To the Graduate Council:

I am submitting herewith a thesis written by Alan T. Gibby entitled "Solving an Eigenvalue-Eigenvector Problem in Minimax Algebra." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

Yueh-er Kuo

Yueh-er Kuo, Major Professor

We have read this thesis
and recommend its acceptance:

Accepted for the Council:

Vice Chancellor
Graduate Studies and Research

SOLVING AN EIGENVALUE-EIGENVECTOR PROBLEM IN MINIMAX ALGEBRA

A Thesis

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Alan T. Gibby

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ABSTRACT

The purpose of this paper is to solve an example of an eigenvalue-eigenvector problem which is formulated in Minimax Algebra. This example is motivated by the Industrial Machine example which is explained in detail in Chapter 1. Also in Chapter 1 we list the basic definition of the binary operations. Next we give the standard axioms on which Minimax Algebra is based.

In Chapter 2, we give some of the theoretical results that arise in Minimax Algebra. Also, the underlying algebraic structures in which the eigenproblem may be viewed is given. A summary of these structures is given in Definition 2.2.

Next, in Chapter 3, we give the related theorems, lemmas, propositions, and corollaries that are needed to establish the procedure for calculating an eigenproblem. These statements make use of the ideas given in Chapter 2, along with some ideas from graph theory.

Finally, in Chapter 4, we state a particular eigenproblem and solve it using procedures that are established in the previous chapters.

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CHAPTER 1

INTRODUCTORY CONCEPTS OF MINIMAX ALGEBRA

A study of Minimax Algebra begins with a study of an algebra of real numbers. We will introduce this algebra by giving the definition for binary operations, a list of axioms, and an example to which this algebra may be applied.

Definition 1.1. The binary operations for minimax algebra are as follows:

(I) (i) Let $x \otimes y$ mean the arithmetical sum of two numbers i.e. "$x + y$" .

(ii) Let the selection of the larger of two numbers be denoted by $x \oplus y$ which is also "$\max(x, y)$" . Also we have $\max(x_1, \ldots, x_n) = x_1 \oplus x_2 \oplus \ldots \oplus x_n$.

(II) (i)' Let $x \otimes y$ be the arithmetical sum of two numbers "$x + y$" .

(ii)' Let the selection of the smaller of two numbers, in symbols, be $x \oplus' y$ or equivalently "$\min(x, y)$" . Again we also have $\min(x_1, \ldots, x_n) = x_1 \oplus' x_2 \oplus' \ldots \oplus' x_n$.

Notice that operations (i) and (i)' are the same operation. The sets of operations (I) and (II) are said to be dual operations.

We next state a list of axioms on which Minimax Algebra is based. Let S be any set of real numbers, then for all $x, y \in S$ we have:

A_1: $x \oplus (y \oplus' x) = x \oplus' (y \oplus x) = x$

A_2: $x \oplus (y \oplus x) = (x \oplus y) \oplus z$

A_3: $x \oplus y = y \oplus x$

A_4: $\forall\ x, y \in S$, $x \oplus y = x$ or $x \oplus y = y$

A_5: $x \oplus x = x$

A_6: $x \otimes (y \otimes z) = (x \otimes y) \otimes z$

A_7: $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$

A_8: $(y \oplus z) \otimes x = (y \otimes x) \oplus (z \otimes x)$

A_9: $x \oplus' (y \oplus' z) = (x \oplus' y) \oplus' z$

A_{10}: $x \oplus' y = y \oplus' x$

A_{11}: $x \oplus' x = x$

A_{12}: $x \otimes' (y \otimes' z) = (x \otimes' y) \otimes' z$

A_{13}: $x \otimes' (y \oplus' z) = (x \otimes' y) \oplus' (x \otimes' z)$

A_{14}: $(y \oplus' z) \otimes' z = (y \otimes' x) \oplus' (z \otimes' x)$

A_{15}: $\exists\ \phi \in S$ such that $\phi \otimes x$ and $x \otimes \phi$ both equal x for all $x \in S$.

A_{16}: $\exists\ \theta \in S$ such that for all $x \in S$, $x \oplus \theta = x$ and $\theta \oplus x$, $x \otimes \theta$ both equal θ .

A_{17}: $\forall x \in S$, $\exists\ x' \in S$ such that $x \otimes x' = \phi$ and $x \otimes' x' = \phi$.

A_{18}: If $E_1 \subset S$ (where E_1 is the set of 1-tuples), then

$$\forall x \in E\ ,\quad [\sum_{a \varepsilon E_1}{}^{\oplus'} (a^* \otimes' a)] \otimes x \geq x \quad \text{and}$$

$$x \otimes [\sum_{a \varepsilon E_1}{}^{\oplus'} (a \otimes' a^*)] \geq x\ .$$

Here the symbol "$\geq$" is defined to be the usual linear ordering relation and "*" means the element which corresponds to a under some bijective mapping.

A_{19}: $x \otimes (y \otimes' z) \leq (x \otimes y) \otimes' z$ and $x \otimes' (y \otimes z) \geq (x \otimes' y) \otimes z$

A_{20}: For all w, x, y, z in a belt V the following hold:

If $w \geq x$ then $y \otimes w \geq y \otimes x$ and $w \otimes y \geq x \otimes y$.

If $w \geq x$ and $y \geq z$ then $w \otimes y \geq x \otimes z$.

A_{21}: $\forall x, y \in S$, $x \otimes y = y \otimes x$.

Next we give an example to which Minimax Algebra may be applied. This example, which we will call the Industrial Machine Example, gives rise to the eigenvalue-eigenvector problem which is the topic of this paper.

To begin with, we have a situation where there are several interdependent machines involved in some industrial process. The relationship between the machines is this: A particular machine cannot begin its cycle until certain others have completed their cycles. Thus, begin by labeling the machines $1, 2, 3, \ldots, n$. The recurrence relations between the machines takes on the form $x_i(r + 1) = \max(x_1(r) + t_1(r), x_2(r) + t_2(r), \ldots, x_i(r) + t_i(r), \ldots, x_n(r) + t_n(r))$. Here x_i denotes the i^{th} machine; $x_i(r)$ is the starting time of the r^{th} cycle for the i^{th} machine. Intuitively what this says is that machine x_i must wait to begin its $(r + 1)$ cycle until the other machines have finished their r^{th} cycle, where the duration of operation for each machine is $t_i(r)$. In general the recurrence relation formula may be stated as follows:

$$(1.1) \qquad x_i(r + 1) = \max(x_1(r) + b_{i1}, x_2(r) + b_{i2}, \ldots, x_n(r) + b_{in})$$

$$\text{where } i = 1, 2, \ldots, n .$$

Now here the terms $b_{ij}(r)$ and $x_j(r)$ may have no physical significance. If this is the case, then the term $-\infty$ is assigned to them, and the max operator simply ignores them. In other words this formulation allows for all possible relationships between the machines to be expressed.

Expression (1.1) can now be stated as $(b_{i1}(r) \otimes x_1(r)) \oplus \ldots \oplus (b_{in}(r) \otimes x_n(r))$ by Definition 1.1(ii). We can now introduce the matrix notation corresponding to these recurrence relations: $\underline{B}(r) = [b_{ij}(r)]$ and $\underline{X}(r) = [x_i(r)]$. Now we have the precise formulation $\underline{X}(r + 1) = \underline{B}(r) \otimes \underline{X}(r)$, which in long notation becomes

$$\begin{bmatrix} x_1(r+1) \\ x_2(r+1) \\ \vdots \\ x_i(r+1) \\ \vdots \\ x_n(r+1) \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{in} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \vdots \\ b_{i1} & b_{i2} & \cdots & b_{in} \\ \vdots & & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \otimes \begin{bmatrix} x_1(r) \\ x_2(r) \\ \vdots \\ x_i(r) \\ \vdots \\ x_n(r) \end{bmatrix} .$$

For an example, see Appendix 4. The formula $\underline{X}(r + 1) = \underline{B}(r) \otimes \underline{X}(r)$ can be thought of in an iterative fashion:

$$\underline{X}(2) = \underline{B}(1) \otimes \underline{X}(1)$$

$$\underline{X}(3) = \underline{B}(2) \otimes \underline{X}(2) = \underline{B}(2) \otimes \underline{B}(1) \otimes \underline{X}(1)$$

$$\underline{X}(4) = \underline{B}(3) \otimes \underline{X}(3) = \underline{B}(3) \otimes \underline{B}(2) \otimes \underline{B}(1) \otimes \underline{X}(1)$$

$$\vdots$$

$$\underline{X}(r + 1) = \underline{B}(r) \otimes \underline{X}(r) = \underline{B}(r) \otimes \underline{B}(r - 1) \otimes \ldots \otimes \underline{B}(1) \otimes \underline{X}(1)$$

If we assume that the matrix $\underline{B}(r)$ is not a function of r (i.e. $\underline{B}$ is the same matrix for $\underline{B}(1)$, $\underline{B}(2), \ldots, \underline{B}(r)$), then $\underline{X}(r + 1) = \underline{B}^r \otimes \underline{X}(1)$. This models the behavior of the systems of machines over a period of time.

The question "how must the system be set in motion to ensure that it progresses in regular intervals?" leads to the eigenvalue-eigenvector problem. More precisely stated, this becomes "for what constant λ will the interval between consecutive cycles on every machine be λ ?" This leads us to solve $\underline{X}(r + 1) = \lambda \otimes \underline{X}(r)$. But $\underline{X}(r + 1) = \underline{B} \otimes \underline{X}(r)$, which implies that we solve $\underline{B} \otimes \underline{X}(r) = \lambda \otimes \underline{X}(r)$ which is an eigenvector-eigenvalue problem. We will later solve this problem. Now we give the needed theoretical background to solve the problem.

CHAPTER 2

THEOREMS AND DEFINITIONS FROM MINIMAX ALGEBRA

In this chapter we present some of the theory behind Minimax Algebra. The motivation here is to list the results that will be needed as background for the resolution of the eigenproblem. The following material may be found in greater detail in Cuninghame-Green "Minimax Algebra", if the reader wishes to investigate further.

We begin by giving a slight modification of the two basic binary operations.

Definition 2.1. $\forall x, y \in S$ let $x \oplus y = x$ if $x \geq y$

$= y$ otherwise.

Here $\oplus$ is denoted as "max". Similarly we have for the dual operation:

$$\forall x, y \in S \quad \text{let} \quad x \oplus' y = x \quad \text{if} \quad y \geq x$$
$$= y \quad \text{otherwise.}$$

Here $\oplus'$ is denoted as "min".

Next, we will look at some of the algebraic structures that are found in Minimax Algebra. These structures may be listed in the following two part definition.

Definition 2.2. Let S be a set of real numbers and let the binary operations $\oplus, \oplus', \otimes, \otimes'$ be defined as in Definition 1.1. The following table (Table 2.1) lists the various algebraic structures in Minimax Algebra and the axioms which they satisfy.

TABLE 2.1

ALGEBRAIC STRUCTURES

	AXIOMS																
Algebraic Structures	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	A_{17}	A_{21}
$(S, \oplus)$		√	√	√	√												
$(S, \oplus, \otimes)$		√	√	√	√	√	√	√									
$(S, \oplus, \otimes)_1$		√	√	√	√	√	√	√									√
$(S, \oplus, \otimes)_2$		√	√	√	√	√	√	√							√		
$(S, \oplus, \otimes)_3$		√	√	√	√	√	√	√							√	√	
$(S, \oplus, \otimes, \oplus', \otimes')$	√	√	√	√	√	√	√	√	√	√	√	√	√	√			
$(S, \oplus, \otimes, \oplus', \otimes')_1$	√	√	√	√	√	√	√	√	√	√	√	√	√	√			√
$(S, \oplus, \otimes, \oplus', \otimes')_2$	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√		
$(S, \oplus, \otimes, \oplus', \otimes')_3$	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√	
$(S, \oplus, \oplus')$		√	√	√	√				√	√	√						

$(S, \oplus)$ is called a <u>commutative band</u>.

$(S, \oplus, \otimes)$ is called a <u>belt</u>. Notice here that $(S, \oplus)$ is implicitly assumed to be a commutative band under the appropriate axioms.

$(S, \oplus, \otimes)_1$ is called a <u>commutative belt</u>.

$(S, \oplus, \otimes)_2$ is called a <u>belt with identity</u>.

$(S, \oplus, \otimes)_3$ is called a <u>division belt</u>.

$(S, \oplus, \otimes, \oplus', \otimes')$ is called a <u>belt with duality</u>. The differences here between a belt and a belt with duality are: (i) the inclusion of the dual operations $\oplus'$, $\otimes'$; (ii) a belt with duality satisfies axioms A_{10} through A_{15} under $\oplus'$, $\otimes'$; (iii) a belt with duality satisfies axiom A_1 which expresses the correlation between $\oplus$ and $\oplus'$.

$(S, \oplus, \otimes, \oplus', \otimes')_1$ is a <u>commutative belt with duality</u>.

$(S, \oplus, \otimes, \oplus', \otimes')_2$ is a <u>belt with duality with identity</u>.

$(S, \oplus, \otimes, \oplus', \otimes')_3$ is a <u>division belt with duality</u>.

$(S, \oplus, \oplus')$ is a <u>commutative band with duality</u>.

As an example consider the two systems $(R, \max, +)$ and $(R, \min, +)$. These two systems are respectively referred to as the <u>principal interpretation</u> and the <u>dual principal interpretation</u>. Both are examples of commutative division belts with identity.

We now list as another definition the idea of a blog. This structure forms the abstract framework in which the eigenproblem can be cast.

<u>Definition 2.3</u>. Let $(G, \oplus, \otimes)$ be a division belt. Now let $W = G \cup \{-\infty\} \cup \{+\infty\}$, i.e., $\forall x \in G$, $+\infty$ and $-\infty$ are the adjoined elements.

We define a new system $(W, \oplus, \otimes, \oplus', \otimes')$ which satisfies the conditions listed in Table 2.2.

TABLE 2.2

BLOG STRUCTURE

x	y	$x \oplus y$	$x \oplus' y$	$x \otimes y$	$x \otimes' y$
$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$-\infty$	$y \in G$	y	$-\infty$	$-\infty$	$-\infty$
$-\infty$	$+\infty$	$+\infty$	$-\infty$	$-\infty$	$+\infty$
$x \in G$	$-\infty$	x	$-\infty$	$-\infty$	$-\infty$
$x \in G$	$y \in G$	$x \oplus y$	$(x^{-1} \oplus y^{-1})^{-1}$	$x \otimes y$	$x \otimes y$
$x \in G$	$+\infty$	$+\infty$	x	$+\infty$	$+\infty$
$+\infty$	$-\infty$	$+\infty$	$-\infty$	$-\infty$	$+\infty$
$+\infty$	$y \in G$	$+\infty$	y	$+\infty$	$+\infty$
$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$

Note: Here $(\ )^{-1}$ stands for the unique two-sided multiplicative inverse of x or y.

Proposition 2.1. $(R, \max, \otimes, \min, \otimes')$ is a blog.

Proof. The proof follows from straightforward verification of the conditions listed in Definition 2.3. ∎

We can now think of $(R, \max, \otimes, \min, \otimes')$ as an extension of the original concepts of the principle interpretation and the dual principle interpretation. From now on $(R, \max, \otimes, \min, \otimes')$ will be considered as the principle interpretation.

The next group of concepts we give will be the functionally related ideas that occur in Minimax Algebra.

Definition 2.4. (1) Let $(S)^U$ be the set of all functions from some given set U to another set S. Now let $S = E_1$ (the set of one-tuples defined over R) and $\hat{E}_1 = (E_1, \oplus, \otimes)$ be a belt. Now suppose $((E_1)^U, \oplus)$ is a commutative band. Then $(E_1)^U$ is called a function space.

(2) Suppose that $(U, \oplus)$ and $(S, \oplus)$ are both commutative bands. For $f \in (E_1)^U$ and $\forall x, y \in E_1$ let $f(x \oplus y) = f(x) \oplus f(y)$. This property is called a band homomorphism.

(3) Suppose f is a band homomorphism; then this with partial ordering for $\oplus$ implies $f(x) \geq f(y)$ if $x \geq y$. Such a function f is said to be isotone.

Let $f \in (E_1)^U$ and $\lambda \in E_1$, and $f \otimes \lambda$, $\lambda \otimes f \in (E_1)^U$. Then define $f \otimes \lambda$ and $\lambda \otimes f$ as:

$$\forall x \in U \begin{cases} (f \otimes \lambda)(x) = f(x) \otimes \lambda \\ \\ (\lambda \otimes f)(x) = \lambda \otimes f(x) \end{cases}$$

(4) Let $S = (S, \oplus, \otimes)$ be a belt and let $T = (T, \oplus)$ be a given system and define a right multiplication of elements of T by elements of S such that the following axioms hold:

B_1: $(T, \oplus)$ is a commutative band

B_2: $\forall x \in T$ and $\forall \lambda, \mu \in S$ $(x \otimes \lambda) \otimes \mu = x \otimes (\lambda \otimes \mu)$

B_3: $\forall x, y \in T$ and $\forall \lambda \in S$ $(x \oplus y) \otimes \lambda = (x \otimes \lambda) \oplus (y \otimes \lambda)$

B_4: $\forall x \in T$ and $\forall \lambda, \mu \in S$ $x \otimes (\lambda \oplus \mu) = (x \otimes \lambda) \oplus (x \otimes \mu)$

B_5: If S has an identity element ϕ then $\forall x \in T$, $x \otimes \phi = x$.

Then $(T, \oplus)$ is a <u>right band space over</u> S . If $(Q, \oplus)$ and $(T, \oplus)$ are both spaces over S then $g: (Q, \oplus) \to (T, \oplus)$ is a <u>right linear homomorphism</u> if and only if $\forall x \in Q$, and $\forall \lambda \in S$, $g(x \otimes \lambda) = g(x) \otimes \lambda$. A <u>left band space</u> and a left linear homomorphism may be defined analogously.

(5) A <u>2 sided space</u> is a triple (L, T, R) such that the following axioms hold:

C_1: L is a belt and T is a left band space over L .

C_2: R is a belt and T is a right band space over R .

C_3: $\forall \lambda \in L$, $\forall x \in T$ and $\forall \mu \in R$: $\lambda \otimes (x \otimes \mu) = (\lambda \otimes x) \otimes \mu$.

Next, we turn our attention to the idea of matrices. To begin with, let $(E_1, \oplus, \otimes, \oplus', \otimes')$ be a division belt with duality. Using elements from E_1, we can form the system $(M_{mn}, \oplus, \otimes, \oplus', \otimes')$ which is also a division belt with duality. (Here the notation M_{mn} stands for the set of all $(m \times n)$ matrices over E_1) . We can further extend $(E_1, \oplus, \otimes, \oplus', \otimes')$ by adjoining the elements $+\infty$, $-\infty$ such that the conditions of Definition 2.3 are met. Then, we will have the algebraic structure $(W, \oplus, \otimes, \oplus', \otimes')$ which is a blog. Again, using elements from W , we can form the system $(W_{mn}, \oplus, \otimes, \oplus', \otimes')$. What this means in terms of the eigenproblem is that we will look at the system $(R_{mn}, \oplus, \otimes, \oplus', \otimes')$ which is interpreted as $(R_{mn}, \max, \otimes, \min, \otimes')$. It must be noted here that the operations of $(W, \oplus, \otimes, \oplus', \otimes')$ and of $(W_{mn}, \oplus, \otimes, \oplus', \otimes')$ are different. The matrix operations are defined as follows:

Definition 2.5. Let a_{ij} be any element in W and let $[a_{ij}]$, $[b_{ij}]$ be any elements of W_{mn} ; then we have the following:

(1) $\forall [a_{ij}], [b_{ij}] \in W_{mn}$, $[a_{ij}] \oplus' [b_{ij}] = [a_{ij} \oplus' b_{ij}] \in W_{mn}$

$\forall [a_{ij}], [b_{ij}] \in W_{mn}$, $[a_{ij}] \oplus [b_{ij}] = [a_{ij} \oplus b_{ij}] \in W_{mn}$

(2) $\forall [a_{ij}], [b_{ij}] \in W_{mn}$,

$$[a_{ij}] \otimes' [b_{ij}] = [\sum_{r=1}^{p}{}_{\oplus'} (a_{ij} \otimes' b_{ij})] \in W_{mn}$$

$\forall [a_{ij}], [b_{ij}] \in W_{mn}$,

$$[a_{ij}] \otimes [b_{ij}] = [\sum_{r=1}^{p}{}_{\oplus}(a_{ir} \otimes b_{rj})] \in W_{mn} .$$

To give justification for the preceding paragraph, we will now establish the $(\underline{G}, \underline{B}, S)$-hom-rep-statement. Let $\underline{G}$, $\underline{B}$ stand for matrices such that given m, n , then G_{mn}, B_{mn} represents classes of $m \times n$ matrices. The phrase (G, B, S) means $\forall m,n$, $p \geq 1$ $(G_{mn}, \oplus)$ is a commutative band and $(B_{mp}, \oplus)$ is a function space over $(S, \oplus, \otimes)$; for each $A \in G_{mn}$, the left multiplication g_A: $\underline{B} \longmapsto \underline{A} \otimes \underline{B}$ is a right linear homomorphism of $(B_{np}, \oplus)$ into $(B_{mp}, \oplus)$ over S and is isotone. The mapping τ: $\underline{A} \longmapsto g_{\underline{A}}$ is a representation of $(G_{mn}, \oplus)$ as a commutative sub-band $(LG_{mn}, \oplus)$ of $(\mathrm{Hom}_S(B_{np}, B_{mp}), \oplus)$. (Here Hom_S stands for the set of homomorphisms between B_{np} and B_{mp}) . If $m = n$ then (G_{nn}, B_{np}, S) is a two-sided space and τ is a representation of the belt $(G_{nn}, \oplus, \otimes)$ as a sub-belt $(LG_{nn}, \oplus, \otimes)$ of $(\mathrm{Hom}_S(B_{np}, B_{np}), \oplus, \otimes)$. We now state a theorem.

Theorem 2.1. Let E_1 be a belt. Then the (M, M, E_1)-hom-rep statement is true where M stands for a class of $m \times n$ matrices over E_1.

Proposition 2.2. Let E_1 be a belt with duality. Then $(M_{nn}, \oplus, \otimes, \oplus', \otimes')$ is a belt with duality.

Proof. The proof follows from Theorem 2.1 since the (M, M, E_1)-hom-rep-statement is true. ∎

The concept of conjugate is now defined in terms of Minimax Algebra.

Definition 2.6. Let $(V, \oplus)$ and $(W, \oplus')$ be given commutative bands. $(W, \oplus')$ is conjugate to $(V, \oplus)$ if $\exists$ a function $g: V \to W$ such that:

Q_1: g is bijective.

Q_2: $\forall x, y \in s$, $g(x \otimes y) = g(x) \oplus' g(y)$.

Q_3: $\forall_{x,y} \in V$, $g(x \otimes y) = g(y) \otimes' g(x)$.

Also, if $(S, \oplus, \oplus')$ is a commutative band with duality, then we say that $(V, \oplus, \oplus')$ is self conjugate if $(V, \oplus')$ is conjugate to $(V, \oplus)$.

In terms of belts with duality, we say $(S, \oplus, \otimes, \oplus', \otimes')$ is self conjugate if $(V, \oplus', \otimes')$ is conjugate to $(V, \oplus, \otimes)$. Here we also have the idea of the conjugate for a matrix. It is defined as $(\{\underline{A}^*\}_{ij}) = (\{\underline{A}\}_{ji})^*$ $(1 \leq i \leq m \text{ , } 1 \leq j \leq n)$. The symbol "*" stands for the conjugate.

Proposition 2.3. If S and S^* are conjugate systems, then $x \geq y$ in S if and only if $x^* \leq y^*$ in S^* . Similarly, $x > y$ in S if and only if $x^* < y^*$ in S^* .

Proposition 2.4. Every division belt is self-conjugate under the bijection $x^* = x^{-1}$ and every blog is self-conjugate under the bijection $(-\infty)^* = +\infty$, $(+\infty)^* = -\infty$, $x^* = x^{-1}$ if x is finite.

We will now state and prove a theorem about conjugate systems.

Theorem 2.2. Let $(E_1, \oplus)$ and $(E_1^*, \otimes')$ be conjugate. Then so are $(M_{mn}, \oplus)$ and $(M_{mn}^*, \oplus')$ for any integers $m, n \geq 1$. If multiplications are also conjugate, then multiplications $\otimes, \otimes'$ are also defined for M_{mn}, M_{mn}^* respectively, for each integer $n \geq 1$, such that $(M_{nn}, \oplus, \otimes)$ and $(M_{nn}^*, \oplus', \otimes')$ are conjugate.

Proof: Start with V as a set of matrices over E_1 and let $g\colon V \to V^*$ be defined by $g\colon \underline{A} \longmapsto \underline{A}^*$ as $A^* = A^{-1}$. Consider the restriction of g to M_{mn} . Since the inverse of a matrix is unique this mapping is 1-1 and onto.

$\underline{A}, \underline{B} \in M_{mn}$ we have

$$\{g(\underline{A} \oplus \underline{B})\}_{ij} = \{(\underline{A} \oplus \underline{B})^*\}_{ij} = (\{\underline{A}\}_{ji} \oplus \{\underline{B}\}_{ji})^*$$

$$= (\{\underline{A}\}_{ji}) \oplus' (\{\underline{B}\}_{ji}) = \{\underline{A}^*\}_{ij} \oplus' \{\underline{B}^*\}_{ij}$$

$$= \{\underline{A}^* \oplus' \underline{B}^*\}_{ij} = \{g(\underline{A}) \oplus' g(\underline{B})\}_{ij} .$$

So we now have $g(\underline{A} \oplus \underline{B}) = g(\underline{A}) \oplus' g(\underline{B})$. So Q_2 is satisfied by all M_{mn} matrices. Now $\{g(\underline{A} \otimes \underline{B})\}_{ij} = \{(\underline{A} \otimes \underline{B})^*\}_{ij}$ by Q_3 .

$$\{(\underline{A} \otimes \underline{B})^*\}_{ij} = (\{\underline{A} \otimes \underline{B}\}_{ji})^* =$$

$$= (\sum_{k=1}^{n}{}_{\oplus} (\{\underline{A}\}_{jk} \otimes \{\underline{B}\}_{ki}))^* = \sum_{k=1}^{n}{}_{\oplus'} (\{\underline{A}\}_{jk} \otimes \{\underline{B}\}_{ki})^*$$

$$= \sum_{k=1}^{n}{}_{\oplus'} ((\{\underline{B}\}_{ki})^* \otimes' (\{\underline{A}\}_{jk})^*)$$

$$= \sum_{k=1}^{n}{}_{\oplus'} (\{\underline{B}^*\}_{ik} \otimes' \{\underline{A}^*\}_{kj}) = \{\underline{B}^* \otimes' \underline{A}^*\}_{ij}$$

$$= \{g(\underline{B}) \otimes' g(\underline{A})\}_{ij} = g(\underline{A} \otimes \underline{B}) = g(\underline{B}) \otimes' g(\underline{A}) .$$

So Q_3 is satisfied for $(M_{mn}, \oplus, \otimes)$, $(M_{mn}, \oplus', \otimes')$. ∎

We now look at the idea of pre-residuation.

Definition 2.7. Let E_1 be a self conjugate belt. We say E_1 is pre-residuated if it satisfies A_{18}: If $J \subset E_1$ is a finite set of elements, then:

$$\forall x \in E_1 , \quad [\sum_{a \in J}{}_{\oplus'} (a^* \otimes' a)] \otimes x \geq x$$

and

$$x \otimes [\sum_{a \in J}{}_{\oplus'} (a \otimes' a^*)] \geq x \quad .$$

Theorem 2.3. All division belts and all blogs are presiduated. In particular, the principal interpretation is pre-residuated.

Proof. By Proposition 2.4 all division belts and blogs are self conjugate. Let J be a finite subset of a division belt or blog

E_1 and let $a \in J$, $x \in E_1$ be arbitrary. If E_1 is a division belt $a^* \otimes' a = \phi$ (since $a^* = a^{-1}$, then $a^{-1} \otimes' a = \phi$). If E_1 is a blog then $a^* \otimes' a = \phi$ or $+\infty$ by Proposition 2.3. So we have $\sum_{a \in J}{}_{\oplus'} (a^* \otimes' a) \geq \phi$. Now by Axiom 20 we have

$\sum_{a \in J}{}_{\oplus'} (a^* \otimes' a) \otimes x \geq \phi \otimes x$ implies $\sum_{a \in J}{}_{\oplus'} (a^* \otimes' a) \otimes x \geq x$. Also, by Axiom 20 we have $\sum_{a \in J}{}_{\oplus'} (a^* \otimes' a) \geq \phi$ implies

$x \otimes \sum_{a \in J}{}_{\oplus'} (a^* \otimes' a) \geq x \otimes \phi$ which implies $x \otimes \sum_{a \in J}{}_{\oplus'} (a^* \otimes' a) \geq x$.

Thus all blogs are pre-residuated so the principal interpretation is pre-residuated. ∎

<u>Theorem 2.4</u>. Let E_1 be a pre-residuated belt and J a given finite set of matrices over E_1. Let $\underline{X}$ be an arbitrary matrix over E_1. Then $\underline{P} \otimes' \underline{X} \leq \underline{X} \leq \underline{Q} \otimes \underline{X} \;—\!\!|$. Also, $\underline{X} \otimes' \underline{P} \leq \underline{X} \leq \underline{X} \otimes \underline{Q} \;—\!\!|$ where $P = \sum_{\underline{A} \in J}{}_{\oplus} (\underline{A} \otimes \underline{A}^*) —\!\!|$, $Q = \sum_{\underline{A} \in J}{}_{\oplus'} (\underline{A}^* \otimes' \underline{A}) —\!\!|$.

Here the symbol "$—\!\!|$" means that the foregoing statement applies only when the matrices involved are <u>conformable</u> for the relevant operations. The meaning of conformable is that if $\underline{A} \in M_{mn}$ and $\underline{B} \in M_{pq}$, then $\underline{A}$ and $\underline{B}$ are conformable for addition whenever both $m = p$ and $n = q$; and conformable for $\underline{A} \otimes \underline{B}$ whenever $n = p$.

<u>Theorem 2.5</u>. If E_1 is a pre-residuated belt, then M_{nn} is a pre-residuated belt for each integer $n \geq 1$.

<u>Theorem 2.6</u>. Let E_1 be a pre-residuated belt satisfying Axiom A_{19}. The following inequalities apply for arbitrary matrices $\underline{A}$, $\underline{X}$ over E_1.

(1) $\underline{A}^* \otimes (\underline{A} \otimes' \underline{X}) \leq \underline{X} \leq \underline{A}^* \otimes' (\underline{A} \otimes \underline{X})$ —|

(2) $\underline{A} \otimes (\underline{A}^* \otimes' \underline{X}) \leq \underline{X} \leq \underline{A} \otimes' (\underline{A}^* \otimes \underline{X})$ —|

(3) $(\underline{X} \otimes' \underline{A}) \otimes \underline{A}^* \leq \underline{X} \leq (\underline{X} \otimes \underline{A}) \otimes' \underline{A}^*$ —|

(4) $(\underline{X} \otimes' \underline{A}^*) \otimes \underline{A} \leq \underline{X} \leq (\underline{X} \otimes \underline{A}^*) \otimes' \underline{A}$ —| .

Proof. We will use Theorem 2.4 and Axiom A_{19} to prove the above statements.

(1) $\underline{A}^* \otimes (\underline{A} \otimes' \underline{X}) \leq (\underline{A}^* \otimes \underline{A}) \otimes' X \leq X$ and $\underline{A}^* \otimes' (\underline{A} \otimes \underline{X}) \geq (\underline{A}^* \otimes' \underline{A}) \otimes \underline{X} \geq X$.

(2) $\underline{A} \otimes (\underline{A}^* \otimes \underline{X}) \leq (\underline{A} \otimes \underline{A}^*) \otimes' X \leq X$ and $\underline{A} \otimes' (\underline{A}^* \otimes \underline{X}) \geq (\underline{A} \otimes' \underline{A}^*) \otimes \underline{X} \geq \underline{X}$.

(3) $(\underline{X} \otimes' \underline{A}) \otimes \underline{A}^* \leq X \otimes' (\underline{A} \otimes \underline{A}^*) \leq \underline{X}$ and $(\underline{X} \otimes \underline{A}) \otimes' \underline{A}^* \geq \underline{X} \otimes (\underline{A} \otimes' \underline{A}^*) \geq \underline{X}$.

(4) $(\underline{X} \otimes' \underline{A}^*) \otimes \underline{A} \geq \underline{X} \otimes' (\underline{A}^* \otimes \underline{A}) \geq \underline{X}$ and $(\underline{X} \otimes \underline{A}^*) \otimes' \underline{A} \leq \underline{X} \otimes (\underline{A}^* \otimes \underline{A}') \leq \underline{X}$. ■

Here we assume $\underline{A} \in M_{mn}$ and $\underline{X} \in M_{nr}$.

We now list some ideas from schedule algebra which are related to the eigenvalue-eigenvector problem. Let $\underline{A}$ be a matrix which defines a process for which the vector of earliest beginning times for the cycle 0 is $\underline{X} \in E_n$. Next, we look at the vector of actual start times. Let this vector be $\underline{t}(0)$. Evidently, we have $\underline{t}(0) \geq x$. This implies that we have $\underline{A} \otimes \underline{t}_0$ for the vector for earliest start times for cycle one, and the vector of actual start times for the first cycle will be $\underline{t}(1) \geq \underline{A} \otimes \underline{X}$. Continuing in this manner, we have:

$$\underline{t}(r) \geq \underline{A} \otimes \underline{t}(r - 1) \geq \dots \underline{A}^r \otimes \underline{t}(0) \geq \underline{A}^r \otimes \underline{X} .$$

Now the criterion we wish to impose is that we want the process to stop at or before a given vector of completion times $\underline{Z}$. We also assume the cycle will run up to and including cycle N. In symbols this requires $\underline{A}^{n+1} \otimes \underline{X} \leq \underline{Z}$. We say that the pair $\underline{X}, \underline{Z}$ is compatible for $(\underline{A}, N)$ if $\underline{A}^{N+1} \otimes \underline{X} \leq \underline{Z}$.

Definition 2.7. A sequence $\{t(r)\}$ $(r = 0, \ldots, N)$ will be called feasible for $(\underline{X}, \underline{Z}, N)$ if

(1) $\underline{t}(r+1) \geq \underline{A} \otimes \underline{t}(r)$

(2) $\underline{t}(N+1) \leq \underline{Z}$

(3) $\underline{t}(0) \geq \underline{X}$.

Theorem 2.7. Let E_1 be a pre-residuated belt which satisfies Axiom A_{18}. If $\underline{A} \in M_{mn}$, $\underline{X} \in E_n$ and $\underline{Z} \in E_m$, we have $\underline{Z} \geq \underline{A} \otimes \underline{X}$ if and only if $\underline{X} \leq \underline{A}^* \otimes' \underline{Z}$.

Theorem 2.8. Let E_1 receive the principal interpretation. Let $\underline{A} \in M_{mn}$ and $\underline{X}, \underline{Z} \in E_n$ for some integer $n \geq 1$. Let $N \geq 0$ be a given integer. Then the following are equivalent:

(i) $\underline{X}, \underline{Z}$ are compatible for $(\underline{A}, N)$

(ii) $\exists$ a sequence $\{t(r)\}$ which is feasible for $(\underline{X}, \underline{Z}, N)$

(iii) $\forall\, r = 0, 1, \ldots, (N+1)$ then $\underline{A}^r \otimes \underline{X} = \underline{A}^{(N-r-1)} \otimes' \underline{Z}$ where $\underline{A}^r \otimes \underline{X} = \underline{X}$ and $\underline{A}^{s*} \otimes \underline{Z} = \underline{Z}$ where $r = 0$, $s = 0$.

Proof. (i) and (iii) are equivalent by Theorem 2.7. Also $\underline{Z} \geq \underline{t}(N+1) \geq A \otimes \underline{t}(N) \geq \ldots \geq \underline{A}^{N+1} \otimes \underline{t}(0)$ so (ii) implies (i). Also if $\underline{Z} \geq \underline{A}^{N+1} \otimes \underline{X}$ then the sequence $\{\underline{t}(r)\}$ defined by $\underline{t}(r) = \underline{A}^r \otimes \underline{X}$ $(r = 0, \ldots, (N+1))$ is a feasible sequence for $(\underline{X}, \underline{Z}, N)$ so (i) implies (ii). ∎

Theorem 2.9. With the ideas from Theorem 2.8, any sequence $t(r)$ which is feasible for $(\underline{X}, \underline{Z}, N)$ satisfies $\underline{A}^r \otimes \underline{X} \leq t(r) \leq \underline{A}^{(N-r+1)*} \otimes' \underline{Z}$ $(r = 0,\ldots, (N+1))$.

Definition 2.8. $(\underline{X}, \underline{Z}, N)$ - feasible y continuation of a sequence is a sequence $t(r)$, $(r = 0,\ldots, (s-1))$ and satisfies $\underline{t}(s) = \underline{Y}$.

Theorem 2.10. With Definition 2.9, a sequence $\{\underline{t}(r)\}$ $(r = 0,\ldots, (s-1))$ satisfying:

(i) $\underline{t}(0) \geq \underline{X}$

(ii) If $S \geq 2$ then $\underline{t}(r+1) \geq \underline{A} \otimes \underline{t}(r)$ $(r = 0,\ldots, (s-2))$

has an $(\underline{X}, \underline{Z}, N)$-feasible Y continuation if and only if $\underline{A} \otimes \underline{t}(s-1) \leq Y \leq \underline{A}^{(N-s+1)} \otimes' \underline{Z}$.

Proof. From Definition 2.7 and Theorem 2.9 we have $\underline{A}^r \otimes \underline{X} \leq t(r) \leq \underline{A}^{(N-r+1)*} \otimes' \underline{Z}$ and $\underline{A} \otimes \underline{t}(s-1) \geq \underline{A}^s \otimes \underline{X}$ where $\underline{t}(s) \geq A \otimes \underline{t}(s-1)$. Thus $\underline{A} \otimes \underline{t}(s-1) \leq \underline{Y} \leq \underline{A}^{(N-s+1)*} \otimes' \underline{Z}$ where we let $\underline{t}(s) = \underline{y}$. Now suppose that $\underline{A} \otimes \underline{t}(s-1) \leq \underline{Y} \leq A^{(N-s+1)*} \otimes' \underline{Z}$. Now by Theorem 2.7 we have $\underline{A}^{(N-s+1)} \oplus \underline{Y} \leq \underline{Z}$. Now from the conditions that

(i) $\underline{t}(0) \leq \underline{X}$

and

(ii) If $S \geq 2$, then $\underline{t}(r+1) \geq \underline{A} \otimes \underline{t}(r)$, we have $\underline{t}(0),\ldots, \underline{t}(s-1)$, $\underline{Y}, \underline{A} \otimes \underline{Y},\ldots, \underline{A}^{(N-s+1)} \otimes \underline{Y}$ is feasible for $(\underline{X}, \underline{Z}, N)$. ∎

Definition 2.9. The vector of component-wise differences between $\underline{Y}$ and $\underline{A}^{(N-s+1)*} \otimes' \underline{Z}$ is the total float vector. The vector of component wise differences between $\underline{Y}$ and $\underline{A}^* \otimes' (\underline{A} \otimes \underline{Y})$ is called the free float vector. We now give an example of Definition 2.9, but first we need to introduce a new notation. Here the symbol "-" means the component-wise differences between any two matrices $\underline{A}$ and $\underline{B}$.

Example 2.1. Let

$$\underline{A} = \begin{bmatrix} 4 & -\infty & 6 \\ -\infty & 2 & 2 \\ 4 & 6 & 3 \end{bmatrix},$$

$$\underline{A}^2 = \begin{bmatrix} 4 & -\infty & 6 \\ -\infty & 2 & 2 \\ 4 & 6 & 3 \end{bmatrix} \otimes \begin{bmatrix} 4 & -\infty & 6 \\ -\infty & 2 & 2 \\ 4 & 6 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 12 & 10 \\ 6 & 8 & 5 \\ 8 & 9 & 10 \end{bmatrix}.$$

For details of the calculations, see Appendix 1.

Now, let

$$\underline{x} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \quad \underline{z} = \begin{bmatrix} 28 \\ 20 \\ 10 \end{bmatrix}, \quad \underline{y} = \begin{bmatrix} 17 \\ 13 \\ 17 \end{bmatrix}$$

$$\underline{A}^2 \otimes \underline{X} = \begin{bmatrix} 10 & 12 & 10 \\ 6 & 8 & 5 \\ 8 & 9 & 10 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 16 \\ 12 \\ 16 \end{bmatrix}$$

$$\underline{A}^{2*} \otimes' \underline{z} = \begin{bmatrix} -10 & -6 & -8 \\ -12 & -8 & -9 \\ -10 & -5 & -10 \end{bmatrix} \otimes \begin{bmatrix} 28 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 18 \\ 16 \\ 18 \end{bmatrix}$$

So $\underline{A}^2 \otimes \underline{X} < \underline{Y} \leq \underline{A}^{2*} \otimes' \underline{z}$. By Theorem 2.10 completion time $\underline{z}$ is achievable. Therefore, the total float vector is

$$\begin{bmatrix} 18 \\ 16 \\ 18 \end{bmatrix} - \begin{bmatrix} 17 \\ 13 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} .$$

Here

$$\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

represents the time in which cycle 2 activities may be delayed.

$$\underline{A} \otimes \underline{Y} = \begin{bmatrix} 4 & -\infty & 6 \\ -\infty & 2 & 2 \\ 4 & 6 & 3 \end{bmatrix} \otimes \begin{bmatrix} 17 \\ 13 \\ 17 \end{bmatrix} = \begin{bmatrix} 23 \\ 19 \\ 21 \end{bmatrix} .$$

$$\underline{A}^* \otimes' (\underline{A} \otimes \underline{Y}) = \begin{bmatrix} 4 & -8 & 6 \\ -\infty & 2 & 2 \\ 4 & 6 & 3 \end{bmatrix} \otimes \begin{bmatrix} 23 \\ 19 \\ 21 \end{bmatrix} = \begin{bmatrix} 27 \\ 23 \\ 27 \end{bmatrix} \quad ;$$

So the free float vector is

$$\begin{bmatrix} 17 \\ 13 \\ 17 \end{bmatrix} - \begin{bmatrix} 27 \\ 23 \\ 27 \end{bmatrix} = \begin{bmatrix} -10 \\ -10 \\ -10 \end{bmatrix} \quad .$$

Now, we will explain some different types of matrices that will be used in Chapter 3 to develop the theory behind the solution of the eigenproblem.

A row stochastic matrix is a non-negative matrix in which the sum of the elements in each row equals zero. A column-stochastic matrix has the sum of the elements in each colum equal to zero. Double stochastic matrices are both row-stochastic and column-stochastic.

In Minimax Algebra we can formulate similar ideas.

Definition 2.10. Let $(E_1, \oplus, \otimes, \oplus', \otimes')$ be a belt with duality and let $(\sigma_\phi, \oplus, \otimes, \oplus', \otimes')$ be a sub-belt with duality. Then $S \subset E_1$ is σ_ϕ-astic if $\sum^{\oplus}_{x \in S} x \in \sigma_\phi$. If σ_ϕ is $\{\phi\}$, then

$\sum^{\oplus}_{x \in S} x =$ the identity ϕ .

Definition 2.11. A matrix over E_1 is row-σ_ϕ-astic if the elements in each row form a σ_ϕ-astic set. Analogously there exists column-σ_ϕ-astic and double σ_ϕ-astic matrices. If σ_ϕ is a group then we use the terminology G-astic. If $G = \{\phi\}$ then use the term ϕ-astic.

We give examples of these ideas.

Example 2.2. Let $E_1 = R$ (the extended reals) and let $\sigma_\phi = I_*$ (the set of integers including $\pm\infty$) , and look at

$$\underline{A} = \begin{bmatrix} 2 & 4 & 6 & 8 & 10 \\ 3 & 7 & 9 & 12 & 15 \end{bmatrix}$$

so

$$S_1 = [2, 4, 6, 8, 10]$$

and

$$S_2 = [3, 7, 9, 12, 15]$$

for row 1: $\sum\limits_{x \in S_1}^{\oplus} x \in \sigma_\phi$ since $\sum\limits_{x \in S_1}^{\oplus} x = 10$ which is an element of σ_ϕ ;

for row 2: $\sum\limits_{x \in S_2}^{\oplus} x = 15$ which is an element of σ_ϕ .

So $\underline{A}$ is row-σ_ϕ-astic. Similarly, it is easy to see that $\underline{A}$ is also column-σ_ϕ-astic. Notice that I_* is a group so $\underline{A}$ is G-astic.

We now discuss the linear equation $\underline{A} \otimes \underline{X} = \underline{b}$, where $\underline{A} \in M_{mn}$, $\underline{b} \in E_m$. Here, we have to find a vector $\underline{x}$ such that $\underline{A} \otimes \underline{X} = \underline{b}$. This may be interpreted as the solution set of simultaneous linear equations in Minimax Algebra.

Proposition 2.4. The set of solutions of $\underline{A} \otimes \underline{x} = \underline{b}$ is either empty or is a commutative band.

Proof. Assume $\underline{x}, \underline{y} \in E_n$ are solutions to $\underline{A} \otimes \underline{x} = \underline{b}$. Then we have $\underline{A} \otimes (\underline{x} \oplus \underline{y}) = (\underline{A} \otimes \underline{x}) \oplus (\underline{A} \otimes \underline{y}) = \underline{b} \oplus \underline{b} = \underline{b}$. ∎

Theorem 2.11 (Cunningham-Green [4]). Let E_1 be a pre-residuated belt satisfying A_{18}. Then $\underline{A} \otimes \underline{x} = \underline{b}$ has at least one solution if and only if $\underline{x} = \underline{Z}^* \otimes' \underline{b}$ is a solution and $\underline{x} = \underline{A}^* \otimes' \underline{b}$ is the greatest solution.

Proof. For the "if" suppose $\underline{x} = \underline{A}^* \otimes' \underline{b}$. Then $\underline{A} \otimes (\underline{A}^* \otimes' \underline{b}) = (\underline{A} \otimes \underline{A}^*) \otimes' \underline{b} = \underline{b}$ so $\underline{x}$ is a solution. For the "only if" use $\underline{\mu} \in E_n$ as a solution of $\underline{A} \otimes \underline{x} = \underline{b}$. Then $\underline{A} \otimes (\underline{A}^* \otimes' \underline{b}) = \underline{A} \otimes (\underline{A}^* \otimes' (\underline{A} \otimes \underline{\mu})) = \underline{A} \otimes \underline{\mu} = \underline{b}$. So $\underline{x} = \underline{A}^* \otimes' \underline{b}$ is a solution of $\underline{A} \otimes \underline{x} = \underline{b}$. Now by Theorem 2.6 we have $\underline{\mu} \leq \underline{A}^* \otimes' (\underline{A} \otimes \underline{\mu}) = \underline{A}^* \otimes' \underline{b} = \underline{x}$ so $\underline{x}$ is the greatest solution.

Note: Here "greatest solution" is in the context of the property of being linearly ordered. ∎

We will now list a set of equations P and their principal solutions Q in Table 2.3.

TABLE 2.3

EQUATIONS OVER A BLOG

P	Q
$\underline{A} \otimes \underline{x} = \underline{B}$	$\underline{A}^* \otimes' \underline{B}$
$\underline{A}^* \otimes \underline{x} = \underline{B}$	$\underline{A} \otimes' \underline{B}$
$\underline{A} \otimes' \underline{x} = \underline{B}$	$\underline{A}^* \otimes \underline{B}$
$\underline{A}^* \otimes' \underline{x} = \underline{B}$	$\underline{A} \otimes \underline{B}$
$\underline{x} \otimes \underline{A} = \underline{B}$	$\underline{B} \otimes' \underline{A}^*$
$\underline{x} \otimes \underline{A}^* = \underline{B}$	$\underline{B} \otimes' \underline{A}$
$\underline{x} \otimes' \underline{A} = \underline{B}$	$\underline{B} \otimes \underline{A}^*$
$\underline{x} \otimes' \underline{A}^* = \underline{B}$	$\underline{B} \otimes \underline{A}$

To give some justification for the solutions listed in Table 2.3, look at $\underline{A} \otimes \underline{x} = \underline{B}$. Since each column of $\underline{B}$ is produced by the action of matrix $\underline{A}$ on the corresponding column of $\underline{x}$, the truth of this statement follows from Theorem 2.11. (Note: Here $\underline{x}$ is a matrix). The other statements follow similarly.

We now give some further results concerning the equation $\underline{A} \otimes \underline{x} = \underline{b}$.

<u>Theorem 2.12</u>. Let E_1 be a blog. If all elements of $\underline{b}$ are $-\infty$, then $\underline{A} \otimes (\underline{A}^* \otimes' \underline{b})$ has $\underline{b}$ as its unique solution. If all elements of $\underline{b}$ are $+\infty$, then solutions to $\underline{A} \otimes (\underline{A}^* \otimes' \underline{b})$ are exactly those elements of E_n which have at least one element equal to $+\infty$.

<u>Theorem 2.13</u>. Let E_1 be a blog and let $\underline{A} \in M_{mn}$ be doubly G-astic and $b \in E_m$ be finite. Then a necessary and sufficient condition that the equation $\underline{A} \otimes \underline{x} = b$ have at least one solution is that we can find a finite number of elements $x_1, x_2, \ldots, x_n$ such that matrix $\underline{B} = [(\{b\}_i)^{-1} \otimes \{\underline{A}\}_{ij} \otimes x_j]$ is doubly ϕ-astic.

<u>Proof</u>. $\underline{A} \otimes \underline{x} = \underline{b}$ has a solution if and only if it has the <u>principal solution</u> $\underline{x} = \underline{A}^* \otimes' \underline{b}$ which is finite, i.e., if and only if we can find finite elements $x_1, \ldots, x_n$ such that:

$$x_j = \sum_{i=1}^{m}{}_{\oplus'} (\{\underline{A}^*\}_{ji} \otimes' \{\underline{b}\}_i) \quad (j = 1, \ldots, n)$$

$$\{\underline{b}\}_i = \sum_{j=1}^{n}{}_{\oplus} (\{\underline{A}\}_{ij} \otimes x_j) \quad (i = 1, \ldots, m)$$

which are the same as

$$\Big(\sum_{i=1}^{m}{}_{\oplus'} (\{\underline{A}^*\}_{ji} \otimes' \{\underline{b}\}_i)\Big)^{-1} \otimes x_j = \phi \qquad (j = 1, \ldots, n) ,$$

$$(\{\underline{b}\}_i)^{-1} \otimes \sum_{j=1}^{n}{}_{\oplus} (\{\underline{A}\}_{ij} \otimes x_j) = \phi \quad (i = 1, \ldots, m) .$$

Now we will have

$$\Big(\sum_{i=1}^{m}{}_{\oplus'} (\{\underline{A}^*\}_{ji} \otimes' \{\underline{b}\}_i)\Big)^{-1} = \Big(\sum_{i=1}^{m}{}_{\oplus'} (\{\underline{A}^*\}_{ji} \otimes' \{\underline{b}\}_i)\Big)^* =$$

$$\sum_{i=1}^{m}{}_{\oplus} (\{\underline{A}^*\}_{ji} \otimes' \{\underline{b}\}_i)^* = \sum_{i=1}^{m}{}_{\oplus}((\{\underline{b}\}_i^*) \otimes (\{\underline{A}^*\}_{ji})^*) =$$

$$\sum_{i=i}^{m}{}_{\oplus}((\{\underline{b}\}_i)^{-1} \otimes \{\underline{A}\}_{ij})$$

which gives

$$\sum_{i=1}^{m}{}_{\oplus}((\{b\}_i)^{-1} \otimes \{\underline{A}\}_{ij} \otimes x_j) = \phi \qquad (j = 1,\ldots, n)$$

and

$$\sum_{j=1}^{n}{}_{\oplus}((\{\underline{b}\}_i^{-1}) \otimes \{\underline{A}\}_{ij} \otimes x_j) = \phi \quad (i = 1,\ldots, m)$$

which implies that these are respectively

$$\sum_{i=1}^{m}{}_{\oplus} \{\underline{B}\}_{ij} = \phi \qquad (j = 1,\ldots, n)$$

and

$$\sum_{j=1}^{n}{}_{\oplus} \{\underline{B}\}_{ij} = \phi \quad (i = 1,\ldots, m) \ . \qquad \blacksquare$$

The last topic in this chapter deals with linear dependence. We will give the appropriate definition and then two examples.

Definition 2.12. If $\underline{A} \in M_{mn}$, and $\underline{A}$ has n columns and each column is an m-tuple i.e. $\underline{a}(j) \in E_m$, then the equation $\underline{A} \otimes \underline{x} = \underline{b}$ may be written as $\sum_{j=1}^{n}{}_{\oplus} \underline{a}(j) \otimes x_j = \underline{b}$. This situation is right linear dependence. Analogously we can define

$$x_j \otimes \sum_{j=1}^{n} \underline{a}(j) \otimes \underline{x}_j = \underline{b}$$

to be left linear dependence.

Example 2.3. Let E_1 be a non-commutative division belt, and let a_1, c, $x \in E_1$ be chosen so that $c \otimes x \neq x \otimes c$ and define:

$$\left.\begin{aligned} a_2 &= a_1 \otimes c \\ b_1 &= a_1 \otimes x \\ b_2 &= a_2 \otimes x \end{aligned}\right\} \qquad \underline{b} = \begin{bmatrix} b_1 \\ \\ b_2 \end{bmatrix}, \quad \underline{a} = \begin{bmatrix} a_1 \\ \\ a_2 \end{bmatrix}.$$

Then we have $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \otimes x$; hence $\underline{b} \in E_2$ is right linearly dependent on $\underline{a} \in E_2$. Suppose $\underline{b}$ were also left linearly dependent on $\underline{a}$ i.e., for some $y \in E_1$ we have:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = y \otimes \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} .$$

Then $a_2 \otimes x = y \otimes a_2$. Substituting for a_2 we have $a_1 \otimes c \otimes x = y \otimes a_1 \otimes c$. But $y \otimes a_1 = b_1 = a_1 \otimes x$. So $a_1 \otimes c \otimes x = a_1 \otimes x \otimes c$ which implies $c \otimes x = x \otimes c$ (since E_1 is a division belt) which is a contradiction of the hypothesis that we had a non-commutative division belt. So left linear independence and right linear independence do not imply one another.

Example 2.4. Let

$$\underline{A} = \begin{bmatrix} 0 & 20 & 100 \\ 20 & 4 & 10 \\ 2 & 18 & 10 \end{bmatrix} \qquad b = \begin{bmatrix} 110 \\ 30 \\ 38 \end{bmatrix}$$

We will show that we have right linear dependence over E_1. The equation $\sum_{j=1}^{n}{}_{\oplus}\ \underline{a}(j) \otimes x(j) = \underline{b}$ is the following where we pick

$$x_{(j)} = \begin{bmatrix} 10 \\ 20 \\ 10 \end{bmatrix}$$

Thus,

$$\begin{bmatrix} 0 & 20 & 100 \\ 20 & 4 & 10 \\ 2 & 18 & 10 \end{bmatrix} \otimes \begin{bmatrix} 10 \\ 20 \\ 10 \end{bmatrix} =$$

$$\begin{bmatrix} (0 \otimes 10) \oplus (20 \otimes 20) \oplus (100 \otimes 10) \\ (20 \otimes 10) \oplus (4 \otimes 20) \oplus (10 \otimes 10) \\ (1 \otimes 10) \oplus (18 \otimes 20) \oplus (10 \otimes 10) \end{bmatrix} = \begin{bmatrix} 110 \\ 30 \\ 38 \end{bmatrix} .$$

CHAPTER 3

THEORY OF THE EIGENVALUE-EIGENVECTOR PROBLEM

In this chapter we give several definitions, statements, and theorems which will supply the needed theoretical background for the solution of the eigen-problem. Again, as in Chapter 2, we refer the reader to Cuninghame-Green "Minimax Algebra" for further study.

Definition 3.1. Let $\underline{A} \in M_{mn}$, $x \in E_n$ (n tuples). Then x is an eigenvector of $\underline{A}$ and $\underline{\lambda} \in E_1$ is an eigenvalue of $\underline{A}$ if and only if $\underline{A} \otimes x = \underline{\lambda} \otimes x$.

Proposition 3.1. Let E_1 be a blog and let $\underline{A} \in M_{mn}$ for given integers $m, n \geq 1$. Then there exists a matrix $\underline{B} \in M_{mm}$ for which every m-tuple in the column-space of $\underline{A}$ is an eigenvector with corresponding eigenvalue ϕ . Namely $\underline{B} = \underline{A} \otimes' \underline{A}^*$.

Proof. y is an eigenvector of $\underline{B}$ with corresponding eigenvalue ϕ (identity) means that $\underline{B} \otimes \underline{y} = \phi \otimes \underline{y}$. The column space of $\underline{A}$ means that for $\underline{A} \otimes \underline{x} = \underline{b}(1)$ we look at the set of all $\underline{b} \in E_m$ for which (1) is soluble for $\underline{x}$. Let $\underline{A} = [a(1), \ldots, a(n)]$. The notation $\langle a(1), \ldots, a(n) \rangle$ is the column space of $\underline{A}$ generated by $a(1), \ldots, a(n)$. Of course each $a(i) \in \langle a(1), \ldots, a(n) \rangle$. Let $\underline{B} = (\underline{A} \otimes' \underline{A}^*)$ then $(\underline{A} \otimes' \underline{A}^*) \otimes \underline{A} = \underline{A} =$

$$\begin{bmatrix} \phi & \theta & \cdots & \theta \\ \theta & \phi & & \vdots \\ \vdots & & \ddots & \vdots \\ \theta & \cdots & \cdots & \phi \end{bmatrix} \otimes \underline{A}$$

where this is an identity matrix and θ is the null (zero). So

$$(\underline{A} \otimes' \underline{A}^*) \otimes [a(1),\ldots, a(n)] =$$

$$\begin{bmatrix} \phi & \theta & \cdots & \theta \\ \theta & \phi & & \vdots \\ \vdots & & \ddots & \vdots \\ \theta & \cdots & \cdots & \phi \end{bmatrix} \otimes [a(1),\ldots, a(n)]$$

$$[(\underline{A} \otimes' A^*) \otimes a(1),\ (\underline{A} \otimes' A^*) \otimes a(2),\ldots,$$

$$(\underline{A} \otimes' A^*) \otimes a(n)] = [\phi \otimes a(1),\ldots, \phi \otimes a(n)] \; .$$

So $(\underline{A} \otimes' \underline{A}^*) \otimes a(i) = \phi \otimes a(i)$ for every i. Hence $a(i)$ is an eigenvector of $\underline{B} = (\underline{A} \otimes' \underline{A}^*)$ with corresponding eigenvalue ϕ. If $\underline{b} \in \langle a(1),\ldots, a(n)\rangle$, then $\underline{b}$ is an eigenvector of $\underline{B}$ with corresponding eigenvalue ϕ. ∎

Lemma 3.1. Let E_1 be a blog, and let $\underline{B} \in M_{nn}$ for given integer $n \geq 1$. Let σ be a circuit in $\Delta(\underline{B})$, of circuit product $p(\sigma)$ and let τ, of circuit product $p(\tau)$, be any circuit obtained from σ by cyclic permutation of the nodes on σ. Then $p(\tau) \sim \phi$ if and only if $p(\sigma) \sim \phi$, where $\sim$ is any one of the symbols $=$, $\geq$, or $\leq$.

Lemma 3.2. Let E_1 be a blog and let $\underline{B} \in M_{nn}$, for given integer $n \geq 1$. Let $\underline{B}$ be definite. Then $\Gamma(\underline{B})$ is definite and if j is an eigennode of $\Delta(\underline{B})$, then $\{\Gamma(\underline{B})\}_{jj} = \phi$. Conversely, if E_1 is linear, and $\{\underline{\Gamma}(\underline{B})\}_{jj} = \phi$ for some index j $(1 \leq j \leq n)$ then j is an eigen-node.

<u>Definition 3.2</u>. The eigenproblem is <u>finitely</u> <u>soluble</u> (when E_1 is a blog) if a finite λ and $\underline{x}$ can be found satisfying $\underline{A} \otimes \underline{x} = \lambda \otimes \underline{x}$.

<u>Lemma 3.3</u>. Let E_1 be a blog and let $\underline{B} \in M_{nn}$ for given integer $n \geq 1$. A necessary condition that the eigenproblem for $\underline{B}$ be finitely soluble is that $\underline{B}$ be row-G-astic. A sufficient condition is that $\underline{B}$ be row-ϕ-astic.

<u>Proof</u>. If $\underline{B} \otimes \underline{x} = \lambda \otimes \underline{x}$ with λ , $\underline{x}$ finite then $\sum_{j=1}^{n}{}_{\oplus} (\{\underline{B}\}_{ij} \otimes \{\underline{x}\}_j) = \lambda \otimes \{\underline{x}\}_i$ is finite, $(i = 1,\ldots, n)$. No $\{\underline{B}\}_{ij}$ can be $+\infty$ and no row of $\underline{B}$ can be made up of $-\infty$ entries. So $\underline{B}$ is row-G-astic. If $\underline{B}$ is row-ϕ-astic let $\underline{x} \in E_n$ be defined by $\{\underline{x}\}_i = \phi$ $(i = 1,\ldots, n)$ and let $\lambda = \phi$. Then

$$\{\underline{B} \otimes \underline{x}\}_i = \sum_{j=1}^{r}{}_{\oplus} (\{\underline{B}\}_{ij} \otimes \phi) = \phi = \{\lambda \otimes \underline{x}\}_i \ . \qquad \blacksquare$$

We now need some concepts from graph theory in order to solve the eigenproblem.

<u>Definition 3.3</u>. A directed graph consists of a pair (N, F) where N is a finite set consisting of $n \geq 1$ elements called <u>nodes</u>, denoted by numerials $1, \ldots, n$ and F is a mapping which associates to each node $i \in N$ a subset $F(i)$ of N . A pair of nodes (i, j) with $j \in F(i)$ is an <u>arc</u>. To each arc (i, j) we associate a weight $a_{ij} \in E_1$, where $(E_1, \oplus, \otimes)$ is a given belt. We say that (N, F)

is a <u>weighted graph</u>. A <u>path</u> (from $i_0 \in N$ to $i_t \in N$) is a sequence $(i_0, i_1, \ldots, i_t)$ of nodes $i_k \in N$ such that $i_k \in F(i_k - 1)$, $k = 1, 2, \ldots, t$. The <u>length</u> of the path is $\underline{t}$. A path is a circuit if $i_0 = i_t$. A path is an <u>elementary path</u> if the nodes $i_0, \ldots, i_t$ are pairwise distinct. A circuit is elementary if $i_0 = i_t$ and the path $(i_0, i_1, \ldots, i_{t-1})$ is elementary.

<u>Definition 3.4</u>. The path product (if $i_0 = i_t$) the circuit product) is defined by $a_{i_0 i_1} \otimes \ldots \otimes a_{i_{t-1} i_t}$.

<u>Definition 3.5</u>. A graph (N, F) is <u>strongly complete</u> if for every pair of nodes $i, j \in F$, then $j \in F(i)$. We can also define the associated matrix of the directed graph.

<u>Definition 3.6</u>. The matrix of the directed graph is denoted by $\underline{A} \in M_{nn}$ where $\{\underline{A}\}_{ij} = [a_{ij}]$, $i, j = 1, \ldots, n$. (Here a_{ij} is the weight assigned to each arc).

<u>Definition 3.7</u>. Let $\Delta(\underline{B})$ denote the graph associated with the given matrix $\underline{A}$.

<u>Definition 3.8</u>. Suppose E_1 is a blog, and $\underline{B} \in M_{nn}$. $\underline{B}$ is <u>definite</u> if it satisfies the following condition: Each circuit σ in $\Delta(\underline{B})$ has product $p(\sigma) \leq \phi$ and at least one such circuit has $p(\sigma) = \phi$.

<u>Definition 3.9</u>. For any given matrix $\underline{B} \in M_{nn}$, the metric matrix generated by $\underline{B}$ is $\underline{\Gamma}(\underline{B}) = \underline{B} \otimes \underline{B}^2 \otimes \ldots \otimes \underline{B}^n$ or dually $\underline{\Gamma}^*(\underline{B}) = \underline{B}^* \otimes' \underline{B}^{2*} \otimes' \ldots \otimes' \underline{B}^{n*}$.

$\underline{\Gamma}(\underline{B})$ may be thought of as the least distance from node N_i to node N_j - Cuningham-Green [3].

Definition 3.10. Suppose a certain matrix $\underline{B}$ over a blog E_1 is definite. Then $\Delta(\underline{B})$ contains at least one circuit with circuit product equal to ϕ. An eigen-node of $\Delta(\underline{B})$ is any node on such a circuit. Two eigenvectors are equivalent if they are on the same circuit whose circuit product is equal to ϕ. Now we give some more theorems which will help in solving the eigenproblem.

Theorem 3.1. Let E_1 be a blog and let $\underline{B} \in M_{nn}$ for any given integer $n \geq 1$. If $\underline{B}$ is definite then $\underline{B}^n < \underline{\Gamma}(\underline{B})$ for every integer $n \geq 1$.

Theorem 3.2. Let E_1 be a blog and let $\underline{B} \in M_{nn}$, for a given integer $n \geq 1$ let $\underline{B}$ be definite. Then $\underline{\Gamma}(\underline{B})$ is definite and if j is an eigennode of $\Delta(\underline{B})$ then: $\{\underline{\Gamma}(\underline{B})\}_{jj} = \phi$. Conversely, if E_1 is linear, and $\{\underline{\Gamma}(\underline{B})\}_{jj} = \phi$ for some index j $(1 \leq j \leq n)$ then j is an eigennode.

Theorem 3.3. Let E_1 be a blog and let $\underline{B} \in M_{nn}$ for a given integer $n \geq 1$. Also, let $\underline{B}$ be definite. If j is an eigen-node of $\Delta(\underline{B})$ then $\underline{B} \otimes \underline{\xi}(J) = \underline{\xi}(J)$ where $\underline{\xi}(j)$ is the $j^{\underline{th}}$ column of $\underline{\Gamma}(\underline{B})$.

Proof.

$\underline{B} \otimes \underline{\Gamma}(\underline{B}) \leq \underline{\Gamma}(B)$ for the $j^{\underline{th}}$ column.

$\underline{B} \otimes \underline{\xi}(J) \leq \xi(J)$ so $\{\underline{B} \otimes \underline{\Gamma}(\underline{B})\}_{ij} = \{\underline{B}^2 \oplus \ldots \oplus \underline{B}^n \oplus \underline{B}^{n+1}\}_{ij}$

$i = (1, \ldots, n)$

$\geq \{\underline{B}^2 \oplus \ldots \oplus \underline{B}^n\}_{ij}$ $(i = 1, \ldots, n)$. Also

$\{\underline{B} \otimes \underline{\Gamma}(\underline{B})\}_{ij} = \sum_{k=1}^{n}{}_{\oplus} \{\underline{B}\}_{ik} \otimes \{\underline{\Gamma}(\underline{B})\}_{kj}$ $(i = 1, \ldots, n)$

$\geq \{\underline{B}\}_{ij} \otimes \{\underline{\Gamma}(\underline{B})\}_{jj}$ $(i = 1, \ldots, n)$. So

$$\{\underline{B} \otimes \underline{\Gamma}(\underline{B})\}_{ij} \geq \{\underline{B}\}_{ij} \quad (i = 1,\dots, n) .$$

$$\{\underline{B} \otimes \underline{\Gamma}(\underline{B})\}_{ij} \geq \{\underline{B} \otimes \underline{B}^2 \otimes \dots \otimes \underline{B}^n\}_{ij} \quad (i = 1,\dots, n) ,$$

i.e., $\underline{B} \otimes \underline{\xi}(J) \geq \underline{\xi}(J)$ and also $\underline{B} \otimes \underline{\xi}(J) \leq \underline{\xi}(J)$, therefore $\underline{B}(x) \otimes \underline{\xi}(\underline{J}) = \underline{\xi}(\underline{J})$.

This theorem states that the columns of $\Gamma(\underline{B})$ which correspond to eigen-nodes furnish eigenvectors for $\underline{B}$ with eigenvalue ϕ . The columns of $\underline{\Gamma}(\underline{B})$ are called the <u>fundamental eigenvectors</u> of $\underline{B}$.

<u>Definition 3.11</u>. If $\underline{B}$ is a definite square matrix over a blog E_1 , let $\underline{\xi}(J_1), \underline{\xi}(J_2),\dots, \underline{\xi}(J_n)$ be a maximal set of non-equivalent fundamental eigenvectors of $\underline{B}$. The space $\langle\underline{\xi}(J_1),\dots, \underline{\xi}(J_n)\rangle$ generated by these eigenvectors will be called the <u>eigenspace</u> of $\underline{B}$.

<u>Theorem 3.4</u>. Let E_1 be a linear blog and let $\underline{B} \in M_{nn}$; for a given integer $n \geq 1$, let $\underline{B}$ be a ϕ-astic definite matrix. If $\underline{n} \in E_n$ is a finite eigenvector having corresponding eigenvalue ϕ , then $\underline{n}$ lies in the eigenspace of $\underline{B}$.

<u>Proof</u>. Let $\underline{\xi}(J_1),\dots, \underline{\xi}(J_s)$ be a set of maximal non-equivalent eigenvectors for $\underline{B}$. Show $\underline{n} = \sum_{k=1}^{s}{}_{\oplus} (\underline{\xi}(j_k) \otimes \{\underline{n}\}_{j_k})$, where $\underline{\xi}(j_1),\dots, \underline{\xi}(j_s)$ form a subset of the columns of $\underline{\Gamma}(\underline{B})$. Now $\sum_{k=1}^{s}{}_{\oplus} (\underline{\xi}(j_k) \otimes \{\underline{n}\}_{j_k}) \leq \underline{\Gamma}(\underline{B}) \otimes \underline{n} = \underline{n}$ because

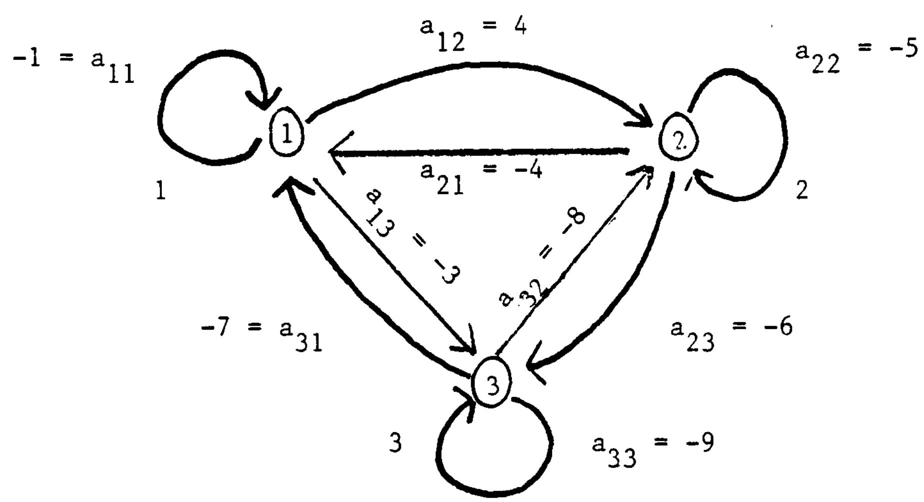
$\underline{\Gamma}(\underline{B}) \otimes \underline{n} = (\underline{B} \oplus \dots \oplus \underline{B}^n) \otimes \underline{n} = \underline{n} \oplus \underline{n} \oplus \dots \oplus \underline{n} = \underline{n}$. Let j be an eigennode of $\Delta(\underline{B})$. Then one of $j_1,\dots, j_s$ is equivalent to j : say j_h . We have then

$\{\sum_{k=1}^{s}{}_{\oplus}(\underline{\xi}(j_k) \otimes \{\underline{n}\}_{j_k})\}_j \geq \{\underline{\xi}(j_h)\}_j \otimes \{\underline{n}\}_{j_h} = \phi \otimes \{\underline{n}\}_j = \{\underline{n}\}_j$. Now let j be a non-eigennode of $\underline{\Delta}(\underline{B})$. Then $\exists$ eigen-node d such that $\{\underline{n}\}_j = \{\underline{\Gamma}(\underline{B})\}_{jd} \otimes \{\underline{n}\}_d$. One of $j_1,\dots, j_s$ is equivalent to d : say j_h is. Then $\{\sum_{k=1}^{s}\oplus(\underline{\xi}(j_k)\otimes\{\underline{n}\}_{j_k})\}_j \geq \{\underline{\xi}(j_h)\}_j \otimes \{\underline{n}\}_{j_h} =$ $\{\underline{\Gamma}(\underline{B})\}_{jd}\otimes\{\underline{n}\}_d = \{\underline{n}\}_j$. So $\underline{n} = \sum_{k=1}^{s}\oplus(\underline{\xi}(j_k)\otimes\{\underline{n}\}_{j_k})$. ■

Theorem 3.5. Let E_1 be a linear blog and let $\underline{A}\in M_{mn}$ for a given integer $n \geq 1$. If the eigen-problem for $\underline{A}$ is finitely soluble then every finite eigenvector has the same unique corresponding finite eigenvalue λ . The matrix $\lambda^{-1}\otimes\underline{A}$ is definite, and all finite eigenvectors of $\underline{A}$ lie in the eigenspace of $\lambda^{-1}\otimes\underline{A}$. The non-equivalent fundamental eigenvectors which generate this space have the property that no one of them is linearly dependent on any subspace of the others.

We now give an example which will serve to illustrate the ideas presented thus far. Here we are considering the principal interpretation where the identity ϕ is 0 .

Example 3.1. Let the following be a diagraph.



Let the arcs be: $a_{11} = -1$, $a_{12} = 4$, $a_{22} = -5$, $a_{23} = -6$, $a_{32} = -8$, $a_{33} = -9$, $a_{21} = -4$, $a_{31} = -7$, and $a_{13} = -3$. Here (1) , (2) , and (3) are the nodes

TABLE 3.1

CIRCUIT TABLE FOR MATRIX $\underline{A}$

POSSIBLE ELEMENTARY CIRCUIT PRODUCTS	POSSIBLE ELEMENTARY CIRCUITS
$a_{12} \otimes a_{23} \otimes a_{31} = 4 - 6 - 7 = -9$	(1, 2, 3, 1)
$a_{23} \otimes a_{31} \otimes a_{12} = -6 - 7 + 4 = -9$	(2, 3, 1, 2)
$a_{31} \otimes a_{12} \otimes a_{23} = -7 + 4 - 6 = -9$	(3, 1, 2, 3)
$a_{13} \otimes a_{32} \otimes a_{21} = -3 - 8 - 4 = -15$	(1, 3, 2, 1)
$a_{32} \otimes a_{21} \otimes a_{13} = -8 - 4 - 3 = -15$	(3, 2, 1, 3)
$a_{21} \otimes a_{13} \otimes a_{32} = -4 - 3 - 8 = -15$	(2, 1, 3, 2)
$a_{11} = -1$	(1, 1)
$a_{22} = -5$	(2, 2)
$a_{33} = -9$	(3, 3)
$a_{12} \otimes a_{21} = 0$	(1, 2, 1)
$a_{23} \otimes a_{32} = -14$	(2, 3, 2)
$a_{31} \otimes a_{13} = -10$	(3, 1, 3)
$a_{21} \otimes a_{12} = 0$	(2, 1, 2)
$a_{32} \otimes a_{23} = -14$	(3, 2, 3)
$a_{13} \otimes a_{31} = -10$	(1, 3, 1)

Now the matrix corresponding to $\Delta(\underline{A})$ is formed as follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} -1 & 4 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{bmatrix} \quad \text{which is } \underline{A}$$

$$\underline{A}^2 = \begin{bmatrix} -1 & 4 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{bmatrix} \otimes \begin{bmatrix} -1 & 4 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & -2 \\ -5 & 0 & -7 \\ -8 & -11 & -10 \end{bmatrix} .$$

For details of the calculation see Appendix 2.

$$\underline{A}^3 = \begin{bmatrix} 0 & -1 & -2 \\ -5 & 0 & -7 \\ -8 & -11 & -10 \end{bmatrix} \otimes \begin{bmatrix} -1 & 4 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 4 & -3 \\ 4 & -1 & -6 \\ -7 & -4 & -11 \end{bmatrix} .$$

For details of the calculation see Appendix 3.

Now, by Definition 3.9 we have:

$$\underline{\Gamma}(\underline{A}) = \underline{A} \otimes \underline{A}^2 \otimes \underline{A}^3 = \begin{bmatrix} -1 & 4 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 & -2 \\ -5 & 0 & -7 \\ -8 & -11 & -10 \end{bmatrix}$$

$$\otimes \begin{bmatrix} -1 & 4 & -3 \\ 4 & -1 & -6 \\ -7 & -4 & -11 \end{bmatrix} = \begin{bmatrix} -2 & -7 & -8 \\ -5 & -6 & -19 \\ -22 & -23 & -30 \end{bmatrix} .$$

By Definition 3.8 $\underline{A}$ is definite since each circuit product is less than or equal to 0. Hence the circuits $(1, 2, 1)$ and $(2, 1, 2)$ have circuit products equal to 0. Since an eigen-node is any node in a circuit whose circuit product is 0, then the eigen-nodes are 1 and 2. Theorem 3.3 says that if j is an eigen-node of $\Delta(\underline{A})$ then the eigenvectors of $\underline{A}$ are just the $j^{\underline{th}}$ columns of $\underline{\Gamma}(\underline{A})$.

So the eigenvectors of $\underline{A}$ here are $\begin{bmatrix} -2 \\ -5 \\ -22 \end{bmatrix}$ and $\begin{bmatrix} -7 \\ -6 \\ -23 \end{bmatrix}$, each with eigenvalue equal to 0. Now since both eigen-nodes 1 and 2 lie in the same circuit, they are equivalent by Definition 3.10. So a maximal set of non-equivalent eigenvectors, corresponding to the eigen-value 0, is either $\left\{\begin{bmatrix} -2 \\ -5 \\ -22 \end{bmatrix}\right\}$ or $\left\{\begin{bmatrix} -7 \\ -6 \\ -22 \end{bmatrix}\right\}$. Now by Definition 3.11 the eigenspace is generated by $\left\langle\begin{bmatrix} -2 \\ -5 \\ -22 \end{bmatrix}\right\rangle$ or $\left\langle\begin{bmatrix} -7 \\ -6 \\ -23 \end{bmatrix}\right\rangle$.

Next we introduce the idea of a circuit mean. This concept will play an important role in the eigenproblem we solve.

<u>Definition 3.12</u>. E_1 is radicable if for each $a \in E_1$ and integer $t \geq 1$ there is a unique $x \in E_1$ such that $x^t = a$.

<u>Theorem 3.6</u>. Let E_1 be a linear radicable blog and for a given integer $n \geq 1$, let $x, y, a, b \in E_1$ satisfy $x^n = a \geq b = y^n$, then $x \geq y$. $x \sim \phi$ if and only if $a \sim \phi$ where $\sim$ is $\geq$, $=$, or $\leq$.

<u>Definition 3.13</u>. Suppose E_1 is a radicable belt and $\underline{A} \in M_{nn}$ for given $n \geq 1$. Then for each circuit σ in $\Delta(\underline{A})$, of length $t(\sigma)$ and circuit product $p(\sigma)$, we can compute a unique <u>circuit mean</u> $\mu(\sigma) \in E_1$ by $(\mu(\sigma))^{t(\sigma)} = p(\sigma)$. Define $\lambda(\underline{A}) = \sum_{\oplus} \mu(\sigma)$ where the sum is overall elementary circuits in $\Delta(\underline{A})$.

We now use information from Example 3.1 to establish Table 3.2.

TABLE 3.2

EXTENDED CIRCUIT TABLE FOR MATRIX $\underline{A}$

ELEMENTARY CIRCUIT	CIRCUIT PRODUCT	CIRCUIT LENGTH	CIRCUIT MEAN
(1, 2, 3, 1)	-9	3	-3
(2, 3, 1, 2)	-9	3	-3
(3, 1, 2, 3)	-9	3	-3
(1, 3, 2, 1)	-15	3	-5
(3, 2, 1, 3)	-15	3	-5
(2, 1, 3, 2)	-15	3	-5
(1, 1)	-1	1	-1
(2, 2)	-5	1	-5
(3, 3)	-9	1	-9
(1, 2, 1)	0	2	0
(2, 3, 2)	-14	2	-7
(3, 1, 3)	-10	2	-5
(2, 1, 2)	0	2	0
(3, 2, 3)	-14	2	-7
(1, 3, 1)	-10	2	-5

As an example of the computation for (3, 2, 1, 3) we have $-8 \otimes -4 \otimes -3 = -15$. To get the corresponding circuit mean we must solve

$X^t = X \otimes X \otimes X = -15$ or $3X = -15$ or $X = -5$.

<u>Definition 3.14</u>. $\underline{A}, \underline{B} \in M_{nn}$ over a blog are directly similar if there are n finite elements $n_1, \ldots, n_n \in E_1$ such that $\{B\}_{ij} = n_i^{-1} \otimes \{\underline{A}\}_{ij} \otimes n_j$. The notation for this is $\underline{A} \approx \underline{B}$.

Next we state several results that will be used in the proof of Theorem 3.8, which proves the necessary and sufficient conditions for the solution of the eigenproblem.

Theorem 3.7. Let E_1 be a commutative linear radicable blog and let $\underline{A} \in M_{nn}$ for a given integer $n \geq 1$. If the eigenproblem for $\underline{A}$ is finitely soluble then $\lambda(\underline{A})$ is finite and $\lambda(\underline{A})$ is then the only possible value for the eigenvalue in any finite solution to the eigenproblem for $\underline{A}$.

Lemma 3.3. Let E_1 be a blog and let $\underline{A}, \underline{B} \in M_{nn}$ for a given integer $n \geq 1$ such that $\underline{A} \approx \underline{B}$. Let $j_1, \ldots, j_{s+1}$ $(s \geq 1)$ be indices $(1 \leq j_k \leq n$, $k = 1, \ldots, s+1)$. The column j_{s+1} in $\underline{B}$ is linearly dependent on columns $j_1, j_2, \ldots, j_s$ in $\underline{B}$ if and only if the same holds in $\underline{A}$. A similar result holds for the rows.

Lemma 3.4. Let E_1 be a blog. Then $\approx$ is an equivalence relation on M_{nn} for any given integer $n \geq 1$. If $\underline{A}, \underline{B} \in M_{nn}$ and $\underline{A} \approx \underline{B}$ then $\exists$ $\underline{P}, \underline{Q} \in M_{nn}$ such that $\underline{B} = \underline{P} \otimes \underline{A} \otimes \underline{Q}$ and $\underline{P} \otimes \underline{Q} = \underline{Q} \otimes \underline{P} = \underline{I}_n$, the identity matrix. Then $\underline{B}^r = \underline{P} \otimes \underline{A}^r \otimes \underline{Q}$ for all integers $r \geq Q$ and $\underline{\Gamma}(\underline{B}) = \underline{P} \otimes \underline{\Gamma}(\underline{A}) \otimes Q$.

Lemma 3.5. Let E_1 be a blog and let $\underline{A}, \underline{B} \in M_{nn}$, for given integers $n \geq 1$. Suppose $\underline{A}, \underline{B}$ are directly similar. Then $\underline{A}$ is definite if and only if $\underline{B}$ is definite, and then a set of indices $j_1, \ldots, j_s$ give a set of non-equivalent eigen-nodes in $\Delta(\underline{A})$ if and only if they give a set of non-equivalent eigen-nodes in $\Delta(\underline{B})$.

Lemma 3.6. Let E_1 be a blog and let $\underline{B} \in M_{nn}$. For a given integer $n \geq 1$, let $\underline{B}$ be row ϕ astic. Let $\underline{\xi}(j_1), \ldots, \underline{\xi}(j_s)$ be a maximal set of non-equivalent fundamental eigenvectors for $\underline{B}$ and let $\underline{\Phi}(\underline{B}) \in M_{ns}$ be the matrix whose columns are $\underline{\xi}(j_1),\ldots, \underline{\xi}(j_s)$ in that order. Then $\underline{\Phi}(\underline{B})$ is doubly ϕ-astic.

Lemma 3.7. Let E_1 be a commutative linear radicable blog and let $\underline{A} \in M_{nn}$ for a given integer $n \geq 1$. If $\lambda(\underline{A})$ is finite then the matrix $(\lambda(\underline{A}))^{-1} \otimes \underline{A})$ is definite.

Theorem 3.8. Let E_1 be a commutative linear radicable blog, and let $\underline{A} \in M_{nn}$ for a given integer $n \geq 1$. Then the eigenproblem for $\underline{A}$ is finitely soluble if and only if $\lambda(\underline{A})$ is finite and $\underline{\Phi}((\lambda(\underline{A}))^{-1} \otimes \underline{A})$ is doubly G-astic, where $\underline{\Phi}((\lambda(\underline{A}))^{-1} \otimes \underline{A})$ is any matrix whose columns form a maximal set of non-equivalent fundamental eigenvectors for the definite matrix $(\lambda(\underline{A}))^{-1} \otimes \underline{A})$.

Proof. If the eigenproblem for $\underline{A}$ is finitely soluble then by Theorem 3.7 $\lambda(\underline{A})$ is finite and using Lemma 3.2, the matrix $(\lambda(\underline{A}))^{-1} \otimes \underline{A})$ is directly similar to a certain row-ϕ-astic matrix $\underline{B}$ and hence $\underline{\Gamma}((\lambda(\underline{A}))^{-1} \otimes \underline{A})$ is directly similar to $\underline{\Gamma}(\underline{B})$ by Lemma 3.4. Now the matrix $\underline{\Phi}(\underline{B})$ which has the same columns from $\underline{\Gamma}(\underline{B})$ as $\underline{\Phi}((\lambda(\underline{A})^{-1} \otimes \underline{A})$ has from $\underline{\Gamma}(\underline{A})$ is by Lemma 3.5 and Lemma 3.6 doubly-ϕ-astic so doubly G-astic. $\underline{\Phi}((\lambda(\underline{A}))^{-1} \otimes \underline{A})$ comes from $\underline{\Phi}(\underline{B})$ by multiplying matrix elements by finite scalars, so we have that $\underline{\Phi}((\lambda(\underline{A}))^{-1} \otimes \underline{A})$ is also doubly-G-astic.

If $\lambda(\underline{A})$ is finite, then $(\lambda(\underline{A}))^{-1} \otimes \underline{A})$ is definite by Lemma 3.7. So the definition of $\underline{\Phi}((\lambda(\underline{A}))^{-1} \otimes \underline{A})$ given in the theorem makes sense. If this matrix has s columns, and $x \in E_s$ is finite then so is $\underline{u} \in E_n$ where $\underline{\mu} = \underline{\Phi}((\lambda(\underline{A}))^{-1} \otimes \underline{A}) \otimes \underline{x}$. Hence $\underline{\mu}$ is a finite element of the eigenspace of $(\lambda(\underline{A}))^{-1} \otimes \underline{A})$. So $(\lambda(\underline{A})^{-1} \otimes A) \otimes \underline{\mu} = \underline{\mu}$ and $A \otimes \mu = \lambda(\underline{A}) \otimes \underline{\mu}$. Thus the eigenproblem for $\underline{A}$ is finitely soluble. ∎

Corollary 3.1. Let E_1 be a commutative linear radicable blog and let $\underline{A} \in M_{nn}$ for given integers $n \geq 1$ be finite. Then the eigenproblem for $\underline{A}$ is finitely soluble .

CHAPTER 4

CALCULATION OF THE EIGENPROBLEM

We will now briefly outline the procedure to solve the eigenproblem in general, and proceed to calculate the solution to a particular eigenproblem of the form $\underline{A} \otimes \underline{X}(r) = \lambda \otimes \underline{X}(r)$. Here $\underline{X}(r)$ will be some eigenvector which we will call d and λ will be $\lambda(\underline{A})$, the eigenvalue.

Theorem 4.1 - Cuninghame-Green [2] . The only possible value of $\lambda(A)$ is given by the greatest circuit mean in $\underline{A}$.

The steps to calculate the eigenproblem are as follows:

(1) Find $\lambda(\underline{A})$;

(2) Evaluate $\underline{\Gamma}(\lambda(\underline{A})^{-1} \otimes \underline{A})$; where the j^{th} columns having $\{\underline{\Gamma}_{jj}\} = \phi$ are the fundamental eigenvectors ;

(3) Eliminate equivalent eigenvectors;

(4) If the remaining columns do not form a doubly-G-astic matrix, then the finite eigenproblem has no solution. If they do, then we have the finite eigenspace by taking finite linear combinations of these columns.

These steps are presented in the flow chart form in Figure 3.2. We label each of the above steps at the appropriate place in the flow chart.

To illustrate the procedure for solving an eigenproblem, we give the following figure (Figure 4.1), Table 4.1 (circuit table for Matrix $\underline{A}$), and appropriate calculations.

Example 4.1.

$$\underline{A} = \begin{bmatrix} 1 & 2 & -2 & 4 \\ 1 & 3 & -6 & -5 \\ 1 & -5 & 1 & -4 \\ -2 & -4 & 3 & -3 \end{bmatrix} \quad \text{and} \quad \underline{b} = \begin{bmatrix} 7 \\ 8 \\ 10 \\ 20 \end{bmatrix} .$$

We should note here that we are considering the blog which we called the principal interpretation. The idea for $\underline{b}$ is that of the specified times at which the next following cycles must be completed for the machine example discussed in Chapter 1. The idea here for $\underline{A}$ is just the description of the interaction of four specific machines (as mentioned in Chapter 1).

So we want to minimize the maximum earliness relative to $\underline{b}$. We now proceed to solve the problem using Figure 4.1 and Table 4.1.

First we note that $\underline{A}$ is clearly row-G-astic.

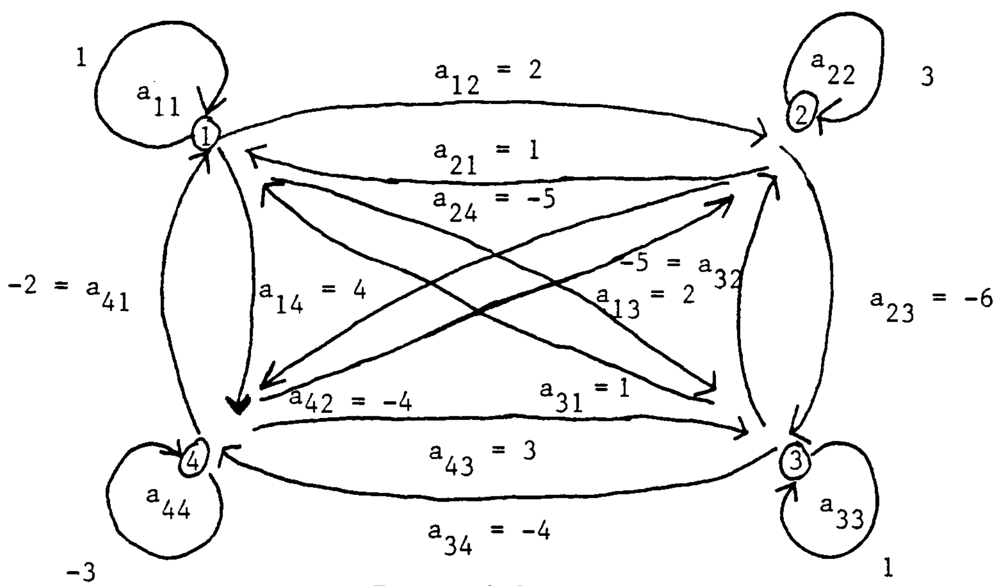


FIGURE 4.1

GRAPH ASSOCIATED WITH $\underline{A}$

TABLE 4.1

CIRCUIT TABLE FOR MATRIX $\underline{A}$

ELEMENTARY CIRCUIT	ELEMENTARY CIRCUIT PRODUCT	CIRCUIT LENGTH	CIRCUIT MEAN
(1, 1)	1	1	1
(2, 2)	3	1	3
(3, 3)	1	1	1
(4, 4)	-3	1	-3
(1, 2, 1)	$a_{12} \otimes a_{21} = 2 + 1 = 3$	2	-1/2
(2, 3, 2)	$a_{23} \otimes a_{32} = -6 - 5 = -11$	2	-1/2
(3,1, 3)	$a_{31} \otimes a_{13} = -2 + 1 = -1$	2	-1/2
(1, 2, 3)	$a_{12} \otimes a_{23} \otimes a_{31} = 2 - 6 + 1 = -3$	3	-1
(3, 2, 1)	$a_{32} \otimes a_{21} \otimes a_{13} = -5 + 1 - 2 = -6$	3	-2
(3, 4, 3)	$a_{34} \otimes a_{43} = -4 + 3 = -1$	2	-1/2
(4, 1, 4)	$a_{41} \otimes a_{14} = -2 + 4 = 2$	2	1
(2, 3, 4)	$a_{23} \otimes a_{34} \otimes a_{42} = -6 + 4 - 4 = -14$	3	-14/3
(4, 3, 2)	$a_{43} \otimes a_{32} \otimes a_{24} = 3 - 5 - 5 = -13$	3	-13/3
(3, 4, 1)	$a_{34} \otimes a_{41} \otimes a_{13} = -4 + 2 - 2 = -8$	3	-8/3
(1, 4, 3)	$a_{14} \otimes a_{43} \otimes a_{31} = 8$	3	8/3
(4, 1, 2)	$a_{41} \otimes a_{12} \otimes a_{24} = -5$	3	-5/3
(2, 1, 4)	$a_{21} \otimes a_{14} \otimes a_{42} = 1$	3	1/3
(1, 2, 3, 4)	$a_{12} \otimes a_{23} \otimes a_{34} \quad a_{41} = -10$	4	-5/2
(4, 3, 2, 1)	$a_{43} \otimes a_{32} \otimes a_{21} \quad a_{14} = -3$	4	-3/4
(2, 4, 2)	$a_{24} \otimes a_{42} = 9$	2	-9/2

$\lambda(\underline{A}) = \sum_{\oplus} \mu(\sigma) = 3$ by Definition 3.13 and by Theorem 4.1:

$$(\lambda(\underline{A}))^{-1} \otimes \underline{A}) = \begin{bmatrix} -2 & -1 & -5 & 1 \\ -2 & 0 & -9 & -8 \\ -2 & -8 & -2 & -7 \\ -5 & -7 & 0 & -6 \end{bmatrix} = \underline{B}$$

$$\underline{B}^2 = \begin{bmatrix} -3 & -1 & 1 & -1 \\ -2 & 0 & -7 & -1 \\ -4 & -3 & -4 & -1 \\ -2 & -6 & -2 & -4 \end{bmatrix} \qquad \underline{\Gamma}(\underline{B}) = \begin{bmatrix} -1 & -1 & 1 & 1 \\ -2 & 0 & -1 & -1 \\ -2 & -3 & -1 & -1 \\ -2 & -3 & 0 & -1 \end{bmatrix}$$

$$\underline{B}^3 = \begin{bmatrix} -1 & -1 & -1 & -2 \\ -2 & 0 & -1 & -1 \\ -5 & -3 & -1 & -3 \\ -4 & -3 & -4 & -1 \end{bmatrix} \qquad \underline{B}^4 = \begin{bmatrix} -3 & -1 & -2 & 0 \\ -2 & 0 & -1 & -1 \\ -3 & -3 & -3 & -4 \\ -5 & -3 & -1 & -3 \end{bmatrix}$$

Here $\underline{\Gamma}(\underline{B})$ has one column with diagonal equal to ϕ, so it is the fundamental eigenvector. The other three columns may be written as linear combinations of these. So $\underline{\Phi}(\underline{B}) = \begin{bmatrix} -1 \\ 0 \\ -3 \\ -3 \end{bmatrix}$ which generates the eigenspace since it is doubly-G-astic. Now

$$\underline{\Phi}(\underline{B}) \otimes (\underline{\Phi}(\underline{B})^* \otimes' \underline{b}) = \begin{bmatrix} -1 \\ 0 \\ -3 \\ -3 \end{bmatrix} =$$

$$= \begin{bmatrix} -1 \\ 0 \\ -3 \\ -3 \end{bmatrix} \left([1\ 0\ 3\ 3] \otimes' \begin{bmatrix} 7 \\ 8 \\ 10 \\ 20 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -1 \\ 0 \\ -3 \\ -3 \end{bmatrix} \left((1 \otimes' 7) \oplus' (0 \otimes' 8) \oplus' (3 \otimes' 10) \oplus' (3 \otimes' 20) \right)$$

$$= \begin{bmatrix} -1 \\ 0 \\ -3 \\ -3 \end{bmatrix} \otimes [8]$$

$$= \begin{bmatrix} 7 \\ 8 \\ 5 \\ 5 \end{bmatrix} = \underline{c} \quad .$$

Now to calculate the times when the next following cycles must be initiated, we subtract $\lambda(\underline{A})$ from each entry of $\underline{c}$ to get

$$\begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix} = \underline{d} \quad .$$

Also

$$\underline{A} \otimes \underline{d} = \begin{bmatrix} 1 & 2 & -2 & 4 \\ 1 & 3 & -6 & -5 \\ 1 & -5 & 1 & -4 \\ -2 & -4 & 3 & -3 \end{bmatrix} \otimes \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 8 \\ 5 \\ 5 \end{bmatrix} = 3 \otimes \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix} = \lambda(\underline{A}) \otimes \underline{d} \ .$$

Hence $\underline{A} \otimes \underline{d} = \lambda(\underline{A}) \otimes \underline{d}$.

So this is indeed in the form for an eigenvalue-eigenvector problem as stated in Chapter 1 , where $\lambda(\underline{A})$ is an eigenvalue and $\underline{d}$ is an eigenvector.

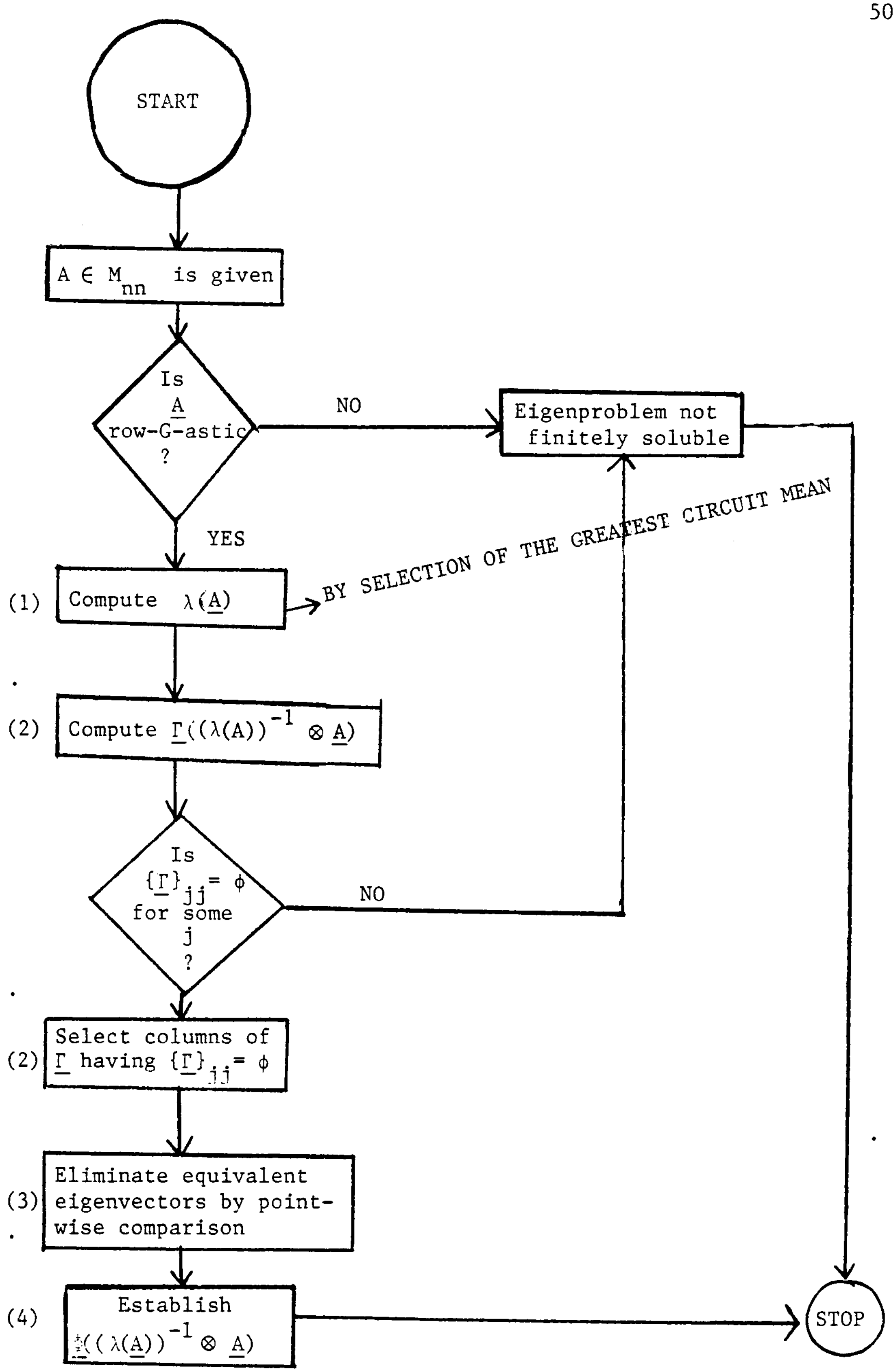


Figure 4.2

Flow Chart for Non-Equivalent Fundamental Eigenvectors

REFERENCES

REFERENCES

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APPENDICES

APPENDIX 1

$$\begin{bmatrix} (4 \otimes 4) \oplus (-\infty \otimes -\infty) \oplus (6 \otimes 4) & (4 \otimes -\infty) \oplus (-\infty \otimes 2) \oplus (6 \otimes 6) & (4 \otimes 6) \oplus (-\infty \otimes 2) \oplus (6 \otimes 3) \\ (-\infty \otimes 4) \oplus (2 \otimes -\infty) \oplus (2 \otimes 4) & (-\infty \otimes -\infty) \oplus (2 \otimes 2) \oplus (2 \otimes 6) & (-\infty \otimes 6) \oplus (2 \otimes 2) \oplus (2 \otimes 3) \\ (4 \otimes 4) \oplus (6 \otimes -\infty) \oplus (3 \otimes 4) & (4 \otimes -\infty) \oplus (6 \otimes 2) \oplus (3 \otimes 6) & (4 \otimes 6) \oplus (6 \otimes 2) \oplus (3 \otimes 3) \end{bmatrix}$$

APPENDIX 2

$$\begin{bmatrix} (-1 \otimes -1) \oplus (4 \otimes -4) \oplus (-3 \otimes -7) & (-1 \otimes 4) \oplus (4 \otimes -5) \oplus (-3 \otimes -8) & (-1 \otimes -3) \oplus (4 \otimes -6) \oplus (-3 \otimes -9) \\ (-4 \otimes -1) \oplus (-5 \otimes -4) \oplus (-6 \otimes -7) & (-4 \otimes 4) \oplus (-5 \otimes -5) \oplus (-6 \otimes -8) & (-4 \otimes -3) \oplus (-5 \otimes -6) \oplus (-6 \otimes -9) \\ (-7 \otimes -1) \oplus (-8 \otimes -4) \oplus (-9 \otimes -7) & (-7 \otimes 4) \oplus (-8 \otimes -5) \oplus (-9 \otimes -8) & (-7 \otimes -3) \oplus (-8 \otimes -6) \oplus (-9 \otimes -9) \end{bmatrix}$$

APPENDIX 3

$$\begin{bmatrix} (C \otimes -1) \oplus (-1 \otimes -4) \oplus (-2 \otimes -7) & (0 \otimes 4) \oplus (-1 \otimes -5) \oplus (-2 \otimes -8) & (0 \otimes -3) \oplus (-1 \otimes -6) \oplus (-2 \otimes -9) \\ (-5 \otimes -1) \oplus (0 \otimes -4) \oplus (-7 \otimes -7) & (-5 \otimes 4) \oplus (0 \otimes -5) \oplus (-7 \otimes -8) & (-5 \otimes -3) \oplus (0 \otimes -6) \oplus (-7 \otimes -9) \\ (-8 \otimes -1) \oplus (-11 \otimes 4) \oplus (-10 \otimes -7) & (-8 \otimes 4) \oplus (-11 \otimes -5) \oplus (-10 \otimes -8) & (-8 \otimes -3) \oplus (-11 \otimes -6) \oplus (-10 \otimes -9) \end{bmatrix}$$

APPENDIX 4

The following are examples of calculations involving the matrix operations $\oplus$, $\oplus'$, $\otimes$, and $\otimes'$.

Example 1. Suppose we have the relationship $\underline{X}(r+1) = \underline{B} \otimes \underline{X}(r)$ where $r = 1, 2, \ldots, n$.

Let $\underline{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and let $X_1(1) = 2$, $X_2(1) = 3$ then we have:

$$\begin{bmatrix} X_1(2) \\ X_2(2) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} (1 \otimes 2) & (2 \otimes 3) \\ (3 \otimes 2) & (4 \otimes 3) \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 7 \end{bmatrix} .$$

Now suppose we have the relationship $\underline{X}(r+1) = \underline{B} \otimes' \underline{X}(r)$; then

$$\begin{bmatrix} X_1(2) \\ X_2(2) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes' \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} (1 \otimes' 2) \otimes' (2 \otimes' 3) \\ (3 \otimes' 2) \otimes' (4 \otimes' 3) \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 5 \end{bmatrix} .$$

<u>Example 2</u>. Suppose we have $\underline{X}(r + 1) = \lambda \otimes' \underline{X}(r)$ where $r = 1, 2, \ldots,$ n, and let $\lambda = 3$ and let $X_1(1), X_2(1)$ be defined as in Example 1. Then

$$\begin{bmatrix} X_1(2) \\ X_2(2) \end{bmatrix} = \lambda \otimes \begin{bmatrix} X_1(1) \\ X_2(1) \end{bmatrix} = 3 \otimes \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} .$$

Note: Here $\lambda \otimes \underline{X}(r)$ is analogous to the conventional operation of multiplying a matrix by a scalar.

<u>Example 3</u>. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. Then we have

$$A \oplus B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \oplus \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 \oplus 5 & 2 \oplus 6 \\ 3 \oplus 7 & 4 \oplus 8 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$A \oplus' B = \begin{bmatrix} 1 & 2 \\ & \\ 3 & 4 \end{bmatrix} \oplus' \begin{bmatrix} 5 & 6 \\ & \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 \oplus' 5 & 2 \oplus' 6 \\ & \\ 3 \oplus' 7 & 4 \oplus' 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ & \\ 3 & 4 \end{bmatrix} .$$

VITA

Alan Thomas Gibby, son of Mr. and Mrs. James H. Gibby, was born in Bryson City, North Carolina on December 6, 1954. He lived in that city and attended elementary and high schools there.

In June 1972 he entered Western Carolina University and in June 1976 received a Bachelor of Science degree in Mathematics. In September 1980 he accepted a teaching assistantship at The University of Tennessee and began study toward a Master's degree. This degree was awarded in June 1982.