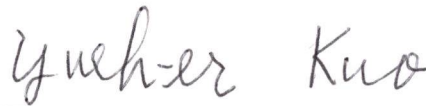


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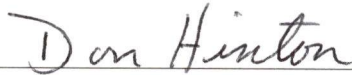
I am submitting a thesis written by Roy Michael Markham entitled "The Transportation Problem: Theory, Methods and Applications." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.



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Yueh-er Kuo, Major Professor

We have read this thesis  
and recommend its acceptance:



Accepted for the Council:



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Associate Vice Chancellor and  
Dean of the Graduate School

**THE TRANSPORTATION PROBLEM:  
THEORY, METHODS AND APPLICATIONS**

The Thesis  
Presented for the  
Master of Science  
Degree

The University of Tennessee, Knoxville

Roy Michael Markham

May 1997

## DEDICATION

This thesis is dedicated to my wife

Deborah J. Markham

for all her love and support,

to my parents

Roy Daniel Markham

and

Alma R. Markham

for giving me life and teaching me love and respect

and my Lord and Savior

Jesus Christ.

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## ABSTRACT

An important part of linear programming is the transportation problem.

The transportation problem minimizes costs by selecting the optimal path for transporting goods from a set of origins to a set of destinations. A discussion of linear programming, and the simplex method are presented. This is followed by theory supporting the balanced transportation problem, methods of finding the optimal solution, variations of the transportation problem and selected applications.

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# CHAPTER 1

## INTRODUCTION

In this paper, we will discuss a specific problem in linear programming called the transportation problem. We will begin by giving theoretical support for the simplex method; a method for solving the classical linear programming problem. This will be followed by an example using the simplex method. We will then give theoretical support to show that a basic feasible solution to the transportation problem exists followed by several methods for determining a basic feasible solution. We will continue with giving theoretical support that an optimal solution to the transportation problem exists followed with two techniques for finding the optimal solution. We will conclude with looking at some special cases involving the transportation problem. The main two sources in this paper are Gass [1] and Dantzig [2].

The classical transportation problem arises in situations in which we must determine an optimum schedule of distribution that:

1. originate at sources where fixed stockpiles of a commodity are available;
2. are sent directly to their final destinations where fixed amounts are required;
3. exhaust the stockpiles and fulfill the demand such that total demand equals total supply; and finally the cost must

4. satisfy a linear objective function; that is, the cost of each shipment is proportional to the amount shipped, and the total cost is the sum of the individual costs [2].

L. V. Kantorovich, in 1939, showed that a class of problems very similar to the classical transportation problem has a wide variety of applications concerned typically with the allotment of tasks to machines whose costs and rates of production vary by task and machine type. The transportation problem was first described in its original form by F. L. Hitchcock in 1941. It was developed in more detail by T. C. Koopsmans in 1949. The linear programming formulation of the transportation problem was completed and first stated by George B. Dantzig in 1951. Several variations of solving the transportation problem have been developed since. Among these are the northwest-corner method, the short-cut method, and Russell's approximation [3].

T. C. Koopmans, as a member of the Combined Shipping Board during World War II, used solutions of the transportation problem to help reduce overall shipping times, for the shortage of cargo ships constituted a critical bottleneck.

The conditions required for the transportation problem make it useful over a wide range of topics. These include finance, electrical power distribution, economics, nutrition, waste disposal, and of course transportation.

The following notational conventions will be used throughout this paper: All matrices will be denoted by capital letters in bold type and vectors will be

denoted by small case and capital letters in bold type. The symbols  $I$  and  $O$  represent the identity matrix and the matrix with all zero entries respectively. The  $i$ th component of a vector  $x$  will be denoted  $x_i$ . Throughout the paper, we will use the convention of treating a vector as a row when it appears on the left of a multiplication, and as a column when it appears on the right of a multiplication. Finally an equation or system may be named by use of parenthesis and numbered, i.e. (1.1a).

Now let us begin by defining a few terms. Unless otherwise stated all of the following terms comes from Gass [1].

**Definition 1.1** An optimal solution is a solution that satisfies both the conditions of the problem and the given objective.

**Definition 1.2** A convex combination of the points  $U_1, U_2, \dots, U_k$  is a point  $U = \alpha_1 U_1 + \alpha_2 U_2 + \dots + \alpha_k U_k$  where the  $\alpha_i$  are scalars,  $\alpha_i \geq 0$  and  $\sum_i \alpha_i = 1$ .

**Definition 1.3** A subset  $C$  of  $E_n$  is convex if for all pairs of points  $U_1$  and  $U_2$  in  $C$ , any convex combination  $U = \alpha_1 U_1 + \alpha_2 U_2$  is also in  $C$ .

**Definition 1.4** A point  $U$  in a convex set  $C$  is an extreme point if  $U$  cannot be expressed as a convex combination of any other two distinct points in  $C$ .

**Definition 1.6** A simplex is an  $n$ -dimensional convex polyhedron having exactly  $n+1$  vertices.

**Definition 1.8** The linear-programming problem is a vector  $(x_1, x_2, \dots, x_j, \dots, x_n)$  which minimizes the linear equation

$$\mathbf{Z} = \mathbf{C}_1\mathbf{X}_1 + \mathbf{C}_2\mathbf{X}_2 + \dots + \mathbf{C}_j\mathbf{X}_j + \dots + \mathbf{C}_n\mathbf{X}_n \quad (1.1a)$$

subject to the following constraints:

$$\begin{aligned} & \mathbf{a}_{11}\mathbf{x}_1 + \mathbf{a}_{12}\mathbf{x}_2 + \dots + \mathbf{a}_{1j}\mathbf{x}_j + \dots + \mathbf{a}_{1n}\mathbf{x}_n = \mathbf{b}_1 \\ & \mathbf{a}_{21}\mathbf{x}_1 + \mathbf{a}_{22}\mathbf{x}_2 + \dots + \mathbf{a}_{2j}\mathbf{x}_j + \dots + \mathbf{a}_{2n}\mathbf{x}_n = \mathbf{b}_2 \\ & \dots\dots\dots \\ & \mathbf{a}_{i1}\mathbf{x}_1 + \mathbf{a}_{i2}\mathbf{x}_2 + \dots + \mathbf{a}_{ij}\mathbf{x}_j + \dots + \mathbf{a}_{in}\mathbf{x}_n = \mathbf{b}_i \\ & \dots\dots\dots \\ & \mathbf{a}_{m1}\mathbf{x}_1 + \mathbf{a}_{m2}\mathbf{x}_2 + \dots + \mathbf{a}_{mj}\mathbf{x}_j + \dots + \mathbf{a}_{mn}\mathbf{x}_n = \mathbf{b}_m \end{aligned} \tag{1.2a}$$

and

$$x_j \geq 0 \quad j = 1, 2, \dots, n \quad (1.3a)$$

where  $a_i$ ,  $b_i$ , and  $c_i$  are given constants and  $m < n$ .

The linear-programming problem will also be referred to using one of the next three forms.

1. Minimize

$$z = \sum_{j=1}^n c_j z_j \quad (1.1b)$$

subject to the following constraints:

$$\sum_{j=1}^n a_{ij} x_j = b_i \quad i = 1, 2, \dots, m \quad (1.2b)$$

and

$$x_j \geq 0 \quad j = 1, 2, \dots, n \quad (1.3b).$$

2. Minimize

$$z = cX \quad (1.1c)$$

$$\text{where } c = (c_1 \ c_2 \ \dots \ c_n) \text{ and } X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

subject to the following constraints:

$$AX = b \quad (1.2c)$$

$$\text{where } A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix} \text{ and } b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{pmatrix}$$

and

$$X \geq 0 \quad (1.3c).$$

3. Minimize

$$z = cX \quad (1.1d)$$

subject to the following constraints:

$$x_1 P_1 + x_2 P_2 + x_3 P_3 + \dots + x_n P_n = P_0 \quad (1.2d)$$

where  $P_j$  is the  $j$ th column of the  $A$  matrix and  $P_0 = b$

and

$$X \geq 0 \quad (1.3d).$$

**Definition 1.9** A feasible solution to the linear-programming problem is a vector  $X = (x_1 \ x_2 \ \cdots \ x_n)^T$  which satisfies (1.2) and (1.3).

**Definition 1.10** A basic solution to (1.2),  $AX = b$ , is a solution obtained by setting  $n - m$  variables equal to zero and solving for the remaining  $m$  variables. The  $m$  variables are called basic variables.

**Definition 1.11** A basic feasible solution is a basic solution which also satisfies (1.3),  $X \geq 0$ ; that is, all basic variables are nonnegative.

**Definition 1.12** A nondegenerate basic feasible solution is a basic feasible solution with exactly  $m$  positive  $x_i$ ; that is all basic variables are positive.

**Definition 1.13** A minimum feasible solution is a feasible solution which also minimizes (1.1),  $z = cX$ .

**Definition 1.14** An optimal basic feasible solution is a basic solution that satisfies conditions (1.1), (1.2), and (1.3).

scalars  $\alpha$  and  $\beta$ . For example, let  $U = (9 \ 3)$ ,  $V = (6 \ 6)$ ,  $\alpha = \frac{1}{3}$ ,  $\beta = -\frac{2}{3}$  and  $f(X) = 2x_1 + x_2$ .  $X = \alpha U + \beta V = \frac{1}{3}(9 \ 3) + \left(-\frac{2}{3}\right)(6 \ 6) = (3 \ 1) + (-4 \ -4) = (-1 \ -3)$ . So  $f(X) = 2(-1) + (-3) = -2 - 3 = -5$ . Also  $\alpha f(U) = 7$  and  $\beta f(V) = -12$ .

The transportation problem is actually a more specific linear-programming problem. Its definition is as stated below:

**Definition 1.16** The transportation problem is stated as follows:

Find values for the variables  $x_{ij}$  which minimize

$$z = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} \quad (1.4)$$

subject to the following restraints:

$$\sum_{j=1}^n x_{ij} = a_i \quad i = 1, 2, \dots, m \quad (1.5)$$

$$\sum_{i=1}^m x_{ij} = b_j \quad j = 1, 2, \dots, n \quad (1.6)$$

and

$$x_{ij} \geq 0 \quad (1.7)$$

where each  $c_{ij}$ ,  $a_i$ , and  $b_j$  are given constants.

## CHAPTER 2

### THE LINEAR-PROGRAMMING PROBLEM AND THE SIMPLEX METHOD

Before we can adequately cover the transportation problem, it is necessary to discuss the linear-programming problem and the simplex method, the most famous method for solving the linear-programming problem. The simplex method involves using one basic feasible solution to the linear programming problem and systematically moving toward the optimal basic feasible solution. The basic feasible solutions can also be thought of as extreme points. We begin with one extreme point and "hop" from extreme point to extreme point until the optimal solution is found. The simplex method, actually, is one technique which can be used to solve the transportation problem, but can be long and tedious.

The linear programming problem as stated in Definition 1.8 is again as follows:

minimize

$$Z = c_1x_1 + c_2x_2 + \dots + c_jx_j + \dots + c_nx_n \quad (1.1a)$$

subject to the following constraints:



## Section 2.1 Preliminary Theorems

**Theorem 2.1** The set of all feasible solution of the linear-programming problem is a convex set.

**Proof:** If the set of solutions has only one element, the theorem obviously holds.

Assume that the set of solutions has more than one element. It is sufficient to show that every convex combination of any two feasible solutions is also a feasible solution. Let  $X_1$  and  $X_2$  be any two feasible solutions of the linear-programming problem such that  $AX_1 = b$  and  $AX_2 = b$  for  $X_1 \geq 0$  and  $X_2 \geq 0$ .

Now for any  $\alpha \in [0, 1]$ , let  $X = \alpha X_1 + (1 - \alpha)X_2$ . It follows that  $X \geq 0$  since  $X_1$  and  $X_2 \geq 0$  and  $X$  falls along the straight line between  $X_1$  and  $X_2$ . Now all that is left to show is that  $X$  is a feasible solution. Multiplying  $A$  and  $X$  gives  $AX = A[\alpha X_1 + (1 - \alpha)X_2] = A\alpha X_1 + AX_2 - A\alpha X_2 = \alpha AX_1 + AX_2 - \alpha AX_2 = \alpha b + b - \alpha b = b$ , and we are finished.

As stated before,  $K$  is the set of solutions, for  $X$ , to the linear-programming problem.  $K$  is a convex set and is determined by the intersection of the equations or constraints (1.2) and (1.3).  $K$  is void, a bounded convex polyhedron, or a convex polyhedron which is unbounded in some direction. We will assume  $K$  is bounded for the rest of the chapter.

**Theorem 2.2** The objective function,  $z = c_1x_1 + c_2x_2 + \dots + c_jx_j + \dots + c_nx_n$ , assumes its minimum at an extreme point of the convex set  $K$  generated by the set of feasible solutions to the linear-programming problem. If it assumes its minimum at more than one extreme point, then it takes on the same value for every convex combination of the particular points.

**Proof:** Since we have assumed  $K$  to be a bounded convex polyhedron,  $K$  has a finite number of extreme points. In two dimensions, it may appear like Figure 2.1.

Let's denote the objective function by  $f(X)$ , the extreme points by  $X_1, X_2, \dots, X_p$  and the minimum feasible solution by  $X_0$ . That is  $f(X_0) \leq f(X)$  for all  $X \in K$ .

- A. (i) If  $X_0$  is an extreme point, the first part of the theorem is true.
- (ii) Suppose  $X_0$  is not an extreme point, then we can write  $X_0$  as a combination of extreme points of  $K$  since . So we have  $X_0 =$

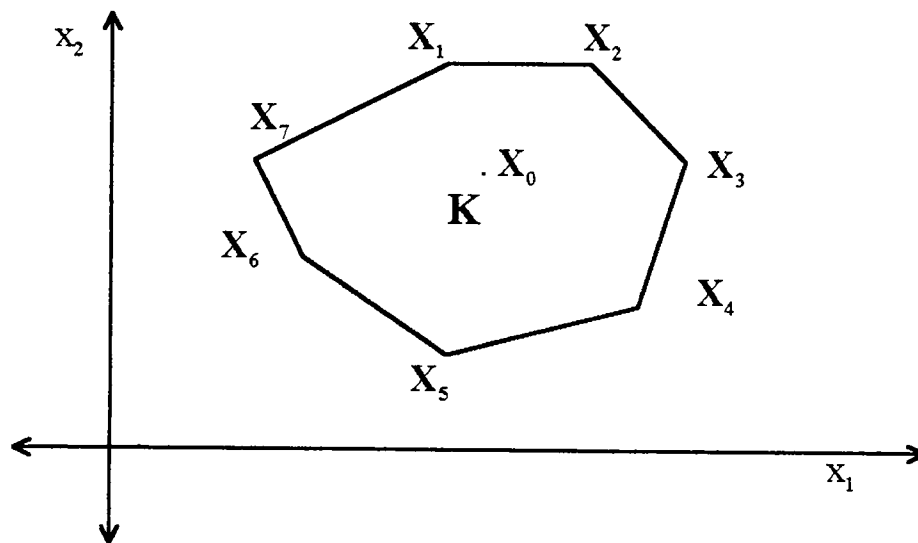


Figure 2.1 The Convex Set  $K$ .

$\sum_{i=1}^P \alpha_i X_i$  for  $\alpha_i \geq 0$  and  $\sum_{i=1}^P \alpha_i = 1$ . Now since  $f(X)$  is linear function, we have  $f(X_0) = f\left(\sum_{i=1}^P \alpha_i X_i\right) = f(\alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_P X_P) = \alpha_1 f(X_1) + \alpha_2 f(X_2) + \dots + \alpha_P f(X_P) = m$  (2.1), where  $m$  is the minimum of all  $X$  in  $K$ . Now let  $f(X_m) = \min f(X_i)$ . Since all  $\alpha_i \geq 0$ , we do not increase the sum of (2.1) if we substitute  $f(X_m)$  for each  $f(X_i)$ . Now since  $\sum_{i=1}^P \alpha_i = 1$ , we have  $f(X_0) \geq \alpha_1 f(X_m) + \alpha_2 f(X_m) + \dots + \alpha_P f(X_m) = f(X_m)$ . Since we assumed  $f(X_0) \leq f(X)$  for all  $X \in K$ ,  $f(X_0) = f(X_m) = m$ . Therefore there is an extreme point,  $X_m$ , at which  $f(X)$  obtains its minimum value. This completes the proof of the first part of the theorem.

B. Let  $f(X)$  assume its minimum value at more than one extreme point, say at  $X_1, X_2, \dots, X_q$ . That is  $f(X_1) = f(X_2) = \dots = f(X_q) = m$ . If  $X$  is any convex combination of  $X_j$  ( $j = 1, 2, \dots, q$ ), then  $X = \sum_{j=1}^q \alpha_j X_j$ ,  $\alpha_j \geq 0$ ,  $\sum_{j=1}^q \alpha_j = 1$ , by definition of a convex combination. Now by definition of a linear function and substitution,  $f(X) = f(\alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_q X_q) = \alpha_1 f(X_1) + \alpha_2 f(X_2) + \dots + \alpha_q f(X_q) = \sum_{j=1}^q \alpha_j m = m$ . This completes the second part of the proof for the theorem.

For the next theorem, we will use the fourth notation for the linear-programming problem. We recall the linear-programming problem in this notation is as follows:

Minimize

$$z = c X \quad (1.1d)$$

subject to

$$x_1 P_1 + x_2 P_2 + x_3 P_3 + \dots + x_n P_n = P_0 \quad (1.2d)$$

where  $P_j$  is the  $j$ th column of the  $A$  matrix and  $P_0 = b$

and

$$X \geq 0 \quad (1.3d).$$

**Theorem 2.3** If a set of  $k \leq m$  vectors  $P_1, P_2, \dots, P_k$  can be found that is linearly independent and such that  $x_1 P_1 + x_2 P_2 + \dots + x_k P_k = P_0$  and all  $x_i \geq 0$ , then the point  $X = (x_1, x_2, \dots, x_k, 0, \dots, 0)$  is an extreme point of the convex set of feasible solutions. Here  $X$  is an  $n$ -dimensional vector whose last  $n - k$  elements are zero.

**Proof:** Let  $K$  be the convex set of feasible solutions. Suppose  $X$  is a feasible solution, that is  $X \in K$ , but not an extreme point. Since  $X$  is not an extreme point then it can be written as a convex combination of two other points  $X_1$ , and  $X_2$  in  $K$ . So we have  $X = \alpha X_1 + (1 - \alpha) X_2$  for  $0 < \alpha < 1$ . Since all the elements  $x_i$  of  $X$  are nonnegative and since  $0 < \alpha < 1$ , the last  $n - k$  elements of  $X_1$  and  $X_2$  must also equal zero; that is  $X_1 = (x_{11}, x_{12}, \dots, x_{1k}, 0, \dots, 0)$  and  $X_2 = (x_{21}, x_{22}, \dots, x_{2k}, 0, \dots, 0)$ . Now since  $X_1$  and  $X_2$  are feasible solutions, we have  $AX_1 = b$  and  $AX_2 = b$ . Rewriting the equations in the vector notation we are using, we have  $x_{11} P_1 + x_{12} P_2 + \dots + x_{1k} P_k = P_0$  and  $x_{21} P_1 + x_{22} P_2 + \dots + x_{2k} P_k = P_0$ . Now it is given that  $P_1, P_2, P_3, \dots, P_k$  is a linearly independent set and hence  $P_0$  can be written as a unique linear combination in terms of  $P_1, P_2, \dots, P_k$ . This implies that  $x_i = x_{1i} = x_{2i}$ . Therefore,  $X$  cannot be expressed as a convex combination of two distinct points in  $K$  and must be an extreme point of  $K$ .

**Theorem 2.4** If  $X = (x_1, x_2, \dots, x_n)$  is an extreme point of  $K$ , then the vectors associated with positive  $x_i$  form a linearly independent set. From this it follows that, at most,  $m$  of the  $x_i$  are positive.

**Proof:** Let the nonzero coefficients be the first  $k$  coefficients, so that  $\sum_{i=1}^k x_i P_i = P_0$ . We want to prove that the vectors with  $x_i \geq 0$  form a linearly independent set first.

(by contradiction) Assume that  $P_1, P_2, \dots, P_k$  is a linearly dependent set.

Then there exists a linear combination of these vectors which equals the zero vector,

$$d_1 P_1 + d_2 P_2 + d_3 P_3 + \dots + d_k P_k = 0 \quad (2.2)$$

with at least one  $d_i \neq 0$ . It is given that

$$x_1 P_1 + x_2 P_2 + x_3 P_3 + \dots + x_k P_k = 0 \quad (2.3).$$

For some  $d > 0$ , we multiply (2.2) by  $d$  and add and subtract the result from (2.3) to obtain  $\sum_{i=1}^k x_i P_i + d \sum_{i=1}^k d_i P_i = P_0$ ,  $\sum_{i=1}^k x_i P_i - d \sum_{i=1}^k d_i P_i = P_0$ . This will give us two solutions to (1.2), which may or may not be feasible solutions:

$$X_1 = (x_1 + dd_1, x_2 + dd_2, \dots, x_k + dd_k, 0, \dots, 0) \text{ and}$$

$$X_2 = (x_1 - dd_1, x_2 - dd_2, \dots, x_k - dd_k, 0, \dots, 0).$$

Since all  $x_i \geq 0$ , we can let  $d$  be sufficiently small such that the first  $k$  components of both  $X_1$  and  $X_2$  are positive. Then  $X_1$  and  $X_2$  are feasible solutions.

Also  $X_1 + X_2 = X$  from above and  $X$  is an extreme point. This contradicts the definition of extreme point. Therefore, there exist a set of vectors  $P_1, P_2, \dots, P_m, P_{m+1}$  that are linearly independent.

Now since  $n$  is finite and  $x_i \geq 0$  for each  $i = 1, 2, \dots, n$ , it follows that at most  $m$  of the  $x_i$  are positive. This completes the proof.

**Theorem 2.5** Associated with every extreme point of  $K$  is a set of  $m$  linearly independent vectors from the given set  $P_1, P_2, \dots, P_n$ .

**Proof:** There are  $m$  vectors which form an independent set and at most  $k$  of these are positive from Theorem 2.4, that is  $k \leq m$ . For  $k = m$ , we are finished. Let  $k < m$  and that we can find only  $r - k$  additional vectors  $P_{k+1}, \dots, P_r$  such that the set  $P_1, P_2, \dots, P_k, P_{k+1}, \dots, P_r$  for  $r < m$  is linearly independent. That is the remaining  $n - r$  vectors are dependent on  $P_1, P_2, \dots, P_r$ . But this contradicts that we always have a set of  $m$  linearly independent vectors in the given set of  $P_1, P_2, \dots, P_n$ .

Therefore there must be  $m$  linearly independent vectors  $P_1, P_2, \dots, P_m$  associated with every extreme point, such that  $\sum_{j=1}^k x_j P_j + \sum_{j=k+1}^m 0 \cdot P_j = P_0$ .

Now the  $m$  of Theorems 2.4 and 2.5 is the rank of matrix  $A$  of (1.2), that is  $AX = b$ . Also  $P_i$  is the  $i$ th column vector of  $A$  and  $P_0 = b$ .

**Theorem 2.6**  $X = (x_1, x_2, \dots, x_n)$  is an extreme point of  $K$  if and only if the positive  $x_i$ 's are coefficients of linearly independent vectors  $P_j$  in  $\sum_{j=1}^n x_j P_j = P_0$ .

**Proof:** (1) Assume  $X = (x_1, x_2, \dots, x_n)$  is an extreme point of  $K$ , then by

Theorem 2.4, the positive  $x_j$ 's are coefficients of linearly independent vectors  $P_j$  in  $\sum_{j=1}^n x_j P_j = P_0$ .

(2) Assume the positive  $x_j$ 's are coefficients of linearly independent vectors  $P_j$  in  $\sum_{j=1}^n x_j P_j = P_0$  and note that all  $x_j \geq 0$ ,  $j = 1, 2, \dots, n$ , then by Theorem 2.3,  $X = (x_1, x_2, \dots, x_n)$  is an extreme point of  $K$ .

From the preceding set of theorems, we can conclude that we need only to consider the extreme points, or the basic feasible solutions generated by  $m$  linearly independent vectors to find the optimum solution of the linear-programming problem. There are at most  $\binom{n}{m}$ , or  $\frac{n!}{(n-m)!m!}$  sets of  $m$  independent vectors. The number of basic feasible solutions can be very large. The simplex method devised by Dantzig [1] is a procedure which begins at one basic feasible solution and systematically progresses towards the optimum solution.

## Section 2.2 The Simplex Method

The simplex method begins by generating an initial basic feasible solution and tableau. This can easily be achieved by augmenting matrices  $A$  and  $b$  and putting the matrix in row reduced echelon form. It is not necessary that the resulting matrix be in row reduced echelon form. However, the resulting matrix should have  $m$  unique column identity vectors. As we have shown the  $(A | b)$  matrix should have  $m$  linearly independent vectors, where  $m$  is the number of rows of  $(A | b)$ , the resulting 0 row vectors may be deleted.  $(A | b)$  altered to row reduced echelon form yields matrix  $D$  which will be as follows:

$$D = \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & d_{1,m+1} & \cdots & d_{1,j} & \cdots & d_{1,k} & \cdots & d_{1,n} & x_{10} \\ 0 & 1 & \cdots & 0 & \cdots & 0 & d_{2,m+1} & \cdots & d_{2,j} & \cdots & d_{2,k} & \cdots & d_{2,n} & x_{20} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 & d_{l,m+1} & \cdots & d_{l,j} & \cdots & d_{l,k} & \cdots & d_{l,n} & x_{l0} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 & d_{m,m+1} & \cdots & d_{m,j} & \cdots & d_{m,k} & \cdots & d_{m,n} & x_{m0} \end{pmatrix} \quad (2.4)$$

We will now redefine  $P_j$  to be the  $j$ th column vector of  $D$ ,  $j = 1, 2, \dots, n$  with the last column being  $P_0$ .  $c$  will be defined as per the linear-programming problem, which as we recall is to minimize  $z = cX$  (1.1d). For our initial basic feasible solution, let  $X_0 = P_0$ . For the tableau, we need to define

$$z_0 = \sum_{i=1}^m c_i x_{i0} \quad (2.5)$$

and  $z_j = \sum_{i=1}^m c_i d_{ij}$ ,  $j = 1, 2, \dots, n$ . The initial tableau can now be set up as per Table 2.1.

Now that we have the initial tableau, we can proceed to search for the optimal solution. This achieved by following the simplex procedure:

1. From the tableau, view the bottom row ( $z_j - c_j$ ) to determine if there exist any positive elements. If not, that is  $z_j - c_j \leq 0$  for all  $j$ , then we have found the optimum solution. If there exist a positive element,  $z_j - c_j > 0$ , proceed to step 2.
2. Select the largest element in the bottom row,  $\max_j (z_j - c_j)$ . This is our pivot column ( $P_j$ ).
3. Determine the  $\min_j \left( \frac{x_{i0}}{d_{ij}} \right)$  for those  $d_{ij} > 0$  and  $i = 1, 2, \dots, m$ . If there is no element in the pivot column, excluding the bottom element, that is positive then the solution is unbounded and we are finished. The value  $d_{ij}$  which yield the  $\min_j \left( \frac{x_{i0}}{d_{ij}} \right)$  becomes our pivot element.

**Table 2.1** General Initial Tableau For The Simplex Method

| i       | Basis   | c       | $P_0$    |         | $c_1$   | $c_2$   | $\cdot$ | $c_l$   | $\cdot$ | $c_m$   | $c_{m+1}$           | $\cdot$ | $c_j$       | $\cdot$ | $c_k$       | $\cdot$ | $c_n$       |
|---------|---------|---------|----------|---------|---------|---------|---------|---------|---------|---------|---------------------|---------|-------------|---------|-------------|---------|-------------|
|         |         |         |          |         | $P_1$   | $P_2$   | $\cdot$ | $P_l$   | $\cdot$ | $P_m$   | $P_{m+1}$           | $\cdot$ | $P_j$       | $\cdot$ | $P_k$       | $\cdot$ | $P_n$       |
| 1       | $P_1$   | $c_1$   | $x_{10}$ | 1       | 0       | 0       | $\cdot$ | 0       | $\cdot$ | 0       | $d_{1,m+1}$         | $\cdot$ | $d_{1,j}$   | $\cdot$ | $d_{1,k}$   | $\cdot$ | $d_{1,n}$   |
| 2       | $P_2$   | $c_2$   | $x_{20}$ | 0       | 1       | $\cdot$ | $\cdot$ | 0       | $\cdot$ | 0       | $d_{2,m+1}$         | $\cdot$ | $d_{2,j}$   | $\cdot$ | $d_{2,k}$   | $\cdot$ | $d_{2,n}$   |
| $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$  | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$             | $\cdot$ | $\cdot$     | $\cdot$ | $\cdot$     | $\cdot$ | $\cdot$     |
| $l$     | $P_l$   | $c_l$   | $x_{l0}$ | 0       | 0       | $\cdot$ | $\cdot$ | 1       | $\cdot$ | 0       | $d_{l,m+1}$         | $\cdot$ | $d_{l,j}$   | $\cdot$ | $d_{l,k}$   | $\cdot$ | $d_{l,n}$   |
| $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$  | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$             | $\cdot$ | $\cdot$     | $\cdot$ | $\cdot$     | $\cdot$ | $\cdot$     |
| $m$     | $P_m$   | $c_m$   | $x_{m0}$ | 0       | 0       | $\cdot$ | $\cdot$ | 0       | $\cdot$ | 1       | $d_{m,m+1}$         | $\cdot$ | $d_{m,j}$   | $\cdot$ | $d_{m,k}$   | $\cdot$ | $d_{m,n}$   |
| $m+1$   |         |         | $z_0$    | 0       | 0       | $\cdot$ | $\cdot$ | 0       | $\cdot$ | 0       | $z_{m+1} - c_{m+1}$ | $\cdot$ | $z_j - c_j$ | $\cdot$ | $z_k - c_k$ | $\cdot$ | $z_n - c_n$ |

4. Replace the  $P_i$  in our basis with  $P_j$ , and multiply the row of our pivot element with reciprocal of the pivot element; such that the pivot element will be transformed to 1.
5. Use row addition and multiplication such that each element in the pivot column is 0 with the exception of the pivot element which will remain 1.
6. The transformed tableau becomes our new tableau. Now proceed to step 1 and start over.

The flow chart in Figure 2.2, which is from Kwak [3], may make this procedure seem simpler.

When the optimum solution has determined,  $Z_j - C_j \leq 0$  for all  $j$ , then  $z_0$  is the optimum solution,  $z = z_0$ , and for each  $P_i$  in the basis  $x_i = x_{i0}$  and for each  $P_j$  not in the basis  $x_j = 0$ . These values will yield our optimum solution.

### Section 2.3 Theoretical Proof of the Simplex Method

Now before we cover a couple of examples involving the simplex method, let's give some theoretical support for the simplex method. For the remaining theorems in this chapter we will assume that the linear-programming problem is feasible, that every basic feasible solution is nondegenerate, and that we are given a basic feasible solution. Unless otherwise stated the theorems and proofs are close variations of Gass[1].

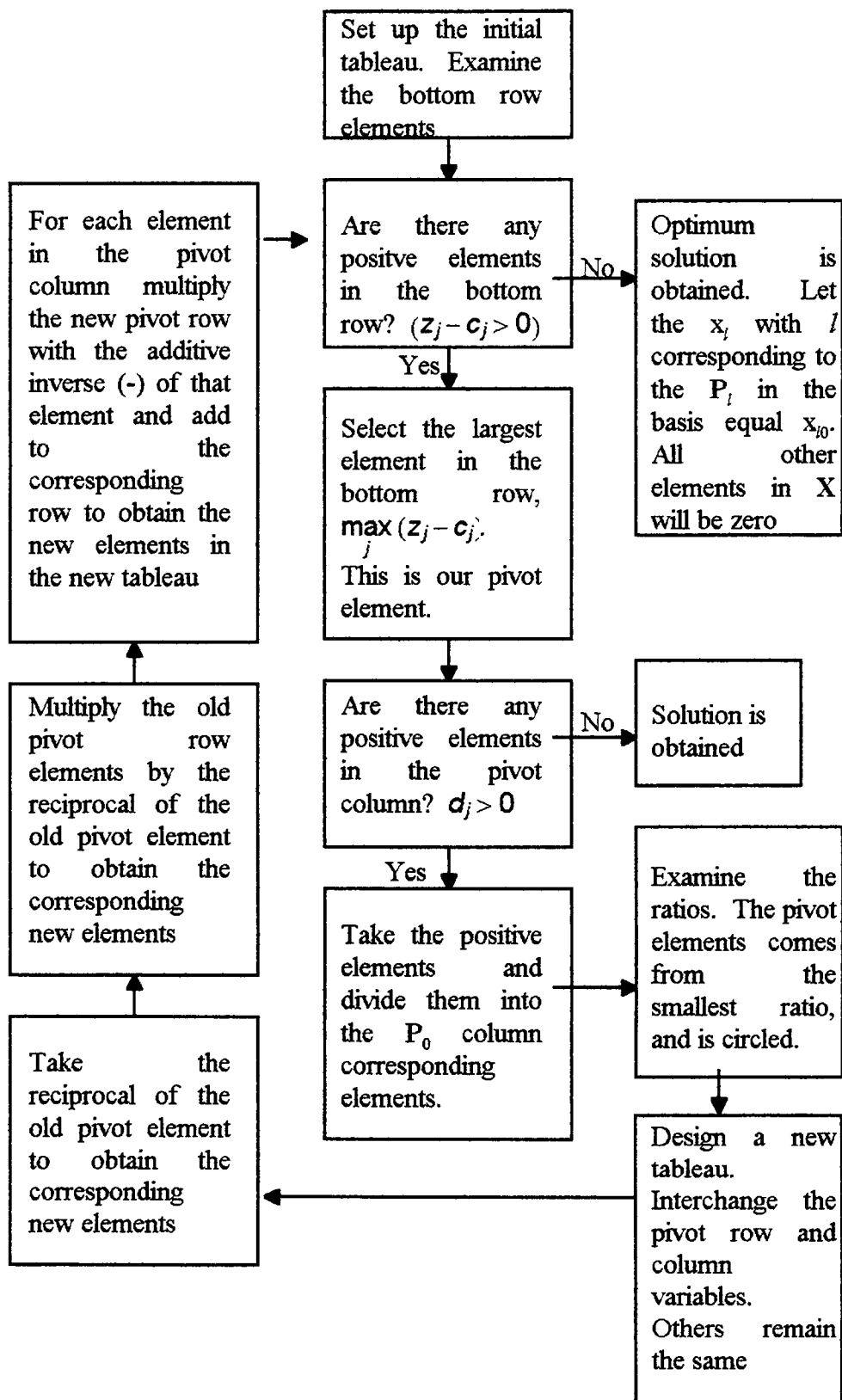


Figure 2.2 The Simplex Method

Referring to D (2.4), let the given basic feasible solution be

$\mathbf{X}_0 = (x_{10} \ x_{20} \ \dots \ x_{m0})^T$  and the linearly independent vectors be  $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_m$ .

Now we can write

$$x_{10}\mathbf{P}_1 + x_{20}\mathbf{P}_2 + \dots + x_{m0}\mathbf{P}_m = \mathbf{P}_0 \quad (2.5)$$

$$\text{and} \quad x_{10}\mathbf{c}_1 + x_{20}\mathbf{c}_2 + \dots + x_{m0}\mathbf{c}_m = \mathbf{z}_0 \quad (2.6)$$

where all  $x_{i0} > 0$  and  $c_i$  are from the linear-programming problem. Now the set

$\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_m$  are linearly independent vectors and thus form a basis. Therefore

any vector from the set  $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_m$  can be expressed in terms of  $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_m$ .

Let  $\mathbf{P}_j$  be given by

$$d_{1j}\mathbf{P}_1 + d_{2j}\mathbf{P}_2 + \dots + d_{mj}\mathbf{P}_m = \mathbf{P}_j, \quad j = 1, 2, \dots, n \quad (2.7)$$

$$\text{and define} \quad d_{1j}c_1 + d_{2j}c_2 + \dots + d_{mj}c_m = z_j, \quad j = 1, 2, \dots, n \quad (2.8).$$

**Theorem 2.7** If for any fixed  $j$ , the condition  $z_j - c_j > 0$  holds, then a set of

feasible solutions can be constructed such that  $z < z_0$  for any member of the set,

where the lower bound of  $z$  is either finite or infinite.

**Proof:** Multiply (2.7) by some number  $\theta$  and subtract from (2.5). Also multiply

(2.8) by the same  $\theta$  and subtract from (2.6). We get

$$(x_{10} - \theta d_{1j})\mathbf{P}_1 + (x_{20} - \theta d_{2j})\mathbf{P}_2 + \dots + (x_{m0} - \theta d_{mj})\mathbf{P}_m + \theta\mathbf{P}_j = \mathbf{P}_0 \quad (2.9)$$

and

$$(x_{10} - \theta d_{1j})c_1 + (x_{20} - \theta d_{2j})c_2 + \dots + (x_{m0} - \theta d_{mj})c_m + \theta c_j = z_0 - \theta(z_j - c_j) \quad (2.10).$$

Note that  $\theta c_j$  was added to both sides of (2.10). If all the coefficients of the

vectors  $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_m, \mathbf{P}_j$  in (2.9) are nonnegative, then we have determined a new

feasible solution whose value of the objective function is, by (2.10),

$z = z_0 - \theta(z_j - c_j)$ . Since the variables in (2.9) are all positive, then there is a value

$\theta > 0$  (finite or infinite) for which the coefficients of the vectors in (2.9) remain

positive. From the assumption that, for a fixed  $j$ ,  $z_j - c_j > 0$ , we have

$z = z_0 - \theta(z_j - c_j) < z_0$  for  $\theta > 0$ . Now there exist two cases: Case 1,  $d_{ij} > 0$  in (2.7) for some  $i = 1, 2, \dots, m$ ; or Case 2,  $d_{ij} \leq 0$  for all  $i = 1, 2, \dots, m$  for some fixed  $j$ .

Case 1 If for the fixed  $j$ , at least one  $d_{ij} > 0$  in (2.7) for  $i = 1, 2, \dots, m$ , the largest value of  $\theta$  for which all coefficients of (2.9) remain nonnegative is given by

$$\theta_0 = \min_j \frac{x_{i0}}{d_{ij}} > 0 \text{ for } d_{ij} > 0. \text{ We have assumed that the problem is}$$

nondegenerate, therefore all basic feasible solutions have  $m$  positive elements; thus,  $\theta_0$  can be obtained for a unique  $i$ . If  $\theta_0$  is substituted for  $\theta$  in (2.9) and (2.10), the coefficient corresponding to this unique  $i$  will disappear.

We then have constructed a new basic feasible solution consisting of  $P_j$  and  $m - 1$  vectors of the original basis. This new basis can be used as the previous one. If a new  $z_j - c_j > 0$  and a corresponding  $d_{ij} > 0$ , another solution can be obtained which has an even smaller value of the objective function. This process will continue either until all  $z_j - c_j \leq 0$ , or until, for some  $z_j - c_j > 0$  all  $d_{ij} \leq 0$ . If all  $z_j - c_j \leq 0$ , the process ends.

Case II If we have for some  $j$ ,  $z_j - c_j > 0$ , and all  $d_{ij} \leq 0$ , then there is no upper bound to  $\theta$  and the objective function has a lower bound of  $-\infty$ . We see for this case, for an  $\theta > 0$ , all the coefficients of (2.9) are positive. We then have a solution of  $m + 1$  positive elements. Hence by taking  $\theta$  large enough, the corresponding value of the objective function given by the right hand side of (2.10) can be made arbitrarily small. This completes the proof.

**Theorem 2.8** If for any basic feasible solution  $\mathbf{X} = (x_{10}, x_{20}, \dots, x_{m0})$  the conditions  $z_j - c_j \leq 0$  hold for all  $j = 1, 2, \dots, n$ , then

$$x_{10}\mathbf{P}_1 + x_{20}\mathbf{P}_2 + \dots + x_{m0}\mathbf{P}_m = \mathbf{P}_0 \quad (2.5),$$

$$x_{10}c_1 + x_{20}c_2 + \dots + x_{m0}c_m = z_0 \quad (2.6)$$

constitute a minimum feasible (optimum) solution.

**Proof:** Let

$$y_{10}\mathbf{P}_1 + y_{20}\mathbf{P}_2 + \dots + y_{m0}\mathbf{P}_m = \mathbf{P}_0 \quad (2.11)$$

and

$$y_{10}c_1 + y_{20}c_2 + \dots + y_{m0}c_m = z^* \quad (2.12)$$

be any other feasible solution with  $z^*$  the corresponding value of the objective function.

We need to show  $z_0 \leq z^*$ . By our assumption  $z_j - c_j \leq 0$  for all  $j$ , so that replacing  $c_j$  by  $z_j$  in (2.12) yields

$$y_{10}z_1 + y_{20}z_2 + \dots + y_{m0}z_m \leq z^* \quad (2.13).$$

For each  $j$ , we substitute the expression for  $\mathbf{P}_j$  given by (2.7) into (2.11) and we get

$$y_{10}\left(\sum_{i=1}^m d_{i1}\mathbf{P}_1\right) + y_{20}\left(\sum_{i=1}^m d_{i2}\mathbf{P}_2\right) + \dots + y_{m0}\left(\sum_{i=1}^m d_{in}\mathbf{P}_n\right) = \mathbf{P}_0 \quad \text{or}$$

$$\left(\sum_{j=1}^n y_{j0}d_{1j}\right)\mathbf{P}_1 + \left(\sum_{j=1}^n y_{j0}d_{2j}\right)\mathbf{P}_2 + \dots + \left(\sum_{j=1}^n y_{j0}d_{mj}\right)\mathbf{P}_m = \mathbf{P}_0 \quad (2.14).$$

Now we substitute for each  $j$ , the expression for  $z_j$  given by (2.8) into (2.13). We now have

$$\left(\sum_{j=1}^n y_{j0}d_{1j}\right)c_1 + \left(\sum_{j=1}^n y_{j0}d_{2j}\right)c_2 + \dots + \left(\sum_{j=1}^n y_{j0}d_{mj}\right)c_m \leq z^* \quad (2.15).$$

Now the set of vectors  $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_m$  is linearly independent, therefore

$$x_{k0} = \sum_{j=1}^n x_{j0}d_{kj}, \quad x_{10}c_1 + x_{20}c_2 + \dots + x_{m0}c_m \leq z^* \quad \text{and} \quad z \leq z^*.$$

This not only completes our proof, it verifies the simplex method.

## Section 2.4 Examples

Now let's look at a couple of examples of using the simplex method.

### Example 2.1 Maximize

$$h = 3x_1 + 4x_2 + 2x_3$$

subject to the following constraints

$$x_1 - 2x_2 + 4x_3 \leq 36$$

$$2x_1 + 3x_2 - 5x_3 \leq 40$$

$$3x_1 + 2x_2 + x_3 \leq 28$$

and

$$x_1, x_2, x_3 \geq 0 \quad [7]$$

We have defined the linear-programming problem as a maximization problem. This problem can easily be turned into a minimization problem by multiplying  $h$  by  $-1$ , or letting  $z = -h$ . Also the inequalities can be transformed into equations by the use of slack variables ( $x_4$ ,  $x_5$ , and  $x_6$ ). The initial basic solution generally are these slack variables. With these changes the problem becomes

Minimize

$$z = -3x_1 - 4x_2 - 2x_3$$

subject to the following constraints

$$x_1 - 2x_2 + 4x_3 + x_4 = 36$$

$$2x_1 + 3x_2 - 5x_3 + x_5 = 40$$

$$3x_1 + 2x_2 + x_3 + x_6 = 28$$

and

$$x_i \geq 0 \quad \text{for } i = 1, 2, \dots, 6.$$

We now have the problem in our general linear-programming problem.

From this we can set up the initial tableau in the form of Table 2.2.

Referring to Table 2.2 below,  $P_4, P_5, P_6$  are in our initial basis. We now go through our simplex procedure. There are three elements on the bottom row which are positive, therefore we do not have an optimal solution.  $P_2$  becomes our pivot column with  $d_{22} = 3$  becoming our pivot element. Performing matrix operations with  $R_i$  representing the respective rows, we pivot. The matrix operations are as follows, in order:  $\frac{1}{3}R_2 \Rightarrow R_2$ ,  $2R_2 + R_1 \Rightarrow R_1$ ,  $-R_2 + R_3 \Rightarrow R_3$ ,  $-4R_2 + R_4 \Rightarrow R_4$ . The result will be Table 2.3.

Now referring to Table 2.3 on the following page, column  $P_3$  has the largest positive number of the bottom row and our test determines  $d_{33}$  to be our pivot element. So we do the pivot operations and the result is Table 2.4

**Table 2.2 Initial Tableau of Example 2.1**

| i | Basis | c | $P_0$ | -3    | -4    | -2    | 0     | 0     | 0     | $\theta$     |
|---|-------|---|-------|-------|-------|-------|-------|-------|-------|--------------|
|   |       |   |       | $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ | $P_6$ |              |
| 1 | $P_4$ | 0 | 36    | 1     | -2    | 4     | 1     | 0     | 0     |              |
| 2 | $P_5$ | 0 | 40    | 2     | 3     | -5    | 0     | 1     | 0     | 13 1/3<br>XX |
| 3 | $P_6$ | 0 | 28    | 3     | 2     | 1     | 0     | 0     | 1     | 14           |
| 4 |       |   | 0     | 3     | 4     | 2     | 0     | 0     | 0     |              |

Pivot Column

**Table 2.3    Tableau 2 of Example 2.1**

| i | Basis | c  | $P_0$             | -3              | -4    | -2               | 0     | 0                | 0     | $\theta$             |
|---|-------|----|-------------------|-----------------|-------|------------------|-------|------------------|-------|----------------------|
|   |       |    |                   | $P_1$           | $P_2$ | $P_3$            | $P_4$ | $P_5$            | $P_6$ |                      |
| 1 | $P_4$ | 0  | $62 \frac{2}{3}$  | $2 \frac{1}{3}$ | 0     | $\frac{2}{3}$    | 1     | $\frac{2}{3}$    | 0     | 94                   |
| 2 | $P_2$ | -4 | $13 \frac{1}{3}$  | $\frac{2}{3}$   | 1     | $-1 \frac{2}{3}$ | 0     | $\frac{1}{3}$    | 0     |                      |
| 3 | $P_6$ | 0  | $1 \frac{1}{3}$   | $1 \frac{2}{3}$ | 0     | $4 \frac{1}{3}$  | 0     | $-\frac{2}{3}$   | 1     | $\frac{4}{13}$<br>XX |
| 4 |       |    | $-53 \frac{1}{3}$ | $\frac{1}{3}$   | 0     | $8 \frac{2}{3}$  | 0     | $-1 \frac{1}{3}$ | 0     |                      |



Pivot Column

**Table 2.4    Optimal Tableau (Tableau 3) for Example 2.2**

| i | Basis | c  | $P_0$              | -3               | -4    | -2    | 0     | 0               | 0               | $\theta$ |
|---|-------|----|--------------------|------------------|-------|-------|-------|-----------------|-----------------|----------|
|   |       |    |                    | $P_1$            | $P_2$ | $P_3$ | $P_4$ | $P_5$           | $P_6$           |          |
| 1 | $P_4$ | 0  | $62 \frac{6}{13}$  | $2 \frac{1}{13}$ | 0     | 0     | 1     | $\frac{10}{13}$ | $-\frac{2}{13}$ |          |
| 2 | $P_2$ | -4 | $13 \frac{11}{13}$ | $1 \frac{4}{13}$ | 1     | 0     | 0     | $\frac{1}{13}$  | $\frac{5}{13}$  |          |
| 3 | $P_3$ | -2 | $\frac{4}{13}$     | $\frac{5}{13}$   | 0     | 1     | 0     | $-\frac{2}{13}$ | $\frac{3}{13}$  |          |
| 4 |       |    | -56                | -3               | 0     | 0     | 0     | 0               | -2              |          |

(Table 2.4).

We now focus on Table 2.4. Since there are no positive numbers on the bottom row,  $z_j - c_j \leq 0$  for all  $j = 1, 2, \dots, 6$ , we have an optimal solution with values  $x_1 = 0$ ,  $x_2 = 13 \frac{11}{13}$ ,  $x_3 = \frac{4}{13}$ ,  $z = -56$  and  $h = 56$ . The values of the slack variables are not important, but are  $x_4 = 62 \frac{6}{13}$ ,  $x_5 = 0$ ,  $x_6 = 0$ .

### Example 2.2 Minimize

$$z = 2x_1 + 2x_2 + 4x_3$$

subject to the following constraints

$$\begin{aligned} 2x_1 + 3x_2 + 5x_3 &\geq 2 \\ 3x_1 + x_2 + 7x_3 &\leq 3 \\ x_1 + 4x_2 + 6x_3 &\leq 5 \end{aligned}$$

and

$$x_1, x_2, x_3 \geq 0 \quad [8]$$

For this example we will need to use an artificial basis. Our initial solution will have  $x_4 = -2$  a nonbasic solution since  $x_4 < 0$ . This is resolved by adding another slack variable  $x_7$  and putting  $x_7$  in the  $z$  equation with a constant  $w$ . We will have an additional row for  $w$ , where  $w = 0$ . When  $x_7$  is removed from the basis then the bottom row can be dropped. Column 7 ( $P_7$ ) can also be dropped and we can proceed as before. So our linear-programming problem is as follows:

Minimize

$$z = 2x_1 + 2x_2 + 4x_3 + wx_7 \quad \text{where } w = 0$$

subject to the following constraints

$$2x_1 + 3x_2 + 5x_3 - x_4 + x_7 = 2$$

subject to the following constraints

$$2x_1 + 3x_2 + 5x_3 - x_4 + x_7 = 2$$

$$3x_1 + x_2 + 7x_3 + x_5 = 3$$

$$x_1 + 4x_2 + 6x_3 + x_6 = 5$$

and

$$x_1, x_2, x_3 \geq 0$$

Now we can set up our initial tableau (Table 2.5).

Referring to Table 2.5, column  $P_3$  has the largest positive number in the bottom row so this is our pivot column with  $d_{13}$  determined to be our pivot element. Now with the pivot, since  $P_3$  is replacing  $P_7$  in the basis,  $P_7$  can be dropped. Also note that row five has no positive numbers and thus can be dropped as we refer to Table 2.6.

Now from Table 2.6 to Table 2.7, row 5 is essentially dropped from our tableau and we use row 4 to check  $z_j - c_j$ . Column  $P_2$  is our pivot column with  $d_{12}$  becoming our pivot element. Working through the pivot gives us Tableau 3 (Table 2.7).

Tableau 3 (Table 2.7) gives us our optimal solution which is as follows:

$x_1 = 0$ ,  $x_2 = \frac{2}{3}$ ,  $x_3 = 0$  and  $z = 2\frac{1}{3}$ . Our slack variables have solutions  $x_4 = 0$ ,  $x_5 = 2\frac{1}{3}$ ,  $x_6 = 2\frac{1}{3}$ .

Table 2.5 Initial Tableau of Example 2.2

| i | Basis          | c | P <sub>0</sub> | 2              | 2              | 4              | 0              | 0              | 0              | w              | θ               |
|---|----------------|---|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|-----------------|
|   |                |   |                | P <sub>1</sub> | P <sub>2</sub> | P <sub>3</sub> | P <sub>4</sub> | P <sub>5</sub> | P <sub>6</sub> | P <sub>7</sub> |                 |
| 1 | P <sub>7</sub> | w | 2              | 2              | 3              | 5              | -1             | 0              | 0              | 1              | 2/5 = .40<br>XX |
| 2 | P <sub>2</sub> | 0 | 3              | 3              | 1              | 7              | 0              | 1              | 0              | 0              | 3/7 = .42       |
| 3 | P <sub>3</sub> | 0 | 5              | 1              | 4              | 6              | 0              | 0              | 1              | 0              | 5/6 = .83...    |
| 4 |                |   | 0              | -2             | -2             | -4             | 0              | 0              | 0              | 0              |                 |
| 5 |                |   | 2w             | 2              | 3              | 5              | -1             | 0              | 0              | 0              |                 |

↑  
Pivot Column

**Table 2.6    Tableau 2 of Example 2.2**

| i | Basis | c | $P_0$ | -3    | -4    | -2    | 0     | 0     | 0     | $\theta$ |
|---|-------|---|-------|-------|-------|-------|-------|-------|-------|----------|
|   |       |   |       | $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ | $P_6$ |          |
| 1 | $P_3$ | 0 | 36    | 1     | -2    | 4     | 1     | 0     | 0     | XX       |
| 2 | $P_5$ | 0 | 40    | 2     | 3     | -5    | 0     | 1     | 0     |          |
| 3 | $P_6$ | 0 | 28    | 3     | 2     | 1     | 0     | 0     | 1     | 14       |
| 4 |       |   | 0     | 3     | 4     | 2     | 0     | 0     | 0     |          |

Pivot Column

**Table 2.7    Optimal Tableau (Tableau 3) of Example 2.2**

| i | Basis | c | $P_0$    | 2         | 2     | 4        | 0        | 0     | 0     | $\theta$ |
|---|-------|---|----------|-----------|-------|----------|----------|-------|-------|----------|
|   |       |   |          | $P_1$     | $P_2$ | $P_3$    | $P_4$    | $P_5$ | $P_6$ |          |
| 1 | $P_2$ | 2 | $2/3$    | $2/3$     | 1     | $1\ 2/3$ | $-1/3$   | 0     | 0     |          |
| 2 | $P_5$ | 0 | $2\ 1/3$ | $2\ 1/3$  | 0     | $5\ 1/3$ | $1/3$    | 1     | 0     |          |
| 3 | $P_6$ | 0 | $2\ 1/3$ | $-1\ 2/3$ | 0     | $-2/3$   | $1\ 1/3$ | 0     | 1     |          |
| 4 |       |   | $1\ 1/3$ | $-2/3$    | 0     | $-2/3$   | $-2/3$   | 0     | 0     |          |

## Section 2.5 The Transportation Problem as a Linear-Programming Problem

Now let's look at the transportation problem set up as a linear-programming problem. Recall from Chapter 1, that the transportation problem is as follows:

Minimize

$$z = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} \quad (1.4)$$

subject to the following restraints

$$\sum_{j=1}^n x_{ij} = a_i \quad i = 1, 2, \dots, m \quad (1.5)$$

$$\sum_{i=1}^m x_{ij} = b_j \quad j = 1, 2, \dots, n \quad (1.6)$$

and

$$x_{ij} > 0$$

where  $a_i$ ,  $b_j$ , and  $c_{ij}$  are given constants.

This can also be written as follows:

Minimize

$$\begin{aligned} z = & c_{11} x_{11} + c_{12} x_{12} + \dots + c_{1n} x_{1n} + c_{21} x_{21} + c_{22} x_{22} + \dots + c_{2n} x_{2n} + \dots + c_{ij} x_{ij} + \\ & \dots + c_{m1} x_{m1} + c_{m2} x_{m2} + \dots + c_{mn} x_{mn}. \end{aligned}$$

subject to the following restraints

$$x_{11} + x_{12} + \dots + x_{1n} = a_1$$

$$x_{21} + x_{22} + \dots + x_{2n} = a_2$$

.....

$$x_{i1} + x_{i2} + \dots + x_{in} = a_i$$

.....

$$x_{m1} + x_{m2} + \dots + x_{mn} = a_m$$

$$x_{11} + \quad \quad \quad x_{21} + \quad \dots \quad + x_{i1} + \dots + \quad x_{m1} \quad = b_1$$

$$\quad \quad \quad x_{12} + \quad \quad \quad x_{22} + \quad \quad \dots + \quad x_{i2} + \dots + \quad x_{m2} \quad = b_2$$

.....

$$\quad \quad \quad x_{1j} + \quad \quad \quad x_{2j} + \quad \dots + \quad x_{ij} + \dots + \quad x_{mj} \quad = b_j$$

.....

$$\quad \quad \quad x_{1n} + \quad \quad \quad x_{2n} + \dots + x_{in} + \dots + \quad x_{mn} \quad = b_n$$

From this it can be seen that the transportation problem does have an optimal solution if the constants are bounded below. It can also be seen that using the simplex algorithm to solve the transportation problem will be time consuming and tedious.

# CHAPTER 3

## THE TRANSPORTATION PROBLEM AND THE INITIAL BASIC FEASIBLE SOLUTION

In this chapter we will look at the transportation problem, define what makes a balanced transportation problem, present the transportation tableau form, theoretically verify that the transportation problem has a feasible and basic feasible solution, and present several techniques for obtaining the initial basic feasible solution.

The transportation problem is stated as follows:

Find values for the variables  $x_{ij}$  which minimize

$$Z = x_{00} = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} \quad (3.1)$$

subject to the following restraints

$$\sum_{j=1}^n x_{ij} = a_i \quad i = 1, 2, \dots, m \quad (3.2)$$

$$\sum_{i=1}^m x_{ij} = b_j \quad j = 1, 2, \dots, n \quad (3.3)$$

and

$$x_{ij} > 0 \quad (3.4)$$

where each  $a_i$ ,  $b_j$ , and  $c_{ij}$  are given constants.

**Definition 3.1** A balanced transportation problem is a transportation problem in which  $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$ .

We will assume that every transportation problem is balanced for the rest of this chapter.

The cost constraints  $c_{ij}$  can be arranged in an  $m \times n$  matrix such that

$$C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mn} \end{pmatrix}.$$

The transportation problem can be placed in a tableau such as in Table 3.1.

**Theorem 3.1** The transportation problem has a feasible solution.

**Proof:** Note that by definition of the transportation problem, each  $a_i$  and  $b_j$  are nonnegative. Also note that we are assuming that the transportation problem is balanced; that is,  $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$ . Let  $A = \sum_{i=1}^m a_i = \sum_{j=1}^n b_j$ , and let  $x_{ij} = \frac{a_i b_j}{A}$  for all  $(i, j)$ . Now  $x_{ij} \geq 0$  for all  $(i, j)$  and  $\sum_{j=1}^n x_{ij} = \sum_{j=1}^n \frac{a_i b_j}{A} = \frac{a_i}{A} \sum_{j=1}^n b_j = \frac{a_i}{A} A = a_i$ . Also  $\sum_{i=1}^m x_{ij} = \sum_{i=1}^m \frac{a_i b_j}{A} = \frac{b_j}{A} \sum_{i=1}^m a_i = \frac{b_j}{A} A = b_j$ .

Therefore, by the definition of a feasible solution, we have shown that the transportation problem has a feasible solution.

Now before we get into determining the initial basic feasible solution, let's look at a specific example of the transportation problem. The following example comes from Kwak [3].

**Table 3.1 The General Transportation Tableau**

| O<br>R<br>I<br>G<br>I<br>N<br>S | DESTINATIONS |          |          |   |          |   |          |
|---------------------------------|--------------|----------|----------|---|----------|---|----------|
|                                 |              | 1        | 2        | . | $j$      | . | $n$      |
|                                 | 1            | $x_{11}$ | $x_{12}$ | . | $x_{1j}$ | . | $x_{1n}$ |
|                                 | 2            | $x_{21}$ | $x_{22}$ | . | $x_{2j}$ | . | $x_{2n}$ |
|                                 | .            | .        | .        | . | .        | . | .        |
|                                 | $i$          | $x_{i1}$ | $x_{i2}$ | . | $x_{ij}$ | . | $x_{in}$ |
|                                 | .            | .        | .        | . | .        | . | .        |
|                                 | $m$          | $x_{m1}$ | $x_{m2}$ | . | $x_{mj}$ | . | $x_{mn}$ |
|                                 |              | $b_1$    | $b_2$    | . | $b_j$    | . | $b_n$    |
|                                 |              |          |          |   |          |   | $x_{00}$ |

$x_{00}$  can also be represented by  $z$ .

**Example 3.1** The Burns Manufacturing Company has plants in Chicago, Detroit, and St. Louis. The company produces and distributes sewing machines to dealers in various cities. On a particular day, the company has 30 sewing machines in Chicago, 40 in Detroit, and 30 in St. Louis. The company plans to ship 20 sewing machines to Atlanta, 20 to Houston, 25 to Los Angeles, and 35 to New York, following the orders received from dealers. The transportation costs per unit between cities are given in Table 3.2

The problem facing the company is to distribute the available sewing machines to the dealers in each city in such a manner as to minimize the total transportation cost. From the above information, we can see that the cost matrix associated with the example is

$$C = \begin{pmatrix} 7 & 10 & 14 & 8 \\ 7 & 11 & 12 & 6 \\ 5 & 8 & 15 & 9 \end{pmatrix}. \quad (3.5)$$

Other values are  $a_1 = 30$ ,  $a_2 = 40$ ,  $a_3 = 30$ ,  $b_1 = 20$ ,  $b_2 = 20$ ,  $b_3 = 25$ , and  $b_4 = 35$ .

**Table 3.2 Cost to Transport Goods of Example 3.1**

| <div>From \ To</div> | Atlanta | Houston | L A | New York | Supply |
|----------------------|---------|---------|-----|----------|--------|
| Chicago              | 7       | 10      | 14  | 8        | 30     |
| Detroit              | 7       | 11      | 12  | 6        | 40     |
| St. Louis            | 5       | 8       | 15  | 9        | 30     |
| Demand               | 20      | 20      | 25  | 35       |        |

### Section 3.1 The Northwest Corner Method

The northwest corner method to obtain the initial basic feasible solution was proposed by Dantzig and named by Charnes and Cooper [1]. The northwest corner method is the simplest method for finding the initial basic feasible solution. The steps are as follows.

1. Let  $x_{11} = \min [a_1, b_1]$ .
2. If  $a_1 < b_1$ , then all the other variables in the first row are to be given the value zero and designated as nonbasic. Next delete the first row, reduce the value of  $b_1$  to  $(b_1 - a_1)$ , and proceed in the same manner to evaluate a variable in the reduced array composed of the  $m - 1$  rows and  $n$  columns remaining.
3. If  $a_1 > b_1$ , then, similarly, the first column is to be deleted and  $a_1$  replaced by  $(a_1 - b_1)$ .
4. If  $a_1 = b_1$ , then delete the row or column but not both.
5. Now move downward (as per step 2), rightward (as per step 3), or downward or rightward (as per step 4).
6. Continue the procedure setting  $x_{ij} = \min [a_i, b_j]$  and applying steps 2, 3, 4 as they apply. The procedure will conclude at the bottom right cell.

Using the northwest corner method on Example 3.1 gives us the initial tableau of Table 3.3

**Table 3.3 Initial Transportation Tableau of Example 3.1**

| From \ To | Atlanta | Houston | L A | New York | $a_i$ |
|-----------|---------|---------|-----|----------|-------|
| Chicago   | 20      | 10      | 0   | 0        | 30    |
| Detroit   | 0       | 10      | 25  | 5        | 40    |
| St. Louis | 0       | 0       | 0   | 30       | 30    |
| $b_j$     | 20      | 20      | 25  | 35       | 950   |

This tableau gives us the values  $x_{11} = 20$ ,  $x_{12} = 10$ ,  $x_{22} = 10$ ,  $x_{23} = 25$ ,  $x_{24} = 5$ ,  $x_{34} = 30$  and the total cost  $x_{00} = 950$ .

The northwest corner method may be the easiest way to find an optimal basic feasible solution but it is not likely to give the optimal solution, since cost is not even considered in determining the initial solution; thus, requiring more iterations to obtain the optimal solution. We've shown this method first, because this method is useful in proving that the transportation problem has a basic feasible solution.

**Theorem 3.2** The transportation problem has a basic feasible solution. A solution of at most  $m + n - 1$  positive  $x_{ij}$ 's exist.

**Proof:** Observe the transportation tableau of Table 3.1. Begin with variable  $x_{11}$ , and make it as large as possible, consistent with row and column totals, ie. set  $x_{11} = \min[a_1, b_1]$ .

Case 1. If  $a_i < b_j$ , then all the other variables in the first row are to be given the value zero and designated as nonbasic. Next delete the first row, reduce the value of  $b_j$  to  $(b_j - a_i)$ , and proceed in the same manner to evaluate a variable in the reduced array composed of the  $m - 1$  rows and  $n$  columns remaining.

Case 2. If  $a_i > b_j$ , then similarly, the first column is to be deleted and  $a_i$  replaced by  $(a_i - b_j)$ .

Case 3. If  $a_i = b_j$ , then delete either the row or the column but not both. However if several columns, but only one row remain in the reduced array, then drop the  $j$ th column, and conversely, if several rows and one column, drop the  $i$ th row.

After completing each step move downward (as per Case 1), or rightward (as per Case 2), or downward or rightward (as per Case 3) and continue the procedure setting  $x_{ij} = \min [a_i, b_j]$  and applying each case as applies.

The rule will select as many variables for the basic set as there are rows plus columns minus one,  $m + n - 1$ , since on the last step when only one row and one column remain both will be dropped. This will also satisfy that there are at most  $m + n - 1$   $x_{ij}$ 's that are positive, since all other  $x_{ij}$ 's are equal to zero.

The resultant tableau will look something like Table 3.4 on page 40. Now we need to show that the solution (3.6) is a basic feasible solution. Since the rank

**Table 3.4    The Initial Basic Feasible Solution to the Transportation Problem Using  
The Northwest Corner Method**

|                      |                              |                                          |              |                                                                                   |                                                                                   |                |       |
|----------------------|------------------------------|------------------------------------------|--------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|----------------|-------|
| $x_{11} = a_1$       | $x_{12} = 0$                 | $x_{13} = 0$                             | ...          | $x_{1j} = 0$                                                                      | ...                                                                               | $x_{1n} = 0$   | $a_1$ |
| $x_{21} = b_1 - a_1$ | $x_{22} = a_2 - (b_1 + a_1)$ | $x_{23} = a_2 - (b_1 + b_2 + a_1) \dots$ | $x_{2j} = 0$ | $x_{2n} = 0$                                                                      |                                                                                   |                | $a_2$ |
| $x_{31} = 0$         | $x_{32} = 0$                 | $x_{33} = 0$                             | ...          | $x_{3j} = 0$                                                                      | ...                                                                               | $x_{3n} = 0$   | $a_3$ |
| .....                |                              |                                          |              |                                                                                   |                                                                                   |                |       |
| $x_{i1} = 0$         | $x_{i2} = 0$                 | $x_{i3} = 0$                             | ...          | $x_{ij} = b_j - \left( \sum_{k=1}^{i-1} a_k - \sum_{l=1}^{j-1} b_l \right) \dots$ |                                                                                   | $x_{in} = 0$   | $a_i$ |
| .....                |                              |                                          |              |                                                                                   |                                                                                   |                |       |
| $x_{m1} = 0$         | $x_{m2} = 0$                 | $x_{m3} = 0$                             | ...          | $x_{mj} = 0$                                                                      | $\dots x_{mn} = a_m - \left( \sum_{l=1}^{n-1} b_l - \sum_{k=1}^{m-1} a_k \right)$ |                | $a_m$ |
| $b_1$                | $b_2$                        | $b_3$                                    | ...          | $b_j$                                                                             | ...                                                                               | $b_n = x_{mn}$ |       |

(3.6)

of the system is  $m + n - 1$ , a set of variables,  $x_{ij}$ , constitutes a basic set of its values, satisfying all  $m + n$  equations, are given uniquely as linear combinations of any  $m + n - 1$  of the  $m + n$  marginals  $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n$  when the remaining values are set equal to zero. The value given each basic variable by the starting rule is the same as a marginal total of some reduced ray; but this total, because of the way in which it was derived, is equal to the difference between partial sums of the original row and column totals. This shows the values are uniquely determined by some particular set of  $m + n - 1$  marginals. However, because of the condition  $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$ , any one of these totals may be expressed as a linear combination of the others. Also  $b_n = \sum_{i=1}^m a_i - \sum_{j=1}^{n-1} b_j$ . Each  $x_{ij} \geq 0$ ,  $\sum_{j=1}^n x_{ij} = a_i$ , and  $\sum_{i=1}^m x_{ij} = b_j$  as per how the solution was derived. Therefore, this solution is a basic feasible solution ([1],[2]).

Similar proofs can be written to verify the other methods used to obtain the initial basic feasible solution.

Now before we cover other methods to obtain the initial basic feasible solution we need to alter the form of the transportation tableau. We need to include the cost constraints in the tableau to obtain the general form of Table 3.5 on page 42.



## Section 3.2 Hartley's Current Line Method

The following method is from Roger Hartley [8]. The steps are as follows:

1. The current line initially is defined to be row 1. Choose the smallest cost in row 1 (the current line) and assign  $\min [a_i, b_j]$  to  $x_{ij}$  corresponding to that cost. Adjust  $a_i$  and  $b_j$  accordingly.
2. Is the current line exhausted?
3. If no, the current line is unchanged. Choose the smallest unused cost in the current line, and assign the value  $\min (a_i, b_j)$  to the  $x_{ij}$  corresponding to that cost. Adjust  $a_i$  and  $b_j$ .
4. If yes, cross out the current line, and the current line will change as follows:  
Call the most recent cost used  $c_{ij}$ . If the current line was row  $i$ , then the current line changes to column  $j$ . Conversely, if the current line was column  $j$ , the current line now changes to row  $i$ . Return to step 2, and continue until all supplies have been depleted, and all demands filled.

When we apply this method to Example 3.1, we get the initial tableau of Table 3.6. So the values are  $x_{12} = \$10$ ,  $x_{14} = \$20$ ,  $x_{23} = \$25$ ,  $x_{24} = \$15$ ,  $x_{31} = \$20$ , and  $x_{32} = \$10$  with the remaining  $x_{ij}$ 's equal to zero. The  $x_{00}$  value is equal to \$830, which is \$120 less than the initial basic feasible solution determined by the northwest corner method. As a matter of fact this initial solution is the optimal solution, as will be shown in the next chapter.

**Table 3.6 Initial Tableau of Example 3.1 using Hartley's  
Current Line Method**

|           | Atlanta | Houston  | L A      | New York | $a_i$ |
|-----------|---------|----------|----------|----------|-------|
| Chicago   | 7<br>0  | 10<br>10 | 14<br>0  | 8<br>20  | 30    |
| Detroit   | 7<br>0  | 11<br>0  | 12<br>25 | 6<br>15  | 40    |
| St. Louis | 5<br>20 | 8<br>10  | 15<br>0  | 9<br>0   | 30    |
| $b_j$     | 20      | 20       | 25       | 35       | 830   |

### Section 3.3 The Minimum Entry Method or the Shortcut Method

Before we look at this method, let's introduce another example from Winston [4].

**Example 3.2** Powerco has three electric power plants that supply the needs of four cities. Each power plant can supply the following number of kilowatt hours (kwh) of electricity: plant 1, 35 million; plant 2, 50 million; plant 3, 40 million. The peak power demands in these cities, which occur at the same time (2 pm) are as follows (in kwh): city 1, 45 million; city 2, 20 million; city 3, 30 million; city 4, 30 million. The cost of sending 1 million kwh of electricity from plant to city depend on the distance the electricity must travel. Minimize the cost of meeting each city's peak power demand.

The cost to supply 1 million kwh to each city is given in Table 3.7. From Table 3.7 we can see that the cost matrix associated with the example is

$$C = \begin{pmatrix} 8 & 6 & 10 & 9 \\ 9 & 12 & 13 & 7 \\ 14 & 9 & 10 & 5 \end{pmatrix} \quad (3.7)$$

The shortcut method is taken from Kwak [3].

1. Begin with the minimum cost in C.
2. Let the minimum cost be in the  $(i, j)$  position in C. Assign to  $x_{ij}$  the value  $\min(a_i, b_j)$ .
3. Adjust  $a_i$  and  $b_j$  to reflect the amount assigned. Cross out the row or column where the supply or demand has been exhausted. If both  $a_i$  and  $b_j$  have been exhausted, cross out the remaining of each row and column.
4. Find the minimum cost from among the remaining entries in C. Go to step 2, and continue until all supplies and demands are exhausted.

When we apply this method to Example 3.2, we get the initial transportation tableau of Table 3.8. This method is so effective in finding an initial basic feasible solution that often the initial basic feasible solution is the optimal solution. The shortcut method gives the initial values of  $x_{ij}$  as follows:  $x_{11} = \$15$ ,  $x_{12} = \$20$ ,  $x_{21} = \$30$ ,  $x_{23} = \$20$ ,  $x_{33} = \$10$ ,  $x_{34} = \$30$ , with the remaining  $x_{ij}$  values equal to zero  $x_{00}$  is equal to \$1080.

**Table 3.7 Example 3.2 Cost Table**

| From \ To | City 1 | City 2 | City 3 | City 4 | Supply |
|-----------|--------|--------|--------|--------|--------|
| Plant 1   | \$8    | \$6    | \$10   | \$9    | 35     |
| Plant 2   | \$9    | \$12   | \$13   | \$7    | 50     |
| Plant 3   | \$14   | \$9    | \$16   | \$5    | 40     |
| Demand    | 45     | 20     | 30     | 30     |        |

**Table 3.8 Initial Tableau of Example 3.2 Using the Shortcut Method**

| From \ To | City 1 | City 2 | City 3  | City 4 | $a_i$  |
|-----------|--------|--------|---------|--------|--------|
| Plant 1   | 8 / 15 | 6 / 20 | 10 / 0  | 9 / 0  | 35     |
| Plant 2   | 9 / 30 | 12 / 0 | 13 / 20 | 7 / 0  | 50     |
| Plant 3   | 14 / 0 | 9 / 0  | 16 / 10 | 5 / 30 | 40     |
| $b_j$     | 45     | 20     | 30      | 30     | \$1080 |

### Section 3.4 The Vogel Advanced Start Method

The Vogel advanced start method (VAM) is named after W. R. Vogel and is taken here from Strayer [10]. The steps are as follows:

1. Compute the difference of the two smallest entries in every row and column of the tableau and write the difference opposite the row or column. If there is only one entry in any row or column, write that entry.
2. Choose the largest difference and use the smallest cost in the corresponding row or column to empty a warehouse ( $a_i$ ) or completely fill a market demand ( $b_j$ ). If there is a tie for the largest difference, use the smallest entry in the corresponding row and/or column. If there is a tie for the smallest entry, use any such entry. Circle the cost used and write above the circle the amount of goods shipped by that route. Reduce the supply and demand in the row and column containing the cost used.
3. Delete the row or column corresponding to the emptied warehouse ( $a_i$ ) or fully supplied market ( $b_j$ ); if both happen simultaneously, delete the row unless that row is the only row remaining in which case delete the column.
4. If all tableau entries are deleted, stop; otherwise go to step 1.

This method uses the cost matrix to get an advanced start in finding the optimal solution. This method is more effective in finding an initial basic feasible solution closer to the optimal solution than the shortcut method. Using Example 3.2 and applying the first run through the procedure we get Table 3.9. Applying

**Table 3.9 Example 3.2 Using VAM (First Run)**

| Difference | 1  | 3  | 3  | 2  | $a_i$ |
|------------|----|----|----|----|-------|
| 2          | 8  | 6  | 10 | 9  | 35    |
| 2          | 9  | 12 | 13 | 7  | 50    |
| 4          | 14 | 9  | 16 | 5  | 40    |
| $b_j$      | 45 | 20 | 30 | 30 |       |

the procedure a second time will give Table 3.10. Continuing to run through the VAM procedure will yield the initial basis feasible solution of Table 3.11.

Table 3.11 shows the initial basic feasible solution of Example 3.2 using the VAM method. The values are as follows:  $x_{12} = \$10$ ,  $x_{13} = \$25$ ,  $x_{21} = \$45$ ,  $x_{23} = \$5$ ,  $x_{32} = \$10$ ,  $x_{34} = \$30$  and all other  $x_{ij} = 0$ . The value for  $x_{00}$  is equal to \$1020. This value is \$60 less than the initial value obtained using the shortcut method.

### Section 3.5 Russell's Approximation Method (RAM)

Before we cover this method, let's present another example. The next example is from Kwak [3].

**Example 3.3** The M&P Company plans to ship its product manufactured in three plants, to four customers in different locations. The amounts supplied and

**Table 3.10 Example 3.2 Using VAM (Second Run)**

| Difference | 1       | 3       | 3        | -       | $a_i$  |
|------------|---------|---------|----------|---------|--------|
| 2          | 8<br>0  | 6<br>10 | 10<br>9  | 9<br>0  | 35     |
| 3          | 9<br>14 | 12<br>9 | 13<br>16 | 7<br>5  | 50     |
| 5          | 0<br>10 | 10<br>0 | 0<br>30  | 30<br>0 | 10(40) |
| $b_j$      | 45      | 20      | 30       | 0 (30)  |        |

**Table 3.11 Example 3.2 Initial Tableau Using VAM**

| From \ To | City 1  | City 2  | City 3   | City 4  | $a_i$  |
|-----------|---------|---------|----------|---------|--------|
| Plant 1   | 8<br>0  | 6<br>10 | 10<br>25 | 9<br>0  | 35     |
| Plant 2   | 9<br>45 | 12<br>0 | 13<br>5  | 7<br>0  | 50     |
| Plant 3   | 14<br>0 | 9<br>10 | 16<br>0  | 5<br>30 | 40     |
| $b_j$     | 45      | 20      | 30       | 30      | \$1020 |

**Table 3.12 Example 3.3 Cost Table**

| From \ To | I  | II | III | IV | Supply |
|-----------|----|----|-----|----|--------|
| A         | 9  | 6  | 8   | 4  | 50     |
| B         | 6  | 4  | 3   | 4  | 40     |
| C         | 5  | 5  | 6   | 7  | 30     |
| D         | 6  | 5  | 7   | 6  | 40     |
| Demand    | 40 | 50 | 35  | 35 | 160    |

demand, together with the unit transportation costs (\$), are given the cost table of Table 3.12.

From the information of Table 3.12, we can see that the cost matrix associated with the example is

$$C = \begin{pmatrix} 9 & 6 & 8 & 4 \\ 6 & 4 & 3 & 4 \\ 5 & 5 & 6 & 7 \\ 6 & 5 & 7 & 6 \end{pmatrix} \quad (3.8).$$

The objective is to minimize cost to transport the product to the four markets.

The Russell approximation method is named for Edward J. Russell [3].

The procedure for the Russell approximation method is as follows:

1. Select the maximum unit transportation costs of each row and each column.

2. Formulate a table consisting of new cost elements. A new cost element is obtained by subtracting the cost element in the table from the sum of the maximum cost of row and the column used.
3. Allocate as many units as feasible to the cell with the largest cost in the new table.
4. Delete any row or column in which either supply or demand is exhausted. Then formulate a new table by following steps 1 and 2.
5. Repeat steps 3 and 4.
6. When all the available resources are fully allocated as required, evaluate the empty cells to test for optimality.

With our first run through Example 3.3 using Russell's approximation method we get Table 3.13. Applying the procedure a second time we get Table 3.14. There is no need to do a third run. Filling in the missing elements is fixed due to the requirements. The result is the initial transportation tableau of Table 3.15. From the information of Table 3.15, we can assign initial values:  $x_{12} = \$15$ ,  $x_{14} = \$35$ ,  $x_{22} = \$5$ ,  $x_{23} = \$35$ ,  $x_{31} = \$30$ ,  $x_{41} = \$10$ ,  $x_{42} = \$30$ , and with the remaining  $x_{ij} = \$0$ . The value for  $x_{00}$  is equal to \$715.

Russell's Approximation Method may be long and drawn out by hand. However, it can be easily programmed into a computer and often generates an initial basic feasible solution closer to the optimal solution than VAM [9]. This has

**Table 3.13 Example 3.3 Using RAM (First Run)**

| Maxima | 9  | 6  | 8  | 7  | $a_i$ |
|--------|----|----|----|----|-------|
| 9      | 9  | 9  | 9  | 12 | 50    |
|        |    |    |    | 35 |       |
| 6      | 9  | 8  | 11 | 9  | 40    |
|        |    |    |    | 0  |       |
| 7      | 11 | 10 | 9  | 7  | 30    |
|        |    |    |    | 0  |       |
| 7      | 11 | 8  | 8  | 8  | 40    |
|        |    |    |    | 0  |       |
| $b_j$  | 40 | 50 | 35 | 35 |       |

**Table 3.14 Example 3.3 Using RAM (Second Run)**

| Maxima | 9  | 6  | 8  | $a_i$ |
|--------|----|----|----|-------|
| 9      | 9  | 9  | 9  | 50    |
|        | 0  |    | 0  |       |
| 6      | 9  | 8  | 10 | 40    |
|        | 0  |    | 35 |       |
| 6      | 10 | 7  | 8  | 30    |
|        | 30 | 0  | 0  |       |
| 7      | 10 | 8  | 8  | 40    |
|        | 10 |    | 0  |       |
| $b_j$  | 40 | 50 | 35 |       |

**Table 3.15 Initial Transportation Tableau of Example 3.3 Using  
RAM**

| From \ To | I       | II      | III     | IV      | $a_i$ |
|-----------|---------|---------|---------|---------|-------|
| A         | 9<br>0  | 6<br>15 | 8<br>0  | 4<br>35 | 50    |
| B         | 6<br>0  | 4<br>5  | 3<br>35 | 4<br>0  | 40    |
| C         | 5<br>30 | 5<br>0  | 6<br>0  | 7<br>0  | 30    |
| D         | 6<br>10 | 5<br>30 | 7<br>0  | 6<br>0  | 40    |
| $b_j$     | 40      | 50      | 35      | 35      | \$715 |

advantages when dealing with large problems. Since the initial basic feasible solution is closer to the optimal solution, it will require less pivots to generate the optimal solution. This will reduce the time to generate the optimal solution. Also fewer pivots when using a computer means less roundoff error, and thus a more accurate answer.

## CHAPTER 4

### FINDING AN OPTIMAL SOLUTION TO THE BALANCED TRANSPORTATION PROBLEM

In this chapter, we will present the loop, or cycle, in the transportation problem tableau, verify theorems regarding the algorithm to find an optimal solution, verify that the transportation problem always has an optimal solution, and present the stepping stone method and u-v method of determining an optimal solution.

#### Section 4.1 Verification That the Transportation Problem Always has an Optimal Solution

Let us again present the transportation problem. The transportation problem is to minimize

$$x_{00} = \sum_{i=1}^m \sum_{j=1}^n c_{ij}x_{ij} \quad (4.1)$$

subject to the following constraints

$$\sum_{j=1}^n x_{ij} = a_i \quad i = 1, 2, \dots, m \quad (4.2)$$

$$\sum_{i=1}^m x_{ij} = b_j \quad j = 1, 2, \dots, n \quad (4.3)$$

and

$$x_{ij} \geq 0 \quad (4.4).$$

The constraints can also be written in the form:

$$\begin{array}{rcl}
 x_{11} + x_{12} + \dots + x_{1n} & & = a_1 \\
 & & \\
 x_{21} + x_{22} + \dots + x_{2n} & & = a_2 \\
 & & \dots \\
 & & \\
 & x_{m1} + x_{m2} + \dots + x_{mn} & = a_m \quad (4.5) \\
 & & \\
 x_{11} & + & x_{21} & + \dots + & x_{m1} & = b_1 \\
 & x_{12} & + & x_{22} & + \dots + & x_{m2} & = b_2 \\
 & & \dots & & & \\
 & x_{1n} & + & x_{2n} & + \dots + & x_{mn} & = b_n .
 \end{array}$$

Now let's write the system (4.5) in matrix notation (from Cooper and Steinberg [7]). We define

$$\hat{y} = [x_{11}, x_{12}, \dots, x_{1n}, x_{21}, x_{22}, \dots, x_{2n}, \dots, x_{m1}, x_{m2}, \dots, x_{mn}]^T,$$

$$\hat{b} = [a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n]^T, \text{ and}$$

$$A = [\hat{a}_{11}, \hat{a}_{12}, \dots, \hat{a}_{1n}, \hat{a}_{21}, \hat{a}_{22}, \dots, \hat{a}_{2n}, \dots, \hat{a}_{m1}, \hat{a}_{m2}, \dots, \hat{a}_{mn}]$$

where  $\hat{a}_{ij}$  is the column of the coefficient matrix  $A$  corresponding to the variable  $x_{ij}$ . Then (4.5) becomes

$$A\hat{y} = \hat{b} \quad (4.5a).$$

Note that all of the coefficients are either 1 or 0. If  $\hat{e}_k$  denotes the  $k$ th column of the identity matrix  $I$  ( $m + n$ ) then  $\hat{a}_{ij} = \hat{e}_i + \hat{e}_{m+j}$ , since as we can see from (4.5)  $x_{ij}$  appears only in the two constraints  $i$  and  $m + j$ .

**Definition 4.1** The  $(i, j)$  cell of the transportation tableau is the "box" in row  $i$  and column  $j$  of the tableau.

**Definition 4.2** A loop or cycle is a sequence of cells such that:

- (a) each adjacent pair of cells lies in either the same row or the same column of the tableau;
- (b) no group of three or more consecutive cells in the sequence lies in the same row or in the same column; and
- (c) the first and last cells in the sequence are in either the same row or the same column.

Every loop is described by a sequence of cells of the form  $(i, j), (k, j), \dots, (u, v), (i, v)$ .

The theorems and proofs which follow unless otherwise stated are from Cooper and Steinberg [7].

**Theorem 4.1** Every loop contains an even number of cells.

**Proof:** Suppose we have a loop containing  $N$  cells. Choose any one of the cells as the starting point, and either of its two adjacent cells in the sequence as the second cell. Number these cells 1 and 2, and continue numbering the remaining cells in the loop, in order, from 3 to  $N$ . Now, cells 1 and 2 either lie in the same row or in the same column. Without loss of generality assume they are in the same row.

Then from cell 2 to cell 3 must be a column change, the step from cell 3 to cell 4 must be a row change, etc. Alternating in this manner, we see that the step from any cell  $t$  to cell  $t + 1$  is a column change if  $t$  is even, and is a row change if  $t$  is

odd. But, the step from cell N to cell 1 must be a column change, since the next change is a row change, from cell 1 to cell 2. Hence, N must be even.

**Theorem 4.2** Let T denote any subset of columns  $\hat{a}_{ij}$  of coefficient matrix A of a transportation problem. Then the columns of T are linearly dependent if and only if the corresponding cells, or a subset of them can be arranged in a sequence S which forms a loop.

**Proof:**

(a) ( $\leftarrow$ ) Assume a sequence of cells forms a loop. Then the sequence is of the form:  $(i, j), (k, j), (k, l), \dots, (u, v), (i, v)$ . The corresponding columns of T are  $\hat{a}_{ij}, \hat{a}_{kj}, \hat{a}_{kl}, \dots, \hat{a}_{uv}, \hat{a}_{iv}$ .

Suppose we alternately add and subtract these columns, forming the linear combination

$$\mathbf{w} = \hat{a}_{ij} - \hat{a}_{kj} + \hat{a}_{kl} - \dots + \hat{a}_{uv} - \hat{a}_{iv} \quad (4.7).$$

Substitution of (4.6) into (4.7) yields

$$\begin{aligned} \mathbf{w} &= (\hat{\mathbf{e}}_i + \hat{\mathbf{e}}_{m+j}) - (\hat{\mathbf{e}}_k + \hat{\mathbf{e}}_{m+j}) + (\hat{\mathbf{e}}_k + \hat{\mathbf{e}}_{m+l}) + \dots + (\hat{\mathbf{e}}_u + \hat{\mathbf{e}}_{m+v}) - (\hat{\mathbf{e}}_i + \hat{\mathbf{e}}_{m+v}) = \\ &= (\hat{\mathbf{e}}_i - \hat{\mathbf{e}}_i) + (\hat{\mathbf{e}}_{m+j} - \hat{\mathbf{e}}_{m+j}) + (\hat{\mathbf{e}}_k - \hat{\mathbf{e}}_k) + \dots + (\hat{\mathbf{e}}_{m+v} - \hat{\mathbf{e}}_{m+v}) = \mathbf{0} \end{aligned}$$

Since there are an even number of terms in the right-hand side of (4.7), a nontrivial linear combination of the columns  $\hat{a}_{ij}, \hat{a}_{kj}, \hat{a}_{kl}, \dots, \hat{a}_{uv}, \hat{a}_{iv}$  equals the null vector, and the set is therefore linearly dependent.

(b) ( $\rightarrow$ ) We must now show that any linearly dependent set T of columns of A has a corresponding sequence of cells which forms a loop. For such a set T, there exist scalars  $\alpha_{ij}$  not all zero such that

$$\sum_{\hat{a}_{ij} \in T} \alpha_{ij} \hat{a}_{ij} = \mathbf{0} \quad (4.8).$$

Henceforth, we shall say that a vector  $\hat{a}_{ij}$  "appears" in the sum of (4.8) if and only if its corresponding scalar  $\alpha_{ij} \neq 0$ .

Suppose that  $\hat{\mathbf{a}}_{ij}$  appears in (4.8); hence the term  $\alpha_{ij}\hat{\mathbf{a}}_{ij} = \alpha_{ij}(\hat{\mathbf{e}}_i + \hat{\mathbf{e}}_{m+j})$  appears. This implies that at least one other  $\hat{\mathbf{a}}_{ij}$  corresponding to row  $i$  of the tableau must also appear in (4.8); if this were not the case, then the  $i$ th component of the vector  $\sum \alpha_{ij}\hat{\mathbf{a}}_{ij}$  would be equal to  $\alpha_{ij} \bullet 1 \neq 0$ , which contradicts equation (4.8). Let  $\hat{\mathbf{a}}_{ik}$  be one such vector appearing in the sum of (4.8),  $k \neq j$ . Then by the same reasoning, there must be at least one other vector appearing in the sum of (4.8) corresponding to column  $j$  and one corresponding to column  $k$ , of the tableau. These conditions can be met only in one of two ways: (1) The vectors  $\hat{\mathbf{a}}_{pj}$  and  $\hat{\mathbf{a}}_{pk}$  both appear in the sum (4.8),  $p \neq i$ . Then a loop is formed by the cells corresponding to the four vectors  $\hat{\mathbf{a}}_{ij}, \hat{\mathbf{a}}_{ik}, \hat{\mathbf{a}}_{pk}, \hat{\mathbf{a}}_{pj}$ . (2) The vectors  $\hat{\mathbf{a}}_{pj}$  and  $\hat{\mathbf{a}}_{qk}$  appear in the sum of (4.8),  $p \neq q$  and  $p, q \neq i$ . In this case the sum of (4.8) must also contain at least one additional vector corresponding to row  $p$  and one corresponding to row  $q$ . These conditions can also be met only in one of two possible ways: either the vectors  $\hat{\mathbf{a}}_{pr}$  and  $\hat{\mathbf{a}}_{qr}$  are in the sum of (4.8), in which case a loop is formed by the cells corresponding to the vectors  $\hat{\mathbf{a}}_{ij}, \hat{\mathbf{a}}_{ik}, \hat{\mathbf{a}}_{qk}, \hat{\mathbf{a}}_{qr}, \hat{\mathbf{a}}_{pr}, \hat{\mathbf{a}}_{pj}$ ; otherwise, two vectors  $\hat{\mathbf{a}}_{pr}$ , and  $\hat{\mathbf{a}}_{qs}$  must appear in the sum of (4.8),  $r \neq s$ . In this case, the same reasoning leads us to the conclusion that either an eight cell loop is formed by the appearance of at least two more vectors corresponding to columns  $r$  and  $s$  of the tableau,  $\hat{\mathbf{a}}_{tr}$  and  $\hat{\mathbf{a}}_{tw}$ ; or that at least four more vectors must appear which, in turn, either leads to the conclusion that a ten

cell loop is formed or else more vectors must appear in the sum of (4.8)

We can continue the above argument only a finite number of times, because there are only a finite number of vectors which can appear in the sum of (4.8). In each step of this argument, we find that the required addition of two vectors either forms a loop, or, in turn requires the addition of at least two more vectors. Finally, we must reach a stage at which we have either formed a loop or must add the last two vectors. Clearly, when all vectors in the tableau are included, a loop is formed.

Hence we have shown that any linearly dependent set of columns of  $A$  has a corresponding sequence of cells which forms a loop.

Now let's define  $y_{kt}$  as the coefficient of the  $k$ th basis vector in the representation of the nonbasic vector  $\hat{a}_{ij}$  in the terms of the current basis vectors.

Also let's define  $\hat{b}_k = k$ th current basis vector,  $k = 1, 2, \dots, (m + n - 1)$ .

**Theorem 4.3** Given any set of basis vectors for the transportation problem, the elements

$$y_{ij}^k = \left\{ \begin{array}{l} \pm 1, \text{ if } \hat{b}_k \text{ is one of the vectors appearing in the} \\ \text{right-hand side of (4.12)} \\ 0, \text{ if } \hat{b}_k \text{ is not one of the vectors in 4.12} \end{array} \right\} \quad (4.9)$$

and

$$y_{ij}^k = \left\{ \begin{array}{l} 1, \text{ if } t \text{ is even} \\ 0, \text{ if } t \text{ is odd} \end{array} \right\} \quad (4.10)$$

where  $\hat{a}_{ij}$  is the only nonbasic vector in a loop and is number 1 and the basic vector

$\vec{b}_k$  is the  $t$ th vector in this list.

**Proof:** Now we have

$$\hat{a}_{ij} = \sum_{k=1}^{m+n-1} y_{ij}^k \vec{b}_k \quad (4.11).$$

The  $m + n$  vectors  $\hat{a}_{ij}, \vec{b}_1, \vec{b}_2, \dots, \vec{b}_{m+n-1}$  are linearly dependent; hence these vectors have a corresponding sequence of cells which forms a loop. Furthermore, the cell  $(i, j)$  must be in this loop, since the vectors  $\vec{b}_1, \vec{b}_2, \dots, \vec{b}_{m+n-1}$  are linearly independent, and their corresponding cells cannot form a loop by Theorem 4.2.

Let this loop consist of the cells  $(i, j), (i, q), (p, q), (p, r), \dots, (u, v), (u, j)$ . This implies that the vectors  $\hat{a}_{iq}, \hat{a}_{pq}, \hat{a}_{pr}, \dots, \hat{a}_{uv}$  and  $\hat{a}_{uj}$  are basic vectors. Then as we have seen in part (a) of the proof of Theorem 4.2,

$$\hat{a}_{ij} - \hat{a}_{iq} + \hat{a}_{pq} - \hat{a}_{pr} + \dots + \hat{a}_{uv} - \hat{a}_{uj} = 0$$

or,

$$\hat{a}_{ij} = \hat{a}_{iq} - \hat{a}_{pq} + \hat{a}_{pr} - \dots - \hat{a}_{uv} + \hat{a}_{uj} \quad (4.12)$$

So when we compare and fit (4.12) into (4.11) we get (4.9).

Furthermore, if we number the vectors corresponding to the cells in the loop in sequence, starting with the nonbasic vector  $\hat{a}_{ij}$  as number 1, so that basic vector  $\vec{b}_k$  is the  $t$ th vector in this list, then

$$y_{ij}^k = \begin{cases} 1, & \text{if } t \text{ is even} \\ 0, & \text{if } t \text{ is odd} \end{cases} \quad (4.10)$$

Now suppose we know which  $(m + n)$  variables are basic and their current values. Let's define  $x_{Bk}, k = 1, 2, \dots, m + n - 1$ , be these current values.

Now let's cover the Transportation Algorithm before we cover the next two theorems.

### Transportation Problem Algorithm

1. Find an initial basic feasible solution.
2. Calculate the quantities  $(z_{pq} - c_{pq})$  for each nonbasic variable. If  $z_{pq} - c_{pq} \leq 0$  for all nonbasic variables, the solution is optimal.
3. Choose the variable  $x_{ij}$  to enter the basis, according to
$$z_{ij} - c_{ij} = \max_{pq} \{z_{pq} - c_{pq}\}.$$
4. Determine the loop formed by the nonbasic cell  $(i, j)$  and the basic cells. Number the cells in this loop consecutively, starting with  $(i, j)$ ; let  $y_{ij}^k = 1$  for even numbered cells, and  $y_{ij}^k = -1$  for odd numbered cells.
5. Choose the variable  $x_{Br}$  to leave the basis, according to
$$x_{Br} = \min_k \{x_{Br} \mid y_{ij}^k = 1\}.$$
6. Compute the new basic feasible solution, by adding the quantity  $x_{Br}$  to each basic variable whose cell is odd numbered in the loop determined in step 4, and by subtracting the quantity  $x_{Br}$  from each basic variable whose cell is even numbered. Also set  $x_{ij} = x_{Br}$ .
7. Return to step 2 [7].

We will cover this in more detail, particularly steps 2 and 3, in the next section. Steps 4, 5, and 6 are important for the next theorem.

**Theorem 4.4** If the  $a_i$  and  $b_j$  for the transportation problem are all nonnegative integers, then every basic feasible solution obtained by the transportation problem algorithm has integral parts [1].

**Proof:** Now we can determine an initial basic feasible solution by using the northwest corner method. Using this method will give integer values for each of the initial basic variables. Now by Theorem 3.2, we know that at most  $m + n - 1$  variables are basic. If less than  $m + n - 1$  variables are basic, we can treat some nonbasic variables as basic to have  $m + n - 1$  basic variables. Let  $x_{Bk}$ ,  $k = 1, 2, \dots, m + n - 1$  be these current values.

Using the transportation algorithm, we choose a nonbasic vector to enter the basis. Suppose we have chosen vector  $\hat{a}_{ij}$ . We then choose the vector to leave  $\hat{b}_r$  from

$$\frac{x_{Br}}{y_{ij}^r} = \min_k \left\{ \frac{x_{Bk}}{y_{ij}^k} \mid y_{ij}^k > 0 \right\} \quad (4.13).$$

Now from Theorem 4.3, we know that if  $y_{ij}^k > 0$ , then  $y_{ij}^k = 1$ . Therefore (4.13) becomes

$$x_{Br} = \min_k \{x_{Bk} \mid y_{ij}^k = 1\} \quad (4.14).$$

Once the vector to enter the basis,  $\hat{a}_{ij}$ , and the vector to leave the basis,  $\hat{b}_r$ , have been determined, the values of the new basic variables are computed by

$$\left\{ \begin{array}{l} x'_{Bk} = x_{Bk} - y_{ij}^k x_{Br}, \quad k \neq r \\ x'_{Br} = x_{Br} \end{array} \right\} \quad (4.15).$$

Now since  $x_{Bk}$  and  $x_{Br}$  are integers for all  $k$  then the new basic variables will be integers, since integers are closed under addition and multiplication. If another nonbasic variable is chosen to enter the basis, the same argument ensures that the new basic variables will be integers. Therefore every basic feasible solution obtained by the transportation problem algorithm has integral parts [7].

Now let's look at one last theorem. This theorem along with the accompanying proof comes from Gass [1].

**Theorem 4.5** A finite minimum feasible solution to the transportation problem always exist.

**Proof:** By Theorem 3.1, the problem is feasible. Since coefficients for equations (4.2) and (4.3) are nonnegative and since we assume that all  $a_i$  and  $b_j$  are nonnegative and finite, no  $x_{ij}$  can be made arbitrarily large. In fact the  $x_{ij}$  cannot be greater than the corresponding  $a_i$  or  $b_j$ . If the  $a_i$  and  $b_j$  are integral, then a basis minimum feasible solution has integral parts. Now we note that, for any basic minimum feasible solution, at least one  $x_{ij}$  equals its corresponding  $a_i$  or  $b_j$ .

In summary, every transportation problem has a finite minimum basic feasible, or optimal, solution. If each  $a_i$  and  $b_j$  have integral values, then the optimal solution will be integral. Also for each element in the transportation tableau, there exist a loop of basic variables.

## Section 4.2 The Stepping Stone Method

Now that we have verified that an optimal solution exists let's look at the algorithm in more detail. We first need to make two adjustments to the transportation tableau of Table 3.5. Since the nonbasic variables are all set equal to zero we can replace them with  $z_{ij} - c_{ij}$ . we can circle the basic variable to

distinguish them from the  $z_{ij} - c_{ij}$  values. The result is the tableau of Table 4.1, which we will use to determine the optimal solution for the transportation problem.

Now when we fill in the tableau on each pivot, we begin by writing in the  $c_{ij}$ 's and basic variables. For the cells of the nonbasic variables, we find a loop of basic variables or cells. Now  $z_{ij} - c_{ij}$  is determined by alternately adding and subtracting each constant value in each respective cell beginning with the cell following the nonbasic variable. To illustrate let's take Example 3.1 with the initial basic feasible solution obtained by the northwest corner method as shown in Table 4.2 on page 66.

Now  $z_{31} = c_{11} - c_{12} + c_{22} - c_{24} + c_{34}$  and  $z_{31} - c_{31} = c_{11} - c_{12} + c_{22} - c_{24} + c_{34} - c_{31} = 7 - 10 + 11 - 6 + 9 - 5 = 6$ . Now when each cell is determined by its respective loop we get Table 4.3. Note that for the basic cells  $z_{ij} - c_{ij} = 0$ .

Now  $z_{ij} - c_{ij}$  is like the  $z_j - c_j$  used in the simplex method. If  $z_{ij} - c_{ij} \leq 0$  for all  $i$  and  $j$ , then the solution is optimal. If there is a cell in which  $z_{ij} - c_{ij} > 0$  then we must pivot; that is pick a new variable to enter the basis. This can be verified in proofs similar to those in Theorem 2.7 and 2.8. This procedure satisfies the second step of the transportation algorithm.

The nonbasic cell with the most positive value will determine the variable to enter the basis. In the case of a tie, just pick one of the variables. In this case, cell (3, 1) will be the new basic variable. This will satisfy the third step of the transportation algorithm.

Table 4.1 General Transportation Tableau with  $z_{ij} - c_{ij}$

| DESTINATIONS |       |                             |                             |                             |   |                             |   |                             |          |  |
|--------------|-------|-----------------------------|-----------------------------|-----------------------------|---|-----------------------------|---|-----------------------------|----------|--|
| $j$<br>$i$   |       | 1                           | 2                           | 3                           | . | $j$                         | . | $n$                         | $a_i$    |  |
| O            | 1     | $c_{11}$<br>$x_{11}$        | $c_{12}$<br>$x_{12}$        | $c_{13}$<br>$z_{13}-c_{13}$ | . | $c_{1j}$<br>$z_{1j}-c_{1j}$ | . | $c_{1n}$<br>$z_{1n}-c_{1n}$ | $a_1$    |  |
|              | 2     | $c_{21}$<br>$z_{21}-c_{21}$ | $c_{22}$<br>$x_{22}$        | $c_{23}$<br>$z_{23}-c_{23}$ | . | $c_{2j}$<br>$z_{2j}-c_{2j}$ | . | $c_{2n}$<br>$z_{2n}-c_{2n}$ | $a_2$    |  |
|              | 3     | $c_{31}$<br>$z_{31}-c_{31}$ | $c_{32}$<br>$x_{32}$        | $c_{33}$<br>$x_{33}$        | . | $c_{3j}$<br>$z_{3j}-c_{3j}$ | . | $c_{3n}$<br>$z_{3n}-c_{3n}$ | $a_3$    |  |
|              | .     | .                           | .                           | .                           | . | .                           | . | .                           | .        |  |
| I            | $i$   | $c_{i1}$<br>$z_{i1}-c_{i1}$ | $c_{i2}$<br>$z_{i2}-c_{i2}$ | $c_{i3}$<br>$z_{i3}-c_{i3}$ | . | $c_{ij}$<br>$x_{ij}$        | . | $c_{in}$<br>$z_{in}-c_{in}$ | $a_i$    |  |
|              | .     | .                           | .                           | .                           | . | .                           | . | .                           | .        |  |
|              | $m$   | $c_{m1}$<br>$z_{m1}-c_{m1}$ | $c_{m2}$<br>$z_{m2}-c_{m2}$ | $c_{m3}$<br>$z_{m3}-c_{m3}$ | . | $c_{mj}$<br>$z_{mj}-c_{mj}$ | . | $c_{mn}$<br>$x_{mn}$        | $a_m$    |  |
| N            | $b_j$ | $b_1$                       | $b_2$                       | $b_3$                       | . | $b_j$                       | . | $b_n$                       | $x_{00}$ |  |

**Table 4.2 Initial Tableau of Example 3.1 with Cost Constraints**

| From \ To |                  | Atlanta | Houston  | L A | New York | $a_i$ |
|-----------|------------------|---------|----------|-----|----------|-------|
|           | $i \backslash j$ | 1       | 2        | 3   | 4        |       |
| Chicago   | 1                | 7<br>20 | 10<br>10 | 14  | 8        | 30    |
| Detroit   | 2                | 7<br>7  | 11<br>10 | 12  | 6<br>5   | 40    |
| St. Louis | 3                | 5<br>6  | 8        | 15  | 9<br>30  | 30    |
| $b_j$     |                  | 20      | 20       | 25  | 35       | 950   |

**Table 4.3 Initial Tableau of Example 3.1 with  $z_{ij} - c_{ij}$**

| From \ To |                  | Atlanta  | Houston  | L A      | New York | $a_i$ |
|-----------|------------------|----------|----------|----------|----------|-------|
|           | $i \backslash j$ | 1        | 2        | 3        | 4        |       |
| Chicago   | 1                | 7<br>20  | 10<br>10 | 14<br>-3 | 8<br>-3  | 30    |
| Detroit   | 2                | 7<br>1   | 11<br>10 | 12<br>25 | 6<br>5   | 40    |
| St. Louis | 3                | 5<br>* 6 | 8<br>6   | 15<br>0  | 9<br>30  | 30    |
| $b_i$     |                  | 20       | 20       | 25       | 35       | 950   |

Steps 4, 5, and 6 can be satisfied by letting the variable entering the basis to be  $\theta$  and subtracting and adding  $\theta$  to each cell in the loop alternately as shown in Table 4.4.  $\theta$  is determined by the value that will cause the first basic element to equal zero; In this case cell (2, 2) or variable  $x_{22}$ . In case two or more cells decrease to zero, pick only one to leave the basis. The transportation algorithm is illustrated in Figure 4.1 as a flow chart on page 69.

Now adjust the basic variables for Example 3.1 with  $x_{22}$  leaving the basis. Then determine each  $z_{ij} - c_{ij}$ . The result can be shown in Table 4.5.

As we can see from Table 4.5, there is a tie with cells (1, 3) and (1, 4). Let's choose  $x_{14}$  to enter the basis. The  $\theta$  value for this pivot will be 10 with  $x_{11}$  determined to leave the basis. Running through the algorithm a third time yields Table 4.6.

Now cell (3, 2) has the most positive  $z_{ij} - c_{ij}$ , therefore  $x_{32}$  will enter the basis.  $x_{34}$  will leave the basis. As we can see from each pivot  $x_{00}$  will decrease each time. The  $\theta$  value for this pivot will again be 10. Another run through the transportation algorithm will give Table 4.7.

Table 4.7 gives the optimal solution since all  $z_{pj} - c_{pj} \leq 0$ . Note that this tableau gives us the initial basic feasible solution for Example 3.1 using Hartley's current line method. Also if  $z_{pq} - c_{pq} = 0$  for a nonbasic cell. This corresponding variable can be substituted into the basis, for a different optimal solution. So for Example 3.1 the optimal solution assigns values as follows:  $x_{12} = \$10$ ,  $x_{14} = \$20$ ,  $x_{23} = \$25$ ,  $x_{24} = \$15$ ,  $x_{31} = \$20$ , and  $x_{32} = \$10$  with the cost,  $x_{00}$ , being \$830.

**Table 4.4 Determining The New Basic Element for Example 3.1**

| From \ To |                  | Atlanta       | Houston       | L A | New York      | $a_i$ |
|-----------|------------------|---------------|---------------|-----|---------------|-------|
|           | $i \backslash j$ | 1             | 2             | 3   | 4             |       |
| Chicago   | 1                | $20 - \theta$ | $10 + \theta$ |     |               | 30    |
| Detroit   | 2                |               | $10 - \theta$ | 25  | $5 + \theta$  | 40    |
| St. Louis | 3                | $\theta$      |               |     | $30 - \theta$ | 30    |
|           |                  | 20            | 20            | 25  | 35            |       |

**Table 4.5 Second Transportation Tableau of Example 3.1**

| From \ To |                  | Atlanta | Houston  | L A      | New York | $a_i$ |
|-----------|------------------|---------|----------|----------|----------|-------|
|           | $i \backslash j$ | 1       | 2        | 3        | 4        |       |
| Chicago   | 1                | 7<br>10 | 10<br>20 | 14<br>3  | 8<br>*3  | 30    |
| Detroit   | 2                | 7<br>-5 | 11<br>-6 | 12<br>25 | 6<br>15  | 40    |
| St. Louis | 3                | 5<br>10 | 8<br>0   | 15<br>0  | 9<br>20  | 30    |
| $b_j$     |                  | 20      | 20       | 25       | 35       | 890   |

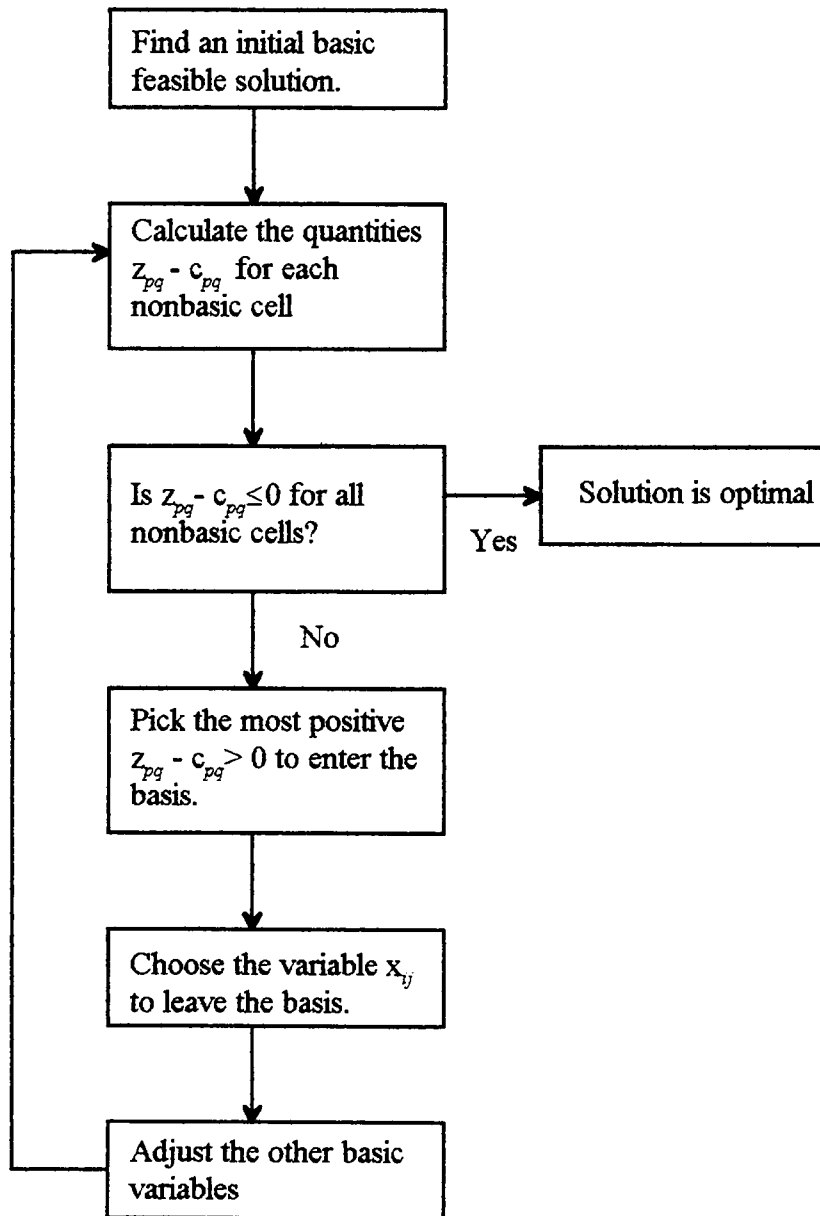


**Figure 4.1** Transportation Problem Algorithm

**Table 4.6 Third Transportation Tableau of Example 3.1**

| To<br>From | $i \backslash j$ | Atlanta | Houston  | L A      | New York | $a_i$ |
|------------|------------------|---------|----------|----------|----------|-------|
|            |                  | 1       | 2        | 3        | 4        |       |
| Chicago    | 1                | 7<br>-3 | 10<br>20 | 14<br>0  | 8<br>10  | 30    |
| Detroit    | 2                | 7<br>-5 | 11<br>-3 | 12<br>25 | 6<br>15  | 40    |
| St. Louis  | 3                | 5<br>20 | 8<br>*3  | 15<br>0  | 9<br>10  | 30    |
| $b_j$      |                  | 20      | 20       | 25       | 35       | 860   |

**Table 4.7 Optimal Transportation Tableau of Example 3.1**

| To<br>From | $i \backslash j$ | Atlanta | Houston  | L A      | New York | $a_i$ |
|------------|------------------|---------|----------|----------|----------|-------|
|            |                  | 1       | 2        | 3        | 4        |       |
| Chicago    | 1                | 7<br>0  | 10<br>10 | 14<br>0  | 8<br>20  | 30    |
| Detroit    | 2                | 7<br>-2 | 11<br>-3 | 12<br>25 | 6<br>15  | 40    |
| St. Louis  | 3                | 5<br>20 | 8<br>10  | 15<br>-3 | 9<br>-3  | 30    |
| $b_j$      |                  | 20      | 20       | 25       | 35       | 830   |

### Section 4.3 The U-V Method

The u-v method is very similar to the stepping stone method. The only difference is with determining each  $z_{pq} - c_{pq}$  for the nonbasic cells. An advantage in using the u-v method is that it does not require finding a loop for each nonbasic variable. Finding a loop requires a little intuition and can be time consuming or even hard to understand. This method is taken from Gass [1] and Kwak [3].

We let a constant  $u_i$  be assigned to each corresponding row and a constant  $v_j$  represent each corresponding column. Now for each constant  $c_{ij}$  corresponding to the basic variable  $x_{ij}$ , we let  $u_i + v_j = c_{ij}$ . We assign one of the u-v values to zero, normally  $u_1$  is chosen. We can then solve for the remaining u-v values. We then define each  $z_{pj}$  for the nonbasic cell to be  $z_{pj} = u_p + v_j$ . Therefore for  $z_{pj} - c_{pj}$  we have

$$z_{pj} - c_{pj} = u_p + v_j - c_{pj}.$$

Now Example 3.2 with the initial basic values as determined by the shortcut method is shown Table 4.8.

Using the information from Table 4.8 we have the following:

$$\begin{array}{lll} u_1 + v_1 = 8 & u_2 + v_1 = 9 & u_3 + v_3 = 16 \\ u_1 + v_2 = 10 & u_2 + v_3 = 13 & u_3 + v_4 = 5. \end{array}$$

Now if we assign  $u_1 = 0$ , then the other values are assigned as follows:  $v_1 = 8$ ,  $v_2 = 10$ ,  $u_2 = 1$ ,  $v_3 = 12$ ,  $u_3 = 4$ , and  $v_4 = 1$ . With this information we can complete the first tableau as shown in Table 4.9.

**Table 4.8 Initial Transportation Tableau of Example 3.2**

| From \ To |       | City 1    | City 2     | City 3     | City 4    | $a_i$ |
|-----------|-------|-----------|------------|------------|-----------|-------|
|           | $v_j$ | $v_1$     | $v_2$      | $v_3$      | $v_4$     |       |
| Plant 1   | $u_1$ | 8<br>(15) | 10<br>(20) | 10         | 9         | 35    |
| Plant 2   | $u_2$ | 9<br>(30) | 12         | 13<br>(20) | 7         | 50    |
| Plant 3   | $u_3$ | 14        | 9          | 16<br>(10) | 5<br>(30) | 40    |
| $b_j$     |       | 45        | 20         | 30         | 30        | 1080  |

**Table 4.9 Initial Transportation Tableau of Example 3.2 Using  
Shortcut Method**

| From \ To |       | City 1    | City 2    | City 3     | City 4    | $a_i$ |
|-----------|-------|-----------|-----------|------------|-----------|-------|
|           | $v_j$ | 8         | 6         | 12         | 1         |       |
|           | $u_i$ |           |           |            |           |       |
| Plant 1   | 0     | 8<br>(15) | 6<br>(20) | 10<br>*2   | 9<br>-8   | 35    |
| Plant 2   | 1     | 9<br>(30) | 12<br>-5  | 13<br>(20) | 7<br>-5   | 50    |
| Plant 3   | 4     | 14<br>-2  | 9<br>1    | 16<br>(10) | 5<br>(30) | 40    |
|           |       | 45        | 20        | 30         | 30        | 1080  |

We can see from Table 4.9 that the initial tableau does not yield the optimal solution. Now cell (1, 3) has the most positive value; therefore,  $x_{13}$  will enter the basis. We must find the loop for this cell to adjust the other basic values and to determine the basic variable to leave the basis. The result is that  $x_{11}$  will leave the basis and the other values are as shown in Table 4.10.

Again from Table 4.10, we do not have the optimal solution. We now have  $x_{32}$  entering the basis and  $x_{33}$  leaving the basis. Another run through the transportation algorithm will give Table 4.11.

The tableau of Table 4.11 gives the optimal solution since  $z_{pj} - c_{pj} \leq 0$  for all the nonbasic cells. This solution is the same initial basic feasible solution obtained for Example 3.2 using the Vogel advanced start method. The basic values for the optimal solution is as follows:  $x_{12} = \$10$ ,  $x_{13} = \$25$ ,  $x_{21} = \$45$ ,  $x_{23} = \$5$ ,  $x_{32} = \$10$ ,  $x_{34} = \$30$  and  $x_{00} = \$1020$ .

The initial basic feasible solution for Example 3.3 using the Russell approximation method is the optimal solution. This can be verified by using the stepping stone method or the u-v method as shown in Table 4.12.

The basic values for Example 3.3 as verified from Table 4.12 are as follows:  $x_{12} = \$15$ ,  $x_{14} = \$35$ ,  $x_{22} = \$5$ ,  $x_{23} = \$35$ ,  $x_{31} = \$30$ ,  $x_{41} = \$10$ ,  $x_{42} = \$30$ , and  $x_{00} = \$160$ .

**Table 4.10 Second Transportation Tableau of Example 3.2**

| From \ To |       | City 1   | City 2   | City 3   | City 4   | $a_i$ |
|-----------|-------|----------|----------|----------|----------|-------|
|           | $v_j$ | 6        | 6        | 10       | -1       |       |
|           | $u_i$ |          |          |          |          |       |
| Plant 1   | 0     | 8<br>-2  | 6<br>20  | 10<br>15 | 9<br>-10 | 35    |
| Plant 2   | 3     | 9<br>45  | 12<br>-3 | 13<br>5  | 7<br>-5  | 50    |
| Plant 3   | 6     | 14<br>-2 | 9<br>*3  | 16<br>10 | 5<br>30  | 40    |
| $b_j$     |       | 45       | 20       | 30       | 30       | 1050  |

**Table 4.11 Third (Optimal) Tableau for Example 3.2**

| From \ To |       | City 1   | City 2   | City 3   | City 4  | $a_i$ |
|-----------|-------|----------|----------|----------|---------|-------|
|           | $v_j$ | 6        | 6        | 10       | 2       |       |
|           | $u_i$ |          |          |          |         |       |
| Plant 1   | 0     | 8<br>-2  | 6<br>10  | 10<br>25 | 9<br>-7 | 35    |
| Plant 2   | 3     | 9<br>45  | 12<br>-3 | 13<br>5  | 7<br>-2 | 50    |
| Plant 3   | 3     | 14<br>-5 | 9<br>10  | 16<br>-3 | 5<br>30 | 40    |
| $b_j$     |       | 45       | 20       | 30       | 30      | 1020  |

**Table 4.12 Optimal Solution For Example 3.3**

| From \ To |       | I       | II      | III     | IV      | $a_i$ |
|-----------|-------|---------|---------|---------|---------|-------|
|           | $v_j$ | 7       | 6       | 5       | 4       |       |
|           | $u_i$ |         |         |         |         |       |
| A         | 0     | 9<br>-2 | 6<br>15 | 8<br>-3 | 4<br>35 | 50    |
| B         | -2    | 6<br>-1 | 4<br>5  | 3<br>35 | 4<br>-2 | 40    |
| C         | -2    | 5<br>30 | 5<br>-1 | 6<br>-3 | 7<br>-5 | 30    |
| D         | -1    | 6<br>10 | 5<br>30 | 7<br>-3 | 6<br>-3 | 40    |
| $b_j$     |       | 40      | 50      | 35      | 35      | 715   |

## CHAPTER 5

### SPECIAL CASES INVOLVING THE TRANSPORTATION PROBLEM

In this chapter we will look at some special cases using the transportation problem. These special cases include the maximization problem, unbalanced transportation problem, restricted routes, dual problems, and the transshipment problem.

#### Section 5.1 The Maximization Problem

The transportation problem involving a maximization of profits is not used nearly as often as the minimization problem we have been covering. The maximization problem is as follows:

Find values for the variables  $x_{ij}$  which maximize

$$Z = x_{00} = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} \quad (5.1)$$

subject to the constraints

$$\sum_{j=1}^n x_{ij} = a_i \quad i = 1, 2, \dots, m \quad (5.2)$$

$$\sum_{i=1}^m x_{ij} = b_j \quad j = 1, 2, \dots, n \quad (5.3)$$

and

$$x_{ij} > 0 \quad (5.4)$$

where each  $a_i$ ,  $b_j$ , and  $c_{ij}$  are given constraints.

Finding the initial basic feasible solution for the maximization problem is similar to those for the minimization problem with the emphasis being placed on the higher cost constraints. The transportation problem algorithm is the same except we want every  $z_{pq} - c_{pq} \geq 0$  and if we must pivot we choose the most negative  $z_{pq} - c_{pq}$  to enter the basis as shown below.

### Transportation Problem Algorithm for the Maximization Problem

1. Find an initial basic feasible solution.
2. Calculate the quantities  $(z_{pq} - c_{pq})$  for each nonbasic variable. If  $z_{pq} - c_{pq} \geq 0$  for all nonbasic variables the solution is optimal. If at least one  $z_{pq} - c_{pq} < 0$ , proceed to step 3.
3. Choose the variable  $x_{ij}$  to enter the basis, according to 
$$z_{ij} - c_{ij} = \min_{pq} \{z_{pq} - c_{pq}\}.$$
4. Determine the loop formed by the nonbasic cell  $(i, j)$  and the basic cells. Number the cells in this loop consecutively, starting with  $(i, j)$ ; let  $y_{ij}^k = 1$  for even numbered cells, and  $y_{ij}^k = -1$  for odd numbered cells.
5. Choose the variable  $x_{Br}$  to leave the basis, according to 
$$x_{Br} = \min_k \{x_{Br} \mid y_{ij}^k = 1\}.$$
6. Compute the new basic feasible solution, by adding the quantity  $x_{Br}$  to each basic variable whose cell is odd numbered in the loop determined in step 4, and by subtracting the quantity  $x_{Br}$  from each basic variable whose cell is even numbered. Also set  $x_{ij} = x_{Br}$ .
7. Return to step 2 [7].

**Example 5.1** The Vol-Mart Company plans to ship its product, manufactured in 3 plants, to four customers in different locations. The amounts supplied and demanded, together with the unit profits (\$) are given in Table 5.1.

The Hartley line method is used to determine the initial basic feasible solution. The first run through the transportation algorithm gives the initial tableau of Table 5.2.

As we can see from Table 5.2, cell (3, 3) is chosen to enter the basis. Cell (3, 4) will leave the basis. A second run through the transportation algorithm will give us the optimal tableau of Table 5.3.

The optimal tableau of Table 5.3 could have been determined initially using the shortcut method. The optimal values then are as follows:  $x_{13} = \$20$ ,  $x_{14} = \$30$ ,  $x_{22} = \$30$ ,  $x_{23} = \$10$ ,  $x_{31} = \$40$ ,  $x_{33} = \$20$ , and  $z = \$930$ .

**Table 5.1 Example 5.1 Cost Table**

| <div>From \ To</div> | I  | II | III | IV | Supply |
|----------------------|----|----|-----|----|--------|
| A                    | 5  | 4  | 5   | 7  | 50     |
| B                    | 4  | 8  | 6   | 5  | 40     |
| C                    | 6  | 3  | 4   | 5  | 60     |
| Demand               | 40 | 30 | 50  | 30 | 150    |

**Table 5.2 Initial Tableau for Example 5.1**

| From \ To | I       | II      | III      | IV      | $a_i$ |
|-----------|---------|---------|----------|---------|-------|
| A         | 5<br>3  | 4<br>3  | 5<br>40  | 7<br>10 | 50    |
| B         | 4<br>5  | 8<br>30 | 6<br>10  | 5<br>3  | 40    |
| C         | 6<br>40 | 3<br>2  | 4<br>*-1 | 5<br>20 | 60    |
| $b_j$     | 40      | 30      | 50       | 30      | 910   |

**Table 5.3 Optimal Tableau for Example 5.1**

| From \ To | I       | II      | III     | IV      | $a_i$ |
|-----------|---------|---------|---------|---------|-------|
| A         | 5<br>2  | 4<br>3  | 5<br>20 | 7<br>30 | 50    |
| B         | 4<br>4  | 8<br>30 | 6<br>10 | 5<br>3  | 40    |
| C         | 6<br>40 | 3<br>3  | 4<br>20 | 5<br>1  | 60    |
| $b_j$     | 40      | 30      | 50      | 30      | 930   |

## Section 5.2 Unbalanced Transportation Problem

**Definition 5.1** An unbalanced transportation problem is the transportation problem with unequal supply and demand.

There are two types of unbalanced transportation problems; (1) supply exceeds demand, and (2) demand exceeds supply. The problem of uneven supply and demand is easily remedied by adding a "dummy" column, where supply exceeds demand; or a "dummy" row, where demand exceeds supply, with cost 0. When we find the initial basic feasible solution, the "dummy" column or row is ignored until the other rows and columns are filled. Another way to handle a situation where demand exceeds supply and where overtime is possible to remedy the problem, additional rows can be added with overtime and transport costs to balance the problem.

The following is from Kwak [3] and is an example where supply exceeds demand.

**Example 5.2** Let's return to the Burns Manufacturing Company which has plants in Chicago, Detroit, and St. Louis. On a particular day, the company has 30 sewing machines in Chicago, 40 in Detroit, and 30 in St. Louis. The company plans to ship 20 sewing machines to Atlanta, 20 to Houston, 25 to Los Angeles, and 30 to New York. The cost matrix is the same as for Example 3.1 (see

Table 3.2, page 34). The first run through the transportation algorithm with the shortcut method used to find the initial solution gives the initial tableau of Table 5.4.

The initial tableau shows that the initial solution is also the optimal solution with the following basic values:  $x_{12} = \$10$ ,  $x_{13} = \$15$ ,  $x_{15} = \$5$ ,  $x_{23} = \$10$ ,  $x_{24} = \$30$ ,  $x_{31} = \$20$ ,  $x_{32} = \$10$ , and  $z = \$790$ .

**Table 5.4 Initial and Optimal Tableau to Example 5.2**

| From \ To | Atlanta | Houston  | L A      | New York | Dummy   | $a_i$ |
|-----------|---------|----------|----------|----------|---------|-------|
| Chicago   | 7<br>0  | 10<br>10 | 14<br>15 | 8<br>0   | 0<br>5  | 30    |
| Detroit   | 7<br>-2 | 11<br>-3 | 12<br>10 | 6<br>30  | 0<br>-2 | 40    |
| St. Louis | 5<br>20 | 8<br>10  | 15<br>-3 | 9<br>-3  | 0<br>-2 | 30    |
| $b_j$     | 20      | 20       | 25       | 30       | 5       | 790   |

### Section 5.3 Prohibited Routes

Prohibited routes pose another problem for the transportation problem. Reasons for prohibited routes include road construction, weight limits on bridges, unexpected floods, and local traffic ordinances.

One way to handle prohibited routes in the cost matrix is to assign an exceptionally large value to that route which we will call M. This will ensure that the prohibited route is not taken on the transportation tableau. An alternative to using M on the transportation tableau is simply cross out that route with an X. This removes this route from consideration [3].

**Example 5.3** The St. Louis Metropolitan Transportation Company has five relief buses in Garage A, five in Garage B, and eight in Garage C. During rush hours, the traffic manager wishes to dispatch four relief buses to Clayton, five to Crestwood, five to Ferguson, and four to Overland. The dispatch of relief buses from Garage A to Overland and from Garage C to Ferguson is temporarily impossible because of road repair work. The expected number of minutes for a relief bus to reach each of four destinations from each garage is given in Table 5.5.

The resulting initial tableau using the Hartley line method is as shown in Table 5.6. As we can see from Table 5.6 on page 80, the initial tableau is again the optimal tableau. Therefore the basic optimal solution is  $x_{11} = 4$  minutes,  $x_{13} = 1$  minute,  $x_{23} = 4$  minutes,  $x_{24} = 1$  minute,  $x_{32} = 5$ ,  $x_{34} = 3$  minutes and  $z = 261$  minutes.

## Section 5.4 Transshipment Problem

The transportation problem allows only shipments from supply points to demand points. The transshipment problem allows some points (called transshipment points) to act as both a supply point and a demand point; that is,

**Table 5.5 Time Cost Table for Example 5.3**

| From \ To | Clayton | Crestwood | Ferguson | Overland | Supply |
|-----------|---------|-----------|----------|----------|--------|
| Garage A  | 12      | 17        | 20       | M        | 5      |
| Garage B  | 18      | 20        | 12       | 15       | 5      |
| Garage C  | 20      | 10        | M        | 16       | 8      |
| Demand    | 4       | 5         | 5        | 4        | 18     |

**Table 5.6 Initial and Optimal Tableau for Example 5.3**

| From \ To | Clayton  | Crestwood | Ferguson      | Overland      | $a_i$ |
|-----------|----------|-----------|---------------|---------------|-------|
| Garage A  | 20<br>4  | 17<br>0   | 20<br>1       | <del>15</del> | 5     |
| Garage B  | 18<br>-6 | 20<br>-5  | 12<br>4       | 15<br>1       | 5     |
| Garage C  | 20<br>-7 | 10<br>5   | <del>12</del> | 16<br>3       | 8     |
| $b_j$     | 4        | 5         | 5             | 4             | 261   |

points that can both give and receive goods [4]. The transshipment problem has algorithms used to solve the problem that are different from the transportation problem. However, the transshipment problem can also be treated as a transportation problem as Example 5.4 shows.

**Example 5.4** Widgetco manufactures widgets at two factories, one in Memphis and one in Denver. The Memphis factory can produce up to 150 widgets per day, and the Denver factory can produce up to 200 widgets per day. Widgets are shipped by air to customers in Los Angeles and Boston. The customers in each city require 130 widgets per day. Because of the deregulation of air fares, Widgetco believes that it may be cheaper to first fly some widgets to New York and Chicago and then fly them to their final destinations. Figure 5.1 shows the air shipping costs between each city per widget.

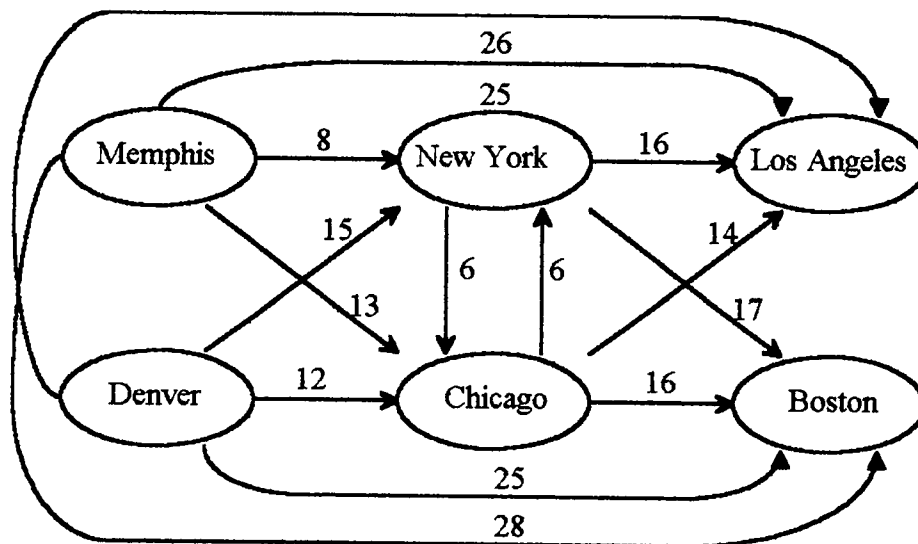


**Figure 5.1** Air Routes and Costs for Example 5.4

To set this problem up as a transportation problem, we establish the transshipment points as supply points and destination points with supply and demand equal to the total supply. The transportation cost for the transshipment point to itself is of course zero. Also since supply exceeds demand, that is the problem is unbalanced, a dummy column is necessary. The amounts supplied and demanded also with the cost per item to transport is given in the cost table in Table 5.7 and the initial transportation tableau is shown in Table 5.8 using the shortcut method.

Two runs through the transportation algorithm gives the optimal solution of Table 5.9. The optimal route is given in Figure 5.2 with 20 widgets remaining in Memphis and 70 widgets remaining in Denver. The optimal cost is \$6370.

**Table 5.7 Cost Table for Example 5.4**

| From \ To | New York | Chicago | L A | Boston | Dummy | Supply |
|-----------|----------|---------|-----|--------|-------|--------|
| Memphis   | 8        | 13      | 25  | 28     | 0     | 150    |
| Denver    | 15       | 12      | 26  | 25     | 0     | 200    |
| New York  | 0        | 6       | 16  | 17     | 0     | 350    |
| Chicago   | 6        | 0       | 14  | 16     | 0     | 350    |
| Demand    | 350      | 350     | 130 | 130    | 90    |        |

**Table 5.8 Initial Tableau for Example 5.4**

| From \ To | New York   | Chicago    | L A         | Boston      | Dummy     | $a_i$ |
|-----------|------------|------------|-------------|-------------|-----------|-------|
| Memphis   | 8<br>(0)   | 13<br>1    | 25<br>(130) | 28<br>-3    | 0<br>(20) | 150   |
| Denver    | 15<br>-7   | 12<br>2    | 26<br>-1    | 25<br>(130) | 0<br>(70) | 200   |
| New York  | 0<br>(350) | 6<br>(0)   | 16<br>1     | 17<br>0     | 0<br>-8   | 350   |
| Chicago   | 6<br>-12   | 0<br>(350) | 14<br>-1    | 16<br>-3    | 0<br>-12  | 350   |
| $b_j$     | 350        | 350        | 130         | 130         | 90        | 6500  |

**Table 5.9 Optimal Tableau For Example 5.4**

| From \ To | New York   | Chicago    | L A         | Boston      | Dummy     | $a_i$ |
|-----------|------------|------------|-------------|-------------|-----------|-------|
| Memphis   | 8<br>(130) | 13<br>-1   | 25<br>-1    | 28<br>-3    | 0<br>(20) | 150   |
| Denver    | 15<br>-7   | 12<br>(0)  | 26<br>-2    | 25<br>(130) | 0<br>(70) | 200   |
| New York  | 0<br>(220) | 6<br>-2    | 16<br>(130) | 17<br>0     | 0<br>-8   | 350   |
| Chicago   | 6<br>-2    | 0<br>(350) | 14<br>-2    | 16<br>-3    | 0<br>-12  | 350   |
| $b_j$     | 350        | 350        | 130         | 130         | 90        | 6370  |

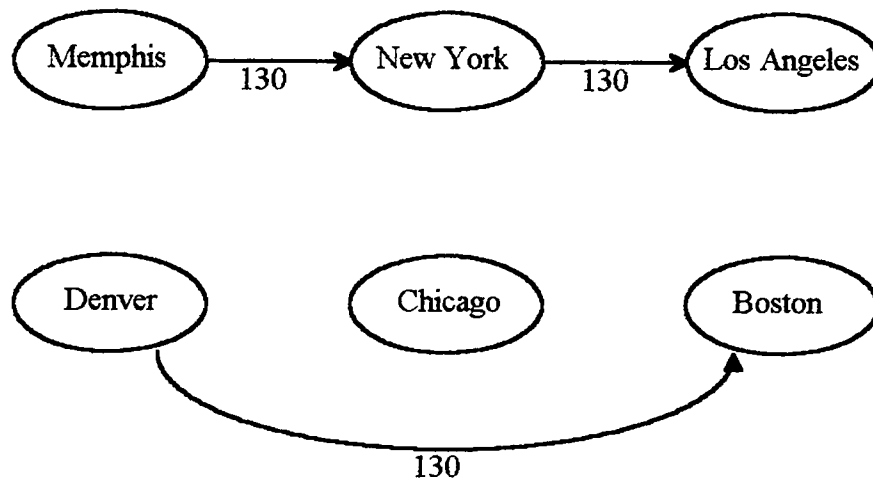


**Figure 5.2 Optimal Route and Quantities for Example 5.4**

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