To the Graduate Council:

I am submitting herewith a thesis written by Evangelia Spanakis entitled "Construction of Wavelet Bases for $L^2(\mathbb{R})$" I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

D. Samuel Jordan

S. Jordan, Major Professor

We have read this thesis
and recommend its acceptance:

Steven M. Serbin

Don Hinton

Accepted for the Council:

C W Minkel

Associate Vice Chancellor
and Dean of The Graduate School

STATEMENT OF PERMISSION TO USE

In presenting this thesis in partial fullfillment of the requirements for a *Master's* degree at The University of Tennessee, Knoxville, I agree that the Library shall make it available to borrowers under rules of the Library. Brief quotations from this thesis are allowable without special permission, provided that accurate acknowledgment of the source is made.

Permission for extensive quotation from or reproduction of this thesis may be granted by my major professor, or in his absence, by the Head of Interlibrary Services when, in the opinion of either, the proposed use of the material is for scholarly purposes. Any coping or use of the material in this thesis for financial gain shall not be allowed without my written permission.

Signature E. Spanakis

Date July 19, 1994

Construction of Wavelet Bases for $L^2(\mathbb{R})$

A Thesis

Presented for the

Master of Science

Degree

The University of Tennessee, Knoxville

Evangelia Spanakis

August, 1994

Στους γονεις μου,

Lillian και Νικο Σπανακη

ACKNOWLEDGEMENTS

I am particularly indebted to my thesis advisor, Professor Samuel G. Jordan, for his valuable help and continuous support. I would like to thank Professors Don Hinton and Steven Serbin for serving in my committee. I would also like to thank the Department of Mathematics of UTK for their financial support and the encouraging academic enviroment they provided throughout my graduate studies.

ABSTRACT

Classical orthonormal bases usually consist of functions whose supports are not local and which, therefore, are not very efficient for representing functions that are local both in time and frequency or in both space and wave number. Wavelet bases overcome this deficiency matching the supports of their elements to their frequencies. In addition, the elements of a wavelet basis are obtained by translation and dilation from a single function and concequently, are especially suitable for use in computationally efficient algorithms.

The purpose of this thesis is to present a readable development of orthonormal bases of wavlets. This development is approached through the concept of a multiresolution analusis. It leads to the derivation of a few key-conditions, whose solutions, as we prove, produce wavelet bases.

This thesis is roughly based on a recent presentation by R. Strichartz [8], but seeks to provide greater rigor and clarity.

TABLE OF CONTENTS

Chapter 1

Introduction

The usefulness of various bases for the L^2 space of square integrable functions on the real line or an interval (with Lebesgue measure) is well known. Recently a new kind of basis was introduced – the "wavelet" bases. These bases are used in many applications, such as audio and image processing, quantum mechanics, and harmonic analysis, to mention just a few. We refer the reader to Strang [6] and the references therein for a list of applications for such bases.

Each wavelet basis is created from a single function ψ through dilations and integer translations. This fact makes computing with them simple and efficient. In addition, ψ may have compact support so that the resulting basis is local. Also, all wavelet bases have a nice scaling property. To be more precise, we suppose that ψ has compact support and generates a wavelet basis $\{\psi_{j,n}\}$ $j, n \in \mathbb{Z}$. Then, as for any basis, each $f \in L^2(\mathbb{R})$ has a representation

$$f = \sum_{j\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} c_j(n)\psi_{j,n} \, .$$

This basis is local in the sense that to represent f on any restricted interval $[a, b]$, one needs only those relatively few basis elements the supports of which intersect $[a, b]$. With Fourier expansions, for instance, this simplification is impossible. Also, as j increases, the support of $\psi_{j,n}$ decreases, so that f can be approximated in finer scale.

How wavelet bases arise is also very interesting. Most classical orthonormal systems are assosiated with differential equations. However, no differential equation is involved with a wavelet basis; instead, there is a dilation equation, $\varphi(x) = \sum_{j\in\mathbb{Z}} a_j\varphi(2x - j)$, with a related ladder of subspaces of L^2.

These aspects of wavelet bases will be revealed in the example of Haar wavelets in the next chapter. The Haar wavelets are not smooth, but they show clearly the fundamental structure associated with wavelet systems.

In Chapters 3-5 we will show how to construct wavelet bases in a more general setting. First in Chapter 3 we define and analyze a fundamental tool for our purpose: a multiresolution analysis (MRA). Roughly, a MRA is a sructure consisting of a function φ and a corresponding sequence of nested spaces $\cdots \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots$ related by a certain scaling property. In Chapter 4 we give conditions for obtaining a MRA and in Chapter 5 we show that there exist functions satisfying the conditions discussed in Chapters 3 and 4. The existence of these functions implies, in turn, the existence of a wavelet basis. In Chapter 6 we present a result on the smoothness of wavelets. Finally, in Chapter 7 we develop an algorithm for constructing wavelets.

Several authors have presented developements of wavelet bases and multiresolution analysis. The presentation here roughly follows that of Robert S. Strichartz in his paper "How To Make Wavelets" [8], but the organization is different, proofs are given in greater detail, and additional theoretical and numerical results are included. It is hoped that this thesis is a readable introduction to wavelet bases and will encourage the reader to learn more about them and their uses.

Chapter 2

The Haar Example

This example leads to discontinuous wavelets, but it is sufficiently simple so that the important structural features will be quite clear. Everything is built upon just one function, the characteristic function φ of $(0,1]$. Note that (integrals without limits are over $\mathbb{R}$)

$$\int \varphi(x)dx = 1\,,$$

$$\{\varphi(x-\gamma)\}_{\gamma\in\mathbb{Z}} \text{ is an othonormal set,}$$

$$\varphi(x) = \varphi(2x) + \varphi(2x-1)\,.$$

These are very important properties of φ, that will arise again in the following chapters. The last one is the reason we call φ the *scaling function*: its graph can be reproduced from the compressed graphs of the functions $\varphi(2x-\gamma)$ (here $\gamma = 0$ and 1).

Let V_0 denote the closure in $L^2(\mathbb{R})$ of the linear span of the functions $\{\varphi(x-\gamma)\}$, $\gamma \in \mathbb{Z}$. This subspace consists of all piecewise constant functions with jump discontinuities in $\mathbb{Z}$ and the set $\{\varphi(x-\gamma)\}_{\gamma\in\mathbb{Z}}$ is an orthonormal basis (ONB) for V_0. If we now allow jump discontinuities in $\frac{1}{2}\mathbb{Z}$, we get a bigger subspace $V_1 \supseteq V_0$, which is clearly related to V_0 by the condition

$$f(x) \in V_1 \Longleftrightarrow f(2^{-1}x) \in V_0\,.$$

A simple change of variables shows that $\{2^{1/2}\varphi(2x-\gamma)\}_{\gamma\in\mathbb{Z}}$ forms an ONB for V_1. Similarly $\left\{2^{\frac{j}{2}}\varphi(2^jx-\gamma)\right\}_{\gamma\in\mathbb{Z}}$ forms an ONB for the subspace V_j of $L^2(\mathbb{R})$

determined by the condition $f(x) \in V_j \Leftrightarrow f(2^{-j}x) \in V_0$. Clearly V_j is the subspace of piecewise constant functions with jump discontinuities in $2^{-j}\mathbb{Z}$.

We now have a sequence of nested closed subspaces of $L^2(\mathbb{R})$,

$$\cdots \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots$$

such that

$$\bigcap_j V_j = \{0\} \quad \text{and} \quad \overline{\bigcup_j V_j} = L^2(\mathbb{R}),$$

(these properties are not difficult to check — proofs in a more general setting will be given in Chapter 4). However, combining the bases of all the V_j 's does not produce an ONB for $L^2(\mathbb{R})$. In fact, the set $\{2^{j/2}\varphi(2^jx-\gamma)\}_{\gamma\in\mathbb{Z},j\in\mathbb{Z}}$ is not even orthogonal since, for example, $\int \varphi(x)\overline{2^{1/2}\varphi(2x)}dx = \sqrt{2}/2$. There exists, though, another "hidden" sequence of subspaces, whose combined bases *do* form an ONB of $L^2(\mathbb{R})$.

Note that for each $j \in \mathbb{Z}$, V_j is properly contained in V_{j+1}. Define W_j to be the orthogonal complement of V_j in V_{j+1}. Then we have the Hilbert space direct sums

$$V_{j+1} = V_j \oplus W_j \tag{2.1}$$

We claim that

$$L^2(\mathbb{R}) = \bigoplus_{j=-\infty}^{\infty} W_j \,. \tag{2.2}$$

Fix $j \in \mathbb{Z}$ and take $k < j$. Then repeated use of (2.1) and the nesting of the V_j give

$$V_j = V_k \oplus \bigoplus_{n=k}^{j-1} W_n = \left(\bigcap_{n=k}^{\infty} V_n\right) \oplus \bigoplus_{n=k}^{j-1} W_n \,.$$

Since $\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$, we also have

$$V_j = \{0\} \oplus \bigoplus_{n=-\infty}^{j-1} W_n = \bigoplus_{n=-\infty}^{j-1} W_n \,.$$

Combining this result and $\overline{\bigcup_j V_j} = L^2(\mathbb{R})$, we conclude that if $f \in L^2(\mathbb{R})$ is orthogonal to $\bigoplus_{n=-\infty}^{\infty} W_n$, then $f = 0$. Finally, since for $k < j$, $W_k \subseteq V_j$ and $V_j \perp W_j$, $\{W_n\}_{n=-\infty}^{\infty}$ is an orthogonal family of closed subspaces. Hence $\bigoplus_{n=-\infty}^{\infty} W_n$ is closed, [4]. Thus (2.2) holds, and combining the bases of all the W_n gives an ONB of $L^2(\mathbb{R})$.

The Haar wavelet ψ is a function whose integer translates form an ONB of W_0. It may be defined by

$$\psi(x) = \begin{cases} 1, & x \in (0, \frac{1}{2}] \\ -1, & x \in (\frac{1}{2}, 1]. \end{cases}$$

The integer translates of ψ are obviously orthonormal; the basis assertion follows from $V_1 = V_0 \oplus W_0$ and the fact that each basis function of V_1 can be expressed as a linear combination of a basis function of V_0 and an integer translate of ψ :

$$\varphi(2x-k) = \begin{cases} 2^{-1}\left(\varphi(x-\frac{k}{2}) + \psi(x-\frac{k}{2})\right), & k \text{ even}, \\ -2^{-1}\left(\psi(x-\frac{(k-1)}{2}) - \varphi(x-\frac{(k-1)}{2})\right), & k \text{ odd}. \end{cases}$$

We can obtain a basis for each W_j from ψ in the same way we get a basis for each V_j from φ. Since

$$\begin{aligned} W_j &= \{f \ : \ f(x) \in V_{j+1},\ f \perp V_j\} \\ &= \left\{f \ : \ f(x) \in V_{j+1},\ \int 2^{j/2}\varphi(2^j x - k)\overline{f(x)}dx = 0,\ \forall k \in \mathbb{Z}\right\} \\ &= \left\{f \ : \ f(2^{-j}x) \in V_1,\ \int \varphi(y-k)\overline{f(2^{-j}y)}dy = 0,\ \forall k \in \mathbb{Z}\right\} \\ &= \{f \ : \ f(2^{-j}x) \in W_0\}, \end{aligned}$$

and $\{\psi(x-\gamma)\}_{\gamma\in\mathbb{Z}}$ is an ONB for W_0, then $\left\{2^{j/2}\psi(2^j x - \gamma)\right\}_{\gamma\in\mathbb{Z}}$ is an ONB for W_j. So finally, we have that the set

$$\left\{2^{j/2}\psi(2^j x - \gamma)\right\}_{\gamma\in\mathbb{Z},\ j\in\mathbb{Z}}$$

is an ONB of $L^2(\mathbb{R})$.

This basis is generated through dilations and translations of a single function ψ, it is local and for each representation

$$f = \sum_{j\in\mathbb{Z}} \sum_{\gamma\in\mathbb{Z}} c_j(\gamma) 2^{j/2} \psi(2^j x - \gamma),$$

the partial sum taken over all $j \leq N$ represents an approximation of f involving details on the order of magnitude 2^{-N} or greater. It is an example of a wavelet basis. In subsequent chapters we will obtain wavelet bases for $L^2(\mathbb{R})$ which not only have these properties, but which also have smooth basis elements with compact support. However, the symmetric property of the Haar φ function, is not shared by the scaling functions corresponding to any other wavelet basis, [2, p.971].

Chapter 3

Conditions for Obtaining a MRA

As the Haar example demonstrates, everything begins with and is based on a scaling function φ and a corresponding sequence of nested subspaces $\cdots \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots$ satisfying certain properties. This structure (defined properly later), is called a *multiresolution analysis* (MRA). The whole theory of wavelets is based upon the fact that whenever a collection of closed subspaces is a MRA, there exists a wavelet basis $\{\psi_{j,\gamma}\}_{j,\gamma\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ generated by dilations and translations of a function ψ and such that for each $j \in \mathbb{Z}$

$$P_{j+1}f = P_j f + \sum_{\gamma\in\mathbb{Z}} \langle f, \psi_{j,\gamma}\rangle \psi_{j,\gamma}\,,$$

where P_j is the orthogonal projection onto V_j and $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(\mathbb{R})$ [1].

In what follows we will construct this kind of basis explicitly. To do so, we will consider φ and ψ simultaneously. First we will derive conditions these functions should satisfy. Next, we will prove that if these conditions are satisfied, then $\{2^{j/2}\psi(2^j x - \gamma)\}_{\gamma\in\mathbb{Z},j\in\mathbb{Z}}$ is an ONB of $L^2(\mathbb{R})$. Finally, we will construct functions that satisfy these conditions.

DEFINITION 3.1. *A multiresolution analysis (MRA)* $\cdots \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots$ *with scaling function* $\varphi \in V_0$ *is an increasing sequence of closed subspaces of* $L^2(\mathbb{R})$ *satisfying the following conditions:*

(i) *(density)* $\bigcup_j V_j$ *is dense in* $L^2(\mathbb{R})$

(ii) *(separation)* $\bigcap_j V_j = \{0\}$

(iii) *(scaling)* $f(x) \in V_j \Leftrightarrow f(2^{-j}x) \in V_0$

(iv) *(orthonormality)* $\{\varphi(x-\gamma)\}_{\gamma\in\mathbb{Z}}$ *is an ONB for* V_0.

We note that because of the scaling identity (iii) we get an ONB for every V_j, given by the family of functions $\left\{2^{j/2}\varphi(2^jx-\gamma)\right\}_{\gamma\in\mathbb{Z}}$. For orthonormality it is sufficient to note that

$$\int 2^{j/2}\varphi(2^jx)\overline{2^{j/2}\varphi(2^jx-\gamma)}dx = \int \varphi(y)\overline{\varphi(y-\gamma)}dy$$
$$= \delta(\gamma,0),$$

where $\delta(i,j)$ is Kronecker's delta function. For the basis property note that if $f(x) \in V_j$, then $f(2^{-j}x) \in V_0$, so that $f(2^{-j}x) = \sum_{\gamma\in\mathbb{Z}} c(\gamma)\varphi(x-\gamma)$ for some constants $c(\gamma)$. Another change of variable brings us back to the space V_j and establishes $\left\{2^{j/2}\varphi(2^jx-\gamma)\right\}_{\gamma\in\mathbb{Z}}$ as a basis for V_j.

We are looking for conditions involving both φ and ψ that will guide the construction of the wavelets. For convenience put $\psi_0 = \varphi$, $\psi_1 = \psi$. The function ψ_1 must be such that the collection $\{\psi_k(x-\gamma)\}_{\gamma\in\mathbb{Z},k=0,1}$ is an ONB for V_1. Therefore ψ_0, ψ_1 must both be elements of V_1. Using the basis we already have, we can represent the functions ψ_0, ψ_1 in the form

$$\psi_k(x) = \sum_{\gamma\in\mathbb{Z}} a_k(\gamma)\varphi(2x-\gamma) \quad (k=0,1), \tag{3.1}$$

where $\sum_{\gamma\in\mathbb{Z}} |a_k(\gamma)|^2 = 1$ for $k = 0, 1$. We refer to (3.1) with $k = 0$ as the *scaling identity.* Of course, convergence in (3.1) is in the L^2 sense and (3.1) holds pointwise only a.e.

We assume

$$\psi_0 \in L^1(\mathbb{R}), \quad \int \psi_0(x)dx \neq 0. \tag{3.2}$$

It is possible to have a MRA with scaling function $\psi_0 \notin L^1(\mathbb{R})$ [1]; however, the scaling function we construct will be bounded with compact support. If the integral of ψ_0 were zero, then we would have $\int \psi_0(2^j x - \gamma)dx = 0$ for all $j, \gamma \in \mathbb{Z}$, and the density condition (i) could not hold. This condition on ψ_0 allows us to obtain a critical condition on the coefficients $a_k(\gamma)$. We get it by integrating (3.1):

$$\begin{aligned}\int \psi_0(x)dx &= \sum_{\gamma\in\mathbb{Z}} a_0(\gamma) \int \psi_0(2x-\gamma)dx \\ &= \sum_{\gamma\in\mathbb{Z}} a_0(\gamma) 2^{-1} \int \psi_0(y)dy\end{aligned}$$

and hence

$$\sum_{\gamma\in\mathbb{Z}} a_0(\gamma) = 2\,. \tag{3.3}$$

(Changing the order of integration and summation here and in several later computations is justified by Fubini's Theorem if, for example, the sequences of coefficients $\{a_k(\gamma)\}$, $k = 0, 1$, are in ℓ^1. In our construction only finitely many of the $a_k(\gamma)$ will be nonzero, so this condition will hold.)

Another condition involves the coefficients of both ψ_0 and ψ_1.

PROPOSITION 3.2. *Suppose ψ_0, ψ_1 satisfy (3.1). Then $\{\psi_k(x-\gamma)\}_{\gamma\in\mathbb{Z}, k=0,1}$ is an orthonormal set, i.e.,*

$$\int \psi_j(x)\overline{\psi_k(x-\gamma)}dx = \delta(j,k)\delta(\gamma,0) \quad (j,k = 0,1\,;\, \gamma \in \mathbb{Z}), \tag{3.4}$$

if, and only if,

$$\sum_{\gamma'\in\mathbb{Z}} a_j(\gamma')\overline{a_k(2\gamma+\gamma')} = 2\delta(j,k)\delta(\gamma,0) \quad (j,k = 0,1; \gamma \in \mathbb{Z}). \tag{3.5}$$

PROOF. By (3.1)

$$
\begin{aligned}
\int \psi_j(x)\overline{\psi_k(x-\gamma)}dx &= \int \sum_{\gamma'\in\mathbb{Z}} a_j(\gamma')\varphi(2x-2\gamma-\gamma')\overline{\sum_{\gamma''\in\mathbb{Z}} a_k(\gamma'')\varphi(2x-\gamma'')}dx \\
&= \sum_{\gamma'\in\mathbb{Z}}\sum_{\gamma''\in\mathbb{Z}} a_j(\gamma')\overline{a_k(\gamma'')}\int \varphi(2x-2\gamma-\gamma')\overline{\varphi(2x-\gamma'')}dx \\
&= \frac{1}{2}\sum_{\gamma'\in\mathbb{Z}}\sum_{\gamma''\in\mathbb{Z}} a_j(\gamma')\overline{a_k(\gamma'')}\int \varphi(y-2\gamma-\gamma')\overline{\varphi(y-\gamma'')}dy \\
&= \frac{1}{2}\sum_{\gamma'\in\mathbb{Z}}\sum_{\gamma''\in\mathbb{Z}} a_j(\gamma')\overline{a_k(\gamma'')}\delta(2\gamma+\gamma',\gamma'') \\
&= \frac{1}{2}\sum_{\gamma'\in\mathbb{Z}} a_j(\gamma')\overline{a_k(2\gamma+\gamma')},
\end{aligned}
$$

so, clearly, (3.4) and (3.5) are equivalent. □

We shall show that finding two sequences $\{a_k(\gamma)\}$, $k = 0, 1$, for which the basic conditions (3.3) and (3.5) hold is essentially sufficient, via (3.1), to obtain ψ_0 and ψ_1 and, hence, a wavelet basis. However, we will not try to solve (3.5) and (3.3) directly. Because of the convolutional structure of most of our identities and the fact that the Fourier transform of a convolution is the *product* of the Fourier transforms of the individual factors, it is much easier to work with Fourier transforms.

For any $f \in L^1(\mathbb{R})$ and $x \in \mathbb{R}$ we define the Fourier transform $\widehat{f}$ of f at x to be

$$\widehat{f}(x) = \int e^{2\pi i x y} f(y)dy \quad (-\infty < x < \infty).$$

The Plancherel Theorem [5], extends the Fourier transform to functions in $L^2(\mathbb{R})$ and includes the relations

$$\hat{\widehat{f}}(x) = f(-x), \quad \|\widehat{f}\|_2 = \|f\|_2 .$$

We now state and prove a lemma regarding orthogonality which will be useful in obtaining the Fourier transform conditions.

LEMMA 3.3. *For any $\varphi \in L^2(\mathbb{R})$, $\{\varphi(x-\gamma)\}_{\gamma\in\mathbb{Z}}$ is an orthonormal set in $L^2(\mathbb{R})$ if and only if*

$$\sum_{\gamma\in\mathbb{Z}} |\widehat{\varphi}(\xi+\gamma)|^2 = 1 \quad \text{a.e.}$$

PROOF. Assume first that $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and put $\widetilde{\varphi}(x) = \overline{\varphi(-x)}, x \in \mathbb{R}$. Since $\widetilde{\varphi} \in L^1(\mathbb{R})$, the convolution $\varphi * \widetilde{\varphi}$ of φ and $\widetilde{\varphi}$, defined by

$$(\varphi * \widetilde{\varphi})(x) = \int \varphi(y)\widetilde{\varphi}(x-y)dy\,,$$

exists for almost all $x \in \mathbb{R}$, and satisfies

$$(\varphi * \widetilde{\varphi})\widehat{\ }(\xi) = \widehat{\varphi}(\xi)\ (\widetilde{\varphi})\widehat{\ }(\xi), \quad \xi \in \mathbb{R}.$$

An easy computation then gives

$$(\varphi * \widetilde{\varphi})\widehat{\ }(\xi) = \widehat{\varphi}(\xi)\overline{\widehat{\varphi}(\xi)} = |\widehat{\varphi}(\xi)|^2, \quad \xi \in \mathbb{R}. \tag{3.6}$$

Now note that the orthonormality of $\{\varphi(x-\gamma)\}_{\gamma\in\mathbb{Z}}$ is equivalent to

$$\int \varphi(y)\overline{\varphi(y+\gamma)}dy = \delta(\gamma,0), \quad \gamma \in \mathbb{Z}.$$

By (3.5), (3.6), the periodicity of e^z, and Fubini's Theorem [5],

$$\begin{aligned}
\int \varphi(y)\overline{\varphi(y+\gamma)}dy &= \int \varphi(y)\widetilde{\varphi}(-\gamma-y)dy \\
&= (\varphi * \widetilde{\varphi})\,(-\gamma) \\
&= (\varphi * \widetilde{\varphi})\widehat{\ }\widehat{\ }(\gamma) \\
&= \int e^{2\pi i\gamma\xi}|\widehat{\varphi}(\xi)|^2 d\xi \\
&= \sum_{n\in\mathbb{Z}} \int_0^1 e^{2\pi i\gamma\xi}|\widehat{\varphi}(\xi+n)|^2 d\xi \\
&= \int_0^1 e^{2\pi i\gamma\xi} \sum_{n\in\mathbb{Z}} |\widehat{\varphi}(\xi+n)|^2 d\xi\,.
\end{aligned}$$

Thus, orthonormality of $\{\varphi(x-\gamma)\}_{\gamma\in\mathbb{Z}}$ is equivalent to

$$(3.7)\qquad \int_0^1 e^{2\pi i\gamma\xi}\sum_{n\in\mathbb{Z}}|\widehat{\varphi}(\xi+n)|^2 d\xi=\delta(\gamma,0),\quad \gamma\in\mathbb{Z}\,.$$

The left-hand side of (3.7) represents the Fourier coefficients of the function $h(\xi)=\sum_{n\in\mathbb{Z}}|\widehat{\varphi}(\xi+n)|^2$ $(0\le\xi\le 1)$, which belongs to $L^1([0,1])$ by Fubini's Theorem, since

$$\sum_{n\in\mathbb{Z}}\int_0^1|\widehat{\varphi}(\xi+n)|^2 d\xi=\int|\widehat{\varphi}(y)|^2 dy<\infty\,.$$

The right-hand side of (3.7) gives the Fourier coefficients of the constant function 1, which also belongs to $L^1([0,1])$. Thus, the two functions are equal a.e. on [0,1], and by periodicity, on $\mathbb{R}$.

For general $\varphi\in L^2(\mathbb{R})$ it is not difficult to approximate φ with elements of $L^1(\mathbb{R})\cap L^2(\mathbb{R})$ and thereby obtain $\int\varphi(y)\overline{\varphi(y+\gamma)}\,dy=\int e^{2\pi i\gamma\xi}|\widehat{\varphi}(\xi)|^2 d\xi$ from which the general result follows by the argument presented above. Thus, the proof is complete. □

We now can obtain the needed Fourier transform conditions. First, recall that the series in (3.1) converges in the L^2 sense. We wish to apply the Fourier transform to the scaling equation (3.1) for $k=0$. Since $\varphi\in L^1(\mathbb{R})$ (as well as $L^2(\mathbb{R})$), we have for each $\xi\in\mathbb{R}$

$$\begin{aligned}\Big(\sum_{|\gamma|\le N}a_k(\gamma)\varphi(2x-\gamma)\Big)\widehat{\ }(\xi)&=\sum_{|\gamma|\le N}a_k(\gamma)\,(\varphi(2x-\gamma))\widehat{\ }(\xi)\\&=\sum_{|\gamma|\le N}a_k(\gamma)\int e^{2\pi i\xi x}\varphi(2x-\gamma)dx\\&=\frac{1}{2}\sum_{|\gamma|\le N}a_k(\gamma)e^{\pi i\xi\gamma}\int e^{2\pi i(\frac{\xi}{2})y}\varphi(y)dy\\&=\frac{1}{2}\sum_{|\gamma|\le N}a_k(\gamma)e^{\pi i\xi\gamma}\widehat{\varphi}(\frac{\xi}{2})\,.\end{aligned}$$

Thus from (3.1) and the continuity of the Fourier transform on $L^2(\mathbb{R})$ we obtain

$$\begin{aligned}\widehat{\psi}_k(\xi) &= \sum_{\gamma\in\mathbb{Z}} a_k(\gamma)e^{\pi i\xi\gamma}\widehat{\varphi}(\frac{\xi}{2}) \quad (k=0,1) \\ &=: A_k(\frac{\xi}{2})\widehat{\varphi}(\frac{\xi}{2}) \quad (k=0,1),\end{aligned} \tag{3.8}$$

where

$$A_k(\xi) = \frac{1}{2}\sum_{\gamma\in\mathbb{Z}} a_k(\gamma)e^{2\pi i\xi\gamma} \quad (k=0,1). \tag{3.9}$$

Here the infinite series representing $\widehat{\psi}_k$ converges in the L^2 sense and the equality in (3.8) holds pointwise a.e. Also, $A_k(\xi) \in L^2([0,1]), k = 0,1$, since the sequence of the coefficients is square summable. The $A_k(\xi)$ will be continuous if these sequences are summable; for example, in our construction $A_k(\xi)$ will reduce to a trigonometric polynomial. Notice that $A_k(\xi)$ is 1-periodic on $\mathbb{R}$.

Condition (3.2) is easily transcribed as

$$A_0(0) = 1. \tag{3.10}$$

However, condition (3.5), which is equivalent to the orthonormality of the set $\{\psi_k(x-\gamma)\}_{\gamma\in\mathbb{Z},k=0,1}$, is easier to rederive in the transform setting than to transcribe. The first step is to prove

PROPOSITION 3.4. *For $\psi_0, \psi_1 \in L^2(\mathbb{R})$*

$$\sum_{\gamma\in\mathbb{Z}} \widehat{\psi}_k(\xi+\gamma)\overline{\widehat{\psi}_j(\xi+\gamma)} = \delta(j,k) \quad \text{a.e.} \qquad (j,k=0,1) \tag{3.11}$$

is equivalent to the orthonormality of $\{\psi_k(x-\gamma)\}_{\gamma\in\mathbb{Z},k=0,1}$.

PROOF. By Parseval 's identity [5], the orthonormality of $\{\psi_k(x-\gamma)\}_{\gamma\in\mathbb{Z},k=0,1}$ is equivalent to

$$\int \widehat{\psi}_k(\xi)\overline{[\psi_j(x-\gamma)]\widehat{\ }(\xi)}d\xi = \delta(j,k)\delta(\gamma,0) \quad (\gamma\in\mathbb{Z}, j,k\in\{0,1\}). \tag{3.12}$$

Fix $j,k \in \{0,1\}$. Then

$$\begin{aligned}
\int \widehat{\psi}_k(\xi)\overline{(\psi_j(x-\gamma))\widehat{\ }(\xi)}d\xi &= \int \widehat{\psi}_k(\xi)\overline{\left\{\int e^{2\pi i\xi y}\psi_j(y+\gamma)dy\right\}}d\xi \\
&= \int \widehat{\psi}_k(\xi)\overline{\left\{e^{-2\pi i\xi\gamma}\int e^{2\pi i\xi z}\psi_j(z)dz\right\}}d\xi \\
&= \int e^{2\pi i\xi\gamma}\widehat{\psi}_k(\xi)\overline{\widehat{\psi}_j(\xi)}d\xi \\
&= \sum_{n\in\mathbb{Z}}\int_0^1 e^{2\pi i\tau\gamma}\widehat{\psi}_k(\tau+n)\overline{\widehat{\psi}_j(\tau+n)}d\tau \\
&= \int_0^1 e^{2\pi i\tau\gamma}\left(\sum_{n\in\mathbb{Z}}\widehat{\psi}_k(\tau+n)\overline{\widehat{\psi}_j(\tau+n)}\right)d\tau\,.
\end{aligned}$$

This last quantity is the γ^{th} Fourier coefficient of the 1-periodic function $h(\tau) = \sum_{n\in\mathbb{Z}}\widehat{\psi}_k(\tau+n)\overline{\widehat{\psi}_j(\tau+n)}$ which belongs to $L^1([0,1])$. This last fact follows from Fubini's Theorem, since the product of two L^2 functions is an L^1 function and, hence,

$$\sum_{n\in\mathbb{Z}}\int_0^1\left|\widehat{\psi}_k(\tau+n)\overline{\widehat{\psi}_j(\tau+n)}\right|d\tau = \int\left|\widehat{\psi}_k\overline{\widehat{\psi}_j}\right|d\tau < \infty\,.$$

Interchanging the order of summation and integration shows that

$$\sum_{n\in\mathbb{Z}}\left|\widehat{\psi}_k(\tau+n)\overline{\widehat{\psi}_j(\tau+n)}\right|$$

converges a.e. and is in $L^1([0,1])$. Because the γ^{th} Fourier coefficient of $h(\tau)$ also is equal to $\delta(j,k)\delta(\gamma,0)$ which is the γ^{th} Fourier coefficient of the $L^1([0,1])$ function identically $\delta(j,k)$, we may conclude that (3.12) is equivalent to (3.11) and the proof of Proposition 3.4 is complete. □

We now can obtain (3.5) in the Fourier transform setting.

PROPOSITION 3.5. *Suppose ψ_0, ψ_1 satisfy (3.1) and $\{\varphi(x-\gamma)\}_{\gamma\in\mathbb{Z}}$ is an orthonormal set, where $\varphi = \psi_0$. Then (3.11) is equivalent to*

$$\sum_{p=1,2} A_k\left(\frac{\xi}{2}+\eta_p\right)\overline{A_j\left(\frac{\xi}{2}+\eta_p\right)} = \delta(j,k) \quad \text{a.e.} \qquad (j,k=0,1), \tag{3.13}$$

where the $A_k(\xi)$ are defined in (3.9) and $\eta_1 = 0,\ \eta_2 = \frac{1}{2}$.

PROOF. Every element of $\mathbb{Z}$ can be written as $2(\gamma + \eta_p)$ for some $\gamma \in \mathbb{Z}$ and $p = 1$ or 2. Thus, since the sum in (3.11) is absolutely convergent for almost all ξ, it may be rewritten as

$$\sum_{p=1,2} \sum_{\gamma \in \mathbb{Z}} \widehat{\psi}_k(\xi + 2(\gamma + \eta_p))\overline{\widehat{\psi}_j(\xi + 2(\gamma + \eta_p))},$$

or, using (3.8), as

$$\sum_{p=1,2} \sum_{\gamma \in \mathbb{Z}} A_k\left(\frac{\xi}{2} + \gamma + \eta_p\right) \widehat{\varphi}\left(\frac{\xi}{2} + \gamma + \eta_p\right) \overline{A_j\left(\frac{\xi}{2} + \gamma + \eta_p\right) \widehat{\varphi}\left(\frac{\xi}{2} + \gamma + \eta_p\right)}$$
$$= \sum_{p=1,2} A_k\left(\frac{\xi}{2} + \eta_p\right) \overline{A_j\left(\frac{\xi}{2} + \eta_p\right)} \sum_{\gamma \in \mathbb{Z}} \left|\widehat{\varphi}\left(\frac{\xi}{2} + \gamma + \eta_p\right)\right|^2 .$$

The orthonormality of $\{\varphi(x - \gamma)\}_{\gamma \in \mathbb{Z}}$ makes the inner sum equal to 1 a.e., so that we finally obtain that the condition (3.11) is equivalent to (3.13) and the proof of Proposition 3.5 is complete. □

The final condition arises from the need for $\{\psi_0(x - \gamma)\}_{\gamma \in \mathbb{Z}}$ to be an orthonormal set. The result we state requires that $A_0(\xi)$ be defined at all points ξ, $|\xi| \le \frac{1}{4}$; this condition holds if, for example, $< a_0(\gamma) > \in \ell^1$, as will be the case in our construction in Chapter 5.

THEOREM 3.6. *Suppose* $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ *satisfies* $\widehat{\varphi}(0) = \int \varphi(x)\,dx = 1$ *and the scaling equation (3.1) with* $k = 0$ *for some sequence* $< a_0(\gamma) >$. *Define* $A_0(\xi)$ *as in (3.9) and suppose that* $< a_0(\gamma) >$ *is such that the infinite product* $\prod_{j=1}^{\infty} A_0(2^{-j}\xi)$ *converges absolutely and uniformly on compact sets. Finally, suppose (3.13) holds for* $j = k = 0$ *and*

$$A_0(\xi) \neq 0 \qquad \text{for} \quad |\xi| \le \frac{1}{4}. \tag{3.14}$$

Then $\{\varphi(x-\gamma)\}_{\gamma\in\mathbb{Z}}$ is an orthonormal set in $L^2(\mathbb{R})$.

PROOF. The idea is to construct a sequence of functions φ_j such that $\{\varphi_j(x-\gamma)\}$, $\gamma\in\mathbb{Z}$, is an orthonormal set for every $j\in\mathbb{Z}$ and $\varphi_j \longrightarrow \varphi$ in $L^2(\mathbb{R})$. Then $\{\varphi(x-\gamma)\}_{\gamma\in\mathbb{Z}}$ will also be an orthonormal set.

We start with φ_0 such that $\widehat{\varphi}_0(\xi)=\chi_{[-1/2,1/2]}(\xi)$ (i.e., $\varphi_0(x)=\sin\pi x/\pi x$). Then, for every integer $j\geq 1$, we define φ_j by the requirement

$$\widehat{\varphi}_j(\xi)=A_0(\frac{\xi}{2})\widehat{\varphi}_{j-1}(\frac{\xi}{2}).$$

We will use induction to prove the orthonormality of the set $\{\varphi_j(x-\gamma)\},\gamma\in\mathbb{Z}$, for each $j\geq 0$.

Since $\sum_{\gamma\in\mathbb{Z}}|\widehat{\varphi}_0(\xi+\gamma)|^2=1$ a.e., it follows from Lemma 3.3 that $\{\varphi_0(x-\gamma)\}$, $\gamma\in\mathbb{Z}$, is orthonormal. By the same lemma, we can show that for fixed $j\geq 1$ the set $\{\varphi_j(x-\gamma)\}$, $\gamma\in\mathbb{Z}$, is orthonormal by showing

$$\sum_{\gamma\in\mathbb{Z}}|\widehat{\varphi}_j(\xi+\gamma)|^2=1 \quad \text{a.e.} \tag{3.15}$$

Since each $\widehat{\varphi}_j$ can be written as

$$\widehat{\varphi}_j(\xi)=A_0(\tfrac{1}{2}\xi)A_0(\tfrac{1}{4}\xi)\cdots A_0(2^{-j}\xi)\chi_{(-1/2,1/2]}(2^{-j}\xi),$$

it vanishes except possibly for $\xi\in[-2^{j-1},2^{j-1}]$. Since $\mathbb{Z}$ has measure zero, we may consider only non-integer ξ. For every such ξ there are exactly 2^j successive integer translates of ξ in $(-2^{j-1},2^{j-1}]$. Let λ be the integer translate of ξ in $(-2^{j-1},-2^{j-1}+1)$. Then (3.15) is equivalent to

$$\sum_{\gamma=0}^{2^j-1}|\widehat{\varphi}_j(\lambda+\gamma)|^2=1 \quad \text{a.e.}$$

Now fix j, and assume $\{\varphi_{j-1}(x-\gamma)\}_{\gamma\in\mathbb{Z}}$ is orthonormal, i.e., that

$$\sum_{\gamma\in\mathbb{Z}} |\widehat{\varphi}_{j-1}(\xi+\gamma)|^2 = 1 \quad \text{a.e.}$$

Since A_0 has period 1, we have by the induction hypothesis and (3.13) that for almost all ξ

$$\begin{aligned}
\sum_{\gamma\in\mathbb{Z}} |\widehat{\varphi}_j(\xi+\gamma)|^2 &= \sum_{\gamma=0}^{2^j-1} |\widehat{\varphi}_j(\lambda+\gamma)|^2 \\
&= \sum_{\mu=0}^{2^{j-1}-1} |\widehat{\varphi}_j(\lambda+2\mu)|^2 + \sum_{\nu=0}^{2^{j-1}-1} |\widehat{\varphi}_j(\lambda+2\nu+1)|^2 \\
&= \sum_{\mu=0}^{2^{j-1}-1} |A_0(\frac{\lambda}{2}+\mu)|^2 |\widehat{\varphi}_{j-1}(\frac{\lambda}{2}+\mu)|^2 \\
&\qquad + \sum_{\nu=0}^{2^{j-1}-1} |A_0(\frac{\lambda}{2}+\frac{1}{2}+\nu)|^2 |\widehat{\varphi}_{j-1}(\frac{\lambda}{2}+\frac{1}{2}+\nu)|^2 \\
&= |A_0(\frac{\lambda}{2})|^2 \sum_{\mu=0}^{2^{j-1}-1} |\widehat{\varphi}_{j-1}(\frac{\lambda}{2}+\mu)|^2 \\
&\qquad + |A_0(\frac{\lambda}{2}+\frac{1}{2})|^2 \sum_{\nu=0}^{2^{j-1}-1} |\widehat{\varphi}_{j-1}(\frac{\lambda}{2}+\frac{1}{2}+\nu)|^2 \\
&= |A_0(\frac{\lambda}{2})|^2 + |A_0(\frac{\lambda}{2}+\frac{1}{2})|^2 \\
&= 1\,.
\end{aligned}$$

Thus, $\{\widehat{\varphi}_j(x-\gamma)\}_{\gamma\in\mathbb{Z}}$ is orthonormal.

To show that $\varphi_j \longrightarrow \varphi$ in $L^2(\mathbb{R})$, it is sufficient, by Parseval's identity, to show instead that $\widehat{\varphi}_j \longrightarrow \widehat{\varphi}$ in $L^2(\mathbb{R})$. First we note that the functions $\widehat{\varphi}_j$ can be written as

$$\widehat{\varphi}_j(\xi) = \left(\prod_{k=1}^{j} A_0(2^{-k}\xi)\right) \chi_{[-2^{j-1},2^{j-1}]}(\xi)\,. \tag{3.16}$$

On the other hand iteration of (3.8) for $k = 0$, $\widehat{\varphi}(0) = 1$, and the continuity of $\widehat{\varphi}$ give

$$\widehat{\varphi}(\xi) = \prod_{j=1}^{\infty} A_0(2^{-j}\xi)\,. \tag{3.17}$$

(Convergence of this product is discussed after the proof is completed.) Hence $\widehat{\varphi}_j$ converges pointwise to $\widehat{\varphi}$.

Now we use the Dominated Convergence Theorem [5] to complete the proof that $\widehat{\varphi}_j \longrightarrow \widehat{\varphi}$ in $L^2(\mathbb{R})$. First, $|\widehat{\varphi}|^2 \in L^1(\mathbb{R})$. Also,

$$\widehat{\varphi}_j(\xi) = \begin{cases} \dfrac{\widehat{\varphi}(\xi)}{\widehat{\varphi}(2^{-j}\xi)} & \text{if } |\xi| \le 2^{j-1} \\ 0 & \text{otherwise} \end{cases} \tag{3.18}$$

(compare (3.16) and (3.17)). Next note that if $\xi \in (-1/2, 1/2)$, (3.14) ensures that each factor $A_0(2^{-j}\xi) \neq 0$ and therefore that the denominator $\widehat{\varphi}_j(2^{-j}\xi) = \prod_{k=1}^{\infty} A_0(2^{-(j+k)}\xi)$ in (3.18) is never zero. Thus, since $\widehat{\varphi}$ is a continuous function, $c := \left(\inf_{(-1/2,1/2]} |\widehat{\varphi}|\right)^{-1}$ is finite, and we can deduce from (3.18) that $|\widehat{\varphi}_j(\xi)| \le c|\widehat{\varphi}(\xi)|$. It follows that $|\widehat{\varphi}_j(\xi) - \widehat{\varphi}(\xi)|^2 \le (c+1)^2|\widehat{\varphi}(\xi)|^2$. Thus, the Dominated Convergence Theorem yields

$$\int |\widehat{\varphi}_j(\xi) - \widehat{\varphi}(\xi)|^2 d\xi \longrightarrow 0\,,$$

so that $\widehat{\varphi}_j \longrightarrow \widehat{\varphi}$ in $L^2(\mathbb{R})$ and the proof is complete. $\square$

Remark. If only finitely many of the $a_0(\gamma)$ are nonzero, as will be the case in our construction, then the infinite product in (3.17) converges absolutely and uniformly on compact sets. To establish this claim, note first that the relation $A_0(0) = 1 = \frac{1}{2}\sum_{\gamma\in\mathbb{Z}} a_0(\gamma)$ (a finite sum now), the mean value theorem and the double-angle formulas give for some $C < \infty$

$$\begin{aligned}
|A_0(\xi)| =& \left| \frac{1}{2} \sum_{\gamma \in \mathbb{Z}} a_0(\gamma) e^{2\pi i \gamma \xi} \right| \\
\leq& 1 + \left| \frac{1}{2} \sum_{\gamma \in \mathbb{Z}} a_0(\gamma)(e^{2\pi i \gamma \xi} - 1) \right| \\
\leq& 1 + \frac{1}{2} \sum_{\gamma \in \mathbb{Z}} |a_0(\gamma)|(|2 \sin^2 \pi i \gamma \xi| + |2 \sin \pi i \gamma \xi \, \cos \pi i \gamma \xi|) \\
\leq& 1 + 2 \sum_{\gamma \in \mathbb{Z}} |a_0(\gamma)| \, |\pi i \gamma \xi| \\
\leq& 1 + C|\xi| \\
\leq& e^{C|\xi|} .
\end{aligned}$$

Thus,

$$\begin{aligned}
|\widehat{\varphi}(\xi)| =& \prod_{j=1}^{\infty} |A_0(2^{-j}\xi)| \\
\leq& \exp\left\{ C \sum_{j=1}^{\infty} 2^{-j} |\xi| \right\} \\
=& e^{C|\xi|} .
\end{aligned}$$

Similar estimates can be obtained if infinitely many of the $a_0(\gamma)$ are nonzero, but decay sufficiently rapidly, [1].

Chapter 4

Sufficiency of the Conditions

In this chapter we show that the key conditions (3.9), (3.12) and (3.13) lead to a MRA and to a function ψ_1 for which the set

$$\{2^{\frac{j}{2}}\psi_1(2^jx-\gamma)\}_{\gamma\in\mathbb{Z}} \tag{4.1}$$

is an ONB for $L^2(\mathbb{R})$. We impose rather strong simlifying assumptions, but they will hold in our construction in the next chapter.

Suppose $< a_k(\gamma) >$, $k = 0,1$ are sequences in ℓ^2 for which the associated functions A_k in (3.9) satisfy

$$A_0(0)=1, \tag{4.2}$$

$$\sum_{p=1,2} A_k\left(\frac{\xi}{2}+\eta_p\right)\overline{A_j\left(\frac{\xi}{2}+\eta_p\right)}=\delta(j,k) \quad \text{a.e.} \quad (j,k=0,1), \tag{4.3}$$

$$A_0(\xi)\neq 0 \quad \text{for } |\xi|\leq \frac{1}{4}. \tag{4.4}$$

(Of course, these conditions are precisely (3.9), (3.12), and (3.13); in particular, $\eta_1=0, \eta_2=\frac{1}{2}$ in (4.3).) Suppose further that $< a_0(\gamma) >$ is such that

$$\prod_{j=1}^{\infty} A_0(2^{-j}\xi) \quad \text{converges absolutely and uniformly on compact sets.} \tag{4.5}$$

Finally, suppose

$$\begin{cases} \psi_0=\varphi & \text{is a bounded, measurable function of compact} \\ & \text{support which satisfies (3.1) with k=0, and} \\ & \widehat{\varphi}(0)=\int \varphi\, dx = 1. \end{cases} \tag{4.6}$$

We want to associate ψ_0 with a MRA. By Theorem 3.2, $\{\psi_0(x-\gamma)\}_{\gamma\in\mathbb{Z}}$ is an orthonormal set. Let V_0 be the closed linear span in $L^2(\mathbb{R})$ of these functions. Also, for each $j \in \mathbb{Z}$, let V_j be the *closed* subspace of $L^2(\mathbb{R})$ defined by

$$f(x) \in V_j \Longleftrightarrow f(2^{-j}x) \in V_0 .$$

Clearly, conditions (iii) and (iv) of the definition of a MRA are satisfied by the ladder of closed subspaces $\cdots \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots$ and the scaling function $\psi_0 = \varphi$ (Definition 3.1.) We must show that conditions (i) and (ii) of the definition also hold.

If $f(x) \in V_0$ then

$$\begin{aligned} f(x) &= \sum_{\gamma\in\mathbb{Z}} \int f(y)\overline{\varphi(y-\gamma)}\,dy\,\varphi(x-\gamma) \\ &= \int K(x,y)f(y)\,dy , \end{aligned}$$

where $K(x,y) := \sum_{\gamma\in\mathbb{Z}} \varphi(x-\gamma)\overline{\varphi(y-\gamma)}$. Since φ has compact support, each of these sums is actually a finite sum. Clearly, then, for each fixed x, we have by the orthonormality of the translates of φ that $\int |K(x,y)|^2\,dy = \sum_{\gamma\in\mathbb{Z}} |\varphi(x-\gamma)|^2$. In addition, since φ is bounded with compact support, $\sum_{\gamma\in\mathbb{Z}} |\varphi(x-\gamma)|^2$ is bounded *uniformly* in x. Therefore by Hölder's inequality [5]

$$|f(x)| \leq \left(\sum_{\gamma\in\mathbb{Z}} |\varphi(x-\gamma)|^2\right)^{1/2} \|f\|_2 \leq C\|f\|_2$$

for some finite C. Condition (ii) now follows from the following general lemma, which does not require that the subspaces V_j form a ladder.

LEMMA 4.1. *Let V_0 be a subspace of $L^2(\mathbb{R})$ which is contained in $L^\infty(\mathbb{R})$ and which has the property that*

$$\|f\|_\infty \leq C\|f\|_2 \quad \textit{for all } f \in V_0 .$$

Then, if V_j is defined by the scaling condition (iii), the separation condition $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ holds.

PROOF. Fix $j \in \mathbb{Z}$, let $f \in V_j$ and define $g(x) = f(2^{-j}x)$, $x \in \mathbb{R}$. Then $g \in V_0$ and

$$\|f\|_\infty = \|g\|_\infty \leq C\|g\|_2 = C2^{j/2}\|f\|_2 .$$

Thus, if $f \in \bigcap_{j\in\mathbb{Z}} V_j$, then taking the limit as $j \to -\infty$ yields $\|f\|_\infty = 0$ and therefore $f = 0$. □

Weaker conditions on φ also yield the separation condition (ii) [1], but the φ we construct will be bounded with compact support. Only the latter assumption is needed to obtain the density condition.

LEMMA 4.2. *Assume φ is as in (4.6) (but not necessarily bounded). Then the density condition $\overline{\bigcup_j V_j} = L^2(\mathbb{R})$ holds.*

SKETCH OF PROOF. Fix $f \in L^2(\mathbb{R})$. For each $j \in \mathbb{Z}$

$$P_j f(x) = 2^j \sum_{\gamma\in\mathbb{Z}} \varphi(2^j x - \gamma) \int f(y)\overline{\varphi(2^j y - \gamma)}dy$$

is the orthogonal projection of f onto V_j. Since $V_j \subseteq V_{j+1}$ for all $j \in \mathbb{Z}$, it suffices to show that $\|f - P_j f\|_2 \longrightarrow 0$ as $j \to \infty$. But $P_j f \perp (f - P_j f)$, so the Pythagorean theorem yields $\|f\|_2^2 = \|(f - P_j f) + P_j f\|_2^2 = \|(f - P_j f)\|_2^2 + \|P_j f\|_2^2$, and we may equivalently show that $\|P_j f\|_2 \longrightarrow \|f\|_2$ as $j \to \infty$. Moreover, by a standard density argument, it suffices to consider only $f = \chi_A$, where A may be any finite interval of $\mathbb{R}$.

For such f note that

$$\begin{aligned}\|P_j f\|_2^2 &= 2^j \sum_{\gamma\in\mathbb{Z}} \left| \int_A \varphi(2^j y - \gamma)dy \right|^2 \\ &= 2^{-j} \sum_{\gamma\in\mathbb{Z}} \left| \int_{2^j A} \varphi(y - \gamma)dy \right|^2 .\end{aligned}$$

As $j \to \infty$, the interval $2^j A$ becomes very large, so except for integers γ "near" the endpoints of $2^j A$, we have that if $\gamma \in 2^j A$, then supp $\varphi(y-\gamma) \subseteq 2^j A$ and if $\gamma \notin 2^j A$, supp $\varphi(y-\gamma) \cap 2^j A = \varnothing$. Thus for large j,

$$\begin{aligned} \|P_j f\|_2^2 &\approx 2^{-j} \sum_{\gamma\in\mathbb{Z}, \gamma\in 2^j A} \left| \int_{2^j A} \varphi(y-\gamma) dy \right|^2 \\ &\approx 2^{-j} \#\{\gamma \in 2^j A\} \approx \text{length}(A) \\ &= \|\chi_A\|_2^2 , \end{aligned}$$

and as $j \to \infty$, $\|P_j f\|_2 \longrightarrow \|f\|_2$, for $f = \chi_A$. $\square$

This result completes the task of showing that there is a MRA with φ as its scaling function.

Now define $\psi_1 = \psi$ by (3.1) with $k = 1$. (Since $< a_1(\gamma) >$ is in ℓ^2, $\psi_1 \in L^2(\mathbb{R})$) Our next goal is to obtain this chapter's main result which establishes the set in (4.1) as an ONB of $L^2(\mathbb{R})$.

First, recall that $\psi_0 = \varphi$ and $\psi_1 = \psi$ satisfy (3.1) and that $\{\varphi_1(x-\gamma)\}_{\gamma\in\mathbb{Z}}$ is an orthonormal set. Since (4.3) is the same as (3.12), it follows from Propositions 3.2, 3.4, and 3.5 that $\{\psi_k(x-\gamma)\}_{\gamma\in\mathbb{Z},k=0,1}$ also is an orthonormal set in $L^2(\mathbb{R})$.

Next we establish

PROPOSITION 4.3. *The set* $\{\psi_k(x-\gamma)\}_{\gamma\in\mathbb{Z},k=0,1}$ *is a basis of* V_1.

PROOF. By the fact that $\{2^{\frac{1}{2}}\psi_0(2x-\gamma)\}$, $\gamma \in \mathbb{Z}$, is a basis for V_1, and the scaling property, it is sufficient to establish for fixed $\tilde{\gamma} \in \mathbb{Z}$ the representation

$$\psi_0(2x-\tilde{\gamma}) = \sum_{k=0,1} \sum_{\gamma\in\mathbb{Z}} c_{k,\gamma} \psi_k(x-\gamma), \tag{4.7}$$

where $c_{k,\gamma}$ will be the Fourier coefficients of $\psi_0(2x-\tilde{\gamma})$.

It suffices to show (4.7) holds only for $\widetilde{\gamma}=0,1$. For if $\widetilde{\gamma}=2\lambda$, $\lambda\in\mathbb{Z}$, and (4.7) holds for $\widetilde{\gamma}=0$, then for $z=x-\lambda$

$$\begin{aligned}\psi_0(2x-\widetilde{\gamma})=\psi_0(2z)&=\sum_{k=0,1}\sum_{\gamma\in\mathbb{Z}}c_{k,\gamma}\psi_k(z-\gamma)\\&=\sum_{k=0,1}\sum_{\gamma\in\mathbb{Z}}c_{k,\gamma}\psi_k(x-(\gamma+\lambda)\\&=\sum_{k=0,1}\sum_{n\in\mathbb{Z}}b_{k,n}\psi_k(x-n)\end{aligned}$$

where $b_{k,n}=c_{k,n-\lambda}$. Similarly if (4.7) holds for $\widetilde{\gamma}=1$, it will also hold for any odd $\widetilde{\gamma}$ in $\mathbb{Z}$.

The coefficients $c_{k,\gamma}$ in (4.7) are given by

$$\begin{aligned}c_{k,\gamma}&=\int\varphi(2x-\widetilde{\gamma})\overline{\psi_k(x-\gamma)}dx\\&=\int\varphi(2x-\widetilde{\gamma})\sum_{\gamma'\in\mathbb{Z}}\overline{a_k(\gamma')\varphi(2x-2\gamma-\gamma')}dx\\&=\sum_{\gamma'\in\mathbb{Z}}\overline{a_k(\gamma')}\int\varphi(2x-\widetilde{\gamma})\overline{\varphi(2x-2\gamma-\gamma')}dx\\&=\sum_{\gamma'\in\mathbb{Z}}\overline{a_k(\gamma')}\frac{1}{2}\delta(\widetilde{\gamma},2\gamma+\gamma')\\&=\frac{1}{2}\overline{a_k(\widetilde{\gamma}-2\gamma)}.\end{aligned}$$

Thus, we must show that for $\widetilde{\gamma}=0,1$

$$\psi_0(2x-\widetilde{\gamma})=\frac{1}{2}\sum_{k=0,1}\sum_{\gamma\in\mathbb{Z}}\overline{a_k(\widetilde{\gamma}-2\gamma)}\psi_k(x-\gamma)$$

in $L^2(\mathbb{R})$ or, by (3.1), that

$$\psi_0(2x-\widetilde{\gamma})=\frac{1}{2}\sum_{k=0,1}\sum_{\gamma\in\mathbb{Z}}\overline{a_k(\widetilde{\gamma}-2\gamma)}\sum_{\gamma'\in\mathbb{Z}}a_k(\gamma')\varphi(2x-2\gamma-\gamma').$$

Changing indices in the right-hand side of the last equation yields

$$
\begin{aligned}
&\frac{1}{2}\sum_{k=0,1}\sum_{u'\in\mathbb{Z}}\overline{a_k(\widetilde{\gamma}+2u')}\sum_{\gamma'\in\mathbb{Z}}a_k(\gamma')\varphi(2x-(\gamma'-2u'))\\
&\quad=\frac{1}{2}\sum_{k=0,1}\sum_{u'\in\mathbb{Z}}\overline{a_k(\widetilde{\gamma}+2u')}\sum_{u\in\mathbb{Z}}a_k(u+2u')\varphi(2x-u)\\
&\quad=\sum_{\gamma\in\mathbb{Z}}\left(\frac{1}{2}\sum_{k=0,1}\sum_{\gamma'\in\mathbb{Z}}\overline{a_k(\widetilde{\gamma}+2\gamma')}a_k(\gamma+2\gamma')\right)\varphi(2x-\gamma),
\end{aligned}
$$

which simplifies to $\varphi(2x-\widetilde{\gamma})$, if we establish

LEMMA 4.4. *The sequences* $\langle a_k(\gamma)\rangle$, $k=0,1$, *satisfy for* $\widetilde{\gamma}=0,1$,

$$
\sum_{k=0,1}\sum_{\gamma'\in\mathbb{Z}}\overline{a_k(\widetilde{\gamma}+2\gamma')}a_k(\gamma+2\gamma')=2\delta(\gamma,\widetilde{\gamma}). \tag{4.8}
$$

PROOF. We prove this algebraic statement in the Fourier transform setting. For $\xi\in\mathbb{R}$ fixed, and $\eta_1=0$, $\eta_2=\frac{1}{2}$, consider the matrix

$$
\begin{bmatrix} A_0(\xi+\eta_1) & A_0(\xi+\eta_2)\\ A_1(\xi+\eta_1) & A_1(\xi+\eta_2)\end{bmatrix}.
$$

Condition (4.8) implies that for almost all ξ the ℓ^2 norm of each row is 1, while the inner product of the two rows is zero, and hence the matrix is unitary by rows. Equivalently the matrix must be unitary by columns, so that for $p,q\in\{0,1\}$ and for almost all ξ

$$
\sum_{k=0,1}A_k(\xi+\eta_p)\overline{A_k(\xi+\eta_q)}=\delta(p,q). \tag{4.9}
$$

By (3.9), equation (4.9) can be written as

$$
\sum_{k=0,1}\left(\frac{1}{2}\sum_{\gamma\in\mathbb{Z}}a_k(\gamma)e^{2\pi i\gamma(\xi+\eta_p)}\right)\left(\frac{1}{2}\sum_{\gamma'\in\mathbb{Z}}\overline{a_k(\gamma')}e^{-2\pi i\gamma'(\xi+\eta_q)}\right)=\delta(p,q).
$$

Rewriting the product of the two sums over the integers as an iterated sum (with the outer sum over γ'), replacing γ by $\gamma+\gamma'$ and then interchanging the order of summation, we obtain

$$\sum_{\gamma\in\mathbb{Z}}\left(\frac{1}{4}\sum_{k=0,1}\sum_{\gamma'\in\mathbb{Z}} a_k(\gamma+\gamma')\overline{a_k(\gamma')}e^{2\pi i(\gamma+\gamma')(\xi+\eta_p)}e^{-2\pi i\gamma'(\xi+\eta_q)}\right)=\delta(p,q),$$

which simplifies to

$$\sum_{\gamma\in\mathbb{Z}}\left(\frac{1}{4}\sum_{k=0,1}\sum_{\gamma'\in\mathbb{Z}} a_k(\gamma+\gamma')\overline{a_k(\gamma')}e^{2\pi i\gamma\eta_p}e^{2\pi i\gamma'(\eta_p-\eta_q)}\right)e^{2\pi i\gamma\xi}=\delta(p,q).$$

We note that since $\langle a_0(\gamma')\rangle$ and $\langle a_1(\gamma')\rangle$ are in ℓ^2, for fixed γ the term in parentheses in the left-hand side is absolutely convergent.

Viewing the last equation as a Fourier series expansion for $\delta(p,q)$, we see that if $p\neq q$, so that $\delta(p,q)=0$, then all the coefficients in the above sum must be zero, and if $p=q$ so that $\delta(p,q)=1$, then the only non-zero coefficient will be the one corresponding to $\gamma=0$, and it will be equal to 1. We may summarize these results as

$$\frac{1}{4}\sum_{k=0,1}\sum_{\gamma'\in\mathbb{Z}} a_k(\gamma+\gamma')\overline{a_k(\gamma')}e^{2\pi i\gamma\eta_p}e^{2\pi i\gamma'(\eta_p-\eta_q)}=\delta(p,q)\delta(\gamma,0). \tag{4.10}$$

Note that

$$\sum_{q=1,2} e^{-2\pi i\mu\eta_q}=\begin{cases}2 & \text{if } \mu\in 2\mathbb{Z}\\ 0 & \text{otherwise.}\end{cases} \tag{4.11}$$

Thus if we set $p=1$ so that $\eta_p=0$ in (4.10) and we take the sum over q, we obtain

$$\frac{1}{4}\sum_{q=1,2}\sum_{k=0,1}\sum_{\gamma'\in\mathbb{Z}} a_k(\gamma+\gamma')\overline{a_k(\gamma')}e^{-2\pi i\gamma'\eta_q}=\sum_{q=1,2}\delta(p,q)\delta(\gamma,0)$$

which, when we use (4.11) and put $\gamma' = 2\lambda$, becomes the case $\widetilde{\gamma} = 0$ of (4.8):

$$\sum_{k=0,1} \sum_{\lambda\in\mathbb{Z}} a_k(2\lambda+\gamma)\overline{a_k(2\lambda)} = 2\delta(\gamma,0)\,.$$

To obtain the case $\widetilde{\gamma} = 1$ of (4.8), we set $p = 2$ so that $\eta_p = 1/2$ in (4.10), multiply (4.10) by $e^{2\pi i\eta_q}$ and then sum over q. We obtain

$$\begin{aligned}\frac{1}{4} \sum_{k=0,1} \sum_{\gamma'\in\mathbb{Z}} a_k(\gamma+\gamma')\overline{a_k(\gamma')}e^{\pi i(\gamma+\gamma')} \sum_{q=1,2} e^{2\pi i(1-\gamma')\eta_q} \\ = \sum_{q=1,2} e^{2\pi i\eta_q}\delta(p,q)\delta(\gamma,0) \\ = -\,\delta(\gamma,0)\end{aligned}$$

By (4.11) $\sum_{q=1,2} e^{-2\pi i(1-\gamma')\eta_q} = 2$ for $\gamma' = 2\lambda+1$ for $\lambda\in\mathbb{Z}$, (and zero otherwise). Setting $\gamma' = 2\lambda+1$ for $\lambda \in \mathbb{Z}$, we get

$$2\frac{1}{4} \sum_{k=0,1} \sum_{\lambda\in\mathbb{Z}} a_k(2\lambda+1+\gamma)\overline{a_k(2\lambda+1)}e^{\pi i(\gamma+2\lambda+1)} = -\delta(\gamma,0).$$

For $\nu := \gamma+1$ the latter equation becomes

$$\sum_{k=0,1} \sum_{\lambda\in\mathbb{Z}} a_k(2\lambda+\nu)\,\overline{a_k(2\lambda+1)}e^{\pi i(\nu-1)} = 2\delta(\nu,1),$$

which is clearly equivalent to (4.8) with $\widetilde{\gamma} = 1$. □

Therefore, $\{\psi_k(x-\gamma)\}$, $\gamma\in\mathbb{Z}$, $k = 0,1$ is an ONB of V_1, and we are now in a position to show that the set of functions $\left\{2^{j/2}\psi_1(2^j x-\gamma)\right\}_{\gamma\in\mathbb{Z},j\in\mathbb{Z}}$ is indeed an ONB of $L^2(\mathbb{R})$.

THEOREM 4.5. *Suppose (4.2)-(4.6) hold and* ψ_1 *is determined by (3.1) with* $k = 1$. *Then the collection of sets in (4.1) is an ONB of* $L^2(\mathbb{R})$.

PROOF. We have shown that φ generates a MRA $\cdots \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots$. We also have that $\{\psi_0(x-\gamma)\}_{\gamma\in\mathbb{Z}}$ is an ONB of V_0, and $\{\psi_k(x-\gamma)\}_{\gamma\in\mathbb{Z},k=0,1}$ is an ONB of V_1. Let W_0 denote the orthogonal complement of V_0 in V_1. Then $V_1 = V_0 \oplus W_0$ and $\{\psi_1(x-\gamma)\}_{\gamma\in\mathbb{Z}}$ must be an ONB for W_0. Since $\{V_j\}_{j\in\mathbb{Z}}$ is a MRA, the scaling identity ($f(x) \in V_j \Leftrightarrow f(2^{-j}x) \in V_0$) and the relation

$$\begin{aligned}\int 2^{j/2}\psi_1(2^j x-\gamma)\overline{\psi_1(2^j x)}dx &= \int \psi_1(y-\gamma)\overline{\psi_1(y)}dy \\ &= \delta(\gamma,0)\end{aligned}$$

imply that $\left\{2^{j/2}\psi_1(2^j x-\gamma)\right\}_{\gamma\in\mathbb{Z}}$ is an ONB for W_j, where W_j is the orthogonal complement of V_j in V_{j+1} —i.e., $V_{j+1} = V_j \oplus W_j$.

The closed subspaces W_j are mutually orthogonal, since if $k < j$, then $W_k \subseteq V_j$ and V_j is orthogonal to W_j. In addition, each W_j is orthogonal to each V_k, $k \leq j$. As a result we have for $k < j$,

$$V_j = V_k \oplus \bigoplus_{m=0}^{j-k-1} W_{k+m}\,.$$

Now it follows from the density and separation properties of a multiresolution analysis that $L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j$ and, hence, the combined set

$$\left\{2^{j/2}\psi_1(2^j x-\gamma)\right\}_{\gamma\in\mathbb{Z},j\in\mathbb{Z}},$$

consisting of the orthonormal bases of all the W_j, is an orthonormal basis of $L^2(\mathbb{R})$. □

A basis for $L^2(\mathbb{R})$ of the form (4.1) is called a wavelet basis. Usually ψ_1 (and, hence, each basis element) has compact support, so that the basis is local.

Chapter 5

Finding Wavelets

Finding wavelets depends on finding functions

$$A_k(\xi) = \frac{1}{2} \sum_{\gamma \in \mathbb{Z}} a_k(\gamma) e^{2\pi i \gamma \xi}, \quad k = 0, 1,$$

which satisfy the three basic conditions we derived in a previous section, namely (4.2), (4.3) and (4.4). Since substituting (3.17) into (3.8) yields

$$\widehat{\psi}_k(\xi) = A_k(\tfrac{1}{2}\xi) \prod_{j=2}^{\infty} A_0(2^{-j}\xi) \quad (k = 0, 1), \tag{5.1}$$

the wavelets are completly determined by the functions A_k.

We begin by finding A_0 of the form

$$A_0(\xi) = \frac{1}{2} \sum_{\gamma \in \mathbb{Z}} a_0(\gamma) e^{2\pi i \gamma \xi} \quad \text{(finite sum)}$$

satisfying (4.2), (4.4) and (4.3) for $j = k = 0$, which then reads

$$|A_0(\xi)|^2 + |A_0(\xi + \tfrac{1}{2})|^2 = 1\,. \tag{5.2}$$

It is easy to check that

$$A_0(\xi) = \frac{1}{2}\left(1 + e^{2\pi i \xi}\right) = e^{\pi i \xi} \cos \pi \xi$$

is a solution: (4.2) and (4.4) are obvious, while

$$|e^{\pi i \xi} \cos \pi \xi|^2 + |e^{\pi i (\xi + \frac{1}{2})} \cos(\pi(\xi + \tfrac{1}{2}))|^2 = (\cos \pi \xi)^2 + (-\sin \pi \xi)^2 = 1$$

establishes (5.2). Unfortunately this solution corresponds to the Haar wavelets, which as we have seen are not continuous. To correct this deficiency we might try to take $A_0(\xi) = \left(e^{\pi i \xi}\cos\pi\xi\right)^N$, since products of Fourier transforms correspond to convolutions, and convolutions typically increase smoothness. However, then (5.2) would not hold. The reason why (5.2) was so easily proved in the Haar case lies in the fact that $\cos(\pi(\xi+\frac{1}{2})) = -\sin\pi\xi$ so that $|\cos\pi\xi|^2 + |\cos(\pi(\xi+\frac{1}{2}))|^2 = 1$. Trying to make use of these identities motivates the next "guesswork" steps, which are due to Daubechies [1].

First we raise $\cos^2\pi\xi + \sin^2\pi\xi = 1$ to an odd power, in this case 5:

$$\begin{aligned} 1 &= \left(\cos^2\pi\xi + \sin^2\pi\xi\right)^5 \\ &= \cos^{10}\pi\xi + 5\cos^8\pi\xi\sin^2\pi\xi + 10\cos^6\pi\xi\sin^4\pi\xi \\ &\quad + 10\cos^4\pi\xi\sin^6\pi\xi + 5\cos^2\pi\xi\sin^8\pi\xi + \sin^{10}\pi\xi\,. \end{aligned}$$

Next we let $|A_0(\xi)|^2$ to be the first half of the above trigonometric polynomial:

$$|A_0(\xi)|^2 = \cos^{10}\pi\xi + 5\cos^8\pi\xi\sin^2\pi\xi + 10\cos^6\pi\xi\sin^4\pi\xi\,. \tag{5.3}$$

Then

$$\begin{aligned} |A_0(\xi+\tfrac{1}{2})|^2 &= \cos^{10}\pi(\xi+\tfrac{1}{2}) + 5\cos^8\pi(\xi+\tfrac{1}{2})\sin^2\pi(\xi+\tfrac{1}{2}) \\ &\quad + 10\cos^6\pi(\xi+\tfrac{1}{2})\sin^4\pi(\xi+\tfrac{1}{2}) \\ &= \sin^{10}\pi\xi + 5\cos^2\pi\xi\sin^8\pi\xi + 10\cos^4\pi\xi\sin^6\pi\xi, \end{aligned}$$

which is the second half of our trigonometric polynomial. Hence for this choice of $|A_0(\xi)|^2$, condition (5.2) follows immediately, while (4.4) is obviously true.

It remains to show that there is a trigonometric polynomial $A_0(\xi)$ for which (5.3) holds. Moreover, we ask that the coefficients $a_0(\gamma)$ be real, in order to get real-valued ψ_k's. Since the right hand side of (5.3) is a positive trigonometric polynomial and an even function, a result of F. Riesz (cf. [1, p.172]) guarantees that there is a trigonometric polynomial with real coefficients for which (5.3) holds. In our particular case it is possible to determine $A_0(\xi)$ by inspection and we now do so. Note that

$$\begin{aligned}|A_0(\xi)|^2 &= \cos^6\pi\xi\left(\cos^4\pi\xi + 5\cos^2\pi\xi\sin^2\pi\xi + 10\sin^4\pi\xi\right)\\ &= \cos^6\pi\xi\left((\cos^2\pi\xi - \sqrt{10}\sin^2\pi\xi)^2 + (5+2\sqrt{10})\cos^2\pi\xi\sin^2\pi\xi\right).\end{aligned}$$

Now take $A_0(\xi)$ to be

$$\begin{aligned}A_0(\xi) &= \left(e^{\pi i\xi}\cos\pi\xi\right)^3\left\{\cos^2\pi\xi - \sqrt{10}\sin^2\pi\xi + i\sqrt{5+2\sqrt{10}}\cos\pi\xi\sin\pi\xi\right\}\\ &= \frac{1}{8}\left(e^{2\pi i\xi}+1\right)^3\left\{(\frac{1}{2}+\frac{1}{2}\cos^2\pi\xi - \frac{1}{2}\sqrt{10}\sin^2\pi\xi)\right.\\ &\qquad\left. + \frac{\sqrt{10}}{2}(-1+\cos^2\pi\xi - \sin^2\pi\xi)\frac{1}{2}i\sqrt{5+2\sqrt{10}}\sin(2\pi\xi)\right\}\\ &= \frac{1}{8}\left(e^{2\pi i\xi}+1\right)^3\left\{\frac{1-\sqrt{10}}{2} + \frac{1+\sqrt{10}}{2}(\cos^2\pi\xi + \sin^2\pi\xi)\right.\\ &\qquad\left. + i\sqrt{5+2\sqrt{10}}\frac{1}{2}\sin(2\pi\xi)\right\}\\ &= \frac{1}{8}\left(e^{2\pi i\xi}+1\right)^3\left\{\frac{1-\sqrt{10}}{2} + \frac{1+\sqrt{10}}{4}2\cos(2\pi\xi)\right.\\ &\qquad\left. + \frac{1}{4}\sqrt{5+2\sqrt{10}}\,i\,\sin(2\pi\xi)\right\}\\ &= \frac{1}{8}\left(e^{2\pi i\xi}+1\right)^3\left\{\frac{1-\sqrt{10}}{2} + \frac{1+\sqrt{10}}{4}\left(e^{2\pi i\xi}+e^{-2\pi i\xi}\right)\right.\\ &\qquad\left. + \frac{1}{4}\sqrt{5+2\sqrt{10}}\left(e^{2\pi i\xi}-e^{-2\pi i\xi}\right)\right\}.\end{aligned}$$

The first line here defines a suitable $A_0(\xi)$ which clearly satisfies (4.2). The last two lines establish that it is of the desired form $A_0(\xi) = \frac{1}{2}\sum_{\gamma\in\mathbb{Z}} a_0(\gamma)e^{2\pi i\gamma\xi}$

where

$$\begin{aligned}
a_0(-1) &= \frac{1}{4}\Big(\frac{1+\sqrt{10}}{4} - \frac{5+2\sqrt{10}}{4}\Big),\\
a_0(0) &= \frac{1}{4}\Big(\frac{1-\sqrt{10}}{2} + 3\frac{1+\sqrt{10}}{4} - 3\frac{5+2\sqrt{10}}{4}\Big),\\
a_0(1) &= \frac{1}{4}\Big(3\frac{1-\sqrt{10}}{2} + 4\frac{1+\sqrt{10}}{4} - 2\frac{5+2\sqrt{10}}{4}\Big),\\
a_0(2) &= \frac{1}{4}\Big(3\frac{1-\sqrt{10}}{2} + 4\frac{1+\sqrt{10}}{4} + 2\frac{5+2\sqrt{10}}{4}\Big),\\
a_0(3) &= \frac{1}{4}\Big(\frac{1-\sqrt{10}}{2} + 3\frac{1+\sqrt{10}}{4} + 3\frac{5+2\sqrt{10}}{4}\Big),\\
a_0(4) &= \frac{1}{4}\Big(\frac{1+\sqrt{10}}{4} + \frac{5+2\sqrt{10}}{4}\Big)\,.
\end{aligned}$$

The coefficients $a_0(\gamma)$ are all real and $a_0(\gamma) \neq 0$ only if $-1 \leq \gamma \leq 4$.

We still need to find $A_1(\xi)$. It must be of the same form as $A_0(\xi)$ and satisfy

$$|A_1(\xi)|^2 + |A_1(\xi + \tfrac{1}{2})|^2 = 1, \tag{5.4}$$

$$A_0(\xi)\overline{A_1(\xi)} + A_0(\xi + \tfrac{1}{2})\overline{A_1(\xi + \tfrac{1}{2})} = 0\,. \tag{5.5}$$

Both these conditions are special cases of (4.3) and they are the only conditions involving $A_1(\xi)$. We can take

$$A_1(\xi) = e^{2\pi i\xi}\,\overline{A_0(\xi + \tfrac{1}{2})}\,,$$

for then the fact that A_0 is 1-periodic allows us to deduce (5.4) from (5.2) and to obtain (5.5):

$$\begin{aligned}
A_0(\xi)\overline{A_1(\xi)} &+ A_0(\xi + \tfrac{1}{2})\overline{A_1(\xi + \tfrac{1}{2})}\\
&= e^{-2\pi i\xi}A_0(\xi)A_0(\xi + \tfrac{1}{2}) + e^{-2\pi i(\xi + \frac{1}{2})}A_0(\xi + \tfrac{1}{2})A_0(\xi + 1)\\
&= A_0(\xi)A_0(\xi + \tfrac{1}{2})\left\{e^{-2\pi i\xi} + e^{-2\pi i\xi}e^{-\pi i}\right\} = 0\,.
\end{aligned}$$

Note that our choice of $A_1(\xi)$ leads to

$$a_1(\gamma) = (-1)^{\gamma+1}\overline{a_0(1-\gamma)}, \tag{5.6}$$

so the coefficients $a_1(\gamma)$ are real and satisfy $a_1(\gamma) \neq 0$ only if $-3 \leq \gamma \leq 2$.

The developement so far shows that there exists $\psi_1 = \psi \in L^2(\mathbb{R})$ for which the set in (4.1) is an ONB for $L^2(\mathbb{R})$. Rather than obtaining ψ_1 from formula (5.1) for $\widehat{\psi_1}$, it is easier first to find $\psi_0 = \varphi$ and then to obtain ψ_1 from (3.1) with $k = 1$. Since φ is a solution of (3.1) with $k = 0$, it is a fixed point of the operator

$$Sf(x) = \sum_{\gamma\in\mathbb{Z}} a_0(\gamma) f(2x-\gamma) \tag{5.6}$$

and one can seek φ through iteration: $\varphi = \lim_{n\to\infty} S^n f$ for a reasonable choice of an initial function f.

In the first part of this chapter we exhibited suitable trigonometric polynomials $A_0(\xi)$, $A_1(\xi)$ from which one can obtain the (finite) sequences $< a_k(\gamma) >$. In Chapter 7 a program for finding φ and ψ numerically is described and graphs of its output are presented.

We do not discuss convergence of the iterative process. However, we do note here some properties of S. By (4.2), $\frac{1}{2}\sum_{\gamma\in\mathbb{Z}} a_0(\gamma) = 1$, so if $\int f(x)\,dx = 1$, then $\int Sf(x)\,dx = 1$. Thus, one can easily obtain $\int \varphi(x)\,dx = 1$. In addition, when the $A_k(\xi)$ are trigonometric polynomials, as in our case, φ and ψ will have compact support if f is chosen to have compact support. In particular, if $a_k(\gamma) = 0$ for $|\gamma| > N$, then for f with support in $[-N, N]$, it is easy to see from (5.5) that Sf also has support in $[-N, N]$ and hence, so will φ and ψ. The computer-generated functions presented in Chapter 7 have compact support.

Chapter 6

Smoothness

In many cases we are interested in considering wavelets which are as smooth as possible. The simple Haar wavelet is not even continuous, but if we follow the construction described in Chapter 5, starting with $(\cos^2(\pi\xi)+\sin^2(\pi\xi))^M = 1$ for $M = 5(N+1)$, it turns out that our wavelet ψ will be in $C^N(\mathbb{R})$, cf. [7]. However as we shall show, compactly supported wavelets cannot be in C^∞.

We still assume that only a finite number of the coefficients $a_0(\gamma)$ appearing in the scaling identity

$$\varphi(x) = \sum_{\gamma\in\mathbb{Z}} a_0(\gamma)\varphi(2x-\gamma)$$

are non-zero. Since the wavelet function is given by

$$\psi(x) = \sum_{\gamma\in\mathbb{Z}} a_1(\gamma)\varphi(2x-\gamma)$$

where $a_1(\gamma) = (-1)^{\gamma+1}\overline{a_0(1-\gamma)}$ (Eq.(5.5)), $\psi(x)$ inherits the smoothness of $\varphi(x)$.

Also notice that if $\varphi(x) = \sum_{\gamma=-M}^{N} a_0(\gamma)\varphi(2x-\gamma)$ where $M, N \in \mathbb{N}$, and $g(x) := \varphi(x-M)$, then

$$\begin{aligned} g(x) = \varphi(x-M) &= \sum_{\gamma=-M}^{N} a_0(\gamma)\varphi(2x-2M-\gamma) \\ &= \sum_{k=0}^{N+M} a_0(k-M)\varphi(2x-M-k) \\ &= \sum_{k=0}^{N+M} b_0(k)g(2x-k)\ . \end{aligned}$$

Since smoothness is translation invariant, we may assume that φ satisfies

$$\varphi(x) = \sum_{n=0}^{N} a(n)\varphi(2x-n) \tag{6.1}$$

for some $N \in \mathbb{N}$, with $a(N) \neq 0$.

We now state and prove a theorem regarding the differentiability of φ, cf. [3].

THEOREM 6.1. *If $\varphi \in L^1(\mathbb{R})$, $\varphi \neq 0$, φ satisfies (6.1) and is in $C^m(\mathbb{R})$, then*

$$m < N - 1.$$

By [3, p.1395], φ has compact support. Notice that if $[a,b]$ is the smallest closed interval containing $\operatorname{supp}\varphi(x)$, then $\operatorname{supp}\varphi(2x-n) \subset [\frac{a}{2}+\frac{n}{2}\ ,\ \frac{b}{2}+\frac{n}{2}]$, and $[\frac{a}{2}+\frac{n}{2}\ ,\ \frac{b}{2}+\frac{n}{2}]$ has exactly half the length of $[a,b]$. Also, as n increases by 1, the support of $\varphi(2x-n)$ moves to the right by $\frac{1}{2}$. Since (6.1) gives φ as a linear combination of these "half functions", we see that the larger the support of φ is, the bigger the value of N must be in (6.1) and vice versa. So the above theorem actually states that if φ is compactly supported, it cannot be infinitely differentiable.

PROOF. Suppose φ satisfies (6.1) and $\varphi \in C^m$ for some $m \in \mathbb{N}$. Let $[a,b]$ be the smallest interval containing $\operatorname{supp}\varphi$. Since supp $(\sum_{n=0}^{N} a(n)\varphi(2x-n)) \subseteq [\frac{a}{2}\ ,\ \frac{b}{2}+\frac{N}{2}]$, and φ satisfies (6.1), it must be the case that $\frac{a}{2} \leq a$ and $\frac{b}{2}+\frac{N}{2} \geq b$, which imply $a \geq 0$ and $b \leq N$. Moreover, since φ is continuous, $\varphi(0) = \varphi(N) = 0$. Hence, $\varphi(j) = 0$ for $j \in \mathbb{Z} - \{1, \ldots, N-1\}$.

Now define $M \in \mathbb{R}^{(N-1)\times(N-1)}$ by $M_{ij} = a(2i-j), \quad j = 1, \cdots N-1$, and $v^0 \in \mathbb{R}^{N-1}$ by $v^0_j = \varphi(j), \quad j = 1, \cdots, N-1$.

We claim that $v^0 \neq 0$; otherwise $\varphi(j) = 0$ for all $j \in \mathbb{N}$ and (6.1) used repeatedly would imply $\varphi(\frac{n}{2^k}) = 0$ for all $n \in \mathbb{Z},\ k \in \mathbb{N}$, so by the continuity of φ we would have $\varphi \equiv 0$ which is a contradiction.

Also for $i \in \{1, \cdots, N-1\}$ fixed, $\varphi(2i-n) \neq 0$ only for $0 < 2i - n \leq N-1$ or, equivalently, for $2i - N + 1 \leq n \leq 2i - 1$, therefore

$$\begin{aligned} v_i^0 =& \varphi(i) = \sum_{n=0}^{N} a(n)\varphi(2i-n) = \sum_{n=2i-N+1}^{2i-1} a(n)\varphi(2i-n) \\ =& \sum_{j=1}^{N-1} a(2i-j)\varphi(j) = (Mv^0)_i. \end{aligned}$$

Consequently $Mv^0 = v^0$, and since $v^0 \neq 0$, 1 is an eigenvalue of M.

For $l = 1, \cdots, m$ define vectors $v^l \in \mathbb{R}^{N-1}$ by $v^l(j) = \varphi^{(l)}(j), \quad j = 1, \cdots, N-1$.

We claim $v^l \neq 0$ - otherwise $\varphi^{(l)} = 0$ as before, which would imply $\varphi^{(l-1)} =$ constant, and since $\varphi^{(l-1)}$ is compactly supported, $\varphi^{(l-1)}$ would be also identically zero. By induction we would have $\varphi = 0$ which is a contradiction.

Again for i fixed,

$$\begin{aligned} v_i^l =& \varphi^{(l)}(i) = 2^l \sum_{n=0}^{N} a(n)\varphi^{(l)}(2i-n) = 2^l \sum_{j=1}^{N-1} a(2i-j)\varphi^{(l)}(j) \\ =& 2^l (Mv^l)_i \ ; \end{aligned}$$

hence $v^l = 2^l M v^l$, and since $v^l \neq 0$, 2^l is also an eigenvalue of M, for each $l = 1, \cdots, m$.

Since the number of eigenvalues of M cannot exceed $N-1$, it must hold that $m + 1 \leq N - 1$ or that $m < N - 1$. $\square$

Chapter 7

Implementation

In this chapter we consider briefly the numerical implementation of an iterative scheme for solving

$$f(x) = \sum_{\gamma \in \mathbb{Z}} a_0(\gamma) f(2x - \gamma). \tag{7.1}$$

We operate under the assumption that there exists a function φ, generating an MRA, which solves (7.1) for a given set of coefficients $a_0(\gamma)$, $\gamma \in \mathbb{Z}$. The wavelet ψ then can be constructed according to

$$\psi(x) = \sum_{\gamma \in \mathbb{Z}} a_1(\gamma) \varphi(2x - \gamma), \tag{7.2}$$

where $a_1(\gamma) = (-1)^{\gamma+1}\, \overline{a_0(1-\gamma)}$, $\gamma \in \mathbb{Z}$. In practice the sums in (7.1) and (7.2) are finite with real coefficients a_0, a_1.

The algorithm proceeds as follows : We start with a suitable function f_0 with $\int f_0(x)\,dx = 1$ and compact support $\subset [a,b]$, $a, b \in \mathbb{N}$. The interval $[a,b]$ is subdivided with $N = (b-a)2^J$, $J \in \mathbb{N}$, equally spaced points $\{x_i\}_{i=1}^N$, where f_0 is evaluated. Next we construct the various integer translates of $f_0(2x_i)$, $i = 1, \ldots, N$. The new iterate is now computed by

$$f_1(x_i) = \sum_{\gamma \in \mathbb{Z}} a_0(\gamma) f_0(2x_i - \gamma), \quad i = 1, \ldots, N,$$

for a given set of coefficients $a_0(\gamma)$, $\gamma \in \mathbb{Z}$. In general, at the $(\ell+1)$−th step, the algorithm computes $f_\ell(2x_i - \gamma)$, $i = 1, \ldots, N$, $\gamma \in \mathbb{Z}$ and obtains the new iterate

$f_{\ell+1}(x_i),\ i = 1, \ldots, N$ from

$$f_{\ell+1}(x_i) = \sum_{\gamma \in \mathbb{Z}} a_0(\gamma) f_\ell(2x_i - \gamma), \quad i = 1, \ldots, N.$$

The algorithm terminates either if a certain number of iterations is performed or if the discrete ℓ_2 relative error $\|f_{\ell+1} - f_\ell\|_2 / \|f_0\|_2$, $\ell = 0, 1, \ldots$, is less than a prescribed tolerance ϵ.

The last iterate is taken as the scaling function φ and is used to compute the wavelet ψ according to (7.2).

Several initial functions f_0 are implemented in the code: the characteristic function of [0,1], two hat functions and a sinusoidal function. All of them are evaluted in $[a, b]$. Furthermore, 7 different sets of coefficients a_0 are included. In particular we have included those derived in Chapter 5, two different sets of Daubechies, [1], and those reported in Strang, [6]. Different initial functions and coefficient sets can be included in the code. However, if f_0 is evaluated on $[a, b]$ and $a_0(\gamma) \neq 0$ for $\gamma \in \{m, m+1, \ldots, n\}$, then it must be that $m \geq a$ and $n \leq b$ to ensure that $\operatorname{supp} f_\ell \subset \operatorname{supp} f_0$ for $\ell = 1, 2, \ldots$.

On the interval $[a, b] = [-1, 8]$ we put a uniform mesh consisting of N=9216 points (J=10). A maximum of 100 iterations was allowed, although the algorithm converged in all test cases in less than 30 iterations with $\epsilon = 10^{-8}$. On the output the graphs of f_0, φ, ψ are presented. All the computations were performed on a SUN SPARC 1 workstation using double precision arithmetic. The required computational time was less than *1 min.* for each test case.

First we took f_0 to be the characteristic function of [0,1] along with $a_0(0) = 1$, $a_0(1) = 1$. In this case f_0 remains invariant, so that $\varphi \equiv \chi_{[0,1]}$, and ψ turns out to be the Haar wavelet mentioned in Chapter 2, as one would expect. The graph of the characteristic function as well as the resulting φ, ψ are presented in FIGURE 1. Next we computed φ and ψ using the characteristic function of [0,1] and two different sets of 4 and 10 coefficients reported in [1]. The corresponding graphs are shown in FIGURES 2 and 3. These graphs are also presented in [1]. Notice that the set of 10 coefficients produces an obviously smoother wavelet, as one would expect from the discussion in Chapter 6. In FIGURE 4 the graphs of φ and ψ are shown for the box function and the 6 coefficients derived in Chapter 5. FIGURE 5 shows the plots of φ and ψ with f_0 being a sinusoidal function, and using again Daubechies' set of 10 coefficients. The graphs of ϕ and ψ are the same as those in FIGURE 3.

Each set of coefficients was tested for all the different functions f_0 mentioned above. For each set the resulting wavelet was always the same.

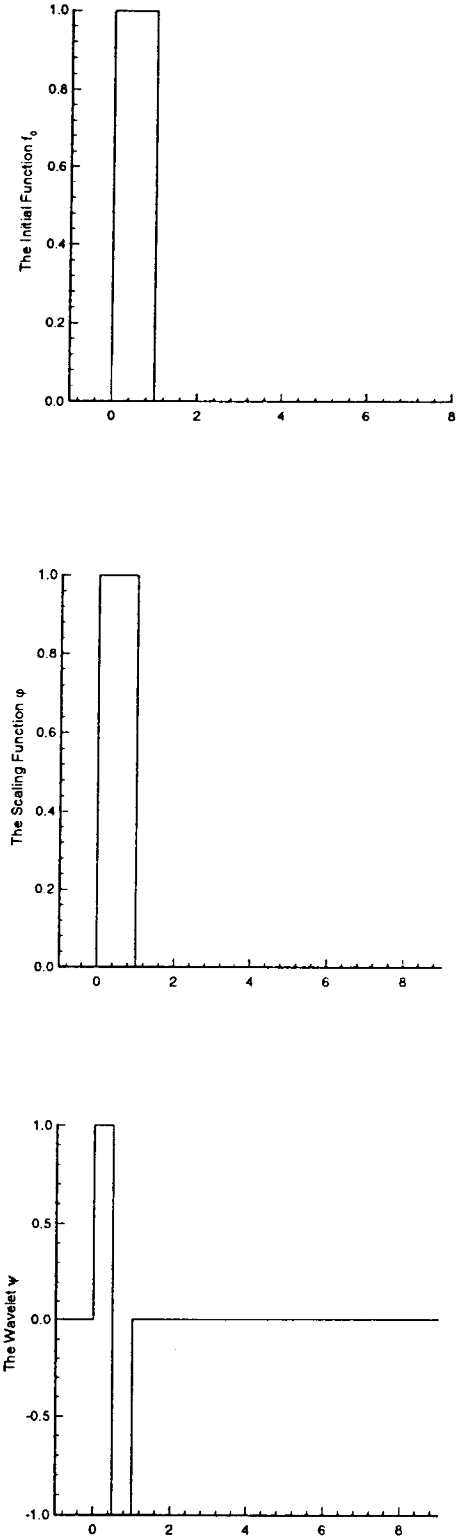

FIGURE 1 . The Haar wavelet .

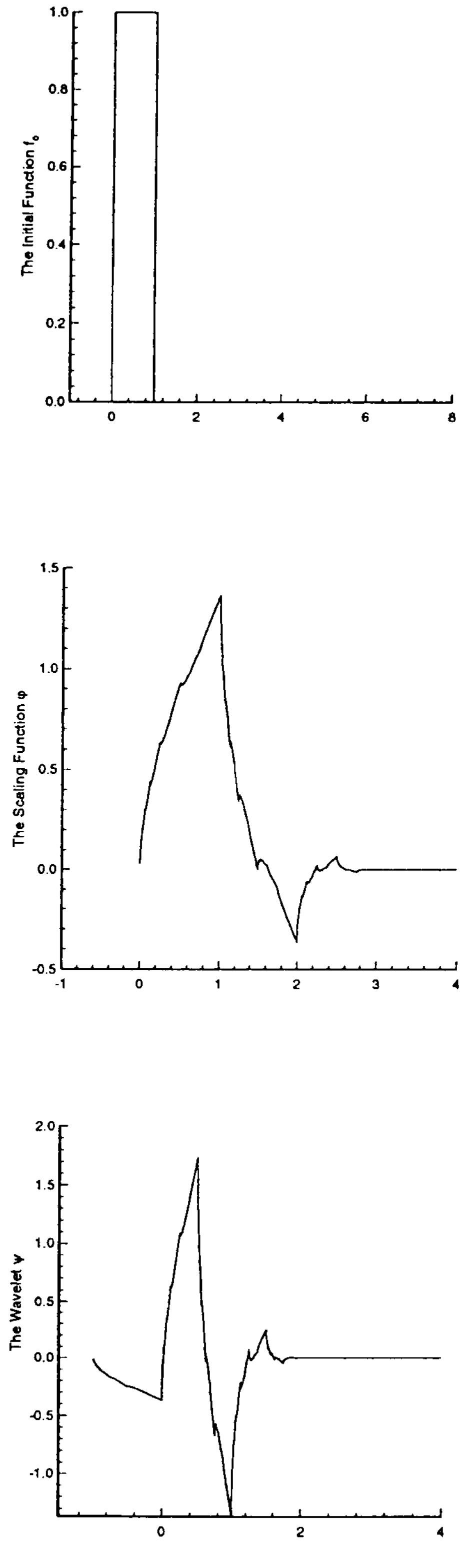

FIGURE 2 . The D_4 wavelet .

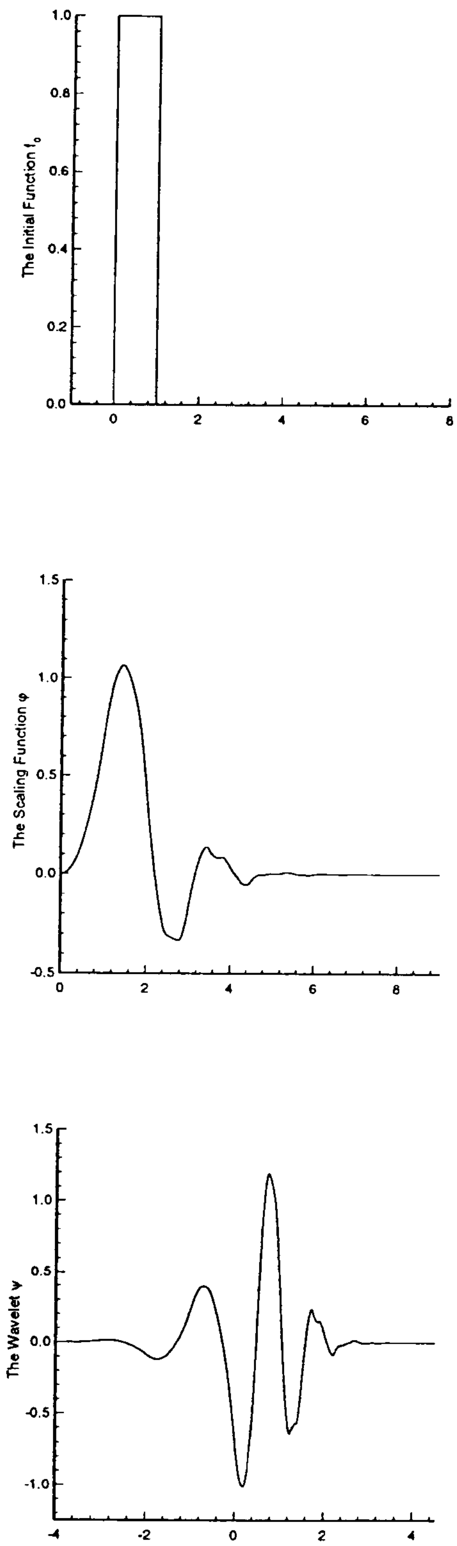

FIGURE 3 . The D_{10} wavelet .

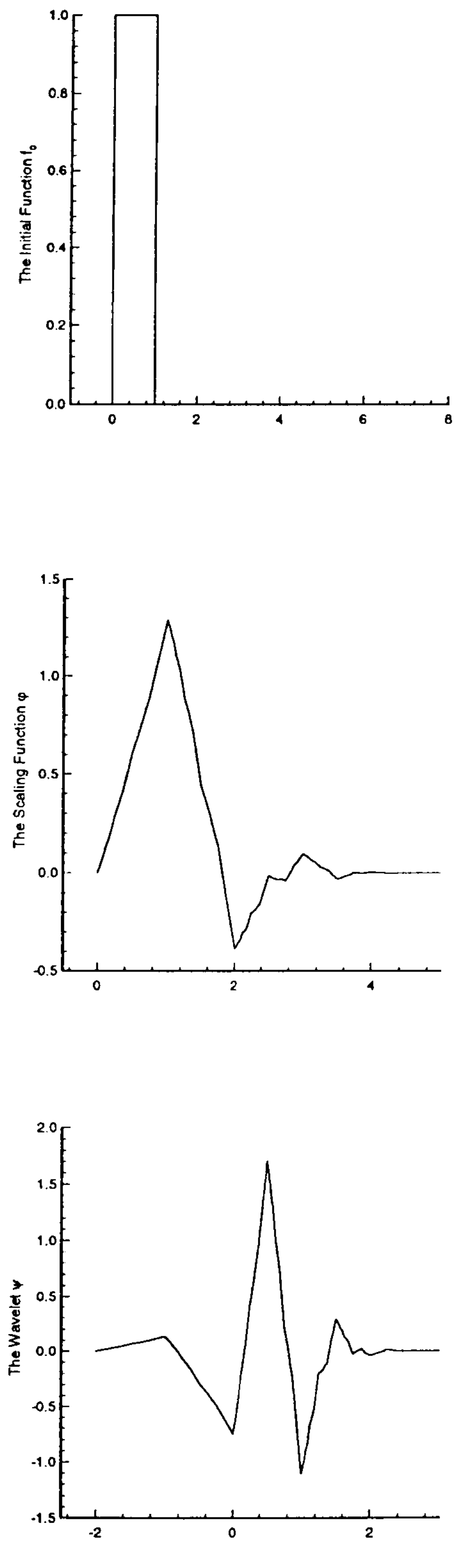

FIGURE 4 . The Strichartz–D_6 wavelet . (Box)

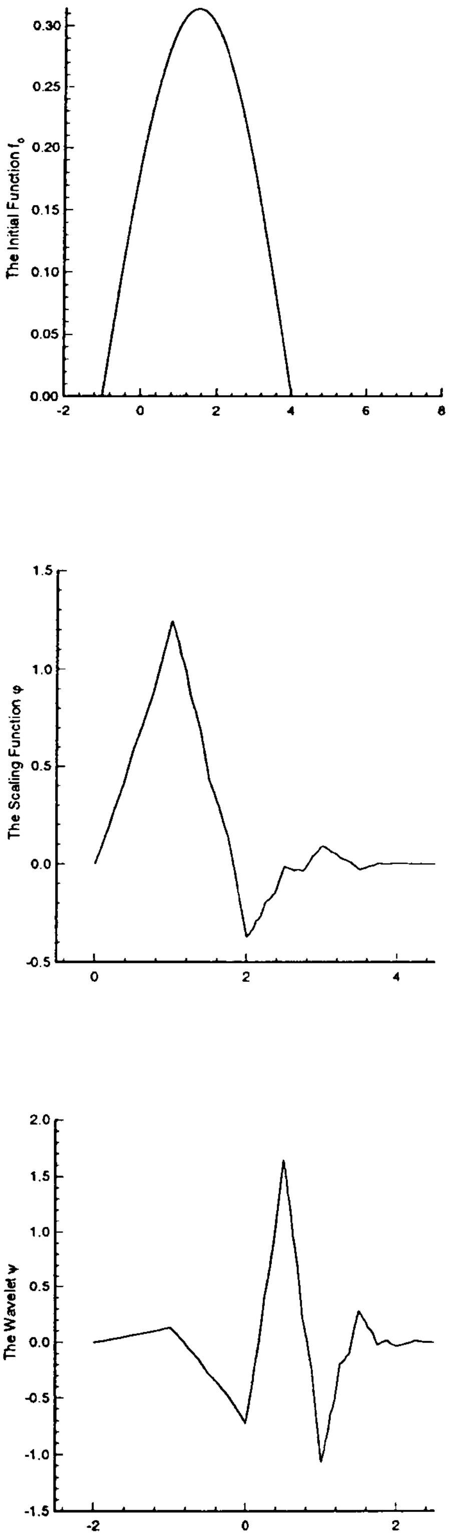

FIGURE 5 . The D_6 wavelet . (Sinusoidal)

LIST OF REFERENCES

LIST OF REFERENCES

[1] I.Daubechies, *Ten Lectures on Waveletes*, Siam, CBMS 61, 1992.

[2] ———, *Orthonormal Bases of Compactly Supported Wavelets*, Communications on Pure and Applied Mathematics **41** (1988), 909–996.

[3] I.Daubechies & J.Lagarias, *Two-scale difference equations I*, Siam Journal of Mathematical Analysis **22** (1991), 1388–1410.

[4] P.R.Halmos, *Introduction to Hilbert Spaces*, Chelsea Publishing Company, New York, 1957.

[5] W. Rudin, *Real and Complex Analysis*, Mc Graw–Hill Book Company, New York, 1974.

[6] G.Strang, *Wavelets and Dilation Equations : A Brief Introduction*, Siam Review **31** (1989), 614–627.

[7] ———, *Wavelet Transforms Versus Fourier Transforms*, Bulletin of the American Mathematical Society **28** (1993), 288–305.

[8] R.Strichartz, *How to Make Wavelets*, American Mathematical Monthly **100** (1993), 539–556.

VITA

Evangelia Spanaki was born in Athens, Greece on April 18, 1968. After graduating from high–school in June of 1986 she enrolled in the University of Crete, Greece, where she received a Bachelor of Science degree in Mathematics in June 1991. In the Spring of 1992 she started graduate studies at the University of Maryland at College Park. In September of the same year she transfered to the University of Tennessee at Knoxville, where, in August of 1994, she was awarded a Master of Science degree with major in Mathematics.