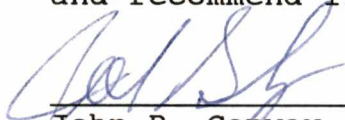


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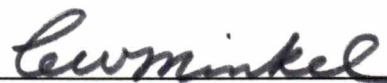

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GENERALIZED DISCRETE SCHWARZ-PICK LEMMA
FOR CIRCLE PACKINGS

A Thesis
Presented for the
Master of Science
Degree
The University of Tennessee, Knoxville

Jeffrey Alan Van Eeuwen

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ABSTRACT

In a paper titled "The Schwarz-Pick Lemma for Circle Packings," Alan Beardon and Kenneth Stephenson proved a discrete analogue of the classical Schwarz-Pick Lemma for circle packings. This paper generalizes this discrete analogue. The circle packings considered by Beardon and Stephenson are restricted in that neighboring circles have overlap angles equal to zero (i.e. neighboring circles are tangent). This restriction is lifted to allow neighboring circles to have overlap angles between 0 and $\pi/2$. Also included is a proof, that is independent of Andreev's work, of the existence of circle packings of triangulations of the closed disc in the hyperbolic plane with prescribed overlap angles.

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1. INTRODUCTION

An intuitive definition of circle packing is a configuration of circles in the euclidean or hyperbolic plane in which each circle has a prescribed set of neighbors, and a given circle touches each of its neighbors in a prescribed way, i.e. they overlap one another at a given angle. A simple example of this in the euclidean plane would be three circles each of which is tangent to the other two. Another example, in the hyperbolic plane, is the circle packing P in Figure 1. Also given in Figure 1 is the circle packing P_a , the extremal version of P . (The numbers in Figure 1 show three circles which correspond to one another in the two packings). P_a is termed an Andreev packing. An exact definition of circle packing and Andreev packing will be given in Definition V, but for now the intuitive definition and examples in Figure 1 will be sufficient.

In a recent paper titled "The Schwarz-Pick Lemma for Circle Packings", Alan Beardon and Kenneth Stephenson prove what they consider a discrete analogue for circle packings of the classical Schwarz Lemma in the invariant form given by Pick. They called this result the "Discrete Schwarz-Pick Lemma" and stated it as follows,

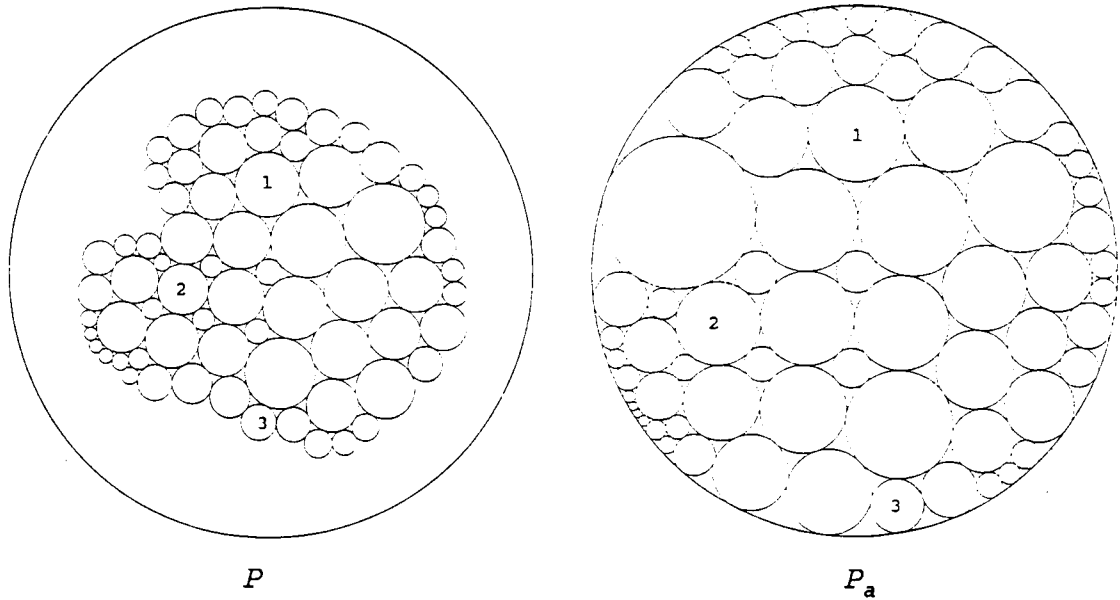


Figure 1. Packing P and the associated Andreev Packing P_a .

Theorem: Let P be a circle packing in the hyperbolic plane, P_a , its Andreev packing. Then

- (i) Each circle of P has a hyperbolic radius which is less than or equal to that of the corresponding circle in P_a .
- (ii) The hyperbolic distance between (centers of) circles in P is less than or equal to the hyperbolic distance between the corresponding circles in P_a .

Moreover, a single instance of (finite) equality in either (i) or (ii) implies equality in every instance, i.e. P and P_a are hyperbolically congruent.

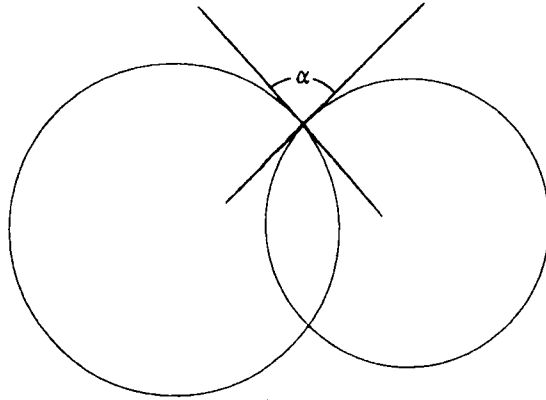


Figure 2. The overlap angle of two circles.

After proving the Discrete Schwarz-Pick Lemma, Beardon and Stephenson go on to suggest the beginnings of discrete analytic function theory. However, it is the Discrete Schwarz-Pick Lemma that we are most interested in. The object of this paper is to generalize the Discrete Schwarz-Pick Lemma. In [3], neighboring circles in a packing have an overlap angle equal to zero (i.e. $\alpha=0$ in Fig. 2). We want to extend this to let neighboring circles have overlap angles anywhere from 0 to $\pi/2$.

Although the existence of circle packings for triangulations of the closed disc was proved by Andreev (interpreted by Thurston), we give an independent proof of this. In the proof, two arguments arise. One follows the Perron method of Philip Bowers in [4]; the other, an inductive argument given in [3].

This paper will follow the basic outline of [3]. The terminology and statement of theorems will also follow closely

that of [3]. The differences lie primarily in the addition of statements for overlap angles, and a few changes in terminology. The proofs of Lemma VI, Theorem I, and Theorem III follow [3] and are included for completeness. Lemma VIII comes from [4].

2. SIMPLICIAL COMPLEXES

Circle packings have a combinatoric foundation that tells which circles are to be neighbors of one another. A method of representing these combinatorics is necessary. One simple way to do this is with abstract 2-dimensional simplicial complexes. It is done using the following rules. To each circle there will correspond a single vertex. These will be the only vertices of the complex. There will be an edge joining two vertices if and only if the two circles corresponding to those vertices are meant to overlap. The interiors of the triangles of centers of neighboring triples of circles will correspond with the triangular faces of the complex. As noted in [3], a benefit of using simplicial complexes is that the topology and terminology given to these are available for use.

Our attention will be placed on "special" complexes K and the associated packings. The conditions to be "special" are given in

Definition I: Let K be a simplicial 2-complex which is isomorphic to a finite triangulation of a closed topological disc. If K has the following properties,

- (i) Every boundary vertex shares an edge
with at least one interior vertex.

- (ii) Any pair of interior vertices can be connected by an edge path whose edges have only interior vertices as end points.

then K will be called special. $|K|$ will denote the realization of K .

As K is isomorphic to a finite triangulation of a closed disc, $|K|$ is homeomorphic to a closed disc. Hence $|K|$ is an orientable 2-manifold and its boundary is a simple closed curve. Every interior vertex is a vertex for at least three faces and the boundary of $|K|$ has at least three vertices.

The connected interior of the unit disc is needed in the proof of the classical Schwarz-Pick lemma, since it allows the Maximum Modulus Theorem to be used in the case of equality. Conditions (i) and (ii) above will allow for an analogous situation in the generalized version of the Discrete Schwarz-Pick Lemma. Hence conditions (i) and (ii) can be thought of as the discrete analogue of connected interior ([3], p.581).

We want to place a hyperbolic structure onto the complex K . In order to do this some basic facts about hyperbolic geometry should be known. The hyperbolic plane will be represented by the unit disc Δ with the Poincare metric, $\rho(,)$. Circles in (Δ, ρ) are euclidean circles in Δ , but their centers are generally not the euclidean centers. Euclidean circles in Δ that are

tangent to $\partial\Delta$ are called Horocycles and will be thought of as hyperbolic circles having infinite radius with center at their point of tangency. The isometries of (Δ, ρ) are Mobius transformations of Δ and their complex conjugates. Hyperbolic geodesics are arcs or parts of arcs of euclidean circles that are orthogonal to the unit circle. Lastly, the area of a polygon in Δ with n sides is $\pi(n-2)$ minus the sum of the interior angles. For example, the area of a triangle is $\pi - \alpha - \beta - \gamma$ where α, β, γ are its interior angles. From here on, whenever a circle's radius(center) is referred to it will be the hyperbolic radius(center), unless noted otherwise.

Briefly, the hyperbolic structure is to be placed on K in the following manner. Hyperbolic radii will be specified for each vertex of K and an overlap angle will be given to each edge. To each face of K there will correspond a triple of radii and overlap angles. From these triples comes three circles in Δ having the given radii and overlap angles. The hyperbolic triangle formed by joining the centers of these circles is associated to the face of K , thus inducing a hyperbolic structure onto this face. Doing this to each face of K gives a hyperbolic structure to K . This is done specifically starting with

Lemma I: Given a triple of hyperbolic radii $\{r_0, r_1, r_2\}$, $0 < r_i \leq \infty$, and overlap angles $\{\alpha_{0,1}, \alpha_{0,2}, \alpha_{1,2}\}$, $0 \leq \alpha_{i,j} \leq \pi/2$, there exist a

unique triple of circles C_0, C_1, C_2 such that

(i) C_i has radius r_i ,

(ii) C_i and C_j overlap at an angle of $\alpha_{i,j}$ ($i < j$),

and a unique hyperbolic triangle formed by the centers of these circles.

The uniqueness of the triple of circles and hyperbolic triangle is up to isometries, but the hyperbolic structure of the triangle (i.e. angles, edge lengths, and area) is unique. A proof of the finite case is suggested in [10].

Proof: Assume r_0, r_1, r_2 are all finite. By looking at Figure 3 and the triangle inequality, we have $F_{0,1} \leq r_0 + r_1$ where $F_{0,1}$ is the length of the geodesic connecting z_0 to z_1 . We claim that $\sup(r_0, r_1) < F_{0,1}$. Suppose without loss of generality (WLOG) that $r_0 \geq F_{0,1}$. The hyperbolic cosine rule and $\cosh r_1 > 1$ give

$$\cos(\pi - \alpha_{0,1}) = \frac{\cosh r_0 \cosh r_1 - \cosh F_{0,1}}{\sinh r_0 \sinh r_1}$$

$$\geq \frac{\cosh r_0 \cosh r_1 - \cosh r_0}{\sinh r_0 \sinh r_1} > 0.$$

But $\cos(\pi - \alpha_{0,1}) \leq 0$, a contradiction. Since there was nothing special about r_0, r_1 and $\alpha_{0,1}$, we have the same inequality for $r_0, r_2, F_{0,2}$ and $r_1, r_2, F_{1,2}$. This implies

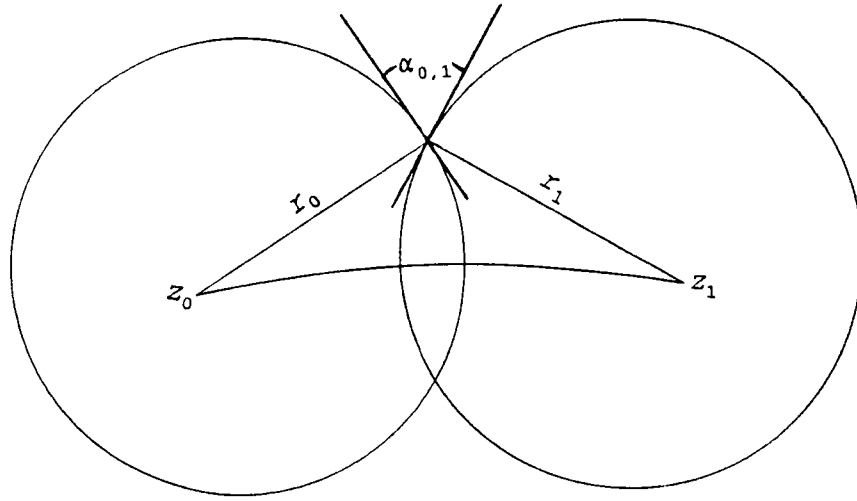


Figure 3. Finding the distance between centers.

$$F_{0,1} \leq r_0 + r_1 \leq F_{0,2} + F_{1,2}$$

$$F_{0,2} \leq r_0 + r_2 \leq F_{0,1} + F_{1,2}$$

$$F_{1,2} \leq r_1 + r_2 \leq F_{0,1} + F_{0,2}.$$

Forming the triangle with side lengths $F_{0,1}$, $F_{0,2}$, $F_{1,2}$ and placing a circle of the appropriate radius r_0 , r_1 , r_2 at each vertex gives the existence part of the lemma. Uniqueness of the triple of circles comes from a simple Mobius transformation argument. The uniqueness of the triangle of centers is evident.

If one of the radii is infinite, then assume that $r_2 = \infty$. Replace r_2 by a finite radius $\hat{r}_2$. There exist circles $C_0, C_1, \hat{C}_2$ in Δ with the given overlap angles, and radii $r_0, r_1, \hat{r}_2$ respectively. Scale this using the map $z \rightarrow z/2$. Note this does

not change any of the overlap angles. Fix the scaled C_0 and C_1 where they are in Δ . It is evident that without changing any of the overlap angles the radius of the scaled $\hat{C}_2$ can be increased until the scaled $\hat{C}_2$ is tangent to a euclidean circle of radius $1/2$ that is centered at the origin. Now rescaling by $z \rightarrow 2z$ gives the desired triple of circles. Again a Mobius transformation argument gives uniqueness. The last two cases are proved in a similar fashion. ■

Definition II: Let E denote the set of edges of K . A function Φ from E into $[0, \pi/2]$ that satisfies

- (i) If $e_1, e_2, e_3 \in E$ are distinct, form a closed edge path and $\Phi(e_1) + \Phi(e_2) + \Phi(e_3) \geq \pi$, then e_1, e_2, e_3 are the edges of a face of K .
- (ii) If $e_1, e_2, e_3, e_4 \in E$ are distinct, form a closed edge path and $\Phi(e_i) = \pi/2$, $i=1,2,3,4$ then e_1, e_2, e_3, e_4 are the edges of two faces that share a common edge e .

will be called an angle assignment for K . Let V be the set of vertices of K . A function R from V into $(0, \infty]$ will be called a label for K . Whenever R and Φ are used they will denote a label and angle assignment respectively.

It will be assumed that $R(v) < \infty$ for all interior vertices of K unless otherwise noted. If R_1 and R_2 are labels for K , $R_1 \leq R_2$ will mean $R_1(v) \leq R_2(v)$ for every $v \in V$. Respectively, Φ and R assign overlap angles and hyperbolic radii to the edges and vertices of K .

Let R and Φ be given. Let f_j be a face of K . The functions R and Φ along with the vertices and edges of f_j determine a triple of both radii and overlap angles. By Lemma I, a unique hyperbolic triangle T_j is determined by these triples. Identifying f_j and T_j pointwise so corresponding vertices match, a hyperbolic structure is induced on f_j from the hyperbolic structure of T_j . If two faces f_1 and f_2 share an edge, they share an overlap angle and two out of three radii. Hence the triples of circles corresponding to f_1 and f_2 can be placed in Δ so they have two circles in common. By doing this, the triangles T_1 and T_2 have a common edge. Near this edge, the identifications of f_1 and f_2 with T_1 and T_2 may be adjusted so that the hyperbolic structures on f_1 and f_2 are compatible across their common edge ([3], p.582). In this fashion, a hyperbolic structure, that is dependent on R and Φ , is placed on K .

Notice that although the hyperbolic structure is compatible across shared edges, it may or may not be at shared vertices.

In other words, R and Φ determine a hyperbolic metric of constant curvature -1 on K with singularities at the vertices ([3], p.582).

Definition III: The complex K along with the hyperbolic structure determined by R and Φ is called a hyperbolic complex and will be denoted by $K(R, \Phi)$.

With this hyperbolic structure on K , we are interested in how $K(R, \Phi)$ compares both locally and globally to the hyperbolic plane. We can examine this with the use of locally isometric immersions. Here, a locally isometric immersion from $K(R, \Phi)$ to Δ will be a continuous map that is a local isometry (except possibly at boundary vertices). First consider the local situation. Let p be any point of $K(R, \Phi)$ that is not a vertex. Since the hyperbolic structure on K is compatible across edges of faces, it is evident that there is an isometric imbedding of a sufficiently small neighborhood of p into Δ .

Let v be an interior vertex of K . Then v may be a singularity of the curvature on $K(R, \Phi)$. If there exists an isometric imbedding of a neighborhood of v into Δ , then v is a removable singularity with curvature -1 . The converse of this is also true i.e. v being a removable singularity with curvature -1 would be enough to insure an isometric imbedding

of a neighborhood of v into Δ . When will v be a removable singularity? To answer this, some definitions are needed.

Definition IV: Let $K(R, \Phi)$ be a hyperbolic complex. Let v be a vertex of K . Each face f of K that is in the star of v has the structure of a hyperbolic triangle. Hence f forms an angle $\theta(v, R, \Phi, f)$ at v . Define the angle sum at v to be

$$\sum_{j=1}^n \theta(v, R, \Phi, f_n)$$

where $f_1, \dots, f_n$ make up the star at v . Denote this by $\theta_R(v)$. (Note that $\theta_R(v) \geq 0$ with equality if and only if $R(v) = \infty$.)

Notice the notation $\theta_R(v)$ reflects that the angle sum is dependent on the labeling R , but does not reflect the dependence of the angle sum on Φ . The reason for this is that interest will lie in what happens to the angle sum as the label is changed and Φ remains fixed. So, for simplicity of notation, Φ is left off.

Let v be an interior vertex of K with $f_1, \dots, f_n$ making up the star at v . Since K is an orientable manifold, $f_1, \dots, f_n$ form a linearly ordered chain with f_i sharing an edge with f_{i-1} , ($i=2, \dots, n$), and f_1 sharing an edge with f_n . Put v in Δ and lay down the triangles associated with the faces of the star

at v in order, starting with f_1 , such that if the faces share an edge, the triangles share the corresponding edge. There are three possible outcomes in doing this: the triangles fit together perfectly, the last triangle laid down can only share an edge with one of its neighbors, or there are two or more triangles that overlap. (See Figure 4). These outcomes correspond to $\theta_R(v) = 2\pi$, $\theta_R(v) < 2\pi$ and $\theta_R(v) > 2\pi$ respectively, and give rise to

Definition V: The hyperbolic complex $K(R, \Phi)$ is said to be a local packing at a vertex v if $\theta_R(v) = 2\pi$, a local subpacking if $\theta_R(v) \geq 2\pi$, or a local superpacking if $\theta_R(v) \leq 2\pi$. $K(R, \Phi)$ is a packing (subpacking, superpacking) if it is a local packing (local subpacking, local superpacking) at every interior vertex. $K(R, \Phi)$ is an Andreiev packing if it is a packing and $R(v) = \infty$ for every boundary vertex v of K . If $K(R, \Phi)$ is a packing (subpacking, superpacking), then R will be referred to as a packing (subpacking, superpacking) label.

It should be clear that only when we have a local packing at an interior vertex v do we have a removable singularity at v with curvature -1 . Specifically, this comes as a result of considering the circumferences of infinitesimal circles about v in the metric of $K(R, \Phi)$ ([3], p.583). Hence $K(R, \Phi)$ being a local packing at v would allow an isometric imbedding of a

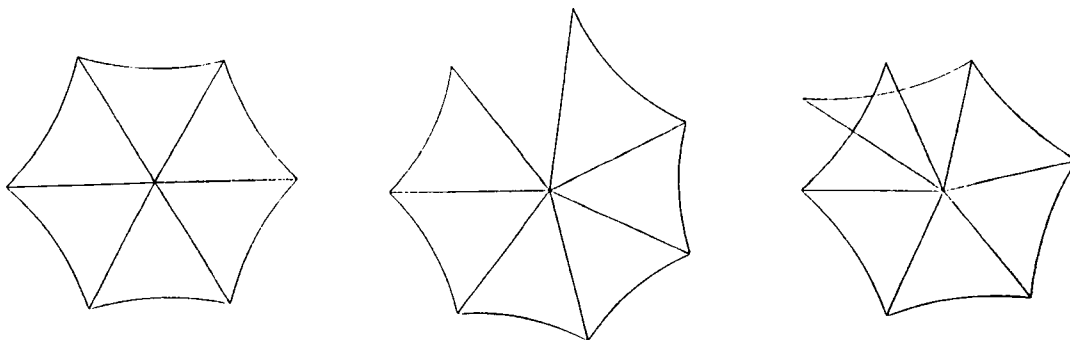


Figure 4. Laying down the triangles in the star of v .

neighborhood of v into Δ . Thus a necessary and sufficient condition to have an isometric imbedding of a neighborhood of v into Δ is that $K(R, \Phi)$ is a local packing at v . It would seem natural then that $K(R, \Phi)$ being a packing would be necessary and sufficient to have a locally isometric immersion from $K(R, \Phi)$ into Δ . This is so and is given as

Theorem I: Let $K(R, \Phi)$ be a hyperbolic complex. A necessary and sufficient condition for the existence of a locally isometric immersion of $K(R, \Phi)$ into Δ is that $K(R, \Phi)$ is a packing. This immersion is unique up to isometries.

If v is a boundary vertex such that $R(v) = \infty$, the immersion will map v to a point of the ideal boundary of Δ rather than into Δ . This should not cause any confusion ([3], p.584).

Proof: By the earlier discussion, necessity has been proved.

Assume $K(R, \Phi)$ is a packing. Since each face of $K(R, \Phi)$ is identified with a triangle in Δ that is unique up to isometries, it will be enough to describe the placement of these triangles in Δ .

The underlying idea of the proof is the following. Suppose $C = \{f_1, \dots, f_n\}$ is a chain of faces of K , where a chain is defined to be a sequence of faces of K such that each face shares an edge with the previous. Choose a location in Δ for the triangle, T_1 , associated to f_1 . Because T_2 , the triangle associated to f_2 , shares an edge and orientation with T_1 , there is now a unique location in Δ for T_2 . Continuing in this manner, triangles $T_3, \dots, T_n$ are uniquely placed in Δ . This process of laying down triangles will be termed the development along C . It will be said that "the location of T_n was obtained from that of T_1 by a development along C " ([3], p.584).

This development process can be used to define the locations of all the triangles T_i associated to the faces f_i of K . Choose one face of K to be f_0 , and specify a location for its triangle T_0 in Δ . To place any other triangle associated to a face of K , choose any chain C whose initial face is f_0 and whose final face is the associated face. Then use the development along that chain to place the triangle. Since K

is connected, the proof will be complete if it can be shown that the location for T_n is independent of the chain that is chosen. To prove this, it will be enough to show that given a chain $C=\{f_0, \dots, f_n\}$ such that $f_0=f_n$ the triangle T_n is the same as T_0 .

Consider the class of closed chains whose initial face is f_0 . Let C be in this class of chains. If the faces of a subchain of C belong to a star at a vertex v , the subchain will be described as local at v . A new chain C' can be formed by replacing a subchain of C that is local at v by another subchain that is local at v and that has the same first and last faces. This switching of subchains will be termed a local modification. An example of a local modification is given in Figure 5. If C_1 and C_2 are in the given class of chains, and the chain C_1 can be obtained from the chain C_2 by using local modifications, then C_1 and C_2 will be called homotopic.

Suppose $C=\{g_1, \dots, g_m\}$ is a chain that is local at a vertex v . Let t_1 and t_m be the triangles associated with g_1 and g_m . Since $K(R, \Phi)$ is a local packing at v , the locations of all the triangles in the star at v are uniquely determined by placing one. So by placing t_1 , the location of t_m is independent of the chain c . This implies if c were a

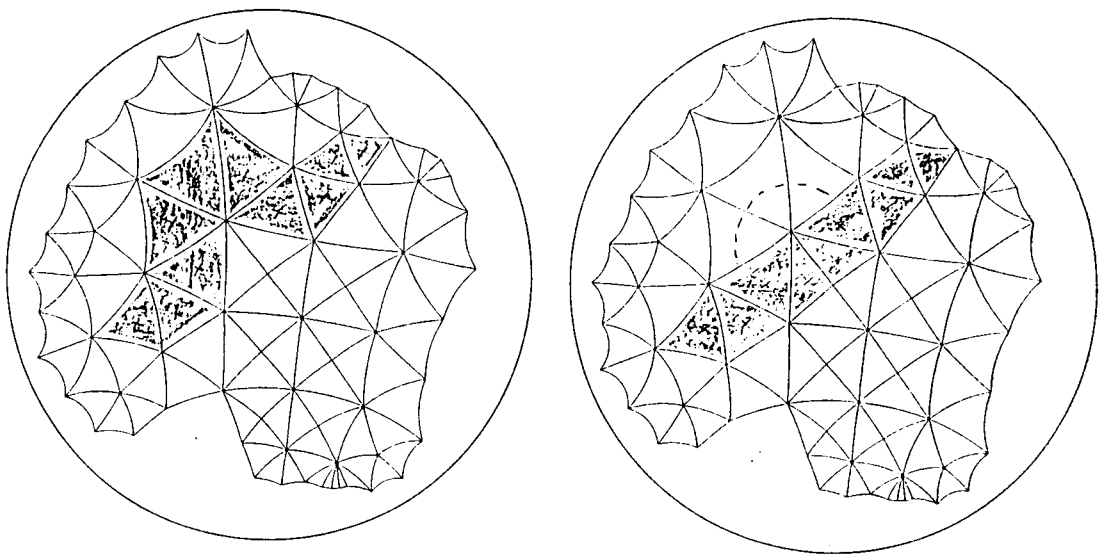


Figure 5. A local modification.

subchain of a closed chain C then local modifications of C will not effect the development along C . Thus the locations of the final triangles of homotopic chains are the same. If it can be shown that all closed chains with initial face f_0 are homotopic to the "null" chain $\{f_0\}$, the proof of sufficiency will be complete.

Let C be a closed chain. A closed curve γ can be drawn that starts in f_0 , passes through the remaining faces of C in order, and does not hit any vertices. Since K is isomorphic to a triangulation of the closed disc, the number of vertices, $N(C)$, that γ separates from ∂K can be counted. If $N(C) > 0$, then using local modifications, a new chain C' that is homotopic to C can be obtained such that $N(C') < N(C)$. Hence

a chain that is homotopic to C can be acquired that does not separate any vertices from ∂K . It is not hard to see that this chain is homotopic to $\{f_0\}$. Hence C is homotopic to $\{f_0\}$.

The placement and orientation of the initial triangle T_0 in Δ determines the placement of all other triangles. This implies that the immersion is determined up to isometry. ■

3. MONOTONICITY RESULTS

A collection of monotonicity results has to be established so that the Perron method can be used in the proof of the existence and boundary value theorem. These results will also be essential in the proof of the Generalized Discrete Schwarz-Pick Lemma. The main focus of this collection will be on the change of the hyperbolic structure of the star at a specified vertex when a radius at a vertex of the star is altered. The easiest way to achieve the desired results for a star is to first look at a single triangle.

See Figure 3 for

Lemma II: Given hyperbolic radii $r_0, r_1 \in (0, \infty)$ and $\alpha_{0,1} \in [0, \pi/2]$, the length of the geodesic between the centers z_0, z_1 of the two circles having radius r_0, r_1 respectively and overlap angle $\alpha_{0,1}$ is $\operatorname{arccosh}(\cosh r_0 \cosh r_1 + \sinh r_0 \sinh r_1 \cos \alpha_{0,1})$, and will be denoted $F_{0,1}$. Furthermore, $F'_{0,1}(r_0)$ is positive and continuous.

Proof: Using the hyperbolic cosine rule and looking at Figure 3, $\cosh F_{0,1} = \cosh r_0 \cosh r_1 - \sinh r_0 \sinh r_1 \cos(\pi - \alpha_{0,1})$. Since $\cos(\pi - \alpha_{0,1}) = -\cos(\alpha_{0,1})$, the first statement is proven.

Differentiating the first result with respect to r_0 ,

$$F'_{0,1}(r_0) = \frac{\sinh r_0 \cosh r_1 + \cosh r_0 \sinh r_1 \cos \alpha_{0,1}}{[(\cosh r_0 \cosh r_1 + \sinh r_0 \sinh r_1 \cos \alpha_{0,1})^2 - 1]^{1/2}} > 0.$$

This finishes the lemma. ■

Notice that the denominator of $F'_{0,1}$ is $\sinh F_{0,1}$. This will be used in later proofs.

The hyperbolic cosine rule has been and is going to be used frequently to prove the needed results about the angles α , β , γ in Figure 6 as functions of r_0 . If r_0, r_1, r_2 are finite the hyperbolic cosine rule gives

$$\cos \alpha = \frac{\cosh F_{0,1} \cosh F_{0,2} - \cosh F_{1,2}}{\sinh F_{0,1} \sinh F_{0,2}}$$

and a similar equation for both $\cos \beta$ and $\cos \gamma$. But, if either r_1, r_2 or both are infinite then two or more of $F_{0,1}, F_{0,2}, F_{1,2}$ are infinite. Hence the hyperbolic cosine rule cannot be used as is. This can be dealt with by using a limit as r_1 and r_2 go to infinity. Doing this, when $r_1 = \infty$,

$$\cos \alpha = \frac{\cosh F_{0,2} (\cosh r_0 + \sinh r_0 \cos \alpha_{0,1}) - (\cosh r_2 + \sinh r_2 \cos \alpha_{1,2})}{\sinh F_{0,2} (\cosh r_0 + \sinh r_0 \cos \alpha_{0,1})}$$

$$\cos \gamma = \frac{\cosh F_{0,2} (\cosh r_2 + \sinh r_2 \cos \alpha_{1,2}) - (\cosh r_0 + \sinh r_0 \cos \alpha_{0,1})}{\sinh F_{0,2} (\cosh r_2 + \sinh r_2 \cos \alpha_{1,2})}.$$

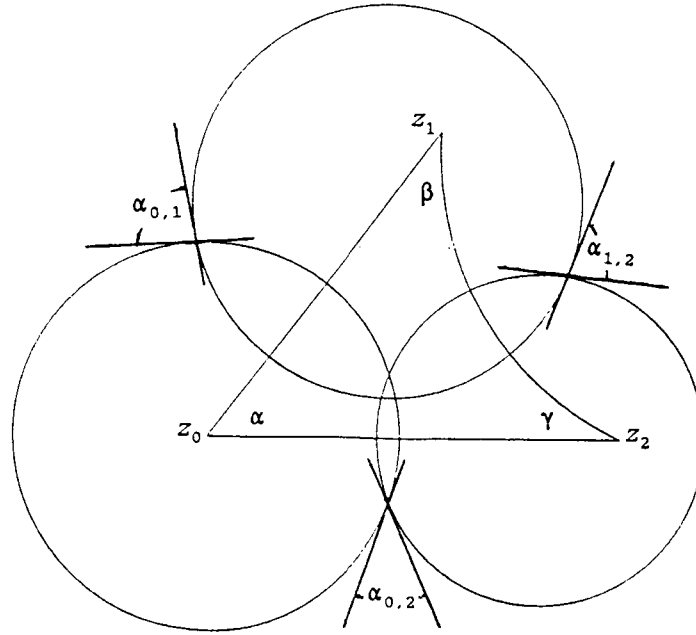


Figure 6. Triangle of centers and notation.

Symmetry gives formulas for $\cos \alpha$ and $\cos \beta$ when $r_2 = \infty$. When $r_1 = r_2 = \infty$ new notation is used for simplicity. Let

$$x = e^{z_0}, \quad A = 1 + \cos \alpha_{0,1}, \quad A' = 1 - \cos \alpha_{0,1},$$

$$B = 1 + \cos \alpha_{0,2}, \quad B' = 1 - \cos \alpha_{0,2}, \quad C = 1 + \cos \alpha_{1,2}.$$

Then

$$\cos \alpha = 1 - \frac{4Cx^2}{(Ax^2 + A')(Bx^2 + B')}.$$

Refer to Figure 6 for

Lemma III: Let $r_0, r_1, r_2 \in (0, \infty]$ be the radii for the circles with centers z_0, z_1, z_2 , and overlap angles $\alpha_{0,1}, \alpha_{0,2}, \alpha_{1,2} \in [0, \pi/2]$.

Then

- (i) $\alpha(r_0)$ is a continuous strictly decreasing function.
- (ii) $\alpha(r_0)$ takes on every value in $[0, \pi - \alpha_{1,2})$ with
 $\alpha(r_0) \rightarrow (\pi - \alpha_{1,2})$ as $r_0 \rightarrow 0$.
- (iii) $\beta(r_0)$ ($\gamma(r_0)$) is a continuous strictly increasing function when $r_1 < \infty$ ($r_2 < \infty$) except when
 $\alpha_{0,1} = 0, \alpha_{0,2} = \alpha_{1,2} = \pi/2$ ($\alpha_{0,2} = 0, \alpha_{0,1} = \alpha_{1,2} = \pi/2$), in which case it is constant.
- (iv) The area, A , of the triangle of centers is a strictly increasing function of r_0 .

Proof: Assuming (i) and (iii) we first prove (iv). If $r_1 = r_2 = \infty$ then $A(r_0) = \pi - \alpha(r_0)$ and (i) gives the result. By a Mobius transformation z_1 can be placed at the origin and z_2 on the positive real axis. If both $r_1, r_2 < \infty$ then since $F_{1,2}$ is fixed and $F_{0,1}, F_{0,2}, \gamma, \beta$ are increasing, it is geometrically evident that A is strictly increasing as r_0 increases. Suppose, WLOG, that $r_2 = \infty$ and $r_1 < \infty$. If it can be shown that the geodesic segment, $g_{0,2}$, from z_0 to z_2 does not "fall" in towards the origin as r_0 increases then the above reasoning gives the monotonicity of A .

By replacing r_2 with a sufficiently large finite radius r'_2 ,

it is evident that $g_{0,2}$ can be approximated by a new geodesic segment, $g'_{0,2}$, that is within a given euclidean distance of $g_{0,2}$, and such that $g'_{0,2}$ separates $g_{0,2}$ from the origin. When r_0 is increased, $g'_{0,2}$ pivots away from the origin (i.e. γ' increases), and remains between the origin and $g_{0,2}$. A limiting argument shows that $g_{0,2}$ cannot "fall" towards the origin as r_0 increases. This finishes (iv). Next, we prove (i).

Note that (ii) is geometrically evident once (i) is proved. First assume r_1 and r_2 are finite. From the hyperbolic cosine rule,

$$\cos \alpha = \frac{\cosh F_{0,1} \cosh F_{0,2} - \cosh F_{1,2}}{\sinh F_{0,1} \sinh F_{0,2}}.$$

Differentiating both sides with respect to r_0 ,

$$\begin{aligned} -\sin \alpha \frac{d\alpha}{dr_0} &= \frac{\sinh F_{0,1} \sinh F_{0,2} (\cosh F_{0,1} \sinh F_{0,2} F'_{0,2} + \cosh F_{0,2} \sinh F_{0,1} F'_{0,1})}{(\sinh F_{0,1} \sinh F_{0,2})^2} \\ &\quad - \frac{(\cosh F_{0,1} \cosh F_{0,2} - \cosh F_{1,2}) (\sinh F_{0,1} \cosh F_{0,2} F'_{0,2} + \sinh F_{0,2} \cosh F_{0,1} F'_{0,1})}{(\sinh F_{0,1} \sinh F_{0,2})^2}. \end{aligned}$$

Since $\alpha \in [0, \pi)$ and $\alpha = 0$ only if $r_0 = \infty$, it is enough to show that the numerator of the right hand side (RHS) is positive. Multiplying and using the identity $\cosh^2 x - \sinh^2 x = 1$, the

numerator becomes

$$\sinh F_{0,1} F'_{0,2} (\cosh F_{1,2} \cosh F_{0,2} - \cosh F_{0,1}) + \sinh F_{0,2} F'_{0,1} (\cosh F_{1,2} \cosh F_{0,1} - \cosh F_{0,2}) .$$

Since both $\sinh F_{0,i}$ and $F'_{0,i}$, ($i=1,2$), are positive, all that needs to be shown is that the differences in parentheses are positive. By symmetry, this need only be shown for one of the differences.

Using the formulas for $F_{0,1}$, $F_{0,2}$ and $F_{1,2}$ from Lemma II and letting $x=e^{r_0}$, $y=e^{r_1}$, $z=e^{r_2}$,

$$\begin{aligned} & \cosh F_{1,2} \cosh F_{0,2} - \cosh F_{0,1} = \\ & \left[\frac{x}{4xyz} ((1+\cos \alpha_{1,2}) y^2 z^2 + (1-\cos \alpha_{1,2}) z^2 + (1-\cos \alpha_{1,2}) y^2 + (1+\cos \alpha_{1,2})) \right] \\ & \left[\frac{y}{4xyz} ((1+\cos \alpha_{0,2}) x^2 z^2 + (1-\cos \alpha_{0,2}) x^2 + (1-\cos \alpha_{0,2}) z^2 + (1+\cos \alpha_{0,2})) \right] - \\ & \left[\frac{z}{4xyz} ((1+\cos \alpha_{0,1}) x^2 y^2 + (1-\cos \alpha_{0,1}) x^2 + (1-\cos \alpha_{0,1}) y^2 + (1+\cos \alpha_{0,1})) \right] . \end{aligned}$$

Factoring $\frac{1}{4xyz}$ and multiplying out, the RHS becomes

$$\begin{aligned} & \frac{1}{4xyz} [(1-\cos \alpha_{1,2}) (1-\cos \alpha_{0,2}) x^3 y^3 + (1+\cos \alpha_{1,2}) (1-\cos \alpha_{0,2}) x^3 y \\ & + (1-\cos \alpha_{1,2}) (1+\cos \alpha_{0,2}) x y^3 + (1+\cos \alpha_{1,2}) (1+\cos \alpha_{0,2}) x y \\ & + x^2 y^2 z [(1+\cos \alpha_{1,2}) (1+\cos \alpha_{0,2}) x y z^3 \\ & + (1-\cos \alpha_{1,2}) (1+\cos \alpha_{0,2}) x y z + (1+\cos \alpha_{1,2}) (1-\cos \alpha_{0,2}) x y z \\ & - 1 - \cos \alpha_{0,1}] + x^2 z [(1-\cos \alpha_{1,2}) (1+\cos \alpha_{0,2}) x y z^3 \end{aligned}$$

$$\begin{aligned}
& + (1 + \cos \alpha_{1,2}) (1 + \cos \alpha_{0,2}) xyz + (1 - \cos \alpha_{1,2}) (1 - \cos \alpha_{0,2}) xyz \\
& \quad - 1 + \cos \alpha_{0,1}] + y^2 z [(1 + \cos \alpha_{1,2}) (1 - \cos \alpha_{0,2}) xyz^3 \\
& + (1 - \cos \alpha_{1,2}) (1 - \cos \alpha_{0,2}) xyz + (1 + \cos \alpha_{1,2}) (1 + \cos \alpha_{0,2}) xyz \\
& \quad - 1 + \cos \alpha_{0,1}] + z [(1 - \cos \alpha_{1,2}) (1 - \cos \alpha_{0,2}) xyz^3 + \\
& (1 + \cos \alpha_{1,2}) (1 - \cos \alpha_{0,2}) xyz + (1 - \cos \alpha_{1,2}) (1 + \cos \alpha_{0,2}) xyz - 1 - \cos \alpha_{0,1}]].
\end{aligned}$$

If it can be shown that

$$\begin{aligned}
& x^2 y^2 z [(1 + \cos \alpha_{1,2}) (1 + \cos \alpha_{0,2}) xyz^3 + (1 - \cos \alpha_{1,2}) (1 + \cos \alpha_{0,2}) xyz \\
& \quad + (1 + \cos \alpha_{1,2}) (1 - \cos \alpha_{0,2}) xyz - 1 - \cos \alpha_{0,1}]
\end{aligned}$$

and

$$\begin{aligned}
& z [(1 - \cos \alpha_{1,2}) (1 - \cos \alpha_{0,2}) xyz^3 + (1 + \cos \alpha_{1,2}) (1 - \cos \alpha_{0,2}) xyz \\
& \quad + (1 - \cos \alpha_{1,2}) (1 + \cos \alpha_{0,2}) xyz - 1 - \cos \alpha_{0,1}]
\end{aligned}$$

added together is nonnegative, then, because the remaining terms are either nonnegative or positive, this case will be done. After multiplying, these two expressions become

$$\begin{aligned}
& x^2 y^2 z [xyz^3 + (\cos \alpha_{1,2} xyz^3 - \cos \alpha_{1,2} \cos \alpha_{0,2} xyz) + \cos \alpha_{0,2} xyz^3 \\
& + (\cos \alpha_{1,2} \cos \alpha_{0,2} xyz^3 - \cos \alpha_{1,2} \cos \alpha_{0,2} xyz) + (xyz - 1) + (xyz - \cos \alpha_{0,1})]
\end{aligned}$$

and

$$\begin{aligned}
& z [(xyz^3 - \cos \alpha_{1,2} xyz^3) - \cos \alpha_{0,2} xyz^3 + (\cos \alpha_{1,2} \cos \alpha_{0,2} xyz^3 \\
& - \cos \alpha_{1,2} \cos \alpha_{0,2} xyz) + (xyz - 1) + (xyz - \cos \alpha_{0,1}) - \cos \alpha_{1,2} \cos \alpha_{0,2} xyz].
\end{aligned}$$

In each expression, the terms in parentheses are nonnegative. The sum of the terms from both expressions that are not in parentheses is easily seen to be nonnegative.

Assume WLOG that $r_1 = \infty$ and $r_2 < \infty$. Using the hyperbolic cosine rule given earlier,

$$\cos \alpha = \frac{\cosh F_{0,2} (\cosh r_0 + \sinh r_0 \cos \alpha_{0,1}) - (\cosh r_2 + \sinh r_2 \cos \alpha_{1,2})}{\sinh F_{0,2} (\cosh r_0 + \sinh r_0 \cos \alpha_{0,1})}.$$

Taking the derivative of both sides with respect to r_0 and showing the numerator of the RHS is positive will again give the result. After differentiating, the numerator of the RHS is

$$\begin{aligned} & -(\cosh r_0 + \sinh r_0 \cos \alpha_{0,1})^2 F'_{0,2} \\ & + \sinh F_{0,2} (\sinh r_0 + \cosh r_0 \cos \alpha_{0,1}) (\cosh r_2 + \sinh r_2 \cos \alpha_{1,2}) \\ & + (\cosh r_0 + \sinh r_0 \cos \alpha_{0,1}) (\cosh r_2 + \sinh r_2 \cos \alpha_{1,2}) \cosh F_{0,2} F'_{0,2}. \end{aligned}$$

From Lemma II,

$$F'_{0,2}(r_0) = \frac{\sinh r_0 \cosh r_2 + \sinh r_2 \cosh r_0 \cos \alpha_{0,2}}{\sinh F_{0,2}}.$$

Substituting this into the above, the above becomes

$$\begin{aligned} & \frac{1}{\sinh F_{0,2}} [-(\cosh r_0 + \sinh r_0 \cos \alpha_{0,1})^2 (\cosh r_2 \sinh r_0 + \sinh r_2 \cosh r_0 \cos \alpha_{0,2}) \\ & + (\cosh r_0 \cosh r_2 + \sinh r_0 \sinh r_2 \cos \alpha_{0,2})^2 (\sinh r_0 \\ & + \cosh r_0 \cos \alpha_{0,1}) (\cosh r_2 + \sinh r_2 \cos \alpha_{1,2}) \\ & - (\sinh r_0 + \cosh r_0 \cos \alpha_{0,1}) (\cosh r_2 + \sinh r_2 \cos \alpha_{1,2}) \\ & + (\cosh r_0 + \sinh r_0 \cos \alpha_{0,1}) (\cosh r_2 + \sinh r_2 \cos \alpha_{1,2}) (\cosh r_0 \cosh r_2 \\ & + \sinh r_0 \sinh r_2 \cos \alpha_{0,2}) (\cosh r_2 \sinh r_0 + \sinh r_2 \cosh r_0 \cos \alpha_{0,2})]. \end{aligned}$$

Multiplying and combining the terms in the square brackets

gives

$$\begin{aligned}
& 2\cosh r_0 \cosh^2 r_2 \sinh^2 r_0 \sinh r_2 \cos \alpha_{0,2} + 2\cosh r_0 \cosh r_2 \sinh^2 r_0 \sinh^2 r_2 \cos \alpha_{0,2} \cos \alpha_{1,2} \\
& + 2\cosh^2 r_0 \cosh r_2 \sinh r_0 \sinh^2 r_2 \cos \alpha_{0,1} \cos \alpha_{0,2} \cos \alpha_{1,2} + \cosh r_2 \sinh^3 r_0 \sinh^2 r_2 \cos^2 \alpha_{0,2} \\
& + \sinh^3 r_0 \sinh^3 r_2 \cos^2 \alpha_{0,2} \cos \alpha_{1,2} + \cosh r_0 \cosh r_2 \sinh^2 r_0 \sinh^2 r_2 \cos \alpha_{0,1} \cos^2 \alpha_{0,2} \\
& + \cosh r_0 \sinh^2 r_0 \sinh^3 r_2 \cos \alpha_{0,1} \cos^2 \alpha_{0,2} \cos \alpha_{1,2} + \cosh^2 r_0 \cosh r_2 \sinh^2 r_0 \sinh r_2 \cos \alpha_{0,2} \\
& + \cosh^2 r_0 \cosh r_2 \sinh r_0 \sinh^2 r_2 \cos^2 \alpha_{0,2} + \cosh^2 r_0 \cosh^2 r_2 \sinh r_0 \sinh r_2 \cos \alpha_{1,2} \\
& + \cosh^3 r_0 \cosh r_2 \sinh^2 r_2 \cos \alpha_{0,2} \cos \alpha_{1,2} + \cosh^2 r_0 \sinh^2 r_0 \sinh^2 r_2 \cos \alpha_{0,2} \cos \alpha_{1,2} \\
& + \cosh^2 r_0 \sinh r_0 \sinh^3 r_2 \cos^2 \alpha_{0,2} \cos \alpha_{1,2} + \cosh r_0 \cosh r_2 \sinh^3 r_0 \sinh r_2 \cos \alpha_{0,1} \cos \alpha_{0,2} \\
& + \cosh r_0 \cosh r_2 \sinh^2 r_0 \sinh^2 r_2 \cos \alpha_{0,1} \cos^2 \alpha_{0,2} \\
& + \cosh r_0 \cosh^2 r_2 \sinh^2 r_0 \sinh r_2 \cos \alpha_{0,1} \cos \alpha_{1,2} \\
& + \cosh^2 r_0 \cosh r_2 \sinh r_0 \sinh^2 r_2 \cos \alpha_{0,1} \cos \alpha_{0,2} \cos \alpha_{1,2} \\
& + \cosh r_0 \sinh^3 r_0 \sinh^2 r_2 \cos \alpha_{0,1} \cos \alpha_{0,2} \cos \alpha_{1,2} \\
& + \cosh r_0 \sinh^2 r_0 \sinh^3 r_2 \cos \alpha_{0,1} \cos^2 \alpha_{0,2} \cos \alpha_{1,2} \\
& + (\cosh^3 r_0 \cosh^2 r_2 \sinh r_2 \cos \alpha_{0,2} - \cosh^3 r_0 \sinh r_2 \cos \alpha_{0,2}) \\
& + (\cosh r_0 \cosh^3 r_2 \sinh^2 r_0 \cos \alpha_{0,1} - \cosh r_0 \cosh r_2 \sinh^2 r_0 \cos \alpha_{0,1}) \\
& + (\cosh r_0 \cosh^3 r_2 \cos \alpha_{0,1} - \cosh r_0 \cosh r_2 \cos \alpha_{0,1}) + (\cosh r_0 \cosh^3 r_2 \sinh^2 r_0 \cos \alpha_{0,1} \\
& - \cosh r_0 \cosh r_2 \sinh^2 r_0 \cos \alpha_{0,1}) + (2\cosh^2 r_0 \cosh^2 r_2 \sinh r_0 \sinh r_2 \cos \alpha_{0,1} \cos \alpha_{0,2} \\
& - 2\cosh^2 r_0 \sinh r_0 \sinh r_2 \cos \alpha_{0,1} \cos \alpha_{0,2}) + (\cosh^3 r_0 \cosh^2 r_2 \sinh r_2 \cos \alpha_{0,1} \cos \alpha_{1,2} \\
& - \cosh r_0 \sinh r_2 \cos \alpha_{0,1} \cos \alpha_{1,2}) + (\cosh^2 r_0 \cosh^2 r_2 \sinh r_0 \sinh r_2 \cos \alpha_{1,2} \\
& - \sinh r_0 \sinh r_2 \cos \alpha_{1,2}) + (\cosh^3 r_2 \sinh r_0 - \cosh r_2 \sinh r_0) + (\cosh^3 r_2 \sinh^3 r_0 \\
& - \cosh r_2 \sinh^3 r_0 \cos^2 \alpha_{0,1}) + \cosh r_0 \sinh r_0 \sinh r_2 \cos \alpha_{0,1} \cos \alpha_{0,2} (\cosh r_0 \cosh^2 r_2 \\
& - \sinh r_0 \cos \alpha_{0,1}) + (\cosh^2 r_0 \cosh^3 r_2 \sinh r_0 - \cosh^2 r_0 \cosh r_2 \sinh r_0) .
\end{aligned}$$

Since the last term is positive and all the others are

nonnegative, the sum is positive. This finishes this case and leaves one to prove.

Assume $r_1=r_2=\infty$. From the hyperbolic cosine rules given earlier,

$$\cos\alpha = 1 - \frac{4Cx^2}{(Ax^2+A')(Bx^2+B')}.$$

If it can be shown that $\frac{d\alpha}{dx} < 0$ then the proof of (i) will be complete. Now

$$-\sin\alpha \frac{d\alpha}{dx} = \frac{-C(8A'B'x - 8ABx^5)}{(Ax^2+A')^2 (Bx^2+B')^2}.$$

Since $8ABx^5 - 8A'B'x > 0$, $\frac{d\alpha}{dx} < 0$.

The same approach used in the proof of (i) is used to prove (iii). First assume $r_1, r_2 < \infty$. The hyperbolic cosine rule and differentiating with respect to r_0 give

$$-\sin\beta \frac{d\beta}{dr_0} = \frac{\sinh F_{1,2} (\cosh F_{0,2} \cosh F_{0,1} F'_{0,1} - \cosh F_{1,2} F'_{0,1} - \sinh F_{0,1} \sinh F_{0,2} F'_{0,2})}{(\sinh F_{1,2} \sinh F_{0,1})^2}.$$

It is enough to show that what is in parentheses in the numerator is negative.

Substituting $\cosh F_{0,1} \cosh F_{0,2} - \sinh F_{0,1} \sinh F_{0,2} \cos\alpha$ for $\cosh F_{1,2}$, the terms in parentheses become $\sinh F_{0,1} \sinh F_{0,2} (\cos\alpha F'_{0,1} - F'_{0,2})$.

Now it only needs to be shown that $(\cos\alpha F'_{0,1} - F'_{0,2}) \leq 0$. Putting $F'_{0,1}$

and $F'_{0,2}$ in terms of r_0, r_1, r_2 , and $\cos \alpha$ in terms of $F_{0,1}, F_{0,2}, F_{1,2}$, $\cos \alpha F'_{0,1} - F'_{0,2}$ becomes

$$\frac{(\cosh F_{0,1} \cosh F_{0,2} - \cosh F_{1,2}) (\cosh r_1 \sinh r_0 + \cosh r_0 \sinh r_1 \cos \alpha_{0,1})}{\sinh^2 F_{0,1} \sinh F_{0,2}}$$

$$- \frac{(\cosh^2 F_{0,1} - 1) (\cosh r_2 \sinh r_0 + \cosh r_0 \sinh r_2 \cos \alpha_{0,2})}{\sinh^2 F_{0,1} \sinh F_{0,2}}.$$

Putting $F_{0,1}, F_{0,2}, F_{1,2}$ in terms of r_0, r_1, r_2 , the numerator becomes

$$\begin{aligned} & \cosh r_0 \sinh r_2 \cos \alpha_{0,2} (1 - \cosh^2 r_1) + \cosh r_2 \sinh r_0 \sinh^2 r_1 (\cos^2 \alpha_{0,1} - 1) \\ & - \cosh r_1 \sinh r_0 \sinh r_1 \sinh r_2 \cos \alpha_{1,2} - \cosh r_0 \sinh^2 r_1 \sinh r_2 \cos \alpha_{0,1} \cos \alpha_{1,2} \\ & - \cosh r_1 \sinh r_0 \sinh r_1 \sinh r_2 \cos \alpha_{0,1} \cos \alpha_{0,2}. \end{aligned}$$

This is easily seen to be less than or equal to zero with the inequality being strict except when $\alpha_{0,1} = 0$, $\alpha_{0,2} = \alpha_{1,2} = \frac{\pi}{2}$.

Suppose $r_2 = \infty$. From the hyperbolic cosine rule,

$$\sin \beta = \frac{\cosh F_{0,1} (\cosh r_1 + \sinh r_1 \cos \alpha_{1,2}) - (\cosh r_0 + \sinh r_0 \cos \alpha_{0,2})}{\sinh F_{0,1} (\cosh r_1 + \sinh r_1 \cos \alpha_{1,2})}.$$

Differentiating both sides with respect to r_0 and showing that the numerator of the RHS is negative will again give the result. The numerator of the RHS after differentiating is

$$(\cosh r_1 + \sinh r_1 \cos \alpha_{1,2})^2 \sinh^2 F_{0,1} \frac{dF_{0,1}}{dr_0} - \sinh F_{0,1} (\cosh r_1 + \sinh r_1 \cos \alpha_{1,2})$$

$$(\sinh r_0 + \cosh r_0 \cos \alpha_{0,2}) - (\cosh r_1 + \sinh r_1 \cos \alpha_{1,2})^2 \cosh^2 F_{0,1} \frac{dF_{0,1}}{dr_0}$$

$$+ (\cosh r_0 + \sinh r_0 \cos \alpha_{0,2}) (\cosh r_1 + \sinh r_1 \cos \alpha_{1,2}) \cosh F_{0,1} \frac{dF_{0,1}}{dr_0}.$$

Putting $\sinh F_{0,1}$ and $\cosh F_{0,1}$ in terms of $r_0, r_1, \alpha_{0,1}$, and $\frac{dF_{0,1}}{dr_0}$

in terms of $\sinh F_{0,1}, r_0, r_1, \alpha_{0,1}$, the above becomes

$$\frac{1}{\sinh F_{0,1}} [-(\cosh r_1 + \sinh r_1 \cos \alpha_{1,2})^2 (\cosh r_1 \sinh r_0 + \sinh r_1 \cosh r_0 \cos \alpha_{0,1})$$

$$- (\cosh r_0 \cosh r_1 + \sinh r_0 \sinh r_1 \cos \alpha_{0,1})^2 (\cosh r_1 +$$

$$\sinh r_1 \cos \alpha_{1,2}) (\sinh r_0 + \cosh r_0 \cos \alpha_{0,2})$$

$$+ (\cosh r_1 + \sinh r_1 \cos \alpha_{1,2}) (\sinh r_0 + \cosh r_0 \cos \alpha_{0,2})$$

$$+ (\cosh r_0 + \sinh r_0 \cos \alpha_{0,2}) (\cosh r_1 + \sinh r_1 \cos \alpha_{1,2})$$

$$(\cosh r_0 \cosh r_1 + \sinh r_0 \sinh r_1 \cos \alpha_{0,1}) (\cosh r_1 \sinh r_0 + \cosh r_0 \sinh r_1 \cos \alpha_{0,1})] .$$

Showing that what is in the square brackets is negative will finish the proof. Multiplying and simplifying, the terms in the square brackets become

$$\begin{aligned} & -\cosh r_1 \sinh r_0 \sinh^2 r_1 \cos^2 \alpha_{1,2} - \cosh r_0 \sinh^3 r_1 \cos \alpha_{0,1} \cos^2 \alpha_{1,2} + \\ & \cosh r_1 \sinh r_0 \sinh^2 r_1 (\cos^2 \alpha_{0,1} - 1) - \cosh r_0 \cosh r_1 \sinh^2 r_1 \cos \alpha_{0,1} \cos \alpha_{1,2} \\ & + \cosh r_0 \cosh r_1 \cos \alpha_{0,2} (1 - \cosh^2 r_1) - \cosh^2 r_1 \sinh r_0 \sinh r_1 \cos \alpha_{0,1} \cos \alpha_{0,2} \\ & + \sinh r_0 \sinh r_1 \cos \alpha_{1,2} (\sinh^2 r_1 \cos^2 \alpha_{0,1} - \cosh^2 r_1) - \sinh r_0 \sinh^3 r_1 \cos \alpha_{1,2} \\ & - \cosh r_1 \sinh r_0 \sinh^2 r_1 \cos \alpha_{0,1} \cos \alpha_{0,2} \cos \alpha_{1,2} \end{aligned}$$

which is less than or equal to zero. Equality holds only when

$$\alpha_{0,1}=0, \alpha_{0,2}=\alpha_{1,2}=\frac{\pi}{2}. \blacksquare$$

Lemma IV: Using Figure 6 and the hypotheses of Lemma III, let T be the triangle of centers. Let p be a point on the edge of T opposite z_0 . Let $l_p(r_0)$ denote the distance from p to z_0 . If at most one of r_1, r_2 is infinite then $l'_p(r_0)$ is positive and continuous. Furthermore, if both r_1, r_2 are finite then there exists a continuous function, $\tau(r_0)$, such that for any p , $l'_p(r_0) > \tau(r_0) > 0$.

Proof: Assume first that both r_1 and r_2 are finite. Let d_i be the fixed distance from z_i to p . The hyperbolic cosine rule gives

$$\cosh l_p = \cosh F_{0,1} \cosh d_1 - \cos \beta \sinh F_{0,1} \sinh d_1.$$

Replacing $\cos \beta$ with $\frac{\cosh F_{1,2} \cosh F_{0,1} - \cosh F_{0,2}}{\sinh F_{1,2} \sinh F_{0,1}}$,

$$\cosh l_p = \cosh F_{0,1} \cosh d_1 - \frac{(\cosh F_{1,2} \cosh F_{0,1} - \cosh F_{0,2}) \sinh d_1}{\sinh F_{1,2}}.$$

Hence

$$\sinh l_p \frac{dl_p}{dr_0} = \sinh F_{0,1} \frac{dF_{0,1}}{dr_0} \cosh d_1$$

$$-\frac{(\cosh F_{1,2} \sinh F_{0,1} \frac{dF_{0,1}}{dr_0} - \sinh F_{0,2} \frac{dF_{0,2}}{dr_0}) \sinh d_1}{\sinh F_{1,2}}$$

$$= \frac{\sinh F_{0,1} \frac{dF_{0,1}}{dr_0} (\cosh d_1 \sinh F_{1,2} - \cosh F_{1,2} \sinh d_1) + \sinh F_{0,2} \sinh d_1 \frac{dF_{0,2}}{dr_0}}{\sinh F_{1,2}}$$

$$= \frac{\sinh F_{0,1} \frac{dF_{0,1}}{dr_0} \sinh(F_{1,2} - d_1) + \sinh F_{0,2} \sinh d_1 \frac{dF_{0,2}}{dr_0}}{\sinh F_{1,2}} > 0.$$

This implies $\frac{dl_p}{dr_0} > 0$.

Let $m = \sup\{F_{0,1}, F_{0,2}\}$. Then $\sinh(l_p) \leq \sinh(m)$. This implies

$$\frac{dl_p}{dr_0} \geq \frac{\sinh F_{0,1} F'_{0,1} \sinh(F_{1,2} - d_1) + \sinh F_{0,2} F'_{0,2} \sinh d_1}{\sinh m \sinh F_{1,2}}$$

$$\geq \inf\left\{\frac{\sinh F_{0,i} F'_{0,i}}{\sinh m} : i=1,2\right\} \frac{\sinh(F_{1,2} - d_1) + \sinh d_1}{\sinh F_{1,2}}.$$

Notice that $\frac{\sinh(F_{1,2} - d_1) + \sinh d_1}{\sinh F_{1,2}}$ is continuous and strictly

greater than zero as a function of d_1 on $[0, F_{1,2}]$. So there exist a constant κ independent of d_1 such that

$$\frac{\sinh(F_{1,2} - d_1) + \sinh d_1}{\sinh F_{1,2}} > \kappa > 0.$$

Then

$$\tau(r_0) = \kappa \inf \left\{ \frac{\sinh F_{0,i} F'_{0,i}}{\sinh m} : i=1,2 \right\}$$

is the desired function.

Assume WLOG that $r_1 = \infty$. Using the hyperbolic cosine rule twice,

$$\begin{aligned} \cosh l_p &= \cosh F_{0,2} \cosh d_2 - \sinh F_{0,2} \sinh d_2 \cos \gamma = \cosh F_{0,2} \cosh d_2 \\ &\quad - \frac{\sinh d_2 [\cosh F_{0,2} (\cosh r_2 + \sinh r_2 \cos \alpha_{1,2}) - (\cosh r_0 + \sinh r_0 \cos \alpha_{0,1})]}{\cosh r_2 + \sinh r_2 \cos \alpha_{1,2}} \\ &= \cosh F_{0,2} (\cosh d_2 - \sinh d_2) + \frac{\sinh d_2 (\cosh r_0 + \sinh r_0 \cos \alpha_{0,1})}{\cosh r_2 + \sinh r_2 \cos \alpha_{1,2}}. \end{aligned}$$

Hence,

$$\sinh l_p \frac{dl_p}{dr_0} = \sinh F_{0,2} F'_{0,2} (\cosh d_2 - \sinh d_2) + \frac{\sinh d_2 (\sinh r_0 + \cosh r_0 \cos \alpha_{0,1})}{\cosh r_2 + \sinh r_2 \cos \alpha_{1,2}} > 0.$$

This implies $\frac{dl_p}{dr_0} > 0$. ■

Let Φ be an angle assignment for K . If $\Phi(e) \neq 0$ for any edge e such that e is the common edge of two faces like those referred to in (ii) of Definition II, then Φ will be called a strict angle assignment.

Lemma V: Let v be a vertex of $K(R, \Phi)$, Φ being a strict angle

assignment, with neighboring vertices $w_1, w_2, \dots, w_n$. Let $r, 0 < r < \infty$, and $r_1, r_2, \dots, r_n, 0 < r_i < \infty$ denote their radii respectively. Let $\theta = \theta(r, r_1, r_2, \dots, r_n)$ be the resulting angle sum at v . Then

- (i) θ is continuous and strictly decreasing in r .
- (ii) θ is continuous and strictly increasing in $r_i, 1 \leq i \leq n$.
- (iii) Fix $r_1, r_2, \dots, r_n$. Given any $\omega, 0 \leq \omega < \sum (\pi - \alpha_{i,j})$ there exist a unique radius $r', 0 < r' < \infty$, for which $\theta(r', r_1, r_2, \dots, r_n) = \omega$.

Proof: This comes easily from Lemma III. ■

The only change in Lemma V that has to be made if Φ is not a strict angle assignment is "strictly increasing" in (ii) must be replaced by "increasing." Here increasing is taken to be nondecreasing. The simple example in Figure 7 shows this. In this example, the edge joining v to w_i is the edge e from above and $\Phi(e) = 0$. The angle sum at v remains constant as the radius of w_i is changed. This is a consequence of the special case in (iii) of Lemma III. Both forms of Lemma V will be utilized in upcoming proofs. Unless otherwise stated, it will not be assumed that Φ is a strict angle assignment.

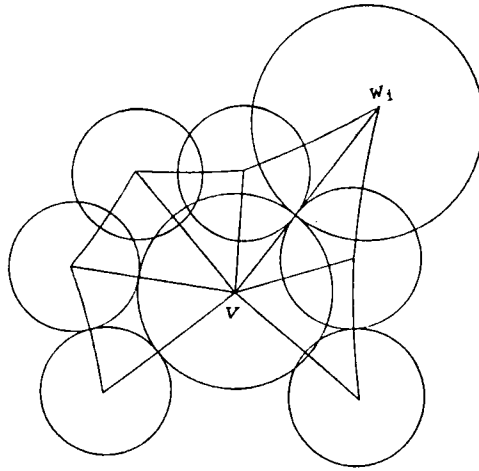


Figure 7. Showing the need for a strict angle assignment.

It should also be pointed out that (iii) of Lemma V along with conditions (i) and (ii) of Definition II on Φ allow us to always be able to find a radius for a given interior vertex such that we have a local packing at that vertex. Simple examples show that without conditions (i) and (ii) of Definition II this is not always true. Now we want to take advantage of these monotonicity results.

Given Φ and R , define the metric, $\rho_R(,)$, on $K(R, \Phi)$ to be the infimum of the lengths of paths between the two points in $K(R, \Phi)$. The notation $\rho_R(,)$ does not show that this metric is dependent on Φ . Again, this is because interest lies in what happens to $\rho_R(,)$ as the label is changed and Φ remains fixed.

Lemma VI: Let Φ be fixed. If $K(R, \Phi)$ and $K(R', \Phi)$ are hyperbolic complexes such that $R' \leq R$ then $\rho_{R'}(v_1, v_2) \leq \rho_R(v_1, v_2)$ for every pair of vertices $v_1, v_2 \in K$. Also if v_1 satisfies $R'(v_1) < R(v_1)$ and $\rho_R(v_1, v_2) < \infty$ then the inequality is strict.

Proof: For notation, everything associated to $K(R', \Phi)$ will be marked with a prime ('). It will be enough to assume that R and R' differ at a single vertex, v . Let v_1 and v_2 be fixed vertices of K , and let Γ be the collection of paths between them. Let $r = R(v)$ and $r' = R'(v)$, $0 < r' < r \leq \infty$. Fix $\gamma \in \Gamma$. The proof will be complete if it can be shown that there is a path $\gamma' \in \Gamma$ such that $l'(\gamma') \leq l(\gamma)$. This will be accomplished by building γ' from γ . As pointed out in [3], it may be assumed that γ is simple and passes through a finite number of faces.

Let $C = \{f_0, \dots, f_m\}$ be the chain of faces that γ passes through. Let $T_0, \dots, T_m$ and $T'_0, \dots, T'_m$ be the corresponding triangles. The developments along this chain will be referred to as D and D' . Immerse the path γ in Δ in D . Let S be the closed star of v . γ can be broken into "good" and "bad" segments. A maximally connected segment will be called good if it lies in the complement of S , and will be called bad otherwise. The triangles in $K(R, \Phi)$ and $K(R', \Phi)$ associated to the faces of C

that are not in S are congruent. This allows, by use of an isometry, the good segments in D to be placed in the corresponding triangles of D' . This means only the bad segments have to be replaced in D' .

Let γ_j be a bad segment of γ , and $C=\{g_1, \dots, g_n\}$ be the subchain of C that it passes through. Let $t_1, \dots, t_n$ and $t'_1, \dots, t'_n$ be the triangles corresponding to $\{g_1, \dots, g_n\}$. These triangles have, in their respective developments, the vertex corresponding to v in common in Δ . Laying these triangles out about this vertex results in a "fan." (See Figure 8). In this fan, the bad segment γ_j either connects v to an end point, p , of a good segment, or it connects the end points, x, y , of two good segments. These end points have well defined locations in D' since the good segments have already been placed in D' . These will be the end points for a new segment γ'_j .

Case 1. Suppose γ_j connects v to p . The point p lies on the edge of t_n opposite of v . First assume that $R(v) < \infty$ and at least one of the other two circles that make up t_n has finite radius. The length of γ_j is at least the length of the geodesic in t_n that joins p to v . Take γ'_j to be the corresponding geodesic in t'_n . By Lemma IV, $l'(\gamma'_j) \leq l(\gamma_j)$.

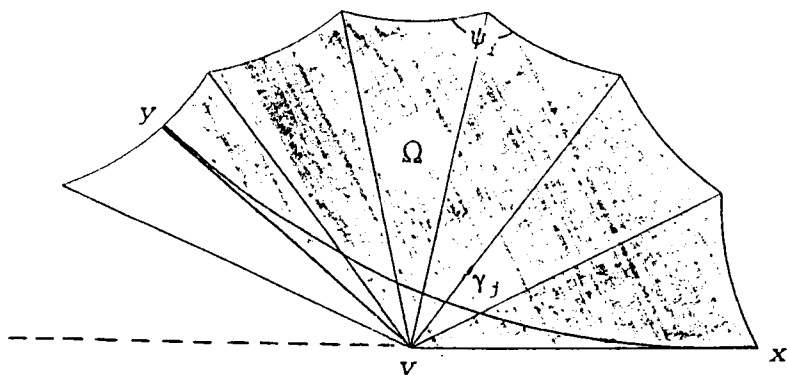


Figure 8. The fan at v and the region Ω .

If $R(v) < \infty$ and the other two radii are both infinite, then, by the assumption made after Definition II, both of the other two vertices are boundary vertices. This implies that γ_j passes through one of these boundary vertices and $l(\gamma_j) = \infty$. If $R(v) = \infty$ then again $l(\gamma_j) = \infty$. So, for both of these cases, take γ'_j to be the geodesic joining v to p .

Case 2. Assume now that all the radii in the star S are finite. Suppose γ_j connects x to y , where x lies on the edge of t_1 opposite v , and y lies similarly on t_n . It can be assumed that the two developments of c are in Δ such that v is at the origin and x is on the positive real axis. If γ_j passes through the origin, two applications of Case 1 will give the new segment γ'_j .

If γ_j does not pass through the origin, consider a point traveling γ_j from x to y as a complex number. If the change in the argument of that point is greater than or equal to π then the shortest path from x to y passing through each of the t_i 's consecutively would pass through the origin. Again, two applications of Case 1 will give the new segment γ_j' . Consider what happens if the argument of the point changes by less than π .

Let $P=[x, w_0, \dots, w_n, y]$ be the polygonal path from x to y along the edges of $t_1, \dots, t_n$ that are opposite v . Let Ω be the polygonal region made up of the rays from the origin to the points of P (Fig. 8). It is apparent that the shortest path between x and y lies in Ω .

Letting r decrease to r' , Ω changes continuously to Ω' in the following ways: the angles of the triangles at v increase; the distances $\rho(0, x)$ and $\rho(0, y)$ decrease; the interior angles ψ_i formed by the edges of P (Fig. 8) decrease; the lengths of the edges opposite v remain unchanged.

The locations of x and y as end points of good segments determine the locations of x' and y' in D' . All that needs to be done is to find a path γ_j' that joins x' to y' , and has length less than or equal to γ_j .

The simplest situation is that the geodesic $[x,y]$ lies, except for x and y , in the interior of Ω . It can be assumed that $\gamma_j = [x,y]$. By the convexity of triangles, the corner points $w_0, \dots, w_k$ of P lie on one side of $[x,y]$ and are a positive distance away. If r is decreased, the edge lengths of P do not change although the interior angles ψ_i decrease. Using the hyperbolic cosine rule and induction on the number of edges of P , it can be shown that the distance between the end points of P (i.e. $\rho(x,y)$) decreases and, for small changes of r , $[x,y]$ stays in Ω . This last statement is dependent on the fact that P lies on one side of the geodesic through its end points, and the assumption of finite radii which gives that each ψ_i is positive. Hence there exists $\delta > 0$ such that $r - \delta < r' < r$ implies $\gamma'_j = [x',y']$ lies in Ω' , and

$$l'(\gamma'_j) = \rho(x',y') \leq \rho(x,y) = l(\gamma_j).$$

Suppose $[x,y]$ does not lie in Ω or $r' \leq r - \delta$. Then paths with geodesic segments which pass through the interior of Ω as well as segments (points) in $\partial\Omega$ must be considered. Using the above argument, the geodesic segments in the interior of Ω can be replaced with segments in D' that have length less than or equal to the corresponding segments in D , and have corresponding end points. The segments in $\partial\Omega$ may be left alone since their lengths are the same in both D and D' . Join

these segments together in D' to get a new segment γ'_j with $l'(\gamma'_j) \leq l(\gamma_j)$. Case 2 will be complete once the situation that one or more vertices of S has infinite radius is dealt with.

The triangles $t_1, \dots, t_n$ and $t'_1, \dots, t'_n$ are determined by circles in Δ with assigned radii. Recall that a circle with infinite radius is a horocycle. It is evident that by replacing an infinite radius with a large enough finite one, the immersion of γ_j can be altered by no more than some given euclidean amount. Applying the previous result to this finite case and using an approximation argument, a new segment γ'_j can be found in D' with length less than or equal to that of γ_j in the infinite case.

Now by joining these new segments to the good segments in D' , a path γ' is acquired with $l(\gamma') \leq l(\gamma)$. The result, $\rho_{R'}(v_1, v_2) \leq \rho_R(v_1, v_2)$, follows.

For the case of equality, it can be assumed that $v=v_1$. The assumption that $\rho_R(v_1, v_2) < \infty$ implies $r=R(v) < \infty$, and that only paths $\gamma \in \Gamma$ with finite length need be considered. The previous method will be used again to get a new path γ' , but this time the length of γ' will be strictly less than that of γ .

The initial segment γ_1 of γ connects v to a point p on the opposite edge of some triangle (i.e. Case 1). If the radii of

the three circles of that triangle are all finite then, by the last statement of Lemma IV, there exists $\eta > 0$, that is independent of p and therefore γ , so that the geodesic segment γ'_1 in D' satisfies $l'(\gamma'_1) < l(\gamma_1) - \eta$. If one of the radii for the circles of the triangle is infinite then, since $\rho_R(v_1, v_2) < \infty$, an upper bound can be placed on $l_p(r_0)$. This makes it possible to find a lower bound on $l'_p(r_0)$, and, hence, an η as above. The case when two radii of the three circles of the triangle are infinite will not arise. Continuing as above, a new path $\gamma' \in \Gamma$ can be constructed such that $l'(\gamma') < l(\gamma) - \eta$. Now, taking the infimum over Γ gives $\rho_{R'}(v_1, v_2) \leq \rho_R(v_1, v_2) - \eta < \rho_R(v_1, v_2)$. ■

We want to find an upper bound on the radius at an interior vertex of K with n (≥ 3) neighbors. Before this can be done three simple propositions must be proved. (Refer to Figure 6).

Proposition I: For $0 < r_0 < \infty$, $0 < r_1, r_2 \leq \infty$, α is a decreasing function of $\alpha_{1,2}$ on $[0, \pi/2]$.

Proof: First assume $r_1, r_2 < \infty$, then

$$\cosh F_{1,2} = \cosh r_1 \cosh r_2 + \sinh r_1 \sinh r_2 \cos \alpha_{1,2}.$$

Hence

$$\sinh F_{1,2} \frac{dF_{1,2}}{d\alpha_{1,2}} = \sinh r_1 \sinh r_2 (-\sin \alpha_{1,2}).$$

This implies $\frac{dF_{1,2}}{d\alpha_{1,2}} \leq 0$, $\alpha_{1,2} \in [0, \pi/2]$. Differentiating both sides

of the hyperbolic cosine rule for α with respect to $\alpha_{1,2}$ gives

$$-\sin \alpha \frac{d\alpha}{d\alpha_{1,2}} = \frac{-\sinh F_{1,2} \frac{dF_{1,2}}{d\alpha_{1,2}}}{\sinh F_{0,1} \sinh F_{0,2}} \geq 0.$$

Since $\sin \alpha > 0$, $\frac{d\alpha}{d\alpha_{1,2}} \leq 0$.

Now, WLOG, let $r_1 = \infty$.

$$\cos \alpha = \frac{\cosh F_{0,2} (\cosh r_0 + \sinh r_0 \cos \alpha_{0,1}) - (\cosh r_2 + \sinh r_2 \cos \alpha_{1,2})}{\sinh F_{0,2} (\cosh r_0 + \sinh r_0 \cos \alpha_{0,1})}$$

implies

$$-\sin \alpha \frac{d\alpha}{d\alpha_{1,2}} = \frac{\sinh r_2 \sin \alpha_{1,2}}{\sinh F_{0,2} (\cosh r_0 + \sinh r_0 \cos \alpha_{0,1})}.$$

Hence for $\alpha_{1,2} \in [0, \pi/2]$, $\frac{d\alpha}{d\alpha_{1,2}} \leq 0$.

Letting $r_1 = r_2 = \infty$, the hyperbolic cosine rule for α is

$$\cos \alpha = 1 - \frac{4(1 + \cos \alpha_{1,2})x^2}{(Ax^2 + A')(Bx^2 + B')}.$$

Thus

$$-\sin\alpha \frac{d\alpha}{d\alpha_{1,2}} = \frac{4x^2 \sin\alpha_{1,2}}{(Ax^2+A')(Bx^2+B')}.$$

Again, $\frac{d\alpha}{d\alpha_{1,2}} \leq 0$. ■

Proposition II: If $0 < r_0 < \infty$, $0 < r_1, r_2 \leq \infty$ then γ is an increasing function of $\alpha_{1,2}$ on $[0, \pi/2]$.

Proof: If $r_2 = \infty$ then $\gamma = 0$, and the result is trivial. Assume first that $r_1, r_2 < \infty$. Differentiating both sides of the hyperbolic cosine rule,

$$-\sin\gamma \frac{d\gamma}{d\alpha_{1,2}} = \frac{\frac{dF_{1,2}}{d\alpha_{1,2}} \sinh F_{0,2} (\cosh F_{0,1} \cosh F_{1,2} - \cosh F_{0,2})}{(\sinh F_{1,2} \sinh F_{0,2})^2}.$$

From the proof of Proposition 1, $\frac{dF_{1,2}}{d\alpha_{1,2}} \leq 0$, and from the proof

of Lemma III, $(\cosh F_{0,1} \cosh F_{1,2} - \cosh F_{0,2}) > 0$. Hence $\frac{d\gamma}{d\alpha_{1,2}} \geq 0$.

Assume that $r_1 = \infty$. Again differentiating both sides of the hyperbolic cosine rule,

$$-\sin\gamma \frac{d\gamma}{d\alpha_{1,2}} = \frac{-\sinh F_{0,2} \sinh r_2 \sin \alpha_{1,2} (\cosh r_0 + \sinh r_0 \cos \alpha_{0,1})}{(\sinh F_{0,2} (\cosh r_2 + \sinh r_2 \cos \alpha_{1,2}))^2}.$$

Hence $\frac{d\gamma}{d\alpha_{1,2}} \geq 0$. ■

Given three euclidean circles with overlap angles between 0 and $\pi/2$, either the intersection of the three circles (including their interiors) is empty, a single point, or more than a single point. (See Figure 9). It is not hard to see using simple geometry that these occur when the sum of the overlap angles is less than, equal to, or greater than π respectively. (Notice this gives a geometric reason for condition (i) of Definition II). The next proposition is now evident.

Proposition III: When $\alpha_{0,1} = \alpha_{0,2} = \pi/2$, $\alpha_{1,2} = 0$, $0 < r_0 < \infty$ and $r_1 = r_2$, the circle of radius r_0 intersects the other two circles at their point of tangency.

Lemma VII: Let v be an interior vertex of $K(R, \Phi)$ with neighbors $w_1, w_2, \dots, w_n$ ($n \geq 3$). Let $R(v) = r < \infty$ and $R(w_i) = r_i \leq \infty$. If $\theta(v) \geq 2\pi$ then $r \leq \sqrt{n}$.

Proof: By Propositions I and II, decreasing the overlap

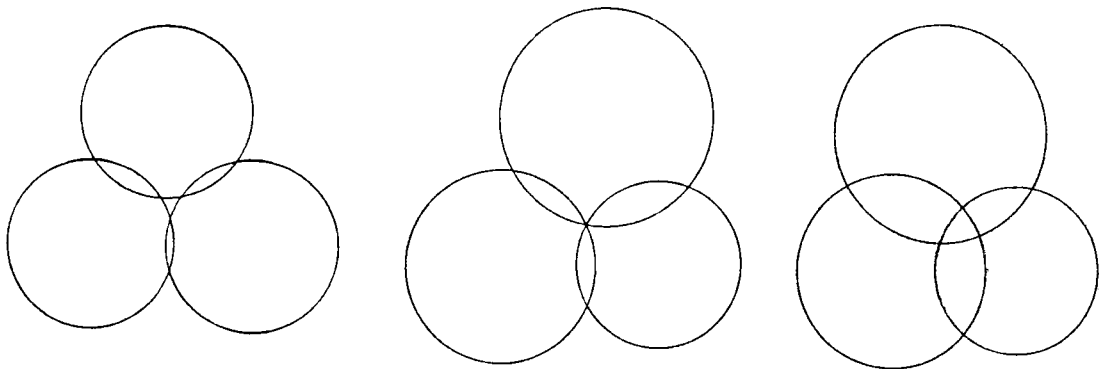


Figure 9. The intersection of three circles.

angles between pairs of neighboring circles $w_1, w_2, \dots, w_n$ to zero and increasing the overlap angles between the circles v and w_i to $\pi/2$ will increase θ . By Lemma V, increasing each r_i to ∞ will again increase θ . By (iii) of Lemma V, r can be increased to some r' until $\theta=2\pi$. Let P be the polygon formed by connecting neighboring w_i 's with geodesics. The area of P , $A(P)$, is equal to $(n-2)\pi$. The circle of radius r' at v has area equal to $4\pi\sinh^2(r'/2) \leq A(P)$. By Proposition III, the circle of radius r' is contained in P . Hence

$$\pi(r')^2 \leq 4\pi(r'/2)^2 \leq 4\pi\sinh^2(r'/2) \leq (n-2)\pi \leq n\pi.$$

This finishes the lemma. ■

4. THE EXISTENCE AND BOUNDARY VALUE THEOREM

In this section we want to give an existence proof for circle packings of triangulations of the unit disc. The proof will be split into two cases; special complexes and nonspecial complexes. (A nonspecial complex is a finite triangulation of the closed disc that is not special). In both cases, we want to prove more than just existence. We want to be able to specify boundary radii and get a packing label with these specified boundary radii.

Theorem II: Let K , Φ and $f:V_\partial \rightarrow (0,\infty]$, where V_∂ is the set of boundary vertices of K , be given. Then there exist a label R such that $K(R,\Phi)$ is a packing, and $R|_{V_\partial}=f$. Furthermore, R is unique.

Proof: If K is a complex with only three vertices the theorem reduces to Lemma I. Suppose that K has $n \geq 4$ vertices and that the theorem is true for complexes having k , $3 \leq k < n$, vertices. Suppose K is a nonspecial complex. Then we can find two boundary vertices v_1, v_2 that are not consecutive on the boundary but are joined by a single edge. Splitting K at that edge gives two complexes K_1, K_2 , each having fewer than n vertices. By the inductive step there exist labels for these

complexes that have boundary radii equal to the radii given to the corresponding boundary vertices of K by f . Using a Mobius transformation, the circles corresponding to v_1, v_2 can be placed with their centers at the same locations on the real axis for both complexes. Superimposing these circles onto one another and taking orientation into consideration shows that the labels for K_1 and K_2 give a packing label for K with boundary radii the same as given by f . This proves existence.

For uniqueness, suppose there exist two packing labels R, R' for K . Split K into K_1 and K_2 again and look at the restrictions R_i and R'_i of R and R' to K_i . These restrictions are packing labels for K_i with the same boundary radii. By uniqueness in the inductive step, $R_i = R'_i$. Hence $R = R'$.

Let K be a special complex. Let V represent the vertex set of K . Then $V = V_\partial \cup V_{int}$ where $V_\partial = \partial V$ and $V_{int} = V \setminus V_\partial$. Let $\mathfrak{R}$ denote the set of all labels R such that $K(R, \Phi)$ is a superpacking and $R(v) \geq f(v)$, $v \in V_\partial$. Recall it was pointed out in Definition IV that if $R(v) = \infty$ then $\theta_R(v) = 0$. Hence $\mathfrak{R}$ is nonempty since the constant map sending each vertex to infinity is in $\mathfrak{R}$. Define $\inf \mathfrak{R}(v) = \inf \{R(v) : R \in \mathfrak{R}\} \forall v \in V$. By the monotonicity results in Lemma V it is clear that $\inf \mathfrak{R}(v) < \infty \forall v \in V_{int}$, and $\inf \mathfrak{R}(v) = f(v) \forall v \in V_\partial$. All that is left to show is that

$\inf \mathfrak{R}(v) > 0 \ \forall v \in V_{int}$ and $K(\inf \mathfrak{R}, \Phi)$ is a packing.

Lemma VIII: Let R and R' be labels for K . Then

(i) If $R \leq R'$ and $R(v) = R'(v)$, $\forall v \in V_\partial$, then

$$\sum_{v \in V_\partial} \theta_R(v) \leq \sum_{v \in V_\partial} \theta_{R'}(v).$$

(ii) If $R''(v) = \min(R(v), R'(v))$, $\forall v \in V$, then

$\theta_{R''}(v) \leq \max(\theta_R(v), \theta_{R'}(v))$, $\forall v \in V$. In particular,

$R, R' \in \mathfrak{R}$ implies $R'' \in \mathfrak{R}$.

(iii) There exist a sequence $\{R_i\} \subset \mathfrak{R}$ such that $R_i \rightarrow \inf \mathfrak{R}$ as $i \rightarrow \infty$.

Proof: (i) comes from Lemma V.

(ii). Let v be a vertex of K such that, WLOG,
 $R''(v) = \min(R(v), R'(v)) = R(v)$. Since $R''(v) = R(v)$ and $R''(w) \leq R(w)$ for any neighboring vertex, $\theta_{R''}(v) \leq \theta_R(v)$. The rest of (ii) is evident.

(iii). For each $i \in \mathbb{N}$ and vertex v of K there exist $R_{i,v} \in \mathfrak{R}$ such that $R_{i,v}(v)$ is within 2^{-i} of $\inf \mathfrak{R}(v)$. Then $R_i = \min\{R_{i,v} : v \in V\}$ is such that $|\inf \mathfrak{R}(v) - R_i(v)| < 2^{-i} \ \forall v \in V$ and (ii) implies $R_i \in \mathfrak{R}$. ■

Claim: If $v \in V_{int}$ and $inf\mathfrak{R}(w) > 0, \forall w \in V$ then $\theta_{inf\mathfrak{R}}(v) = 2\pi$.

Suppose there exists an interior vertex v such that $\theta_{inf\mathfrak{R}}(v) \neq 2\pi$. Because angle sums are continuous functions of the radii and (iii) of Lemma VIII, the angle sum at any interior vertex of $K(inf\mathfrak{R}, \Phi)$ is less than or equal to 2π . So, in particular, $\theta_{inf\mathfrak{R}}(v) < 2\pi$. Since $inf\mathfrak{R}(v) > 0$, Lemma V says the radius at v can be decreased by a small amount while holding all other radii constant and the angle sum at v will remain less than 2π . Also the angle sum at every other vertex in $K(inf\mathfrak{R}, \Phi)$ will remain less than or equal to 2π . This implies the existence of a superpacking label for K whose radius at v is strictly less than $inf\mathfrak{R}(v)$. This is a contradiction. Hence $\theta_{inf\mathfrak{R}}(v) = 2\pi$.

It is now enough to show $inf\mathfrak{R}(v) > 0 \forall v \in V_{int}$. To do this a stronger version of the above claim is needed, i.e. if $v \in V_{int}$ and $inf\mathfrak{R}(v) > 0$, then $\theta_{inf\mathfrak{R}}(v) = 2\pi$. Defining the angle α in Figure 6 in the obvious manner when $r_0 > 0$, and one (or both) of r_1, r_2 is zero, it is easily seen that α is a continuous function of r_1 and r_2 on $[0, \infty]$. With this, the stronger version of the claim can be proved in the same manner as above.

Let $R: V \rightarrow (0, \infty]$ be any map and $H \subseteq V$. Define the angle sum of H determined by R to be

$$\theta_R(H) = \sum_{v \in H} \theta_R(v) .$$

The excess of H determined by R is

$$e_R(H) = \sum_{v \in H} (\theta_R(v) - 2\pi)$$

and the shortage is

$$s_R(H) = \sum_{v \in H} (2\pi - \theta_R(v)) = -e_R(H).$$

If $H = \emptyset$, the angle sum, excess and shortage are defined to be zero. Define total excess and total shortage to be $e_R = e_R(V_{int})$ and $s_R = s_R(V_{int})$ respectively. The total positive excess is $e_R^+ = e_R(\{v \in V_{int} : \theta_R(v) \geq 2\pi\})$ and total positive shortage is $s_R^+ = s_R(\{v \in V_{int} : \theta_R(v) \leq 2\pi\})$. Notice $s_R^+ = 0$ if and only if $K(R, \Phi)$ is a subpacking, and $e_R^+ = 0$ if and only if $K(R, \Phi)$ is superpacking. A_R is the total area of the faces of $K(R, \Phi)$ as hyperbolic triangles.

Let F' , E' and V' denote the number of faces, edges and vertices of K respectively. The number of vertices in V_∂ and V_{int} will be denoted by V'_∂ and V'_{int} . Assume an ordering on the faces of K . Let $\alpha_i, \beta_i, \gamma_i$, $i \in \{1, 2, \dots, F'\}$, denote the angles of the i^{th} face of $K(R, \Phi)$. Then

$$A_R = \sum_{i=1}^{F'} (\pi - \alpha_i - \beta_i - \gamma_i)$$

$$=F'\pi-\theta_R(V)$$

$$=F'\pi-(\theta_R(V_\partial)+e_R+2\pi V'_{int}).$$

Substituting $e_R^+-s_R^+$ for e_R ,

$$A_R+\theta_R(V_\partial)+e_R^+-s_R^+=(F'-2V'_{int})\pi.$$

Notice that $(F'-2V'_{int})\pi$ is independent of both R and Φ . (This invariant was noted by Bowers [4]).

For K , we have $3F'=2E'-V'_\partial$. Hence

$$E'=\frac{3F'+V'_\partial}{2}.$$

This and the Euler characteristic, $\chi(|K|)=F'-E'+V'$, results in

$$F'-2V'+V'_\partial=-2\chi(|K|). \quad \text{Substituting } V'_{int}+V'_\partial \text{ for } V',$$

$$F'-2V'_{int}=-2\chi(|K|)+V'_\partial. \quad \text{Since } |K| \text{ and the closed disc are}$$

homeomorphic, $\chi(|K|)=1$. Therefore, $F'-2V'_{int}=V'_\partial-2$. (Note this

gives $F'=V'_\partial+2(V'_{int}-1)$. Since $V'_\partial \geq 3$,

$$A_R+\theta_R(V_\partial)+e_R^+-s_R^+=(F'-2V'_{int})\pi > 0$$

for any R and Φ given to K .

Define a collection Q of vertices of K to be edge path connected if every two vertices of Q are the end points of an edge path in K in which all the vertices in that path lie in Q . Let $H \subseteq V$. An edge path component of H is a maximal edge

path connected subset of H . Let P be an edge path component of H then $L \subseteq P$ is said to be a loop of P if there exist distinct edges in K having endpoints only in L that form a closed path and every vertex in L is an end point for exactly two edges in that closed path. The interior of a loop will be the interior of the polygon formed by its closed edge path.

Let $W = (\inf \mathfrak{K})^{-1}(0)$ and B be an edge path component of W . Let L be the subcomplex of K determined by the conditions that a (closed) face of K is in L if and only if at least one of its vertices lies in B and every edge and vertex of L lies in a face of L . From this definition we get the following. Every vertex in B is an interior vertex of L , and every vertex in L has a neighbor in B . Since B is edge path connected, $|L|$ must be connected. Restricting $R \in \mathfrak{K}$ and Φ to the edges and vertices of L if necessary, it is clear that $L(R, \Phi) (|L|)$ is a subset of $K(R, \Phi) (|K|)$ and more specifically it is the union of all the faces of $K(R, \Phi) (|K|)$ that hit B . Notice that $|L|$ is an orientable 2-manifold with boundary if $|L|$ is simply connected.

There exist three possibilities for L . Either $L = \emptyset$, $\emptyset \neq L \neq K$, or $L = K$. If it can be shown that $L = K$ and $\emptyset \neq L \neq K$ can not be, then L must be empty. L being empty implies that W is empty and proves that $\inf \mathfrak{K}$ is a packing label for K .

Case I. Suppose $L = K$. First notice that because B is edge

path connected, $L=K$ implies that $W=B$.

Claim: $V_{int}=B=W$.

Suppose $v \in V_{int} \setminus B$. Suppose all v 's neighbors are in B . Let $v_1, v_2, \dots, v_n$ be v 's neighbors. Since $\inf \mathfrak{K}(v) > 0$ and $\inf \mathfrak{K}(v_i) = 0$, it is not hard to see that $\theta_{R_i}(v) \rightarrow 0$ as $i \rightarrow \infty$ where $\{R_i\}$ is the sequence given in (iii) of Lemma VIII. But $\theta_{R_i}(v) \rightarrow \theta_{\inf \mathfrak{K}}(v)$. Since $\inf \mathfrak{K}(v) > 0$, $\theta_{\inf \mathfrak{K}}(v) = 2\pi$. This is a contradiction. So v must have a neighbor in $V \setminus B$.

Now v is an element of an edge path component of $V \setminus B$, call it P . Since $L=K$ and B is edge path connected, P cannot have any loops with vertices from V_{int} . Define an end of P to be a vertex that has only one edge joining it to another vertex in P . P has at least one end in V_{int} , call it p . All the neighbors of p but one are elements of B . So, at most two angles in the angle sum at p may not be going to zero as $R_i \rightarrow \inf \mathfrak{K}$. From overlap angles being in $[0, \pi/2]$ and (ii) of Lemma III, neither of these two angles can be greater than $\pi/2$ in the limit. Hence $\theta_{R_i}(p) \rightarrow m \leq \pi$ as $i \rightarrow \infty$. This results in the same contradiction as before. Therefore $V_{int}=B=W$.

Let $\{R_i\}$ be the sequence given in (iii) of Lemma VIII. Let f be a face of K . Let $A_{R_i}(f)$ be the area of f in $K(R_i, \Phi)$. Since $R_i(v) \rightarrow 0$ as $i \rightarrow \infty$ for every interior vertex v , it is easy

to see that if f has two or more interior vertices, then $A_{R_i}(f) \rightarrow 0$ as $i \rightarrow \infty$. Notice also that if f has exactly two interior vertices, then the angle at the third vertex (a boundary vertex) converges to zero as $i \rightarrow \infty$.

Let $f_1, f_2, \dots, f_m$ be the faces of K with one boundary edge (i.e. two boundary vertices). Let $e_1, e_2, \dots, e_m$ be these boundary edges. Let $\alpha_n(i)$ and $\beta_n(i)$ be the angles at the boundary vertices of f_n in $K(R_i, \Phi)$, and let $\gamma_n(i)$ be the remaining angle. By what was said above, it is not hard to see that

$$\lim_{i \rightarrow \infty} A_{R_i} = \sum_{n=1}^m \lim_{i \rightarrow \infty} A_{R_i}(f_n),$$

and

$$\lim_{i \rightarrow \infty} \theta_{R_i}(V_\partial) = \sum_{n=1}^m \lim_{i \rightarrow \infty} (\alpha_n(i) + \beta_n(i)).$$

Since $A_{R_i}(f_n) = \pi - \alpha_n(i) - \beta_n(i) - \gamma_n(i)$,

$$\lim_{i \rightarrow \infty} (A_{R_i} + \theta_{R_i}(V_\partial)) = \sum_{n=1}^m \lim_{i \rightarrow \infty} \gamma_n(i).$$

By (ii) of Lemma III, $\lim_{i \rightarrow \infty} \gamma_n(i) = \pi - \Phi(e_n)$. Now, it is easy to

see that

$$A_{R_i} + \theta_{R_i}(V_\partial) \rightarrow \mu \text{ where } \begin{cases} \mu \leq \frac{\pi}{2} V'_\partial, & V'_\partial \geq 5 \\ \mu < 2\pi, & V'_\partial = 4 \\ \mu < \pi, & V'_\partial = 3 \end{cases} \text{ as } i \rightarrow \infty.$$

Definition II gives the strict inequality when $V'_\partial = 3$ or 4 .

For $V_\partial \geq 5$, this gives

$$(F' - 2V'_{int})\pi - A_{R_i} - \theta_{R_i}(V_\partial) \rightarrow (F' - 2V'_{int})\pi - \mu$$

$$\geq (F' - 2V'_{int})\pi - V'_\partial \frac{\pi}{2}$$

$$\geq (V'_\partial + 2V'_{int} - 2 - 2V'_{int})\pi - \frac{\pi}{2} V'_\partial$$

$$= \frac{\pi}{2} V'_\partial - 2\pi > 0.$$

The second inequality is a consequence of $F' = V'_\partial + 2(V'_{int} - 1)$. For large enough i 's,

$$(F' - 2V'_{int})\pi - A_{R_i} - \theta_{R_i}(V_\partial) > 0.$$

Since $K(R_i, \Phi)$ is a subpacking, $e_{R_i}^+$ is zero. This implies

$A_{R_i} + \theta_{R_i}(V_\partial) \geq (F' - 2V'_{int})\pi$ for every i . Together, this and the

above inequality result in a contradiction. Using a similar argument, the same contradiction arises for $V_\partial = 3, 4$. Hence

$L \neq K$.

Case II. Suppose $\emptyset \neq L \neq K$. There are two possibilities in this case. $|L|$ is simply connected or it is not. First suppose $|L|$ is simply connected.

It was noted earlier that every vertex of B lies in the interior of L and every vertex of L has a neighbor in B . This says that L satisfies conditions (i) and (ii) of Definition I. Since $|L|$ is connected, $|L|$ can have only one boundary component. Because $|L|$ is simply connected with one boundary component, $\chi(|L|)=1$. Now $\chi(|L|)=1$ and $|L|$ being an orientable 2-manifold with one boundary component implies $|L|$ is homeomorphic to the closed disc. Hence L is a special complex, and every face of L has a vertex that is in B .

Let M be the set of boundary vertices of L . If $v \in M$ then $\inf \mathfrak{X}(v) > 0$. Because if $\inf \mathfrak{X}(v)$ were zero, $v \in W$. Hence $v \in B$, contradicting the fact that v is a boundary vertex of L . Let m be the minimum value of $\inf \mathfrak{X}$ on M and notice that m is positive. Let $\mathfrak{X}'$ be the set of superpacking labels for L that have values on the boundary vertices greater than or equal to m . Let $R \in \mathfrak{X}$ and define R' to be the restriction of R to L . Clearly $R' \in \mathfrak{X}'$, and hence $\inf \mathfrak{X}'$ is zero on B . With this, Case I results in a contradiction.

Suppose $|L|$ is not simply connected. Then there exists a loop of boundary vertices of L such that the interior of this loop has empty intersection with L . Let H be the edge path

component of vertices of $L \setminus B$ that contains this loop. If the set of loops of H contains the loop of "outermost" boundary vertices of L , then the following statement does not include the loop of "outermost" boundary vertices. The interiors of the loops of H have empty intersection with L , but may have non-empty intersection with W . If this were true, there would be an edge path component of W in the interior of a loop of H . In which case, our attention would focus on it rather than B . So assume the only vertices in the interior of the loop(s) of H are from $V \setminus W$.

By the same argument as in Case I, H cannot have any ends. Hence H consist of loops joined by edges and vertices. Let $H_1, H_2, \dots, H_n$ be the loops of H . There exist an H_i such that at most one vertex of H_i , call it v_i , has a neighbor in H that is not in H_i , and the intersection of the interior of H_i with L is empty. Notice that $H_i \setminus \{v_i\} \subseteq V_{int}$, and every neighbor of $v \in H_i \setminus \{v_i\}$ lying in $L \setminus H_i$ is in B . Assume this loop is H_1 .

Let C be the complex formed by the vertices that are in H_1 , the edges connecting them, and all of the edges, vertices, and faces of K that are in the interior of H_1 . The overlap angles of neighbors in C are the angles given to the corresponding neighbors in K . Every vertex v of C except possibly v_1 has the property that $0 < \inf \mathfrak{A}(v) < \infty$.

Let $\text{inf}\mathfrak{X}_C$ be the restriction of $\text{inf}\mathfrak{X}$ to the vertices of C . Note that $\text{inf}\mathfrak{X}_C$ is a packing label for C . The total angle sum of C with the label $\text{inf}\mathfrak{X}_C$ is less than $\pi F'_C$ where F'_C is the number of faces of C . Hence

$$2\pi V'_{\text{int}_C} + \theta_{\text{inf}\mathfrak{X}_C}(V_{\partial_C}) \leq \pi F'_C.$$

Consider C as being in K and let v be a boundary vertex of C that is not v_1 . Since $\text{inf}\mathfrak{X}_C(v) = \text{inf}\mathfrak{X}(v) > 0$, the angle sum at v in C is equal to 2π minus the sum of the angles at v that are in faces of K that are not in C . Because $\text{inf}\mathfrak{X}(w) = 0$ for all the neighbors w of v not in C , there are at most two of these angles that are not zero. Both of these angles are in a face of K that has two vertices, one of which is v , that are boundary vertices of C . At the vertex v_1 the angle sum in C is again 2π minus the sum of the angles at v_1 that are in faces of K not in C . Since v_1 has neighbors not in W , there may be more than two of these angles that are not zero. But two of these possible nonzero angles are the angles at v_1 that are in a face of K that has two vertices, one of which is v_1 , that are boundary vertices of C . So to each boundary edge of C there corresponds two angles that are possibly not zero, and are not in the boundary angle sum of C . By (ii) of Lemma III, the sum of the two angles corresponding to any boundary edge

is at most $\pi/2$. It is not hard to see then that

$$\theta_{\inf \mathfrak{K}_c}(V_{\partial_c} \setminus \{v_1\}) \geq (V'_{\partial_c} - 1)2\pi - \frac{\pi}{2} V'_{\partial_c}.$$

This implies

$$\begin{aligned} \theta_{\inf \mathfrak{K}_c}(V_{\partial_c}) &\geq 2\pi(V'_{\partial_c} - 1) - V'_{\partial_c} \frac{\pi}{2} \\ &= V'_{\partial_c} 2\pi - \frac{\pi}{2} V'_{\partial_c} - 2\pi. \end{aligned}$$

Thus

$$\begin{aligned} 2\pi V'_{\text{int}_c} + \theta_{\inf \mathfrak{K}_c}(V_{\partial_c}) &\geq 2\pi V'_{\partial_c} - V'_{\partial_c} \frac{\pi}{2} - 2\pi + 2\pi V'_{\text{int}_c} \\ &= 2\pi(V'_{\partial_c} + V'_{\text{int}_c}) - \frac{\pi}{2} V'_{\partial_c} - 2\pi. \end{aligned}$$

This gives $\pi F'_c \geq 2\pi(V'_{\partial_c} + V'_{\text{int}_c}) - \frac{\pi}{2} V'_{\partial_c} - 2\pi$. Hence

$$F'_c \geq 2(V'_{\partial_c} + V'_{\text{int}_c}) - \frac{V'_{\partial_c}}{2} - 2.$$

Let E'_c be the number of edges of C and E'_{∂_c} the number of boundary edges ($E'_{\partial_c} = V'_{\partial_c}$). Notice that $|C|$ is simply connected

with one boundary component, so $\chi(|C|) = 1$. From $3F'_c = 2E'_c - E'_{\partial_c}$

comes $F'_c = \frac{2}{3}E'_c - \frac{E'_{\partial_c}}{3}$. Using this and the Euler characteristic,

$$1 = V'_{\partial_c} + V'_{int_c} - E'_c + F'_c = V'_{\partial_c} + V'_{int_c} - E'_c + \frac{2}{3}E'_c - \frac{E'_{\partial_c}}{3}.$$

So, $V'_{\partial_c} + V'_{int_c} = 1 + \frac{E'_c}{3} + \frac{E'_{\partial_c}}{3}$. Now,

$$\begin{aligned} 1 &= V'_{\partial_c} + V'_{int_c} - E'_c + F'_c \\ &\geq V'_{\partial_c} + V'_{int_c} - E'_c + 2(V'_{\partial_c} + V'_{int_c}) - \frac{V'_{\partial_c}}{2} - 2 \\ &= 3(V'_{\partial_c} + V'_{int_c}) - E'_c - \frac{V'_{\partial_c}}{2} - 2 \\ &= 3 + E'_c + E'_{\partial_c} - E'_c - \frac{V'_{\partial_c}}{2} - 2 \\ &= 1 + \frac{V'_{\partial_c}}{2}. \end{aligned}$$

But, $\frac{V'_{\partial_c}}{2} \geq 1$. Therefore $\emptyset \neq L \neq K$ is not possible. So, $L = \emptyset$.

To prove the uniqueness of $\inf \mathfrak{K}$ suppose there exists another packing label R' for K with $R'|_{V_\partial} = f$. Since $\inf \mathfrak{K} \leq R'$, Lemma III gives $A_{\inf \mathfrak{K}} \leq A_{R'}$ with strict inequality if $\inf \mathfrak{K} \neq R'$. From Lemma VIII, $\theta_{\inf \mathfrak{K}}(V_\partial) \leq \theta_{R'}(V_\partial)$. Also, the total positive

shortages of both packing labels are zero. Because of the
 latter, $A_{inf\mathfrak{K}} + \theta_{inf\mathfrak{K}}(V_\partial) = (F' - 2V'_{int})\pi = A_{R'} + \theta_{R'}(V_\partial)$. Since
 $\theta_{inf\mathfrak{K}}(V_\partial) \leq \theta_{R'}(V_\partial)$, $A_{inf\mathfrak{K}} \geq A_{R'}$. This implies $A_{inf\mathfrak{K}} = A_{R'}$ which implies
 $inf\mathfrak{K} = R'$. ■

5. THE GENERALIZED DISCRETE SCHWARZ-PICK LEMMA

In the proof of Theorem II it was shown that the unique label for K given Φ is the infimum of all the superpacking labels for K . By using just the existence part of Theorem II and defining $\mathcal{C}$ to be the set of all subpacking labels, R , of K such that $R(v) \leq f(v)$, $v \in V_\partial$, for some given f , the Perron method can be used to show that $\sup \mathcal{C}$ (defined in the same manner as $\inf \mathcal{R}$) is also a packing label for K . The only difficulty in doing this is proving the dual of $\inf \mathcal{R}(v) > 0$ for every interior vertex v (i.e. $\sup \mathcal{C}(v) < \infty$, $\forall v \in V_{int}$), but this was done in Lemma VII. By uniqueness, $\sup \mathcal{C} = \inf \mathcal{R}$. This is summed up as

Lemma IX: Given K , Φ and $f: V_\partial \rightarrow (0, \infty]$, the unique packing label for K with boundary radii given by f is the supremum of all subpacking labels for K with boundary radii less than or equal to those given by f .

We are now in a position to prove

Theorem III: (Generalized Discrete Schwarz-Pick Lemma) Let Φ be a strict angle assignment. Let K be a special complex with

Andreev label R_a and let R be any labeling of K such that $K(R, \Phi)$ is a packing. Then

$$(i) \quad R \leq R_a.$$

$$(ii) \quad \rho_R(v_1, v_2) \leq \rho_{R_a}(v_1, v_2) \text{ for any pair of vertices}$$

in K .

$$(iii) \quad \text{Area}_R(f) \leq \text{Area}_{R_a}(f) \text{ for every face } f \text{ of } K.$$

Furthermore, if equality holds for a single interior vertex in (i), a single pair of interior vertices in (ii) or a single face in (iii) then $R = R_a$.

Proof: (i). This comes Lemma IX. If equality holds at a single interior vertex v_0 , then we claim it holds for all the neighboring vertices. Suppose that there exist a neighbor w such that $R(w) < R_a(w)$. Since the angle sum at v_0 is an increasing function of the radius at w and $\theta_R(v_0) = 2\pi$, $\theta_{R_a}(v_0) > 2\pi$. This contradicts that R_a is a packing label.

Hence there must be equality at all of v_0 's neighbors. This equality spreads to all vertices by repeating the above argument.

(ii). Lemma VI and (i) give this result. The case of equality comes from the last part of Lemma VI and (i).

(iii). Every face has at least one interior vertex v . This and Lemma III give the inequality. Since monotonicity is

strict, the area of two faces being equal implies $R(v) = R_a(v)$.

By (i), $R = R_a$. ■

Notice, if Φ is not a strict angle assignment, the statement regarding equality does not hold. The reason for this comes from the example given after Lemma V. Consider the vertex v in Figure 7 to be an interior vertex and all the others to be boundary vertices. Suppose $R(v) = R_a(v)$. This does not imply equality in all of v 's neighbors. As before, the radius of w_i can be changed without altering the angle sum at v .

Theorem III is not stated as generally as it can be. The following changes can be made without altering the proof. Replace "Andreev label R_a and let R be any label of K such that $K(R, \Phi)$ is a packing" with " R a subpacking label and $\hat{R}$ a packing label such that $R(v) \leq \hat{R}(v) \forall v \in V_0$ " and the remaining R_a 's with $\hat{R}$'s.

6. CONCLUSION

Although this generalization of Beardon and Stephenson's work may at first seem geometrically evident, the proofs given for the monotonicity results show that the relationships for overlapping circles proved here are hidden in the geometry. It is the analogies to classical complex analysis made in [3] that remain clear. These analogies refer to the Monodromy Theorem, subharmonic functions, and, of course, the Schwarz-Pick lemma. Note that superharmonic functions can be added to this list.

Theorem II is a special case of Andreev's theorem for circle packings of the sphere with prescribed overlap angles, but the proof of Theorem II suggests an alternative proof of Andreev's theorem. This was pointed out by Bowers and Stephenson. Briefly, the idea is the following.

Circles in Δ and the arcs of euclidean circles that overlap $\partial\Delta$ have constant curvature in the hyperbolic plane. The curvature of the arc of a circle that overlaps $\partial\Delta$ is related to the overlap angle between $\partial\Delta$ and the circle. The angles α, β, γ in Figure 6 can be seen as continuous functions of the curvature of the circles centered at z_0, z_1, z_2 . It seems possible to prove the monotonicity results of section 3 in terms of curvature rather than radii. If this could be accomplished then the Perron method and the invariant used in

the proof of Theorem II might be used to prove the existence of a circle packing in Δ with boundary "circles" being arcs of circles having curvatures corresponding to desired overlap angles. Projecting this onto the sphere would give the desired circle packing of the sphere.

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