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I am submitting herewith a thesis written by James T. Smith entitled "The Natural Factor Formulation of the Matrix Displacement Method of Structural Analysis." I recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

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We have read this thesis
and recommend its acceptance:

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THE NATURAL FACTOR FORMULATION OF THE MATRIX DISPLACEMENT METHOD OF STRUCTURAL ANALYSIS

A Thesis

Presented for the

Master of Science

Degree

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ABSTRACT

The natural factor method proposed by J. H. Argyris for the solution of ill-conditioned structural analysis problems is presented. This method uses the structural relationships that exist at the element level to develop a factorization for the total structural problem. This new method and the classical approach for the solution of structural analysis problems are compared conceptually and numerically. Specific features of the method are examined and several examples are given.

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CHAPTER I

INTRODUCTION

Finite element methods have become standard methods for the solution of structural analysis problems. Fundamental to any such method is the factorization technique employed. There are several possible factorization techniques available, each with strong points and weak points with respect to numerical stability and computational efficiency. Those methods which attempt to utilize the structural properties that exist at the element level show great promise of being much more numerically stable than do those methods employed at the overall structural level. In this paper is discussed such a method, the natural factor approach.

Chapter II describes the basic theory of the matrix displacement methods of structural analysis. Chapter III describes the classical method in terms of a series of matrix transformations. Chapters IV and V use this same series of transformations to describe the natural factor method at both the element and the overall structural levels. In Chapters VI and VII examples of this method at both levels are presented. Some observations concerning the natural factor method are made in Chapter VIII. The Appendix contains figures and tables referenced by the thesis.

CHAPTER II

BACKGROUND

The structural design engineer is responsible for the determination of arrangements and proportions of members of a structure, e.g. trusses of a bridge, that will withstand, economically and efficiently, the conditions anticipated during the lifetime of a structure. Central to this function is the calculation of the distribution of forces within the structure and the displaced state of the system.

Fundamentally, the behavior of all types of structures is described by means of differential equations. In practice, the writing of these differential equations is rarely practical or necessary; rather, numerical solutions are sought by an approach such as the finite element method. The basic concept of the method is that a total structure may be modeled by its subdivision into regions, called finite elements, in each of which the behavior is described by a set of assumed functions representing the stresses or displacements in that region. The formulation of the structural problem thus may be represented by a system of algebraic equations.

Viewed in this way, structural analysis may be broken down into five parts:

1. Basic mechanics. The fundamental relationships of stress and strain, compatibility, and equilibrium.
2. Finite element mechanics. The exact or approximate solution of the differential equations of the element.

3. Equation formulation. The establishment of the governing algebraic equations of the system.
4. Equation solution. Computational methods and algorithms.
5. Solution interpretation. The presentation of results in a form useful in design [7].

In the finite element idealization of a structure each element is idealized as a set of points, called nodes. Each node is assumed to be subject to certain types of action, e.g. displacement, rotation, and twist, which are called degrees of freedom. The description of these freedoms requires some type of reference or coordinate system. To each element is assigned a local coordinate system. For example, a rectilinear coordinate system for a line or axial force element is idealized so that the element lies along one of the coordinate axes as shown in Figure A-1, Appendix. The element shown there consists of two nodes and two degrees of freedom.

A mathematical model of the state of stress of an element is represented by forces at the element nodes. The corresponding degrees of freedom are used to characterize the displaced state of the element. The principles of elasticity and the laws of equilibrium are used to yield relationships between the forces and the displacements of each element.

The relationship of each element to the total structure must be determined. A given node and its freedoms may be part of more than one element, just as a joint in a bridge may join several trusses together. Consequently, the elemental relationships are expressed in some global coordinate system. By the summing of the elemental forces at each node

with respect to each degree of freedom and the equating of the result to the corresponding applied force, the conditions of equilibrium within the total structure are fully accounted for. The solution of this idealized structure is exact within the confines of the assumptions made at each element [7].

As will be shown, the matrix displacement finite element method of structural analysis leads to the basic matrix relation

$$Kr = F \tag{2.1}$$

where

K is the (n × n) global stiffness matrix for the system,

r is the (n × 1) displacement vector,

F is the (n × 1) force vector.

The global stiffness matrix K can be shown to be symmetric positive semi-definite. The symmetry of K follows from Maxwell's reciprocal theorem, or more generally from Betti's law [7]. The positive semi-definiteness of K follows from energy principles in which the work done by the force vector F acting through the displacement vector r is given by

$$W = \frac{1}{2}r \cdot F = \frac{1}{2}r^{t}Kr \geq 0 \ . \tag{2.2}$$

The stiffness matrix K may be formed without consideration of the support of the structure being modeled. The matrix K only expresses the manner in which the degrees of freedom of the idealized structure are related to one another. The interaction of the idealized structure with the rest of the world is modeled by support conditions placed upon the

supported degrees of freedom. After the rows and columns of K that correspond to the support have been removed, the reduced stiffness equation, also denoted by equation (2.1), is then nonsingular and the structure is said to be supported in a statically determinate manner [7].

For example, consider two identical oil derricks, one anchored to the sea floor and the other being floated into position. The global stiffness matrices of these two are identical except that support conditions of the first have been imposed. The first is supported in a statically determinate manner, and its stiffness as idealized has a unique solution. The second as it floats in the water exhibits a number of rigid body motions which are solutions of its singular stiffness equation.

After the support conditions have been imposed, the stiffness matrix K is now symmetric positive definite so that the solution of equation (2.1) may be determined by the application of any of a number of standard procedures, both direct and iterative. There are several ordering schemes, e.g. the minimum degree ordering or the reverse Cuthill McKee ordering, which may be used to decrease the number of operations and the amount of computer storage used to solve the problem [5]. Eventually, one obtains a factorization of K, e.g. the Cholesky factorization of K as

$$K = U^t U \ , \tag{2.3}$$

where U is an upper triangular matrix. The solution to equation (2.1) then is given by

$$r = U^{-1}U^{-t}F \ , \tag{2.4}$$

which may be determined first by solving

$$U^{t}y = F \tag{2.5}$$

by forward substitution and then by solving

$$Ur = y \tag{2.6}$$

by backward substitution.

There are several possible factorization techniques available, each with strong points and weak points, e.g. its numerical stability and its operation count. To evaluate the most effective factorization technique, it is useful to realize that the physical laws and geometrical arguments leading to the formation of the global stiffness matrix K may be expressed (not necessarily so implemented) as a sequence of matrix transformations. In fact, the formation of K may be reduced to a number of congruent matrix transformations which take place mainly on two levels:

1. On the element level for the derivation of the stiffness matrices of the individual elements,
2. On the overall structural level to insure the correct assembly procedure of the individual elements into the complete system [2].

In the classical approach, the reordering and the factoring occur at the overall structural level. On the other hand, J. H. Argyris and his associates at the Imperial College of Science and Technology, University of London and at the Institut für Statik und Dynamik,

Universität Stuttgart in Germany [1, 2, 3, 6] have attempted to take full advantage of the transformations that occur on the element level. The result of this effort, the natural factor method, has generated considerable interest. In the natural factor formulation, the Cholesky factorization of K is derived not via factorization at the overall structural level but directly from the natural stiffness formulation at the elemental level. The intent of this thesis is the examination of the formulation and the implementation of this natural factor method.

CHAPTER III

THE CLASSICAL APPROACH

Both the classical approach and the natural factor approach begin at the element level in local coordinates with a stiffness relationship of the form

$$k_L r_L = F_L \ . \tag{3.1}$$

By means of an orthogonal transformation the element is expressed in the global coordinates of the overall structure, that is

$$T^t k_L T T^t r_L = T^t F_L \ , \tag{3.2}$$

or simply

$$T^t k_L T = k \tag{3.3}$$

where T is the coordinate transformation matrix for the given element.

Now in global coordinates, the element stiffness matrices k_i are then ordered into a block diagonal matrix k of the form

$$k = \begin{bmatrix} k_1 & & \\ & k_2 & \\ & & \ddots \\ & & & k_m \end{bmatrix} \ . \tag{3.4}$$

Finally, an orthogonal transformation with a boolean assembly matrix A yields the global stiffness matrix

$$K = A^t k A \tag{3.5}$$

of equation (2.1). The result is a complete set of coefficients of the global stiffness equations but with no recognition of the support conditions as yet.

At this point the stiffness matrix K is symmetric positive semi-definite as was noted in Chapter II. All relations among the various elements of the structure have been specified. Now the support conditions which relate the structure to the rest of the world are specified. The number of supported degrees of freedom involved in this specification equals the number of rigid body motions of the structure so that the remaining degrees of freedom constitute a positive definite system.

An appropriate permutation matrix P applied as an orthogonal transformation to equation (2.1), i.e.,

$$P^t K P P^t r = P^t F \ , \tag{3.6}$$

induces a partitioning of the form

$$\left[\begin{array}{c|c} K_{NN} & K_{NS} \\ \hline K_{NS}^t & K_{SS} \end{array}\right] \left[\begin{array}{c} r_N \\ \hline r_S \end{array}\right] = \left[\begin{array}{c} F_N \\ \hline F_S \end{array}\right] . \tag{3.7}$$

With K in this partitioned form, the vector r_S contains the supported degrees of freedom. In the simplest case, r_S is set identically to zero. Note that the partition in equation (3.7) may be viewed as being induced by a reordering and partitioning of the assembly matrix A of the form

$$A = [A_N \mid A_S] \ . \tag{3.8}$$

With r_S set to zero, the resulting system

$$K_{NN}r_N = F_N \tag{3.9}$$

is symmetric positive definite so that K_{NN} may be factored and equation (3.9) then may be solved for r_N . From the same partition in equation (3.7) with r_S set to zero is obtained the reaction equation

$$K_{NS}^t r_N = F_S \ . \tag{3.10}$$

The solution r_N of equation (3.9) may be substituted into equation (3.10) to determine F_S , the reactionary forces that are experienced at the supported degrees of freedom due to the imposed support conditions [7].

Thus the factorization of K in the classical approach is performed at the global stiffness matrix level after all matrix transformations have been made and just prior to the actual solution process for r . In contradistinction, the natural factor approach begins with a Cholesky factorization at the element level in local coordinates.

CHAPTER IV

THE NATURAL FACTOR AT THE ELEMENT LEVEL

As previously noted, the elemental stiffness relationship may be expressed in local coordinates in the form

$$k_L r_L = F_L \ .$$

In general, the matrix k_L is symmetric and positive semi-definite. The loss of rank of k_L is equal to the number of rigid body modes which are inherent in the idealization of the element. The intent of the natural factor formulation, according to [2], is to arrive at an effective formation of the factor s_L of the element stiffness matrix k_L for numerical computation such that

$$k_L = s_L^t s_L \ . \tag{4.1}$$

To this end the system in equation (3.1) is permuted, if necessary, and partitioned just as is done at the overall structural level in the classical approach. Thus equation (3.1) may be written in partitioned form as

$$\left[\begin{array}{c|c} k_{NN}^L & k_{NS}^L \\ \hline \left(k_{NS}^L\right)^t & k_{SS}^L \end{array}\right] \left[\begin{array}{c} r_N^L \\ \hline r_S^L \end{array}\right] = \left[\begin{array}{c} F_N^L \\ \hline F_S^L \end{array}\right] . \tag{4.2}$$

The selection of this partition is arbitrary except that the partition must obey the same criterion required of equation (3.6) at the overall structural level, namely that the element is supported in a statically

determinate manner when r_S^L is set equal to zero. Without loss of generality, the element stiffness relationship may be assumed to be expressed in a manner such that the partitioned form in equation (4.2) already exists without the permutation of equation (3.1). The natural stiffness matrix k_{NN}^L is symmetric positive definite and therefore may be Cholesky factored to yield

$$k_{NN}^L = u_N^t u_N \ . \tag{4.3}$$

By means of the factorization of k_{NN}^L in equation (4.3) and the application of basic principles of structural analysis, a factorization of k_L is derived [2]. Elimination of r_N^L from the relations represented by equation (4.2) yields the relationship

$$[k_{SS}^L - (k_{NS}^L)^t (k_{NN}^L)^{-1} k_{NS}^L] r_S^L = F_S^L - (k_{NS}^L)^t (k_{NN}^L)^{-1} F_N^L \ . \tag{4.4}$$

Since a rigid body mode cannot give rise to a deformation, equation (4.4) must be identically zero for any choice of r_S^L. This formulates an equilibrium condition for the nodal forces F_N^L and F_S^L, that is

$$k_{SS}^L = (k_{NS}^L)^t (k_{NN}^L)^{-1} k_{NS} \tag{4.5}$$

must always hold. Substitution of the factorization of k_{NN}^L given by equation (4.3) into equation (4.5) yields the factorization of k_{SS}^L as

$$k_{SS}^L = (u_N^{-t} k_{NS})^t (u_N^{-t} k_{NS}) \ . \tag{4.6}$$

Finally, substitution of equations (4.3) and (4.6) into equation (4.2) leads to the factorization of k_L indicated in equation (4.1) with s_L

given by the partitioned form

$$s_L = [u_N \mid u_N^{-t} k_{NS}^L] \ . \tag{4.7}$$

To obtain the factorization of the element in global coordinates, equation (4.1) is substituted into equation (3.3) to yield

$$k = (s_L T)^t (s_L T) \ , \tag{4.8}$$

that is, in global coordinates

$$k = s^t s \tag{4.9}$$

where s is given by

$$s = [u_N \mid u_N^{-t} k_{NS}^L] T = s_L T \ . \tag{4.10}$$

Each element k_i of the idealized system is factored as indicated in equation (4.9), that is,

$$k_i = s_i^t s_i \ . \tag{4.11}$$

CHAPTER V

THE NATURAL FACTOR AT THE OVERALL STRUCTURAL LEVEL

Once the element stiffness matrices k_i have been expressed in factored form, the same sequence of matrix transformations used to form the overall stiffness matrix K in the classical approach now induces the natural factorization of K from the factorizations of its elements. In the expression of the unassembled k in equation (3.4) the element stiffness matrices k_i are replaced by their factorizations as given by equation (4.11) to yield the natural factorization of the unassembled overall structural matrix k as

$$k = \delta^t \delta \ , \tag{5.1}$$

where the factor δ is given by

$$\delta = \begin{bmatrix} \delta_1 & & & \\ & \delta_2 & & \\ & & \ddots & \\ & & & \delta_m \end{bmatrix} . \tag{5.2}$$

Finally, substitution of equation (5.1) into equation (3.5) yields the "natural" factorization of K as

$$K = S^t S \ , \tag{5.3}$$

where the factor S is given by

$$S = \delta A \ . \tag{5.4}$$

In the classical approach the partitioning of the assembly matrix A as given in equation (3.7) induced the partitioning of equation (2.1) as given in equation (3.6). This same partitioning of A similarly induces a partitioning of S as

$$S = [S_N \mid S_S] \ . \tag{5.5}$$

The nonsingular system of equation (3.9) is then expressed in terms of the natural factorization as

$$S_N^t S_N r_N = F_N \ , \tag{5.6}$$

and the reaction equation (3.10) is expressed as

$$S_S^t S_N r_N = F_S \ . \tag{5.7}$$

Suppose r_N is an $(n \times 1)$ vector. Then S_N is an $(m \times n)$ matrix of rank n. The basic strategy for solving equation (5.6) is to perform a QR factorization of S_N, e.g. to form

$$S_N = Q\tilde{R} \ , \tag{5.8}$$

where Q is an $(m \times m)$ orthogonal matrix and $\tilde{R}$ is an $(m \times n)$ rectangular matrix of the form

$$\tilde{R} = \begin{bmatrix} R \\ \hline 0 \end{bmatrix} \tag{5.9}$$

with R an $(n \times n)$ upper triangular matrix. This factorization may be formed by Householder transformations, Givens rotations, or Gram-Schmidt orthogonalization [5]. Substitution of equations (5.8) and

(5.9) into equation (5.6) finally expresses equation (3.9) in the factored form of

$$R^{t}Rr_{N} = F_{N} \ . \qquad (5.10)$$

CHAPTER VI

EXAMPLES OF THE ELEMENT NATURAL FACTOR

The first example will be the derivation of the natural factorization of the three-dimensional axial force element [7]. An axial force element is an idealized rod subjected to a compressing or stretching force at each end. A representation of an axial force element in local and global coordinates appears in Figure A-1, Appendix.

In local coordinates the axial force element stiffness relationship is given by

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} r_{x_{L,1}} \\ r_{x_{L,2}} \end{bmatrix} = \begin{bmatrix} F_{x_{L,1}} \\ F_{x_{L,2}} \end{bmatrix}, \tag{6.1}$$

where

A is the cross-sectional area of the rod,

E is the elastic modulus,

L is the length of the element.

The element stiffness matrix k_L in local coordinates given by

$$k_L = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{6.2}$$

is of rank 1. If the element is assumed to be supported at node 2, the partition thus induced as in equation (4.2) yields

$$k^L_{NN} = \left[\frac{EA}{L}\right], \tag{6.3}$$

which when factored as indicated in equation (4.3) yields

$$u_N = \left(\frac{EA}{L}\right)^{\frac{1}{2}}. \tag{6.4}$$

The natural factorization of k^{L}_{NN} in local coordinates as indicated in equation (4.1) follows with

$$\delta_L = \left(\frac{EA}{L}\right)^{\frac{1}{2}} \begin{bmatrix} 1 & -1 \end{bmatrix} . \tag{6.5}$$

The coordinate transformation matrix T to transform to global coordinates is given by

$$T = \begin{bmatrix} \lambda & \mu & \nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & \mu & \nu \end{bmatrix} , \tag{6.6}$$

where λ, μ, and ν are direction cosines. Substitution of equation (6.6) into equation (4.10) yields the natural factorization of k in global coordinates with

$$\delta = \left(\frac{EA}{L}\right)^{\frac{1}{2}} \begin{bmatrix} \lambda & \mu & \nu & -\lambda & -\mu & -\lambda \end{bmatrix} . \tag{6.7}$$

The element stiffness matrix k in global coordinates, in contrast to equation (6.7), is given by

$$k = \frac{EA}{L} \begin{bmatrix} \lambda^2 & \lambda\mu & \lambda\nu & -\lambda^2 & -\lambda\mu & -\lambda\nu \\ & \mu^2 & \mu\nu & -\lambda\mu & -\mu^2 & -\mu\nu \\ & & \nu^2 & -\lambda\nu & -\mu\nu & -\nu^2 \\ & & & \lambda^2 & \lambda\mu & \lambda\nu \\ & & & & \mu^2 & \mu\nu \\ \text{Sym.} & & & & & \nu^2 \end{bmatrix} . \tag{6.8}$$

As a second example the natural factorization of the two-dimensional six degree of freedom framework element will be derived. A

framework element is an idealized rod that is subjected not only to axial forces as in the first example but also to a flection, or bending force, at each end node [7]. A representation of a framework element in local and global coordinates appears in Figure A-2, Appendix.

In local coordinates the framework element stiffness relationship is given by

$$\begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ & & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ & & & \frac{EA}{L} & 0 & 0 \\ & & & & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \text{Sym.} & & & & & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} x_{L,1} \\ y_{L,1} \\ \theta_2 \\ x_{L,2} \\ y_{L,2} \\ \theta_2 \end{bmatrix} = \begin{bmatrix} F_{x_{L,1}} \\ F_{y_{L,1}} \\ M_{L,1} \\ F_{x_{L,2}} \\ F_{y_{L,2}} \\ M_{L,2} \end{bmatrix}, \quad (6.9)$$

where

A is the cross-sectional area,

I is the moment of inertia,

x and y are displacements,

θ is a flection,

P is a force,

M is a moment.

Since the local element stiffness matrix k_L is of rank 3, the element may be idealized as supported at node 2 to induce the partition

of k_L as indicated by equation (4.2). The symmetric positive definite submatrix k^L_{NN} is then given by

$$k^L_{NN} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ & \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \text{Sym.} & & \frac{4EI}{L} \end{bmatrix} \tag{6.10}$$

and may be Cholesky factored so that equation (4.3) is satisfied by

$$u_N = \left(\frac{EI}{L}\right)^{\frac{1}{2}} \begin{bmatrix} \left(\frac{A}{I}\right)^{\frac{1}{2}} & 0 & 0 \\ 0 & \frac{2\sqrt{3}}{L} & \sqrt{3} \\ 0 & 0 & 1 \end{bmatrix}. \tag{6.11}$$

According to equation (4.7), the natural factorization of k_L in local coordinates then is determined as

$$s_L = \left(\frac{EI}{L}\right)^{\frac{1}{2}} \left[\begin{array}{ccc|ccc} \left(\frac{A}{L}\right)^{\frac{1}{2}} & 0 & 0 & -\left(\frac{A}{L}\right)^{\frac{1}{2}} & 0 & 0 \\ 0 & \frac{2\sqrt{3}}{L} & \sqrt{3} & 0 & -\frac{2\sqrt{3}}{L} & \sqrt{3} \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]. \tag{6.12}$$

From Figure A-2, Appendix, the coordinate transformation matrix T is seen to be

$$T = \left[\begin{array}{ccc|ccc} \cos\phi & \sin\phi & 0 & & & \\ -\sin\phi & \cos\phi & 0 & & \bigcirc & \\ 0 & 0 & 1 & & & \\ \hline & & & \cos\phi & \sin\phi & 0 \\ & \bigcirc & & -\sin\phi & \cos\phi & 0 \\ & & & 0 & 0 & 1 \end{array}\right] . \quad (6.13)$$

Substitution of equation (6.12) and (6.13) for δ_L and T into equation (3.10) yields the natural factorization of the two-dimensional framework element in global coordinates with the factor δ given by

$$\delta = \left(\frac{EI}{L}\right)^{\frac{1}{2}} \left[\begin{array}{ccc|ccc} \left(\frac{A}{I}\right)^{\frac{1}{2}}\cos\phi & \left(\frac{A}{I}\right)^{\frac{1}{2}}\sin\phi & 0 & -\left(\frac{A}{I}\right)^{\frac{1}{2}}\cos\phi & -\left(\frac{A}{I}\right)^{\frac{1}{2}}\sin\phi & 0 \\ -\frac{2\sqrt{3}}{L}\sin\phi & \frac{2\sqrt{3}}{L}\cos\phi & \sqrt{3} & \frac{2\sqrt{3}}{L}\sin\phi & -\frac{2\sqrt{3}}{L}\cos\phi & \sqrt{3} \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] . \quad (6.14)$$

CHAPTER VII

AN EXAMPLE OF THE NATURAL FACTOR APPROACH AT THE OVERALL STRUCTURAL LEVEL

For an example of the natural factor approach at the overall structural level, consider the rigid framework represented in Figure A-3, Appendix, with member properties given in Table A-1, Appendix [7]. This structure is idealized as consisting of four framework elements joining the five nodes a, b, c, d, and e. From the values in Table A-1, Appendix, the natural factor of each of the elements δ_{ab}, δ_{bc}, δ_{cd}, and δ_{ed} is evaluated according to the closed form given in equation (6.14). The result is given in Table A-2, Appendix.

For this rigid framework to be supported in a statically determinate manner, i.e., so that the structure remains erect, the six degrees of freedom at both nodes a and e are assumed set equal to zero. The boolean assembly matrix A is partitioned, accordingly, with nodes b, c, and d in A_N and with nodes a and e in A_S as suggested by equation (3.8). If the element factors are ordered as indicated by equation (5.2), that is,

$$\delta = \begin{bmatrix} \delta_{ab} & & & \\ & \delta_{bc} & & \\ & & \delta_{cd} & \\ & & & \delta_{de} \end{bmatrix}, \qquad (7.1)$$

then the terms of the boolean assembly matrix A are given in Table A-3, Appendix, with partitioning into A_N and A_S as noted. Finally, the

terms of the assembled natural factor S and its subfactors S_N and S_S determined by equations (5.4) and (5.5) are given in Table A-4 and Table A-5, Appendix.

CHAPTER VIII

OBSERVATIONS

In both the classical approach and the natural factor approach the support conditions that are imposed at the overall structural level are dependent upon the particular structure being modeled. The imposition of support conditions in the development of the natural factor of a given element is not based upon consideration of the conditions of a particular structure. For example, the choice of all degrees of freedom at node 2 as the supported degrees in the framework example of Chapter VI could have been any other choice of nodes for which k^{L}_{NN} is a (3×3) matrix of rank 3.

The derivation of the natural factor of a given element in local coordinates in analytical terms, e.g. equations (6.5) and (6.12), is especially important. Note that the derivation of the natural factor of the axial force element in global coordinates (see equation (6.7)) could have been derived from the global element stiffness matrix k given in equation (6.8). Clearly, the use of local coordinates is preferred over global coordinates. Finally, the analytical derivation of the natural factor of a given element avoids the computer time used and any numerical instability incurred by numerically forming the natural factor of each element during the actual solution of a structural analysis problem.

In general, the dimensions of an element natural factor δ will be $(r \times n)$, where the original element stiffness matrix k is of dimension $(n \times n)$ and of rank $r \leq n$. For example, the element natural

factor Δ of the two-dimensional framework element given in equation (6.14) is (3×6). The dimensions of the overall natural factor S and of its partitions S_N and S_S are determined in a similar manner. Consider the structure in Chapter VII. The structure is modeled by four two-dimensional framework elements, each of dimension (3×6); therefore, the factors S, S_N, and S_S each have twelve rows (four elements times three rows per element). The structure consists of five nodes, each with three degrees of freedom; therefore, the factor S has fifteen columns (five nodes times three degrees of freedom per node). Since two nodes are supported (six degrees of freedom), S_S has six columns and S_N has nine columns.

In general, S_N will be a rectangular matrix of dimension $(m \times n)$ of rank n. In the formation of the overall natural factor S_N, none of the terms of the element natural factors are combined in any way. The modification of the structural model by the insertion or the deletion of elements is easily accomplished.

The solution of equation (5.6) may be accomplished by several methods. The factor R given in equation (5.10) is equivalent to the Cholesky factor U in equation (2.3), except that the rows of R may be the negatives of the corresponding rows in U. Numerically, the computed solution of equation (5.10) is generally more accurate than the computed solution of equation (2.1). The formation of $K = S^t S$ and the subsequent Cholesky factorization of K is equivalent to the solution of a least squares problem by the normal equations approach. Unfortunately, the normal equations method has several drawbacks:

1. Unless extended precision is employed, which could be costly in space and time, there may be a serious loss of information in explicitly forming and processing $K = S^t S$.
2. The condition number of K is the square of the condition number of S, so that the accuracy of the computed solution to the system in equation (2.1) may be questionable, especially if S itself is already poorly conditioned.
3. The sparsity of S does not necessarily imply that K will be comparably sparse. Indeed, if S has a full row then K is full.

Several areas for further research are suggested. The natural factor S is a sparse matrix. The zero structure of a given element natural factor may be determined analytically; the framework element of equation (6.14) always has four zeroes in its third row. The determination of boolean assembly matrices that yield a factor S for which the fill-in during the QR factorization process is minimized should be studied. Several sparse least squares codes are being developed [4, 5]. The development of iterative methods for natural factor QR factorization presently is being persued by J. H. Argyris [3]. The application of the natural factor approach to new types of structural problems is of current interest [1]. In summary, there are many questions involving the natural factor approach that are yet to be investigated.

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REFERENCES

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APPENDIX

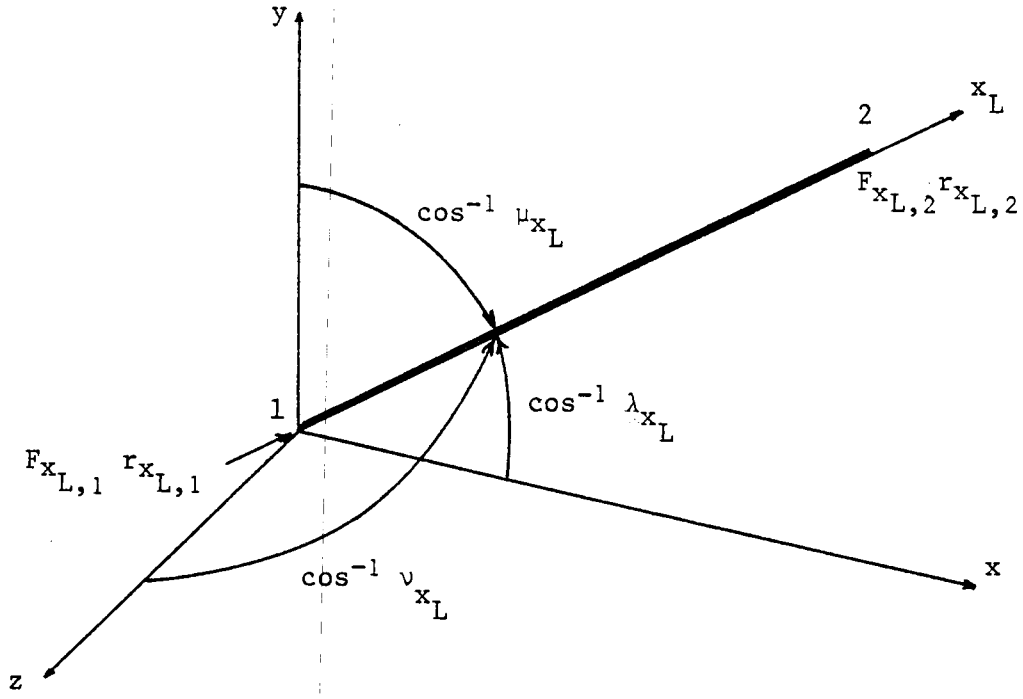

Figure A-1. Three-dimensional Axial Force Member

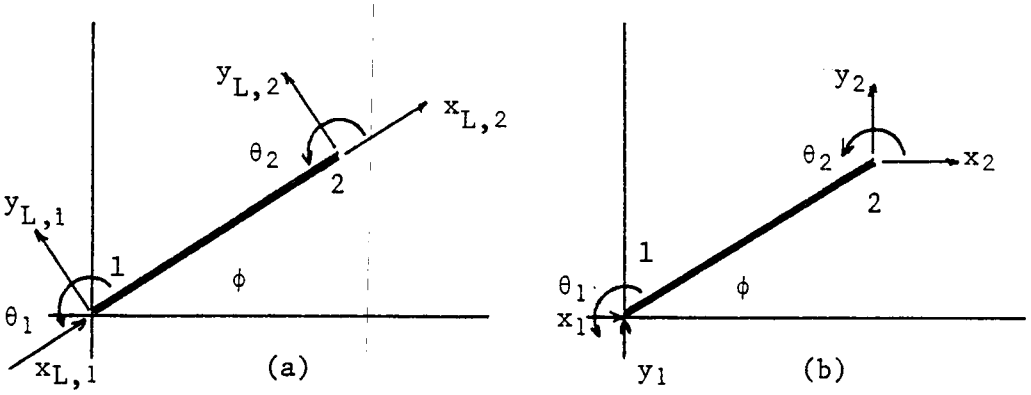

Figure A-2. Two-dimensional Framework Element

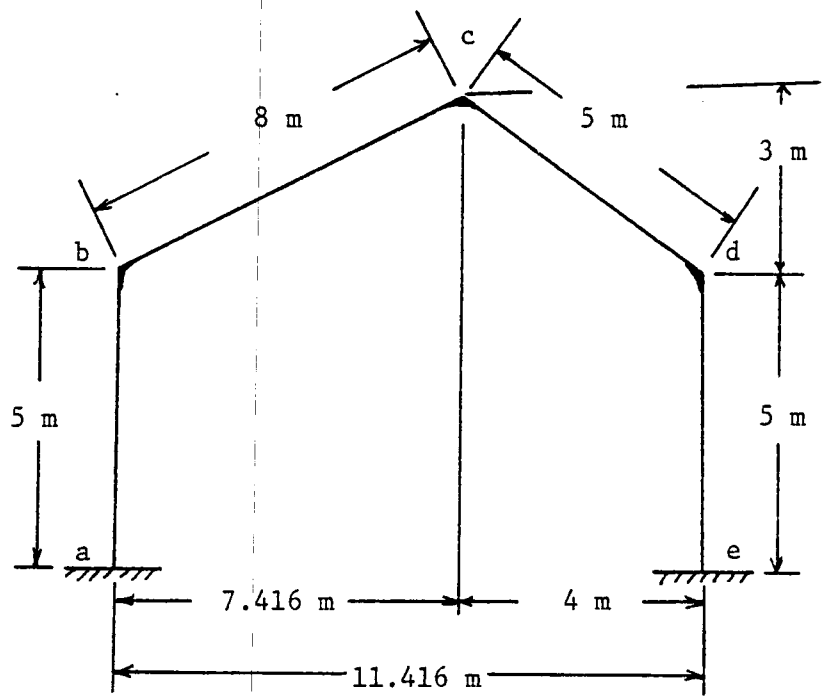

Figure A-3. Rigid Frame

TABLE A-1

MEMBER PROPERTIES FOR THE
STRUCTURE IN FIGURE A-3

δ	A	E	I	L	ϕ
ab	4×10^3 mm^2	2×10^5 MPa	5×10^7 mm^4	5 m	$\frac{\pi}{2}$
bc	6×10^3 mm^2	2×10^5 MPa	2×10^8 mm^4	8m	$\sin^{-1}(\frac{3}{8})$
cd	4×10^3 mm^2	2×10^5 MPa	5×10^7 mm^4	5m	$\sin^{-1}(-\frac{3}{5})$
ed	4×10^3 mm^2	2×10^5 MPa	5×10^7 mm^4	5m	$\frac{\pi}{2}$

TABLE A-2

ELEMENT FACTORIZATIONS

$$\delta_{ab} = \left[\begin{array}{ccc|ccc} 0 & 12.649 & 0 & 0 & -12.649 & 0 \\ -0.97979 & 0 & 2449.5 & 0.97979 & 0 & 2449.5 \\ 0 & 0 & 1414.2 & 0 & 0 & -1414.2 \end{array}\right]$$

$$\delta_{bc} = \left[\begin{array}{ccc|ccc} 11.353 & 4.5928 & 0 & -11.353 & -4.5928 & 0 \\ -0.36309 & 0.89756 & 3873.0 & 0.36309 & -0.89756 & 3873.0 \\ 0 & 0 & 2236.1 & 0 & 0 & -2236.1 \end{array}\right]$$

$$\delta_{cd} = \left[\begin{array}{ccc|ccc} 10.119 & -7.5895 & 0 & -10.119 & 7.5895 & 0 \\ 0.58788 & 0.7838 & 2449.5 & -0.58788 & -0.7838 & 2449.1 \\ 0 & 0 & 1414.2 & 0 & 0 & 1414.2 \end{array}\right]$$

$$\delta_{de} = \left[\begin{array}{ccc|ccc} 0 & 12.649 & 0 & 0 & -12.649 & 0 \\ -0.97979 & 0 & 2449.5 & 0.97979 & 0 & 2449.5 \\ 0 & 0 & 1414.2 & 0 & 0 & -1414.2 \end{array}\right]$$

TABLE A-3

BOOLEAN ASSEMBLY MATRIX A

	A_N									A_S						
	x_b	y_b	θ_b	x_c	y_c	θ_c	x_d	y_d	θ_d	x_a	y_a	θ_a	x_e	y_e	θ_e	
										1	0	0				x_a
										0	1	0				y_a
										0	0	1				θ_a
δ_{ab}	1	0	0													x_b
	0	1	0													y_b
	0	0	1													θ_b
δ_{bc}	1	0	0													x_b
	0	1	0													y_b
	0	0	1													θ_b
				1	0	0										x_c
				0	1	0										y_c
				0	0	1										θ_c
δ_{cd}				1	0	0										x_c
				0	1	0										y_c
				0	0	1										θ_c
							1	0	0							x_d
							0	1	0							y_d
							0	0	1							θ_d
δ_{de}													1	0	0	x_e
													0	1	0	y_e
													0	0	1	θ_e
							1	0	0							x_d
							0	1	0							y_d
							0	0	1							θ_d

TABLE A-4

GLOBAL ASSEMBLED NATURAL FACTOR S_N

$S = \delta A = [\delta_1 \ . \ . \ . \ \delta_4] \cdot A$

S_N

0	-12.649	0						
0.97979	0	2449.5						
0	0	-1414.2						
11.353	4.5928	0	-11.353	-4.5928	0			
-0.36309	0.89756	3873.0	0.36309	-0.89756	3873.0			
0	0	2236.1	0	0	-2236.1			
			10.1193	-7.5895	0	-10.1193	7.5895	0
			0.58788	0.78384	2449.5	-0.58788	-0.78384	2449.5
			0	0	1414.2	0	0	-1414.2
						0	12.649	0
						0.97979	0	2449.5
						0	0	-1414.2

TABLE A-5

GLOBAL ASSEMBLED NATURAL FACTOR S_S

$S = \delta A = [\delta_1 \ . \ . \ . \ \delta_4] \cdot A$

S_S

0	12.649	0			
-0.97979	0	2449.5			
0	0	1414.2			
			0	12.649	0
			0.97979	0	2449.5
			0	0	1414.2

VITA

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