

To the Graduate Council:

I am submitting herewith a dissertation written by Matthew Edward Cathey entitled “Packings of Conformal Preimages of Circles.” I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Kenneth Stephenson

Major Professor

We have read this dissertation
and recommend its acceptance:

Charles Collins

Morwen Thistlethwaite

Bradley Vander Zanden

Accepted for the Council:

Anne Mayhew

Vice Chancellor
and Dean of Graduate Studies

(Original signatures are on file with official student records.)

Packings of Conformal Preimages of Circles

A Dissertation

Presented for the

Doctor of Philosophy Degree

The University of Tennessee, Knoxville

Matthew Edward Cathey

May 2006

Acknowledgments

I would like to thank my advisor, Dr. Kenneth Stephenson, for introducing me to circle packings and for his help in preparing this thesis, as well as Dr. Robert Daverman for the invaluable hints that made this whole thing work. I would like to acknowledge the fantastic instruction I have received that prepared me for this point in my career; without the help and encouragement of teachers like Don Steely, Catherine Cavagnaro, and Clay Ross, I would never have been successful in graduate school. I also wish to express my gratitude to all of my friends and family who have supported me through all of my educational experience. I'd also like to thank the faculty and administration of Wofford College, especially Dean Dan Maulsby and Dr. Lee Hagglund, for their patience and support while I finished this work. Finally, I want to thank my wife, Meghan, for without her love and support I could never have done this.

The diagrams in Chapter 3 were created using Paul Taylor's commutative diagrams package. Many other figures were created using Ken Stephenson's `CirclePack`, available for download at <http://www.math.utk.edu/~kens>.

Abstract

This dissertation provides existence and uniqueness results for packings of conformal preimages of circles in the unit disk. Examples are given showing how these results can be applied in more general situations, such as finite- and infinite-to-one covers of the punctured plane.

Contents

1	Circle Packings and General Packings	1
1	Circle Packings and the Discrete Uniformization Theorem	1
2	General Packings	2
3	Goals	6
2	Some Preliminaries	7
1	Conformal Preimages	7
2	Precircles Are Blunt and Disklike	8
3	$\mathcal{F}$ Is Continuous	8
4	$\mathcal{F}$ is a 3-Manifold	10
5	Precircles Are Compatible	11
6	$\mathcal{F}$ Is a Packable Collection	11
3	Packing Precircles in the Unit Disk	12
1	Some Technical Details	12
2	Proof of Theorem 1.11	23
4	Branched Packings in the Unit Disk	24
1	Non-Univalent Packings	24

2	Defining Precircles	25
3	$\mathcal{F}_R$ Is a Packable Collection	28
4	$\mathcal{F}_B$ Is a Packable Collection	30
5	Creating the Packing	35
5	Locally Univalent Branched Packings	37
1	Defining Precircles	38
2	$\mathcal{F}$ Is a Continuous 3–Manifold	40
6	Finite-to-One Coverings of the Punctured Plane	43
1	The Complex	43
2	Building the Packing	45
3	Doyle Spirals	48
7	Infinite Covers of the Punctured Plane	54
1	The Complex	54
2	Building the Packing	56
8	Packings Using More General Sets	60
1	Conformal Preimages of Other Families	60
	Bibliography	63
	Vita	66

List of Figures

1.1	Examples of sets which are and are not disklike	4
3.1	Building $\mathcal{T}'$	13
3.2	Vertices v_0 and v_1 each have degree three, resulting in a triangulation which violates our hypotheses	15
3.3	Commutative diagrams	18
3.4	Adding corners	19
3.5	Constructing β	22
4.1	Two flowers with nine petals	25
4.2	The effect of f on winding number	26
4.3	Fundamental domains	26
4.4	Defining $\mathcal{K}$	27
4.5	The circle $ z - \frac{1}{2} = \frac{1}{4}$ (dashed), with its two precircles under p	28
4.6	Defining r_A and R_A	30
4.7	The annulus $f(A)$, with the two possibilities for $f(A_j)$	32
4.8	Two annuli	34
5.1	A locally univalent branched surface	38

5.2	A problem defining precircles	39
5.3	A circle in Ω that is not associated with a precircle	40
6.1	The limits of C_i and D_i ; the limit of D_i is not blunt	47
6.2	The Penny Packing	49
6.3	Some Doyle spirals, with corresponding values for A	50

Chapter 1

Circle Packings and General Packings

1 Circle Packings and the Discrete Uniformization Theorem

A *circle packing* is a collection of circles which realize a given pattern of tangencies. Circle packings were first investigated in the 1930s by Koebe [5], but he failed to find an application for his work. Fifty years later, William Thurston [9] speculated that circle packings could be used to approximate conformal maps; in 1987, Burt Rodin and Dennis Sullivan [6] proved that the discrete analytic maps built with circle packings converge to the function whose existence is guaranteed by the Riemann Mapping Theorem. Beardon and Stephenson proved the following *Discrete Uniformization Theorem* [2]:

Theorem 1.1. *Let K be a CP 2-complex which triangulates a (topological) surface*

S. Then there exists a Riemann surface $\mathcal{S}_K$ homeomorphic to S and a circle packing $\mathcal{P}$ for K in the associated intrinsic metric on $\mathcal{S}_K$ such that $\mathcal{P}$ is univalent and fills $\mathcal{S}_K$. The surface $\mathcal{S}_K$ is unique up to conformal equivalence and $\mathcal{P}$ is unique up to conformal automorphisms of $\mathcal{S}_K$.

We have techniques (e.g., [3]) for building circle packings for simply connected domains and certain other special cases. Our aim here is to extend the range of our techniques.

2 General Packings

Oded Schramm has done significant work into proving existence and uniqueness of packings of classes of sets more general than just circles. We will use several of his results; first, however, we will need some definitions (most of which are first seen in [7]).

Definition 1.2. A *packing* is a collection $\mathcal{P} = \{P_v | v \in V\}$ of compact connected sets in the sphere or the plane such that the interior of P_v is disjoint from the interior of P_w for all $v \neq w$. We can construct a graph whose vertex set is V and which has an edge connecting v and w if and only if $P_v \cap P_w$ is not empty; we require that this graph be a CP 2-complex which is isomorphic to a triangulation of a topological surface. We will refer to this graph as the *complex associated with $\mathcal{P}$* .

We have seen one of the most important theorems in the circle packing canon, the Discrete Uniformization Theorem. It would be nice to have an extension of that theorem to our more general sets. That brings up an important question: what are the key properties of circles that enable the existence of circle packings with a

prescribed complex? What “circleish” properties should the sets we want to consider possess?

Definition 1.3. A subset A of the sphere S^2 will be called *disklike* if it is the closure of its interior, its interior is connected, and the complement of any connected component of A^c is a topological disk. A bounded subset of the plane is disklike if it is the image under stereographic projection of a disklike set in S^2 .

Note that all closed topological disks are disklike. An example of a set that is disklike but isn’t a disk is a closed annulus; we’ll come back to this example later. See Figure 1.1 for more examples.

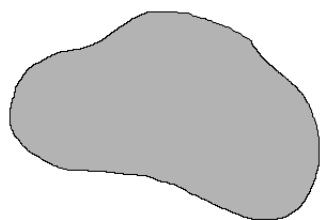
Definition 1.4. A disklike set A is called *blunt* if for every point $p \in \partial A$ there is a set with smooth boundary contained in A and containing p .

Intuitively, this means that A is blunt if it has no outward-pointing cusps.

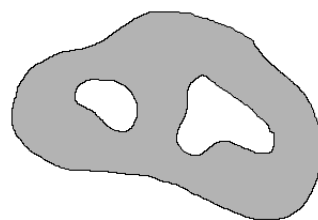
Are these the properties we seek? Schramm [7] proved that these properties are vital in packing with general sets. They are not, however, sufficient. To get the results we need, we must build up some more machinery.

Definition 1.5. Let $\{A_n\}$ be a sequence of closed subsets in either S^2 or $\mathbb{C}$. We define the *limit superior* of A_n (denoted $\overline{\lim} A_n$) to be the set of all accumulation points of sequences $\{x_n\}$, with $x_n \in A_n$. The *limit inferior* of A_n ($\underline{\lim} A_n$) is the set of all limit points of convergent sequences $\{x_n\}$. We will say that A_n *converges* to a set A if $\underline{\lim} A_n = \overline{\lim} A_n = A$ and $A^c = \text{interior}(\overline{\lim} A_n^c)$.

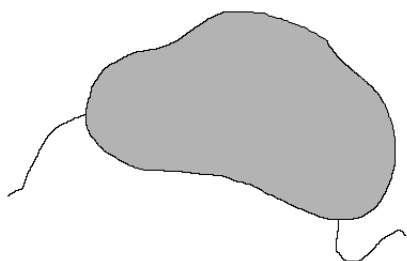
Definition 1.6. Let U be an open subset of S^2 (resp. $\mathbb{C}$); let $\mathcal{F}$ be a collection of subsets of U . $\mathcal{F}$ is called *continuous* if for every sequence of sets $\{A_n\} \subset \mathcal{F}$ contained



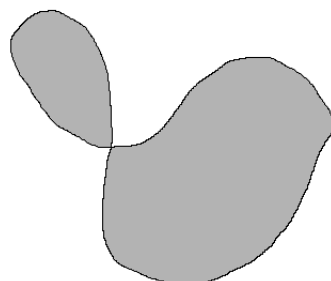
(a) Disklike



(b) Disklike



(c) Not disklike; it is not the closure of its interior



(d) Not disklike; its interior is not connected

Figure 1.1: Examples of sets which are and are not disklike

in some compact proper subset of S^2 (resp. $\mathbb{C}$), there is a subsequence $\{A_{n_i}\}$ such that either $\{A_{n_i}\}$ converges to a set in $\mathcal{F}$ or $\overline{\lim} A_{n_i}$ is a point.

Definition 1.7. A *trilateral* is a closed topological disk with three distinguished points (called *vertices*) on its boundary. A trilateral is called *cornered* if it has no subset with smooth boundary which contains one of its vertices. Further, a trilateral is *decent* if it is cornered and no two of its smooth interiorwise-disjoint subsets meet the boundary at the same point (i.e., its boundary has no inward cusps of angle 2π).

Definition 1.8. Let A and B be two closed topological disks in S^2 . Then A is said to *cut* B if there are two points in $B \setminus \text{interior}(A)$ which are not connected by any curve in $\text{interior}(B) \setminus A$. A and B are called *incompatible* if $A \neq B$ and either A cuts B or B cuts A . Sets that are not incompatible are called *compatible*.

There is a similar but more complicated definition of compatibility for disklikes. We will only need it for a couple of examples, so we will postpone that definition until we need it.

Definition 1.9. Let U be a nonempty open simply connected subset of S^2 . A collection $\mathcal{F}$ of blunt disklikes in U is called *packable* if $\mathcal{F}$ is continuous on U , it is a 3-manifold (in the topology induced by our definition of convergence), and its members are pairwise compatible.

The idea that these collections form a 3-manifold is perhaps the most difficult to grasp. Essentially, saying that these collections form a 3-manifold is equivalent to saying that, given any set P in our collection, we can continuously change it by varying three parameters. In the case of the collection of circles in the plane, we can vary the real and imaginary parts of the center, as well as the radius.

With these concepts and definitions in hand, we can now state one of the main results from [7]:

Theorem 1.10. *Let U be some open subset of S^2 . Let T be a triangulation of a topological sphere, with vertex set V . Let $[a, b, c]$ be a triangle in T , and for $v \in V \setminus \{a, b, c\}$ let $\mathcal{F}_v$ be a packable collection on U . Then, given a decent trilateral D with edges D_a , D_b , and D_c in U , there exists a unique packing P contained in D whose complex is T such that $P_v = D_v$ for $v \in \{a, b, c\}$ and $P_v \in \mathcal{F}_v$ otherwise.*

3 Goals

This thesis will build on Theorem 1.10 to build packings with the preimages of circles under well known and well understood conformal maps. That will give us this theorem, whose proof appears in Chapter 3:

Theorem 1.11. *Let $\mathcal{T}$ be a CP-complex which triangulates a closed topological disk and which has at least six boundary vertices, and let Ω be a bounded, simply connected subset of $\mathbb{C}$ such that $\partial\Omega$ has no inward cusps of angle 2π . Given any interior vertex v in $\mathcal{T}$ and any point z in the interior of Ω , there exists a circle packing $\mathcal{P}$ with complex $\mathcal{T}$ in Ω such that each boundary circle of $\mathcal{P}$ intersects $\partial\Omega$ and the circle associated with v is centered at z .*

We will extend this result to obtain infinite packings, non-univalent packings, and packings that fill unbounded domains. We will also build on a result proven in [1] on Doyle spirals.

Chapter 2

Some Preliminaries

1 Conformal Preimages

Let Ω be a bounded, simply connected domain in $\mathbb{C}$ such that its boundary has no inward cusps of angle 2π (in other words, we cannot find two circles in $\overline{\Omega}$ whose interiors are disjoint which intersect at a point on the boundary). The Riemann Mapping Theorem guarantees the existence of a conformal bijection $f : \mathbb{D} \rightarrow \Omega$, where $\mathbb{D}$ is the (open) unit disk. What we aim to do is this: given a CP-complex $\mathcal{T}$ which triangulates a closed disk, we will build packings in $\mathbb{D}$ of the preimages under f of circles in such a way that, when f is applied, we obtain a circle packing in Ω whose complex is isomorphic to $\mathcal{T}$ and whose boundary circles are tangent to $\partial\Omega$. Theorem 1.10 will be our starting point.

For any point z in $\mathbb{D}$, define $\mathcal{F}_z$ to be the set of all preimages under f of closed disks in Ω centered at $f(z)$. Let $\mathcal{F} = \bigcup_{z \in \mathbb{D}} \mathcal{F}_z$. From this point forward, we will refer to any set whose image under f is a closed circular disk in Ω as a *precircle*. We will also consider sets B such that $\overline{f(B)} \subset \overline{\Omega}$ is a closed circular disk, but are

not precircles; these are sets whose boundary intersects the boundary of $\mathbb{D}$. For this reason, we will call such sets *boundary precircles*. Note that $\mathcal{F}$ contains no boundary precircles.

2 Precircles Are Blunt and Disklike

Suppose C is a circle in Ω . Since f is a continuous bijection, it is a homeomorphism. Thus, we conclude that $f^{-1}(C)$ is a topological disk, which is necessarily disklike.

Now, suppose A is an element of $\mathcal{F}$, and suppose A is not blunt. Then, as noted earlier, the boundary of A must have an outward-pointing cusp. Since conformal maps preserve angles, $f(A)$ must then also have a cusp. This is impossible, since $f(A)$ is a circle. Thus, A is blunt.

3 $\mathcal{F}$ Is Continuous

In order to apply Theorem 1.10, we must first verify that $\mathcal{F}$ has the properties that we need.

Lemma 2.1. *$\mathcal{F}$ is a continuous collection.*

PROOF. For all positive integers n , let $\{A_n\}$ be a sequence in $\mathcal{F}$ such that $\bigcup A_n$ is contained in some compact subset K of $\mathbb{D}$. For each n , choose z_n such that A_n is in $\mathcal{F}_{z_n}$. Then $\{z_n\}$ is contained in K ; hence there is a subsequence $\{z_{n_k}\}$ which converges to a point z in K . Similarly, in Ω , $\widehat{A_n} = f(A_n)$ is a disk centered at $\widehat{z_n} = f(z_n)$, and $\widehat{z_{n_k}} = f(z_{n_k})$ converges to $\widehat{z} = f(z)$. Clearly $\widehat{z}$ is in $\underline{\lim} \widehat{A_{n_k}} \cap \overline{\lim} \widehat{A_{n_k}}$. Let r_{n_k} be the radius of $\widehat{A_{n_k}}$. Since Ω is bounded, there exists a convergent subsequence $\{r_{\alpha}\}$

in $\{r_{n_k}\}$; call the limit r . If we show that $\overline{\lim A_\alpha} = \underline{\lim A_\alpha} = \overline{B(\hat{z}, r)}$, where $B(w, \rho)$ denotes the open ball centered at w with radius ρ , it will follow that $\overline{\lim A_\alpha} = \underline{\lim A_\alpha}$.

First, we show that $\overline{\lim A_\alpha}$ is a subset of $\overline{B(\hat{z}, r)}$. Choose w in $\overline{\lim A_\alpha}$. Choose w_α in $\widehat{A_\alpha}$ such that w is contained in $\overline{\{w_\alpha\}}$. Then there must be a subsequence $\{w_{\alpha_i}\}$ in $\{w_\alpha\}$ which converges to w . We have that $|w_{\alpha_i} - z_{\alpha_i}| \leq r_{\alpha_i}$. Taking limits, we get $|w - z| \leq r$, as desired.

Next, we show that $\overline{B(\hat{z}, r)}$ is contained in $\underline{\lim A_\alpha}$. Let w be an element of $\overline{B(\hat{z}, r)}$. We must find a sequence of points in the $\widehat{A_\alpha}$ which converges to w . Choose w_α in $\widehat{A_\alpha}$ such that $|w_\alpha - w| = \min_{w' \in \widehat{A_\alpha}} |w' - w|$. Note that $\min_{w' \in \widehat{A_\alpha}} |w' - w| \leq r + |z - z_\alpha| - r_\alpha$ for all α . Taking limits, we see that $|w_\alpha - w| \rightarrow 0$, as desired.

Finally, it remains to show that $A^c = \text{interior}(\overline{\lim A_n^c})$, where $A = \overline{\lim A_\alpha} = \underline{\lim A_\alpha}$. This is equivalent to showing that $\widehat{A^c} = \text{interior}(\widehat{\overline{\lim A_n^c}})$. Since $\widehat{A_n^c}$ is a circular disk in the Riemann sphere for all n , we can use the arguments presented above to conclude that $\overline{\lim A_n^c}$ is also a circular disk; in $\mathbb{C}$, $\overline{\lim A_n^c}$ is an unbounded region with a circle as its boundary. Choose any point s in $\mathbb{C}$. Since $s \in \widehat{A_n}$ or $s \in \widehat{A_n^c}$ for any given n , s must be an element of either $\overline{\lim A_n}$ or $\overline{\lim A_n^c}$. Since $\overline{\lim A_n} = \widehat{A}$, it remains to show that $\widehat{A^c}$ is open. We will show that $\widehat{A}$ is closed. Let y be an element of $\widehat{A}$. Then there exists a sequence $\{y_n\}$ of points in $\widehat{A}$ which converges to y . For each point y_n , there exists a sequence $\{y_{n_i}\}$ which converges to y_n such that y_{n_i} is an element of $\widehat{A_i}$ for all i . Now consider the sequence $\{y_{n_n}\}$; note that y_{n_n} is an element of $\widehat{A_n}$ for all n , and that $\{y_{n_n}\}$ must converge to y . Hence $y \in \underline{\lim A_n} = \widehat{A}$, and so $\widehat{A}$ is closed. Since we have shown that $\widehat{A}$ and $\overline{\lim A_n^c}$ cover the plane and that $\widehat{A}$ is closed, it follows that $\widehat{A^c} = \text{interior}(\widehat{\overline{\lim A_n^c}})$. ■

We note here for future reference another fact that follows from this argument.

Lemma 2.2. *The collection of circles in $\mathbb{C}$ is continuous.*

4 $\mathcal{F}$ is a 3-Manifold

Lemma 2.3. *$\mathcal{F}$ is a 3-manifold.*

PROOF. For each $z \in \mathbb{D}$, let $\rho_z = \sup\{r \mid B(f(z), r) \subset \Omega\}$, where $B(f(z), r)$ denotes the open ball centered at $f(z)$ with radius r . To show that $\mathcal{F}$ is a 3-manifold, we must demonstrate that each element of $\mathcal{F}$ has a neighborhood in $\mathcal{F}$ that is homeomorphic to an open set in $\mathbb{R}^3$. Choose $A \in \mathcal{F}$. Then there is a point $z_0 \in \mathbb{D}$ such that $A \in \mathcal{F}_{z_0}$. Choose $\lambda > 0$ such that $\overline{B(z_0, \lambda)} \subset \mathbb{D}$. Since $\overline{B(z_0, \lambda)}$ is compact, $\max\{\rho_z \mid z \in \overline{B(z_0, \lambda)}\}$ exists; call it ρ . Let $U = B(z_0, \lambda) \times (0, \rho)$. We now want show that U is homeomorphic to some neighborhood of A . Define $\phi : U \rightarrow \mathcal{F}$ by letting $\phi(a, b, c) = f^{-1}(B(f(a + bi), c))$. We note that ϕ is injective because f is injective. Also, U was chosen in such a way that $\phi(U)$ is contained in $\mathcal{F}$. Finally, we must show that ϕ is continuous. Let $\{(a_n, b_n, c_n)\}$ be a sequence in U which converges to (a, b, c) . We just saw in Lemma 2.1 that $\mathcal{F}$ is continuous; by the proof of that lemma we see that the collection of balls in Ω is also continuous. Thus, since f is bicontinuous, $\lim f^{-1}(B(f(a_n + b_n i), c_n)) = f^{-1}(\lim B(f(a_n + b_n i), c_n)) = f^{-1}(B(f(a + bi), c))$. Hence, ϕ is continuous. It then follows that $\mathcal{F}$ is a 3-manifold. ■

5 Precircles Are Compatible

Lemma 2.4. *The elements of $\mathcal{F}$ are pairwise compatible.*

PROOF. Choose $A, B \in \mathcal{F}$, and suppose A and B are incompatible. We may then assume without loss of generality that A cuts B . Then there exist $b_0, b_1 \in B \setminus \text{interior}(A)$ that are not connected by a path in $\text{interior}(B) \setminus A$. Since f is injective, it follows that $f(b_0)$ and $f(b_1)$ are not connected by a path in $f(\text{interior}(B)) \setminus f(A)$. However, $f(B)$ and $f(A)$ are disks in the plane; since two different circles in the plane can intersect at most twice, the difference of two disks must always be path connected. Thus we must be able to join $f(b_0)$ and $f(b_1)$ with a path, contradicting our earlier statement about these points. Hence, our supposition that A and B are incompatible must be false; the only other possibility is that A and B are compatible. This completes the proof. ■

6 $\mathcal{F}$ Is a Packable Collection

The previous three lemmas have demonstrated that our collection of sets $\mathcal{F}$ is, in fact, packable. We can then apply Theorem 1.10 to get packings composed of our precircles that are based on a triangulation of the sphere and packed inside decent trilaterals. Our next goal will be to loosen these restrictions so that we can get packings in the unit disk based on triangulations of topological disks.

Chapter 3

Packing Precircles in the Unit Disk

Now that we have a packable collection of sets in $\mathbb{D}$, we can start moving towards building packings. The packings we want need to be maximal, in the sense that precircles on the boundary must intersect $\partial\mathbb{D}$. We also want to have the freedom to force any interior vertex in our triangulation to end up at the origin in our packing. Unfortunately, Theorem 1.10 allows us to pack our precircles only in decent trilaterals; no matter the chosen vertices, $\partial\mathbb{D}$ can never be cornered. Further, we must start with a triangulation of S^2 ; for our packings we are given triangulations of closed topological disks. In this chapter we will work through these problems.

1 Some Technical Details

Let $\mathcal{T}$ be a (finite) CP-complex which triangulates a closed topological disk with vertices $\mathcal{V}(\mathcal{T})$ and edges $\mathcal{E}(\mathcal{T})$. Denote the boundary vertices of $\mathcal{T}$ by $\{a_1, a_2, \dots, a_\ell, b_1, \dots, b_m, c_1, \dots, c_n\}$, listed according to counter-clockwise orientation. We must further assume that ℓ , m , and n are all at least two (which means $\mathcal{T}$ must have

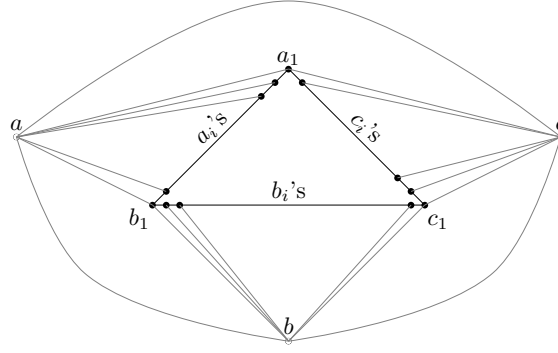


Figure 3.1: Building $\mathcal{T}'$

at least six boundary vertices). Let $\mathcal{T}'$ be the triangulation of S^2 whose vertices are $\mathcal{V}(\mathcal{T}) \cup \{a, b, c\}$ and whose edges are $\mathcal{E}(\mathcal{T})$, along with edges added between a and b ; b and c ; a and c ; $a_1, \dots, b_1$ and a ; $b_1, \dots, c_1$ and b ; and $c_1, \dots, a_1$ and c (see Figure 3.1). Let $\mathcal{F}$ be defined as before. Given three fixed points z_1, z_2, z_3 on $\partial\mathbb{D}$ (numbered according to counterclockwise orientation), let $\{\mathcal{R}_i\}$ be a sequence of decent trilaterals with corners z_1, z_2 , and z_3 such that, for all i , $\mathcal{R}_i \subset \overline{\mathbb{D}}$ and, as i approaches ∞ , $\mathcal{R}_i$ converges to $\partial\mathbb{D}$ in such a way that the edges from z_a to z_b in $\{\mathcal{R}_i\}$ converge to the arc on $\partial\mathbb{D}$ from z_a to z_b . Any sequence of trilaterals that meets these conditions will work. For example, let T be the triangle (including its interior) with vertices z_1, z_2 , and z_3 , and let $\mathcal{R}_i$ be the boundary of the set $T \cup \overline{B(0, 1 - \frac{1}{i+\lambda})}$, where $\lambda > 0$ is chosen so that $\overline{B(0, 1 - \frac{1}{i+\lambda})}$ intersects each edge of T . Let $\{\mathcal{P}_v^i | v \in \mathcal{V}(\mathcal{T})\}$ be a collection of sets in $\mathcal{F}$ which packs $\mathcal{R}_i$, and define $\mathcal{P}_a^i, \mathcal{P}_b^i$, and $\mathcal{P}_c^i$ to be the edges of $\mathcal{R}_i$, with $\mathcal{P}_a^i$ being the edge from z_1 to z_2 , $\mathcal{P}_b^i$ from z_2 to z_3 , and $\mathcal{P}_c^i$ from z_3 to z_1 so that the resulting packing, denoted $\mathcal{P}^i$, has combinatorics given by $\mathcal{T}'$. We now wish to take a limit of these packings; to do so, we must take the limit inside the set of all precircles and boundary precircles (which, by Lemma 2.2, is a continuous set), instead of inside the set of precircles, $\mathcal{F}$. Since $\mathcal{T}$ is finite, it is easy to see that there is a subsequence

of $\{\mathcal{P}^i\}$, for which the sequence of precircles for each vertex in $\mathcal{T}$ converges; for the sake of simplicity, we will re-index the sequence if necessary and continue to call it $\{\mathcal{P}^i\}$.

By continuity of $\mathcal{F}$, $\mathcal{P}_v^i$ has a convergent subsequence if $\bigcup \mathcal{P}_v^i$ is contained in some compact subset of $\mathbb{D}$. There are two possible problems. First, what if our subsequence converges to a set that intersects the boundary of $\mathbb{D}$? This happens whenever v is a boundary vertex; we shall soon see that this only happens when v is a boundary vertex. Also, you may recall that this limit could be a point. The following propositions address these possibilities.

Proposition 3.1. *The sequence $\{\mathcal{P}_v^i\}$ never converges to a point for any vertex v .*

PROOF. Suppose that, for some vertex v_0 , the sequence $\{\mathcal{P}_{v_0}^i\}$ converges to a point z . Consider the neighbors of v_0 in $\mathcal{T}$. Since $\mathcal{T}$ is a CP-complex, there must be at least three of them; call them $\alpha_1, \alpha_2, \dots, \alpha_n$. If the precircle sequence $\{\mathcal{P}_{\alpha_j}^i\}$ does *not* converge to a point for more than two of those neighbors, then the limiting precircles (or boundary precircles) of those sequences must meet at z , since $\mathcal{P}_\alpha^i$ intersects $\mathcal{P}_{v_0}^i$ for every α and i . Also, since $\mathcal{P}_\alpha^i$ and $\mathcal{P}_\beta^i$ have disjoint interiors for all $\alpha \neq \beta$ and for all i , we can conclude that $\mathcal{P}_\alpha$ and $\mathcal{P}_\beta$ have disjoint interiors if $\alpha \neq \beta$. Then we must have more than two precircles or boundary precircles with disjoint interiors meeting at the single point z . This is impossible, since these objects are all blunt. Hence, we conclude that v_0 has at most two neighbors for which the corresponding precircle sequences do not converge to points. Since v_0 has at least three neighbors, there is a vertex v_1 neighboring v_0 such that $\{\mathcal{P}_{v_1}^i\}$ converges to a point. By its incidence with $\mathcal{P}_{v_0}$, that point must be z . By the same reasoning we applied to v_0 , v_1 can have at most two neighbors whose sequences do not converge to a point. But, since $\{\mathcal{P}_{v_1}^i\}$

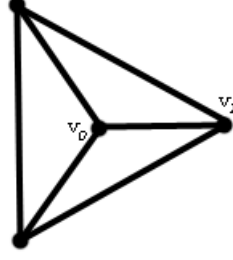


Figure 3.2: Vertices v_0 and v_1 each have degree three, resulting in a triangulation which violates our hypotheses

converges to the same point as $\{\mathcal{P}_{v_0}^i\}$, bluntness assures us that at most two of the neighbors of v_0 and v_1 taken together can have precircle sequences which converge to a point.

What if v_0 and v_1 each have degree three? Then $\mathcal{T}$ looks like Figure 3.2 (up to switching the labels of v_0 and v_1); note that we cannot add more vertices and edges to $\mathcal{T}$ without either changing the degree of v_1 or forcing $\mathcal{T}$ not to be a CP-complex. Thus v_0 or v_1 has degree greater than three; we may conclude that there is a vertex v_2 which shares an edge with either v_0 or v_1 such that $\{\mathcal{P}_{v_2}^i\}$ converges to z , and, as before, at most two of the neighbors of v_0 , v_1 , and v_2 taken together can have precircle sequences which do not converge to points. Can we choose a vertex v_3 such that v_3 is a neighbor of v_0 , v_1 , or v_2 such that $\{\mathcal{P}_{v_3}^i\}$ converges to a point? Let us try to generalize the process we have been describing. Let $\mathcal{V}_D$ denote the set of vertices v in $\mathcal{T}$ such that there exists an edge-connected path from v_0 to $v_p = v$, where $\{\mathcal{P}_{v_k}^i\}$ converges to a point for every k . This set is clearly connected in $\mathcal{T}$, and $\{\mathcal{P}_v^i\}$ must converge to z for every v in $\mathcal{V}_D$. How many vertices in $\mathcal{T}$ can *not* be in $\mathcal{V}_D$? Let $\mathcal{V}_N$ be the set of vertices in $\mathcal{T}$ which share an edge with a vertex in $\mathcal{V}_D$ but are not themselves in $\mathcal{V}_D$. For every vertex w in $\mathcal{V}_N$, $\mathcal{P}_w$ must intersect z ; thus, since $\mathcal{P}_w$ is

blunt for all w in $\mathcal{V}_N$, there can be at most two vertices in $\mathcal{V}_N$. Now, since $\mathcal{T}$ is a CP-complex, removing the two elements of $\mathcal{V}_N$ cannot disconnect $\mathcal{T}$. We conclude that the precircle sequence associated with every vertex in $\mathcal{T}$ except at most two converges to z . Recall that we labelled the boundary vertices of $\mathcal{T}$ $\{a_1, a_2, \dots, a_\ell, b_1, \dots, b_m, c_1, \dots, c_n\}$; it follows from our restrictions on ℓ , m , and n that there exist r , s , and t such that a_r , b_s , and c_t are all in $\mathcal{T}_D$. We also know that $\mathcal{P}_{a_r}$ touches the arc on $\partial\mathbb{D}$ from z_1 to z_2 , $\mathcal{P}_{b_s}$ touches the arc on $\partial\mathbb{D}$ from z_2 to z_3 , and $\mathcal{P}_{c_t}$ touches the arc on $\partial\mathbb{D}$ from z_3 to z_1 . Since z cannot lie on all of these three arcs, we have a contradiction. Hence, $\mathcal{P}_v$ is not a point for every v . ■

Proposition 3.2. *Suppose $\bigcup \mathcal{P}_v^i$ is not contained in any proper compact subset of $\mathbb{D}$ for some $v \in \mathcal{T}$. Then there exists a subsequence of $\{\mathcal{P}_v^i\}$ which converges to a boundary precircle.*

PROOF. Let us consider what happens to these sets under f . Let $\widehat{\mathcal{P}}_v^i$ be the image of $\mathcal{P}_v^i$ in Ω . By our assumption, $(\bigcup \widehat{\mathcal{P}}_v^i) \cap \partial\Omega$ cannot be contained in any compact subset of Ω . So, we have a sequence of circles in $\mathbb{C}$ that are contained in a compact subset of $\mathbb{C}$ (namely, $\overline{\Omega}$). So, by Lemma 2.2, this sequence has a subsequence that converges to either a circle in $\overline{\Omega}$ or a point on $\partial\Omega$. Proposition 3.1 rules out the second case, so the limit must be a circle C in $\overline{\Omega}$. The preimage of $C \cap \Omega$ is a boundary precircle in $\mathbb{D}$, and the proof is complete. ■

Now, since the precircle sequences associated with boundary vertices do not degenerate to points, there exists a continuous closed curve contained in $\bigcup \mathcal{P}_{\mathcal{V}_B}$, where $\mathcal{V}_B$ denotes the boundary vertices of $\mathcal{T}$; this curve encloses all of the interior precircles.

Thus, the precircle sequences associated with interior vertices in $\mathcal{T}$ cannot converge to boundary precircles. Also, this proposition shows that the collection of boundary precircles are the limit points of sequences in $\mathcal{F}$; hence, the boundary precircles are the boundary of $\mathcal{F}$ in its topology as a 3-manifold.

We have thus demonstrated that we can build a non-degenerate precircle packing in the unit disk based on a given triangulation. We must now check that we can place any interior vertex at the origin.

Let $\mathbf{T}$ be the closed triangular region in the plane with vertices $(0, 0)$, $(0, 2\pi)$, and $(2\pi, 2\pi)$. Let D be the closed unit disk, with fixed point $\zeta \in \partial D$. Given a finite triangulation $\mathcal{T}$ and a fixed interior vertex v , define $f_v : \mathbf{T} \rightarrow D$ by letting $f_v(\theta_b, \theta_c)$ be the location of v in the (possibly degenerate) packing created by our techniques by forcing edge b to start at ζ and continue counter-clockwise with length θ_b , and edge c to start where b ends, with length $\theta_c - \theta_b$.

Claim: f_v is surjective.

If the claim is true, we can then choose values for θ_b and θ_c so that, in the resulting packing, vertex v will be at the origin.

PROOF OF CLAIM. First, consider the map $\tilde{f}_v = f_v|_{\partial \mathbf{T}}$. As we have seen, as any of the edges a , b , or c shrink to points, the interior of the packing shrinks to the same point. Thus, as $\theta_b \rightarrow 0$ or $\theta_c \rightarrow 2\pi$, $f_v(\theta_b, \theta_c) \rightarrow \zeta$. Also, if $\theta_b \rightarrow \theta_0$ and $\theta_c \rightarrow \theta_0$, then $f_v(\theta_b, \theta_c)$ approaches the point on ∂D that is at an angle of θ_0 from ζ .

Now, suppose f_v is not onto; say $u \in D \setminus f_v(\mathbf{T})$. Then there exists a retraction $r : f_v(\mathbf{T}) \rightarrow \partial D$. We then have the commutative diagram pictured in Figure 3.3(a). Passing to the first fundamental group level, we get the commutative diagram seen in Figure 3.3(b).

Since the homomorphism induced by $\tilde{f}_v$, $(\tilde{f}_v)_* : \pi_1(\partial \mathbf{T}) \rightarrow \pi_1(\partial D)$, maps a

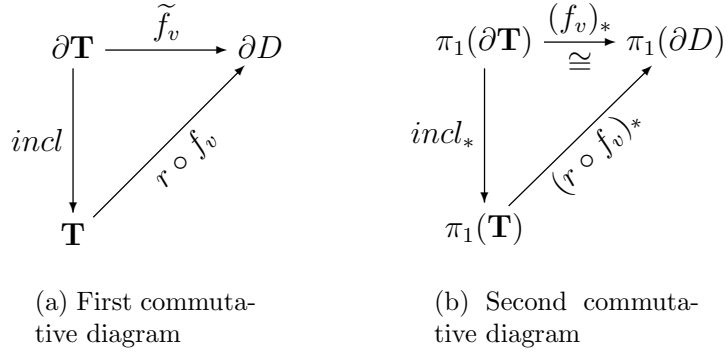


Figure 3.3: Commutative diagrams

generator of $\pi_1(\partial\mathbf{T})$ to a generator of $\pi_1(\partial D)$ as demonstrated above, it must in fact be an isomorphism. Since the other path through the diagram passes through a trivial group, it cannot represent an isomorphism. This contradicts the fact that our diagram is commutative, and the claim is proven. ■

Finally, we wish to conclude that these precircle packings are essentially unique.

Theorem 3.3. *Given any triangulation $\mathcal{T}$ of a closed topological disk with at least six boundary vertices, a bounded, simply-connected region $\Omega \subset \mathbb{C}$, and a conformal map $f : \mathbb{D} \rightarrow \Omega$, there exists a precircle packing $\mathcal{P}$ with combinatorics $\mathcal{T}$ which fills $\mathbb{D}$ and is unique up to a family of transformations which depend on three parameters.*

PROOF. We have already seen techniques which, given $\mathcal{T}$, produce a packing which fills $\mathbb{D}$; we need only show uniqueness. Our proof will depend on a theorem found in [8]:

Theorem 3.4. *Let $T = T(V)$ be an oriented triangulation of a quadrilateral with boundary vertices a, b, c , and d in clockwise order with respect to the other vertices of T (if such exist). Let D be a quadrilateral in S^2 with edges D_1, D_2, D_3 , and*

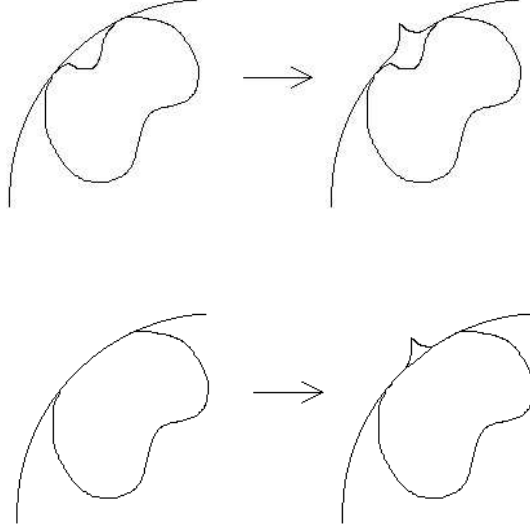


Figure 3.4: Adding corners

D_4 . Suppose that $\mathcal{Q} = \{\mathcal{Q}_v | v \in V\}$ and $\mathcal{P} = \{\mathcal{P}_v | v \in V\}$ are two nondegenerate packings in D with combinatorics T . Suppose also that $\mathcal{Q}_v$ and $\mathcal{P}_v$ are disklike for $v \in V \setminus \{a, b, c, d\}$, that $\mathcal{P}_a \subset D_1$, $\mathcal{Q}_b \subset D_2$, $\mathcal{P}_c \subset D_3$, $\mathcal{Q}_d \supset D_4$, and that $\mathcal{P}_v$ does not intersect $\mathcal{Q}_d$ for any $v \in V \setminus \{a, c, d\}$. Then there is some vertex $v \in V \setminus \{a, b, c, d\}$ for which $\mathcal{Q}_v$ and $\mathcal{P}_v$ are incompatible.

Let $\mathcal{P}$ be a precircle packing filling $\mathbb{D}$ with combinatorics $\mathcal{T}$. We have a couple of cases depending on the nature of the boundary precircles of $\mathcal{P}$:

CASE I. Suppose that the closures of three or more boundary precircles intersect $\partial\mathbb{D}$ at more than one point. Choose three such boundary precircles. We can augment $\mathbb{D}$ to add corners on its boundary to get a cornered trilateral that is packed by $\mathcal{P}$ (see Figure 3.4).

We can now apply Theorem 1.10 to obtain our result.

CASE II. Suppose that the closures of two or fewer boundary precircles in-

intersect $\partial\mathbb{D}$ at more than one point. Choose three distinct boundary precircles $\mathcal{P}_d$, $\mathcal{P}_e$, and $\mathcal{P}_f$ listed according to counter-clockwise orientation. By our supposition, at least one of these must have a closure that intersects $\partial\mathbb{D}$ in one point; without loss of generality, we assume that $\mathcal{P}_d$ has that property. Call that point on $\partial\mathbb{D}$ z_1 . Choose a point in each of $\overline{\mathcal{P}_e} \cap \partial\mathbb{D}$ and $\overline{\mathcal{P}_f} \cap \partial\mathbb{D}$ and label them z_2 and z_3 . Let D_a be the arc from z_1 to z_2 , D_b be the arc from z_2 to z_3 , and D_c be the arc from z_3 to z_1 , where the arcs are taken counterclockwise in $\partial\mathbb{D}$, and let D be the trilateral with edges D_a , D_b , and D_c . We now expand our triangulation $\mathcal{T}$ of a closed topological disk to a triangulation $\mathcal{T}'$ of a sphere by adding a vertex a adjacent to all the boundary vertices of $\mathcal{T}$ from d to e (according to orientation), a vertex b adjacent to all the boundary vertices of $\mathcal{T}$ from e to f , a vertex c adjacent to all the boundary vertices of $\mathcal{T}$ from f to d , and edges connecting a , b , and c . Now, by defining $\mathcal{P}_a = D_a$, $\mathcal{P}_b = D_b$, and $\mathcal{P}_c = D_c$, we have a packing for $\mathcal{T}'$. If we show that this packing is unique, our result will follow. Our proof is inspired by the proof of Theorem 1.10 given in [7].

We will show that we get unique packings for any triangulation of the sphere and any trilateral, so let us forget for the moment that $\mathcal{T}'$ is in any way related to $\mathcal{T}$, as well as all of the naming conventions seen so far, save our triangulations $\mathcal{T}$ and $\mathcal{T}'$.

Suppose τ is any simple closed curve in $\mathbb{C}$ with no inward (that is, toward the bounded component of $\mathbb{C} \setminus \tau$) cusps of angle 2π , and let z_1 , z_2 , and z_3 be distinct points on τ . We define D_1 , D_2 , and D_3 to be the sections of τ between the z_i as above. We have thus associated with τ a trilateral D . Let $\check{\tau}$ be the compact set bounded by τ .

Suppose $\mathcal{P}$ and $\mathcal{Q}$ are packings with combinatorics isomorphic to $\mathcal{T}'$ such that $\mathcal{P}_j = \mathcal{Q}_j = D_j$ for $j = a, b, c$ and $\mathcal{P}_v$ and $\mathcal{Q}_v$ are taken from the same packable

collection for all other vertices v .

We proceed by induction on the number of vertices in $\mathcal{T}'$. If a , b , and c are the only vertices in $\mathcal{T}'$, we are done. So, let us assume that the number of vertices in $\mathcal{T}'$ is at least four and that the result holds for triangulations with fewer vertices than $\mathcal{T}'$.

Let d be the vertex in $\mathcal{T}'$ besides b that forms a face with a and c . (We are secretly identifying this vertex with the vertex d chosen in $\mathcal{T}$ above, but for now we still assume no connection between the two triangulations.) Suppose that $\mathcal{P}_d \neq \mathcal{Q}_d$. Let r and s be points in the intersections of $\mathcal{Q}_d$ with D_a and D_c , respectively; let r' and s' be points in the intersections of $\mathcal{P}_d$ with D_a and D_c , respectively. Note the possibility that $r = s$ and/or $r' = s'$. Since $\mathcal{P}_d$ and $\mathcal{Q}_d$ come from the same packable collection, neither of these sets may cut the other. Thus, we may choose a simple curve $\alpha : [0, 1] \rightarrow \check{\tau}$ with $\alpha(0) = r$, $\alpha(1) = s$, and $\alpha(t)$ in the interior of $\mathcal{Q}_d \setminus \mathcal{P}_d$. Similarly, choose a simple curve $\alpha' : [0, 1] \rightarrow \check{\tau}$ with $\alpha'(0) = r'$, $\alpha'(1) = s'$, and $\alpha'(t)$ in the interior of $\mathcal{P}_d \setminus \mathcal{Q}_d$. Now either α separates $\mathcal{P}_d$ from D_b in $\check{\tau}$ or α' separates $\mathcal{Q}_d$ from D_b in $\check{\tau}$. (We must point out a slight technicality here. If $r = s = r' = s'$, our initial choices of α and α' may not have this property. In this case, compatibility guarantees that suitable choices for α and α' can be made.) Without loss of generality, we may assume that α' separates $\mathcal{Q}_d$ from D_b . This separation implies that $\mathcal{P}_v$ is disjoint from $\mathcal{Q}_d$ for all v besides d , a , and c .

Suppose r and s are not equal. Then α splits D into a quadrilateral and a trilateral. Call the quadrilateral $\widehat{D}$. Define new packings $\widehat{\mathcal{P}}$ and $\widehat{\mathcal{Q}}$ by $\widehat{\mathcal{P}}_v = \mathcal{P}_v \cap \widehat{D}$ and $\widehat{\mathcal{Q}}_v = \mathcal{Q}_v \cap \widehat{D}$ for all v . Then the packings $\widehat{\mathcal{P}}$ and $\widehat{\mathcal{Q}}$, along with the quadrilateral $\widehat{D}$, satisfy the hypotheses for Theorem 3.4. Hence, for some $v \neq a, b, c, d$, $\mathcal{P}_v$ and $\mathcal{Q}_v$ are incompatible for some v . However, $\mathcal{P}_v$ and $\mathcal{Q}_v$ are from the same packable

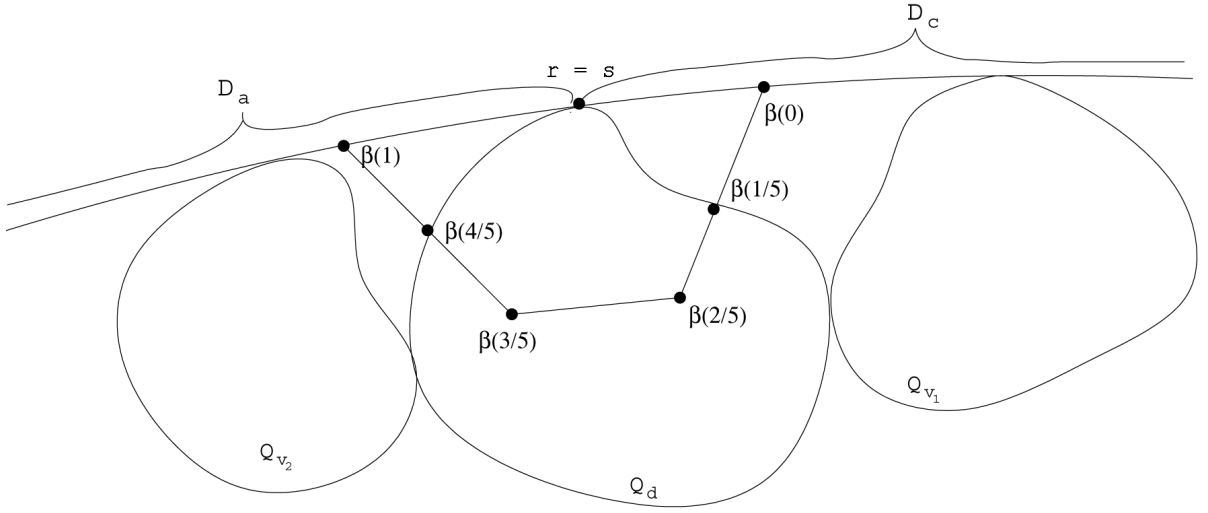


Figure 3.5: Constructing β

collection, which by definition are compatible. This is a contradiction, so either $r = s$ or $\mathcal{P}_d = \mathcal{Q}_d$.

Suppose $r = s$. Let v_1 and v_2 be the neighbors of d with $[v_1, c, d]$ forming an ordered face in $\mathcal{T}'$ and $[v_2, d, a]$ also forming a face. Choose points w_1 and w_2 in τ such that w_1 is between $\mathcal{Q}_{v_1} \cap \tau$ and r and w_2 is between r and $\mathcal{Q}_{v_2} \cap \tau$. We build a simple path $\beta : [0, 1] \rightarrow \tilde{\tau}$ such that $\beta(0) = w_1$, $\beta|_{(0, \frac{1}{5})} \subset \tilde{\tau} \setminus (\bigcup \mathcal{Q}_v)$, $\beta|_{[\frac{1}{5}, \frac{4}{5}]} \subset \mathcal{Q}_d$, $\beta|_{(\frac{4}{5}, 1)} \subset \tilde{\tau} \setminus (\bigcup \mathcal{Q}_v)$, and $\beta(1) = w_2$ (see Figure 3.5). Define $\hat{D}$ to be the quadrilateral with edges $\beta|_{[\frac{3}{5}, 1]}$ glued to the section of τ from w_2 to z_2 , D_b , the section of τ from z_3 to w_1 glued to $\beta|_{[0, \frac{2}{5}]}$, and $\beta|_{[\frac{2}{5}, \frac{4}{5}]}$. Define the packings $\hat{\mathcal{P}}$ and $\hat{\mathcal{Q}}$ as above; we can then apply Theorem 3.4 to these packings in the quadrilateral $\hat{D}$ to obtain a vertex v for which $\mathcal{P}_v$ and $\mathcal{Q}_v$ are incompatible. As before this is a contradiction; we conclude that $\mathcal{P}_d = \mathcal{Q}_d$.

For $v \neq d$, we apply our inductive hypothesis. Remove $\mathcal{P}_d = \mathcal{Q}_d$ from both packings. Redefine D_a and D_c (if necessary) so that the result is a packing whose combinatorics give a triangulation of a sphere. Since the number of vertices is one less

than the number of vertices in $\mathcal{T}'$, the two packings must coincide. Hence, $\mathcal{P}_v = \mathcal{Q}_v$ for all $v \in \mathcal{T}'$.

Now, by reestablishing the connection between $\mathcal{T}'$ and $\mathcal{T}$, it follows that $\mathcal{P}_v = \mathcal{Q}_v$ for all $v \in \mathcal{T}$. Since this result holds for any trilateral, it holds for the trilateral whose edges are arcs of the unit circle we discussed at the beginning of our treatment of Case II.

Finally, by noting that we can build packings such as these in any trilateral by choosing appropriate values for the fixed point ζ and the angles θ_b and θ_c (thereby exhausting our three parameters), the proof is finished. ■

2 Proof of Theorem 1.11

The preceding arguments lead up to the following:

Theorem 3.5. *Let $\mathcal{T}$ be a triangulation of a closed topological disk with at least six boundary vertices. Let $\Omega \subset \mathbb{C}$ be a simply connected bounded open set, where $\partial\Omega$ has no inward cusps of angle 2π . Let $f : \mathbb{D} \rightarrow \Omega$ be a conformal bijection. Then, for any vertex v in $\mathcal{T}$ and a given neighbor w of v , there exists a unique maximal precircle packing $\mathcal{P}$ with complex $\mathcal{T}$ in $\mathbb{D}$ such that the preimage under f of the centers of the circles $f(\mathcal{P}_v)$ and $f(\mathcal{P}_w)$ are at the origin and on the real axis, respectively.*

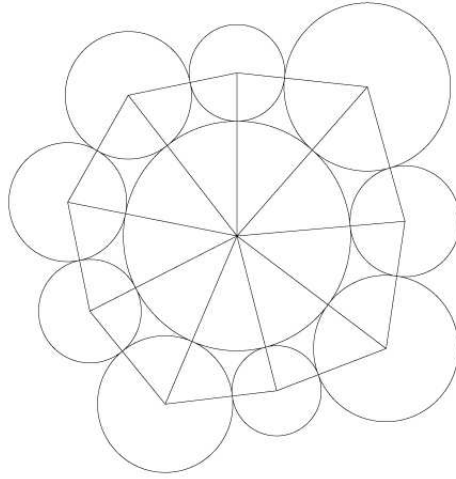
PROOF OF THEOREM 1.11. Let $g : \mathbb{D} \rightarrow \Omega$ be a conformal bijection that sends the origin to z . We can then use Theorem 3.5 to obtain a precircle packing; applying g to this precircle packing gives us our desired circle packing in Ω . ■

Chapter 4

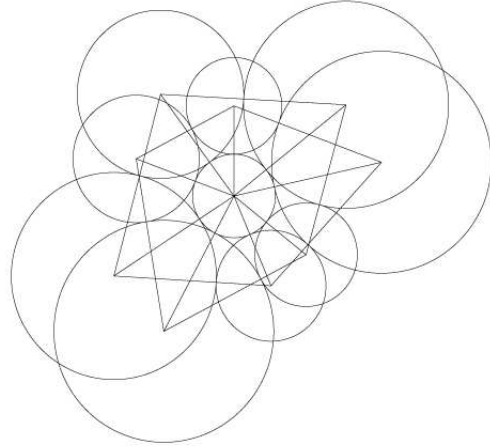
Branched Packings in the Unit Disk

1 Non-Univalent Packings

A univalent circle packing is one in which the interiors of the circles in the packing are pairwise disjoint. Not all circle packings, however, are univalent. In Figure 4.1 we see two flowers of circles with nine petals, with the segments connecting the centers of all circles drawn in. In Figure 4.1(a), the petals wrap around the central circle exactly once. This is most easily seen by studying the angles made by the radial segments at the central circle's center. Now consider Figure 4.1(b). Notice how the petals now wrap around twice. The center circle is called a *branch circle*, and its associated vertex in the triangulation is a *branch vertex*. *Branched packings* are packings that contain at least one branch circle. In practice, branched packings are used to approximate multivalent functions, like the logarithm or root functions. In this chapter, we will build branched packings in the unit disk that have a single



(a) A univalent flower



(b) A branched flower

Figure 4.1: Two flowers with nine petals

branch circle using univalent precircle packings.

2 Defining Precircles

An important question to consider at this point is: What do we mean by precircles in this context? Let $f : \mathbb{D} \rightarrow \Omega$ be a continuous function which is analytic on $\mathbb{D}$, has a nonzero derivative everywhere except at a point z_0 , and has an associated integer n such that, for any closed loop γ in $\mathbb{D} \setminus \{z_0\}$, the winding number of $f \circ \gamma$ in Ω around $f(z_0)$ is n times the winding number of γ in $\mathbb{D}$ around z_0 . Figure 4.2 illustrates this idea; here n is 3. One consequence of this property of f is that, except for $f(z_0)$, each point in Ω has three preimages in $\mathbb{D}$. Figure 4.3 illustrates this. The segment on the right which cuts from $f(z_0)$ to the boundary of Ω gets pulled back to

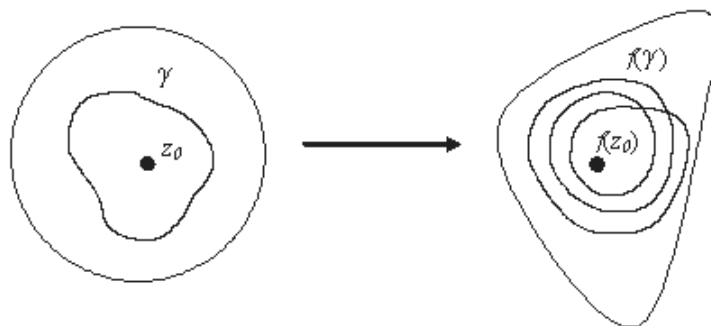


Figure 4.2: The effect of f on winding number

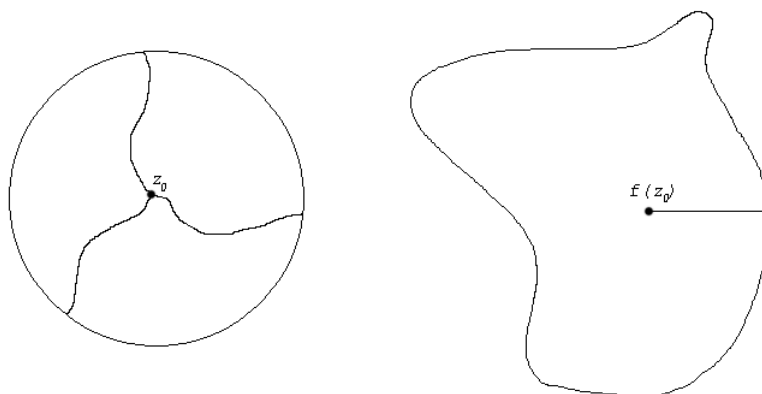


Figure 4.3: Fundamental domains

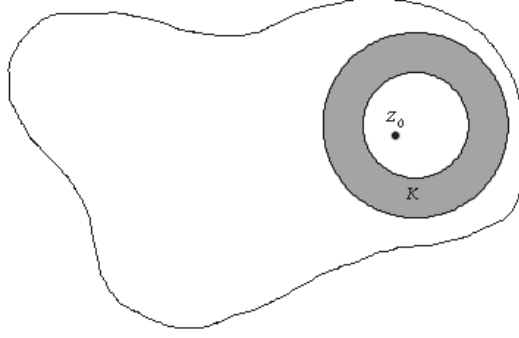


Figure 4.4: Defining $\mathcal{K}$

three paths, each of which cuts $\mathbb{D}$ from z_0 to the boundary of $\mathbb{D}$. These paths divide $\mathbb{D}$ into three sectors, each of which is mapped by f onto Ω , with $f(z_0)$ and the added segment removed. These sectors are called *fundamental domains*. Note that there is a set of fundamental domains for each simple path (called a *branch cut*) from $f(z_0)$ to $\partial\Omega$, and that f restricted to any fundamental domain is a conformal bijection onto its image. Let C be a closed circular disk in Ω which does not contain $f(z_0)$. Then the set $f^{-1}(C)$ will have n connected components in $\mathbb{D}$ (one in each fundamental domain associated with a branch cut that does not intersect C). We will define a precircle associated with a disk C not containing $f(z_0)$ to be any closed topological disk in $\mathbb{D}$ whose image in Ω is C ; let $\mathcal{F}_R$ be the collection of all of these precircles. By this definition, every disk in Ω not containing $f(z_0)$ may have as many as n associated precircles.

Let $\mathcal{K}$ be the collection of all closed (circular) annuli K in Ω whose inner radius is half of its outer radius, for which the distance from the center of K to $f(z_0)$ is less than or equal to half the inner radius of K , and ∂K does not intersect $\partial\Omega$. This ensures that our branch value $f(z_0)$ is inside the hole of each annulus in $\mathcal{K}$ (see Figure 4.4). Define $\mathcal{F}_B$ to be the set of preimages under f of the elements of $\mathcal{K}$.

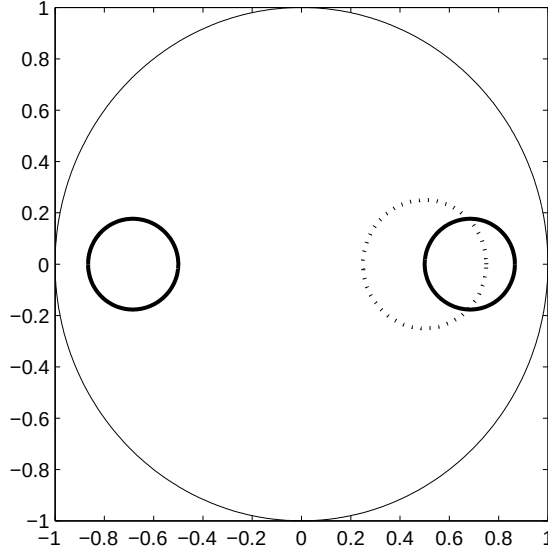


Figure 4.5: The circle $|z - \frac{1}{2}| = \frac{1}{4}$ (dashed), with its two precircles under p

Let us consider another example. Define the function $p : \mathbb{D} \rightarrow \mathbb{D}$ by $z \mapsto z^2$. Then p has nonzero derivative everywhere except at 0, and the winding number around 0 of any closed loop doubles in the image. Figure 4.5 shows the two connected components we get when we pull back the disk $|z - \frac{1}{2}| \leq \frac{1}{4}$ under p ; these components will be the two precircles associated with that disk.

3 $\mathcal{F}_R$ Is a Packable Collection

We must now verify that our collection $\mathcal{F}_R$ is a packable collection of blunt disklikes. Let A be an element of $\mathcal{F}_R$. Let D_A be a fundamental domain containing A . Since the restriction of f to D_A is a conformal bijection, A is blunt and disklike if $f(A)$ is blunt and disklike. This is clearly the case, because $f(A)$ is a circular disk. It remains to show that $\mathcal{F}_R$ is a continuous manifold, whose members are pairwise

compatible. By the same argument we used in Chapter 2, the elements of $\mathcal{F}_R$ are compatible. To verify that $\mathcal{F}_R$ is continuous and a manifold, we must modify our previous arguments somewhat.

Recall that the elements of $\mathcal{F}_R$ were chosen so that each does not contain the branch point z_0 . So, for the moment, let us consider the elements of $\mathcal{F}_R$ as subsets of $\mathbb{D}^* = \mathbb{D} \setminus \{z_0\}$. We shall first demonstrate that $\mathcal{F}_R$ is continuous. Let $\{A_n\}$ be a sequence in $\mathcal{F}_R$ of sets that are contained in some compact subset Q of $\mathbb{D}^*$. The proof follows almost at once from the proof of Lemma 2.1; the only ambiguity is the possibility that our sequence approaches the deleted point z_0 in the limit. However, since we require our sequence to stay within the compact set Q , our sequence is bounded away from z_0 . It then follows that $\mathcal{F}_R$ is continuous.

We now check that $\mathcal{F}_R$ is a 3-manifold. Choose an element A in $\mathcal{F}_R$. We wish to find a neighborhood of A in the topology induced on $\mathcal{F}_R$ by our definition of continuity that is homeomorphic to an open subset of $\mathbb{R}^3$. Consider the image of A in Ω . Let z_A be the center of the disk, and let r_A be the disk's radius. Let R_A be the radius of the smallest disk centered at z_A such that the boundary of $M = B(z_A, R_A)$ intersects the boundary of $\Omega \setminus \{f(z_0)\}$ (note R_A is simply the distance from the point z_A to the boundary of $\Omega \setminus \{f(z_0)\}$). See Figure 4.6. Let U be the collection of disks contained in M ; this is an open neighborhood of A in $\mathcal{F}_R$. We now need to define a homeomorphism between U and an open subset V of $\mathbb{R}^3$. We will say that $\eta = (x, y, z)$ is in V if $D((x, y), z)$ is in U . This defines a bijection between U and V ; since convergence in U implies convergence in V (and vice versa), U and V are homeomorphic. Hence, $\mathcal{F}_R$ is a manifold. It follows that $\mathcal{F}_R$ is a packable collection.

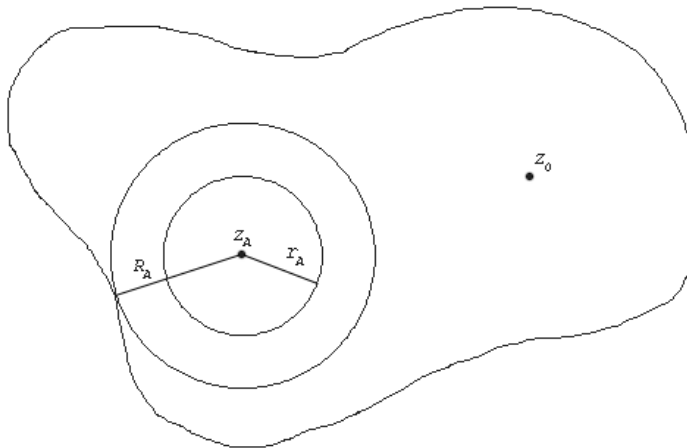


Figure 4.6: Defining r_A and R_A

4 $\mathcal{F}_B$ Is a Packable Collection

We now must check that $\mathcal{F}_B$ is a packable collection of blunt disklikes. Let B be an element of $\mathcal{F}_B$; let $K = f(B)$. We must use Schramm's definition of *disklike* in a new way. Recall Definition 1.3, which says that, in our situation, the set B is disklike if it is the closure of its interior, its interior is connected, and the complement of any connected component of B^c is a topological disk. Since the only bounded component of the complement of B is a topological disk, we see that B is disklike, but not a disk. Our set B is also blunt, since p is conformal away from z_0 and the image of the outer boundary of B is a circle in Ω .

We now must redefine our notion of compatibility. Recall that Definition 1.8 applies only to topological disks.

Definition 4.1. Let A and B be two sets which are disklike, but not topological disks. Let $\{A'_i\}$ be the collection of all connected components of the complement of A , where i is an element of some indexing set I (it can be shown that I must be

finite, but we don't need that fact). Define A_j to be the set

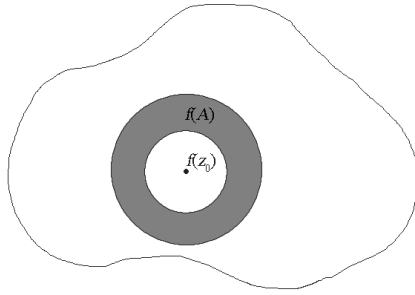
$$A \cup \left(\bigcup_{i \neq j} A'_i \right);$$

define B_k in a similar way. We say that A and B are *disklike-compatible* if $A = B$ or if A_j and B_k are compatible for all possible j and k .

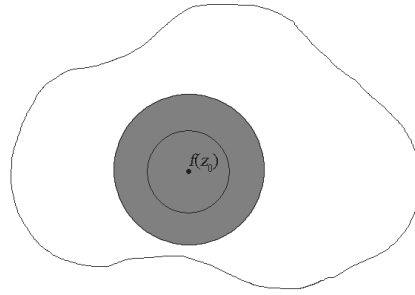
(N.B.— Definition 4.1 is actually Schramm's only definition of *compatible*. Note that, if A is a topological disk, then $A_j = A$ since A^c has only one component; then our Definitions 1.8 and 4.1 coincide. We split the definitions for the sake of clarity in the exposition.)

Suppose $A \neq B$ are elements of $\mathcal{F}_B$. Then $f(A)$ and $f(B)$ are annuli. Consider what $f(A_j)$ must look like. Since A^c has only two components, there are only two possibilities for A_j : it includes the branch point z_0 or it doesn't. If $z_0 \in A_j$, then $\partial f(A_j)$ is simply the outer circle of the annulus $f(A)$; if z_0 is not an element of A_j , then $\partial f(A_j)$ is the outer circle of $f(A)$ (Figure 4.7). We have seen that compatibility depends entirely upon the boundary of the sets in question; in particular, if two different sets are bounded by circles, then one cannot cut the other. Hence, the topological disks A_j and B_k are compatible for all j and k ; it then follows that A and B are disklike-compatible.

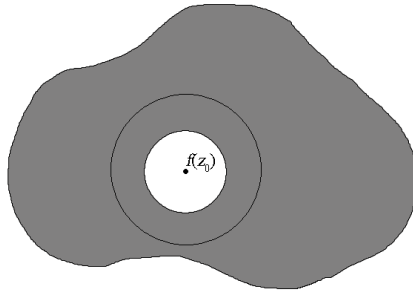
Let B_n be a sequence of elements in $\mathcal{F}_B$ which are all contained in some compact set. By choosing subsequences if necessary, we may assume that c_n , the sequence of centers of the annuli $f(B_n)$, converges to a point c , and that r_n , the sequence of inner radii of $f(B_n)$, converges to r . If $r = 0$, then $\lim B_n$ is a point (namely, z_0). Let us assume, then, that $r > 0$. We wish to show that $\lim B_n$ is the preimage under f of the annulus A centered at c with inner radius r and outer radius



(a) The annulus $f(A)$



(b) $f(A_j)$, if z_0 is in A_j



(c) $f(A_j)$, if z_0 is *not* in A_j

Figure 4.7: The annulus $f(A)$, with the two possibilities for $f(A_j)$

$2r$.

Let x be an element of $\overline{\lim}f(B_n)$. Then there must be some sequence $\{x_n\}$, with $x_n \in f(B_n)$, that has x as an accumulation point. Choose $\epsilon > 0$. Now $|x_n - c_n| \geq r_n$ for all n ; for some sufficiently large N , $|x_N - x| < \frac{\epsilon}{3}$, $|c_N - c| < \frac{\epsilon}{3}$, and $|r_N - r| < \frac{\epsilon}{6}$. Hence, the smallest $|x - c|$ could possibly be is $|x_N - c_N| - |x_N - x| - |c_N - c|$, which is at worst $r_N - \frac{2\epsilon}{3}$; it follows that $|x - c|$ is at least as big as $r - \epsilon$. Since ϵ was chosen arbitrarily, we may conclude that $|x - c| \geq r$. We also have that $|x_n - c_n| \leq 2r_n$ for all n , and $|2r_N - 2r| < \frac{\epsilon}{3}$. Hence

$$\begin{aligned} |x - c| &= |x - x_N + x_N - c_N + c_N - c| \\ &\leq |x - x_N| + |x_N - c_N| + |c_N - c| \\ &< \frac{\epsilon}{3} + 2r_N + \frac{\epsilon}{3} \\ &< \frac{\epsilon}{3} + 2r + 2(\frac{\epsilon}{6}) + \frac{\epsilon}{3} \\ &= 2r + \epsilon. \end{aligned}$$

We then conclude that $|x - c| \leq 2r$. It follows that $\overline{\lim}f(B_n) \subset A$.

Let a be an element of A . We wish to demonstrate that there exists a sequence $\{a_n | a_n \in f(B_n)\}$ which converges to a . It suffices to show that, for any $\epsilon > 0$, there exists an N such that the open disk centered at a with radius ϵ (denoted $B(a, \epsilon)$) intersects $f(B_n)$ for all $n \geq N$. Choose N large enough so that $|c_N - c| < \frac{\epsilon}{4}$ and $|r_N - r| < \frac{\epsilon}{4}$. Figure 4.8 demonstrates our situation. Here, the shaded annulus is centered at x ; the other at y . The figure has been pictured so that the difference in the inner radii of the annuli is the same as the distance from x to y . The two “worst case scenarios” for a are shown. As you can see, if the unshaded annulus is A and the shaded annulus is $f(B_N)$, then the maximum distance from a point in A

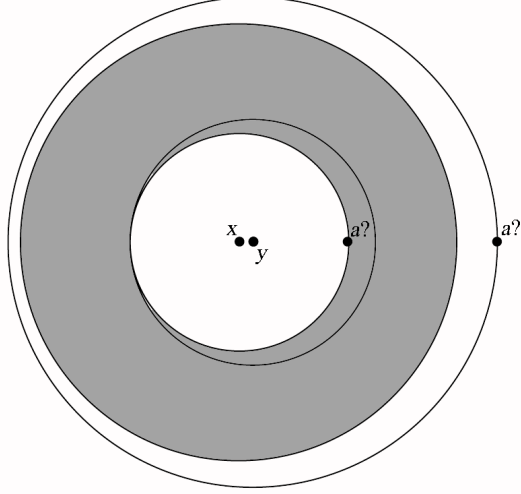


Figure 4.8: Two annuli

to the nearest point in $f(B_N)$ is $|c - c_N| + 2 \cdot |r - r_N| < \frac{3\epsilon}{4}$. Thus, $A \subset \underline{\lim} f(B_n)$. We have thus demonstrated that the images under f of the elements of $\mathcal{F}_B$ comprise a continuous family. Since continuous functions respect limit points and convergent sequences, it follows that $\mathcal{F}_B$ is continuous.

It remains to show that $\mathcal{F}_B$ is a 3-manifold. Choose an element B in $\mathcal{F}_B$. Let c be the center of $f(B)$, and let r be the inner radius of $f(B)$. We wish to show that B has a neighborhood in $\mathcal{F}_B$ which is homeomorphic to an open subset of $\mathbb{R}^3$; two of the coordinates will come from our freedom to move c slightly, and the third will come from our freedom to change r slightly. Let $\delta = |c - f(z_0)|$, and let $\epsilon = r - \delta$ (note that ϵ is positive, since $f(z_0)$ is inside the hole of the annulus $f(B)$). Let $\tau = \min\{|z - c| : z \in \partial\Omega\}$, and let $\sigma = \tau - 2r$ (which is positive since $f(B)$ has outer radius $2r$ and is contained in Ω). Let $\zeta = \frac{1}{3} \min\{\sigma, \epsilon\}$. Define our neighborhood of B to be the set of preimages of annuli centered at c' with inner radius r' and outer radius

$2r'$, where $|c - c'| < \zeta$ and $|r - r'| < \zeta$. By our construction, each of these annuli are contained in Ω , and the point $f(z_0)$ is contained in the hole of each. Also, it is easy to see that this neighborhood is homeomorphic to the open cylinder $B(0, \zeta) \times (-\zeta, \zeta)$ in $\mathbb{R}^3$. It follows then that $\mathcal{F}_B$ is a 3-manifold.

5 Creating the Packing

Let $\mathcal{T}$ be a triangulation as before. Choose an interior vertex v_0 to be our branch vertex. How do we use these two packable collections, $\mathcal{F}_R$ and $\mathcal{F}_B$, to build a single precircle packing? Recall that Theorem 1.10 required only that each vertex in our triangulation have some packable collection of blunt disklikes associated with it; not that all vertices must use the same collection. Therefore, we will associate with v_0 the collection $\mathcal{F}_B$; these are the precircles that surround the branch point of our map f . With all other vertices, we will associate the collection $\mathcal{F}_R$. By our work done in Chapter 3, we can create a *univalent* precircle packing in $\mathbb{D}$ so that the preimage of the center of the annulus associated with vertex v_0 is located at z_0 . By filling in the hole, we get a set whose image is a circular disk in Ω , centered at the branch value $f(z_0)$. It follows that the image of our precircle packing in Ω is in fact a circle packing. Let γ be an edge-connected path of vertices in $\mathcal{T}$ that wraps around v_0 exactly once. The path γ is naturally associated with a path in $\mathbb{D}$ which wraps around z_0 exactly once; the image of the path therefore must wrap around $f(z_0)$ n times. Since this image path goes through the centers of the circles associated with the vertices in γ , we see that our circle packing is branched. We have thus proven the following:

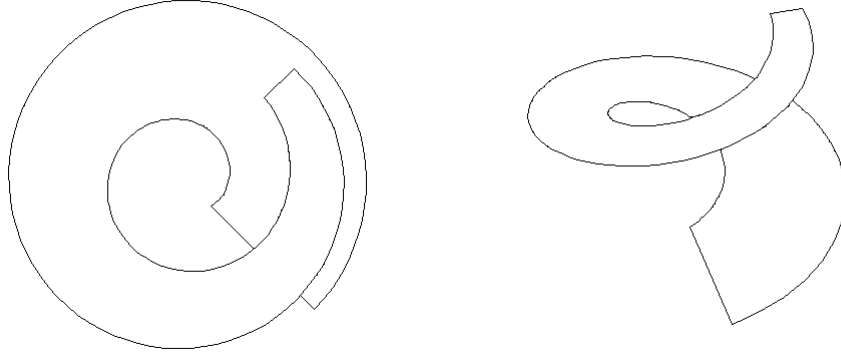
Theorem 4.2. *Let $f : \mathbb{D} \rightarrow \Omega$ be a continuous function which is analytic on $\mathbb{D}$, has*

a nonzero derivative everywhere except at a point z_0 , and has an associated integer n such that, for any closed loop γ in $\mathbb{D} \setminus \{z_0\}$, the winding number of $f \circ \gamma$ in Ω around $f(z_0)$ is n times the winding number of γ in $\mathbb{D}$ around z_0 . For any finite CP-complex T with distinguished vertex v_0 , there exists a maximal packing of sets in $\mathbb{D}$ with combinatorics $\mathcal{T}$ such that the image under f of the packing is a branched circle packing which fills Ω and whose branch circle is associated with v_0 .

Chapter 5

Locally Univalent Branched Packings

We now turn our attention to a related class of packings. Here, our circle packings will fill the conformal image Ω of the disk $\mathbb{D}$, where the conformal map f is locally one-to-one, but not necessarily globally one-to-one. We assume that f extends continuously to a map f^* with domain $\overline{\mathbb{D}}$ in such a way that the image of the closed disk intersects itself. An example of such an image is given in Figure 5.1. In this example, the image appears to be a topological annulus, but the second view shows that the image is actually a topological disk that wraps around a branch point (in the center of the annulus) far enough that it overlaps itself. The circle packing will therefore not be univalent, in the sense that the interiors of the packed circles may overlap; however, the packing will be *locally* univalent, i.e., the interiors of a given circle and its neighbors in the packing will be mutually disjoint.



(a) The region in the plane

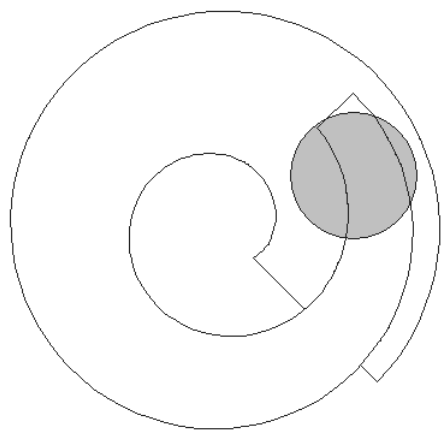
(b) An “exploded” view

Figure 5.1: A locally univalent branched surface

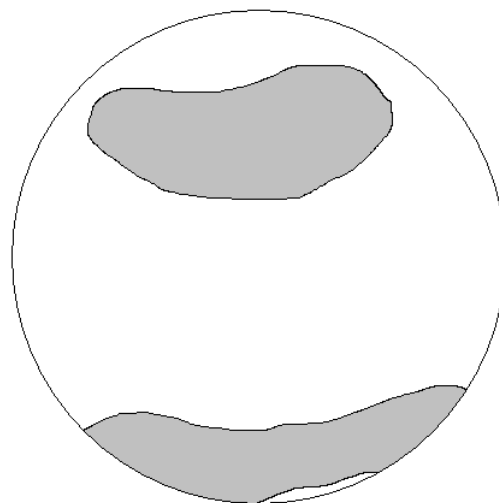
1 Defining Precircles

In Chapter 2, we defined a precircle to be the conformal preimage of a circular disk. The difficulty with defining precircles in this setting is that the preimage of a single circular disk may have more than one component. It is tempting simply to define a precircle to be a component of the preimage of a circular disk; however, the image of each component may not be a circular disk, as desired. See Figure 5.2. Therefore, we define our precircles as follows: let $\mathcal{F}$ be the collection of connected sets A such that the closure of A is contained in $\mathbb{D}$ and the closure of $f(A)$ is a closed circular disk in Ω . Note that if the closure of A in $\overline{\mathbb{D}}$ intersects the boundary of $\mathbb{D}$, then $f(A)$ is not a closed disk in Ω . In this case, $\overline{f(A)}$ will be tangent to $f^*(\partial\mathbb{D})$, the image of the boundary of $\mathbb{D}$ under the extension of f , which may be in the interior of Ω .

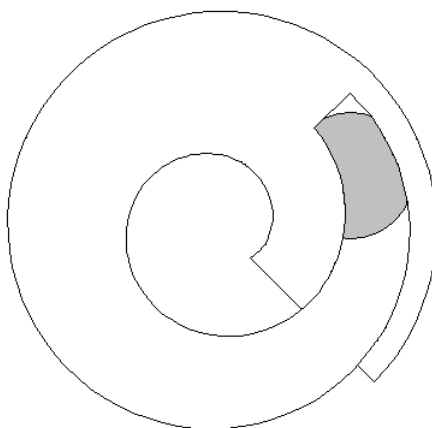
It follows immediately from the arguments above that $\mathcal{F}$ is a collection of blunt



(a) This circular disk...



(b) ...pulls back to two components, one of which...



(c) ...has a non-circular image.

Figure 5.2: A problem defining precircles

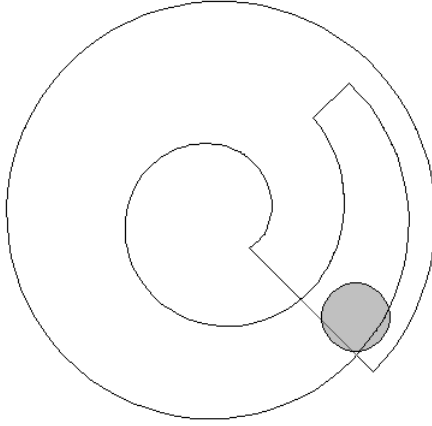


Figure 5.3: A circle in Ω that is not associated with a precircle

disklikes which are pairwise compatible. It remains to show that $\mathcal{F}$ is a continuous 3-manifold.

2 $\mathcal{F}$ Is a Continuous 3-Manifold

Since the justification of the continuity of $\mathcal{F}$ is very similar to the proofs seen already, we will simply outline the differences between this situation and those seen in previous sections. Suppose $\{A_n\}$ is a sequence of sets in $\mathcal{F}$ which are all contained in some compact subset of $\mathbb{D}$; let $C_n = f(A_n)$ for all n , and let z_n and r_n be the center and radius of C_n , respectively. As before, by taking subsequences if necessary, we assume z_n and r_n converge to z_0 and r_0 . Assuming r_0 is not zero, we get a limiting circle C_0 in Ω . It is possible, however, that C_0 has no precircle associated with it; see Figure 5.3. The circle in the figure stretches across the image under f^* of the boundary of $\mathbb{D}$ on every branch of the function; hence the image of either of the components of its preimage will be not be a circle. It is easy to see, however, that

our limiting circle C_0 cannot do this, since none of the circles C_n do. We conclude that $\mathcal{F}$ is a continuous collection.

We now wish to show that $\mathcal{F}$ is a 3-manifold. The proof here is also very similar to those above. Choose an element A in $\mathcal{F}$ such that the boundary of A does not intersect the boundary of $\mathbb{D}$. Let z and r be the center and radius, respectively, of $C = f(A)$. Let R be the distance from z to the image of the boundary of $\mathbb{D}$. Then, by our choice of A , $r < R$; call the difference between them ϵ . Let $U = B(z, \frac{\epsilon}{3})$. Consider a circle C' whose center z' is in U and whose radius r' satisfies $|r' - r| < \frac{\epsilon}{3}$. Then no point in C' can be on the image of the boundary of $\mathbb{D}$: if x is in C' , then

$$\begin{aligned} |x - z| &< |x - z'| + |z' - z| \\ &< \frac{\epsilon}{3} + r' \\ &< \frac{\epsilon}{3} + r + \frac{\epsilon}{3} \\ &< r + \epsilon \\ &= R. \end{aligned}$$

It follows that the set of circles whose centers are within $\frac{\epsilon}{3}$ of z and whose radii are within $\frac{\epsilon}{3}$ of r are all bounded away from the image of the boundary of $\mathbb{D}$. Thus, these circles have precircles associated with them that form a neighborhood of A which is homeomorphic to $U \times (\frac{\epsilon}{3}, \frac{\epsilon}{3})$. We conclude that $\mathcal{F}$ is a 3-manifold with boundary comprised of those precircles whose closure in $\mathbb{C}$ intersects the boundary of $\mathbb{D}$.

Since we now have a packable collection, we can apply the techniques in Chapter 3 to get the following result:

Theorem 5.1. *Let $\mathcal{T}$ be a CP-complex which triangulates a closed topological disk,*

and which has at least six boundary vertices. Let $f : \mathbb{D} \rightarrow \Omega$ be a conformal locally one-to-one function which continuously extends to the boundary of $\mathbb{D}$. Then there exists a maximal precircle packing under f in $\mathbb{D}$ with combinatorics $\mathcal{T}$.

Chapter 6

Finite-to-One Coverings of the Punctured Plane

Next, we will consider a much more complicated situation. We will let $\mathbb{C}^*$ denote the punctured plane $\mathbb{C} \setminus \{0\}$. Given any positive integer n , the map $f_n : \mathbb{C}^* \rightarrow \mathbb{C}^*$ given by $z \mapsto z^n$ is a conformal n -to-one surjection. Our goal in this chapter is to work towards the creation of circle packings on these n -to-one covers of $\mathbb{C}^*$.

1 The Complex

To build the packings we seek, we must of course start with a triangulation of our surface; in this case, the surface is homeomorphic to $\mathbb{C}^*$. This presents a two-fold problem: our surface is not compact and has no boundary, and it is not simply connected. The first part of the problem implies that our triangulation must be infinite (since finite triangulations fill compact surfaces). The second means we

must do some topological wrangling in order to apply our technique. We will attack both of these problems at once.

Let $\mathcal{T}$ be a parabolic triangulation of $\mathbb{C}^*$ with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$ (here, parabolic means that any circle packing in the sphere associated with this triangulation must cover all but at most two points on the sphere; if we omit this requirement, our packing will not fill $\mathbb{C}^*$). Choose a vertex v_1 , then choose a closed edge path γ in $\mathcal{E}$, beginning and ending at v_1 such that the path class $[\gamma]$ generates $\pi_1(\mathbb{C}^*)$ and γ is a path in that class of minimal combinatorial length; denote the vertices that γ passes through by $v_1, v_2, \dots, v_k, v_{k+1} = v_1$ (ordered according to positive counter-clockwise orientation). Define a new triangulation $\mathcal{T}_0$ with vertex set $v_1, \dots, v_k$ along with a designated *plug* vertex v_0 . The edges of $\mathcal{T}_0$ are the edges of γ , with edges added between v_0 and every other vertex v_j . Our plug vertex is placed on the “inside” of γ (according to our orientation, this is the left side).

We now recursively define a sequence of triangulations $\{\mathcal{T}_i\}$ with vertex and edge sets $\mathcal{V}_i$ and $\mathcal{E}_i$, respectively. We have constructed $\mathcal{T}_0$. For $i > 0$, we will build $\mathcal{T}_i$ as follows: Begin with $\mathcal{T}_{i-1}$. Remove v_0 and all of its edges. The resulting triangulation fills a topological annulus; as such it has two boundary components, one of which we will designate the *inside* boundary as above. By identifying each vertex in $\mathcal{V}_{i-1}$ with its corresponding vertex in $\mathcal{V}$, we now build two intermediate triangulations $\mathcal{T}'_i$ and $\mathcal{T}''_i$. $\mathcal{T}'_i$ is constructed by adjoining to $\mathcal{T}_{i-1}$ (with v_0 and its edges removed) all neighbors of boundary vertices not already included along with the corresponding edges. If $\mathcal{T}'_i$ has two boundary components, we set $\mathcal{T}''_i = \mathcal{T}'_i$. If not, consider the components of $\mathcal{T} \setminus \mathcal{T}'_i$. All but two of these components contain a finite number of vertices. Define $\mathcal{T}''_i$ by adding the vertices and edges in the finite components of the complement to $\mathcal{T}'_i$. In either case, we are left with a triangulation $\mathcal{T}''_i$ that fills an

annulus. We now build $\mathcal{T}_i$ by reinserting v_0 inside of $\mathcal{T}_i''$ and adding edges from v_0 to each vertex in the inside boundary component of $\mathcal{T}_i''$.

Our sequence $\{\mathcal{T}_i\}$ has been built in such a way that, as $i \rightarrow \infty$, $\mathcal{T}_i$ exhausts $\mathcal{T}$, with the exception of vertex v_0 and its edges.

2 Building the Packing

For any circle in $\mathbb{D}$ not containing the origin, there exists a branch of the multivalent n^{th} root function g_n that contains that circle in its domain. Let $\mathcal{F}$ be the collection of precircles obtained using this map; let $\mathcal{F}_0$ be the set of circles in $\mathbb{D}$ with center z and radius r such that $|z| < \frac{r}{2}$. Then $\mathcal{F}$ and $\mathcal{F}_0$ are packable collections. Using Theorem 4.2, we will define a sequence of packings $\{\mathcal{P}_i \mid i \geq 1\}$ with complex $\mathcal{T}_i$. Let $\mathcal{P}_v^i$ denote the precircle associated with vertex v in $\mathcal{P}_i$. Our sequence of packings will be chosen so that $\mathcal{P}_v^i$ is an element of $\mathcal{F}$ for $v \neq v_0$, and $\mathcal{P}_{v_0}^i$ is an element of $\mathcal{F}_0$ centered at the origin. Suppose that α_i is a point in $\mathcal{P}_{v_1}^i$ so that $f_n(\mathcal{P}_{v_1}^i)$ is centered at α_i^n . Let $\mathcal{Q}_i$ be the packing obtained by multiplying $\mathcal{P}_i$ by $\frac{1}{\alpha_i}$.

Remark 6.1. If D is the preimage of a circle under f_n , then for any $\beta \in \mathbb{C}^*$, βD is the preimage of a circle under f_n .

PROOF. This is a straightforward computation: $f_n(\beta D) = \beta^n f_n(D)$. Since scaled circles are themselves circles, the result follows. ■

By this remark, $\mathcal{Q}_i$ is a precircle packing in $\mathbb{C}$ for all i .

Conjecture 6.2. For $v \neq v_0$, there exists a subsequence of $\{\mathcal{Q}_v^i\}$ which converges to a nondegenerate precircle $\mathcal{Q}_v$.

Let us first consider the sequence of precircles $\{Q_{v_1}^i\}$. By our construction, for all i , the center of $Q_{v_1}^i$ is at 1 and the origin is not contained in it. Thus, the sequence is contained in a compact set (namely, $f_n^{-1}(\overline{B(1, 1)})$), and so we may find a subsequence which has a limit; we conclude that Q_{v_1} exists. The concern with this conjecture is that, to obtain the packing, we are multiplying a sequence of packings in the unit disk by a sequence of constants which could be unbounded, then taking a limit. Continuity only gives us limits if our sequence of precircles is bounded; for a given vertex, could the associated sequence of precircles fail to be bounded? We suspect that the limit exists, but we have not proven this; there are two potential problems. First, the sequence $\{\frac{1}{\alpha_i}\}$ is unbounded; can we be sure that $\{\frac{1}{\alpha_i}\mathcal{P}_v^i\}$ is unbounded? I believe that the sequence is bounded, but a proof is elusive. Second, is it possible that, for some vertex $v \neq v_0$, $\{Q_v^i\}$ is bounded, but approaches the origin? The problem is that, especially around branch points, precircles can be oddly shaped. For example, let $C_i = \overline{B(1 + \frac{1}{i}, 1)}$ for $i = 1, 2, \dots$; let D_i be the preimage of C_i under the map $z \mapsto z^2$. Note that the sequence $\{D_i\}$ is not contained in any compact subset of $\mathbb{C}^*$, so we cannot appeal to the continuity of our collection of precircles to conclude that the limit set is a precircle. In fact, the limit of this sequence is the preimage of a closed circular disk; however, it is not blunt (see Figure 6.1). I believe that, for a given vertex v , the precircles associated with the vertices between v and v_0 will keep $\mathcal{P}_v^i$ away from the branch point, but we have not yet found a proof. We therefore cannot apply the techniques we have developed to conclude that there is a subsequence of the sequence of precircle packings such that $\lim Q_v^i$ is a nondegenerate precircle for all $v \neq v_0$.

Assuming Conjecture 6.2 is true, let us consider the sequence $\{Q_{v_0}^i\}$. Suppose $\lim Q_{v_0}^i$ is not a point. By our construction, $Q_{v_0}^i$ is a circular disk centered at the origin

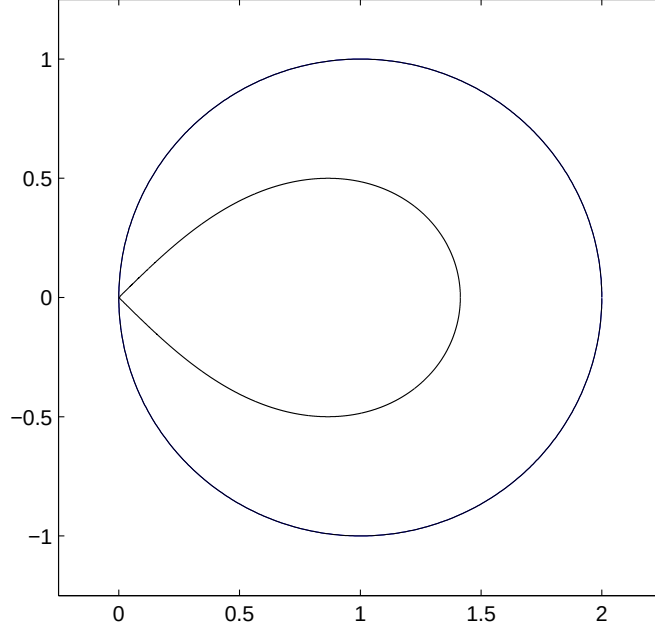


Figure 6.1: The limits of C_i and D_i ; the limit of D_i is not blunt

which never contains the point $z = 1$; hence $\{\mathcal{Q}_{v_0}^i\}$ has a convergent subsequence. Suppose the limit of that subsequence $\mathcal{Q}_{v_0}$ is not a point; by Lemma 2.2, that set must be a circle centered at the origin. Consider the complex associated with the packing $\{\mathcal{Q}_v | v \neq v_0\}$ (we know that this does in fact form a packing by our conjecture). This complex must be isomorphic to $\mathcal{T}$, the triangulation of $\mathbb{C}^*$ with which we began. Consider the image circle packing under f_n . If we apply the Möbius transformation $z \mapsto \frac{1}{z}$, we get an infinite circle packing that fills a disk in $\mathbb{C}$. This implies that $\mathcal{T}$ is not parabolic, contrary to our assumption. Hence it must be the case that $\lim \mathcal{Q}_{v_0}^i$ is a point.

3 Doyle Spirals

A nice extension of this idea relates to work done by Beardon, Dubejko, and Stephenson in [1]. We first need a definition.

Definition 6.3. A circle packing is *k-coherent* for some integer k if it is locally univalent and globally k -valent.

Theorem 6.4. *If $\mathcal{P}$ is a k -coherent circle packing of $\mathbb{C}^*$ with hexagonal combinatorics, then there exists a k -coherent Doyle spiral with the same combinatorics as $\mathcal{P}$.*

Peter Doyle observed that circles with radii a , b , b/a , $1/a$, $1/b$, and a/b (in that order) fit perfectly around the unit circle, i.e., they can be fit together to form a circle packing with six circles surrounding a seventh (a *hex flower*). Of course, this process can be scaled; in general, given any three mutually externally tangent circles C_0 , C_1 , and C_2 with radii r_0 , r_1 , and r_2 respectively, we build the circle externally tangent to C_0 and C_2 and opposite C_1 with radius $\frac{r_0 r_2}{r_1}$. If we work our way around C_0 adding circles according to this rule, we clearly end up with a hex flower of the type described by Doyle. An ordered triple of mutually externally tangent circles (such as our C_0 , C_1 , and C_2) is called a *cell*, and this process of adding successive circles is called *cell continuation*. Applying this process to generate all possible circles in every direction automatically creates a triangulated 2-manifold M in which every vertex has degree six. One of the key results proven in [1] is this theorem:

Theorem 6.5. *Given any cell, there are two possibilities for its associated manifold M : If all three circles have a common radius, then M is $\mathbb{C}$. Otherwise, there exists a point $\zeta \in \mathbb{C}$ such that M is a smooth unlimited covering space of $\mathbb{C} \setminus \{\zeta\}$, and all circles created by this cell continuation lie in $\mathbb{C} \setminus \{\zeta\}$.*

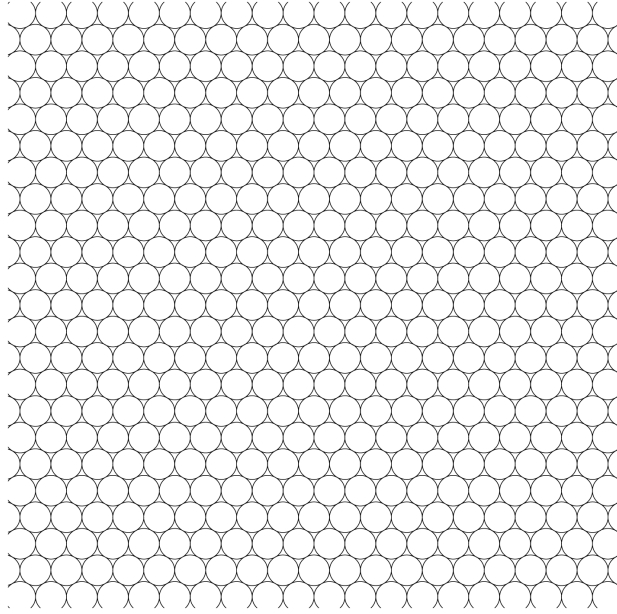


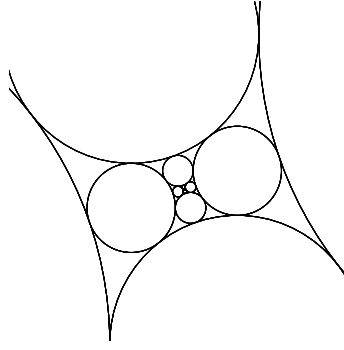
Figure 6.2: The Penny Packing

The first case is known as the *penny packing*; if the common radius is 1 and one of the circles is centered at 0, we will refer to it as $\mathcal{P}_H$ (see Figure 6.2). The packings that comprise the other case are known as Doyle spirals (see Figure 6.3). We will, without loss of generality, always assume that $\zeta = 0$.

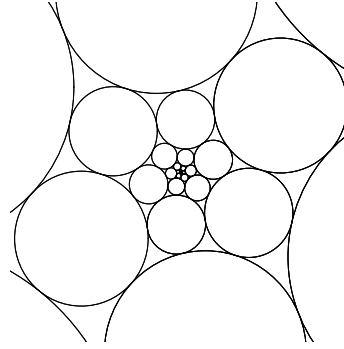
PROOF OF 6.4. Let $\mathcal{P}$ be a k -coherent hexagonal circle packing of $\mathbb{C}^*$ with combinatorics $\mathcal{T}$, normalized so that there is a circle centered at 1. There are two cases:

Case I. $\mathcal{P}$ is univalent in $\mathbb{C}^*$. This case is another of the main results in [1].

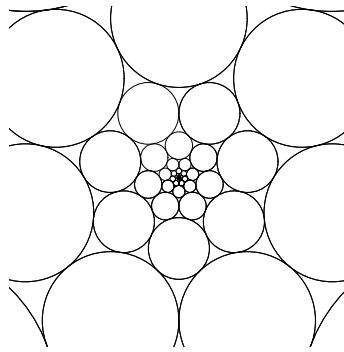
Case II. $\mathcal{P}$ is univalent in an k -to-one cover of $\mathbb{C}^*$. Following the terminology used in [1], given $A \in \mathbb{C}$ and $\mu, \nu \in \mathbb{R}$, we define $\Phi : \mathbb{C} \rightarrow \mathbb{C}$ by $\Phi(x + iy) = x + i(\mu x + \nu y)$, and note that Φ is linear. Further, we define $\Lambda(z) = \frac{2\pi i \Phi(|A|z/A)}{\Phi(|A|)}$ and $E(z) = \exp(\Lambda(z))$. We pick out four particular circle centers in $\mathcal{P}_H$: $w_0 = 0$, $w_1 = 1 - i\sqrt{3}$, $w_2 = 2$, and $w_3 = 1 + i\sqrt{3}$. (The centers of the circles in $\mathcal{P}_H$ form a



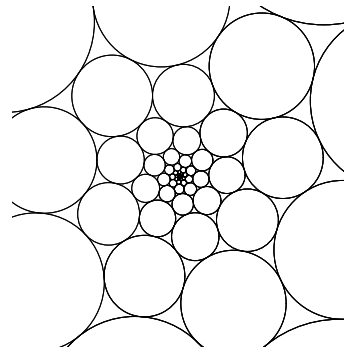
(a) $A = 6 + 2i\sqrt{3}$



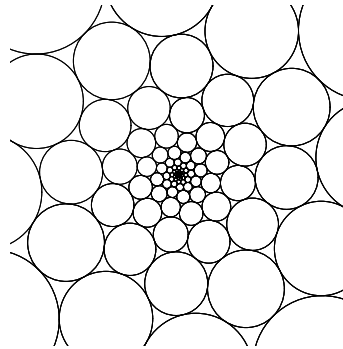
(b) $A = 10 + 2i\sqrt{3}$



(c) $A = 14$



(d) $A = 15 + 3i\sqrt{3}$



(e) $A = 20 + 4i\sqrt{3}$

Figure 6.3: Some Doyle spirals, with corresponding values for A

hexagonal lattice W in $\mathbb{C}$. The number A is the least number which is the difference of two points in the lattice $\Phi(W)$ which are identified under the exponential map E .) This leads us up to a lemma proven in [1]:

Lemma 6.6. *Suppose $\arg(A) \in [0, \frac{\pi}{6}]$ and $|A| > \frac{2\sqrt{3}}{\sin(2\pi/3 - \arg(A))}$. Then there exists a unique pair $(\mu, \nu) \in \mathbb{R} \times (0, \infty)$ such that $\Im(\Lambda(w_j)) \in [0, \pi)$ for $j = 1, 2, 3$, and, given the triple with circles centered at $z_j = E(w_j)$, $j = 0, 1, 2$, the center of the circle obtained by cell continuation is at $z_3 = E(w_3)$. Conversely, given a Doyle spiral normalized so that some circle is centered at $z_0 = 1$, choose circle centers z_1 and z_2 such that the circles centered at z_0 , z_1 , and z_2 form a triple, $|z_1| > 1$, $|z_2| \geq 1$, and $0 \leq \arg(z_1) \leq \arg(z_3) \leq \arg(z_2) < \pi$ (this can always be done). Then there exist a unique A with $\arg(A) \in [0, \frac{\pi}{6}]$ and $|A| > \frac{2\sqrt{3}}{\sin(2\pi/3 - \arg(A))}$ and unique $(\mu, \nu) \in \mathbb{R} \times (0, \infty)$ such that $\Im(\Lambda(w_j)) \in [0, \pi)$ and $z_j = E(w_j)$ for $j = 1, 2, 3$.*

In other words, this lemma says that, if $\{w_n | n = 1, \dots, \infty\}$ are the vertices of $\mathcal{P}_H$, a proper choice of A gives us μ and ν so that $\{z_n = E(w_n)\}$ are the vertices of a Doyle spiral. Conversely, given a normalized Doyle spiral, there exist A , μ , and ν so that $\{E(w_n)\}$ is the set of centers of the spiral.

Consider the precircle packing $\mathcal{P}'$ obtained by taking the logarithm of $\mathcal{P}$. Let v_0 and v_1 be the vertices in $\mathcal{T}'$ associated with the precircles containing 0 and $2n\pi i$, respectively. Construct a *univalent* circle packing in $\mathbb{C}^*$ with combinatorics isomorphic to $\mathcal{T}$ (which means that v_0 and v_1 coincide). By Theorem 7 in [1], this packing must be a Doyle spiral. Choose the value of A associated with this spiral. Let $A_0 = A/k$; assume for now that A_0 satisfies the restrictions given in Lemma 6.6. We can now build a corresponding Doyle spiral; call it $\mathcal{P}_{A_0}$. We must verify that the circles in $\mathcal{P}_{A_0}$ corresponding to v_0 and v_1 coincide. Look at these vertices in $\mathcal{P}_H$; say

they are centered at w_0 and w_1 . By our choice of A ,

$$\frac{2\pi i \Phi(|A|w_0/A)}{\Phi(|A|)} - \frac{2\pi i \Phi(|A|w_1/A)}{\Phi(|A|)} = \pm 2\pi i.$$

Multiplying both sides by k and appealing to the linearity of Φ , we get

$$\frac{2\pi i \Phi(|\frac{A}{k}|w_0/\frac{A}{k})}{\Phi(|\frac{A}{k}|)} - \frac{2\pi i \Phi(|\frac{A}{k}|w_1/\frac{A}{k})}{\Phi(|\frac{A}{k}|)} = \pm 2k\pi i,$$

which means that $E(w_0) = E(w_1)$, as desired. We have therefore created a k -coherent Doyle spiral $\mathcal{P}_D$ whose triangulation is isomorphic to $\mathcal{T}$.

We must now address the possibility that $|A_0| \leq \frac{2\sqrt{3}}{\sin(2\pi/3 - \arg(A_0))}$. In [1], the restriction on $|A|$ is in place to ensure that the spiral doesn't wind so tightly around the origin that its triangulation cannot be realized by packed circles. But, as we have already seen, we have a packing with the given combinatorics. Thus $|A_0|$ must be big enough. ■

The ultimate goal of this line of reasoning is the following conjecture:

Conjecture 6.7. If $\mathcal{P}$ is a k -coherent circle packing of $\mathbb{C}^*$ with hexagonal combinatorics, $\mathcal{P}$ is a Doyle spiral.

In the case $k = 1$, Conjecture 6.7 is known to be true; this is one of the main results in [1]. In order to prove this for all k , one must not only overcome Conjecture 6.2; but also, assuming the limit packing exists, the problem of showing that the packing is unique up to some automorphism. The difficulty in verifying uniqueness is the fact that the image circle packing is not univalent; as noted, in the univalent case, uniqueness is proven. A related, more general statement is also something to

be pursued:

Conjecture 6.8. If $\mathcal{P}$ is a k -coherent circle packing of $\mathbb{C}^*$ with triangulation $\mathcal{T}$, then any other k -coherent circle packing with triangulation $\mathcal{T}$ is Möbius equivalent to $\mathcal{P}$.

In [8], Schramm proves uniqueness up to Möbius transformation of univalent circle packings that fill the plane except possibly at countably many points; his argument defies translation to the non-univalent case because it relies on a normalization that puts the point at infinity in $\widehat{\mathbb{C}}$ inside a trilateral bounded by three circles in the packing (and, more importantly for us, not inside a closed circular disk). In the non-univalent case, however, this normalization may not be possible; a given trilateral bounded by three circles may be covered by a circle on a different sheet. We believe Conjecture 6.8 to be true as well, but altogether different techniques will likely be needed.

Chapter 7

Infinite Covers of the Punctured Plane

Next, we will discuss building packings in $\mathbb{C}^*$ that wrap infinitely many times around the puncture. To this end, we will build a sequence of precircle packings in bounded subsets of the universal covering space, $\mathbb{C}$, using the exponential map. In order to implement this technique, we must make some assumptions about $\mathcal{T}$.

1 The Complex

Let $\mathcal{T}$ be a CP-complex which triangulates a complete, smooth infinite-to-one cover of $\mathbb{C}^*$ with vertex set $\mathcal{V}$ (call this covering space $\widetilde{\mathbb{C}^*}$). Since $\widetilde{\mathbb{C}^*}$ is homeomorphic to $\mathbb{C}$, we can view $\mathcal{T}$ as a triangulation of the plane. We will assume that every vertex in $\mathcal{V}$ has degree at least four; if any vertex has degree three, remove it and its edges from $\mathcal{T}$; the result is still a valid triangulation. After we have our packing, we may pack a precircle in the trilateral formed by the three neighbors of any deleted

vertex, restoring our original triangulation. Finally, let us suppose that there exists a $\mathbb{Z}$ -action on $\mathcal{T}$ and that $\mathcal{T}$ is *pre-parabolic*, a term that we will define soon.

Choose $w_0 \in \mathcal{V}$. Let the group action on $\mathcal{T}$ be generated by the simplicial map $g : \mathcal{T} \rightarrow \mathcal{T}$. We now will build up a sequence of subcomplexes $\mathcal{T}_c$ that exhaust $\mathcal{T}$ which are roughly square, in the sense that we will pick out four vertices on the boundary of $\mathcal{T}_c$ (the “corners” of our square) so that each is about the same combinatorial distance from the next. We will use the action g to define this square in such a way that will keep the “sides” of the square from getting too close together.

Find a shortest path between w_0 and $g^{-1}(w_0)$; let m be its length. Let $n = \lceil \frac{m}{2} \rceil$, and let v_0 be the vertex on this path with $d(w_0, v_0) = n$ (where d is the combinatorial distance). The number n will be the half the length of each of the edges of our first square. We now construct a path γ that is, in some sense, perpendicular to the action of g . We define this path inductively as follows: choose a neighbor v_1 of v_0 so that $|d(v_1, w_0) - d(v_1, g^{-1}(w_0))|$ is minimized (This is meant to mimic the geometric construction of a perpendicular bisector of a line segment; the edge from v_0 to v_1 is a point on the “perpendicular bisector” of the shortest path from w_0 to $g^{-1}(w_0)$). Put v_0 and v_1 in γ . Next, given v_i and v_{i+1} in γ , choose v_{i-1} such that v_{i-1} shares an edge with v_i , $d(v_{i-1}, v_{i+1}) = 2$, and $d(v_{i-1}, w_0) - d(v_{i-1}, g^{-1}(w_0))$ is minimized. Also, choose v_{i+2} such that v_{i+2} shares an edge with v_{i+1} and $d(v_{i+2}, w_0) - d(v_{i+2}, g^{-1}(w_0))$ is minimized. Put these vertices into γ as well; in this way, we have recursively defined the path $\gamma = \dots, v_{-2}, v_{-1}, v_0, v_1, v_2, \dots$. Finally, choose w_1 to be a neighbor of w_0 such that $d(w_1, v_0) - d(w_1, g(v_0))$ is minimized; this makes the edge from w_0 to w_1 also “perpendicular” to the action of g , hence “parallel” to γ .

We now construct the exhaustive sequence of subcomplexes of $\mathcal{T}$. For $c \geq 1$, let $\mathcal{T}_c$ be the subcomplex of $\mathcal{T}$ bounded by:

- $g^{-c}(v_{-cn}) \dots g^{-c}(v_{cn})$,
- $g^{c+1}(v_{-cn}) \dots g^{c+1}(v_{cn})$,
- successive images under g of the shortest path from $g^{-c}(v_{-cn})$ to $g^{-c+1}(v_{-cn})$,
and
- successive images under g of the shortest path from $g^{-c}(v_{cn})$ to $g^{-c+1}(v_{cn})$.

The first and second bounds are in the orbit under g of a piece of γ of length $2cn$; these are the top and bottom of our combinatorial square. The corners of our square are thus $g^{-c}(v_{cn})$, $g^{-c}(v_{-cn})$, $g^{c+1}(v_{-cn})$, and $g^{c+1}(v_{cn})$. The sides of the square are paths from corner to corner, defined to be in the same direction as the action of g . Note also that w_0 is in $\mathcal{T}_c$ for sufficiently large c , since w_0 is between γ and $g^{-1}(\gamma)$.

2 Building the Packing

Given a circle in $\mathbb{C}^*$, there exists a branch of the logarithm with that circle in its domain. Let $\mathcal{F}$ be the collection of precircles under the logarithm. For $c \geq 1$, let $\mathcal{S}_c$ be the square whose vertices are $\pm(2c+1)\pi \pm (2c+1)\pi i$ and $\pm(2c+1)\pi \mp (2c+1)\pi i$. Let f_c be the Riemann map from $\mathcal{S}_c$ to $\mathbb{D}$ with $f_c(0) = 0$ and $f_c(1)$ on the real axis. Define $\mathcal{F}'_c$ to be the collection of images under f_c of elements of $\mathcal{F}$ that are also subsets of $\mathcal{S}_c$. Since the composition of conformal maps is conformal, $\mathcal{F}'_c$ is a continuous collection of precircles for every c (this statement is more fully explored in Theorem 8.1). For large enough c , we can use Theorem 5.1 to build a packing $\mathcal{P}'_c$ in $\mathbb{D}$ of elements of $\mathcal{F}'_c$ with combinatorics $\mathcal{T}_c$ such that the precircle associated with w_0 is centered at 0 and the precircle associated with w_1 is on the real axis. Finally, we define $\mathcal{P}_c$ to be the

inverse image under f_c of $\mathcal{P}'_c$. By our construction, $\mathcal{P}_c$ is a precircle packing in $\mathcal{S}_c$. We are now ready to introduce *pre-parabolicity*:

Definition 7.1. A triangulation with a $\mathbb{Z}$ -action is *pre-parabolic* if there exists a subsequence of the precircle packing sequence $\mathcal{P}_c$ obtained using the construction outlined above which converges and fills the plane.

Since our triangulation by assumption is pre-parabolic, we can now let $\mathcal{P}$ be the limit of a subsequence of this family of packings as c goes to infinity to obtain a precircle packing which fills the plane. As a result, $\exp(\mathcal{P})$ is an infinite-to-one circle packing of $\mathbb{C}^*$.

This leads us to some open questions.

Question. Can we relax the requirement that $\mathcal{T}$ must have a $\mathbb{Z}$ -action?

Here, we used the $\mathbb{Z}$ -action in several ways. We used it to place a sort of loose geometry on $\mathcal{T}$, as well as to define our subcomplexes. The idea is that we wish to build the packing in such a way that the periodicity of the exponential map is at least loosely related to the periodicity of $\mathcal{T}$. In other words, in our final infinite-to-one circle packing of $\mathbb{C}^*$, if we travel along some path in $\mathcal{T}$ from a vertex to its image under g , we should wind approximately once around the origin on the corresponding circles in the final packing. We could have built the subcomplexes in any manner that exhausted the infinite complex, and packed them in any nested sequence of domains that exhausted the plane. However, this will not in general result in a nondegenerate packing. For example, the domains we choose might grow too slowly for our sequence of subcomplexes, more specifically, the ratio of vertices in the subcomplex to area of the domain grows without bound. As a result, some precircles

could shrink at each step, eventually degenerating to points. Our construction helps keep that from happening, since, by the action of g and the definition of the squares $\mathcal{T}_c$, each $2\pi \times 2\pi$ square contains about the same number of precircles, keeping the vertex-to-area ratio in check. I suspect that given any parabolic triangulation, and any exhaustive sequence of subcomplexes, one may find a suitable sequence of subdomains of $\mathbb{C}$ that yields a precircle packing that fills the plane, but that has not been verified. This leads us to our next question.

Question. Can we characterize pre-parabolicity?

We have this result:

Proposition 7.2. *Let $\mathcal{T}$ be a triangulation with bounded degree. If $\mathcal{T}$ is pre-parabolic, then it is also parabolic.*

PROOF. Suppose $\mathcal{T}$ is pre-parabolic but not parabolic. Then $\mathcal{T}$ must be hyperbolic, and it has a maximal circle packing in the unit disk $\mathbb{D}$ (see [2]). Since $\mathcal{T}$ is pre-parabolic, we may build an infinite-to-one circle packing in $\mathbb{C}^*$ with complex $\mathcal{T}$, as outlined above. Having two circle packings with the same triangulations automatically generates a map between the spaces they fill; in this case, we get a κ -quasiconformal map f for some κ . As such, f can be expressed as a composition of a κ -quasiconformal map $g : \mathbb{D} \rightarrow \mathbb{C}$ and the exponential map. But this cannot be; Liouville's theorem asserts that there cannot be any quasiconformal map from $\mathbb{C}$ to a bounded subset of $\mathbb{C}$, while f^{-1} would be just such a map. ■

I suspect that the converse of Proposition 7.2 is also true, but that hasn't been verified.

Question. Can we remove our condition that $\mathcal{T}$ has bounded degree (the bounded degree condition is part of the definition of a CP-complex; it is needed to apply the Ring Lemma from [6])?

Nearly every result in the circle packing literature dealing with parabolicity relies on a bounded degree assumption. We see it not only in the Discrete Uniformization Theorem cited above, but also in more detailed studies of the so-called “type” problem (see [4]). I feel strongly that our need for it here is related to the need for it in these other applications.

Chapter 8

Packings Using More General Sets

1 Conformal Preimages of Other Families

To this point, we have considered packing families of precircles. However, the techniques we have developed apply to many other families as well.

Theorem 8.1. *Let X and Y be bounded, simply-connected domains whose boundaries have no inward-pointing cusps of angle 2π . Suppose that $\mathcal{F}_Y$ is a packable collection of blunt disklikes in Y . If $f : X \rightarrow Y$ is a conformal homeomorphism, then the collection $\mathcal{F}_X$, whose elements are defined to be the preimages under f of each element of $\mathcal{F}_Y$, is also a packable collection of blunt disklikes.*

PROOF. Note that f tacitly defines a map ϕ from $\mathcal{F}_X$ to $\mathcal{F}_Y$ that sends a set A to the set B comprised of the images under f of all the points in A , a set that must be in $\mathcal{F}_Y$. Since f is a conformal homeomorphism, $\mathcal{F}_X$ must be a collection of blunt disklikes. Also, ϕ clearly preserves compatibility. We must therefore verify that $\mathcal{F}_X$ is a continuous collection that comprises a 3-manifold.

Let $\{A_n\}$ be a sequence of elements of $\mathcal{F}_X$ that are contained in some compact subset K of X . Let $B_n = \phi(A_n)$ for all n ; since $f(K)$ is also compact, $\{B_n\}$ is a sequence of elements of $\mathcal{F}_Y$ which are contained in a compact subset of Y . Therefore, by continuity of $\mathcal{F}_Y$, there is an increasing sequence of positive integers $\{n_i\}$ such that the sequence $\{B_{n_i}\}$ either converges to a point y or a set B in $\mathcal{F}_Y$. Consider the subsequence $\{A_{n_i}\}$; by the continuity of f^{-1} , this subsequence must converge to either the point $f^{-1}(y)$ or the set $\{f^{-1}(b) \mid b \in B\}$. Since this latter set is just $\phi^{-1}(B)$, it follows that $\mathcal{F}_X$ is continuous. Now, we check that $\mathcal{F}_X$ is a 3-manifold. We will accomplish this by showing that ϕ is a homeomorphism. It is clear that ϕ is both surjective and injective; we must prove that ϕ is bicontinuous. But this fact follows immediately from our justification that $\mathcal{F}_X$ is continuous: if $\{A_n\}$ is a convergent sequence in $\mathcal{F}_X$, then $\{\phi(A_n)\}$ is a convergent sequence in $\mathcal{F}_Y$ by continuity of f ; this shows that ϕ is continuous. Similarly, by continuity of f^{-1} , ϕ^{-1} is continuous as well. It follows that ϕ is a homeomorphism, and so $\mathcal{F}_X$ is a 3-manifold. ■

This result allows us to consider “pre-precircle” packings, packings of the preimages of circles under a composition of maps. It is easy to see that this idea can be extended using the ideas presented in this thesis to compositions in which the function f is a branched multivalent function.

Another application of the ideas presented here relates to the technique seen in Chapter 4 of assigning different packable collections to different vertices. Let X and Y be domains as before, and suppose f and g are conformal maps from X onto Y . Let $\mathcal{T}$ be a triangulation of a closed disk with vertices $\mathcal{V}$ and edges $\mathcal{E}$. Pick a path of vertices $\gamma = \{v_{-n}, v_{-n+1}, \dots, v_{-1}, v_0, v_1, \dots, v_m\}$ such that $v_i \neq v_j$ for all $i \neq j$, v_i is connected by an edge to v_j if and only if $|i - j| < 2$, v_{-n} and v_m are boundary

vertices, and v_k is an interior vertex for $-n < k < m$. Let $\mathcal{V}_f$ be the set of vertices on the left side of γ , oriented according to increasing subscript, and let $\mathcal{V}_g$ be the set of vertices in and to the right of γ . Let $\mathcal{F}_f$ be the set of precircles in X under f , and let $\mathcal{F}_g$ be the set of precircles in X under g . Then, given a point y in Y and a direction Δ , there exists a packing $\mathcal{P}$ in X with triangulation $\mathcal{T}$, where $\mathcal{P}_v$ is taken from $\mathcal{F}_f$ if v is in $\mathcal{V}_f$, or from $\mathcal{F}_g$ if v is in $\mathcal{V}_g$, so that $g(\mathcal{P}_{v_0})$ is centered at $g(x_0)$ and the center of $g(\mathcal{P}_{v_1})$ is in the direction of Δ from y . Assuming f and g are not equal, $\mathcal{P}$ is not a precircle packing under either of those maps; in fact, the packing $\mathcal{Q}$ in Y obtained by taking the image of $\mathcal{P}$ under g is made up of a mixture of circles (corresponding to the vertices in $\mathcal{V}_g$) and the images of circles under $g \circ f^{-1}$. We may construct a path σ dividing X into two pieces X_f and X_g in such a way that $\sigma = X_f \cap X_g$, σ does not intersect the interior of $\mathcal{P}_v$ for any v , and so that, for v in $\mathcal{V}_f$ ($\mathcal{V}_g$, respectively), $\mathcal{P}_v$ is a subset of X_f (X_g). We may say, then, that the sets in $\mathcal{P}$ are precircles under the piecewise function Γ defined to be f on $X_f \setminus \{\sigma\}$ and g on X_g . If we look at the image of this packing under Γ , however, we will not in general get a circle packing; the path σ may not have the same image under each of f and g . In fact, there may not be any path that has the same image under f and g .

Question. In the case where the images of σ under f and g do not coincide, how do we interpret the packing $\mathcal{P}$? In particular, what happens as we refine $\mathcal{T}$, that is, we add vertices and edges in such a way that we maintain some of the information contained in $\mathcal{T}$?

Bibliography

- [1] Alan F. Beardon, Tomasz Dubejko, and Kenneth Stephenson. Spiral hexagonal circle packings in the plane. *Geometriae Dedicata*, 49:39–70, 1994.
- [2] Alan F. Beardon and Kenneth Stephenson. The uniformization theorem for circle packings. *Indiana Univ. Math. J.*, 39:1383–1425, 1990.
- [3] Charles R. Collins and Kenneth Stephenson. A circle packing algorithm. *Comput. Geom.*, 25:233–256, 2003.
- [4] Zheng-Zu He and Oded Schramm. Hyperbolic and parabolic packings. *Discrete Comput. Geom.*, 14:123–149, 1995.
- [5] Paul Koebe. Kontaktprobleme der konformen abbildung. *Ber. Sächs. Akad. Wiss. Leipzig, Math.-Phys. Kl.*, 88:141–164, 1936.
- [6] Burt Rodin and Dennis Sullivan. The convergence of circle packings to the Riemann mapping. *J. Differential Geometry*, 26:349–360, 1987.
- [7] Oded Schramm. Existence and uniqueness of packings with specified combinatorics. *Israel J. Math.*, 73:321–341, 1991.
- [8] Oded Schramm. Rigidity of infinite (circle) packings. *Journal Amer. Math. Soc.*, 4:127–149, 1991.

- [9] William Thurston. The finite Riemann mapping theorem, March 1985. Invited talk, An International Symposium at Purdue University on the occasion of the proof of the Bieberbach conjecture.

Vita

Matthew Edward Cathey was born May 17, 1976 in Columbia, TN. He grew up in Belfast, TN, and attended school in Lewisburg, TN. Matt graduated from Marshall County High School in 1994. He then matriculated at the University of the South in Sewanee, TN, where, in 1998, he received a B.A. in Mathematics and in Music. Matt is now pursuing a Ph.D. in Mathematics at the University of Tennessee's Knoxville campus. He is currently an instructor of mathematics at Wofford College in Spartanburg, SC.