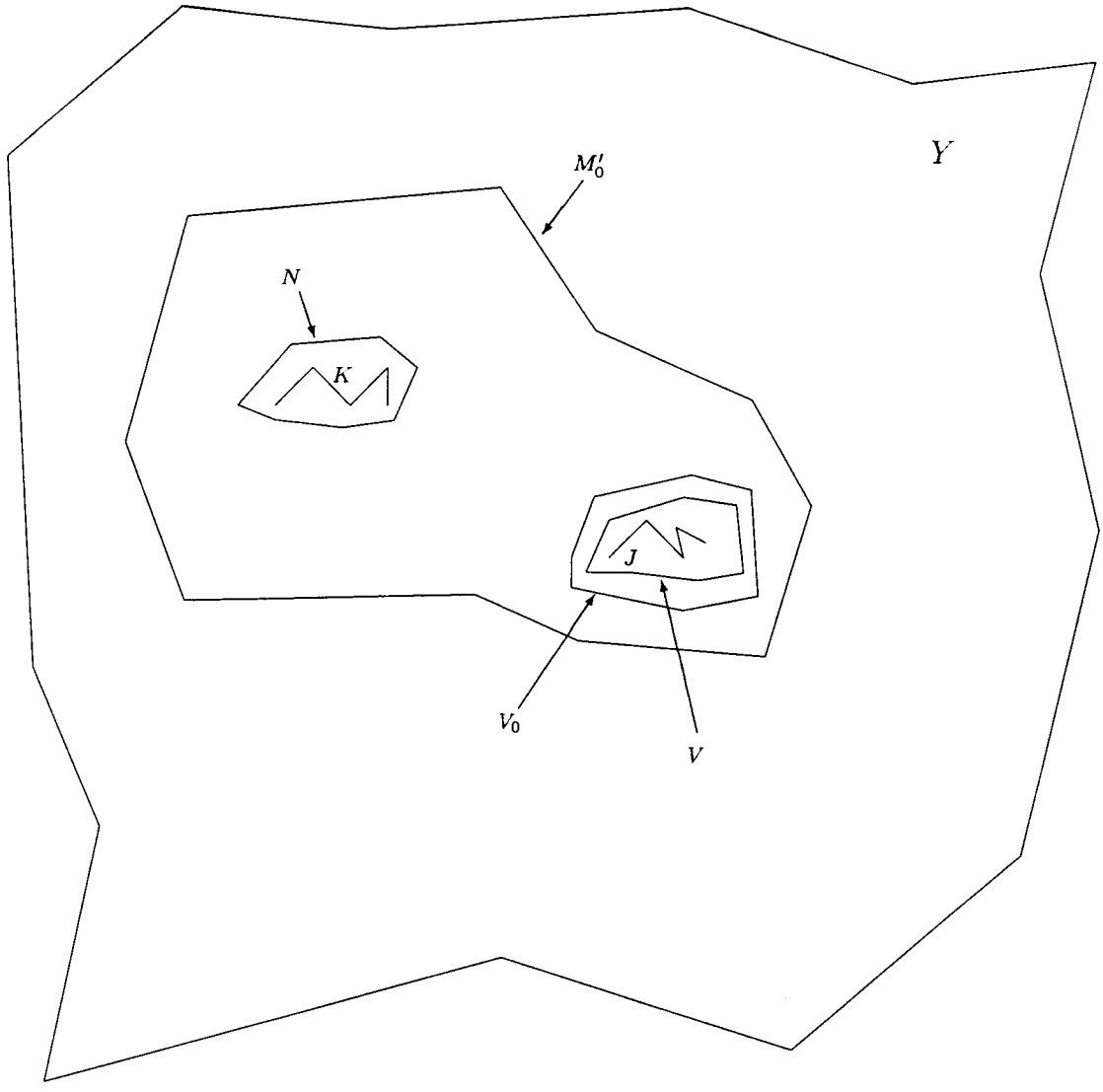


Figure 2.1: Placement Illustration

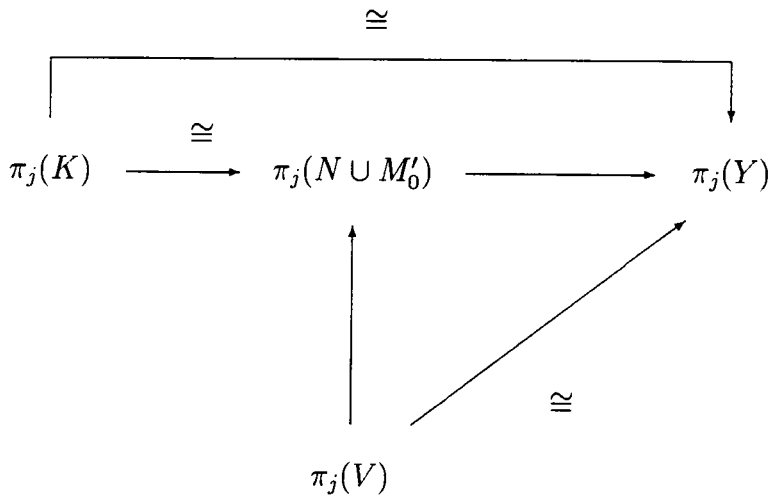


Figure 2.2: Commutative Diagram of Inclusion-Induced Homomorphisms

Now, choose a small regular neighborhood, Ω, of K in Y such that $\Omega \subset intV_0'$. See Figure 2.3.

Claim: Ω is a pseudospine of Y.

It suffices to check that $W = Y - int\Omega$ is an h-cobordism between ∂Y and $\partial\Omega$. Once again, Corollary 2.2 promises that $\pi_j(W, \partial\Omega) \cong 0$ for all j. Therefore, we need only observe that W deforms into ∂Y in W.

Now, $N \cup M_0'$ and Ω are both regular neighborhoods of K in Y and $\Omega \subset N \cup M_0'$. So $W = Y - int\Omega$ deforms to $Y - int(N \cup M_0')$ (in W) by using the collar structure on the nested pair of regular neighborhoods. Furthermore, $V \cup V_0' \subset int(N \cup M_0')$ by design. Since $V \cup V_0'$ is a regular neighborhood of J in Y, it is a spine of Y. (In fact, $Y - int(V \cup V_0')$ is a collar on $\partial(V \cup V_0')$.) Therefore, $Y - int(V \cup V_0')$ deforms to ∂Y in $Y - int(V \cup V_0')$. Since we've already deformed $W = Y - int\Omega$ into $Y - int(N \cup M_0')$ in W and since $Y - int(N \cup M_0') \subset Y - int(V \cup V_0')$, we can deform $Y - int(N \cup M_0')$ into ∂Y in $Y - int(V \cup V_0')$. But, as $\Omega \subset int(V \cup V_0')$, this deformation occurs in $Y - int\Omega$. Hence, we've deformed $W = Y - int\Omega$ into ∂Y in W as desired. So, $(W, \partial Y, \partial\Omega)$ is an h-cobordism and the argument is complete. □

We shall make use of this theorem immediately. However, it is not just the conclusion of the theorem that will be useful: Upon proving that the regular neighborhood Ω of K in $intY$ was a pseudospine of Y, we actually proved something stronger. We state the stronger statement as a corollary so we can make reference to it later.

Corollary 2.2 *In the setting of the previous theorem, if N is any neighborhood of K in $intY$, then $(Y - intN, \partial Y, \partial N)$ is an h-cobordism.*

Proof: The fact that the result holds for any regular neighborhood of K follows from isotopy uniqueness of regular neighborhoods. □

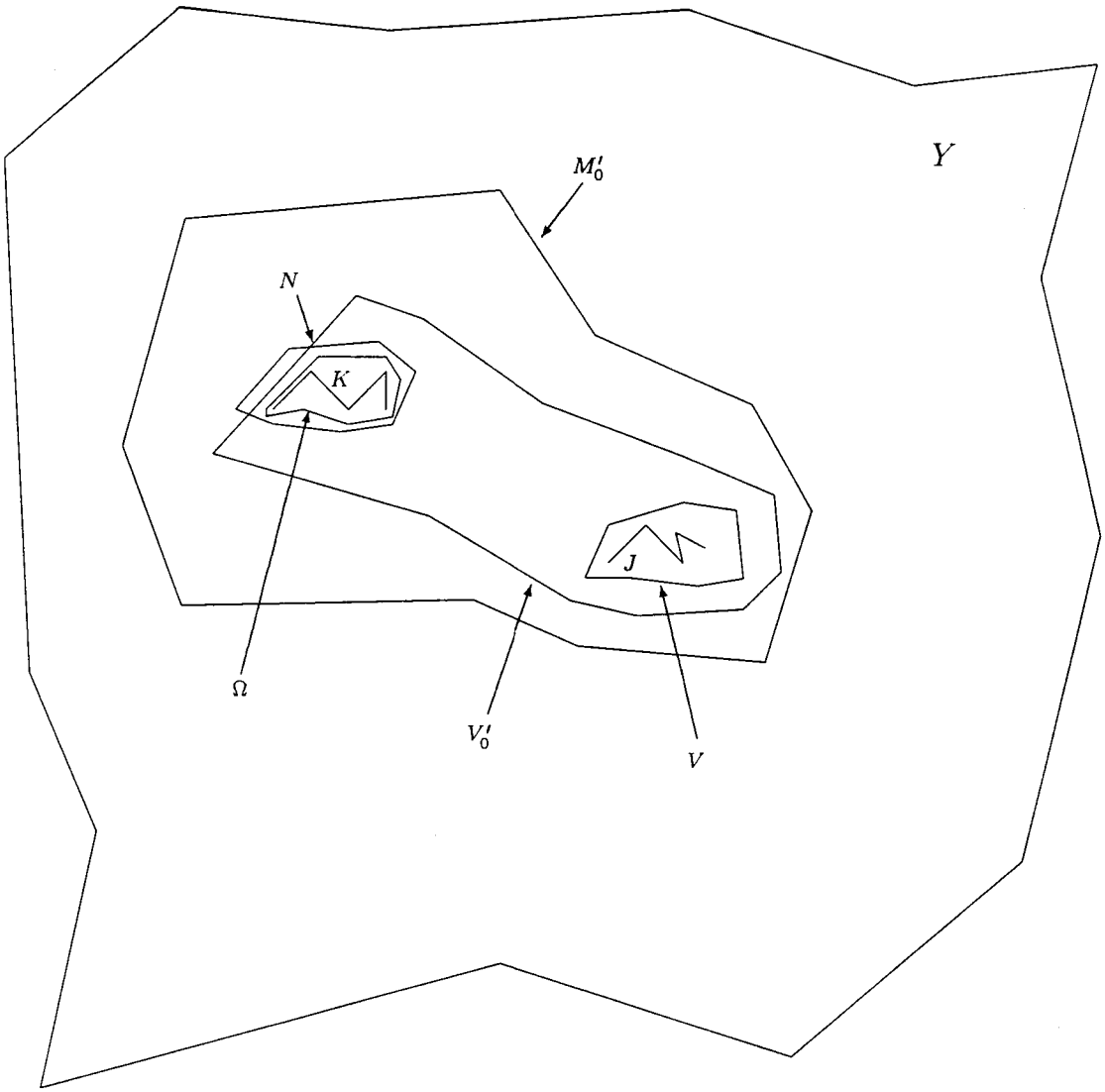


Figure 2.3: Completed Engulfing Steps Illustration

The proof of the following theorem makes use of both Theorem 2.1 as well as the stronger statement provided in Corollary 2.2.

Theorem 2.2 *Let M be a regular neighborhood of a k-complex K in R^n with $n-k \geq 3$ and $n \geq 2k$. Suppose that A is an a-dimensional complex with $n-a \geq 3$ too. Suppose $a+k<n$. Let $X = M \times I$. Suppose that K_1, K_2 are copies of A embedded in ∂X with disjoint regular neighborhoods N_1, N_2 respectively. Suppose further that $K_i \hookrightarrow X$ is a homotopy equivalence for $i = 1,2$. Then, if $W = \partial X - intN_1 - intN_2$, $(W, \partial N_1, \partial N_2)$ is an h-cobordism.*

Proof: By general position, assume that K_1 and K_2 do not intersect $K \times \{1\}$. By isotopy uniqueness of regular neighborhoods, we may assume that $K_1 \subset int(M \times \{0\})$ with $K_1 \cap (K \times \{0\}) = \emptyset$. Moreover, assume that N_1 is contained in the interior of $M \times \{0\}$ as well.

By collapsing $X = M \times I$ to $M \times \{0\}$ using the product structure, observe that $K \times \{0\}$ is a spine of $M \times \{0\}$. Moreover, $K_1 \hookrightarrow M \times \{0\}$ is a homotopy equivalence. The dimension hypotheses insure that both K_1 and $K \times \{0\}$ have codimension at least 3. Consequently, Theorem 2.1 applies to show that K_1 (and hence, N_1) is a pseudospine of $M \times \{0\}$.

Using the open collar structure $M \times \{0\} - N_1 \approx \partial(M \times \{0\}) \times [0,1)$, we can assume now that $K_2 \subset int(M \times \{1\})$ via an isotopy of ∂X which fixes N_1. In fact, we can assume further that N_2 is contained in the interior of $M \times \{1\}$.

A collapsing map $M \times I \searrow M \times \{1\}$ which sends $K \times \{\frac{1}{2}\}$ to $K \times \{1\}$ shows that $K_2 \hookrightarrow M \times \{1\}$ is a homotopy equivalence. Corollary 2.2 then shows that $(M \times \{1\}) - intN_2$ is an h-cobordism between $\partial(M \times \{1\})$ and ∂N_2. (See Figure 2.4.)

Similarly the same corollary shows that $(M \times \{0\}) - intN_1$ is an h-cobordism between $\partial(M \times \{0\})$ and ∂N_1. Combining these h-cobordisms in the natural way,

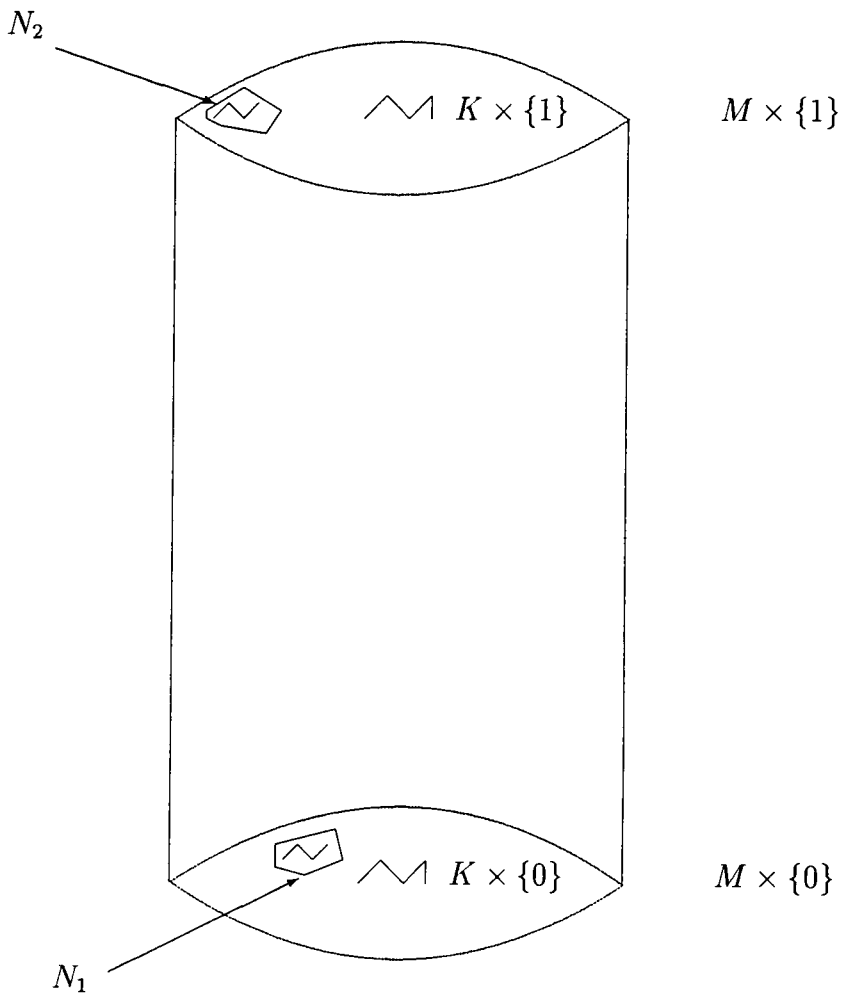


Figure 2.4: Location Illustration

it follows that $\partial X - intN_1 - intN_2$ is an h-cobordism between ∂N_1 and ∂N_2 as desired. □

An application of the last theorem will be made in the main result of this chapter (which will be presented shortly). Its utility will be displayed in Chapter III as well. However, we require several more facts first.

Lemma 2.2 [Gui, Lemma 4] *Let M be a compact manifold containing a PL pseudospine P, and let N be a neighborhood of P in M. Then, there is a (PL) spine K of M contained in N.*

Proof: Let $h : \partial M \times [0,1) \to M - P$ be a (PL) homeomorphism. Choose $0 < r < 1$ so that $h(\partial M \times [r,1)) \subset N$. Let $K = h(\partial M \times [r,1)) \cup P$. □

Recall the notation M_f. It is used to denote the mapping cylinder of f given some specified map f. Relating to mapping cylinders, we make the following definition. In this section, A and K are complexes.

Definition 7 *Let $f : K \to A$ be a map. Then, the top of the mapping cylinder M_f is defined to be the copy of A embedded by the quotient map $q : (K \times I) \oplus A \to M_f$. The base of the mapping cylinder is the image of $K \times \{0\}$ in M_f under q.*

Lemma 2.3 *Let X^{n+1} be a manifold with a spine A such that $dim A = a$. Let K be a k-dimensional complex and $f : K \to A$ a map. If $2 \cdot dim(M_f) < n+1$, then there exists an embedding of M_f in X with base in ∂X and the top precisely A.*

Proof: Note that since $dim(M_f) = max\{k+1, a\}$ the requirement that $2 \cdot dim(M_f) < n+1$ insures that both $2(k+1) < n+1$ and $2a < n+1$ which show that $k + a < n$. Now, M_f is trivially mapped into $A \subset X$; the map sends the fiber of M_f corresponding to $x \in K$ to $f(x)$. Put this map into general position rel the top

of M_f. Then, the embedded base misses the spine A and we can homotope the base out to ∂X. This homotopy gives rise to a new map of M_f into X with the features that the top goes homeomorphically onto A and the base is mapped into ∂X. Adjusting the map on the base in ∂X via general position allows us to assume the base is embedded in ∂X as well since $2k < n = \dim \partial X$. One final application of general position applied to this final map of M_f into X rel the top and base provides the embedded copy of M_f desired when $2 \cdot dim(M_f) < n + 1$. □

The essential ideas in the following two propositions are from [Gui]. We copy the techniques used in the main theorem (Theorem 4.1) there. Our statements are adapted so as to apply in the setting of acyclic complexes. The original arguments took place in a manifold setting. The propositions are important for our results and so the proofs are adapted and provided here as well.

Proposition 2.3 *Let K be an acyclic k-complex. Let $p \in K^{(0)}$. Let cK denote the cone on K. If $K_1 = K \times \{p\}$ and $K_{-1} = \{p\} \times K$, then $K_i \hookrightarrow (K \times K) \cup_{K_{-i}} cK$ is a homotopy equivalence for $i = 1, -1$.*

Proof: We check the case $i = 1$. The other case follows symmetrically. Let $A = (K \times K) \cup_{K_{-1}} cK$. It suffices to check that $\pi_j(A, K_1) \cong 0$ for all j.

Let $\widetilde{K}$ denote the universal cover of K. Let $q : \widetilde{K} \to K$ be the covering map. Then, $(q \times id) : \widetilde{K} \times K \to K \times K$ is also a covering map. Note that $(q \times id)^{-1}(K \times \{p\}) = \widetilde{K} \times \{p\}$. Moreover, the preimage of $K_{-1} = \{p\} \times K$ is a discrete collection $\{K_\alpha\}$ of copies of K, one for each element of $\pi_1(K_1)$. We can attach cK to each of these in the natural way. For specificicity, attach $(cK)_\alpha$ to K_α. The resulting space is simply connected. Using the natural covering map, call it ρ, observe that this space is the universal cover of $A = (K \times K) \cup_{K_{-1}} cK$. Denote this universal cover by $\tilde{A}$.

Moreover, observe that $\rho^{-1}(K \times \{p\}) = \widetilde{K} \times \{p\}$. Hence, $\pi_j(A, K \times \{p\}) \cong \pi_j(\tilde{A}, \widetilde{K} \times \{p\})$ for every j. Because both spaces in the latter pair are simply connected, the Hurewicz Theorem shows that we need only observe that $H_j(\tilde{A}, \widetilde{K} \times \{p\}) \cong 0$ to obtain the desired result.

Using the long exact sequence for pairs, we will show that $H_j(\widetilde{K} \times \{p\}) \to H_j(\tilde{A})$ is an isomorphism for each j.

As both spaces are simply connected, this fact follows immediately for $j = 0, 1$. When $j \geq 2$, consider the following inclusion induced factorization:

$$H_j(\widetilde{K} \times \{p\}) \to H_j(\widetilde{K} \times K) \to H_j(\tilde{A}) \tag{2.1}$$

Because K is acyclic, the Künneth Formula shows that the first homomorphism is an isomorphism for each j. For the second homomorphism, consider the Mayer-Vietoris sequence:

$$\ldots \to H_j(\cup K_\alpha) \to H_j(\widetilde{K} \times K) \oplus H_j(\cup (cK)_\alpha) \to H_j(\tilde{A}) \to H_{j-1}(\cup K_\alpha) \ldots$$

Since K_α is acyclic and $(cK)_\alpha$ is contractible, it follows, from this Mayer-Vietoris sequence, that the second inclusion induced homomorphism in Equation 2.1 is an isomorphism when $j \geq 2$ as well. □

Along similar lines, we need to know that the *diagonal* map becomes a homotopy equivalence when a cone is attached to an acyclic product. The following proposition illustrates the point.

Proposition 2.4 *Let K be an acyclic k-complex. Let $p \in K^{(0)}$. Let cK denote the cone on K. Then, if $\Delta : K \to K \times K$ is the diagonal map, $\Delta : K \to (K \times K) \cup cK$ is a homotopy equivalence where the adjunction space allows for the attachment of the cone along either copy of K (i.e., $K \times \{p\}$ or $\{p\} \times K$).*

Proof: By symmetry of argument, it suffices to assume that we attach cK to $K \times K$ along $\{p\} \times K$. We use notation similar to the proof of Proposition 2.3. In particular, $q : \widetilde{K} \to K$ is the universal covering of K, $\rho : \widetilde{K} \times K \cup cK_\alpha \to (K \times K) \cup cK$ is the universal covering of $(K \times K) \cup cK$, a space we denote by A for convenience.

Let Δ_K denote the diagonal of $K \times K$. Then, we can regard Δ_K as a subset of A in a natural way. We have, $\rho^{-1}(\Delta_K) = \{(u \times q(u)) : u \in \widetilde{K}\}$. In point, $\tau : \widetilde{K} \to (\widetilde{K} \times K) \cup cK_\alpha$ defined by $\tau(u) = (u \times q(u))$ takes $\widetilde{K}$ homeomorphically onto $\rho^{-1}(\Delta_K)$. Then, denote $\rho^{-1}(\Delta_K)$ by $\widetilde{\Delta_K}$ and note that it is the universal cover of Δ_K.

We wish to observe that $\pi_j((K \times K) \cup cK, \Delta_K) \cong 0$ for each j. It suffices to check that $H_j(\widetilde{\Delta_K}) \to H_j((\widetilde{K} \times K) \cup cK_\alpha)$ is an isomorphism for each j to complete the proof. For $j < 2$, the result is immediate as both spaces are simply connected. If $j > 2$, factor the map,

$$H_j(\widetilde{\Delta_K}) \to H_j(\widetilde{K} \times K) \to H_j((\widetilde{K} \times K) \cup cK_\alpha) \tag{2.2}$$

It was observed that the second map was an isomorphism for every j during the proof of Proposition 2.3. Let $\varphi : \widetilde{K} \times K \to \widetilde{K}$ be projection. Then, $\widetilde{\Delta_K} \to \widetilde{K} \times K \xrightarrow{\varphi} \widetilde{K}$ is a homeomorphism and, therefore, induces an isomorphism of homology. The Künneth formula shows that $\varphi_* : H_j(\widetilde{K} \times K) \to H_j(\widetilde{K})$ is an isomorphism as well. Therefore, the following diagram commutes and shows that $H_j(\widetilde{\Delta_K}) \to H_j(\widetilde{K} \times K)$

$$\begin{array}{ccc} & H_j(\widetilde{K} \times K) & \\ \nearrow & & \searrow^{\varphi_*} \\ H_j(\widetilde{\Delta_K}) & \xrightarrow{\cong} & H_j(\widetilde{K}) \end{array}$$

is an isomorphism as desired. Thus, Δ is, indeed, a homotopy equivalence. □

2.3 Techniques in Dimension $n = 4k$

In this section, we will obtain the main result of the chapter. The uniqueness of regular neighborhoods in the stable range allows us to exploit a product structure which exists in a preferred copy of a certain regular neighborhood which arises.

Theorem 2.3 *Let K be a finite, acyclic, k-complex with nontrivial fundamental group. Then, $New(K, n)$ has disjoint spines provided $n \geq 4k$.*

Proof: Let J_1 and J_2 be copies of K. As $J_1 \times J_2$ is acyclic, we can embed $J_1 \times J_2$ in R^{4k} using Corollary 2.1. Let N be a regular neighborhood of the image. Let $X = N \times I$. Let N_1 be a regular neighborhood of $J_1 \times 1 \subset N \times 1$. Let N_2 be a regular neighborhood of $J_2 \times 0 \subset N \times 0$. ($J_1$ and J_2 now designate copies we see in $J_1 \times J_2$, namely $J_1 \times \{pt\}$ and $\{pt\} \times J_2$.)

Now, let $\Delta : K \to J_1 \times J_2$ be the diagonal map. Apply Lemma 2.3 to embed the mapping cylinder M_Δ in X with the feature that the *base* of the mapping cylinder is embedded in ∂X and the top goes homeomorphically onto $(J_1 \times J_2) \times \frac{1}{2}$. Moreover, by general position, we may assume that this base (which is homeomorphic to a copy of K) lies in $\partial N \times (0, 1)$ and, therefore, misses $N_1 \cup N_2$.

Attaching a ball B_1 to X along N_1 gives a $(4k + 1)$-manifold (call it X_1) which has a spine homeomorphic to $J_1 \times J_2 \cup_{J_1} cJ_1$. Name this spine J. That is, by collapsing X along mapping cylinder lines and collapsing the attached ball first to cN_1 and then to cJ_1 (from the sides) and further collapsing out the fibers corresponding to $N_2 - (J_2 \times \{0\})$, we obtain a spine homeomorphic to $J_1 \times J_2 \cup_{J_1} cJ_1$. This is what we call J. Note that this is a $2k$-dimensional complex (as $k \geq 2$). Since $X \cup_{N_1} B_1$ is a $(4k + 1)$-manifold which collapses to J, Proposition 2.1 promises

that this regular neighborhood is unique up to homeomorphism. Observe that the mapping cylinder M_Δ is also embedded naturally in X_1. Let K_2 denote $J_2 \times 0$ and let K_3 denote the base of the embedded cylinder. Choose a regular neighborhood N_3 of K_3 which misses N_2.

Because of the uniqueness of the regular neighborhood, we can form a preferred regular neighborhood of J as follows: Embed J in R^{4k}. Take a regular neighborhood, Q, of the image and then let $P = Q \times I$. Now, identifying P with X_1 via some homeomorphism h, we observe $h(N_2)$ and $h(N_3)$ in ∂P. The idea is to use the previous theorem to utilize the natural product structure between $Q \times 0$ and $Q \times 1$ in P to obtain an open product structure between $\partial h(N_2)$ and $\partial h(N_3)$. More precisely, the claim is that $(\partial P - int(h(N_2)) - int(h(N_3)), \partial h(N_2), \partial h(N_3))$ is an h-cobordism. In order to proceed, note that the inclusions of K_2 and K_3 into X_1 are homotopy equivalences (Propositions 2.3 and 2.4). The dimensional requirements of Theorem 2.2 are met (as well as the other hypothetical requirements) and so, it follows that $(\partial P - int(h(N_2)) - int(h(N_3)), \partial h(N_2), \partial h(N_3))$ is indeed an h-cobordism. Pulling back to ∂X_1 via h^{-1}, we observe an h-cobordism $(\partial X_1 - intN_2 - intN_3, \partial N_2, \partial N_3)$.

To finish the argument, note that attaching another ball to X_1 along N_2 provides a PL homology $(4k+1)$-cell with simply connected boundary. By virtue of the High Dimensional PL Poincaré Theorem, this boundary is homeomorphic to S^{4k}. Furthermore, N_3 lies in this space in a natural way. Hence, we can identify $New(K, n = 4k)$ by removing $intN_3$ from this sphere. But, the above h-cobordism implies that N_2 is then a pseudospine of $New(K, n = 4k)$. An analogous argument can be used to show that N_1 is a pseudospine as well by producing an h-cobordism involving ∂N_1 and ∂N_3 (by attaching the ball to X along N_2 first this time). Thus, $New(K, 4k)$ has a disjoint pair of pseudospines.

Lemma 2.2 then shows that $New(K, n = 4k)$ has a pair of disjoint PL spines as desired. □

This last result provides for the existence of contractible 8-manifolds with disjoint spines. While this is a step in the direction of producing the lower dimensional examples we are in search of, we shall do much better than this in Chapter III.

CHAPTER III

PRODUCING LOWER DIMENSIONAL EXAMPLES

In this chapter, we introduce techniques that prove the existence of interesting contractible manifolds containing disjoint spines in all dimensions $n \geq 5$. As mentioned near the outset, the existence of such manifolds has been known for dimensions $n \geq 9$. The results of this chapter will be based on constructions involving non-simply connected acyclic 2-complexes. It should be pointed out that the existence of interesting contractible manifolds with disjoint spines of dimension n which are already known with $9 \leq n \leq 12$ all involve such a 2-complex. We will mimic the construction technique used in [Gui] for as long as possible in a general sense; however, new methods will be utilized to avoid the general position failure encountered there.

The first method that allows us to produce examples in dimensions $n < 9$ is by use of a construction which, essentially, allows the known argument to proceed to capture dimensions 7 and 8 in a manner similar to that of Chapter II. That is, with the construction mentioned above, we could apply techniques reminiscent of Chapter II to produce examples in dimensions 7 and above. However, in order to obtain examples in dimensions 5 and 6, we need other techniques as well. These new techniques, which produce output in all dimensions $n \geq 5$, provide a slightly stronger result than that of the known argument when $n \geq 9$. Therefore, we apply these techniques in all cases when $n \geq 6$. The case $n = 5$ will be treated separately for technical reasons.

The results in this chapter are inherently based on 3-dimensional complexes which are constructed from given acyclic 2-complexes. We'll begin by providing this construction.

A common practice is to let $A \vee B$ denote the wedge of spaces A and B (assuming there is a natural wedge-point).

3.1 A Construction on 2-complexes

Theorem 3.1 *Let K be a finite acyclic 2-dimensional PL CW complex with 1 0-cell. Let J_1 and J_2 denote copies of K. Then, there exists an acyclic PL CW 3-complex A consisting of $J_1 \vee J_2$ together with some additional 2-cells and 3-cells so that $\pi_1(A) \cong \pi_1(K) \times \pi_1(K)$ and so that $\pi_j(A \cup_{J_1} cJ_1, J_2)$ and $\pi_j(A \cup_{J_2} cJ_2, J_1)$ are trivial for each j. (Here, $J_1 \vee J_2$ should be taken as the wedge of J_1 and J_2 along the 0-cell of each and $A \cup_{J_i} cJ_i$ is the adjunction space formed by gluing A and the cone cJ_i together along the natural copy of J_i in each.)*

Proof: Form the wedge $J_1 \vee J_2$ by wedging at the 0-cell of each copy of K. Let L denote the complex obtained from $J_1 \vee J_2$ by adding "commutator 2-cells" by attaching 2-cells to $J_1 \vee J_2$ along their boundaries by maps of the form $a_i b_j \overline{a_i} \overline{b_j}$ for $i, j \in \{1, 2, \ldots, t\}$ where a_i, b_i are the 1-cells of J_1, J_2 respectively for $i = 1, 2, \ldots t$ as in the construction of $J_1 \times J_2$. Observe that all of the 2-cells of $J_1 \times J_2$ which were not already present in the wedge $J_1 \vee J_2$ are attached in this fashion.

The first claim is that $\widetilde{H}_j(L) = 0$ if $j \neq 2$ and $H_2(L)$ is free on t^2 generators. For convenience, let $[a_i, b_j]$ denote the 2-cell which is to be attached to $J_1 \vee J_2$ by the map of the form $a_i b_j \bar{a}_i \bar{b}_j$.

The claim will be verified by using an inductive argument along with Mayer-Vietoris. Let $L_0 = J_1 \vee J_2$, $L_1 = L_0 \cup [a_1, b_1]$, $L_2 = L_1 \cup [a_1, b_2]$, $\ldots L_t = L_{t-1} \cup [a_1, b_t]$, $L_{t+1} = L_t \cup [a_2, b_1]$, $\ldots L_{t^2} = L_{t^2-1} \cup [a_t, b_t]$.

To use Mayer-Vietoris, consider each 2-cell to be attached along a simple closed curve in the interior of that 2-cell to see that the attaching curve is collared. This attaching curve is homeomorphic to S^1. See figure 3.1.

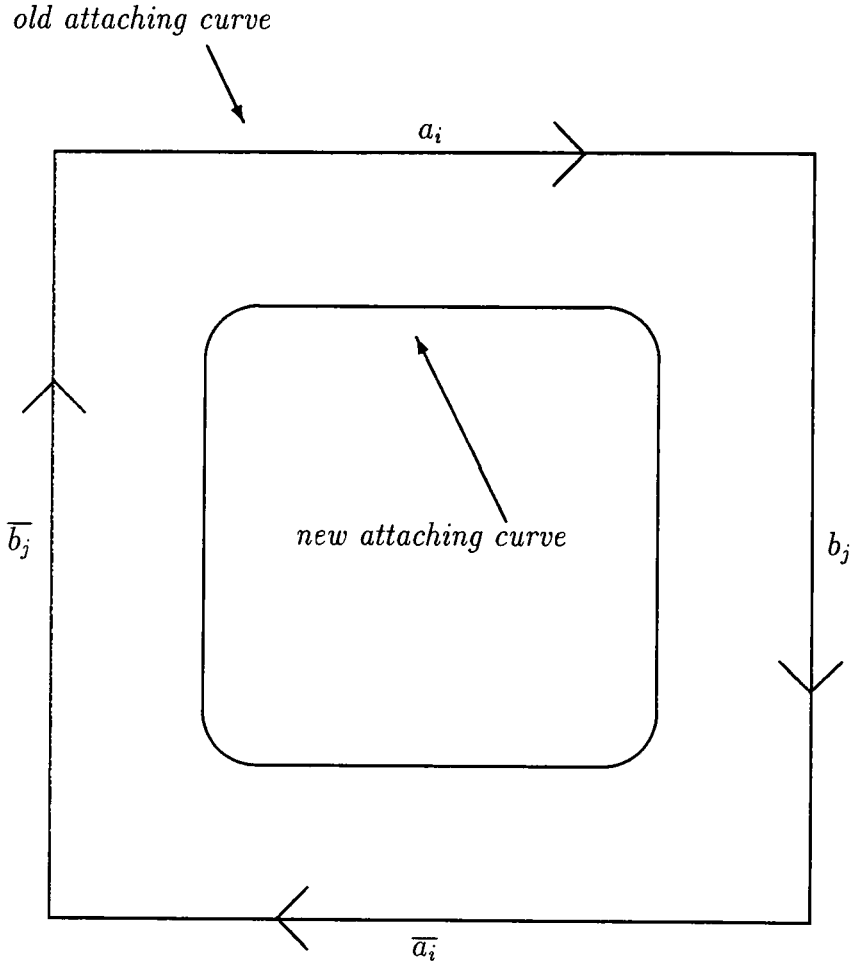


Figure 3.1: Altered Attachings

Let L_i' denote L_i together with the extended collar onto which the $i+1^{st}$ 2-cell, call it δ_{i+1}, is to be attached for $i = 0, 1, \ldots, t^2 - 1$. (So, $L_i' \searrow L_i$.) Let A_i be the attaching curve of δ_i. Then $A_i \approx S^1$ for each i.

From Mayer-Vietoris, we have the exact sequence ...

$$\ldots \widetilde{H}_j(A_i) \to \widetilde{H}_j(L_{i-1}') \oplus \widetilde{H}_j(\delta_i) \to \widetilde{H}_j(L_i) \to \widetilde{H}_{j-1}(A_i) \ldots$$

Since L was obtained from $J_1 \vee J_2$ by attaching 2-cells and J_1 and J_2 are each acyclic, it follows that $H_j(L) \cong 0$ when $j \geq 3$. Since L is connected, $\widetilde{H}_0(L) \cong 0$ as well. When $j = 1$, we have (since δ_i is acyclic)

$$Z \to H_1(L_{i-1}') \oplus 0 \to H_1(L_i) \to 0$$

and because $L_{i-1}' \searrow L_{i-1}$, we have $H_j(L_{i-1}') \cong H_j(L_{i-1})$ for all j. Since $H_1(L_0) \cong 0$ as J_1 and J_2 are acyclic, $H_1(L_0') \cong 0$ too. Induction shows that $H_1(L_i) \cong 0$ for each i. Since $L_{t^2} = L$, $H_1(L) \cong 0$ as claimed.

In a similar fashion, we can observe that $H_2(L)$ is free on t^2 generators: When $j = 2$, we have

$$0 \to H_2(L_{i-1}') \oplus 0 \to H_2(L_i) \to Z \to H_1(L_{i-1}') \oplus 0$$

and as just verified, $H_1(L_{i-1}') \cong H_1(L_{i-1}) \cong 0$. Hence, the sequence splits and after t^2 steps (or induction) we observe that $H_2(L_{t^2})$ is free on t^2 generators noting that $H_2(L_0) \cong H_2(L_0') \cong 0$ (because J_1, J_2 are acyclic) to begin the induction. Since $L_{t^2} = L$, the claim is verified.

Now, the goal is to attach t^2 3-cells to L to form a PL CW complex A which will be acyclic and which will satisfy the desired relative homotopy conditions. To do this, we'll observe that the generating cycles of $H_2(L)$ are spherical. That is, each is obtainable as a map of a 2-sphere into L. Hence, it will be possible to attach t^2 3-cells to L so that each generator bounds precisely one of the attached 3-cells

homologically and, therefore, obtain an acyclic space. However, we will need to be explicit in how we attach the 3-cells for some later computations.

Since G is a perfect group, G is its own commutator subgroup. In particular, each element of G can be represented as a finite product of commutators of G. Let $E_1, E_2, \ldots, E_{t^2}$ be t^2 3-cells which are to be attached to L.

To describe the attaching map for E_i, think of ∂E_i ($\approx S^2$) as the union of 2 hemispherical disks B_i^+, B_i^- where $B_i^+ \cap B_i^- \approx S^1$. Let D_i denote the i^{th} 2-cell attached to the wedge. Then, B_i^+ is attached by mapping its interior homeomorphically onto D_i and taking the boundary to the boundary. So, the boundary of B_i^+ is mapped in according to the formula $a_p b_q \bar{a}_p \bar{b}_q$ for the appropriate pair of indices corresponding to D_i. Now, to attach B_i^-, write $[a_p]$ and $[b_q]$ as finite products of commutators ... Say, $[a_p] = [g_1, g_2][g_3, g_4] \ldots [g_{p'}, g_{p'+1}]$ and $[b_q] = [h_1, h_2][h_3, h_4] \ldots [h_{q'}, h_{q'+1}]$ where g_α, h_β are elements of $\pi_1(J_1), \pi_1(J_2)$ respectively (with no requirement that the $g'_\alpha s$ or $h'_\beta s$ be pairwise distinct).

To map in B_i^-, note the attaching map on the boundary is predetermined by $a_p b_q \bar{a}_p \bar{b}_q$. We'll construct an interesting null homotopy of this loop in L in order to map in this disk. To do this, first homotope the attaching map to the "product-of-commutators" representation $[g_1, g_2][g_3, g_4] \ldots [g_{p'}, g_{p'+1}]\ [h_1, h_2][h_3, h_4] \ldots [h_{q'}, h_{q'+1}]\ \overline{[g_{p'}, g_{p'+1}]} \ldots \overline{[g_3, g_4][g_1, g_2][h_{q'}, h_{q'+1}]} \ldots \overline{[h_3, h_4][h_1, h_2]}$.

Now, we use the added commutator 2-cells to homotope this map to a map which first runs through all the elements g_α which are used and then through the elements h_β. To do this, choose, now and forever, specific null homotopies (which are singular disks in L) of the elements $g_\alpha h_\beta \overline{g_\alpha} \overline{h_\beta}$ for each $g_\alpha \in \pi_1(J_1), h_\beta \in \pi_1(J_2)$. Call the singular disk which corresponds to this null homotopy $H_{\alpha,\beta}$. See Figures 3.2 and 3.3 for an illustration.

The result of these homotopies is that the attaching map is homotoped to

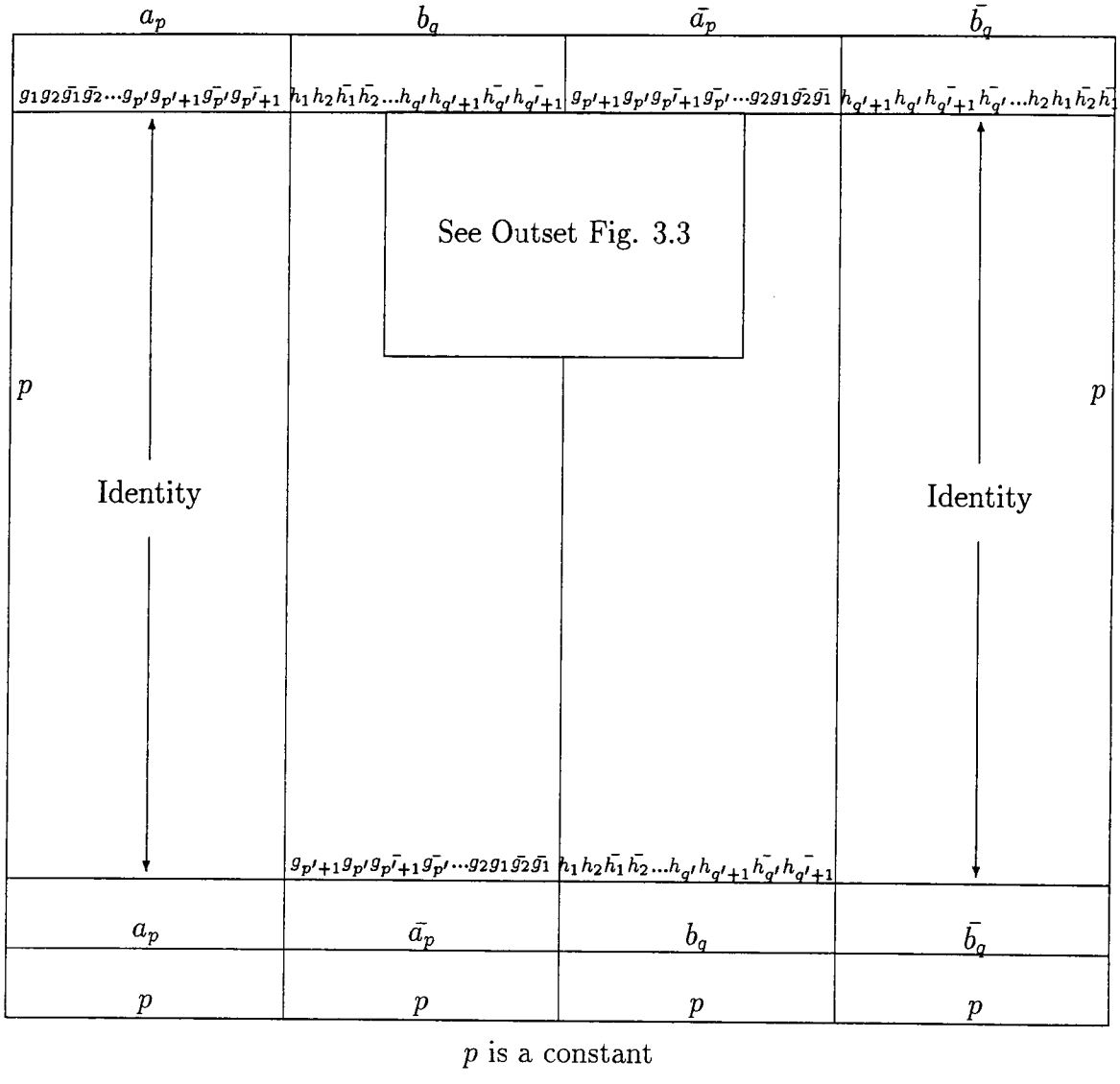


Figure 3.2: Attaching map for B_i^-

The map is constant on vertical segments

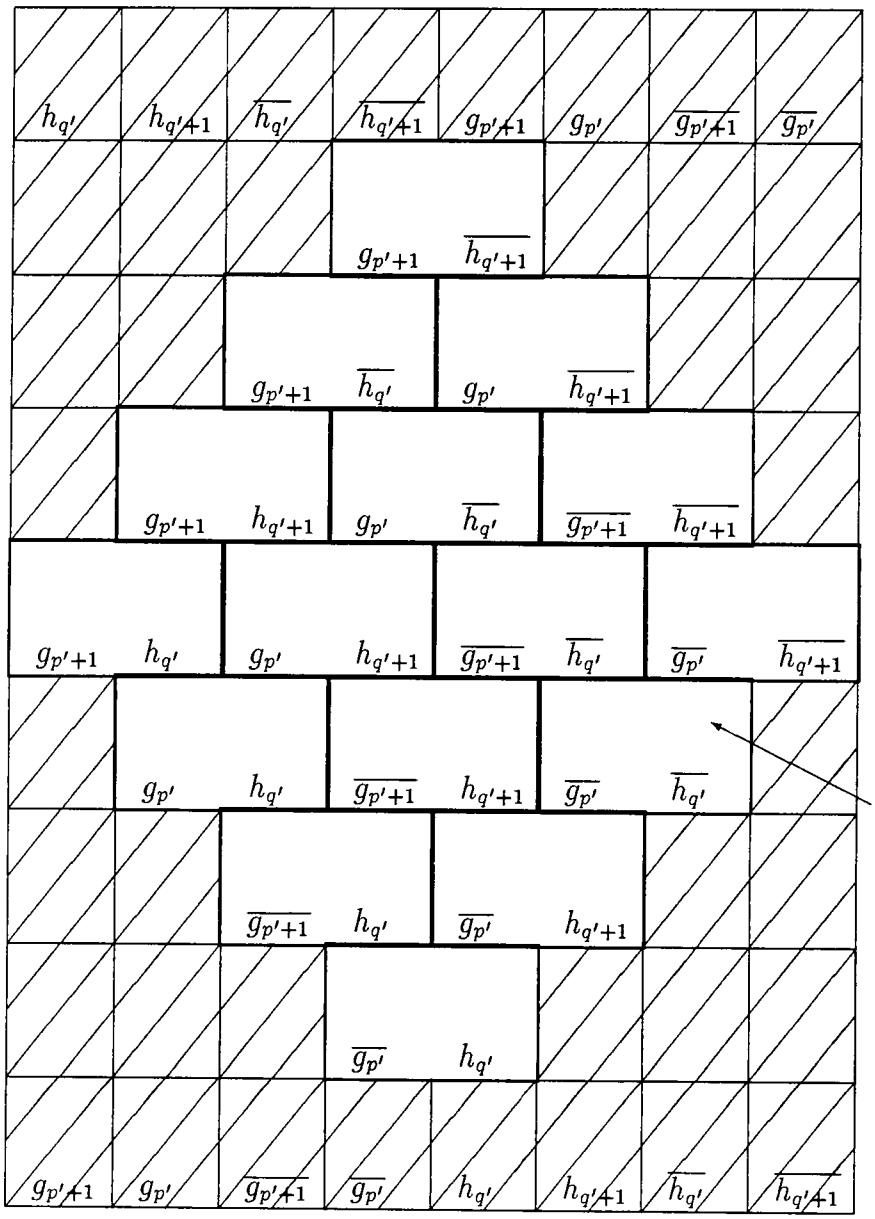


Figure 3.3: Outset Portion of Attaching Map

a map with an especially nice form. Namely, we see the rearranged commutators $[g_1, g_2][g_3, g_4] \dots [g_{p'}, g_{p'+1}]\ \overline{[g_{p'}, g_{p'+1}]} \dots \overline{[g_3, g_4][g_1, g_2]}\ [h_1, h_2][h_3, h_4] \dots [h_{q'}, h_{q'+1}]$ $\overline{[h_{q'}, h_{q'+1}]} \dots \overline{[h_3, h_4][h_1, h_2]}$. Now, homotope back to $a_p\overline{a_p}b_q\overline{b_q}$ using the same homotopies used before (i.e., the homotopy that takes each element to its product-of-commutators representation which was used above). Finally, homotope $a_p\bar{a}_pb_q\bar{b}_q$ to a constant *in the 1-skeleton.*

The above procedure shows how to map in B_i^- and, thereby, provides a method of attaching a 3-cell. We attach 3-cells using this method and require that the attaching maps be PL in order to obtain the desired complex. At this point, it is not difficult to observe directly that attaching 3-cells in this manner results in an acyclic complex. However, this fact follows from the Mayer-Vietoris sequence once the desired relative homotopy conditions are checked: Let A denote the complex obtained by attaching $E_1, E_2, \dots, E_{t^2}$ in the manner described. Suppose the relative homotopy groups in the statement of the theorem are trivial for each j. Then, the Mayer-Vietoris exact sequence

$$\dots \widetilde{H}_j(J_1) \to \widetilde{H}_j(cJ_1) \oplus \widetilde{H}_j(A) \to \widetilde{H}_j(A \cup_{J_1} cJ_1) \to \widetilde{H}_{j-1}(J_1) \dots$$

implies that $\widetilde{H}_j(A)$ is trivial for each j because the triviality of the relative homotopy groups, $\pi_j(A \cup_{J_1} cJ_1, J_2) \cong 0$ for every j, shows that $A \cup_{J_1} cJ_1$ is acyclic since it is homotopy equivalent to the acyclic space J_2.

Therefore, we need only verify the relative homotopy conditions to complete the argument. We'll observe that $\pi_j(A \cup_{J_2} cJ_2, J_1) \cong 0$ for each j. The proof for the other relative homotopy group follows similarly.

To begin, we note that we have, in essence, the wedge of J_1 and cJ_2 together with some attached 2-cells and 3-cells. It will be useful to break this space into pieces as follows. The complex $A \cup_{J_2} cJ_2$ is constructed from $L \cup_{J_2} cJ_2$ by attaching the 3-cells in the obvious way. Moreover, it follows from Mayer-Vietoris that $H_2(L \cup_{J_2} cJ_2)$

is free on t^2 generators, in fact, the image of a generator of $H_2(L)$ under inclusion provides a generator in $H_2(L \cup cJ_2)$.

Now, because the attaching maps of the 2-cells and 3-cells can be simultaneously homotoped (rel their restriction to J_1) to maps with image missing J_2, it follows that $A \cup_{J_2} cJ_2$ is homotopy equivalent (rel J_1) to a complex A^* consisting of J_1 together with some 2-cells and 3-cells where the cells are attached to J_1 so that the t 2-cells which were attached to $J_1 \vee J_2$ by maps of the form $a_p b_q \bar{a}_p \bar{b}_q$ for a fixed p, are now attached to J_1 according to $a_p \bar{a}_p$ as q ranges over $\{1, 2, \ldots t\}$ and the cone, cJ_2, has been collapsed away. Also, the 3-cells are now mapped in so that any point which was attached to J_2 before is now attached to the 0-cell in J_1.

Let L^* denote the space obtained from $L \cup cJ_2$ by homotoping the attachings of the 2-cells into J_1 rel J_1 and collapsing cJ_2 (using a restriction of the same homotopy mentioned above). That is, we take L^* to be the 2-skeleton of A^*. Notice that the images of the commutator 2-cells (attached to J_1 after the homotopy) still "correspond to" generators of $H_2(L^*)$ and, in fact, are now homotopically trivially attached to J_1. Furthermore, the spherical generators observed earlier still represent generators of $H_2(L^*)$ and each dies in A^* as each bounds a 3-chain there. More precisely, each spherical generator of $H_2(L^*)$ bounds a 3-chain which is single 3-cell attached to $L^* \subset A^*$. It will be convenient to continue to refer to the 2-cells of L^* obtained from the homotoped commutator 2-cells as, *still*, commutator 2-cells.

Since the homotopy equivalence mentioned above is rel J_1, it suffices to check that $J_1 \hookrightarrow A^*$ is a homotopy equivalence to know that $J_1 \hookrightarrow A \cup_{J_2} cJ_2$ is a homotopy equivalence as well (which provides triviality of the desired relative homotopy groups).

To do this, we'll move to the universal cover where we have $\widetilde{J_1} \subset \widetilde{L^*} \subset \widetilde{A^*}$ because $J_1 \hookrightarrow L^* \hookrightarrow A^*$ are π_1-isomorphisms as L^* is obtained from J_1 by attaching

2-cells which are homotopically trivially attached and A^* is obtained from L^* by attaching 3-cells.

Now, $H_2(\widetilde{L^*}, \widetilde{J_1})$ is free on $t^2\ Z\pi_1$ generators (i.e., $Z\pi_1(J_1)$ generators). Also, $H_3(\widetilde{A^*}, \widetilde{L^*})$ is free on $t^2\ Z\pi_1$ generators. We claim that the boundary operator, ∂, is an isomorphism $\partial : H_3(\widetilde{A^*}, \widetilde{L^*}) \to H_2(\widetilde{L^*}, \widetilde{J_1})$ in the exact sequence of the triple $(\widetilde{A^*}, \widetilde{L^*}, \widetilde{J_1})$. If so, then because

1. $H_j(\widetilde{A^*}, \widetilde{J_1}) = 0$ when $j \neq 2, 3$,
2. $H_j(\widetilde{L^*}, \widetilde{J_1}) = 0$ when $j \neq 2$, and
3. $H_j(\widetilde{A^*}, \widetilde{L^*}) = 0$ when $j \neq 3$

all because of dimensional considerations, then, in the exact sequence of the triple we'll have,

$$0 \to H_3(\widetilde{A^*}, \widetilde{J_1}) \to H_3(\widetilde{A^*}, \widetilde{L^*}) \xrightarrow{\cong} H_2(\widetilde{L^*}, \widetilde{J_1}) \to H_2(\widetilde{A^*}, \widetilde{J_1}) \to 0$$

and so the isomorphism will, in particular, show that $H_3(\widetilde{A^*}, \widetilde{J_1}) \cong 0 \cong H_2(\widetilde{A^*}, \widetilde{J_1})$. Therefore, $H_j(\widetilde{A^*}, \widetilde{J_1}) \cong 0$ for each j. Hence, the inclusion map $i : \widetilde{J_1} \hookrightarrow \widetilde{A^*}$ induces a homology equivalence and, therefore, a homotopy equivalence by the Whitehead Theorem as both spaces are simply connected. In particular, $\pi_j(\widetilde{A^*}, \widetilde{J_1}) \cong \pi_j(A^*, J_1)$ will be trivial for each j so that the inclusion of J_1 into A^* will be a homotopy equivalence as desired.

At this point, we require the specificity of the attaching maps of the 3-cells. We shall notice that the homotoped attaching maps (i.e., homotoped according to the equivalence $A \cup_{J_2} cJ_2 \to A^*$) take on an especially nice form again. Figure 3.4 illustrates changes made from Figure 3.3 in showing how E_i is attached after the homotopy $A \cup_{J_2} cJ_2 \to A^*$. In particular, the integral incidence numbers between

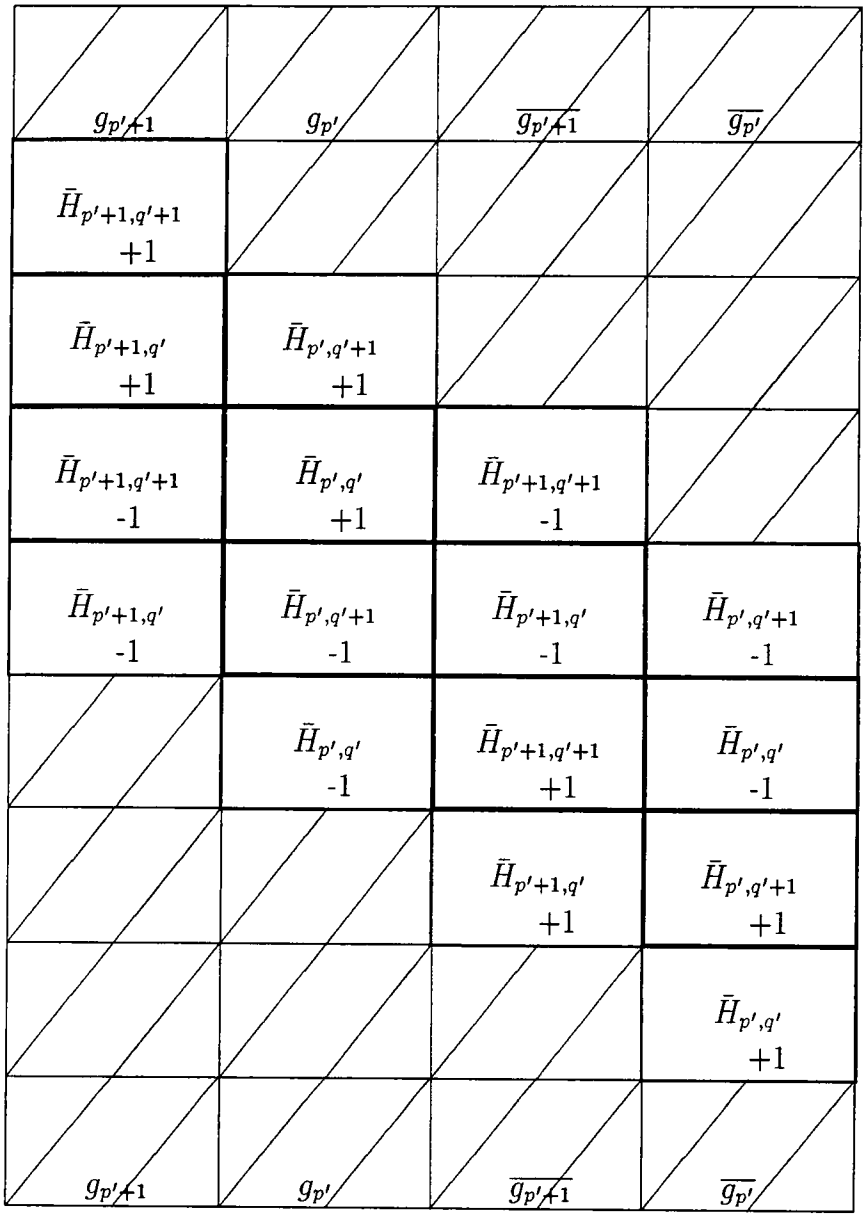


Figure 3.4: Outset Figure after Homotopy

the 2-cells and 3-cells are preserved. Moreover, at the π_1-level, there are no complications regarding the use of the singular disks, $H_{\alpha,\beta}$ (after homotopy). That is, if we think about the universal covers, observe that a spherical generator which "corresponds to" a commutator 2-cell in L^* as before lifts to a cycle which "corresponds to" a lift of that commutator 2-cell as is checked by observing the incidence numbers. That is, because of the specific attachings of the 3-cells, the incidence number of the lift of a 3-cell is 1 on the same lift of that commutator 2-cell which it was designed to kill homologically in A^* and is 0 on all other lifted 2-cells. The point is that the matrix of $Z\pi_1$ incidence numbers given by the boundary operator ∂ is precisely the identity matrix (when the cells are based wisely and we allow the interchange of rows: Any choice of basing coupled with the allowance of interchanging rows gives a matrix with trivial units of $Z\pi_1$ on the main diagonal. But of course these differences in resulting matrices are determined by pure choice of base paths, and therefore, can be negated by choosing paths more judiciously.) The key observation is that $\partial : H_3(\widetilde{A^*}, \widetilde{L^*}) \to H_2(\widetilde{L^*}, \widetilde{J_1})$ is, indeed, an isomorphism. So, $\pi_j(A \cup_{J_1} cJ_1, J_2) \cong 0$ as required. □

Remark: Note that the strong deformation retractions promised by the vanishing relative homotopy groups represent the trivial Whitehead torsion element in both cases: The pair $(A \cup_{J_2} cJ_2, J_1)$ is simple-homotopy equivalent to the pair (A^*, J_1) because we've just homotoped some attaching maps rel J_1 and collapsed cJ_2 to a point. Moreover, the latter pair is essentially in simplified form. That is, A^* is obtained from J_1 by attaching cells of dimensions 2 and 3 only, and, moreover, the 2-cells are homotopically trivially attached.

In simplified form, the only condition that is not strictly met is that the 2-cells are attached trivially at a point. But, this can be arranged via homotoping the attaching maps farther. If this is done, then the incidence numbers between

the 2-cells and 3-cells (over $Z\pi_1$) are preserved and we've already observed that this matrix is (Whitehead equivalent to) the identity. It follows from [Coh] that $J_1 \hookrightarrow A^*$ is, in fact, a simple-homotopy equivalence and thus the inclusion $J_1 \hookrightarrow A \cup_{J_2} cJ_2$ is a simple-homotopy equivalence too. Symmetry shows $J_2 \hookrightarrow A \cup_{J_1} cJ_1$ is a simple-homotopy equivalence as well.

A useful feature of the previous construction is supplied by the following proposition. In fact, this feature is crucial in producing the examples which lay ahead.

We point out that we shall frequently make no distinction between a PL CW complex and its underlying point set. In particular, we shall generally use notation like $f : K \to A$ to represent a map from $|K|$ to $|A|$ or say $x \in K$ to mean $x \in |K|$ with the belief that no confusion will arise and, in fact, with the belief that the meaning will be more clear. This convention arises in the statement which follows.

Proposition 3.1 *Given the PL CW 3-complex A constructed from an acyclic 2-complex as in Theorem 3.1, there exists a map $f : K \to A$ so that $f_i : K \to A \cup_{J_i} cJ_i$ defined by $f_i(x) = f(x)$ for $x \in K$ is a homotopy equivalence for $i = 1, 2$.*

Proof: Take a copy of K (still called K) and label the generators of $\pi_1(K)$ by $c_1, c_2, \ldots, c_t$ for distinction. Define a map on the 1-skeleton of K by mapping c_i onto $a_i b_i$ in the 1-skeleton of A (Here, a_i is the generator of $\pi_1(J_1)$ corresponding to c_i, b_i is the generator of $\pi_1(J_2)$ corresponding to c_i where J_1, J_2 are the copies of K used in the construction of A.). With a little checking, we note that this map has an extension to all of K (using the commutator 2-cells of A). It may have many extensions. In fact, we can always arrange it so that at the π_1-level we have the homomorphism $\pi_1(K) \to (\pi_1(J_1) \times \pi_1(J_2)) \cong \pi_1(A)$ given by $g \mapsto g \times g$ by homotoping the diagonal map of $J_1 \times J_2$ into the 2-skeleton. Let $F : K \to A$ denote

some extension obtained from the diagonal map in this fashion.

Unfortunately, an arbitrary extension may not have the desired features. So, we will have to alter the map slightly to satisfy our needs. To do this, let's first consider the maps $r_i : (A \cup_{J_i} cJ_i) \to J_{i'}$ for $i = 1, 2$ ($i' = 1$ if $i = 2$, $i' = 2$ if $i = 1$) where r_i is the finishing retraction promised by the strong deformation retraction $A \cup_{J_i} cJ_i \to J_{i'}$.

Remark: It follows from the proof of Theorem 3.1 that we may assume the retraction r_i sends $J_i \subset A \cup_{J_i} cJ_i$ to $*$, the 0-cell of $J_{i'}$.

We will modify the map F to insure that the desired condition will be met. To do this, let D be a 2-cell of K. The singular disk $(r_1 \circ F)(D) \subset J_1$ has the feature that the attaching map on its boundary is homotopic to the attaching map of the corresponding 2-cell (call it D') of J_1.

Hence, we have a natural map of a 2 sphere into J_1 determined by $(r_1 \circ F)(D)$ and $D' \subset J_1$ and the homotopy between the maps of the disks on their boundaries. We simply wish we knew that the element of $\pi_2(J_1, *)$ determined by this map were trivial (where $*$ denotes the 0-cell of $J_1 \vee J_2$ again). For then, we could extend the map to I^3 and homotope $(r_1 \circ F)(D)$ across the image of the 3-cell so that the image of D was D'.

Remark: Observe that when K is aspherical, this is not an issue. That is, any extension inducing "the diagonal homomorphism" satisfies the requirements of the desired map as everything can be checked at the π_1-level. If K is not aspherical, some adjustments to F will be made.

If we can adjust F to a new map F' so that the elements of $\pi_2(J_1, *)$ are trivial for each 2-cell of K, then $r_1 \circ F'$ will be homotopic to the natural identification of K with J_1. We can alter F so that this is indeed the case.

For a fixed 2-cell D of K, let $\alpha \in \pi_2(J_1, *)$ be the element determined by the

mapped in 2-sphere mentioned above which corresponds to D. Adjust F as follows: Choose a map $\varphi : S^2 \to (J_1, *)$ which represents $-\alpha \in \pi_2(J_1, *)$. In mapping in the disk D via F, alter the map so that it "runs over" this representative. That is, choose a point in D in the preimage of $*$, $F^{-1}(*)$, and wedge a copy of S^2 to D at this point. See figure 3.5. Map this new space into A by using F on the disk D and φ on the copy of S^2 in the natural way to obtain a new map that we'll label F'. We can think of F' as a map on the disk D again without harm. See figure 3.6. There are several crucial observations to be made:

Observation (1): Since the image under φ of the wedged copy of S^2 is contained in J_1, the composition $r_1 \circ F'$ has no effect on this portion of the new map F'. That is, $r_1 \circ F'|$ wedged 2-sphere $\equiv \varphi$. In fact, and more importantly, the strong deformation retraction fixes this portion of the map at each parameter level.

Observation (2): $r_2 \circ F'$ maps this portion of D that "corresponds to the 2-sphere" to a point, namely the 0-cell $* \in J_2$.

The first observation shows that with this adjustment to F', the new element of $\pi_2(J_1, *)$ obtained which corresponds to the augmented 2-cell D of K is the trivial element. Hence, the map extends over a 3-cell and we can homotope $(r_1 \circ F')(D)$ across this mapped-in 3-cell so that it maps onto the corresponding 2-cell D' of J_1. Doing this for each 2-cell of K shows how to adjust F to a better map (still called F'). In this new map, $(r_1 \circ F')$ is homotopic to the natural identification of K with J_1.

However, we must adjust F' farther so that we obtain a homotopy equivalence "on the other side" as well. This is where the second observation comes in. The second observation shows that we may perform the same type of changes to F' so as to obtain a final new map f so that when mapping in the 2-cells of K, the elements

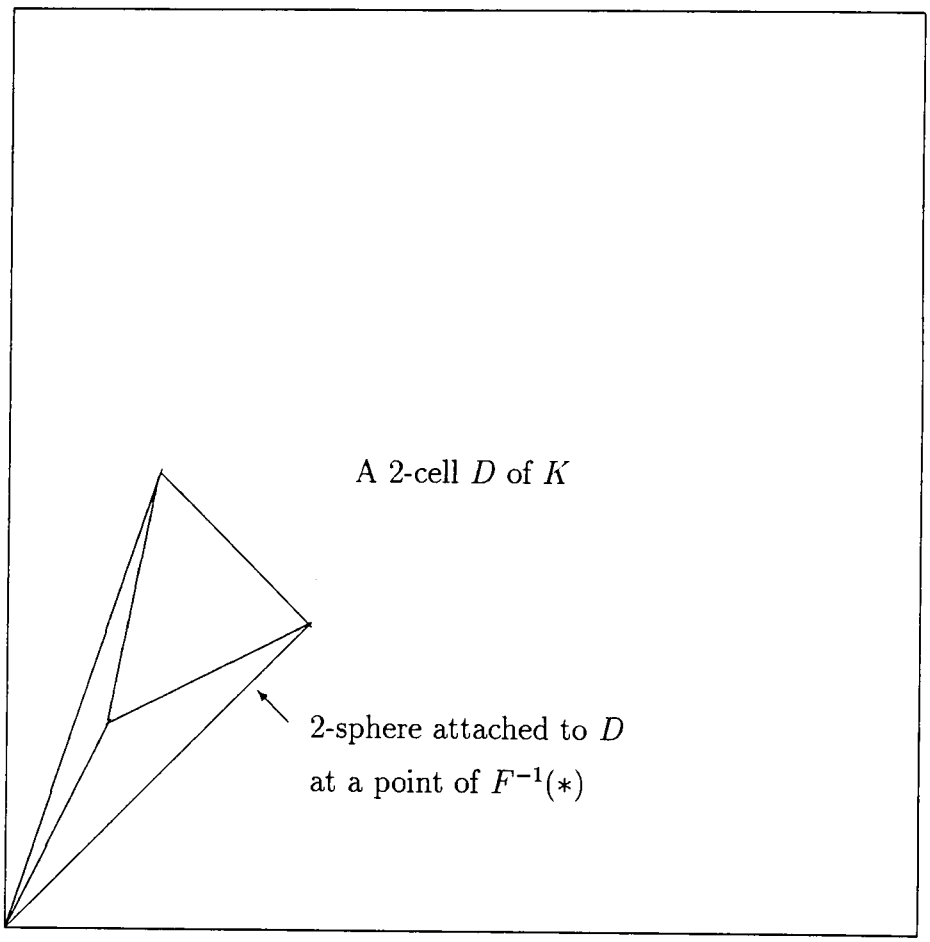


Figure 3.5: Wedging a 2-sphere to D

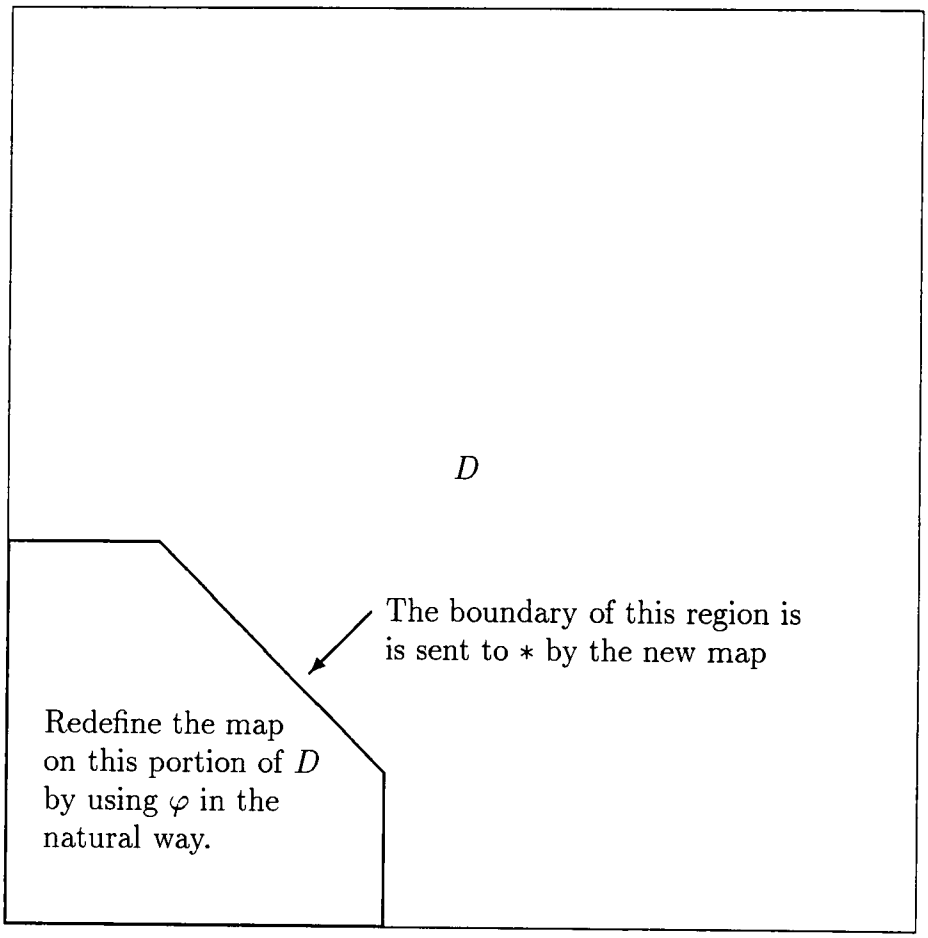


Figure 3.6: A new map on D

of $\pi_2(J_2, *)$ obtained in the symmetrical fashion corresponding to the 2-cell D are trivial when using the target space $A \cup_{J_1} cJ_1$ and *still* have the desired homotopy equivalence with natural identification of K with J_1 when using $A \cup_{J_2} cJ_2$; the point is that because the images of any 2-spheres wedged to the original disk D are each contained in J_1 or J_2 exclusively along with the fact that $r_i(J_i) = *$, no harm is done to the "other side" when making these adjustments. That is, any 2-sphere wedged to D that maps into J_2 has no effect when improving the map so as to obtain a homotopy equivalence to the identification of K with J_1 in $A \cup_{J_2} cJ_2$. Similarly, any wedged 2-sphere with image in J_1 has no effect in deforming $A \cup_{J_1} cJ_1$ to J_2. So, we can alter the map F to a better map, f, in this fashion. Then, f has the desired features. □

3.2 Examples in Dimensions $n \geq 6$

We'll now produce examples of contractible manifolds with disjoint spines in dimensions $n \geq 6$.

Theorem 3.2 *Let K be a finite acyclic 2-complex with nontrivial fundamental group. Then, for $n \geq 6$, the Newman contractible manifold $M = New(K, n)$ has disjoint spines.*

Proof: We assume that K is a finite acyclic 2-complex with 1 0-cell and with $\pi_1 = \pi_1(K) \neq 1$. Let A be the PL CW 3-complex associated with K provided by the construction of Theorem 3.1.

Step 1: Geometrical Setup

Let $n \geq 6$ and let N be a regular neighborhood of A in R^n using Corollary 2.1. Then, $X = N \times I$ is a regular neighborhood of a copy of A in R^{n+1} which

contains copies of A in ∂X at the 0 and 1 levels of $N \times I$. Moreover, X is a PL homology $(n+1)$-cell and so its boundary is a PL homology n-sphere. Let K_1 denote the copy of K corresponding to $J_1 \times \{1\} \subset A \times \{1\}$ and let K_2 denote the copy of K corresponding to $J_2 \times \{0\} \subset A \times \{0\}$ (where J_1 and J_2 are the copies of K used in producing A). Then K_1, K_2 are disjoint and we can choose regular neighborhoods of them $N_1 \subset (N \times \{1\}), N_2 \subset (N \times \{0\})$ respectively.

Now, in ∂X, remove the interiors of N_1 and N_2. Since N_1, N_2 are each regular neighborhoods of an acyclic 2-complex in ∂X, each is a PL homology n-cell whose boundary is a PL homology $(n-1)$-sphere. Let C_1, C_2 be contractible manifolds with boundaries homeomorphic to $\partial N_1, \partial N_2$ respectively. (Take $C_i \approx M$.) Let $\sum = \partial X - int(N_1 \cup N_2) \cup_\partial (C_1 \cup C_2)$ using the usual adjunction space notation where ∂ is the union of the homeomorphisms $\partial_1 : \partial C_1 \to \partial N_1$ and $\partial_2 : \partial C_2 \to \partial N_2$.

We now check that $\pi_1(\sum) = \{1\}$: The inclusion $\partial X \hookrightarrow X$ is a π_1-isomorphism by general position. Furthermore, since any element $g \in \pi_1(A)$ can be homotoped into the 1-skeleton $A^{(1)}$, it follows from the construction of A and X that $\pi_1(\partial X)$ is carried by $N_1 \cup N_2 \cup \alpha$ where α is any arc in $\partial X - (intN_1 \cup intN_2)$ connecting ∂N_1 to ∂N_2. But, since any loop in N_i can be homotoped off the 2-dimensional spine K_i of N_i, actually, $\pi_1(\partial X)$ is carried by $\partial N_1 \cup \partial N_2 \cup \alpha$. Attaching the contractible manifolds C_1 and C_2 via ∂ gives rise to a null homotopy of any preassigned loop in $\partial N_1 \cup \partial N_2 \cup \alpha$. Hence $\pi_1(\sum) = \{1\}$.

Since $\sum$ is a simply-connected PL homology n-sphere with $n \geq 6$, it follows from the High Dimensional PL Poincaré Theorem that $\sum \approx S^n$. Moreover, for $i = 1, 2$, $C_i \cup_{\partial_i} N_i \approx S^n$ as well; we took $C_i \approx M = New(K, n)$. Hence, we can attach an $(n+1)$ ball B_i^{n+1} to X along N_i by a homeomorphism, call it $\hat{\partial}_i$, which sends ∂B_i^{n+1} homeomorphically onto $C_i \cup_{\partial_i} N_i$.

Claim: $K_1 \hookrightarrow X \cup_{\hat{\partial}_2} B_2^{n+1}, K_2 \hookrightarrow X \cup_{\hat{\partial}_1} B_1^{n+1}$ are homotopy equivalences.

proof of claim: We proceed to check the first inclusion. We'll verify the claim by showing that $X \cup_{\widehat{\partial_2}} B_2^{n+1}$ collapses to a complex which is homeomorphic to the mapping cylinder of the inclusion map of J_1 into $A \cup_{J_2} cJ_2$ with K_1 playing the role of the base of the mapping cylinder. Since $A \cup_{J_2} cJ_2$ strong deformation retracts to J_1 by Theorem 3.1, it will follow from this that $X \cup_{\widehat{\partial_2}} B_2^{n+1}$ strong deformation retracts to K_1 too, thus verifying the claim.

First, note that we can collapse the portion of B_2^{n+1} not meeting X to an interior point spine from the boundary by the cone structure of the ball leaving cN_2 attached to X. Also, $A \times \frac{1}{2}$ is a spine of X. By collapsing along fiber lines, $X = N \times I \searrow N \times \frac{1}{2}$. Also, $N \times \frac{1}{2} \searrow A \times \frac{1}{2}$. So, we can collapse X to $(J_1 \times [0, \frac{1}{2}]) \cup (N \times \frac{1}{2}) \cup (N_2 \times [\frac{1}{2}, 1])$. Moreover, because $N_2 \searrow K_2$, there are collapses (from the sides) of cN_2 to $(cK_2 \cup N_2)$ (in what was the ball) and, furthermore, of $(N_2 \times [\frac{1}{2}, 1])$ to $(K_2 \times [\frac{1}{2}, 1])$ (vertically). Finally, the collapse $N \times \frac{1}{2} \searrow A \times \frac{1}{2}$ gives rise to a complex which is homeomorphic to the mapping cylinder described above and strongly deforms to its base, i.e., to K_1.

Through the use of the symmetrical argument for K_2, $K_2 \to X \cup_{\widehat{\partial_1}} B_1^{n+1}$ is a homotopy equivalence as well.

Remark: By collapsing the mapping cylinders obtained above onto their tops, observe that $X \cup_{\widehat{\partial_i}} B_i^{n+1}$ has a spine which is homeomorphic to $A \cup_{J_i} cJ_i$.

Step 2: Constructing a "diagonal" copy of K in ∂X.

Let f denote the map $f : K \to A$ constructed in Proposition 3.1 with respect to K and A. Recall that f has the features that $f_i : K \to A \cup_{J_i} cJ_i$ is a homotopy equivalence (where $f_i(x) = f(x)$ for $x \in K$).

For this map f, apply Lemma 2.3 to assume the mapping cylinder M_f is embedded in X with top precisely $A \times \{\frac{1}{2}\}$ and the base in ∂X. Moreover, we can

assume that the base misses $N_1 \cup N_2$ in ∂X by using general position in ∂X and applying general position in X again to reembed the "fiber-portion" of the mapping cylinder. (In fact, the only important embedding ingredient of this mapping cylinder is that the base is embedded in $\partial X - (N_1 \cup N_2)$ and the top goes homeomorphically onto $A \times \frac{1}{2}$.). Let K_3 denote the copy of K in ∂X which corresponds to the base of the mapping cylinder and let N_3 be a regular neighborhood of K_3 in ∂X which also misses $N_1 \cup N_2$.

Then, we notice the following: The inclusion map of K_3 into $X \cup_{\widehat{\partial 2}} B_2^{n+1}$ is homotopic to the inclusion of $J_1 \times \frac{1}{2}$ into the same space: The inclusion map of K_3 can be homotoped down the fibers of the embedded mapping cylinder, M_f, in $X \subset X \cup_{\widehat{\partial 2}} B_2^{n+1}$ to a map whose image is contained in $A \times \{\frac{1}{2}\}$. Since this space has a spine obtained by attaching cJ_2 to $A \times \frac{1}{2}$ by the remark in Step 1, the inclusion of K_3 into $X \cup_{\widehat{\partial 2}} B_2^{n+1}$ is homotopic to the inclusion of $J_1 \times \frac{1}{2}$ into $X \cup_{\widehat{\partial 2}} B_2^{n+1}$ (recalling the argument of Proposition 3.1). The latter inclusion is a homotopy equivalence. Hence, $K_3 \hookrightarrow X \cup_{\widehat{\partial 2}} B_2^{n+1}$ is a homotopy equivalence too.

Similarly, the inclusion of K_3 into $X \cup_{\widehat{\partial 1}} B_1^{n+1}$ is a homotopic to the inclusion of $J_2 \times \frac{1}{2}$ into the same space and, likewise, is a homotopy equivalence.

Furthermore, it follows from the above construction shows K_1, K_2 are homotopic to the inclusion of $J_1 \times \frac{1}{2}, J_2 \times \frac{1}{2}$ respectively in X, hence in $X \cup_{\widehat{\partial 2}} B_2^{n+1}, X \cup_{\widehat{\partial 1}} B_1^{n+1}$ again respectively. This observation will be of use to us later.

Step 3: Identifying Disjoint Spines

By an argument given in Step 1, $\sum \approx S^n$. By Step 2, K_3 is a copy of K in ∂X with a regular neighborhood N_3 which misses $N_1 \cup N_2$. So, we can think of N_3 as a subset of $\sum$ in a natural way. By definition then, $\sum - int N_3$ is precisely $New(K, n)$. Let M denote $\sum - int N_3$.

Claim: C_1 and C_2 are disjoint spines of M.

Remark: It is here that there is a slight strengthening of the output of the theorem. Using the construction techniques of [Gui], one could proceed to check that $(M - intC_i, \partial C_i, \partial M)$ are h-cobordisms for $i = 1, 2$ when $n \geq 7$ by use of general position arguments. This would show that C_1, C_2 are disjoint pseudospines of M. (Recall that a compactum Y in the interior of a manifold M is a pseudospine of M if $M - Y \approx \partial M \times [0, 1)$.) Since it is known that an arbitrary neighborhood of a pseudospine of M contains a spine of M (Lemma 2.2), this would provide disjoint spines when $n \geq 7$. However, the above claim will verify that C_1, C_2 are not only pseudospines of M, but, in fact, spines of M and works in all dimensions $n \geq 6$.

proof of claim: It suffices by symmetry to verify that C_1 is a spine of M. Let $Y = X \cup_{\widehat{\partial 2}} B_2^{n+1}$. Then recall that Y is a regular neighborhood of a copy of $A \cup_{J_2} cJ_2$ by Step 1. Let N be a regular neighborhood of $J_1 \times \frac{1}{2}$ in the interior of Y. Let $W = Y - intN$. By Corollary 2.2, $(W, \partial Y, \partial N)$ is an h-cobordism. Moreover, the torsion associated with this h-cobordism $\tau(W, \partial N)$ is the torsion $\tau(Y, N)$ (by [Coh, Excision Lemma 20.3]). However, this torsion is the torsion of the pair $(A \cup_{J_2} cJ_2, J_1)$ [Coh, Corr. 22.7]. It was noted that the latter torsion was trivial in the remark following Theorem 3.1. Hence, the PL s-cobordism Theorem applies as $\dim W \geq 7$ and it follows that $W \approx \partial N \times [0, 1]$. Consequently, W is a collar on N and Y is a regular neighborhood of $J_1 \times \frac{1}{2}$ too.

Since Y is a PL homology cell, attaching a contractible manifold along ∂Y gives S^{n+1}. This observation is simply a check that the PL structure of Y is that of a regular neighborhood of K in R^{n+1}.

Now, since any two embeddings of K into R^{n+1} are isotopic when $n + 1 \geq 6$ by trivial range results, regular neighborhoods are unique up to homeomorphism there. We'll use this fact to form a preferred model for Y. Let $e : K \to R^n$ be an embedding of K in R^n. Let $L = e(K)$ and let P be a regular neighborhood of L in

R^n. Crossing with $[0,1]$ gives rise to a preferred model $Z = P \times [0,1]$. Hence, we can identify Y with Z via a homeomorphism h which carries the spine $J_1 \times \frac{1}{2}$ to the spine of Z, call it J. (So, $J \approx K$.) This homeomorphism takes ∂Y to ∂Z.

Note that in ∂Y we have two copies of K, namely K_1 and K_3, each of whose inclusion into Y is homotopic to the inclusion of $J_1 \times \frac{1}{2}$ in Y (Step 2). This data carries over to $Z = h(Y)$.

Now the idea is to use the natural product structure in ∂Z between $\partial(P \times \{0\})$ and $\partial(P \times \{1\})$ to produce a product structure between $h(\partial N_1)$ and $h(\partial N_3)$. To do this note that there exist homotopies between the inclusion maps of any pair of the four copies of K we see in ∂Z: $h(K_1), h(K_3), L \times \{0\}$, and $L \times \{1\}$ (because each inclusion into Z is homotopic to the inclusion of the spine J in Z). General position in Z allows us to push the homotopies between pairs of copies of K into the boundary.

Isotoping a homotopy between $h(K_1)$ and $L \times \{0\}$ off of $L \times \{1\}$ rel $(h(K_1) \cup L \times \{0\})$ and applying trivial range results allows us to assume that $h(K_1)$ is a spine of $P \times \{0\} \cup_{\partial P \times \{0\}} (\partial P \times [0,1])$ using the collar structure on $(P - L) \times \{1\}$. Moreover, after isotoping a homotopy between $h(K_3)$ and $L \times \{1\}$ rel $(h(K_3) \cup L \times \{1\})$ off of a small regular neighborhood Ω of $h(K_1)$, trivial range results show that $h(K_3)$ is a spine of $\partial Z - \Omega$. Hence, choosing regular neighborhoods of $h(K_1)$ and $h(K_3)$ contained in $(h(N_1) \cap \Omega)$ and $h(N_3)$ respectively, we obtain a product structure between the boundaries of these regular neighborhoods.

Isotopy uniqueness of regular neighborhoods then reveals a product structure between $h(\partial N_1)$ and $h(\partial N_3)$. Applying h^{-1}, we obtain a product structure between ∂N_1 and $\partial M (= \partial N_3)$ in $M - intN_1 \subset \partial Y$ showing that C_1 is a spine of M as desired.

Symmetry then shows that C_2 is also a spine of M and so, M has a pair of disjoint spines. □

3.3 Some 5-Dimensional Considerations

We now proceed to develop arguments which produce contractible manifolds with disjoint spines in dimension 5. While many of the underlying ideas and motivations are the same as for the $n \geq 6$ case, one must work slightly harder to develop the arguments needed. Part of the reason for this is that we are no longer in the trivial range for a 2-complex. Hence, even though the preservation of the necessary features for producing examples in dimension 5 will be exposed, the techniques of our previous work must be modified to extract these features. The case is brought to point by use of several results from Chapter II whose underlying role is to approximate the part of the argument for the $n \geq 6$ case which relies on trivial range results. Since we are no longer in the trivial range for a 2-complex when dealing with a 5-manifold, we don't know as yet that the contractible manifold constructed from the Newman Method is unique. However, the known h-cobordism theorems imply that a contractible manifold of dimension at least 5 is uniquely determined up to homeomorphism by its boundary (Theorem 1.2). Coupling this fact together with Proposition 2.1, we observe that if K is a 2-complex, $New(K, 5)$ is well-defined up to homeomorphism.

Regarding notation involving expanding and collapsing of polyhedra, we write $K \nearrow\searrow K'$ to mean that K' can be obtained from K by a finite sequence of expansions and collapses. This situation arises in the 5-dimensional case.

Theorem 3.3 *Let K be a finite acyclic 2-complex with nontrivial fundamental group. Then, $New(K, 5)$ has disjoint spines.*

Proof: Let K be a finite acyclic 2-complex with 1 0-cell and nontrivial fundamental group. Let A denote the 3-complex constructed from K as in Theorem 3.1.

Step 1: Geometrical Setup

Let N be a regular neighborhood of A in R^6 using Corollary 2.1. Regard N as a handle decomposition with 1 handle for each cell of A. Then, the 0,1, and 2 handles form a 6-dimensional regular neighborhood of the 2-skeleton $A^{(2)}$ of A. Since regular neighborhoods of $A^{(2)}$ are unique up to isotopy in R^6 by trivial range results, we can regard the regular neighborhood of $A^{(2)}$ formed from the r-handles with $r \leq 2$ as standard. Denote this regular neighborhood by $N^{(2)}$. Then, embed $A^{(2)}$ in R^5 and let P denote a regular neighborhood of the image. Let $Z = P \times I$. Then, $Z \approx N^{(2)}$. In particular, we see copies of $A^{(2)}$ in the 0 and 1 levels of Z which then carry over to $N^{(2)}$ under some homeomorphism.

Now, to form N, there are some additional 3-handles attached to $N^{(2)}$. Since the attaching spheres of these 3-handles are 2-dimensional, general position in $\partial N^{(2)}$ allows us to isotope the attaching maps of the 3-handles off of the two disjoint copies of $A^{(2)}$ in $\partial N^{(2)}$. Now, attach new 3-handles to $N^{(2)}$ in this fashion. Using the mapping cylinder notion of a regular neighborhood, observe that by collapsing the 3-handles onto their cores and collapsing the mapping cylinder $N^{(2)}$ as much as possible, the resulting manifold X is a regular neighborhood of a 3-complex, call it A', whose 2-skeleton is precisely $A^{(2)}$. Moreover, $X \approx N$ as we've simply isotoped attaching maps of the 3-handles. It follows from simple-homotopy theory that $X \nearrow\searrow N$ rel $(N^{(2)})$ by collapsing the 3-handles of X onto their cores, homotoping the attaching maps of these 3-cells to those of N (with the 3-handles of N collapsed to their cores), and rethickening to obtain N. The useful feature of this is that $A' \nearrow X \nearrow\searrow N \searrow A$ rel $A^{(2)}$. We shall return to this observation shortly.

For now, identify copies of K, say K_1 and K_2 in ∂X so that if $r : \partial X \to A'$ is the map with X identified with M_r, $r|K_1 : K_1 \to A'$ takes K_1 homeomorphically onto $J_1 \subset A^{(2)} \subset A'$ in the natural way. Similarly, choose K_2 so that r takes K_2 homeomorphically onto $J_2 \subset A'$. Finally, choose regular neighborhoods N_1 and N_2 of K_1 and K_2, respectively, in ∂X which each miss the attaching regions of all of the 3-handles. (So, N_1, N_2 are actually regular neighborhoods of K_1 and K_2 in $\partial N^{(2)}$).

Removing $int(N_1 \cup N_2)$ from ∂X and replacing by contractible manifolds C_1, C_2 along their boundaries (as in the $n \geq 6$ case) gives rise to a 6-dimensional PL homology cell whose boundary is simply connected and, hence, is I^6 by the PL Poincaré Theorem. Let $\Sigma = (\partial X - int(N_1 \cup N_2)) \cup_\partial (C_1 \cup C_2)$ (where $\partial_i : \partial C_i \to \partial N_i$ is the identification map on the boundaries and ∂ is the union of the maps ∂_1 and ∂_2). So, $\Sigma \approx S^5$.

Let B_i^6 be a 6-ball attached to X along N_i. Let $\hat{\partial}_i : \partial B_i^6 \to X \cup_{\partial_i} C_i$ be an attaching map for B_i^6 for $i = 1, 2$

Claim: $K_1 \hookrightarrow X \cup_{\hat{\partial}_2} B_2^{n+1}, K_2 \hookrightarrow X \cup_{\hat{\partial}_1} B_1^{n+1}$ are homotopy equivalences.
proof of claim: By collapsing the 3-handles onto their cores and using the mapping cylinder feature of the regular neighborhood $N^{(2)}$, one checks that $X \cup_{\hat{\partial}_2} B_2^6$ PL collapses to a complex homeomorphic with the mapping cylinder of the inclusion of J_1 into $A' \cup_{J_2} cJ_2$ with the role of the base being played by K_1. Since the inclusion of J_1 in A is a simple-homotopy equivalence and $A \nearrow\searrow A'$ rel $(J_1 \cup J_2)$, so is the inclusion of J_1 into A'. Consequently,

(1) $K_1 \hookrightarrow X \cup_{\hat{\partial}_2} B_2^6$ is a homotopy equivalence as claimed.

(2) The inclusion of J_1 into $X \cup_{\hat{\partial}_2} B_2^6$ is a homotopy equivalence as well (by collapsing the mapping cylinder onto its top).

The symmetrical argument for K_2 shows that:

(1) $K_2 \hookrightarrow X \cup_{\widehat{\partial_1}} B_1^6$ is also a homotopy equivalence.

(2) The inclusion of J_2 into $X \cup_{\widehat{\partial_1}} B_1^6$ is a homotopy equivalence as well.

Step 2: Constructing a "diagonal" copy of K in ∂X.

The techniques used to produce a map $f : K \to A$ in Proposition 3.1 with the feature that $f_i : K \to A \cup_{J_i} cJ_i$ defined by $f_i(x) = f(x)$ for $x \in K$ is a homotopy equivalence for $i = 1, 2$ may be used equally well to produce a map $f' : K \to A'$ so that $f_i' : K \to A' \cup_{J_i} cJ_i$ defined by $f_i'(x) = f'(x)$ for $x \in K$ is a homotopy equivalence for $i = 1, 2$. The key elements used in the proof of Proposition 3.1 were that A consisted of $J_1 \vee J_2$ together with some attached 2 and 3-cells and that $A \cup_{J_i} cJ_i$ strongly deformed to cJ_i' (where $i' = 1$ if $i = 2$, $i' = 2$ if $i = 1$). These features are preserved in passing from A to A'. Hence, we can obtain a map f' with the desired features in a likewise manner.

With this map in hand, we apply the techniques of Lemma 2.3 to try to embed $M_{f'}$ in X with the base in ∂X missing $N_1 \cup N_2$ and with the top precisely A'. The last general position application in the lemma fails to provide an embedding of M_f on the fiber-portion of the mapping cylinder for dimensional reasons now; however, the rest of the argument goes through to provide a map $g : M_{f'} \to X$ with the features that the top of $M_{f'}$ goes homeomorphically onto A' and the base is embedded in ∂X and misses $N_1 \cup N_2$ there. (It was mentioned in Step 2 of Theorem 3.2 that this was really all we needed there as well.) Let K_3 denote the copy of K in ∂X which corresponds to the base of the mapped-in mapping cylinder. Let N_3 be a regular neighborhood of K_3 in ∂X which also misses $N_1 \cup N_2$.

Then, as $\Sigma \approx S^5$, we can think of K_3 as embedded in Σ in a natural way.

From this construction, it follows that the inclusion $K_3 \hookrightarrow X \cup_{\widehat{\partial_1}} B_1^6$ is a homotopy equivalence as is the inclusion $K_3 \hookrightarrow X \cup_{\widehat{\partial_2}} B_2^6$ (by homotoping the inclusion map of K_3 down the fibers of the (singular) mapping cylinder and using the fact that $A' \cup_{J_i} cJ_i$ deforms to J_i' where $i' = 1$ if $i = 2$ and $i' = 2$ if $i = 1$ again). In point, the feature that we utilize is not just that the inclusion of K_3 into $X \cup_{\widehat{\partial_i}} B_i^6$ is a homotopy equivalence, but the inclusion is homotopic to the inclusion of $J_{i'}$ into the same space. Note that also $K_{i'} \hookrightarrow X \cup_{\widehat{\partial_i}} B_i^6$ is homotopic to the inclusion of $J_{i'}$ by our construction techniques. We'll make use of these facts shortly.

Step 3: Identifying Disjoint Spines

By Steps 1 and 2, $\Sigma - intN_3 \approx New(K, 5)$. Let M denote $\Sigma - intN_3$. The claim is that C_1 and C_2 are disjoint pseudospines of M. It suffices by symmetry to observe that C_1 is a pseudospine of M.

Let $Y = X \cup_{\widehat{\partial 2}} B_2^{n+1}$. Then recall that Y is a regular neighborhood of a copy of $A' \cup_{J_2} cJ_2$ by Step 1. Let N be a regular neighborhood of $J_1 \subset A'$ in the interior of Y. Let $W = Y - intN$. By Theorem 2.1, $(W, \partial Y, \partial N)$ is an h-cobordism. Moreover, the torsion associated with this h-cobordism, $\tau(W, \partial N)$, is the torsion of the pair $(A' \cup_{J_2} cJ_2, J_1)$, that is, $\tau(A' \cup_{J_2} cJ_2, J_1)$. Because we've already observed that $A \nearrow\searrow A'$ rel $(J_1 \cup J_2)$ and because $\tau(A \cup_{J_2} cJ_2, J_1) = 0$, it follows that $\tau(A' \cup_{J_2} cJ_2, J_1) = 0$ as well. Hence, the PL s-cobordism theorem applies (as $\dim Y = 6$) to show that Y is, in fact, a regular neighborhood of J_1 in R^6.

Because J_1 is 2-dimensional, we can use trivial range results again to see that Y is homeomorphic to $P \times I$ where P is some regular neighborhood of an embedding of K in R^5. The hypotheses of Theorem 2.1 applied to K_1, K_3 in ∂Y are met and we obtain an h-cobordism $(Y - intN_1 - intN_3, \partial N_1, \partial N_3)$. In particular,

then, $(Y - intN_3) - N_1 \approx \partial N_3 \times [0,1)$. Hence, $M - C_1 \approx \partial M \times [0,1)$. So, C_1 is a pseudospine of M.

Then, C_2 is a pseudospine of M as well by a symmetrical argument. Hence, M contains a pair of disjoint pseudospines.

The result is completed by using the fact that an arbitrary neighborhood of a pseudospine contains an actual spine (Lemma 2.2). □

3.4 Some Final Considerations

After obtaining examples in dimensions $n \geq 5$, we finish with a natural conjecture which arises concerning disjoint spines in the stable range for Newman Manifolds...

Conjecture: *If K is a finite, acyclic, non-simply connected, k-complex, then, $New(K,n)$ contains a pair of disjoint spines if $n \geq 2k+1$.*

Of course other issues exist as well. While great effort has been expended in producing examples of compact, contractible manifolds containing disjoint spines, a subtle issue remains...

Question: *Do there exist any contractible manifolds which do not have disjoint spines?*

For now, the answer to this question remains elusive.

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