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To the Graduate Council:

I am submitting herewith a dissertation written by William Lewis Morris entitled "Minimal Weinstein Discs in a Subspace." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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MINIMAL WEINSTEIN DISCS IN A SUBSPACE

A Dissertation
Presented to
the Graduate Council of
The University of Tennessee

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

by
William Lewis Morris
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CHAPTER I

INTRODUCTION

I. NOTATION AND TERMINOLOGY

Let A denote a square matrix of order n over the field of complex numbers. For a nonzero vector y in the total vector space of order n over the complex numbers, consider the two scalar valued functions

$$\gamma(A;y) \equiv (y^H y)^{-1} y^H A y, \text{ and}$$

$$\rho(A;y) \equiv \gamma(A^H A;y) - \gamma(A;y)\gamma(A^H;y)$$

where the superscript H denotes the conjugate transpose. If α is a non-zero scalar, then $\gamma(A;\alpha y) \equiv \gamma(A;y)$ and so also $\rho(A;\alpha y) \equiv \rho(A;y)$. Thus, without loss of generality, one can restrict y to be a unit vector, i.e., $\|y\|^2 \equiv y^H y \equiv 1$. Denoting the complex conjugate of a scalar by a superscript bar, it is clear that $\overline{\gamma(A;y)} \equiv \gamma(A^H;y)$ from which it follows that $\rho(A;y) \equiv \|(A - \gamma(A;y)I)y\|^2$ where I is the identity matrix of order n . Thus $\rho(A;y)$ is real and nonnegative.

A matrix is said to be normalizable in case it is similar to a normal matrix. Since a matrix is normal if and only if it is unitarily similar to a diagonal matrix, it follows that a matrix is normalizable if and only if it is similar to a diagonal matrix. For each normalizable matrix A there is a real scalar $\kappa(A) \geq 1$ such that the disc

$$D(A;y) \equiv \{ \lambda : |\lambda - \gamma(A;y)| \leq \kappa(A) \sqrt{\rho(A;y)} \} \quad (1.1)$$

in the (complex) λ -plane is known to contain at least one root (proper value, eigenvalue, characteristic root) of the matrix A for each unit vector y . If A is Hermitian then this last statement is known as Weinstein's Theorem [6]^{*} and the disc (1.1) is called a Weinstein Disc. The details on extending Weinstein's Theorem to a wider class of matrices are given in Bauer and Householder [1]. Also it is known that if A is normal then $\kappa(A) = 1$.

The Weinstein Disc is a particular example of an inclusion region, i.e., a point set in the complex plane that contains at least one root of A . Other inclusion regions are given by

Theorem 1.1 [1]: Let A be a normalizable matrix and let $\alpha(\lambda)$ and $\beta(\lambda)$ be any two polynomials. For any nonnull vector w ,

$$\left\{ \lambda : \left| \frac{\alpha(\lambda)}{\beta(\lambda)} \right| \leq \kappa(A) \frac{\|\alpha(A)w\|}{\|\beta(A)w\|} \right\} \quad (1.2)$$

is an inclusion region.

Of particular interest is the set of polynomials

$$\varphi_m(\lambda) \equiv \frac{\det(\lambda W^H W - W^H A W)}{\det(W^H W)}$$

where $W \equiv (w, Aw, A^2 w, \dots, A^{m-1} w)$ and $\det$ denotes the determinant.

Theorem 1.2 [1]: Among all polynomials of degree m with leading coefficient unity, the choice $\alpha(\lambda) = \varphi_m(\lambda)$ minimizes the right member of (1.2) for a given $\beta(\lambda)$.

*Numbers in square brackets refer to the bibliography at the end of this dissertation.

Since $\varphi_1(\lambda) \equiv \lambda - \frac{w^H A w}{w^H w} \equiv \lambda - \gamma(A; w)$, it follows that with

$\alpha(\lambda) = \varphi_1(\lambda)$ and $\beta(\lambda) = 1$ the inclusion region (1.2) becomes the Weinstein Disc $D(A; w)$.

Since $\rho(A; y)$ can be expressed as the square of the Euclidean norm of a vector, it follows that $\rho(A; y) = 0$ if and only if $(A - \gamma(A; y)I)y = 0$ so that

Theorem 1.3: $\rho(A; y) = 0$ if and only if y is a proper vector of A .
In this case $\gamma(A; y)$ is a root of A .

At this point it should be noted that the symbol "0" has been used to denote both the scalar zero and the zero vector. Elsewhere it will be used to denote the zero matrix. It will be clear from the context which interpretation is intended.

If B is a matrix with n rows then the subspace of the total vector space of order n that is spanned by the columns of B is called B -space. If the columns of S are an orthonormal basis for S -space and if T is a matrix whose columns are an orthonormal basis for the orthogonal complement of S -space then $S^H T = 0$ and $W = (S, T)$ is a unitary matrix. Also, for a given S and T , every vector in the total vector space has a unique S -space and T -space component.

The matrix $S^H A S$ where $S^H S = I$ is called a section of A . If A and $S^H A S$ have a common property, then this property is said to be preserved by the section S . For example, if A is Hermitian then $(S^H A S)^H = S^H A^H S = S^H A S$ so that the Hermitian property is preserved by any section. One frequently used identity involving a matrix and a section is

$$\gamma(A;Sx) \equiv \gamma(S^H A S; x). \quad (1.3)$$

Here and elsewhere the symbol " $\equiv$ " is used to denote a functional identity in the vector variable over the unit vectors in some total vector space. Also this symbol is used to denote a definition of a new notational form. Some other notational conventions are: matrices are designated by upper case Latin letters, vectors by lower case Latin letters and scalars by lower case Greek letters. Exceptions are made in the case of subscripts, order of matrices and vectors, dimensions of spaces and the like. The i th column of an identity matrix is always denoted by e_i .

II. STATEMENT OF THE PROBLEM

The results to be presented concern the problem of determining for a given matrix the minimal Weinstein Disc over all vectors from a specified subspace. In particular, consider the subspace of the total vector space of order n that is spanned by the $k \leq n$ orthonormal column of the $n \times k$ matrix S . The problem is to minimize $\rho(A;y)$ over all unit vectors of the form $y = Sx$ where x is a vector of order k . The constraint $y^H y = 1$ becomes $1 = x^H S^H S x = x^H I x = x^H x$ since the columns of S are orthonormal. Thus the problem is to find the minimal value of $\rho(A;Sx)$ subject to $x^H x = 1$. The function $\rho(A;Sx)$ is continuous and the k -sphere specified by $x^H x = 1$ is closed and bounded so that some vector on the unit k -sphere minimizes $\rho(A;Sx)$.

Interest in the minimization of $\rho(A;Sx)$ arises from Clanton's algorithm [2]. In this algorithm a sequence of unit vectors $\{y_i\}$ is generated so that the associated sequence $\{\rho(A;y_i)\}$ is nonincreasing. If

$\rho(A; y_1) \rightarrow 0$ then by theorem 1.3 the algorithm produces a proper vector and the associated root of A . The y_i are taken from specified k -dimensional subspaces, i.e., each $y_i = S_i x_i$ where the columns of S_i are orthonormal. Each term in the sequence $\{\rho(A; S_i x_i)\}$ is in general not the minimal value of $\rho(A; S_i x)$.

The analysis of $\rho(A; Sx)$ begins by showing that it is expressible as $\rho(A; Sx) = \rho(S^H A S; x) + \delta(A; S; x)$ where $\delta(A; S; x)$ is a nonnegative function. With this decomposition some of the fundamental properties of $\rho(A; Sx)$ are derived. Also, the decomposition leads to the establishment of upper and lower bounds for the minimum of $\rho(A; Sx)$.

These bounds are examined in Chapter III and are used to define a set of vectors $\mathcal{M}$ that contains a minimizing vector for $\rho(A; Sx)$. Some special cases are examined for normal matrices.

Minimization of $\rho(A; y)$ and $\rho(A; Sx)$ is just a special case of the problem of finding stationary points of these two functions. Necessary and sufficient conditions for a vector to be a stationary point of ρ are derived in Chapter IV. These conditions are compared for the general case of $\rho(A; y)$ and the constrained case of $\rho(A; Sx)$. If A is normal then it is shown that the stationary points of $\rho(A; y)$ have a particularly simple geometric interpretation.

The results of Chapter IV are applied to the problem of improving the bounds developed in Chapter III. The approach used is based upon an iterative method. The behavior of this iterative method is studied for the general case as well as particular cases which can be realized in practical computations.

Finally, the basic features of Clanton's algorithm are reviewed and it is shown that the analysis of $\rho(A;Sx)$ leads to a refinement of the algorithm. A new algorithm to solve $Ay = \lambda y$ is the concluding item in this investigation.

CHAPTER II

A DECOMPOSITION THEOREM

For a unit vector x let Su and Tv be the S -space and T -space components respectively of $(A - \gamma(S^H AS; x)I)Sx$, i.e., let

$$(A - \gamma(S^H AS; x)I)Sx = Su + Tv. \quad (2.1)$$

Multiplication of equation (2.1) by S^H gives

$$u = S^H(A - \gamma(S^H AS; x)I)Sx = (S^H AS - \gamma(S^H AS; x)I)x; \quad (2.2)$$

and multiplication of equation (2.1) by T^H gives

$$v = T^H ASx. \quad (2.3)$$

It follows from equation (2.1) that

$$\|(A - \gamma(S^H AS; x)I)Sx\|^2 = u^H u + v^H v = \|u\|^2 + \|v\|^2, \quad (2.4)$$

so that (2.2), (2.3), and (2.4) imply

$$\|(A - \gamma(S^H AS; x)I)Sx\|^2 \equiv \|(S^H AS - \gamma(S^H AS; x)I)x\|^2 + \|T^H ASx\|^2. \quad (2.5)$$

Let $\delta(A; S; x) \equiv \|T^H ASx\|^2 \equiv x^H (T^H AS)^H (T^H AS)x$. That δ is a function of A , S and x only follows from the observation that $W = (S, T)$ is a unitary matrix and therefore

$$I = WW^H = (S, T) \begin{pmatrix} S^H \\ T^H \end{pmatrix} = SS^H + TT^H. \quad (2.6)$$

Thus $TT^H = I - SS^H$ so that

$$(T^H AS)^H (T^H AS) = S^H A^H TT^H AS = S^H A^H (I - SS^H) AS = S^H A^H AS - S^H A^H SS^H AS \quad (2.7)$$

and thus δ is independent of the particular choice of T .

Using identity (1.3), a restatement of identity (2.5) in functional notation is

Theorem 2.1: $\rho(A; Sx) \equiv \rho(S^H AS; x) + \delta(A; S; x).$

Since ρ and δ are both nonnegative scalar valued functions, an obvious consequence of theorem 2.1 is

Corollary: $\rho(A; Sx) = 0$ if and only if both $\rho(S^H AS; x)$ and
 $\delta(A; S; x) = 0.$

From theorem 1.3 it is clear that $\rho(S^H AS; x) = 0$ if and only if x is a proper vector of $S^H AS$. The solutions of $\delta(A; S; x) = 0$ are characterized by

Theorem 2.2: $\delta(A; S; x) = 0$ if and only if $ASx = Sz$ for some
vector $z.$

Proof: $\delta(A; S; x) = 0$ if and only if $T^H ASx = 0$. The null space of T^H is S -space so that ASx is in the null space of T^H if and only if $ASx = Sz$ for some vector z .

Corollary 1: $\delta(A; S; x) \equiv 0$ if and only if S -space is an invariant subspace of A , i.e., for each x there is a z such that $ASx = Sz.$

Corollary 2: If S -space is an invariant subspace of A then for each x that is a proper vector of $S^H AS$, Sx is a proper vector of A and conversely.

Proof: By corollary 1 (above), $\rho(A;Sx) \equiv \rho(S^H A;Sx)$. The result follows from theorem 1.3.

Another way to express δ is $\delta(A;S;x) \equiv \rho(TT^H A;Sx)$. This follows since $\gamma(TT^H A;Sx) \equiv \gamma(S^H TT^H A;Sx) \equiv \gamma(0;x) \equiv 0$, and $\gamma(A^H TT^H TT^H A;Sx) \equiv \gamma(S^H A^H TT^H A;Sx) \equiv \delta(A;S;x)$. Similarly, it is easy to verify that $\rho(SS^H A;Sx) \equiv \rho(S^H A;Sx)$. Hence an equivalent formulation of theorem 1.1 is

$$\rho(A;Sx) \equiv \rho(SS^H A;Sx) + \rho(TT^H A;Sx).$$

This last identity is a sort of additivity property since by (2.6) $A = IA = (SS^H + TT^H)A = SS^H A + TT^H A$. However, this property does not hold in general; a simple calculation shows that

$$\rho(SS^H A;y) + \rho(TT^H A;y) \equiv \rho(A;y) + 2\text{Re}[\gamma(SS^H A;y) \bar{\gamma}(TT^H A;y)] ,$$

where Re denotes the real part of a scalar.

A useful identity in two vector variables is given by

Lemma 2.3: For all unit vectors y and z ,

$$\rho(A;y) + |\gamma(A;y) - \gamma(A;z)|^2 \equiv y^H (A - \gamma(A;z)I)^H (A - \gamma(A;z)I) y.$$

Proof: This identity can be verified directly.

$$\begin{aligned} (A - \gamma(A;z)I)^H (A - \gamma(A;z)I) &\equiv A^H A - \bar{\gamma}(A;z)A - \gamma(A;z)A^H + |\gamma(A;z)|^2 I \\ &\equiv [A^H A - \bar{\gamma}(A;y)A] + [\bar{\gamma}(A;y)A - \bar{\gamma}(A;z)A - \gamma(A;z)A^H + |\gamma(A;z)|^2 I] . \end{aligned}$$

Thus,

$$y^H (A - \gamma(A;z)I)^H (A - \gamma(A;z)I) y \equiv \rho(A;y) + |\gamma(A;y) - \gamma(A;z)|^2 .$$

All proper vectors of A which are in S-space can be found in a systematic way. Two cases are to be considered. Case one: if $\det(T^H AS)^H(T^H AS) \neq 0$ then $\delta(A:S;x) > 0$ and so by theorem 2.1 and its corollary there is no proper vector of A in S-space. In geometric terminology this case implies that AS-space and S-space are disjoint as stated in theorem 2.2. Case two: if on the other hand $\det(T^H AS)^H(T^H AS) = 0$ then let the columns of Q be an orthonormal basis for the null space of $(T^H AS)^H(T^H AS)$. Since $\delta(A:S;x) = 0$ if and only if $x = Qz$, it follows that $x = Qz$ is a necessary condition for a solution of $\rho(A;Sx) = 0$. Hence a solution of $\rho(S^H AS;Qz) = 0$ will yield a proper vector of A that is in S-space. By theorem 2.1

$$\rho(S^H AS;Qz) \equiv \rho(Q^H S^H ASQ;z) + \delta(S^H AS;Q;z). \quad (2.8)$$

Now the argument can be centered on the identity (2.8) and the same two cases applied to the δ term in this equation. Continuing in this manner it can be determined that either there is no proper vector of A in S-space or all proper vectors that are in S-space will be isolated.

CHAPTER III

BOUNDS AND SPECIAL CASES

Let $\rho(A:S) = \min \{\rho(A;Sx) : x^H x = 1\}$; let $\mathcal{C}$ denote the set of all unit proper vectors of $S^H A S$; and let $\delta(A:S) = \min \{\delta(A:S;x) : x \in \mathcal{C}\}$. Since $(T^H A S)^H (T^H A S)$ is nonnegative Hermitian it has k nonnegative (not necessarily distinct) roots $0 \leq \alpha_1 \leq \dots \leq \alpha_k$.

Theorem 3.1: $\alpha_1 \leq \rho(A:S) \leq \delta(A:S)$.

Proof: For any x , $\delta(A:S;x) \geq \alpha_1$ so that $\alpha_1 \leq \delta(A:S;x) \leq \rho(S^H A S;x) + \delta(A:S;x) = \rho(A;Sx)$. However, for some x , $\rho(A:S) = \rho(A;Sx)$. Therefore $\alpha_1 \leq \rho(A:S)$. On the other hand, $\rho(A:S) \leq \rho(A;Sx) = \rho(S^H A S;x) + \delta(A:S;x)$ for all x . In particular, for $x \in \mathcal{C}$, $\rho(A:S) \leq \delta(A:S;x)$ by theorem 1.3, and so $\rho(A:S) \leq \delta(A:S)$.

These bounds give rise to the following possibilities:

$$\left. \begin{array}{ll} \text{(a)} & \alpha_1 < \rho(A:S) < \delta(A:S) \\ \text{(b)} & \alpha_1 < \rho(A:S) = \delta(A:S) \\ \text{(c)} & \alpha_1 = \rho(A:S) = \delta(A:S) \\ \text{(d)} & \alpha_1 = \rho(A:S) < \delta(A:S) \end{array} \right\} \quad (3.1)$$

If as in cases (c) and (d) $\alpha_1 = \rho(A:S)$ then for some x , $\alpha_1 = \rho(A;Sx) = \rho(S^H A S;x) + \delta(A:S;x)$. But since α_1 is the smallest root of $(T^H A S)^H (T^H A S)$ then $\delta(A:S;x) \geq \alpha_1$ so that $\alpha_1 = \rho(S^H A S;x) + \delta(A:S;x) \geq \rho(S^H A S;x) + \alpha_1$. This means that $\rho(S^H A S;x) = 0$

which by theorem 1.3 implies that $x \in \mathcal{C}$ and so $\alpha_1 = \delta(A:S;x) = \delta(A:S)$.

Therefore, case (d) is impossible, and case (c) implies that $\rho(A:S)$ occurs at a common proper vector of $(T^H AS)^H (T^H AS)$ and $S^H AS$. To summarize,

Theorem 3.2: $\alpha_1 < \rho(A:S) \leq \delta(A:S)$ if and only if $\alpha_1 \neq \delta(A:S)$.

The bounds developed above can be applied to the problem of determining the vectors which minimize $\rho(A;Sx)$. If $\rho(A:S) = \rho(A;Sm)$ then by theorem 3.1,

$$\alpha_1 \leq \rho(A;Sm) = \rho(S^H AS;m) + \delta(A:S;m) \leq \delta(A:S) . \quad (3.2)$$

In addition,

$$\alpha_1 \leq \rho(S^H AS;m) + \alpha_1 \leq \rho(S^H AS;m) + \delta(A:S;m) . \quad (3.3)$$

Combining (3.2) and (3.3) yields $\alpha_1 \leq \rho(S^H AS;m) + \alpha_1 \leq \delta(A:S)$, or $0 \leq \rho(S^H AS;m) \leq \delta(A:S) - \alpha_1$.

Let $\mathcal{M} = \{x : \rho(S^H AS;x) \leq \delta(A:S) - \alpha_1\}$. The last inequality establishes

Theorem 3.3: $\rho(A:S) = \rho(A;Sm)$ for some $m \in \mathcal{M}$.

Remark 3.1: It is possible that theorem 3.3 may give no new information, i.e., $\mathcal{M}$ could be the entire k -sphere. The content of theorem 3.3 depends not only on the numerical value of $\delta(A:S) - \alpha_1$ but also on the behavior of $\rho(S^H AS;x)$ and in particular on the maximum value of $\rho(S^H AS;x)$. However, it is convenient to defer a discussion of this general problem.

Remark 3.2: Of the three legitimate possibilities (3.1) it is clear that (b) and (c) hold if and only if a minimizing vector is a member of $\mathcal{C}$, and therefore case (a) holds if and only if a minimizing vector is a member of the relative complement of $\mathcal{C}$ in $\mathcal{M}$.

The bounds in theorem 3.1 hold for an arbitrary matrix. Additional information relating to these bounds can be developed under the assumption that A is normal.

Lemma 3.4: If A is normal with $Av = \lambda v$, $Aw = \mu w$, $w^H v = 0$, $\|v\| = \|w\| = 1$, and $t = \alpha v + \beta w$ where $\|t\| = 1$, then $\rho(A; t) = |\alpha|^2 |\beta|^2 |\lambda - \mu|^2$.

Proof: Since $\|t\|^2 = t^H t = (\bar{\alpha} v^H + \bar{\beta} w^H)(\alpha v + \beta w) = |\alpha|^2 + |\beta|^2$, it follows that $1 = \|t\| = \|t\|^2 = |\alpha|^2 + |\beta|^2$. Now,

$$\rho(A; t) = [A(\alpha v + \beta w)]^H [A(\alpha v + \beta w)] - |(\alpha v + \beta w)^H A(\alpha v + \beta w)|^2$$

which simplifies to

$$\begin{aligned} \rho(A; t) &= |\alpha|^2 |\lambda|^2 - |\alpha|^4 |\lambda|^2 + |\beta|^2 |\mu|^2 - |\beta|^4 |\mu|^2 - |\alpha|^2 |\beta|^2 (\bar{\lambda} \mu + \lambda \bar{\mu}) \\ &= |\alpha|^2 |\beta|^2 |\lambda|^2 - |\alpha|^2 |\beta|^2 (\bar{\lambda} \mu + \lambda \bar{\mu}) + |\alpha|^2 |\beta|^2 |\mu|^2 \\ &= |\alpha|^2 |\beta|^2 |\lambda - \mu|^2. \end{aligned}$$

Let $\mathcal{L}$ be the set of all unit proper vectors of $(T^H A S)^H (T^H A S)$.

Theorem 3.5: If A is Hermitian, $\rho(A; S) = \rho(A; Sv)$ and $v \in \mathcal{C}$ then $v \in (\mathcal{C} \cap \mathcal{L}) \neq \emptyset$.

Proof: Since $v \in \mathcal{C}$ there is a λ such that $S^H A S v = \lambda v$. Let w be such that $S^H A S w = \mu w$ where $w^H v = 0$. Let t be as in the lemma so that $\rho(S^H A S; t) = |\alpha|^2 |\beta|^2 (\lambda - \mu)^2$ where $|\alpha|^2 + |\beta|^2 = 1$.

$$\begin{aligned} \delta(A; S; t) &= |\alpha|^2 \delta(A; S; v) + |\beta|^2 \delta(A; S; w) + 2 \operatorname{Re} \{ \bar{\alpha} \beta v^H (T^H A S)^H (T^H A S) w \} , \\ &= |\beta|^2 [\delta(A; S; w) - \delta(A; S; v)] + \delta(A; S; v) + 2 \operatorname{Re} \{ \bar{\alpha} \beta v^H (T^H A S)^H (T^H A S) w \}. \end{aligned}$$

By hypothesis,

$$0 \leq \rho(A; S t) - \rho(A; S v) = \rho(S^H A S; t) + \delta(A; S; t) - \delta(A; S; v),$$

so that

$$|\alpha|^2 |\beta|^2 (\lambda - \mu)^2 - |\beta|^2 [\delta(A; S; w) - \delta(A; S; v)] + 2 \operatorname{Re} \{ \bar{\alpha} \beta v^H (T^H A S)^H (T^H A S) w \} \geq 0.$$

Since

$$\rho(A; S w) - \rho(A; S v) = \delta(A; S; w) - \delta(A; S; v) \geq 0,$$

the above inequality is of the form

$$|\beta|^2 (|\alpha|^2 |\eta|^2 + |\xi|^2) + 2 \operatorname{Re}(\bar{\alpha} \beta \epsilon) \geq 0$$

with

$$|\alpha|^2 + |\beta|^2 = 1$$

and

$$0 < |\alpha|^2 < 1.$$

Next, it is shown that $\epsilon = 0$ is a necessary and sufficient condition for this inequality to hold over all admissible α and β . Sufficiency is trivial. To establish necessity, let $\epsilon = |\epsilon| \exp(i\psi)$, $\alpha = |\alpha| \exp(i\theta)$

and $\beta = |\beta| \exp(i\varphi)$ so that $2\operatorname{Re}(\bar{\alpha}\beta\epsilon) = 2|\alpha||\beta||\epsilon|\cos(\varphi - \theta + \psi)$. By taking φ and θ such that $\cos(\varphi - \theta + \psi) = -1$ the inequality becomes

$$|\beta|^2(|\alpha|^2|\eta|^2 + |\xi|^2) \geq 2|\alpha||\beta||\epsilon|.$$

Assume that $|\eta|^2 + |\xi|^2 \neq 0$ since otherwise it is trivially true that $\epsilon = 0$. Then

$$(|\alpha|^{-2}-1)^{\frac{1}{2}} = \frac{|\beta|}{|\alpha|} \geq \frac{2|\epsilon|}{|\alpha|^2|\eta|^2 + |\xi|^2} \geq \frac{2|\epsilon|}{|\eta|^2 + |\xi|^2}.$$

Since

$$\inf\{(|\alpha|^{-2}-1)^{\frac{1}{2}} : 0 < |\alpha|^2 < 1\} = 0,$$

it follows that $\epsilon = 0$. But $\epsilon = v^H(T^H A S)^H(T^H A S)w = 0$ if and only if $(T^H A S)^H(T^H A S)v$ is orthogonal to w . This argument holds for each proper vector that is orthogonal to v . Also since $S^H A S$ is Hermitian it has proper vectors which are orthogonal to v and are an orthogonal basis for the orthogonal complement of v -space. Thus $(T^H A S)^H(T^H A S)v$ is in v -space, i.e., v is a proper vector of $(T^H A S)^H(T^H A S)$. This completes the proof.

Corollary: If A is Hermitian and $\mathcal{C} \cap \mathcal{H} = \emptyset$ then $\alpha_1 < \rho(A:S) < \delta(A:S)$.

Proof: Cases (b) and (c) in (3.1) are excluded by theorem 3.5.

The only remaining possibility is case (a).

Theorem 3.5 could have been established under the weaker assumption that $S^H A S$ is normal. However, in general, normality is not preserved by a section. The following results give necessary and sufficient conditions for a section of a normal matrix to be normal.

Lemma 3.6: The matrix A is normal if and only if $\rho(A;y) \equiv \rho(A^H;y)$.

Proof: If A is normal then $A^H A = A A^H$ so that for all unit vectors y, $\rho(A;y) \equiv y^H A^H A y - (y^H A y)(y^H A^H y) \equiv y^H A A^H y - (y^H A^H y)(y^H A y) \equiv \rho(A^H;y)$. On the other hand, $\rho(A;y) \equiv \rho(A^H;y)$ implies that $y^H A A^H y \equiv y^H A^H A y$ or $y^H (A A^H - A^H A) y \equiv 0$. Since $A A^H - A^H A$ is Hermitian, there exists a unitary matrix U and a diagonal matrix D such that

$$A A^H - A^H A = U^H D U$$

and therefore

$$y^H (A A^H - A^H A) y \equiv (Uy)^H D (Uy) \equiv 0.$$

The choice $y = U^H e_i$ implies that the i th diagonal element of D is zero.

Thus D is a zero matrix and therefore $A A^H - A^H A = U^H O U = 0$, or A is normal.

Theorem 3.7: If A is normal then $S^H A S$ is normal if and only if $\delta(A:S;x) \equiv \delta(A^H:S;x)$.

Proof: Since A is normal, the above lemma and theorem 2.1 imply that $\rho(A;Sx) \equiv \rho(S^H A S;x) + \delta(A:S;x) \equiv \rho(S^H A^H S;x) + \delta(A^H:S;x) \equiv \rho(A^H;Sx)$. If $S^H A S$ is normal then by lemma 3.6 we have

$$\rho(S^H A S;x) \equiv \rho(S^H A^H S;x)$$

so that

$$\delta(A:S;x) \equiv \delta(A^H:S;x).$$

If

$$\delta(A:S;x) \equiv \delta(A^H:S;x)$$

then

$$\rho(S^H A S;x) \equiv \rho(S^H A^H S;x)$$

so that $S^H AS$ is normal by lemma 3.6.

Corollary: If A is normal then $S^H AS$ is normal if and only if

$$(T^H AS)^H (T^H AS) = (T^H A^H S)^H (T^H A^H S).$$

At this point it is possible to resolve a particular case of the open question contained in remark 3.1. Let $k = 2$ and let $S^H AS$ be normal with $S^H ASv = \lambda v$, $SASw = \mu w$, $w^H v = 0$ and $\|v\| = \|w\| = 1$. Any vector x on the 2-sphere has a unique representation as $x = \alpha v + \beta w$ where $|\alpha|^2 + |\beta|^2 = 1$. Furthermore, by lemma 3.4 it follows that

$$\rho(S^H AS; x) = |\alpha|^2 |\beta|^2 |\lambda - \mu|^2 = |\alpha|^2 (1 - |\alpha|^2) |\lambda - \mu|^2. \quad (3.4)$$

Since

$$\max\{|\alpha|^2 (1 - |\alpha|^2) : 0 \leq |\alpha|^2 \leq 1\} = \frac{1}{4},$$

it follows that

$$\max\{\rho(S^H AS; x) : \|x\| = 1\} = \left|\frac{\lambda - \mu}{2}\right|^2.$$

In this case $\mathcal{N}$ is a proper subset of the 2-sphere if and only if

$$\left|\frac{\lambda - \mu}{2}\right|^2 > \delta(A:S) - \alpha_1$$

where

$$\delta(A:S) = \min [\delta(A:S; v), \delta(A:S; w)].$$

Also from (3.4) it follows that $x \in \mathcal{N}$ if and only if

$$|\alpha|^2 (1 - |\alpha|^2) |\lambda - \mu|^2 \leq \delta(A:S) - \alpha_1.$$

Thus, in this simple case, members of $\mathcal{H}_1$ are completely characterized by the quadratic inequality (in $|\alpha|^2$)

$$|\alpha|^4 - |\alpha|^2 + \frac{\delta(A:S) - \alpha_1}{|\lambda - \mu|^2} \geq 0$$

with $0 \leq |\alpha|^2 \leq 1$.

CHAPTER IV

STATIONARY POINTS

I. THE GENERAL CASE

Minimization of $\rho(A;y)$ and $\rho(A;Sx)$ is just a special case of the problem of finding all stationary points of these two functions. It is shown that such points exist and necessary and sufficient conditions for a vector to be a stationary point of ρ are derived. These results are needed for an analysis of an algorithm to find $\rho(A:S)$ and are used to analyze various algorithms of the Clanton variety.

Two applications of a general lemma due to Forsythe and Golub [3] give the desired results.

Lemma 4.1 [3]: Let $\varphi(z,w)$ and $\theta(z,w)$ be regular analytic functions of the two complex vector variables $z = (\delta_1, \delta_2, \dots, \delta_n)^T$ and $w = (\epsilon_1, \epsilon_2, \dots, \epsilon_n)^T$ with the property that $\varphi(z,\bar{z})$ and $\theta(z,\bar{z})$ are real. Then $\varphi(z,\bar{z})$ is stationary at $z = y$ with respect to all z such that $\theta(z,\bar{z}) = \theta(y,\bar{y})$ if and only if there exists a real constant λ such that

$$\frac{\partial \varphi}{\partial \epsilon_i} - \lambda \frac{\partial \theta}{\partial \epsilon_i} = 0$$

for $z = y$, $w = \bar{y}$ and $i = 1, 2, \dots, n$.

As a first application of the lemma, let

$$\varphi(z, w) = w^T A^H A z - (w^T A z)(w^T A^H z)$$

and

$$\theta(z, w) = w^T z.$$

Both $\varphi(z, w)$ and $\theta(z, w)$ are polynomials in the vector variables and are therefore regular analytic. Also $\varphi(z, \bar{z}) \equiv \rho(A; z)$ and $\|z\|^2$ are both real valued. For notational convenience let

$$\frac{\partial \varphi}{\partial \epsilon} = \left(\frac{\partial \varphi}{\partial \epsilon_1}, \frac{\partial \varphi}{\partial \epsilon_2}, \dots, \frac{\partial \varphi}{\partial \epsilon_n} \right)^T$$

and

$$\frac{\partial \theta}{\partial \epsilon} = \left(\frac{\partial \theta}{\partial \epsilon_1}, \frac{\partial \theta}{\partial \epsilon_2}, \dots, \frac{\partial \theta}{\partial \epsilon_n} \right)^T.$$

By a straightforward calculation,

$$\frac{\partial \varphi}{\partial \epsilon} - \lambda \frac{\partial \theta}{\partial \epsilon} = A^H A z - (w^T A z)(A^H z) - (w^T A^H z) A z - \lambda z.$$

From the lemma it follows that $\varphi(z, \bar{z}) \equiv \rho(A; z)$ is stationary at $z = y$ with respect to all z such that $\theta(z, \bar{z}) = \theta(y, \bar{y})$ if and only if

$$A^H A y - (y^H A y)(A^H y) - (y^H A^H y)(A y) - \lambda y = 0 \quad (4.1)$$

for some real λ .

By taking $\theta(y, \bar{y}) = 1$ and multiplying (4.1) by y^H it follows that

$$\lambda = y^H A^H A y - 2(y^H A y)(y^H A^H y) = \rho(A; y) - \gamma(A; y)\gamma(A^H; y), \quad (4.2)$$

and so λ is real.

Substituting (4.2) in (4.1) gives

$$[A^H A - \gamma(A; y) A^H - \gamma(A^H; y) A + \gamma(A; y) \gamma(A^H; y) I] y = \rho(A; y) y,$$

or

$$(A - \gamma(A; y) I)^H (A - \gamma(A; y) I) y = \rho(A; y) y; \quad y^H y = 1. \quad (4.3)$$

Let

$$M(A; y) \equiv (A - \gamma(A; y) I)^H (A - \gamma(A; y) I).$$

A restatement of (4.3) in this notation is

$$M(A; y) y = \rho(A; y) y; \quad y^H y = 1. \quad (4.4)$$

The stationary points of $\rho(A; Sx)$ can be characterized by a condition that is analogous to (4.3). To do this, let

$$\varphi(z, w) = w^T S^H A^H A S z - (w^T S^H A S z)(w^T S^H A^H S z)$$

and

$$\varphi(z, w) = w^T z.$$

Then $\varphi(z, \bar{z}) = \rho(A; Sz)$. In applying the lemma to this case, the calculations are almost identical to those in the first application and lead to the condition

$$S^H M(A; Sx) Sx = \rho(A; Sx) x; \quad x^H x = 1. \quad (4.5)$$

Notice that the left side of (4.5) is equal to

$$S^H (A - \gamma(S^H A S; x) I)^H (T T^H + S S^H) (A - \gamma(S^H A S; x) I) Sx$$

by (2.6), and this simplifies to

$$[(S^H A S - \gamma(S^H A S; x)I)^H (S^H A S - \gamma(S^H A S; x)I) + (T^H A S)^H (T^H A S)]x.$$

Thus lemma 4.1 implies

Theorem 4.2: (a) A unit vector y is a stationary point of $\rho(A; y)$ if and only if y satisfies (4.4);
 (b) A unit vector x is a stationary point of $\rho(A; Sx)$ if and only if x satisfies (4.5).

A comparison of the solutions of (4.4) and (4.5) shows the essential difference between the general and the constrained minimization problem.

Theorem 4.3: If $y = Sx$ is a solution of (4.4) then x is a solution of (4.5).

Proof: Substitution of Sx for y in (4.4) followed by premultiplication by S^H yields (4.5).

The converse of theorem 4.3 is certainly not true in general. If x is a solution of (4.5) then

$$SS^H M(A; Sx) Sx = \rho(A; Sx) Sx$$

and from (2.6) it follows that

$$M(A; Sx) Sx = \rho(A; Sx) Sx + TT^H M(A; Sx) Sx.$$

Thus Sx is a solution of (4.4) if and only if

$$T^H M(A; Sx) Sx = 0. \quad (4.6)$$

Theorem 4.4: If x is a solution of (4.5) then $y = Sx$ is a solution of (4.4) if and only if (4.6) holds.

II. NORMAL MATRICES

If U is a unitary matrix, then for each unit vector y there exists a unique unit vector z such that

$$y = Uz. \quad (4.7)$$

If (4.7) is substituted in (4.3) then

$$[A - \gamma(U^H AU; z)I]^H U U^H [A - \gamma(U^H AU; z)I] Uz = \rho(U^H AU; z) Uz$$

and on multiplying this by U^H

$$[B - \gamma(B; z)I]^H [B - \gamma(B; z)I] z = \rho(B; z) z \quad (4.8)$$

where $B = U^H AU$ and $z^H z = 1$.

Thus it follows that (4.3) and (4.8) are equivalent in that any solution of the one gives a solution of the other via (4.7).

The above remarks simplify the analysis of (4.3) if A is taken to be a normal matrix. Since a normal matrix is unitarily similar to a diagonal matrix, U can be chosen so that $B = U^H AU$ is diagonal. To be specific, let

$$B = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n),$$

where $\lambda_i \neq \lambda_j$ for $i \neq j$, and take the columns of U to be unit proper vectors ordered so as to produce the stipulated choice of B . With these selections of U and B it is clear that any vector z which satisfies (4.8) yields a vector y which is a solution of (4.3) where y is a linear combination of unit proper vectors of A and the coefficients of this linear combination are just the components of z .

Since B is diagonal with distinct roots it follows that z is a unit proper vector of B if and only if $z = \beta_i e_i$ where $|\beta_i| = 1$. From theorem 1.3 it follows that $\rho(B; z) = 0$ if and only if $z = \beta_i e_i$. Thus if z is a solution of (4.8) with $\rho(B; z) \neq 0$ then z must have two or more nonzero components so that

$$(B - \gamma(B; z)I)^H (B - \gamma(B; z)I)$$

must have a root of multiplicity at least two. If the multiplicity is $r \geq 2$ then, after a possible permutational transformation

$$|\lambda_1 - \gamma(B; z)|^2 = |\lambda_2 - \gamma(B; z)|^2 = \dots = |\lambda_r - \gamma(B; z)|^2 = \rho(B; z); \quad r \geq 2. \quad (4.9)$$

Also, if only the first r components of z are nonzero and z satisfies (4.9) then clearly z is a solution of (4.8).

The geometric interpretation of (4.9) is a disc $D(B; z)$ with roots λ_i , $i = 1, 2, \dots, r$, on the boundary. Since the center of $D(B; z)$ is a convex combination of the λ_i , not all of the λ_i can be on an open semicircle.

Also, if λ_i , $i = 1, 2, \dots, r$, are on a circle with center γ and radius squared ρ and not all λ_i are on an open semicircle then there exists solutions of

$$\sum_{i=1}^r |\eta_i|^2 \lambda_i = \gamma \quad (4.10)$$

$$\sum_{i=1}^r |\eta_i|^2 = 1.$$

Let $z = (\eta_1, \eta_2, \dots, \eta_r, 0, \dots, 0)^T$ where the η_i satisfy (4.10). It follows from (4.10) that

$$\gamma = \gamma(B; z)$$

and by hypothesis

$$|\lambda_1 - \gamma(B; z)|^2 = |\lambda_2 - \gamma(B; z)|^2 = \dots = |\lambda_r - \gamma(B; z)|^2.$$

Let

$$|\lambda_1 - \gamma(B; z)|^2 = \rho^*.$$

Then

$$(B - \gamma(B; z)I)^H (B - \gamma(B; z)I)z = \rho^* z. \quad (4.11)$$

On multiplying (3.6) by z^H

$$\rho(B; z) = z^H (B - \gamma(B; z)I)^H (B - \gamma(B; z)I)z = \rho^*.$$

Thus z satisfies (4.9) and hence is a solution of (4.8). Relating these observations to equation (4.3) implies

Theorem 4.5: Let A be a normal matrix. A unit vector y is a solution of (4.3) if and only if $D(A; y)$ is a point or a disc with two or more roots of A on its boundary, not all of which are on an open semicircle and y is a linear combination of those and only those proper vectors of A associated with the roots on the boundary of $D(A; y)$.

The number of solutions of (4.3) depends upon the relative position of the roots of A in the complex plane. Any two roots can be thought of as diametrically opposed on a circle and therefore for any roots λ_i and λ_j

$$\gamma(A; y) = \frac{1}{2}(\lambda_i + \lambda_j), \quad (4.12)$$

and

$$\rho(A; y) = \left| \frac{1}{2}(\lambda_i - \lambda_j) \right|^2 \quad (4.13)$$

define $D(A; y)$ for some y that satisfies (4.3).

If λ_i , λ_j and λ_k are three noncollinear roots of A that are not on an open semicircle then some elementary calculations give that

$$\gamma(A; y) = \frac{(\lambda_i - \lambda_j)(|\lambda_i|^2 - |\lambda_k|^2) - (\lambda_i - \lambda_k)(|\lambda_i|^2 - |\lambda_j|^2)}{2i[\operatorname{Im}(\overline{\lambda_i} - \overline{\lambda_k})(\lambda_i - \lambda_j)]^2} \quad (4.14)$$

$$\rho(A; y) = \frac{|\lambda_i - \lambda_j|^2 |\lambda_i - \lambda_k|^2 |\lambda_j - \lambda_k|^2}{4[\operatorname{Im}(\overline{\lambda_i} - \overline{\lambda_k})(\lambda_i - \lambda_j)]^2} \quad (4.15)$$

define $D(A; y)$ for some y that satisfies (4.3).

If more than three roots of A are on the same circle, then, in general, a different solution of (4.3) is possible. However, since three points completely determine a circle, the values of γ and ρ will be of the form (4.14) and (4.15) no matter how many additional roots are on the circle.

Thus, one characterization of the solutions of (4.3) in the case of a normal matrix is

Theorem 4.6: If A is normal and y is a stationary point of $\rho(A;y)$ then either $\gamma(A;y)$ and $\rho(A;y)$ are given by (4.12) and (4.13) for some choice of i and j (i = j not excluded) or by (4.14) and (4.15) for some choice of i, j, and k.

Since the number of solutions of (4.3) depends upon the geometry of the roots of A, some special cases specified by particular geometric constraints are of interest. If, for example, all of the roots of A are collinear, then solutions involving three or more roots on a circle are eliminated. In particular,

Corollary: Let A be a Hermitian matrix. If y is a solution of (4.3) then $\gamma(A;y)$ and $\rho(A;y)$ are given by (4.14) and (4.15) for some choice of i and j.

The essence of this corollary was reported by Hufford [4].

Finally, for a normal matrix,

Theorem 4.7: (a) For each choice of i and j there exists a unit vector y such that (4.3), (4.12) and (4.13) are simultaneously satisfied.

(b) For each choice of i, j, and k such that $\lambda_i, \lambda_j, \lambda_k$ determine a circle whose center is in their convex hull there exists a unit vector y such that (4.3), (4.14) and (4.15) are simultaneously satisfied.

As a first application of the results of this chapter we return to the Remark 2.1 and the open question contained therein. To this end suppose that the roots of $S^H A S$ are $\lambda_1, \lambda_2, \dots, \lambda_k$. Let

$$\sigma(S^H A S) \equiv \max \{ |\lambda_i - \lambda_j| : i, j = 1, 2, \dots, k \}.$$

This scalar valued function is called the spread of $S^H A S$. Let

$$\mu(S^H A S) \equiv \max \{ \rho(S^H A S; x) : x^H x = 1 \}$$

and let $S^H A S$ be normal. It follows from theorem 4.7 that

$$\mu(S^H A S) \geq \max \{ |\frac{1}{2}(\lambda_i - \lambda_j)|^2 : i, j = 1, 2, \dots, k \} = [\frac{1}{2}\sigma(S^H A S)]^2. \quad (4.16)$$

Inequality (4.16) could be a strict inequality in the case of a maximal Weinstein disc through three roots of $S^H A S$.

If $\mu(S^H A S) \leq \delta(A; S) - \alpha_1$, then $\mathcal{N}$ is the entire sphere defined by $x^H x = 1$ and so theorem 3.3 is trivially true. On the other hand, if $\mu(S^H A S) > \delta(A; S) - \alpha_1$ then $\mathcal{N}$ is a proper subset of the k -sphere. Therefore theorem 3.3 is nontrivial if and only if

$$\mu(S^H A S) > \delta(A; S) - \alpha_1.$$

A sufficient condition that $\mathcal{N}$ be a proper subset of the k -sphere is that

$$[\frac{1}{2}\sigma(S^H A S)]^2 > \delta(A; S) - \alpha_1. \quad (4.17)$$

Thus, inequalities relating the elements of a matrix to its spread are applicable. For example [5], if $S^H A S \equiv (\sigma_{ij})$ is normal then

$$\sigma(S^H A S) \geq \sqrt{3} \max \{ |\sigma_{ij}| : i \neq j \}. \quad (4.18)$$

Combining (4.17) and (4.18) gives

Theorem 4.8: If $S^H A S \equiv (\sigma_{ij})$ is normal and

$$(3/4) \max \{ |\sigma_{ij}|^2 : i \neq j \} > \delta(A:S) - \alpha_1$$

then $\mathcal{N}$ is a proper subset of the k -sphere.

In much the same way each of the spread inequalities for normal matrices in [5] will yield a theorem of a type similar to theorem 4.1.

Finally, we note that

$$\delta(A:S) - \alpha_1 \leq \sigma[(T^H A S)^H (T^H A S)]$$

with equality if and only if

$$\alpha_k = \delta(A:S).$$

CHAPTER V

IMPROVED BOUNDS

As noted in remark 3.2 there are three possibilities with regard to the bounds for $\rho(A:S)$. If (3.1c) holds then $\rho(A:S)$ is determined. If (3.1c) does not hold then there is a need to determine which of the remaining two cases hold. Finally, if it can be shown that (3.1a) holds then there is a need to improve the upper bound for $\rho(A:S)$. These problems are treated in the course of investigating an iterative method to solve (4.5).

The iteration to be studied is:

$$S^H M(A; Sx_i) Sx_{i+1} = \mu_i x_{i+1} \quad (5.1)$$

where μ_i is the smallest root of $S^H M(A; Sx_i) S$, x_{i+1} is an associated unit proper vector and A is normalizable.

If x_i and x_{i+1} are linearly dependent, i.e., if $x_{i+1} = \beta x_i$ for some scalar β , then

$$\mu_i = x_i^H S^H M(A; Sx_i) Sx_i = \rho(A; Sx_i)$$

so that x_i is a solution of (4.5).

After selecting an initial vector x_1 , the iteration produces a sequence of vectors $\{x_i\}$ and an associated sequence of nonnegative numbers $\{\rho(A; Sx_i)\}$. If the smallest root of $S^H M(A; Sx_i) S$ is simple for each i then for a given initial vector the members of $\{x_i\}$ are unique

up to a scalar factor and since $\rho(A; Sx_i) = \rho(A; S\beta x_i)$ the sequence $\{\rho(A; Sx_i)\}$ is unique.

Theorem 5.1: For any initial vector the sequence $\{\rho(A; Sx_i)\}$ is nonincreasing.

Proof: Since $S^H M(A; Sx_i) S$ is Hermitian, it follows that $\mu_i = \min \{v^H S^H M(A; Sx_i) S v : v^H v = 1\}$ and $\rho(A; Sx_i) \in \{v^H S^H M(A; Sx_i) S v : v^H v = 1\}$. Therefore, $\mu_i \leq \rho(A; Sx_i)$. Also, from theorem 1.2,

$$\rho(A; Sx_{i+1}) = \|(A - \gamma(A; Sx_{i+1})I)Sx_{i+1}\|^2 \leq \|(A - \gamma(A; Sx_i)I)Sx_{i+1}\|^2.$$

But

$$\|(A - \gamma(A; Sx_i)I)Sx_{i+1}\|^2 = x_{i+1}^H S^H M(A; Sx_i) S x_{i+1} = \mu_i.$$

Thus

$$\rho(A; Sx_{i+1}) \leq \mu_i \leq \rho(A; Sx_i) \quad (5.2)$$

for all i . This completes the proof.

Since $\mu_i = x_{i+1}^H S^H M(A; Sx_i) S x_{i+1}$ it follows from lemma 2.3 that $\mu_i = \rho(A; Sx_{i+1}) + |\gamma(A; Sx_{i+1}) - \gamma(A; Sx_i)|^2$. Thus

$$\rho(A; Sx_{i+1}) + |\gamma(A; Sx_{i+1}) - \gamma(A; Sx_i)|^2 = \mu_i \leq \rho(A; Sx_i)$$

or

$$0 \leq |\gamma(A; Sx_{i+1}) - \gamma(A; Sx_i)|^2 \leq \rho(A; Sx_i) - \rho(A; Sx_{i+1}). \quad (5.3)$$

Inequality (5.3) could be used in a proof of theorem 5.1 that is independent of the general theorem 1.2.

This inequality has a nice geometric interpretation. Let α be a point on the boundary of $D(A; Sx_i)$ such that the line segment from $\gamma(A; Sx_{i+1})$ to α is perpendicular to the line segment from $\gamma(A; Sx_i)$ to $\gamma(A; Sx_{i+1})$. Then

$$|\gamma(A; Sx_{i+1}) - \gamma(A; Sx_i)|^2 + |\gamma(A; Sx_{i+1}) - \alpha|^2 = \rho(A; Sx_i) .$$

Thus (5.3) becomes

$$- |\gamma(A; Sx_{i+1}) - \alpha|^2 + \rho(A; Sx_i) \leq \rho(A; Sx_i) - \rho(A; Sx_{i+1}) ,$$

or

$$|\gamma(A; Sx_{i+1}) - \alpha|^2 \geq \rho(A; Sx_{i+1}) .$$

Thus the boundary of $D(A; Sx_{i+1})$ meets the line segment from $\gamma(A; Sx_{i+1})$ to α or a diameter of $D(A; Sx_{i+1})$ is in $D(A; Sx_i)$. An immediate consequence of (5.3) is

Theorem 5.2: For any initial vector x_1 ,

(a) if $\rho(A; Sx_i) = \rho(A; Sx_{i+1})$ for some i then

$$\gamma(A; Sx_{i+1}) = \gamma(A; Sx_i);$$

(b) if $\gamma(A; Sx_{i+1}) \neq \gamma(A; Sx_i)$ then $\rho(A; Sx_{i+1}) < \rho(A; Sx_i)$.

Theorem 5.3: If x_1 is a solution of (4.5) and $\rho(A; S) = \rho(A; Sx_1)$ then $\rho(A; Sx_1)$ is the smallest root of $S^H M(A; Sx_1) S$.

Proof: If $\rho(A; Sx_1)$ is not the smallest root, i.e., if $\mu_i < \rho(A; Sx_1)$, then by (5.2)

$$\rho(A; Sx_{i+1}) \leq \mu_i < \rho(A; Sx_i)$$

and this contradicts the assumption that $\rho(A; S) = \rho(A; Sx_i)$.

If x_i and x_{i+1} are unit vectors such that $x_{i+1} = \beta x_i$ then clearly $\rho(A; Sx_i) = \rho(A; Sx_{i+1})$. Conversely, if $\rho(A; Sx_i) = \rho(A; Sx_{i+1})$ then $\mu_i = \rho(A; Sx_i)$ since $\rho(A; Sx_{i+1}) \leq \mu_i \leq \rho(A; Sx_i)$. Thus x_i and x_{i+1} are both unit proper vectors associated with the smallest root of $S^H M(A; Sx_i) S$. If the smallest root of this matrix is simple then it follows that $x_{i+1} = \beta x_i$. From the Cauchy-Schwarz inequality, $|x_{i+1}^H x_i| \leq |x_{i+1}| |x_i| = 1$, with equality if and only if x_{i+1} and x_i are dependent. Thus,

Theorem 5.4: If the smallest root of $S^H M(A; Sx_i) S$ is simple then $\rho(A; Sx_{i+1}) = \rho(A; Sx_i)$ if and only if $|x_{i+1}^H x_i| = 1$.

Corollary: If the smallest root of $S^H M(A; Sx_i) S$ is simple then $\rho(A; Sx_{i+1}) < \rho(A; Sx_i)$ if and only if $|x_{i+1}^H x_i| < 1$.

If for some i the smallest root of $S^H M(A; Sx_i) S$ has multiplicity $r > 1$ let the columns of R be orthonormal proper vectors of $S^H M(A; Sx_i) S$ associated with μ_i . Since $S^H M(A; Sx_i) S$ is Hermitian, R exists. Then x_{i+1} can be any unit vector from R -space and $\gamma(A; Sx_{i+1}) = \gamma(A; SRv) = \gamma(R^H S^H A S R; v)$ for some unit vector v . If for some unit vector v , $\gamma(R^H S^H A S R; v) \neq \gamma(A; Sx_i)$ then it follows from (5.3) that $\rho(A; Sx_{i+1}) < \rho(A; Sx_i)$ with $x_{i+1} = Rv$. Thus in the case of multiple roots of $S^H M(A; Sx_i) S$ (5.3) gives a sufficient condition for the sequence to be strictly decreasing.

In order to get an improved upper bound for $\rho(A; S)$ as given in (3.1), a natural choice for the initial vector is $x_1 \in \mathcal{B}$ such that

$\delta(A;S;x_1) = \delta(A;S)$. Let $\mathcal{D} = \{x \in \mathcal{C} : \delta(A;S;x) = \delta(A;S)\}$. It follows from theorem 5.1 that with $x_1 \in \mathcal{D}$, $\rho(S^H A S; x_1) + \delta(A;S;x_1) = \rho(A;Sx_1) \leq \delta(A;S)$ or $\rho(S^H A S; x_1) \leq \delta(A;S) - \delta(A;S;x_1)$. But $\delta(A;S;x_1) \geq \alpha_1 = \min \{\delta(A;S;x) : x^H x = 1\}$. Thus $\rho(S^H A S; x_1) \leq \delta(A;S) - \alpha_1$ for all i . This last inequality assures that $x_i \in \mathcal{M}$ for all i and therefore

Theorem 5.5: The iteration (5.1) with $x_1 \in \mathcal{D}$ yields a sequence of vectors $\{x_i\}$ with $x_i \in \mathcal{M}$ and a corresponding bounded nonincreasing sequence $\{\rho(A;Sx_i)\}$ such that the limit of $\rho(A;Sx_i)$ is no greater than $\delta(A;S)$.

Next, conditions under which $\lim \rho(A;Sx_i) < \delta(A;S)$ are developed for the general case, i.e., with no restrictions on A or on S other than orthonormality. To begin this development, let $x_1 \in \mathcal{D}$ and let $U = (x_1, V)$ be a unitary matrix. Then $S^H M(A;Sx_1)S$ has the same roots as

$$U^H S^H M(A;Sx_1) S U = U^H M(S^H A S; x_1) U + U^H (T^H A S)^H (T^H A S) U.$$

But

$$U^H M(S^H A S; x_1) U = \begin{pmatrix} 0 & 0 \\ 0 & V^H M(S^H A S; x_1) V \end{pmatrix}$$

and

$$U^H (T^H A S)^H (T^H A S) U = \begin{pmatrix} \delta(A;S) & x_1^H (T^H A S)^H (T^H A S) V \\ V^H (T^H A S)^H (T^H A S) x_1 & V^H (T^H A S)^H (T^H A S) V \end{pmatrix}$$

so that

$$U^H S^H M(A; Sx_1) S U = \begin{pmatrix} \delta(A:S) & x_1^H (T^H A S)^H (T^H A S) V \\ V^H (T^H A S)^H (T^H A S) x_1 & V^H S^H M(A; Sx_1) S V \end{pmatrix}. \quad (5.4)$$

The submatrix in the lower right block of (5.4) is Hermitian so that $Z^H V^H S^H M(A; Sx_1) S V Z$ is diagonal for some unitary matrix Z . Premultiplication of (5.4) by $\begin{pmatrix} 1 & 0 \\ 0 & Z^H \end{pmatrix}$ and postmultiplication by $\begin{pmatrix} 1 & 0 \\ 0 & Z \end{pmatrix}$ gives a matrix that is unitarily similar to (5.4) and has a diagonal submatrix in the lower right block. Thus without loss of generality $V^H S^H M(A; Sx_1) S V$ is assumed to be diagonal.

Under this assumption the diagonal elements of $V^H S^H M(A; Sx_1) S V$ are $v_i^H S^H M(A; Sx_1) S v_i$ where $V = (v_1, \dots, v_{k-1})$. From lemma 2.3

$$v_i^H S^H M(A; Sx_1) S v_i = \rho(A; S v_i) + |\gamma(A; Sx_1) - \gamma(A; S v_i)|^2.$$

If x_1 is a proper vector of $(T^H A S)^H (T^H A S)$ then both $x_1^H (T^H A S)^H (T^H A S) V$ and $V^H (T^H A S)^H (T^H A S) x_1$ are null vectors so that $\delta(A:S)$ is a root of $S^H M(A; Sx_1) S$. In this case $\delta(A:S)$ is the smallest root if and only if

$$\rho(A; S v_i) + |\gamma(A; Sx_1) - \gamma(A; S v_i)|^2 \geq \delta(A:S)$$

for each column v_i of V . On the other hand, if x_1 is not a proper vector of $(T^H A S)^H (T^H A S)$ and the smallest root of $S^H M(A; Sx_1) S$ is simple then $\delta(A:S) > \mu_1$. Thus, for the iteration (5.1),

Theorem 5.6: Let $x_1 \in \mathcal{D}$. (a) If x_1 is not a proper vector of $(T^H A S)^H (T^H A S)$ and the smallest root of $S^H M(A; Sx_1) S$ is simple then $\rho(A; Sx_2) < \delta(A:S)$.

(b) If x_1 is a proper vector of $(T^H AS)^H (T^H AS)$ and $\rho(A; Sx_2) < \delta(A; S)$ then

$$\rho(A; Sv) + |\gamma(A; Sx_1) - \gamma(A; Sv)|^2 < \delta(A; S)$$

for some v orthogonal to x_1 .

Sharper results can be obtained in the case of $k = 2$, i.e., S has just two orthonormal columns. This case is of interest since it is the only one suitable for practical computations.

Suppose that $\mathcal{D}$ includes at least one proper vector of $(T^H AS)^H (T^H AS)$ and let x_1 be one of these. If the roots of $(T^H AS)^H (T^H AS)$ are $\alpha_1 \leq \alpha_2$ then

(1) $\alpha_1 = \alpha_2$ implies that $\delta(A; S) = \alpha_1$ so that

$$\delta(A; S) = \rho(A; Sx_1) = \rho(A; S);$$

(2) $\alpha_1 < \alpha_2$ and $\delta(A; S) = \alpha_1$ implies that

$$\delta(A; S) = \rho(A; Sx_1) = \rho(A; S);$$

(3) $\alpha_1 < \alpha_2$ and $\delta(A; S) = \alpha_2$ implies that $S^H AS$ has only one linearly independent proper vector.

The conclusion follows in case (3) since if there is a unit vector $z \in \mathcal{D}$ such that $z \neq \beta x_1$ then $\delta(A; S; z) < \alpha_2$ and this contradicts the definition of $\delta(A; S)$. In this case $S^H AS$ is not normalizable.

If the roots of $S^H AS$ are distinct then $S^H AS$ is normalizable. If $S^H AS$ has a double root and is normalizable then $S^H AS$ is a scalar matrix. In this case $\rho(A; S) = \delta(A; S) = \alpha_1$ and this can be found without using the iteration.

If $S^H A S$ has a double root and is not normalizable then if $\alpha_1 \leq \delta(A:S) < \alpha_2$ the iteration can be used to get either $\rho(A:S)$ or an improved upper bound for it. If $\delta(A:S) = \alpha_2$ then this is the exceptional case. However, this case can be treated directly.

Suppose $S^H A S$ is not normalizable, v is a unit vector such that $S^H A S v = \lambda v$ and $\delta(A:S;v) = \alpha_2 > \alpha_1$. Let z be a unit vector such that $v^H z = 0$. Since $(T^H A S)^H (T^H A S)$ is Hermitian it follows that $(T^H A S)^H (T^H A S) z = \alpha_1 z$. Now $U = (v, z)$ is a unitary matrix and

$$U^H S^H A S U = \begin{pmatrix} \lambda & \beta \\ 0 & \alpha_1 \end{pmatrix}$$

where

$$\beta = v^H S^H A S z$$

and

$$U^H (T^H A S)^H (T^H A S) U = \begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_1 \end{pmatrix}.$$

Let $y = (\eta_1, \eta_2)^T$ be a unit vector. Then

$$\gamma(U^H S^H A S U; y) = \lambda |\eta_1|^2 + \beta \bar{\eta}_1 \eta_2 + \alpha_1 |\eta_2|^2 = \lambda + \beta \bar{\eta}_1 \eta_2$$

and

$$U^H S^H A S U - \gamma(U^H S^H A S U; y) I = \begin{pmatrix} -\beta \bar{\eta}_1 \eta_2 & \beta \\ 0 & -\beta \bar{\eta}_1 \eta_2 \end{pmatrix} = \beta \begin{pmatrix} -\eta_1 \eta_2 & 1 \\ 0 & -\bar{\eta}_1 \eta_2 \end{pmatrix}.$$

Thus

$$\rho(U^H S^H_{ASU}; y) = |\beta|^2 y^H \begin{pmatrix} |\eta_1|^2 |\eta_2|^2 & -\eta_1 \bar{\eta}_2 \\ -\bar{\eta}_1 \eta_2 & |\eta_1|^2 |\eta_2|^2 \end{pmatrix} y = |\beta|^2 (|\eta_2|^4 - |\eta_2|^2)$$

and

$$\delta(A; SU; y) = \alpha_2 |\eta_1|^2 + \alpha_1 |\eta_2|^2 = \alpha_2 + |\eta_2|^2 (\alpha_1 - \alpha_2) .$$

Therefore

$$\rho(A; SU; y) = |\beta|^2 |\eta_2|^4 - (|\beta|^2 + \alpha_2 - \alpha_1) |\eta_2|^2 + \alpha_2 \quad (5.5)$$

where

$$0 \leq |\eta_2|^2 \leq 1 .$$

Thus $\rho(A; S)$ can be found by minimizing a second degree real polynomial, namely (5.5). If $x = \eta_1 v + \eta_2 z$ where $|\eta_2|^2$ minimizes (5.5) and $|\eta_1|^2 = 1 - |\eta_2|^2$ then $\rho(A; Sx) = \rho(A; S)$.

The preceding remarks along with part (a) of theorem 5.6 establishes

Theorem 5.7: Let $k = 2$ and $x_1 \in \mathcal{D}$. (a) If the roots of S^H_{AS} are distinct or S^H_{AS} is not normalizable and $\delta(A; S) < \alpha_2$ then the iteration (5.1) gives either $\rho(A; S)$ or an upper bound for $\rho(A; S)$ that is less than $\delta(A; S)$.

(b) If S^H_{AS} is normalizable and has a double root then $\rho(A; S) = \alpha_1$.

(c) If S^H_{AS} is not normalizable and $\delta(A; S) = \alpha_2$ then $\rho(A; S)$ can be found by minimizing (5.5).

If $S^H A S$ is normal and $k = 2$ then theorem 5.7 implies that $\rho(A; Sx_2) < \delta(A; S)$ if and only if x_1 is not a proper vector of $(T^H A S)^H (T^H A S)$. This situation prevails for arbitrary S , i.e., for S with k columns where $n \geq k \geq 2$.

Theorem 5.8: Let $x_1 \in \mathcal{D}$. And let the smallest root of $S^H M(A; Sx_1) S$ be simple. If $S^H A S$ is normal then $\rho(A; Sx_2) < \delta(A; S)$ if and only if x_1 is not a proper vector of $(T^H A S)^H (T^H A S)$.

Proof: Without loss of generality $S^H A S$ can be taken to be diagonal, $S^H A S = \text{diag}(\delta_1, \delta_2, \dots, \delta_k)$, and $B = (\beta_{ij})$ can be taken to be unitarily similar to $(T^H A S)^H (T^H A S)$ with $\beta_{11} = \min \{\beta_{ii} : i=1, 2, \dots, k\}$. Then $x_1 = e_1$ and

$$S^H M(A; Sx_1) S = \begin{pmatrix} \beta_{11} & \beta_{12} & \dots & \beta_{1k} \\ \beta_{21} & \beta_{22} + |\delta_2 - \delta_1|^2 & \dots & \beta_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{k1} & \beta_{k2} & \dots & \beta_{kk} + |\delta_k - \delta_1|^2 \end{pmatrix}.$$

Since $S^H M(A; Sx_1) S$ is Hermitian it follows that the diagonal entries of this matrix, i.e., $e_i^T S^H M(A; Sx_1) S e_i$, are convex combinations of the roots of $S^H M(A; Sx_1) S$ and β_{11} is the smallest diagonal entry. Thus $\beta_{11} \geq \mu_1$ and, since μ_1 is assumed to be a simple root, $\beta_{11} = \mu_1$ if and only if e_1 is a proper vector of $S^H M(A; Sx_1) S$. But e_1 is a proper vector of $S^H M(A; Sx_1) S$ if and only if e_1 is a proper vector of B . Thus $\delta(A; S) = \mu_1$ if and only if x_1 is a proper vector of $(T^H A S)^H (T^H A S)$.

Since a section of a Hermitian matrix is Hermitian, an immediate corollary is

Corollary. If $x_1 \in \mathcal{D}$, the smallest root of $S^H M(A; Sx_1) S$ is simple and A is Hermitian then $\rho(A; Sx_2) < \delta(A; S)$ if and only if x_1 is not a proper vector of $(T^H A S)^H (T^H A S)$.

CHAPTER VI

ALGORITHMS FOR SOLVING $Ay = \lambda y$

In this chapter the results of chapter 5 are applied to the problem of solving $Ay = \lambda y$ where A is a normalizable matrix. The methods which are developed are refinements and extensions of a basic algorithm due to Clanton [2]. A basic theorem in Clanton's development is

Theorem 6.1 [2]: Let A be a normalizable matrix of order n and let q be a unit vector. If S is an n by k ($n \geq k$) matrix of orthonormal columns with $S = (q, V)$ and if $p = Sw$ where w is a unit proper vector associated with the smallest root of

$$Q = [(A - \gamma(A;q)I)S]^H [(A - \gamma(A;q)I)S] .$$

then $\rho(A;p) \leq \rho(A;q)$. If the smallest root of Q is a simple root then equality holds if and only if $p = \beta q$ where $|\beta| = 1$.

Repeated application of theorem 6.1 yields a nonincreasing sequence of Weinstein discs and is the basis for a Clanton algorithm. If the radii of the sequence of discs have limit zero then the algorithm produces an approximate root and proper vector of A .

Consider first the case where $n = k$, i.e., the case where S is a unitary matrix. Assume that λ and w are the smallest root and associated proper vector of Q , i.e.,

$$Qw = S^H M(A;q)Sw = \lambda w , \quad (6.1)$$

Multiplication of (6.1) by S gives

$$M(A;q)p = \lambda p ,$$

where

$$p = Sw .$$

From this it is clear that in this case the algorithm is based on the iteration

$$M(A;y_i)y_{i+1} = \lambda_i y_{i+1} \quad (6.2)$$

where λ_i is the smallest root of $M(A;y_i)$ and y_{i+1} is the associated unit proper vector.

If $y_i = \beta y_{i+1}$ then on multiplying (6.2) by y_{i+1}^H we have that $\lambda_i = \rho(A;y_i)$ and thus y_i is a solution of (4.4). Therefore this algorithm represents an iterative method to solve (4.4). From theorem 4.7 it follows that not all sequences of iterates (6.2) produce a sequence of Weinstein discs that approaches zero radius; for example, if A is normal then the iteration (6.2) could produce a sequence $\{\rho(A;y_i)\}$ that approaches one of the stationary values (4.13) or (4.15). If $y_i = \beta y_{i+1}$ then y_i is called a stationary vector of the algorithm.

In the general case, i.e., $n \geq k$, the matrix Q expressed in partitioned form is

$$Q = \begin{pmatrix} \rho(A;q) & q^H M(A;q)V \\ V^H M(A;q)q & V^H M(A;q)V \end{pmatrix}. \quad (6.3)$$

If q is a solution of (4.4) then it follows that the nondiagonal entries in (6.3) are zero submatrices. In this case $\rho(A;q)$ is a root of Q with

unit proper vector e_1 and the remaining roots of Q are roots of the submatrix in the lower right block of Q . Since $p = Sw = (q, V)w = \beta q$ if and only if $w = \beta e_1$, it follows that if q is a solution of (4.4) then q is a stationary vector of the algorithm if and only if $\rho(A; q)$ is less than every root of $V^H M(A; q) V$.

On the other hand, Clanton proves that

Theorem 6.2 [2]: If q is a stationary vector of the algorithm (for all choices of V) then Q is null in row one and column one except for the diagonal entry.

Not all solutions of (4.4) are stationary vectors of the algorithm. For example,

Theorem 6.3: If A is normal then q is a stationary vector of the Clanton algorithm if and only if q is a solution of (4.4) and no roots of A are interior points of $D(A; q)$.

Proof: If q is a stationary vector of the algorithm then, from theorem 6.2, $V^H M(A; q) q = 0$ for all admissible V . Thus, from theorem 4.4, q is a solution of (4.4). Since q is stationary the smallest root of Q is $\rho(A; q)$ which implies that all of the roots of $V^H M(A; q) V$ are greater than $\rho(A; q)$. But $V^H M(A; q) V$ is a section of a matrix with roots of the form $|\lambda_i - \gamma(A; q)|^2$ where λ_i is a root of A . Thus $\rho(A; q)$ is less than the roots of $V^H M(A; q) V$ for all V if and only if no roots of A are interior points of $D(A; q)$.

If q is a solution of (4.4) and no root of A is an interior point of $D(A; y)$ then, from the observations following (6.3), q is a stationary vector of the algorithm.

Corollary: If A is Hermitian with roots $\lambda_1 < \lambda_2 < \dots < \lambda_n$, q is a stationary vector of the Clanton algorithm and $\rho(A;q) \neq 0$ then

$$\rho(A;q) = [\frac{1}{2}(\lambda_i - \lambda_{i+1})]^2 \text{ and } \gamma(A;q) = \frac{1}{2}(\lambda_i + \lambda_{i+1}) \text{ for some}$$

$$i = 1, 2, \dots, n-1.$$

The mechanics of Clanton's algorithm can be stated quite simply in terms of the iteration (5.1).

Clanton's Algorithm

- (a) Choose S such that $S^H S = I$;
- (b) Apply (5.1) for one iteration with $x_1 = e_1$;
- (c) Choose S_1 such that $S_1 e_1 = S x_2$ and $S_1^H S_1 = I$;
- (d) Replace S with S_1 ;
- (e) Return to step (b).

Thus, this algorithm is based upon a search for a minimum of $\rho(A;y)$ by taking y from an initial subspace, making one iteration of (5.1) and choosing a new subspace so that the next application of (5.1) yields a disc with radius no larger than the preceding one. In each subspace S no attempt is made to find $\rho(A:S)$.

There is no particular reason for using only one application of the iteration (5.1) in step (b). In fact if the required sections of A and $A^H A$ are computed for a first iteration then they need not be recomputed for successive iterations. Also, since the purpose of the algorithm is to find a proper vector of A , knowing $\rho(A:S)$ is of primary interest since A has a proper vector in S -space if and only if $\rho(A:S) = 0$.

As for the initial vector x_1 in (5.1), $x_1 \notin \mathcal{M}$ if and only if

$$\rho(S^H A S; x_1) > \delta(A:S) - \alpha_1$$

which implies that

$$\rho(A; Sx_1) = \rho(S^H A S; x_1) + \delta(A:S; x_1) > \delta(A:S) + \delta(A:S; x_1) - \alpha_1 \geq \delta(A:S) .$$

Thus $x_1 \notin \mathcal{M}$ if and only if

$$\rho(A; Sx_1) > \delta(A:S) .$$

Therefore, if $e_1 \notin \mathcal{M}$ then

$$\rho(A; Se_1) > \delta(A:S) .$$

It follows from the corollary to theorem 2.1 that if there is a proper vector of A in S -space then the choice $x_1 \in \mathcal{D}$ will cause the iteration to terminate after one step and a proper vector of A will be found. A simple example shows that this need not be the case with $x_1 = e_1$.

For the example, let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and let

$$S = (e_1, e_2) .$$

There are two linearly independent proper vectors of A in S -space, yet the algorithm finds neither of them. In fact $\rho(A;Se_1) = 4$ whereas for any $x \in \mathcal{C}$, $\rho(A;Sx) = 0$.

For these reasons, a modification of the Clanton algorithm is in order. The mechanics of a modified algorithm are:

Modified Algorithm

- (a) Choose S such that $S^H S = I$;
- (b) Apply (5.1) until $|x_i^H x_{i+1}| = 1$ with $x_1 \in \mathcal{D}$;
- (c) Choose S_1 such that $S_1 e_1 = Sx_1$ and $S_1^H S_1 = I$;
- (d) Replace S with S_1 ;
- (e) Return to step (b).

Step (c) insures that $\rho(A;S_1) \leq \rho(A;Sx_1)$ since Sx_1 is a member of S_1 -space.

Both of these algorithms are in fact families of algorithms with a different one for each method of implementing step (a) and step (c). In addition, if the algorithms are to be applied to the problem of finding all solutions of $Ay = \lambda y$, then once a proper vector of A is found a procedure for continuing the algorithm without "rediscovering" the known proper vector is needed. These requirements can be satisfied and lead to a version of the modified algorithm which can be realized in practical computations. Interest is restricted to the case of $k = 2$ in order to simplify the calculations and to take advantage of the results in theorem 5.7.

Let A be a normalizable matrix of order n and let $W = (w_1, w_2, \dots, w_n)$ be a unitary matrix with columns arranged so that

$\rho(A;w_n) \leq \rho(A;w_i)$ for $i = 1, 2, \dots, n-1$. The algorithm to be described depends only on W and is called the W -algorithm.

Let $S_i = (w_i, w_n)$ for $i = 1, 2, \dots, n-1$. Clearly w_n is in each S_i -space so that $\rho(A;S_i) \leq \rho(A;w_n)$ for $i = 1, 2, \dots, n-1$. By applying theorem 5.7 for each S_i a set of $n-1$ unit vectors, say x_i , can be found such that $\rho(A;S_i x_i) \leq \rho(A;w_n)$. Suppose that $\rho(A;S_j x_j) \leq \rho(A;S_i x_i)$ for $i = 1, 2, \dots, n-1$. If $x_j = (\eta_1, \eta_2)^T$ then $y_j = (-\bar{\eta}_2, \bar{\eta}_1)$ is a unit vector orthogonal to x_j . Thus by replacing w_n with $S_j x_j$ and w_j with $S_j y_j$ the matrix W is again unitary and the condition $\rho(A;w_n) \leq \rho(A;w_i)$ for $i = 1, 2, \dots, n-1$ is satisfied. Repeated application of this procedure gives a sequence of unitary matrices for which the radius of the disc for the last column of W is nonincreasing. Also at each step the best choice for w_n is made in the sense that the smallest disc over all $n-1$ S_i -spaces is selected.

If $\rho(A;S_j x_j) = \rho(A;w_n) \neq 0$ then it can be shown that either w_n is a stationary point of $\rho(A;y)$ or $\rho(A;S_i) < \rho(A;w_n)$ for some i . To do this, first apply iteration (5.1) for each S_i with e_2 as the initial vector. If for some S_i a vector x_2 is produced by one application of (5.1) such that $\rho(A;S_i x_2) < \rho(A;S_i e_2)$ then $\rho(A;S_i) < \rho(A;w_n)$ and the algorithm can be continued. If this is not the case then by inequality (5.3)

$$|\gamma(A;S_i x_2) - \gamma(A;S_i e_2)|^2 = |\gamma(A;S_i x_2) - \gamma(A;w_n)|^2 = 0. \quad (6.4)$$

By hypothesis

$$S_i^H M(A;S_i x_2) S_i x_2 = \rho(A;S_i x_2) x_2 = \rho(A;w_n) x_2$$

so that

$$\det[S_i^H M(A; Sx_2) S_i - \rho(A; w_n) I] = 0. \quad (6.5)$$

But,

$$S_i^H M(A; Sx_2) S_i = \begin{pmatrix} \rho(A; w_i) + |\gamma(A; Sx_2) - \gamma(A; w_i)|^2 & w_i^H (A - \gamma(A; Sx_2) I)^H (A - \gamma(A; Sx_2) I) w_n \\ w_n^H (A - \gamma(A; Sx_2) I)^H (A - \gamma(A; Sx_2) I) w_i & \rho(A; w_n) + |\gamma(A; Sx_2) - \gamma(A; w_n)|^2 \end{pmatrix}$$

by lemma 2.3. Therefore (6.4) and (6.5) imply that

$$w_i^H (A - \gamma(A; w_n) I)^H (A - \gamma(A; w_n) I) w_n = 0 \quad (6.6)$$

for $i = 1, 2, \dots, n-1$. But from Theorem 4.4 (6.6) implies that w_n is a stationary point of $\rho(A; y)$. In this case w_n is called a stationary vector of the W-algorithm. Thus the sequence of discs generated by the W-algorithm have strictly decreasing radii and in general converge to a disc defined by a stationary vector. If A is normal and w_n is a stationary vector then $\rho(A; w_n)$ and $\gamma(A; w_n)$ are characterized in chapter IV.

If A is normalizable and $\rho(A; w_n) = 0$ then

$$W^H A W = \begin{pmatrix} U^H A U & 0 \\ w_n^H A U & \lambda_n \end{pmatrix} \quad (6.7)$$

where $A w_n = \lambda_n w_n$ and $U = (w_1, w_2, \dots, w_{n-1})$. In this case the W-algorithm can be applied to $U^H A U$ by selecting a new unitary matrix of

order $n-1$. Repeated application of this procedure amounts to reducing A to triangular form by unitary similarity transformations and thus producing all roots of A .

If A is normal and $\rho(A; w_n) = 0$ then it is known that exactly $n-1$ mutually orthogonal proper vectors of A are in the orthogonal complement of w_n -space. In this case the W -algorithm can be continued in the orthogonal complement of w_n -space by first rearranging the columns of W so that $\rho(A; w_{n-1}) \leq \rho(A; w_i)$ $i = 1, 2, \dots, n-2$ and then applying theorem 5.7 to the $n-2$ subspaces defined by $S_i = (w_i, w_{n-1})$ $i = 1, 2, \dots, n-2$. By continuing in this manner A can be reduced to diagonal form by a unitary similarity transformation without calculating sections of A as in the normalizable case. Thus for a normal matrix the W -algorithm represents a way of finding all roots and proper vectors of A .

Finally if w_n is a stationary vector of the W -algorithm then the algorithm can be continued in the orthogonal complement of w_n -space, i.e., by using only the first $n-1$ columns of W , and if a vector w_i is found such that $\rho(A; w_i) < \rho(A; w_n)$ then one can continue using all columns of W in the calculations and w_n is no longer a stationary vector.

Implementation of the W -algorithm in the form of a computer code calls for the solution of some additional problems most of which are machine dependent, i.e., depend upon the characteristics of the particular computer being used. To illustrate the nature of these problems consider the following simplified version of the W -algorithm.

For a normal matrix A the basic step in the W -algorithm are these:

W -algorithm

- (1) Set $m = 0$.
- (2) Find j such that $\rho(A; w_j) \leq \rho(A; w_i)$ for $i = 1, 2, \dots, n-m$.
- (3) Interchange w_j and w_{n-m} .
- (4) Set $S_i = (w_i, w_{n-m})$ for $i = 1, 2, \dots, n-m-1$.
- (5) Apply theorem 5.7 for each S_i to obtain vectors $x_i = (\alpha_i, \beta_i)^T$ for $i = 1, 2, \dots, n-m-1$.
- (6) Set $y_i = (-\overline{\beta_i}, \overline{\alpha_i})^T$ for $i = 1, 2, \dots, n-m-1$.
- (7) Find j such that $\rho(A; S_j x_j) \leq \rho(A; S_i x_i)$ for $i = 1, 2, \dots, n-m-1$.
- (8) Set $w_j = S_j y_j$ and $w_{n-m} = S_j x_j$.
- (9) If $\rho(A; w_{n-m}) = 0$ set $m = m+1$; otherwise go to step (4).
- (10) If $n-m > 1$ go to step (2); otherwise terminate the algorithm.

Techniques for handling stationary vectors have been excluded since they are not central to the discussion.

Each time step (8) is executed a new matrix W is generated. It is essential that each W be a unitary matrix whereas this might not be so due to the effects of rounding errors in the machine arithmetic. Let α be a fixed positive number and suppose that W satisfies

$$\|W^H W - I\| > \alpha \quad (6.8)$$

where $\| \cdot \|$ is any convenient matrix norm. Inequality (6.8) could then be used as a criterion for orthonormalizing the columns of W . However, since orthonormalizing the columns of W entails rounding errors, it is

possible the new W satisfies (6.8). Thus the minimal value of α is machine dependent. Nonetheless condition (6.8) or something equivalent would have to be included in a practical algorithm in order to minimize the effects of W being a nonunitary matrix.

In step (9) the condition $\rho(A; w_{n-m}) = 0$ would, in practice, be replaced by

$$\rho(A; w_{n-m}) < \epsilon \quad (6.9)$$

where ϵ is an assigned positive number. Of course the number of iteration of the W -algorithm necessary to satisfy (6.9) for a given ϵ might be too large to be practical in a given computer. This could be controlled by keeping a tally of the number of times step (8) is executed for a given m and thereby limit the number of iterations to an assigned maximum.

A similar situation prevails in step (5) in that the number of times the iteration (5.1) is computed would have to be limited to an assigned maximum. A criterion for convergence of the iteration (5.1) is given in theorem 5.4, but in practice it would have to be modified to

$$|x_{i+1}^H x_i| > 1 - \delta \quad (6.10)$$

where δ is an assigned positive constant. Thus the iteration (5.1) could be halted when successive iterates satisfy (6.10).

No matter what the criteria for controlling the iterations within the algorithm or for terminating the algorithm, the inclusion

theorem 1.1 assures that some specific information about the roots of A is obtained from the calculations. Therefore, the application of the W -algorithm to a normal matrix always yields a set of Weinstein discs and in the favorable cases, as discussed above, the discs can (within the capabilities of the computer) be made arbitrarily small.

In summary, the problem of calculating $\rho(A:S)$ arises in considering an algorithm to solve $Ay = \lambda y$ and the theoretical treatment of the problem has as its main application a new algorithm to solve $Ay = \lambda y$.

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