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I am submitting herewith a dissertation written by Essam Sima'an Ackleh entitled "TWO PHOTON COUPLINGS OF QUARKONIUM AND POSITRONIUM WITH GENERAL ANGULAR QUANTUM NUMBERS." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degreee of Doctor of Philosophy, with a major in Physics.

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TWO PHOTON COUPLINGS OF QUARKONIUM AND POSITRONIUM WITH GENERAL ANGULAR QUANTUM NUMBERS

A Dissertation

Presented for the

Doctor of Philosophy

Degree

The University of Tennessee, Knoxville

Essam Sima'an Ackleh

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DEDICATION

This dissertation is dedicated to

my parents

Mr. Sima'an Elias Ackleh

سمعان إلياس عاقلة

and

Mrs. Wedad Toufique Ackleh

وداد توفيق عاقلة

without whose patience, encouragement, support,

and above all love,

my education would not have been possible.

To them I owe it all.

بكل حب وإخلاص

عصام

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ABSTRACT

In this dissertation we derive relativistic formulas for the $\gamma\gamma$ widths of $q\bar{q}$ states with general total angular momentum J. Previously, two photon decay widths of $q\bar{q}$ states had been calculated relativistically only for states with $J^{PC} = 0^{-+}$, 0^{++} and 2^{++}, and very recently these results were supplemented by a nonrelativistic calculation of the $l = 2$ 2^{-+} $\gamma\gamma$ width. This dissertation was suggested by the recent observation of a large $\gamma\gamma$ width for the $l = 2$ $q\bar{q}$ state $\pi_2(1670)$, and by the nonrelativistic calculation which predicted a width $2-3$ orders of magnitude smaller than the measured rate. Motivated by this first observation of a state with $l > 1$ in $\gamma\gamma$ collisions, and the numerical inaccuracy of the nonrelativistic prediction for this rate, we derive relativistic quark model results for the $\Gamma_{\gamma\gamma}$ of a general quarkonium state. Our formulas involve a one dimensional overlap integral of the $q\bar{q}$ wavefunction in momentum space multiplied by a function of momentum which depends on the quantum numbers of the system. We have confirmed that our formulas reproduce the existing relativistic and nonrelativistic results. We find large relativistic corrections to the absolute rate for light quark systems, and present new numerical results for the $\Gamma_{\gamma\gamma}$ of many interesting $q\bar{q}$ systems ($q = u, d, s, c, b$ and t) and helicity selection rules. Finally we apply our nonrelativistic results to positronium, and

derive an explicit positronium $\Gamma_{\gamma\gamma}$ for all J^{PC} in closed form.

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CHAPTER 1

INTRODUCTION

1.1 Preface

In the 1950s and 1960s the discovery of more and more "elementary" strongly interacting particles made it appear that there might well be an infinity of such states. This rich spectrum implied that either subnuclear physics is very complex, or these particles are not "elementary" and instead are made from a smaller set of constituents, which combine in different ways to produce an abundance of seemingly different particles, just as protons, neutrons, and electrons bind to produce over a hundred types of atoms. In 1964 a hypothesis put forth by G. Zweig[1], M. Gell-Mann[2] and others accounted for the known strongly interacting particles (hadrons). This was the hypothesis of quarks, which are a set of fundamental spin-$\frac{1}{2}$ strongly interacting particles which formed the known hadrons. Three different types (flavors) of quarks seemed to explain all hadrons discovered before 1974. These are the up (u), down (d) and strange (s) quarks. At present five flavors have been confirmed, adding to the last three the charm quark (c) and the bottom quark (b); a sixth one, the top quark (t), has been widely accepted theoretically, and some experimental evidence appears to support its existence. These quarks have electric charges of $\frac{2}{3}$ for $\{u, c, t\}$ and $-\frac{1}{3}$ for $\{d, s, b\}$, in units of the positron charge. The strongly-interacting bound states are classified according to their spin; integer-spin hadrons are known as mesons, and half integer spin as baryons. In the context of the quark model most mesons can be understood as bound states of a quark and an antiquark $(q\bar{q})$, and baryons as bound states of three quarks (qqq). The quark model historically preceded the formal quantum field theory of the strong interactions, which is known as Quantum Chromodynamics (QCD). QCD introduces gluons, which are elementary vector bosons that mediate the strong forces between quarks, just as photons mediate the electromagnetic forces between charged particles. QCD and the notion of color-singlet physical states motivated theoretical predictions of hadrons outside the standard $q\bar{q}$ and qqq of the quark model. These include meson-meson molecules $(q\bar{q})(q\bar{q})$ and four quark systems $(qq\bar{q}\bar{q})$[3,4], and hybrid systems such as hybrid mesons $(q\bar{q}g)$ and hybrid baryons $(qqqg)$, and glueballs (gg). We shall refer to these as the "*extraordinary*" hadrons versus the ordinary $q\bar{q}$ and qqq hadrons. In the quark potential model the quarks are assumed to obey the Schrödinger equation with a confining potential, which we will discuss in more detail subsequently. For $q\bar{q}$ systems the quark potential model predicts orbital and radial excitations, just as in hydrogenic systems. To date the most well-established mesons have orbital angular momentum $L = 0$ (S-wave) or $L = 1$ (P-wave) and the understanding of the meson spectrum is far from satisfactory. Nonetheless, as much is already known about $q\bar{q}$ and qqq states, the most exciting topic in QCD spectroscopy at present is the search for "*extraordinary*" hadrons. This search concentrates primarily on mesons, where more types of unconventional states are expected. The masses predicted for such hadrons are usually in the $1.5 - 2.5$ GeV region, and therefore a crucial part of the search for such hadrons is to have a very well understood $q\bar{q}$ meson spectrum in that mass region. This involves orbitally and radially excited $q\bar{q}$ systems containing only u, d, and s quarks. Therefore any study that helps one understand the ordinary hadrons and establish a clear spectrum for them, in turn serves to establish the background for "*extraordinary*" ones. One of the most important tools for studying one sector of the meson spectrum, mesons with positive charge conjugation (C=+1), is two photon couplings. This is so because photons couple to electric charges, so measurement of the two photon partial decay width ($\Gamma_{\gamma\gamma}$) of mesons provides a probe of their internal charge structure, and thus of their quark flavor content and wavefunctions. In this project we report a relativistic calculation of the $\Gamma_{\gamma\gamma}$ of $q\bar{q}$ mesons with all allowed J^{PC} quantum numbers in the context of the quark model.

1.2 QCD and the quark model

Quarks and gluons are not seen in nature as free particles; they occur in bound states as hadrons, which are bound by the quark-gluon and gluon-gluon interactions of Quantum Chromodynamics (QCD). Today elementary particles are described by quantum field theories such as QCD and QED (Quantum Electrodynamics, which describes the interaction of photons and charged particles, and which has succeeded in describing electrodynamical processes to very high accuracy). One of the remaining problems is that of bound states; at present there is no widely accepted systematic field theoretic approach to bound states. For this reason one conventionally resorts to other means such as nonrelativistic quantum mechanical models for treating bound states.

The QCD lagrangian is[5]

$$\mathcal{L}_{QCD}(x) =$$

$$\bar{\psi}_f(x)\Big\{i\,\not{\partial} - g\frac{\lambda^a}{2}\,\not{A}^a(x) - m_f\Big\}\psi_f - \frac{1}{4}F^a_{\mu\nu}(x)F^{a\,\mu\nu}(x)\,, \tag{1.1}$$

where repeated indices are summed over. The subscript f runs over all six quark flavors (u, d, ...), and $a \in \{1,2,\ldots,8\}$ runs over all 8 gluonic color degrees of freedom. The 8 λ^a's are the 3×3 Gell-Mann color matrices, which are traceless and hermitian and satisfy the commutation relations of the group $SU(3)$,

$$\left[\lambda^a, \lambda^b\right] = 2\,i\,f^{abc}\,\lambda^c\,. \tag{1.2}$$

The structure constants of this group's Lie algebra, f^{abc}, are antisymmetric and symmetric under odd and even permutations of $\{a, b, c\}$ respectively (for explicit values of these constants, and a brief survey of group theory relevant to QCD and the quark model, see the discussion by Close[6]). The color of the strong interactions is the analog of charge for QED; it is in effect the charge of the strong force. The fact that QCD is a non-Abelian gauge theory results in a feature different from QED, which is gluon self-interactions. Local $SU(3)$ gauge invariance requires that gluons have three-gluon and four-gluon self-interactions. This is readily seen in the form of the gluon field tensor

$$F_a^{\mu\nu}(x) = \partial^\mu A_a^\nu(x) - \partial^\nu A_a^\mu(x) + g\, f_{abc}\, A_b^\mu(x)\, A_c^\nu(x)\,, \tag{1.3}$$

which in (1.1) implies A^3 and A^4 terms in the interaction. QCD also exhibits the property that its effective coupling constant $\alpha_s \equiv \frac{g^2}{4\pi}$ (which includes higher order loop diagrams) depends on the separation of the quarks. At large separations this effective α_s increases and the self-interaction of gluons becomes more important and eventually becomes nonperturbatively large. This is believed to generate a linear potential ($V(r) \propto r$) between quarks which confines the quarks within hadrons. This effect in QCD is termed *confinement.* In contrast at small separations the effective α_s decreases and, as in QED, a Coulomb potential dominates. Another property of quark bound states is that they appear to exist in nature either as $q\bar{q}$ mesons (quarkonia), or as qqq baryons. There are exceptions; there is considerable evidence for the existence of two meson-meson molecules, $(q\bar{q})(q\bar{q})$ which are analogous to the deuteron[3]. Although quarks exist in three different colors, these hadronic bound states are color singlet (color neutral) states. The color wavefunctions of mesons and baryons are

$$|\text{meson}\rangle = \frac{1}{\sqrt{3}}\sum_{i=1}^{3}|q_i\bar{q}_i\rangle \tag{1.4}$$

and

$$|\text{baryon}\rangle = \frac{1}{\sqrt{6}}\sum_{i=1}^{3}\epsilon^{ijk}|q_iq_jq_k\rangle \tag{1.5}$$

respectively, where $i, j, k \in \{1, 2, 3\}$ are the quark color indices and ϵ^{ijk} is the totally antisymmetric Levi-Civita tensor.

One relates all this to the quark model (QM) (in particular the nonrelativistic quark potential model(NQPM)) by neglecting possible valence gluons and assuming that the quarks obey the nonrelativistic Schrödinger equation with a phenomenological potential (Coulomb-plus-linear is the usual choice). The hope in this approach is that with the gluonic degrees of freedom excluded one has an approximately satisfactory description of hadrons using an effective quantum mechanical hamiltonian with a potential in terms of quarks only. Although the quark states have the same quantum numbers as the quark fields in $\mathcal{L}_{QCD}$, they differ from them in some dynamical quantities that are attributed to quarks in the quark model to fit data, such as their masses. For example, whereas the current masses of the u and d light quarks are a few MeV, the masses used in the NQPM are usually about 330 MeV (In a relativistic version Godfrey and Isgur showed that a somewhat smaller but comparable mass of about 220 MeV for these quarks would result in better general agreement with data[7].) It is customary to refer to these quarks and their masses as the *constituent* ones, to distinguish them from the quark fields and current masses of $\mathcal{L}_{QCD}$.

Every parameter in the $\mathcal{L}_{QCD}$ of (1.1) is flavor independent except for the current masses. These masses for light quarks are[5]

$$m_u \approx 4\,\text{MeV}, \qquad m_d \approx 7\,\text{MeV}, \qquad \text{and} \qquad m_s \approx 150\,\text{MeV}\,. \tag{1.6}$$

One can approximate this situation by assuming equal masses for all these light quarks. In this limit the QCD lagrangian (1.1) is invariant under global $SU(3)_f$ transformations;

$$\mathcal{L}_{QCD}(\chi_3 \to e^{i\alpha^a\lambda^a}\,\chi_3) = \mathcal{L}_{QCD}(\chi_3)\,, \tag{1.7}$$

where χ_3 is a flavor-space vector

$$\chi_3 = \begin{pmatrix} u \\ d \\ s \end{pmatrix}\,, \tag{1.8}$$

and a sum over the 8 parameters of $SU(3)_f$ is implied (α^a's are space-time independent constants[5]). In the language of group theory, the light quarks are assigned to the fundamental representation $\mathcal{D}^{(3)}$ of the group $SU(3)$ (special unitary transformations, represented by 3×3 unitary matrices). This representation is denoted by **3**. In turn the antiquarks are assigned to the basic representation $\bar{\chi}_3$ of that group, which is denoted by $\bar{\mathbf{3}}$,

$$\bar{\chi}_3 = \begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix} . \tag{1.9}$$

Exact invariance of the lagrangian $\mathcal{L}_{QCD}$ under the transformation (1.7) would imply that the physical mesons and baryons span degenerate multiplets belonging to the irreducible representations of $SU(3)_f$. The 3 in $SU(3)_f$ is simply the three quark flavors; since our subject involves the light $q\bar{q}$ mesons, we will illustrate the above discussion using these particles.

Since mesons are combinations of a quark and an antiquark, in group theory we are forming a product of states which transforms as the representation **3** and the conjugate one $\bar{\mathbf{3}}$. This product state is reducible, and can be decomposed into a sum of irreducible representations as

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1} . \tag{1.10}$$

These denote 8 degenerate states (an octet), and a singlet. In the experimental meson spectrum, we find these flavor states in both spin-singlet and spin-triplet S-wave mesons (called pseudoscalars and vectors respectively). The pseudoscalars octet and singlet are called $\{\pi^0, \pi^+, \pi^-, K^0, \bar{K}^0, K^+, K^-, \eta_8\}$ and η_1, and the corresponding vector mesons are $\{\rho^0, \rho^+, \rho^-, K^{*0}, \bar{K}^{*0}, K^{*+}, K^{*-}, \omega_8\}$ and ω_1. **Table 1.1** shows the flavor wavefunctions corresponding to

Table 1.1. Unmixed $SU(3)_f$ flavor wavefunctions.

Pseudoscalars (0^{-+})	Vectors (1^{--})	Flavor states
π^0	ρ^0	$\frac{1}{\sqrt{2}}(\|u\bar{u}> -\|d\bar{d}>)$
π^+	ρ^+	$-\|u\bar{d}\rangle$
π^-	ρ^-	$\|d\bar{u}\rangle$
K^0	K^{*0}	$-\|d\bar{s}\rangle$
K^+	K^{*+}	$-\|u\bar{s}\rangle$
K^-	K^{*-}	$\|s\bar{u}\rangle$
$\bar{K}^0$	$\bar{K}^{*0}$	$-\|s\bar{d}\rangle$
η_8	ω_8	$\frac{1}{\sqrt{6}}(\|u\bar{u}> +\|d\bar{d}> -2\|s\bar{s}>)$
η_1	ω_1	$\frac{1}{\sqrt{3}}(\|u\bar{u}> +\|d\bar{d}> +\|s\bar{s}>)$

the irreducible representations of $SU(3)_f$ to which each group of these mesons belongs, with the few exceptions being due to physical mixing between the two octet-singlet pairs which have identical quantum numbers. For example, the "η" resonances observed in nature are not the η_8 octet and the η_1 singlet of the irreducible representations of $SU(3)_f$, but are instead rotated states known as the η and η'. These are known experimentally to have significant octet-singlet mixing, and are written as

$$|\eta\rangle = cos(\theta)\ |\eta_8\rangle + sin(\theta)\ |\eta_1\rangle \tag{1.11}$$

and

$$|\eta'\rangle = -sin(\theta)\ |\eta_8\rangle + cos(\theta)\ |\eta_1\rangle\ , \tag{1.12}$$

where the mixing angle is $\theta \approx 10°$[8]. For the vector mesons the strange component dominates in the ϕ meson, and the lighter ω is mostly nonstrange, so they are the following ω_8-ω_1 linear combinations;

$$|\omega\rangle = \sqrt{\frac{2}{3}}\ |\omega_1\rangle + \frac{1}{\sqrt{3}}\ |\omega_8\rangle = \frac{1}{\sqrt{2}}\{|u\bar{u}\rangle + |d\bar{d}\rangle\} \tag{1.13}$$

and

$$|\phi\rangle = \frac{1}{\sqrt{3}}\ |\omega_1\rangle - \sqrt{\frac{2}{3}}\ |\omega_8\rangle = |s\bar{s}\rangle\ . \tag{1.14}$$

If one excludes the strange quark from consideration, one has a better symmetry, because m_u and m_d are much closer. This symmetry is described by the isospin group $SU(2)_f$. Here we have only two quark flavors, u and d, so the fundamental quark and antiquark representations are $\mathbf{2}$ and $\bar{\mathbf{2}}$, respectively; combining these gives

$$\mathbf{2} \otimes \bar{\mathbf{2}} = \mathbf{3} \oplus \mathbf{1}\ . \tag{1.15}$$

These are symmetric triplet and the antisymmetric singlet states (the symmetry is that found in interchanging the quark and the antiquark). The flavor wavefunctions of these representations can be obtained from those of $SU(3)_f$ by excluding the terms involving the s or $\bar{s}$, and then normalizing the remaining flavor wavefunctions. These correspond to the nearly degenerate pseudoscalar triplet $\{\pi^0,\ \pi^+,\ \pi^-\}$, the singlet η, and the corresponding vector isotriplet ρ and the singlet ω.

It is *a priori* quite surprising that the NQPM successfully describes most of the observed spectrum of resonances, in view of the many simplifications and assumptions and the associated nonrelativistic dynamics. The speeds of the constituent light quarks in hadrons are actually comparable to the speed of light ($v/c \approx 1$), so relativistic effects are naively expected to be very important for light quarks. Regarding this concern we quote a discussion in "Hadron Transitions in the Quark-Model"[5]

"In the face of such a situation, it is possible to propose various directly relativistic schemes as alternatives to the nonrelativistic quark model....... We prefer methods where the interaction is described by an instantaneous potential. These methods are only partially relativistic. Their main relativistic features are a relativistic expression for the kinetic energy, and quark Dirac spinors instead of nonrelativistic Pauli spinors. If one wants to know whether or not these models lead to an improvement, the best and often the only practicable method is to use a systematic v/c expansion, which has the two advantages of starting from the rather successful nonrelativistic model and of yielding an unambiguous algorithm to implement corrections order by order....... In the case of light quarks, even if a v/c expansion seems theoretically unjustified, v/c not being so

small, the corrections do seem to go in the right direction."

Here we shall also adopt this point of view, and will carry out our calculations much as these authors suggest. However instead of carrying out an expansion of the amplitude to a finite order in v/c, we use a fully relativistic amplitude. As we shall see the passage above suggests a very useful comparison, and important differences in our quarkonium $\Gamma_{\gamma\gamma}$ do appear in a comparison of our relativistic and nonrelativistic results. The relativistic corrections do seem to improve agreement between theory and experiment.

1.3 The two-photon reaction

Direct two-photon scattering, the scattering of light by light, is in one sense a fundamentally quantum field theoretic process, since it is not possible in classical electrodynamics, and happens only as a result of a transition of the photons into other particles, such as e^+e^-. The two photon-reaction we will study, shown in **Fig 1**, involves the emission of virtual photons by both the electron and the positron in an e^+e^- machine. These photons have a bremsstrahlung-like distribution, so that most of the scattered electrons and positrons continue down the beam pipe with very small scattering angle θ, and the invariant four-momentum

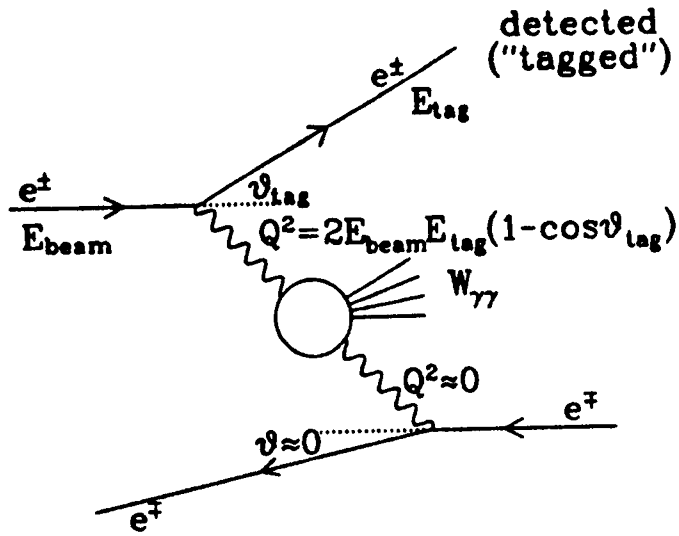

Fig 1. The two-photon process[9].

Q^2 transferred by the photons peaks near zero. Thus the zero-tag mode (neither scattered fermion detected) implicitly selects nearly real ($Q^2 = 0$) photons. In practice one usually assumes that these photons actually are real in their couplings to the hadrons produced. Since a two-photon state has $C = (+)$, it can only couple to $q\bar{q}$ states of positive C-parity; this e^+e^- collision technique was originally suggested by Low[10] for e^-e^- machines as a way to measure the lifetime of the π^0 meson by extracting the $\gamma\gamma-\pi^0$ coupling and thus determining the rate for $\pi^0 \to \gamma\gamma$. Two-photon physics is now one of the most important techniques for extracting information about meson spectroscopy in general and "*extraordinary*" hadrons such as hybrid mesons ($q\bar{q}g$) and $(q\bar{q})(q\bar{q})$ in particular. This technique is especially useful because photons couple to the internal charges of mesons, and thus provide a probe of their internal structure. The light 0^{-+} and 2^{++} states π^0, η, η', $f_2(1270)$, $a_2(1320)$ and $f_2'(1525)$ and the $\eta_c(2980)$ and $\chi_2(3555)$ in particular have all been clearly observed at e^+e^- machines, with $\gamma\gamma$ partial widths consistent with quark model predictions[9–19]. As an illustration of the use of $\gamma\gamma$ couplings to test quark model assignments, consider the $J^{PC} = 2^{++}$ isospin partners $f_2(1270)$ and $a_2(1320)$; these are expected in the quark model to have $\gamma\gamma$ widths related by their flavor wavefunctions and masses as

$$\frac{\Gamma_{\gamma\gamma}(f_2(1270))}{\Gamma_{\gamma\gamma}(a_2(1320))} = \frac{25}{9} \cdot \left(\frac{M_{f_2}}{M_{a_2}}\right)^3 \approx 2.5\,, \tag{1.16}$$

where the factor 25/9 is the ratio of the flavor factor of the f_2 to that of the a_2 (as will be explained in section **1.6** for a general $q\bar{q}$ state), and the mass ratio factor will be explained in section **3.2**. The experimental partial widths are[19]

$$\Gamma_{\gamma\gamma}(f_2(1270)) = 2.8 \pm 0.4\ \mathrm{keV} \tag{1.17}$$

and

$$\Gamma_{\gamma\gamma}(a_2(1320)) = 1.04 \pm 0.09\ \mathrm{keV}\,, \tag{1.18}$$

which gives

$$\frac{\Gamma_{\gamma\gamma}(f_2(1270))}{\Gamma_{\gamma\gamma}(a_2(1320))} = 2.69 \pm 0.45\,, \tag{1.19}$$

consistent with (1.16).

Although spin-1 states cannot be produced from two on-shell photons (according to the Landau-Yang theorem[20]), the 1^{++} $q\bar{q}$ state $f_1(1285)$ has also been observed, with a $\gamma\gamma$ coupling consistent with quark model estimates for $Q^2 \neq 0$ photons[15,19,21–23]. Conversely, mesons which are thought to be non-$q\bar{q}$ states generally have $\gamma\gamma$ widths far from expectations for a $q\bar{q}$ state. These anomalous states include the $f_0(975)$ and $a_0(980)$ $K\bar{K}$-molecule candidates[3,24], the $\eta(1440)$ and the $f_2(1720)$ (which should now be referred to as the $f_0(1710)$ in light of a new analysis[25]). As an illustration of J^P dependence, quark model calculations show that the P-wave 0^{++} and 2^{++} partial widths should be related by

$$\frac{\Gamma_{\gamma\gamma}(0^{++})}{\Gamma_{\gamma\gamma}(2^{++})} = \left(\frac{M_{0^{++}}}{M_{2^{++}}}\right)^3 \cdot \begin{cases} \frac{15}{4}, & \text{nonrelativistically;} \\ \approx 2 & \text{at order } (v/c)^{2}\,[26]. \end{cases} \tag{1.20}$$

Since the $J^{PC} = 2^{++}$ $f_2(1270)$ has an accurately determined $\gamma\gamma$ width of given in (1.17), one expects to find a 0^{++} state with a two-photon width related to this one by (1.20). If one assumes this to be the $f_0(975)$, (1.20) gives $\Gamma_{\gamma\gamma}(f_0(975)) \approx 2.5 - 4.5$ keV. Experimentally $\Gamma_{\gamma\gamma}(f_0(975)) \approx 0.3 - 0.4$ keV[27], which the 1992 PDG has just revised upward to ≈ 0.6 keV (due to the incorporation of a reanalysis of the data by Morgan and Pennington). In part as a result of this discrepancy this resonance is now widely believed to be a $K\bar{K}$ molecule; Barnes calculated the $\gamma\gamma$ width given this assignment in 1985 and found $\Gamma_{\gamma\gamma}(K\bar{K}) \approx$

0.6 keV[8]. Very recently, an $f_0(1250)$ signal has been identified in $\gamma\gamma \to \pi\pi$ with a reported width of $\Gamma_{\gamma\gamma} = 4.3 \pm 0.8 \pm 0.6$ keV[28] and 8 ± 1.67 keV[29]. This is a much more plausible 0^{++} $q\bar{q}$ state, because the quark model expectation for a 0^{++} $q\bar{q}$ partner of the $f_2(1270)$ with a mass of 1250 MeV is, from (1.20),

$$\Gamma_{\gamma\gamma}(f_0(1250)) \approx \begin{cases} 10\ \text{keV}, & \text{nonrelativistically;} \\ 5\ \text{keV}, & \text{at order } (v/c)^2. \end{cases} \tag{1.21}$$

Mesons which have $\gamma\gamma$ widths far from $q\bar{q}$ quark model predictions should evidently be considered candidate non-$q\bar{q}$ states. For this reason it is clearly important to have accurate quark model predictions of $\gamma\gamma$ widths for all experimentally accessible $q\bar{q}$ mesons, in order to identify the non-exotic $q\bar{q}$ resonances.

We should caution the reader that the absolute scale of $\gamma\gamma$ widths of a light $q\bar{q}$ multiplet is a rather sensitive quantity. One can easily be misled into believing that $\gamma\gamma$ width calculations are relatively "stable" by models that reproduce well-established 0^{-+} and 2^{++} widths with accuracies of typically $\sim 30\%$. This however represents the quality of fit possible given good experimental data, and the overall scale is actually quite sensitive to relativistic effects, the quark mass assumed, and the prescription used to incorporate the physical resonance mass. Without experimental data to constrain the overall width scale, order-of-magnitude uncertainties are typically encountered in $\gamma\gamma$ width predictions for $l > 0$ light $q\bar{q}$ states. (See for example the relativistic $q\bar{q}$ calculation of Bergström, Hulth and Snellman[30], which predates accurate data on 2^{++} decays.) For recently observed states such as the light 2^{-+} $q\bar{q}$ system (see section **1.6**), the overall theoretical $\gamma\gamma$ width should only be considered an order-of-magnitude estimate, whereas relative rates such as $I = 0/I = 1$ are of course much more reliably predicted. This uncertainty of overall scale generally increases with increasing L (the orbital angular momentum) and decreases with increasing quark mass; the accuracy of $\gamma\gamma$ partial width predictions will presumably improve as data on more well-established $q\bar{q}$ states becomes available.

1.4 $\Gamma_{\gamma\gamma}$ of quarkonium and positronium

The general calculation of the partial width of $q\bar{q} \to \gamma\gamma$ can be separated into two parts. Since quarks are spin-$\frac{1}{2}$ particles, the total spin of a $q\bar{q}$ bound state is the combination $s = s_1 \otimes s_2$, which produces $s = 0$ and 1 (spin-singlet and spin-triplet respectively). The eigenstates of the spin operators s^2 and s_z in terms of the states $|s_{1z}, s_{\bar{2}z}\rangle$, where the bar over the second entry denotes an antiquark, are as follows: the spin-singlet is

$$\frac{1}{\sqrt{2}}\{|\uparrow\bar{\downarrow}\rangle - |\downarrow\bar{\uparrow}\rangle\} \qquad \Rightarrow \qquad s, s_z = 0, 0 \tag{1.22}$$

and the spin-triplet states are

$$|\uparrow\bar{\uparrow}\rangle \qquad \Rightarrow \qquad s, s_z = 1, +1 \tag{1.23}$$

$$\frac{1}{\sqrt{2}}\{|\uparrow\bar{\downarrow}\rangle + |\downarrow\bar{\uparrow}\rangle\} \qquad \Rightarrow \qquad s, s_z = 1, 0 \tag{1.24}$$

$$|\downarrow\bar{\downarrow}\rangle \qquad \Rightarrow \qquad s, s_z = 1, -1\ , \tag{1.25}$$

The total angular momentum J is composed from s and the orbital angular momentum of the system L as $J = L \otimes s$. Note that for the spin-singlet case $J = L$. However for the spin-triplet there are generally three possible values, $J = L+1, L$ and $L-1$ (except for $L = 0$, for which the only possibility is $J = s = 1$). Since the $\Gamma_{\gamma\gamma}$ of such a bound state depends on the total wavefunction, and on the spin part in particular, our problem is naturally divided into two parts, the spin-singlet and the spin-triplet cases, which have antisymmetric and symmetric spin wavefunctions respectively. The latter involves all three magnetic substates, and the difference between the three possible J values for a given L (for a given spatial wavefunction determined by L) is in the choice of Clebsch-Gordan coefficients. The spatial wavefunctions will enter as a weight function in momentum space which multiplies the free-particle amplitude for $q\bar{q} \rightarrow \gamma\gamma$. We approximate this amplitude by the tree-level Feynman diagrams of **Fig 2**. Here we do not incorporate gluon exchanges in the annihilation amplitude (which is a standard approximation in the QM). The tree-level amplitude is a QED process calculated to order $\alpha \approx \frac{1}{137}$. To this free amplitude in momentum space one attaches a wavefunction for the bound $q\bar{q}$ system, and integrates over all quark momenta $\vec{p}$. This provides an amplitude for the process $q\bar{q}$ (bound

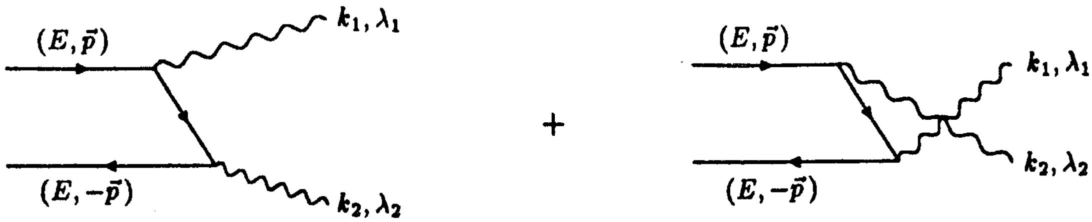

Fig 2. Annihilation diagrams incorporated in our calculation.

state)$\rightarrow \gamma\gamma$. Using this amplitude one proceeds to calculate the physical decay rate $\Gamma_{\gamma\gamma}$.

The $\gamma\gamma$ angular partial width of a $q\bar{q}$ bound state was found by Li, Close, and Barnes to be[26]

$$\frac{d\Gamma}{d\Omega_{\vec{k}}} = \frac{1}{32\pi^2} \sum_{\lambda_1,\lambda_2} \Big|\mathcal{M}(\lambda_1,\lambda_2)\Big|^2 , \tag{1.26}$$

so the integrated partial width is

$$\Gamma_{\gamma\gamma} = \frac{1}{2!} \int d\Omega_{\vec{k}} \, \frac{d\Gamma}{d\Omega_{\vec{k}}} , \tag{1.27}$$

where the sum in (1.26) is over all possible photon polarizations, and $\frac{1}{2!}$ is the standard statistical factor for two identical final photons. The bound state amplitude $\mathcal{M}(\lambda_1,\lambda_2)$ is related to the invariant plane wave amplitude $\mathcal{M}_{fi}$ and the total wavefunction $\Phi^{tot}_{JM}(\vec{p})$ by

$$\mathcal{M}(\lambda_1,\lambda_2) = \frac{1}{\sqrt{(2\pi)^3}} \int d\vec{p} \sqrt{\frac{m^2}{E_1 E_2}}\ \Phi^{tot}_{JM}(\vec{p}) \otimes \mathcal{M}_{fi}(s_{1z}, s_{2z}) , \qquad (1.28)$$

where the dependence on all parts of the wavefunction other than orbital (spin, color and flavor) are included in $\mathcal{M}_{fi}$. As we are dealing with real (massless) photons there are two possible polarizations for each, and hence four for a two photon final state $(++, +-, -+, --)$. Rotational invariance implies however that there are only two independent amplitudes which contribute to these four final states, as will be discussed in more detail in section (1.5).

Since this is a hadronic decay process, there are of course several modifications required to our previous calculation of a positronium decay to $\gamma\gamma$. First the masses and the charges of the constituents are different. Second, the quarks have a color degree of freedom. The dependence of (1.28) on these parameters is trivial, and thus one may relate e^+e^- and $q\bar{q}$ decay rates easily. There are however two less trivial differences: *a)* The meson constituents are relativistic (at least for light quarks), whereas those of positronium are nonrelativistic, and *b)* the radial wavefunctions describing these two different systems are quite different (one arises from a pure Coulomb potential and the other from a Coulomb-plus-linear potential). Given equations (1.26)-(1.28), one can maintain a close relation between the two calculations by deriving analytical results that depend on an unspecified radial spatial wavefunction, and by working to all orders in (v/c), so that our results do not assume nonrelativistic dynamics. This latter approach has only been followed previously in the $\Gamma_{\gamma\gamma}$ calculations of the $J^{PC} = 0^{-+}$ state by Bergström, Snellman and Tengstrand[31] in 1977 and by Hayne and Isgur[32] in 1982, and for 0^{++} and 2^{++} by Bergström, Hulth and Snellman in 1983[30]. (This latter study differs from ours somewhat since it is based on the Bethe-Salpeter equation with various simplifying approximations.)

To proceed from the positronium $\gamma\gamma$ width to that of quarkonium, note first that the amplitude (1.28) for positronium decay depends explicitly on m_e, the J^{PC} quantum numbers of the positronium state, and is proportional to e^2 (e being the electron charge). To transform this amplitude to that for the decay of a $q\bar{q}$ state with a given J^{PC}, one first requires the flavor wavefunction. For example if we are considering the $J^{PC} = 0^{-+}$ π^0 decay, the e^+e^- state is replaced by $(u\bar{u} - d\bar{d})/\sqrt{2}$. Since e_u and e_d, the charges of the u and d quarks, are different from e, the following transformation of charges is required in the amplitude;

$$e^2 \quad \rightarrow \quad \frac{e_u^2 - e_d^2}{\sqrt{2}} = \frac{e^2}{3\sqrt{2}} \,. \tag{1.29}$$

Quarks also have three color states, which enter the calculation in the color wavefunction. The mesonic color wavefunction in (1.4) implies a factor of $\frac{1}{\sqrt{3}}$ in the amplitude for each quark color, and hence $\sqrt{3}$ for the total quarkonium decay amplitude. Combining these results, one obtains color and flavor factors for the π^0 decay in the nonrelativistic quark model from that of $J^{PC} = 0^{-+}$ positronium as follows;

$$\Gamma(\pi^0 \to \gamma\gamma) = 3 \cdot \frac{1}{18} \cdot \Gamma(e^+e^- \to \gamma\gamma)\Big|_{m_e \to m_q} , \tag{1.30}$$

where m_q is the constituent quark mass. More generally, for a state with given J^{PC} one has color and flavor factors as follows;

$$\Gamma_{\gamma\gamma}(q\bar{q}) \;=\; 3 \;\cdot\; \left|\langle (e_q/e)^2 \rangle\right|^2 \;\cdot\; \Gamma_{\gamma\gamma}(e^+e^-)\Big|_{m_e \to m_q} . \tag{1.31}$$

(Note that equations (1.30) and (1.31) are not complete; a phenomenologically motivated treatment of the dependence of the $\gamma\gamma$ partial width on the mass of the decaying resonance to correct the phase space will lead to another multiplicative factor in (1.30) and (1.31). This will be discussed in more detail in chapter 3.)

1.5 Restrictions on quarkonium quantum numbers

A state of N photons has charge conjugation (or C-parity) $C = (-1)^N$; therefore $|\gamma\gamma\rangle$ has $C = +1$. Since QED is invariant under charge conjugation, the initial quarkonium (or positronium) state must have $C = +1$. One can show that for $q\bar{q}$ states

$$C = (-1)^{L+s} , \tag{1.32}$$

and therefore the first restriction we have on the initial state decaying to $\gamma\gamma$ is

$$L + s = even \ . \tag{1.33}$$

Also one has for the parity P of such a $q\bar{q}$ system

$$P = (-1)^{L+1} \ , \tag{1.34}$$

so that for spin-singlet and spin-triplet states (1.33) and (1.34) which can decay to $\gamma\gamma$ we must have

$$s = 0 \Rightarrow L = even \Rightarrow J^{PC} = even^{-+} \ . \tag{1.35}$$

$$s = 1 \Rightarrow \ L = odd \Rightarrow J^{PC} = \begin{cases} (L+1)^{++} = even^{++} \\ \quad L^{++} \quad = \quad odd^{++} \\ (L-1)^{++} = even^{++} \end{cases} . \tag{1.36}$$

Only quarkonium states with the quantum numbers (1.35) and (1.36) can decay to two-photon final states. (The one exception, $J^{PC} = 1^{++}$, is forbidden to couple to two real photons by the Landau-Yang theorem[20].) These quantum number restrictions will serve as a check on our analytical results for $\Gamma_{\gamma\gamma}$. Positronium and quarkonium $\gamma\gamma$ widths have only previously been published for $J^{PC} = 0^{-+}$, 0^{++}, 2^{++} and 2^{-+} states; we will discuss the history of these calculations in the following section.

As photons are vector particles ($S = 1$), each would *a priori* have three possible polarizations (or spin orientations). However, for real photons, the fact that they have zero mass leads to an additional constraint on the photon field due to gauge invariance; this eliminates one of these polarizations, the one along the photon's momentum $\vec{k}$, leaving only two transverse components. Here instead of spin we will use helicity labels, which are simpler to use with massless particles. These are the projections of spin along the particles direction of motion. The magnetic helicity substates are labelled by the eigenvalue $\vec{s} \cdot \hat{k}$, which is called λ. For a spin-one photon, $\lambda = \pm 1$, abbreviated by $(+,-)$, indicates that it is polarized along or opposite to the direction of motion. (For a systematic derivation of the helicity formalism and illustrations, see the article by Jacob and Wick[33].)

For a two-particle state in the C.M. frame, the total helicity can be defined in terms of the two individual particle helicities as

$$\lambda \equiv \lambda_1 - \lambda_2 \,. \tag{1.37}$$

For a two-photon final state $|\gamma_1\gamma_2\rangle$, $\lambda = 0$ is the case in which the photons are polarized in opposite directions; for two-photon states $\lambda = \pm 2$ otherwise. We refer to these as "helicity zero" and "helicity two" $\gamma\gamma$ states. (Note that we refer to both $\lambda = \pm 2$ as "helicity two", since one state turns into the other under interchange of the two photons; as we shall see in chapter 3, their contributions to $\Gamma_{\gamma\gamma}$ are equal.) The helicity states for $|\gamma\gamma\rangle$, when required to be eigenstates of the parity operator $\mathcal{P}$, may be written in terms of the individual photon helicity basis $\{|\lambda_1, \lambda_2\rangle\}$ as

$$\lambda = 0,\ J^{-+} \quad \Rightarrow \quad \frac{1}{\sqrt{2}}\Big\{|++\rangle \ - \ |--\rangle\Big\}, \tag{1.38}$$

$$\lambda = 0,\ J^{++} \quad \Rightarrow \quad \frac{1}{\sqrt{2}}\Big\{|++\rangle \ + \ |--\rangle\Big\}, \tag{1.39}$$

$$\lambda = +2,\ J^{++} \quad \Rightarrow \quad |+-\rangle, \tag{1.40}$$

and

$$\lambda = -2,\ J^{++} \quad \Rightarrow \quad |-+\rangle. \tag{1.41}$$

Note that only the state (1.38) has negative parity. These states with definite J^{PC} and $J_z = M$ have the following angular wavefunctions[33];

$\lambda = 0 \Rightarrow$

$$|\gamma\gamma; J^{-+}M\rangle = \frac{1}{\sqrt{2}} \int d\Omega\, Y_{JM}(\Omega)\Big\{|k,\Omega;++\rangle \ - \ |k,\Omega;--\rangle\Big\}, \tag{1.42}$$

$\lambda = 0 \Rightarrow$

$$|\gamma\gamma; J^{++}M\rangle = \frac{1}{\sqrt{2}} \int d\Omega\, Y_{JM}(\Omega)\Big\{|k,\Omega;++\rangle \ + \ |k,\Omega;--\rangle\Big\}, \tag{1.43}$$

$\lambda = 2 \Rightarrow$

$$|\gamma\gamma; J^{++}M\rangle = \sqrt{\frac{2J+1}{8\pi}} \int d\Omega\, \mathcal{D}^{J*}_{M,2}(\phi,\theta,-\phi)|k,\Omega;+-\rangle, \tag{1.44}$$

and

$\lambda = -2 \Rightarrow$

$$|\gamma\gamma;J^{++}M\rangle = (-1)^J \sqrt{\frac{2J+1}{8\pi}} \int d\Omega\, \mathcal{D}^{J*}_{M,2}(\phi,\theta,-\phi)|k,\Omega;-+\rangle\ , \qquad (1.45)$$

(where the function $\mathcal{D}^{j}_{m,m'}$ is defined and used in section **2.4**).

As we shall see, considerable simplification is possible if we utilize these angular distributions in the calculations performed in chapter 2.

1.6 History of $\Gamma_{\gamma\gamma}$ calculations

Historically, $\Gamma_{\gamma\gamma}$ was first derived for S-wave and then P-wave positronium. In 1944, the first calculation for ${}^1S_0$ parapositronium was reported by Pirenne[34] in his dissertation; he showed that for 0^{-+} positronium to lowest order in α

$$\Gamma_{\gamma\gamma}(n^1S_0) = \frac{\alpha^2}{m_e^2}\left|\psi_n(0)\right|^2 = \frac{1}{2n^3}\, m_e\, \alpha^5\ . \qquad (1.46)$$

Later this result was confirmed by Wheeler[35], who showed in 1946 that to the same order in α

$$\Gamma_{\gamma\gamma} = \sigma \cdot v \cdot \rho\ , \qquad (1.47)$$

where σ is the threshold cross section of the process of **Fig 2**, v is the relative speed of the electron and the positron, and ρ is the density of electrons at the position of the positron. He demonstrated that the nonrelativistic limit of $\sigma(e^+e^- \rightarrow \gamma\gamma)$ derived by Dirac in 1930, together with (1.47), gives the result (1.46). Alekseev rederived (1.46) and gave results for the $\Gamma_{\gamma\gamma}$ of P-wave positronium in 1957[36] (which agree with P-wave results derived in 1953 by Tumanov using configuration space QED). These results are

$$\Gamma_{\gamma\gamma}(0^{++}) = \frac{n^2-1}{8n^5}\, m_e\, \alpha^7\ , \tag{1.48}$$

and

$$\Gamma_{\gamma\gamma}(2^{++}) = \frac{n^2-1}{30n^5}\, m_e\, \alpha^7\ , \tag{1.49}$$

and a vanishing $\Gamma_{\gamma\gamma}$ for the 1^{++} case, in agreement with the Landau-Yang theorem. In these results one sees the well known $\Gamma_{\gamma\gamma}$ ratios attributed to P-wave positronium and quarkonium states in the nonrelativistic quark model,

$$\Gamma_{\gamma\gamma}(0^{++}) : \Gamma_{\gamma\gamma}(1^{++}) : \Gamma_{\gamma\gamma}(2^{++}) = \frac{15}{4} : 0 : 1\ . \tag{1.50}$$

In 1979 Bergström, Snellman and Tengstrand showed that the full $\mathcal{O}(\alpha)$ Feynman amplitude leads to the result[31]

$$\Gamma_{\gamma\gamma}(q\bar{q}(0^{-+})) = 3 \cdot \left|\left\langle \frac{e_q^2}{e^2} \right\rangle\right|^2 \cdot \frac{2\alpha^2}{\pi {m_q}^2}$$
$$\times \left| \int_0^\infty dp\ p^2\ \phi(p)\ \left(\frac{m_q^2}{pE}\right)\ \ln\!\left(\frac{E+p}{m_q}\right) \right|^2 , \tag{1.51}$$

which was later derived independently by Hayne and Isgur in 1982[32]. (They also multiply the overall rate by an $(M_R/M_{ref})^3$ factor to impose an expected phase-space dependence on the mass of the $q\bar{q}$ resonance, M_R.)

In 1983 Bergström, Hulth and Snellman derived results for the $\Gamma_{\gamma\gamma}$ of P-wave quarkonium using the full $\mathcal{O}(\alpha)$ Feynman amplitude[30]. Their approach differs from that which lead to (1.51), and which we will adopt, in that it is based on the Bethe-Salpeter equation with some simplifying approximations. These results are

$$\Gamma_{\gamma\gamma}(q\bar{q}(0^{++})) = 3 \cdot \left|\left\langle\frac{e_q^2}{e^2}\right\rangle\right|^2 \cdot \frac{4\alpha^2}{{m_q}^2 M_0^2}\left|\int \frac{d^3p}{(2\pi)^3} p\ \phi_P(p)\left(\frac{m_q}{E}\right)^2 \right.$$

$$\left. \times\left\{1 - \frac{EM_0 - 2{m_q}^2}{2p^2}\left(1 - \frac{E}{p}\ \ln\left(\frac{E+p}{m_q}\right)\right)\right\}\right|^2 \tag{1.52}$$

and

$$\Gamma_{\gamma\gamma}(q\bar{q}(0^{++})) = 3 \cdot \left|\left\langle\frac{e_q^2}{e^2}\right\rangle\right|^2 \cdot \frac{16\alpha^2}{15{m_q}^2 M_2^2}$$

$$\times\left\{\underbrace{\left|\int \frac{d^3p}{(2\pi)^3} p\ \phi_P(p)\left(\frac{m_q}{E}\right)\left\{1 + \frac{3}{4}I(p)\right\}\right|^2}_{\lambda=0}\right.$$

$$\left. + \frac{1}{24}\underbrace{\left|\int \frac{d^3p}{(2\pi)^3} p\ \phi_P(p)\left(\frac{m_q}{E}\right)\frac{3}{4}\left\{I_1(p) + \frac{M_2}{E} I_2(p)\right\}\right|^2}_{\lambda=2}\right\}, \tag{1.53}$$

where M_0 and M_2 are the masses of the 0^{++} and 2^{++} states respectively, $\Phi_P(p)$ is the P-wave radial wavefunction and $I(p)$, $I_1(p)$ and $I_2(p)$ are the following functions;

$$I(p) = -1 + \frac{{m_q}^2 + 5m_q E}{3p^2} + \left(1 - \frac{m_q}{E}\right)\frac{E^4}{p^4}$$

$$+ \left\{\frac{1}{2}\left(1 + \frac{m_q}{E}\right) - \frac{m_q E}{p^2} - \frac{1}{2}\left(1 - \frac{m_q}{E}\right)\frac{E^4}{p^4}\right\}\frac{E}{p}\ln\left(\frac{E+p}{E-p}\right), \tag{1.54}$$

$$I_1(p) = \frac{12m_q - 8E}{E} \cdot \frac{E^2}{p^2} + \frac{12(E - m_q)}{E} \cdot \frac{E^4}{p^4}$$

$$+ \left(\frac{2m_q}{p} + \frac{6E - 8m_q}{E} \cdot \frac{E^3}{p^3} + \frac{6(m_q - E)}{E} \cdot \frac{E^5}{p^5} \right) \ln\left(\frac{E+p}{E-p} \right) \tag{1.55}$$

and

$$I_2(p) = \frac{4E^2}{p^2} + \frac{6(m_q - E)}{E} \cdot \frac{E^4}{p^4}$$

$$+ \left\{ \frac{m_q - 3E}{E} \cdot \frac{E^3}{p^3} + \frac{3(E - m_q)}{E} \cdot \frac{E^5}{p^5} \right\} \ln\left(\frac{E+p}{E-p} \right) . \tag{1.56}$$

They reported that the $\lambda = 0$ contribution to $\Gamma_{\gamma\gamma}(2^{++})$ is negligible, and that the nonrelativistic limit overestimates the $\gamma\gamma$ width considerably, especially for light quarks.

Li, Close and Barnes[26] recently (in 1991) showed that at order $(v/c)^2$ the ratio (1.50) is modified considerably by relativistic corrections (from $\frac{15}{4}$ to ≈ 2). Their results indicate however that the helicity-zero component of the 2^{++} P-wave decay amplitude, which is zero in the nonrelativistic limit, does not receive large relativistic corrections at this order.

Anderson, Austern and Cahn were the first to calculate $\Gamma_{\gamma\gamma}$ for an $L > 1$ state; they recently (in 1991) derived a nonrelativistic result for the $L = 2$ 2^{-+} state[37],

$$\Gamma_{NR}(q\bar{q}(2^{-+}) \to \gamma\gamma) = 192\alpha^2 \left| \left\langle \frac{e_q^2}{e^2} \right\rangle \right|^2 \frac{|\psi''(0)|^2}{M_R^6} . \tag{1.57}$$

Here M_R is actually twice the fermion mass; they choose to identify this with the mass of the quarkonium resonance in their numerical evaluation. Their calculation was motivated by the recent (1990) measurement of the two-photon width of the $I=1$, $J^{PC}=2^{-+}$ state $\pi_2(1670)$ by the CELLO[38] and Crystal Ball[39] collaborations who found

$$\Gamma(\pi_2(1670) \to \gamma\gamma) = 1.30 \pm 0.3 \pm 0.2 \ \text{keV} \tag{1.58}$$

and

$$\Gamma(\pi_2(1670) \to \gamma\gamma) = 1.41 \pm 0.23 \pm 0.28 \ \text{keV} \tag{1.59}$$

respectively, for a combined value[19]

$$\Gamma(\pi_2(1670) \to \gamma\gamma) = 1.35 \pm 0.26 \ \text{keV}\ . \tag{1.60}$$

A partial width of ≈ 1 keV for an $I=1$ pseudotensor state in this mass region appears plausible *a priori* for a $q\bar{q}$ state, so both experimental groups cited their measurements as evidence that the $\pi_2(1670)$ is a conventional ${}^1D_2$ $(u\bar{u} - d\bar{d})/\sqrt{2}$ quarkonium state. (Compare the Particle Data Group[19] average value for the well-established $I=1$ tensor $a_2(1320)$ in (1.18)). For their parameters, (1.57) led Anderson, Austern and Cahn to the surprisingly small width estimate of

$$\Gamma_{NR}(\pi_2(q\bar{q}) \to \gamma\gamma) \ = \ 1.2 - 9.4 \ \text{eV}, \tag{1.61}$$

which is two to three orders of magnitude smaller than the experimental results for the $\pi_2(1670)$. This led them to a discussion of possible non-$q\bar{q}$ assignments for the $\pi_2(1670)$, and to suggest that the data is insufficient to rule out a $J^{PC} = 0^{-+}$ assignment. Either they are correct, and the experimental results are inconsistent with such a $q\bar{q}$ and J^{PC} assignment, or their estimate (1.61) is too small and the theoretical width requires more careful investigation. As we shall see in chapter 3, we consider it highly unlikely that the $\pi_2(1670)$ is not a $q\bar{q}$ state, so the assumptions which led to the results (1.57) and (1.61) are presumably inaccurate. Although there are expectations of $q\bar{q}g$ hybrid mesons with these quantum numbers[17,40,41] and others[42] in this general mass region, the proximity of the 3^{--} $\rho_3(1690)$ and 1^{--} $\rho(1700)$ strongly supports the identification of the $\pi_2(1670)$ with the ${}^1D_2$ member of an $l = 2$ $q\bar{q}$ multiplet. Its detection in $\gamma\gamma$ is a very useful development, as we must have an accurate understanding of the couplings of higher-mass conventional $q\bar{q}$ states if we are to distinguish them from hybrid mesons or other exotics.

Experimentally, $\gamma\gamma$ widths of $q\bar{q}$ mesons below 2 GeV can be resolved at present at the ≈ 0.2 keV level. Comparing the results (1.18) and (1.60) suggests that there may be little suppression of these widths with orbital excitation, and therefore higher L states may be observable. As we shall see in chapter 3, our relativistic results do support the possibility of little suppression of $\Gamma_{\gamma\gamma}$ for light

$q\bar{q}$ states. At present higher-L $q\bar{q}$ resonances are not easily accessible in $\gamma\gamma$ collisions, because the masses of such states are somewhat large (> 1.5 MeV), these resonances may decay to high-multiplicity final states, and the effective intensity of the two-photon source decreases rapidly with the invariant mass of the two photons[43]. Fortunately, a new experimental approach (if implemented) is expected to solve the luminosity problem. This is the Photon Linear Collider (PLC), otherwise known as a "Compton Collider". By Compton-backscattering laser photons off Linac electrons, high energy photon beams are produced, which are then brought into collision with electron beams or with another photon beam. In the latter case, which is of interest here, one expects to have enough luminosity at higher invariant mass to allow the study of heavy quarkonium states such as bottonium ($b\bar{b}$) and toponium ($t\bar{t}$). Although such a machine would be of greatest interest in searches for the higgs particle, considerable progress in QCD spectroscopy would be expected to follow from hadron production experiments. (For more details of a PLC see the review article by Border, Bauer and Caldwell[44].)

The fact that Anderson, Austern and Cahn underestimated the overall scale of the $\gamma\gamma$ width for the $\pi_2(1670)$ in their nonrelativistic calculation leads one to wonder if nonrelativistic calculations for S and P-waves are generally unreliable for higher L states. (To avoid misleading the reader, we emphasize that the nonrelativistic formulas actually overestimate the overall scale; this conclusion has been reported elsewhere[30]. We will discuss in chapter 3 why Anderson, Austern and Cahn found the opposite result.)

The arguments above, together with the numerical success of the relativistic calculation of Hayne and Isgur for the π^0, lead us to two observations:

1. Theoretical estimates of $\Gamma_{\gamma\gamma}$ for higher-L $q\bar{q}$ states are needed, together with helicity selection rules and ratios, to aid experimentalists in the search for excited $q\bar{q}$ states.

2. These calculations should include higher-order corrections in (v/c), as a measure of the importance of relativistic effects on theoretical results such as the helicity structure and $\Gamma_{\gamma\gamma}$ ratios within multiplets. If such effects are indeed found to be important and provide better agreement with data, this would support the speculation of LeYaouanc *et al* quoted in section **1.2**, and would establish the importance of relativistic corrections in at least one application of the quark model.

Regarding the second suggestion, we will anticipate our results and note that the ratio 15/4, which becomes ≈ 2 at order $(v/c)^2$, does not experience additional large corrections in our relativistic calculations (it becomes ≈ 1.8). The absolute scale of $\Gamma_{\gamma\gamma}$ does however have large corrections relative to the nonrelativistic calculations. As one example, the nonrelativistic limit of the

$\pi_2(1670)$ $\Gamma_{\gamma\gamma}$ is an underestimate by about a factor of 64. The nonleading helicity zero amplitude of the 2^{++} P-wave state however receives only relatively small contributions of $\approx 5\%$ and $\approx 2\%$ to $\Gamma_{\gamma\gamma}$ when corrected to leading order $\mathcal{O}((v/c)^2)$, and to all orders of (v/c) respectively. (Actually the $(v/c)^2$ calculation uses radial wavefunctions arising from simple harmonic oscillator potential, and although $\mathcal{M}_{fi}$ in (1.28) describes a free QED process, which means that in the C.M. frame the energies of the fermions and photons would be equal ($\equiv E$), Li *et al* chose to regard the factor of E appearing in $\mathcal{M}_{fi}$ and coming from the external fermion lines as the energy of the fermion ($E = \sqrt{\vec{p}^2 + m^2}$), and to equate the other E, coming from the photon four-momentum, to half the mass of the resonance. This assumption at higher order in (v/c) leads to problems such as C-parity violation[45].) This indicates that helicity selection rules derived in the nonrelativistic quark potential model may be reasonably accurate. These would be useful because the $\lambda = 0 / \lambda = 2$ ratios are easily derived in the nonrelativistic quark potential model, whereas the relativistic calculations involve detailed knowledge of the wavefunctions, and require numerical integration weighted by complicated functions of quark momentum. However, for relative values of $\Gamma_{\gamma\gamma}$ within a multiplet and especially the absolute scales of such widths, one obtains much better results using the relativistic formulas.

CHAPTER 2

DERIVATION OF $\Gamma_{\gamma\gamma}$ FOR GENERAL J^{PC}

2.1 Setting up the calculation

In this chapter we derive analytical relativistic expressions for the $\Gamma_{\gamma\gamma}$ of quarkonia with general angular momentum J. These expressions will depend explicitly on the radial part of the orbital wavefunction $\phi(p)$, which is obviously model dependent. Here we will use the nonrelativistic Schrödinger equation with a Coulomb-plus-linear potential to generate the $\{\phi(p)\}$ for numerical evaluation of $\Gamma_{\gamma\gamma}$, although other wave equations could also have been used.

To proceed, we first note that (1.31) provides a relation between $\Gamma_{\gamma\gamma}$ of positronia and quarkonia which allows a transformation between them. As the positronium calculations are more straightforward, we will consider them first, and then make use of (1.31) to derive formulas for the corresponding quarkonium states.

For positronium we have no color or flavor dependence in the $\mathcal{M}_{fi}$ of **Fig 2**; the only dependence on the spin and spatial wavefunction of the positronium state. The amplitude in (1.28) may be written as

$$\mathcal{M}(\lambda_1,\lambda_2) = \frac{1}{\sqrt{(2\pi)^3}} \int d\vec{p}\, \sqrt{\frac{m^2}{E_1 E_2}}\; \Phi^{tot}_{JM}(\vec{p}) \otimes \mathcal{M}_{fi}\,, \tag{2.1}$$

where

$$\Phi^{tot}_{JM}(\vec{p}) \otimes \mathcal{M}_{fi}(s_{1z}, s_{2z}) \equiv$$

$$\sum_{m=-l}^{l} \sum_{s_z=-s}^{s} \langle l, m; s, s_z | J, M\rangle\; \Phi_{lm}(\vec{p})\; \chi_{s,s_z} \otimes \mathcal{M}_{fi}(s_{1z}, s_{2z})\,. \tag{2.2}$$

The Clebsch-Gordan coefficient $\langle l, m; s, s_z | J, M\rangle$ is required to combine (l, m) and (s, s_z) to obtain (J, M), and the expression $\chi_{s,s_z} \otimes \mathcal{M}_{fi}(s_{1z}, s_{2z})$ indicates that the invariant amplitude $\mathcal{M}_{fi}$ is a sum over the individual spin orientations of the electron and positron according to the spin wavefunction χ_{s,s_z}. These are given explicitly in the $|s_{1z}, s_{2z}\rangle$ basis in (1.22)-(1.25). Our normalization condition for the positronium wavefunction is $\int_0^\infty dp\; p^2 |\phi(p)|^2 = 1$. Here we combine l and s to form total $J = l \otimes s$, where J, l and s are the total, orbital and spin angular momenta of the positronium state, and $J_z \equiv M$ and $l_z \equiv m$. (We have changed the label L of chapter 1 to l, to occupy less space in our rather intricate results.) We will now separate this calculation into two independent ones, for each positronium spin case.

2.2 Analysis of the invariant amplitude $\mathcal{M}_{fi}$

The invariant amplitude $\mathcal{M}_{fi}$ is approximated by the Feynman diagrams of **Fig 2**, and is given by

$$\mathcal{M}_{fi} = \mathcal{M}_1 + \mathcal{M}_2, \tag{2.3}$$

where

$$\begin{aligned}\mathcal{M}_1 &= \bar{v}_{\vec{p}_2 s_2}(-ie\gamma_\nu)\epsilon_2^{*\nu}\frac{i}{\not{q}_1 - m}(-ie\gamma_\mu)\epsilon_1^{*\mu}u_{\vec{p}_1 s_1} \\ &= -ie^2\,\bar{v}_{\vec{p}_2 s_2}\,\not{\epsilon}_2^*\frac{1}{\not{q}_1 - m}\,\not{\epsilon}_1^* u_{\vec{p}_1 s_1}\end{aligned} \tag{2.4}$$

and

$$\mathcal{M}_2 = -ie^2\,\bar{v}_{\vec{p}_2 s_2}\,\not{\epsilon}_1^*\frac{1}{\not{q}_2 - m}\,\not{\epsilon}_2^* u_{\vec{p}_1 s_1}\,. \tag{2.5}$$

(For notation and method of calculation see Bjorken and Drell[46].)

The momenta satisfy $q_1 = p_1 - k_1 = k_2 - p_2$ and $q_2 = p_1 - k_2 = k_1 - p_2$. Specializing to the CM frame, we define $\vec{p}_1 = -\vec{p}_2 \equiv \vec{p}$, $\vec{k}_1 = -\vec{k}_2 \equiv \vec{k}$, $E_1 = E_2 \equiv E = \sqrt{\vec{p}^{\,2} + m^2}$ and $k_1^0 = k_2^0 = |\vec{k}_1| = |\vec{k}_2| \equiv \omega$. Note that the form of q_1 and q_2 implies that $p_1 + p_2 = k_1 + k_2$ which in the CM frame implies $E_1 + E_2 = k_1^0 + k_2^0$, so that $E = \omega$. Although this is of course correct for the free process, for the bound state decay it introduces an ambiguity. In the CM frame of the quarkonium state (in which the quarkonium is at rest), each photon produced will have an energy of $\omega = M/2$, where M is the initial resonance mass. In the quark model calculation however the electrons are assumed to be on-shell (free) in the Feynman amplitude, and the binding potential only enters the calculation through the initial wavefunction, in an integral over $\vec{p}$. This means that $\vec{p}$, hence E_p is a variable, but ω should be the constant $M/2$. This ambiguity is a result of the neglect of binding in $\mathcal{M}_{fi}$. Note however that this only affects the relativistic calculation, since in the nonrelativistic limit $E_p = \sqrt{\vec{p}^2 + m^2} \approx m$, and $M = 2m + kinetic\ energy \approx 2m$, so that $E = M/2 = \omega$ nonrelativistically.

Introducing explicit Dirac spinors, for a spin-up electron we have

$$u_{\vec{p}\uparrow} = \eta \begin{bmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_+}{E+m} \end{bmatrix} , \tag{2.6}$$

where p_+ and the overall constant η are defined by

$$\eta \equiv \sqrt{\frac{E+m}{2m}} \qquad \text{and} \qquad p_\pm \equiv p_x \pm i p_y \, . \tag{2.7}$$

Note that the first two entries in $u_{\vec{p}\uparrow}$ can be written as

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv I u_\uparrow \, , \tag{2.8}$$

where I is the 2×2 identity and $u_\uparrow$ is the spin-up rest electron spinor. Using the Pauli spin matrices in the standard representation one has

$$\vec{\sigma} \cdot \vec{p} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_x + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} p_y + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p_z$$

$$= \begin{pmatrix} p_z & p_x - i p_y \\ p_x + i p_y & -p_z \end{pmatrix} , \tag{2.9}$$

so the last two entries in $u_{\vec{p}\uparrow}$ are equivalent to

$$\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_+}{E+m} \end{pmatrix} = \frac{\vec{\sigma}\cdot\vec{p}}{E+m} u_\uparrow \,. \tag{2.10}$$

Using these results we can rewrite $u_{\vec{p}\uparrow}$ as

$$u_{\vec{p}\uparrow} = \eta \begin{bmatrix} I \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{bmatrix} u_\uparrow \,. \tag{2.11}$$

Similarly one has

$$u_{\vec{p}\downarrow} = \eta \begin{bmatrix} 0 \\ 1 \\ \frac{p_-}{E+m} \\ \frac{-p_z}{E+m} \end{bmatrix} = \eta \begin{bmatrix} I \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{bmatrix} u_\downarrow \,, \tag{2.12}$$

and for positrons

$$v_{\vec{p}\uparrow} = \eta \begin{bmatrix} \frac{p_-}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{bmatrix} = \eta \begin{bmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ I \end{bmatrix} v_\uparrow \,, \tag{2.13}$$

and

$$v_{\vec{p}\downarrow} = -\eta \begin{bmatrix} \frac{p_z}{E+m} \\ \frac{p_+}{E+m} \\ 1 \\ 0 \end{bmatrix} = \eta \begin{bmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ I \end{bmatrix} v_\downarrow \,, \tag{2.14}$$

where the rest-frame Pauli spinors are

$$u_1 \equiv u_\uparrow = \begin{bmatrix} +1 \\ 0 \end{bmatrix} \quad , \quad u_2 \equiv u_\downarrow = \begin{bmatrix} 0 \\ +1 \end{bmatrix} \,; \tag{2.15}$$

$$v_1 \equiv v_\uparrow = \begin{bmatrix} 0 \\ +1 \end{bmatrix} \quad , \quad v_2 \equiv v_\downarrow = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \,. \tag{2.16}$$

(The minus sign in v_2 is crucial, and derives from the transformation properties of a conjugate representation in $SU(2)$.) These results can be combined as

$$u_{\vec{p}_1 s_1} = u_{\vec{p} s_1} = \eta \begin{bmatrix} I \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{bmatrix} u_{s_1} \tag{2.17}$$

and

$$v_{\vec{p}_2 s_2} = v_{-\vec{p}s_2} = \eta \begin{bmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ I \end{bmatrix} v_{s_2} \, , \tag{2.18}$$

and therefore

$$\bar{v}_{\vec{p}_2 s_2} = \bar{v}_{-\vec{p}s_2} = v_{-\vec{p}s_2}{}^{\dagger}\gamma_0 =$$

$$\eta v_{s_2}^{\dagger} \left[-\frac{\vec{\sigma}\cdot\vec{p}}{E+m} \quad I \right] \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = -\eta v_{s_2}^{\dagger} \left[\frac{\vec{\sigma}\cdot\vec{p}}{E+m} \quad I \right] . \tag{2.19}$$

Given (2.3)-(2.5), the invariant amplitude can be written as

$$\mathcal{M}_{fi} = ie^2 \bar{v}_{\vec{p}_2 s_2} \hat{O} u_{\vec{p}_1 s_1} \, , \tag{2.20}$$

where $\hat{O}$ is a 4×4 matrix

$$\hat{O} \equiv (-1)\left\{ \not{\epsilon}_2^* \left(\frac{\not{q}_1 + m}{q_1^2 - m^2} \right) \not{\epsilon}_1^* + \not{\epsilon}_1^* \left(\frac{\not{q}_2 + m}{q_2^2 - m^2} \right) \not{\epsilon}_2^* \right\} , \tag{2.21}$$

and we have used

$$\frac{1}{\not{q}_i - m} = \frac{\not{q}_i + m}{q_i^2 - m^2} \, . \tag{2.22}$$

Making use of the identity $\not{a} \not{b} = 2a \cdot b - \not{b} \not{a}$ for any two four-vectors a^μ and b^μ, and $q_i = p_1 - k_i \; \forall \, i \in \{1, 2\}$, we have

$$(\not{q}_i + m) \not{\epsilon}_i^* u_{\vec{p}_1 s_1} = \{2p \cdot \epsilon_i^* - 2k_i \cdot \epsilon_i^* - \not{\epsilon}_i^* \not{p} + \not{\epsilon}_i^* \not{k}_i + m \not{\epsilon}_i^* \} u_{\vec{p}_1 s_1}$$

$$= \{2p \cdot \epsilon_i^* + \not{\epsilon}_i^* \not{k}_i\} u_{\vec{p}_1 s_1} \, , \tag{2.23}$$

using $\epsilon_i^* = (0, -\vec{\epsilon}_i^*)$ and the fact that the polarization vector is perpendicular to the direction of propagation for real (massless) photons ($\epsilon_i^* \cdot k_i = 0$), and the

Dirac equation $(\not{p} - m)u_{\vec{p}_1 s_1} = 0$. Substituting (2.21) into (2.20) for $\hat{\mathcal{O}}$, and using (2.23) for $\mathcal{M}_{fi}$ gives

$$\mathcal{M}_{fi} = ie^2 \bar{v}_{\vec{p}_2 s_2} \hat{\mathcal{O}} u_{\vec{p}_1 s_1} = ie^2 \bar{v}_{\vec{p}_2 s_2} \mathcal{O} u_{\vec{p}_1 s_1} \,, \tag{2.24}$$

with $\mathcal{O}$ differing from $\hat{\mathcal{O}}$ by two terms involving the Dirac operator $(\not{p} - m)$,

$$\begin{aligned} \mathcal{O} &\equiv (-1)\left\{ \frac{2p\cdot\epsilon_1^* \,\not{\epsilon}_2^* + \not{\epsilon}_2^* \,\not{\epsilon}_1^* \,\not{k}_1}{q_1^2 - m^2} + \frac{2p\cdot\epsilon_2^* \,\not{\epsilon}_1^* + \not{\epsilon}_1^* \,\not{\epsilon}_2^* \,\not{k}_2}{q_2^2 - m^2} \right\} \\ &= \left\{ \frac{2p\cdot\epsilon_1^* \,\not{\epsilon}_2^* + \not{\epsilon}_2^* \,\not{\epsilon}_1^* \,\not{k}_1}{2E^2(1 - \beta\hat{k}\cdot\hat{p})} + \frac{2p\cdot\epsilon_2^* \,\not{\epsilon}_1^* + \not{\epsilon}_1^* \,\not{\epsilon}_2^* \,\not{k}_2}{2E^2(1 + \beta\hat{k}\cdot\hat{p})} \right\} . \end{aligned} \tag{2.25}$$

Here we used the CM properties $\vec{k}_1 = -\vec{k}_2$ and $E = \omega$, the definition $\beta \equiv p/E$, and conservation of momentum at vertices in the Feynman diagrams of **Fig 2** ($q_i = p - k_i \ \forall \ i \in \{1,2\}$). Now we introduce explicit Dirac gamma matrices $\{\gamma_\mu\}$ using Bjorken and Drell conventions[46] in 2×2 block form,

$$\gamma_0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \quad , \quad \vec{\gamma} = \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix} , \tag{2.26}$$

and the notation $\tilde{a} \equiv \vec{\sigma}\cdot\vec{a}$ to write for any four-vector a^μ

$$\not{a} = \begin{bmatrix} a_0 I & \tilde{a} \\ -\tilde{a} & -a_0 I \end{bmatrix} . \tag{2.27}$$

Using (2.27), and writing $\mathcal{O}$ in 2×2 blocks

$$\mathcal{O} = \begin{pmatrix} \mathcal{O}_{11} & \mathcal{O}_{12} \\ \mathcal{O}_{21} & \mathcal{O}_{22} \end{pmatrix} , \tag{2.28}$$

we find that

$$\mathcal{O}_{11} = -\mathcal{O}_{22} = \frac{1}{2E}\left\{ \frac{(\vec{\sigma}\cdot\vec{\epsilon}_2^{\,*})(\vec{\sigma}\cdot\vec{\epsilon}_1^{\,*})}{1 - \beta\hat{k}\cdot\hat{p}} + \frac{(\vec{\sigma}\cdot\vec{\epsilon}_1^{\,*})(\vec{\sigma}\cdot\vec{\epsilon}_2^{\,*})}{1 + \beta\hat{k}\cdot\hat{p}} \right\} , \tag{2.29}$$

and

$$\mathcal{O}_{12} = -\mathcal{O}_{21} = \frac{1}{2E^2}\left\{\frac{(\vec{\sigma}\cdot\vec{\epsilon}_2^*)(\vec{\sigma}\cdot\vec{\epsilon}_1^*)(\vec{\sigma}\cdot\vec{k}) + 2\vec{p}\cdot\vec{\epsilon}_1^*(\vec{\sigma}\cdot\vec{\epsilon}_2^*)}{1-\beta\hat{k}\cdot\hat{p}}\right.$$

$$\left. - \frac{(\vec{\sigma}\cdot\vec{\epsilon}_1^*)(\vec{\sigma}\cdot\vec{\epsilon}_2^*)(\vec{\sigma}\cdot\vec{k}) - 2\vec{p}\cdot\vec{\epsilon}_2^*(\vec{\sigma}\cdot\vec{\epsilon}_1^*)}{1+\beta\hat{k}\cdot\hat{p}}\right\}. \tag{2.30}$$

We may rewrite the invariant amplitude $\mathcal{M}_{fi}$ in terms of Pauli spinors and 2×2 matrices using (2.17), (2.19), (2.24) and (2.28)-(2.30), which gives

$$\mathcal{M}_{fi} = -\eta^2\, v^\dagger_{s_2}$$

$$\left\{\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\mathcal{O}_{11} + \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\mathcal{O}_{12}\,\frac{\vec{\sigma}\cdot\vec{p}}{E+m} + \mathcal{O}_{21} + \mathcal{O}_{22}\,\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right\}u_{s_1}$$

$$= \eta^2\, v^\dagger_{s_2}\left\{\mathcal{O}_{12} - \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\mathcal{O}_{12}\,\frac{\vec{\sigma}\cdot\vec{p}}{E+m} + \frac{1}{E+m}\left[\mathcal{O}_{11},\vec{\sigma}\cdot\vec{p}\right]\right\}u_{s_1}\,. \tag{2.31}$$

Now, for any two vectors $\vec{a}$, $\vec{b}$, one has the identity

$$\tilde{a}\,\tilde{b} \equiv \sigma\cdot\vec{a}\;\sigma\cdot\vec{b} = \vec{a}\cdot\vec{b} + i\sigma\cdot(\vec{a}\times\vec{b})\,, \tag{2.32}$$

which may be applied successively to derive other identities between more than two arbitrary vectors, such as

$$\tilde{a}\,\tilde{b}\,\tilde{c} = \tilde{a}\;\vec{b}\cdot\vec{c} - \tilde{b}\;\vec{c}\cdot\vec{a} + \tilde{c}\;\vec{a}\cdot\vec{b} + i\vec{a}\cdot(\vec{b}\times\vec{c})\,, \tag{2.33}$$

$$\tilde{a}\,\tilde{b}\,\tilde{a} = 2\vec{a}\cdot\vec{b}\;\tilde{a} - a^2\;\tilde{b}\,, \tag{2.34}$$

and

$$\tilde{d}\,\tilde{a}\,\tilde{b}\,\tilde{c}\,\tilde{d} = 2\tilde{d}\left\{\vec{a}\cdot\vec{b}\;\vec{c}\cdot\vec{d} - \vec{a}\cdot\vec{c}\;\vec{b}\cdot\vec{d} + \vec{a}\cdot\vec{d}\;\vec{b}\cdot\vec{c}\right\} -$$

$$d^2 \left\{ \tilde{a}\, \vec{b} \cdot \vec{c} - \tilde{b}\, \vec{c} \cdot \vec{a} + \tilde{c}\, \vec{a} \cdot \vec{b} - i \vec{a} \cdot (\vec{b} \times \vec{c}) \right\} , \tag{2.35}$$

where it is understood that the 2×2 matrix identity multiplies scalar terms in the last four equations. A straightforward calculation using (2.29), (2.30), and (2.33)-(2.35) gives

$$\begin{aligned} \mathcal{O}_{12} = \frac{1}{E^2(1-\beta^2(\hat{k}\cdot\hat{p})^2)} \Big\{ & -i\vec{k}\cdot(\vec{\epsilon}_1^{\,*} \times \vec{\epsilon}_2^{\,*}) + \vec{p}\cdot\vec{\epsilon}_1^{\,*}\, \tilde{\epsilon}_2^{\,*} \\ & + \vec{p}\cdot\vec{\epsilon}_2^{\,*}\, \tilde{\epsilon}_1^{\,*} + \tilde{k}\, \vec{\epsilon}_1^{\,*}\cdot\vec{\epsilon}_2^{\,*} \beta \hat{k}\cdot\hat{p} + (\vec{p}\cdot\vec{\epsilon}_1^{\,*}\, \tilde{\epsilon}_2^{\,*} - \vec{p}\cdot\vec{\epsilon}_2^{\,*}\, \tilde{\epsilon}_1^{\,*})\beta\hat{k}\cdot\hat{p} \Big\} , \end{aligned} \tag{2.36}$$

$$\begin{aligned} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\, \mathcal{O}_{12}\, \frac{\vec{\sigma}\cdot\vec{p}}{E+m} = & \frac{1}{E^2(E+m)^2(1-\beta^2(\hat{k}\cdot\hat{p})^2)} \\ & \Big\{ -i\vec{k}\cdot(\vec{\epsilon}_1^{\,*} \times \vec{\epsilon}_2^{\,*})p^2 + 4\vec{p}\cdot\vec{\epsilon}_1^{\,*}\, \vec{p}\cdot\vec{\epsilon}_2^{\,*}\tilde{p} - p^2(\vec{p}\cdot\vec{\epsilon}_1^{\,*}\, \tilde{\epsilon}_2^{\,*} + \vec{p}\cdot\vec{\epsilon}_2^{\,*}\, \tilde{\epsilon}_1^{\,*}) \\ & + \left[2\vec{\epsilon}_1^{\,*}\cdot\vec{\epsilon}_2^{\,*}\, \vec{p}\cdot\vec{k}\, \tilde{p} - p^2 \vec{\epsilon}_1^{\,*}\cdot\vec{\epsilon}_2^{\,*}\, \tilde{k} - p^2(\vec{p}\cdot\vec{\epsilon}_1^{\,*}\, \tilde{\epsilon}_2^{\,*} - \vec{p}\cdot\vec{\epsilon}_2^{\,*}\, \tilde{\epsilon}_1^{\,*}) \right] \beta\hat{k}\cdot\hat{p} \Big\} , \end{aligned} \tag{2.37}$$

and finally

$$\frac{1}{E+m}\left[\mathcal{O}_{11}, \vec{\sigma}\cdot\vec{p}\right] = \frac{2\beta\hat{k}\cdot\hat{p}}{E(E+m)(1-\beta^2(\hat{k}\cdot\hat{p})^2)} \left[\vec{p}\cdot\vec{\epsilon}_1^{\,*}\, \tilde{\epsilon}_2^{\,*} - \vec{p}\cdot\vec{\epsilon}_2^{\,*}\, \tilde{\epsilon}_1^{\,*} \right] . \tag{2.38}$$

By substituting (2.36)-(2.38) into (2.31), after some algebra one obtains $\mathcal{M}_{fi}$ in the form

$$\mathcal{M}_{fi}(s_1, s_2) = \mathcal{A}\, v_{s_2}^\dagger I u_{s_1} \quad + \quad \vec{\mathcal{B}}\cdot v_{s_2}^\dagger \vec{\sigma} u_{s_1} , \tag{2.39}$$

where the expressions $\mathcal{A}$ and $\vec{\mathcal{B}}$ are

$$\mathcal{A} = \frac{e^2}{E}\left\{ \frac{\hat{k}\cdot(\vec{\epsilon}_1^{\,*} \times \vec{\epsilon}_2^{\,*})}{1-\beta^2(\hat{k}\cdot\hat{p})^2} \right\} \tag{2.40}$$

and

$$\vec{\mathcal{B}} = -\frac{ie^2}{2m}\,\frac{1}{(1-\beta\hat{k}\cdot\hat{p})}\Bigg\{\hat{k}\,\vec{\epsilon}_1^{\,*}\cdot\vec{\epsilon}_2^{\,*} + \beta\Big(\vec{\epsilon}_1^{\,*}\,\vec{\epsilon}_2^{\,*}\cdot\hat{p} + \vec{\epsilon}_2^{\,*}\,\vec{\epsilon}_1^{\,*}\cdot\hat{p}\Big)$$

$$-\beta^2\frac{E}{E+m}\hat{p}\,\hat{k}\cdot\hat{p}\,\vec{\epsilon}_1^{\,*}\cdot\vec{\epsilon}_2^{\,*} - \beta^3\frac{2E}{(E+m)}\hat{p}\,\vec{\epsilon}_1^{\,*}\cdot\hat{p}\,\vec{\epsilon}_2^{\,*}\cdot\hat{p}\Bigg\}$$

$$+ \quad \Big(\vec{\epsilon}_1^{\,*}\leftrightarrow\vec{\epsilon}_2^{\,*}\ ,\ \hat{k}\to-\hat{k}\Big)$$

$$= -\frac{ie^2}{m}\frac{1}{(1-\beta^2(\hat{k}\cdot\hat{p})^2)}\Bigg\{\beta\Big(\hat{k}\,\vec{\epsilon}_1^{\,*}\cdot\vec{\epsilon}_2^{\,*}\hat{k}\cdot\hat{p} + \vec{\epsilon}_1^{\,*}\,\vec{\epsilon}_2^{\,*}\cdot\hat{p} + \vec{\epsilon}_2^{\,*}\,\vec{\epsilon}_1^{\,*}\cdot\hat{p}\Big)$$

$$-\beta^3\frac{E}{E+m}\,\hat{p}\left(2\,\vec{\epsilon}_1^{\,*}\cdot\hat{p}\,\vec{\epsilon}_2^{\,*}\cdot\hat{p} + \vec{\epsilon}_1^{\,*}\cdot\vec{\epsilon}_2^{\,*}\big(\hat{k}\cdot\hat{p}\big)^2\right)\Bigg\}\,. \tag{2.41}$$

From (2.2), and (2.39) we take matrix elements of the matrices I and $\vec{\sigma}$ between $v_{s_2}^\dagger$ and u_{s_1} using the spin-eigenstates in (1.22)-(1.25). For the singlet ($s=0$) case this gives

$$v_{s_2}^\dagger\, I\, u_{s_1} \Rightarrow \frac{1}{\sqrt{2}}\{v_\downarrow^\dagger\, I\, u_\uparrow \ - \ v_\uparrow^\dagger\, I\, u_\downarrow\} = -\sqrt{2} \qquad \text{for} \qquad (s,s_z)=(0,0)\ , \tag{2.42}$$

and similarly

$$v_{s_2}^\dagger\vec{\sigma}\; u_{s_1} \Rightarrow\ 0\ \ \text{for}\ \ \ (s,s_z)=(0,0)\ ; \tag{2.43}$$

whereas for the triplet case we have

$$v_{s_2}^\dagger\, I\, u_{s_1} \Rightarrow\ 0 \quad \text{for } s=1,\ s_z\in\{-1,0,+1\}\ , \tag{2.44}$$

$$v_{s_2}^\dagger\,\vec{\sigma}\, u_{s_1} \Rightarrow\ -\sqrt{2}\,\hat{e}_+ \quad \text{for}\ \ (s,s_z)=(1,+1)\ , \tag{2.45}$$

$$v_{s_2}^\dagger\,\vec{\sigma}\, u_{s_1} \Rightarrow\ -\sqrt{2}\,\hat{e}_0 \quad \text{for}\ \ (s,s_z)=(1,0)\ , \tag{2.46}$$

and

$$v^{\dagger}_{s_2}\,\vec{\sigma}\,u_{s_1} \Rightarrow -\sqrt{2}\,\hat{e}_- \quad \text{for} \quad (s, s_z) = (1, -1)\,, \tag{2.47}$$

where

$$\hat{e}_{\pm} \equiv \mp\frac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y}) \quad \text{and} \quad \hat{e}_0 \equiv \hat{z}\,. \tag{2.48}$$

These results show that only $\mathcal{A}$ in (2.39) contributes to the singlet decay amplitude and only $\vec{\mathcal{B}}$ contributes to the triplet amplitude. This allows us to separate the singlet and triplet amplitudes as

$$\mathcal{M}^{(0)}_{fi} = \mathcal{A}\,v^{\dagger}_{s_2} I u_{s_1} \qquad \text{and} \qquad \mathcal{M}^{(1)}_{fi} = \vec{\mathcal{B}}\cdot v^{\dagger}_{s_2}\vec{\sigma}\,u_{s_1}\,. \tag{2.49}$$

For the singlet case, we have $J = l$ and $m = M$, so the summations over m and s_z in (2.2) produce only one term, and (2.2) becomes

$$\Phi^{tot}_{JM}(\vec{p}) \otimes \mathcal{M}_{fi}(s=0) = \Phi_{l,m}(\vec{p})\langle l, m; 0, 0 | l, m\rangle \chi_{0,0} \otimes \mathcal{M}^{(0)}_{fi}(s_{1z}, s_{2z})$$

$$= \Phi_{lm}(\vec{p})\,\frac{1}{\sqrt{2}}\left\{\mathcal{M}^{(0)}_{fi}(\uparrow, \bar{\downarrow}) - \mathcal{M}^{(0)}_{fi}(\downarrow, \bar{\uparrow})\right\}\,, \tag{2.50}$$

where we have also used the trivial Clebsch-Gordan coefficient $\langle l, m; 0, 0 | l, m\rangle = 1$. Using (2.42) and (2.49), we can write the final expression for $\Phi^{tot}_{JM}(\vec{p}) \otimes \mathcal{M}_{fi}(s=0)$ as

$$\Phi^{tot}_{JM}(\vec{p}) \otimes \mathcal{M}_{fi}(s=0) = -\sqrt{2}\,\frac{e^2}{E}\left\{\frac{\hat{k}\cdot(\vec{\epsilon}^{\,*}_1 \times \vec{\epsilon}^{\,*}_2)}{1 - \beta^2(\hat{k}\cdot\hat{p})^2}\right\}\Phi_{JM}(\vec{p})\,. \tag{2.51}$$

For the triplet case s_z has 3 possible values $\{+1, 0, -1\}$, which gives 3 possible m values for a given value of M due to the constraint $M = m + s_z$.

Accordingly, the summations in (2.2) yield 3 nonvanishing terms,

$$
\begin{aligned}
&\Phi^{tot}_{JM}(\vec{p}) \otimes \mathcal{M}_{fi}(s=1) = \\
&\qquad \sum_{m=-l}^{l} \sum_{s_z=-s}^{s} \Phi_{lm}(\vec{p}) \langle l, m; 1, s_z | J, M\rangle \; \chi_{1,s_z} \otimes \mathcal{M}^{(1)}_{fi}(s_{1z}, s_{2z}) \\
&\qquad = \sum_{m=-l}^{l} \Phi_{lm}(\vec{p}) \Big\{ \langle l, m; 1, +1 | J, M\rangle \; \chi_{1,+1} \otimes \mathcal{M}^{(1)}_{fi}(s_{1z}, s_{2z}) \\
&\qquad\qquad + \langle l, m; 1, 0 | J, M\rangle \; \chi_{1,0} \otimes \mathcal{M}^{(1)}_{fi}(s_{1z}, s_{2z}) \\
&\qquad\qquad + \langle l, m; 1, -1 | J, M\rangle \; \chi_{1,-1} \otimes \mathcal{M}^{(1)}_{fi}(s_{1z}, s_{2z}) \Big\} \, . \qquad (2.52)
\end{aligned}
$$

Note the Clebsch-Gordan coefficients $\langle l, m; s, s_z | J, M\rangle$ with $M \neq m + s_z$ vanish, which leaves

$$
\begin{aligned}
\Phi^{tot}_{JM}(\vec{p}) \otimes \mathcal{M}_{fi}(s=1) &= \langle l, M-1; 1, +1 | J, M\rangle \mathcal{M}_{fi}(\uparrow, \bar{\uparrow}) \Phi_{l,M-1}(\vec{p}) \\
&\quad + \langle l, M; 1, 0 | J, M\rangle \frac{1}{\sqrt{2}} \Big\{ \mathcal{M}_{fi}(\uparrow, \bar{\downarrow}) + \mathcal{M}_{fi}(\downarrow, \bar{\uparrow}) \Big\} \Phi_{l,M}(\vec{p}) \\
&\quad + \langle l, M+1; 1, -1 | J, M\rangle \mathcal{M}_{fi}(\downarrow, \bar{\downarrow}) \Phi_{l,M+1}(\vec{p}) \Big\} \, . \qquad (2.53)
\end{aligned}
$$

Using the results (2.45)-(2.47) and the form of $\mathcal{M}^{(1)}_{fi}$ in (2.49) we find

$$
\mathcal{M}_{fi}(\uparrow, \bar{\uparrow}) = -\sqrt{2} \; \hat{e}_{+} \cdot \vec{\mathcal{B}} \, , \qquad (2.54)
$$

$$
\frac{1}{\sqrt{2}} \Big\{ \mathcal{M}_{fi}(\uparrow, \bar{\downarrow}) + \mathcal{M}_{fi}(\downarrow, \bar{\uparrow}) \Big\} = -\sqrt{2} \; \hat{z} \cdot \vec{\mathcal{B}} \, , \qquad (2.55)
$$

and

$$
\mathcal{M}_{fi}(\uparrow, \bar{\uparrow}) = -\sqrt{2} \; \hat{e}_{-} \cdot \vec{\mathcal{B}} \, , \qquad (2.56)
$$

which when substituted into (2.53) give

$$\Phi^{tot}_{JM}(\vec{p}) \otimes \mathcal{M}_{fi}(s=1) = -\sqrt{2}\Big\{ \langle l, M-1; 1, +1 | J, M \rangle \, \hat{e}_+ \, \Phi_{l,M-1}(\vec{p})$$

$$+ \langle l, M; 1, 0 | J, M \rangle \, \hat{z} \, \Phi_{l,M}(\vec{p})$$

$$+ \langle l, M+1; 1, -1 | J, M \rangle \, \hat{e}_- \, \Phi_{l,M+1}(\vec{p}) \Big\} \cdot \vec{\mathcal{B}} \ . \qquad (2.57)$$

The equations (2.51) and (2.57) represent a decomposition of the general decay amplitude into separate singlet and triplet amplitudes.

2.3 The singlet case (s=0)

One may rewrite (2.51) as

$$\Phi^{tot}_{JM}(\vec{p}) \otimes \mathcal{M}_{fi}(s=0)$$

$$= -\frac{e^2 \hat{k} \cdot (\vec{\epsilon}_1^{\,*} \times \vec{\epsilon}_2^{\,*})}{\sqrt{2}E} \left\{ \frac{1}{1 - \beta \hat{k} \cdot \hat{p}} + \frac{1}{1 + \beta \hat{k} \cdot \hat{p}} \right\} \phi_J(p) Y_{JM}(\Omega_{\vec{p}}) \ . \qquad (2.58)$$

Using this form and (2.1), we write

$$\mathcal{M}(s=0) = \frac{-me^2}{\sqrt{(2\pi)^3}}$$

$$\times \int d\vec{p} \, \frac{\hat{k} \cdot (\vec{\epsilon}_1^{\,*} \times \vec{\epsilon}_2^{\,*})}{\sqrt{2}E^2} \left\{ \frac{1}{1 - \beta \hat{k} \cdot \hat{p}} + \frac{1}{1 + \beta \hat{k} \cdot \hat{p}} \right\} \Phi_{JM}(\vec{p}) \ . \qquad (2.59)$$

Here we note that the denominators involved in the integrand of (2.59), $(1 \pm \beta \hat{k} \cdot \hat{p})$, make it difficult to carry out the angular integral completely. This suggests expanding the integrand in powers of β and perhaps retaining the leading nonvanishing order[36], or some finite higher order[26]. The angular integrals are then very straightforward and the integral over $p \equiv |\vec{p}|$ is tractable analytically with a hydrogenic wavefunction $\phi(p)$. A general calculation of the $\gamma\gamma$ decay rate using (2.59) without such approximations has only been reported for the case $J = l = 0$ and $s = 0$[31,32]. (Note that the relativistic 0^{++} and 2^{++} calculations are based on a different technique[30].) For a full integration of (2.59) with a $q\bar{q}$ spin-singlet meson of arbitrary total angular momentum $J = l$, it is useful to note that the decay photon angular wavefunction is given by (1.42). From this one can see that the photon angular probability distribution in this state is proportional to $\left|Y_{JM}(\Omega_{\vec{k}})\right|^2$, so the angular distribution of emitted photons in singlet-positronium decay must also be proportional to $\left|Y_{JM}(\Omega_{\vec{k}})\right|^2$;

$$\frac{d\Gamma^{JM}}{d\Omega_{\vec{k}}} = C_{JM} \times \left|Y_{JM}(\Omega_{\vec{k}})\right|^2 . \tag{2.60}$$

This together with (1.27) gives

$$\Gamma^J_{\gamma\gamma} = \frac{1}{2!}\int d\Omega_{\vec{k}} \frac{d\Gamma}{d\Omega_{\vec{k}}} = \frac{1}{2}\int d\Omega_{\vec{k}} C_{JM} \times \left|Y_{JM}(\Omega_{\vec{k}})\right|^2 = \frac{1}{2} C_{JM} , \tag{2.61}$$

which tells us that $C_{JM} = C_J$, since $\Gamma_{\gamma\gamma}$ is independent of J_z. This gives the partial width as a function of the total width and the angular distribution of the photons,

$$\frac{d\Gamma^{JM}}{d\Omega_{\vec{k}}} = 2\Gamma^J_{\gamma\gamma} \times \left|Y_{JM}(\Omega_{\vec{k}})\right|^2 . \tag{2.62}$$

Note that (2.62) holds for any $M = J_z$ and any $\hat{\Omega}_{\vec{k}}$ (i.e, $\hat{k}$). We may therefore obtain $\Gamma^J_{\gamma\gamma}$ directly from the angular distribution (2.62) with a particular choice for M and $\hat{\Omega}_{\vec{k}}$. Here we choose $M = 0$ and $\hat{k} = \hat{z}$, which gives

$$\Gamma^J(e^+e^- \to \gamma\gamma) = \frac{1}{2|Y_{J0}(\hat{z})|^2} \times \frac{d\Gamma^{J0}}{d\Omega_{\vec{k}}}(e^+e^- \to \gamma\gamma)\bigg|_{\hat{k}=\hat{z}} \,. \tag{2.63}$$

On combining (1.26), (2.59) and (2.63), we obtain the $\gamma\gamma$ decay rate

$$\Gamma^J_{\gamma\gamma}(s=0) = \frac{1}{64\pi^2\left|Y_{J0}(\hat{z})\right|^2} \sum_{\lambda_1\lambda_2}\bigg|\int \frac{d\vec{p}}{\sqrt{(2\pi)^3}}$$

$$\times\left\{-\frac{me^2\hat{z}\cdot(\vec{\epsilon}_1^{\,*}\times\vec{\epsilon}_2^{\,*})}{\sqrt{2}E^2}\left\{\frac{1}{1-\beta\cos\theta}+\frac{1}{1+\beta\cos\theta}\right\}\phi_J(p)Y_{JM}(\theta,\phi)\right\}\bigg|^2 \,. \tag{2.64}$$

The angles θ and ϕ in the integrand refer to the direction of $\hat{p}$ within the positronium wavefunction, not to the axis $\hat{k} \equiv \hat{z}$ of the outgoing photons, so that $\hat{k}\cdot\hat{p} = \hat{z}\cdot\hat{p} = \cos\theta \equiv \mu$. Writing the spherical harmonic in terms of the Legendre polynomial $P_J(\mu)$

$$Y_{J0}(\theta,\phi) = \sqrt{\frac{2J+1}{4\pi}}\,P_J(\mu) \tag{2.65}$$

and using $P_J(cos\ 0) = P_J(1) = 1\ \forall\ J$, we have

$$\frac{Y_{J0}(\theta,\phi)}{Y_{J0}(\Omega_{\hat{z}})} = P_J(\mu)\,, \tag{2.66}$$

which may be used to rewrite (2.64) as

$$\Gamma^J_{\gamma\gamma}(s=0) = \frac{e^2}{2^{10}\pi^5} \times \left\{\sum_{\lambda_1\lambda_2}\left|\hat{z}\cdot(\vec{\epsilon}_1^{\,*}\times\vec{\epsilon}_2^{\,*})\right|^2\right\}$$

$$\times\left|\int_0^\infty dp\ \phi_J(p)\frac{m}{E^2}\int_0^{2\pi} d\phi \int_{-1}^{1} d\mu\ P_J(\mu)\left\{\frac{1}{1-\beta\mu}+\frac{1}{1+\beta\mu}\right\}\right|^2 . \quad (2.67)$$

The sum over photon polarizations gives a factor of 2 (see appendix A), and the integral over the polar angle ϕ gives a factor of 2π. For the integral over μ we use the definition of the Legendre function of the second kind,

$$Q_J(x) \equiv \int_{-1}^{+1} \frac{P_J(\mu)}{x-\mu}\, d\mu\ , \quad (2.68)$$

so we obtain

$$\int_{-1}^{+1} d\mu\ P_J(\mu)\left\{\frac{1}{1-\beta\mu}+\frac{1}{1+\beta\mu}\right\}$$
$$= \begin{cases} \frac{2}{\beta}Q_J(\beta^{-1}), & \text{J=even;} \\ 0 & \text{otherwise.} \end{cases} \quad (2.69)$$

Here we have used the fact that $P_J(\mu)$ is odd (even) in μ if J is odd (even). Note that (2.69) is consistent with the expectation $\Gamma_{\gamma\gamma} = 0$ for ($J = L =$ *odd*), because this implies $C = -1$, which is forbidden for two-photon states. Combining (2.67) and (2.69) and using $e^2 = 4\pi\alpha$, we obtain our singlet two-photon width formula[47]

$$\Gamma^J_{\gamma\gamma}(s=0) = \frac{2\alpha^2}{\pi m^2}\left|\int_0^\infty dp\ p^2\ \phi_J(p)\ \left(\frac{1-\beta^2}{\beta}\right)\ Q_J(\beta^{-1})\right|^2 . \quad (2.70)$$

This is an exact relativistic result for singlet positronium, given the diagrams of **Fig 2** and the decay formulae (1.26) and (1.28). Application of this result to quarkonium requires multiplication of (2.70) by a color factor of 3, a flavor factor of $|\langle e_q^2/e^2\rangle|^2$, the replacement of m by m_q in the amplitude, as given by (1.30), and an experimentally motivated treatment of the dependence of $\Gamma_{\gamma\gamma}$ on the resonance mass M_R (which we shall include and discuss in detail in the following chapter). This gives for the $\Gamma_{\gamma\gamma}$ of a singlet quarkonium state

$$\Gamma_{\gamma\gamma}\big(q\bar{q}(J^{-+})\big) = \mathcal{F}_q \cdot \frac{6\alpha^2}{\pi {m_q}^2} \left| \int_0^\infty dp\; p^2\; \phi_J(p) \left(\frac{1-\beta^2}{\beta}\right) Q_J(\beta^{-1}) \right|^2 , \qquad (2.71)$$

where we have defined the flavor factor

$$\mathcal{F}_q \equiv \left|< e_q^2/e^2 >\right|^2. \qquad (2.72)$$

We may use our relativistic result (2.71) to model meson decays to $\gamma\gamma$ under the usual quark model assumption that the initial meson is a pure $q\bar{q}$ state with a wavefunction $\Phi_{lm}(\vec{p})$, which we obtain from a nonrelativistic Coulomb-plus-linear potential model. In principal the decay rate (2.71) could also be used with another $\phi_J(p)$ wavefunction, such as from a relativistic wave equation. Of course this is only an approximate method, and additional important corrections may arise from effects such as gluon exchange, non-$q\bar{q}$ components in the meson wavefunction and corrections to the free quark propagator assumed in $\mathcal{M}_{fi}$. For the 0^{-+} $\pi^0 \to \gamma\gamma$ decay, the flavor factor is $\frac{1}{18}$, so (2.71) gives

$$\Gamma(\pi^0 \to \gamma\gamma) = \frac{\alpha^2}{3\pi {m_q}^2} \left| \int_0^\infty dp\; p^2\; \phi(p) \left(\frac{1-\beta^2}{\beta}\right) Q_0(\beta^{-1}) \right|^2$$

$$= \frac{\alpha^2}{3\pi {m_q}^2} \left| \int_0^\infty dp\; p^2\; \phi(p) \left(\frac{1-\beta^2}{\beta}\right) \frac{1}{2} \ln\!\left(\frac{1+\beta}{1-\beta}\right) \right|^2$$

$$= \frac{\alpha^2}{3\pi {m_q}^2} \left| \int_0^\infty dp\ p^2\ \phi(p)\ \left(\frac{m_q^2}{pE}\right)\ \ln\left(\frac{E+p}{m_q}\right) \right|^2 . \tag{2.73}$$

This is identical to the relativistic results of Bergström, Snellman and Tengstrand[31] and Hayne and Isgur[32], except that Hayne and Isgur then multiply the rate by an *ad hoc* overall factor of $(M_R/M_{ref})^3$ to impose an expected approximate M_R^3 mass dependence. As mentioned in chapter one, this is the only relativistic $\Gamma_{\gamma\gamma}$ result for singlet $q\bar{q}$ states that exists in the literature.

The nonrelativistic limit

To obtain the nonrelativistic limit ($\beta \to 0$) we expand all functions of β or p in (2.70) in powers of β, and retain the leading nonvanishing term (which is of order β^l). Using $\beta \approx p/m$ to lowest order, and

$$Q_J(\beta^{-1}) = \frac{2^J (J!)^2}{(2J+1)!} \beta^{J+1} + \mathcal{O}(\beta^{J+2}) , \tag{2.74}$$

we can write the leading nonvanishing contribution to $\Gamma_{\gamma\gamma}$ in (2.70) as

$$\Gamma^J_{\gamma\gamma}(s=0) = \frac{2^{2J+1}(J!)^4}{\pi 2J+1)!^2} \frac{\alpha^2}{m^{2J+2}} \left| \int_0^\infty dp\ p^{J+2}\ \phi_J(p) \right|^2 . \tag{2.75}$$

This may also be written in terms of derivatives of the position space wavefunction $\psi(r)$ at the origin, using the result

$$\int_0^\infty dp\ p^{J+2} \phi_J(p) = \sqrt{\frac{\pi}{2}}\, (-i)^J \frac{(2J+1)!!}{J!}\ \psi_J^{(J)}(0) . \tag{2.76}$$

After using the identity

$$(2n+1)! = 2^n \; n! \; (2n+1)!! \tag{2.77}$$

and replacing $l = J$ (which is the case for $s = 0$), one finds the remarkably simple result

$$\Gamma^J_{NR}(e^+e^- \to \gamma\gamma) = \frac{\alpha^2}{m^{2l+2}} \left|\psi_l^{(l)}(0)\right|^2, \tag{2.78}$$

where $\psi_l^{(l)}(0)$ is the l^{th} derivative of ψ_l at the origin. For positronium with a pure $-\alpha/r$ potential, the derivatives at contact can be evaluated analytically, and substitution into (2.78) gives our general nonrelativistic result for the $\gamma\gamma$ decay width of singlet positronium with principal quantum number n and angular momentum $J = l$;

$$\Gamma^J_{NR}(e^+e^- \to \gamma\gamma) = \frac{(n+l)!\,(l!)^2}{2[(2l+1)!]^2\;(n-l-1)!\;\;n^{(2l+4)}} \cdot m\alpha^{2l+5} \tag{2.79}$$

which may be written in terms of the binomial coefficients $\binom{a}{b} \equiv \frac{a!}{b!(a-b)!}$ as

$$\Gamma^J_{NR}(e^+e^- \to \gamma\gamma) = \frac{1}{2(l+1)} \frac{\binom{n+l}{2l+1}}{\binom{2l+1}{l}}\, n^{-(2l+4)} \cdot m\alpha^{2l+5}\,. \tag{2.80}$$

In the $l = 0$ case this reduces to the familiar result of (1.46) Pirenne[34],

$$\Gamma_{NR}(n^1S_0 \to \gamma\gamma) = \frac{1}{2n^3} \cdot m\alpha^5\,. \tag{2.81}$$

Again, we use the transcription given in (1.31) to apply this result to quarkonium and obtain our nonrelativistic $\Gamma_{\gamma\gamma}$ for singlet quarkonium

$$\Gamma_{NR}(q\bar{q}(J^{-+}) \to \gamma\gamma) = 3 \cdot \mathcal{F}_q \cdot \frac{\alpha^2}{{m_q}^{2l+2}} \left|\psi_l^{(l)}(0)\right|^2\,. \tag{2.82}$$

For $J = l = 2$ this gives

$$\Gamma_{(NR)}(q\bar{q}(^1D_2) \to \gamma\gamma) = 3 \cdot \mathcal{F}_q \cdot \frac{\alpha^2}{m_q{}^6} \left|\psi''(0)\right|^2 , \tag{2.83}$$

which agrees with the formula (1.57) of Anderson, Austern and Cahn[37] after replacing m_q by $M_R/2$, which was one of their substitutions. (Note that this step is unconventional and that the mass of the $\pi_2(1670)$ is actually much bigger than the $2m_q \approx 440 - 660$ MeV of light constituent quarks.)

2.4 The triplet case (s=1)

Here the calculation is more complicated for two reasons: *a*) the amplitude $\mathcal{M}^{(1)}_{fi}$ has more terms, and *b*) it involves the direction of the fermion momenta (i.e, $\hat{\Omega}_{\vec{p}}$) in the numerators as well. Since s_z has 3 possible values $\{-1, 0, +1\}$, there are 3 terms in $\Phi^{tot}_{JM}(\vec{p}) \otimes \mathcal{M}^{(1)}_{fi}$ given in (2.57); these 3 terms and the form of $\vec{B}$ in (2.41) lead to 15 complicated integrals if we use the full denominator $(1 - \beta^2(\hat{k} \cdot \hat{p})^2)$, or 30 if we use the simpler forms $(1 \pm \beta\hat{k} \cdot \hat{p})$. Obviously, one would not wish to proceed without looking for simplifications. According to the discussion in section (1.4), we see that the photon angular probability distribution in the state $|J^{++}, M\rangle$ with $\lambda = 0$ is proportional to $\left|Y_{JM}(\Omega_{\vec{k}})\right|^2$, and that with $\lambda = \pm 2$ it is proportional to $\left|\mathcal{D}^{J\,*}_{M,\pm 2}\right|^2$. Therefore the angular distribution of emitted photons in our decay formulae must also be proportional to the same functions. Using (1.43), one may write

$$\frac{d\Gamma}{d\Omega_{\vec{k}}}^{J^{++}M}(\lambda=0) = C^0_{JM} \times \left|Y_{JM}(\Omega_{\vec{k}})\right|^2 \tag{2.84}$$

which together with (1.27) give the width in terms of the proportionality constant

$$\Gamma^{J^{++}}_{\gamma\gamma}(\lambda=0) = \frac{1}{2!}\int d\Omega_{\vec{k}} \frac{d\Gamma^{JM}}{d\Omega_{\vec{k}}}(\lambda=0) = \frac{1}{2}C^0_{JM} \equiv \frac{1}{2}C^0_J\,, \tag{2.85}$$

since $\Gamma_{\gamma\gamma}$ is only J dependent. Similarly, using (1.44) and (1.45) we have

$$\frac{d\Gamma^{JM}}{d\Omega_{\vec{k}}}(\lambda=\pm2) = C^{\pm2}_{JM}\times \left|\mathcal{D}^{J*}_{M,\pm2}(\phi,\theta,-\phi)\right|^2 = C^{\pm2}_{JM} \times \left|d^{J}_{M,\pm2}(\theta)\right|^2\,, \tag{2.86}$$

where we have used the definition of the rotation matrix $\mathcal{D}^j_{m',m}$[48]

$$\mathcal{D}^{j}_{m',m}(\alpha,\beta,\gamma) = \langle j,m'|e^{-i\alpha J_z}e^{-i\beta J_y}e^{-i\gamma J_z}|j,m\rangle$$

$$= e^{-im'\alpha}\,\langle j,m'|e^{-i\beta J_y}|j,m\rangle\, e^{-im\gamma} \equiv e^{-im'\alpha}\, d^{j}_{m',m}(\beta)\, e^{-im\gamma}\,. \tag{2.87}$$

$d^{j}_{m',m}$ is a real function with the normalization[33,48]

$$\int d\Omega \left[d^{j}_{m',m}(\theta)\right]^2 = \frac{4\pi}{2j+1}\,. \tag{2.88}$$

Using (1.27), (2.86), and (2.88), we have

$$\Gamma^{J^{++}}_{\gamma\gamma}(\lambda=\pm2) = \frac{1}{2!}C^{\pm2}_{JM}\int d\Omega_{\vec{k}}\left|d^{J*}_{M,\pm2}(\theta)\right|^2$$

$$= \frac{2\pi}{2J+1}\,C^{\pm2}_{JM} \equiv \frac{2\pi}{2J+1}\,C^{\pm2}_{J}\,. \tag{2.89}$$

Again we used the fact that $\Gamma_{\gamma\gamma}$ is independent of $J_z(=M)$. Since (2.84) and (2.86) hold for any value of M and any direction $\hat{\Omega}_{\vec{k}}$, and since the proportionality constants, C_J^0 and $C_J^{\pm 2}$, are independent of M, one can solve for those constants with a particular choice of M and $\hat{\Omega}_{\vec{k}}$. The obvious choice is $M = \lambda$ and $\hat{\Omega}_{\vec{k}} = \hat{z}$ for all three cases. Thus from (2.84) and (2.85) we have

$$\Gamma_{\gamma\gamma}^{J^{++}}(\lambda = 0) = \frac{1}{2} C_J^0 = \frac{2\pi}{2J+1} \frac{d\Gamma^{J^{++}0}}{d\Omega_{\vec{k}}}(\lambda = 0, \hat{k} = \hat{z}) \,, \tag{2.90}$$

where we used $\left|Y(\hat{z})\right|^2 = \frac{2J+1}{4\pi}$. For the other two cases we have from (2.86) and (2.89)

$$\Gamma_{\gamma\gamma}^{J^{++}}(\lambda = \pm 2) = \frac{2\pi}{2J+1} \frac{d\Gamma^{J^{++}\pm 2}}{d\Omega_{\vec{k}}}(\lambda = \pm 2, \hat{k} = \hat{z}) \,, \tag{2.91}$$

where we have used $d_{m,m}^{\,j}(0) = 1$[48]. Since the three two photon states with J^{++} given in (1.39)-(1.41) are linearly independent, one can rewrite the differential decay rate in the J^{++} case (i.e, s=1) using (1.26) as

$$\begin{aligned}
\frac{d\Gamma^{J^{++}M}}{d\Omega_{\vec{k}}} &= \frac{1}{32\pi^2} \sum_{\lambda=0,\pm 2} \left|\mathcal{M}(\lambda_1, \lambda_2; \hat{k})\right|^2 \\
&= \frac{1}{32\pi^2} \left\{ \left|\mathcal{M}(\lambda = 0; \hat{k})\right|^2 + \left|\mathcal{M}(\lambda = +2; \hat{k})\right|^2 + \left|\mathcal{M}(\lambda = -2; \hat{k})\right|^2 \right\} \\
&= \frac{d\Gamma^{J^{++}M}}{d\Omega_{\vec{k}}}(\lambda = 0) + \frac{d\Gamma^{J^{++}M}}{d\Omega_{\vec{k}}}(\lambda = 2) + \frac{d\Gamma^{J^{++}M}}{d\Omega_{\vec{k}}}(\lambda = -2) \,.
\end{aligned} \tag{2.92}$$

Note that since our decay amplitude is symmetric under the interchange of γ_1 and γ_2 (specifically under the exchange of the sets $(\hat{k}, \vec{\epsilon}_1^{\,*})$ and $(-\hat{k}, \vec{\epsilon}_2^{\,*})$), and since the state with $\lambda = -2$ for the two photons can be obtained from $\lambda = +2$ state by interchanging γ_1 with γ_2, the contributions of those two states to the total width must be equal, which allows us to write

$$\frac{d\Gamma^{J^{++}M}}{d\Omega_{\vec{k}}} = \frac{d\Gamma^{J^{++}M}}{d\Omega_{\vec{k}}}(\lambda=0) \ + \ 2 \times \frac{d\Gamma^{J^{++}M}}{d\Omega_{\vec{k}}}(\lambda=2) \, . \tag{2.93}$$

Using (1.27) we then have

$$\begin{aligned}\Gamma^{J^{++}}_{\gamma\gamma} &= \frac{1}{2!}\int d\Omega_{\vec{k}}\left\{\frac{d\Gamma^{J^{++}M}}{d\Omega_{\vec{k}}}(\lambda=0) \ + \ 2\times\frac{d\Gamma^{J^{++}M}}{d\Omega_{\vec{k}}}(\lambda=2)\right\} \\ &= \Gamma^{J^{++}}_{\gamma\gamma}(\lambda=0) \ + \ 2\times\Gamma^{J^{++}}_{\gamma\gamma}(\lambda=2) \, . \end{aligned} \tag{2.94}$$

Using (2.90), (2.91) and (2.94) we obtain

$$\begin{aligned}\Gamma^{J^{++}}_{\gamma\gamma} &= \frac{2\pi}{2J+1}\left\{\frac{d\Gamma^{J^{++}0}}{d\Omega_{\vec{k}}}(\lambda=0,\hat{k}=\hat{z}) \ + \ 2\times\frac{d\Gamma^{J^{++}2}}{d\Omega_{\vec{k}}}(\lambda=2,\hat{k}=\hat{z})\right\} \\ &\equiv \Gamma^{J^{++}}_{\gamma\gamma}(0) \ + \ \Gamma^{J^{++}}_{\gamma\gamma}(2) \, , \end{aligned} \tag{2.95}$$

where the helicity zero and helicity two contributions to $\Gamma^{J^{++}}_{\gamma\gamma}$ are the first and second terms respectively. Now, according to (2.92) and (2.95)

$$\begin{aligned}\Gamma^{J^{++}}_{\gamma\gamma}(0) &= \left(\frac{2\pi}{2J+1}\right)\frac{1}{32\pi^2}\left|\mathcal{M}(\lambda_1,\lambda_2;\hat{k}=\hat{z})\right|^2 \\ &= \frac{1}{16\pi(2J+1)}\left|\frac{1}{\sqrt{2}}\left\{\mathcal{M}(+,+;\hat{z}) \ + \ \mathcal{M}(-,-;\hat{z})\right\}\right|^2 . \end{aligned} \tag{2.96}$$

Inspection of $\mathcal{M}(\lambda_1,\lambda_2;\hat{k})$ in (1.28) and $\Phi^{tot}_{JM}(\vec{p}) \otimes \mathcal{M}_{fi}(s=1)$ in (2.57) shows that $\mathcal{M}(\lambda_1,\lambda_2;\hat{k})$ depends on λ_1,λ_2 and $\hat{k}$ through $\vec{\mathcal{B}}$. Note that exchanging γ_1 with γ_2 changes $|+,+\rangle$ to $|-,-\rangle$, and the fact that $\vec{\mathcal{B}}$ is symmetric under this operation ($\vec{\epsilon}_1^* \leftrightarrow \vec{\epsilon}_2^*$ and $\hat{k} \to -\hat{k}$) leads to $\mathcal{M}(+,+;\hat{k}) = \mathcal{M}(-,-;\hat{k})$, so (2.96) becomes

$$\Gamma_{\gamma\gamma}^{J^{++}}(0) = \frac{1}{8\pi(2J+1)} \Big|\mathcal{M}(+,+;\hat{z})\Big|^2 . \tag{2.97}$$

Using (1.28) and (2.57), one has in the CM frame

$$\mathcal{M}(+,+;\hat{z}) = \frac{-\sqrt{2}}{\sqrt{(2\pi)^3}} \int d\vec{p}\, \frac{m}{E} \Big\{ \mathcal{C}^{l\otimes 1}_{-1,1}(J)\, \Phi_{l,-1}(\vec{p})\, \hat{e}_+$$

$$+\, \mathcal{C}^{l\otimes 1}_{0,0}(J)\, \Phi_{l,0}(\vec{p})\, \hat{z} \,+\, \mathcal{C}^{l\otimes 1}_{1,-1}(J)\, \Phi_{l,+1}(\vec{p})\, \hat{e}_- \Big\} \cdot \vec{\mathcal{B}}(+,+;\hat{z}) \,, \tag{2.98}$$

where $\vec{\mathcal{B}}(+,+;\hat{z})$ denotes $\vec{\mathcal{B}}$ with both photons polarized along their directions of motion. We have taken the photon direction of motion to be the z-axis, $\hat{k}_1 \equiv \hat{k} = \hat{z}$, and have abbreviated the Clebsch-Gordan coefficients for $J = l \otimes s = l \otimes 1$ as

$$\mathcal{C}^{l\otimes 1}_{l_z,s_z}(J) \equiv \langle l, l_z; 1, s_z | J, M\rangle \,. \tag{2.99}$$

Note that although we set $M = 0$, and later for $\lambda = 2$ we will set $M = 2$, it is unnecessary to label $\mathcal{C}^{l\otimes 1}_{m,s_z}(J)$ with M since $M = l_z + s_z = m + s_z$. Similarly, (2.92) and (2.95) give

$$\Gamma_{\gamma\gamma}^{J^{++}}(2) = \Big(\frac{4\pi}{2J+1}\Big) \frac{1}{32\pi^2} \Big|\mathcal{M}(\lambda = +2; \hat{k} = \hat{z})\Big|^2$$

$$= \frac{1}{8\pi(2J+1)} \Big|\mathcal{M}(+,-;\hat{z})\Big|^2 . \tag{2.100}$$

Again using (1.28) and (2.57), and setting $M = \lambda = 2$, we arrive at

$$\mathcal{M}(+,-;\hat{z}) = \frac{-\sqrt{2}}{\sqrt{(2\pi)^3}} \int d\vec{p}\, \frac{m}{E} \Big\{ \mathcal{C}^{l\otimes 1}_{1,1}(J)\, \Phi_{l,1}(\vec{p})\, \hat{e}_+$$

$$+\mathcal{C}^{l\otimes 1}_{2,0}(J)\,\Phi_{l,2}(\vec{p})\,\hat{z}\;+\;\mathcal{C}^{l\otimes 1}_{3,-1}(J)\,\Phi_{l,3}(\vec{p})\,\hat{e}_-\Big\}\cdot\vec{\mathcal{B}}(+,-;\hat{z})\;, \quad (2.101)$$

where $\vec{\mathcal{B}}(+,-;\hat{z})$ denotes a first photon with $\vec{k}_1$ along $+\hat{z}$ and both photons polarized along $+\hat{z}$. So, thus far we have succeeded in solving for the helicity components of the width by determining the decay amplitudes for photons moving only along the z-axis. We now have to evaluate the integrals in (2.98) and (2.101) for both helicities. We first rewrite the vector $\vec{\mathcal{B}}(\lambda_1,\lambda_2)$ (with $\hat{k}=\hat{z}$ implied here and subsequently). Using $\hat{p}\cdot\hat{k}=cos\theta\equiv\mu$, and $\frac{\beta^2 E}{E+m}=1-\frac{m}{E}$ in (2.41), we obtain

$$\vec{\mathcal{B}}(\lambda_1,\lambda_2)=\frac{ie^2}{m}\frac{\beta}{(1-\beta^2\cos^2\theta)}\Big\{\hat{z}\,\vec{\epsilon}_1^{\,*}\cdot\vec{\epsilon}_2^{\,*}\,\mu+\vec{\epsilon}_1^{\,*}\,\vec{\epsilon}_2^{\,*}\cdot\hat{p}\;+\;\vec{\epsilon}_2^{\,*}\,\vec{\epsilon}_1^{\,*}\cdot\hat{p}$$

$$-\hat{p}\left(1-\frac{m}{E}\right)\left(2\,\vec{\epsilon}_1^{\,*}\cdot\hat{p}\,\vec{\epsilon}_2^{\,*}\cdot\hat{p}+\vec{\epsilon}_1^{\,*}\cdot\vec{\epsilon}_2^{\,*}\mu^2\right)\Big\}\,. \quad (2.102)$$

Now for a specific case (λ_1,λ_2) we require the appropriate $\vec{\epsilon}_1^{\,*}$ and $\vec{\epsilon}_2^{\,*}$ for photons travelling along $\pm\hat{z}$. For a photon with $\hat{k}=\hat{z}$, the polarization vector is[33]

$$\vec{\epsilon}_1(\lambda_1=\pm)=\hat{e}_{\pm}\,. \quad (2.103)$$

For the other photon with $\hat{k}=-\hat{z}$, by applying a rotation about $\hat{y}$ by π to the first photon, we obtain[33]

$$\vec{\epsilon}_2(\lambda_2=\pm)=\hat{e}_{\mp}\times e^{i\alpha_{\mp}}\,. \quad (2.104)$$

The factors $e^{i\alpha_{\pm}}$ are possible phases generated by the rotation. Note however that these phases do no affect our calculation, because for each of the two helicity cases we have a specific value for λ_2, and since $\vec{\epsilon}_2$ occurs only once in each term of $\vec{\mathcal{B}}$ (and thus only once in each term of $\mathcal{M}$) we can factor out the overall phase for a given helicity, and can simply use $\hat{e}_\pm$ for $\vec{\epsilon}_2$. The overall phase is irrelevant in the decay rate, which is proportional to $|\mathcal{M}|^2$. So here we may set

$$\vec{\epsilon}_2(\lambda_2 = \pm) = \hat{e}_\mp \, . \tag{2.105}$$

Several properties of $\hat{e}_\pm$ and $\hat{z}$ listed below will be useful in determining $\vec{\mathcal{B}}$ for each helicity;

$$\hat{e}_+ \cdot \hat{e}_+ = \hat{e}_- \cdot \hat{e}_- = \hat{e}_\pm \cdot \hat{z} = 0 \, , \tag{2.106}$$

$$\hat{e}^*_+ = -\hat{e}_-, \; \Rightarrow \; \hat{e}_+ \cdot \hat{e}^*_+ = \hat{e}_- \cdot \hat{e}^*_- = -\hat{e}_+ \cdot \hat{e}_- = 1 \, . \tag{2.107}$$

The helicity zero ($\lambda = 0$) case

Here we are interested in $\vec{\mathcal{B}}(+,+)$. For this we use

$$\vec{\epsilon}_1 = \vec{\epsilon}_1(\lambda_1 = +) = \hat{e}_+ \qquad \text{and} \qquad \vec{\epsilon}_2 = \vec{\epsilon}_2(\lambda_2 = +) = \hat{e}_- \, , \tag{2.108}$$

and therefore

$$\vec{\epsilon}^{\,*}_1 = -\hat{e}_- \qquad \text{and} \qquad \vec{\epsilon}^{\,*}_2 = -\hat{e}_+ \, . \tag{2.109}$$

Using these expressions and the relations (2.106) and (2.107), we obtain

$$\vec{\mathcal{B}}_0 \equiv \vec{\mathcal{B}}(+,+;\hat{z}) =$$

$$\frac{ie^2}{m}\frac{\beta}{(1-\beta^2\cos^2\theta)}\Bigg\{\hat{z}\,\hat{e}_-\cdot\hat{e}_+\,\mu\;+\hat{e}_-\,\hat{e}_+\cdot\hat{p}\;+\;\hat{e}_+\,\hat{e}_-\cdot\hat{p}$$

$$-\hat{p}\left(1-\frac{m}{E}\right)\left(2\,\hat{e}_-\cdot\hat{p}\,\hat{e}_+\cdot\hat{p}+\hat{e}_-\cdot\hat{e}_+\mu^2\right)\Bigg\}\,, \qquad (2.110)$$

which together with the identity

$$-2\hat{p}\cdot\hat{e}_+\,\hat{p}\cdot\hat{e}_-\;\equiv -2\hat{p}_+\cdot\hat{p}_-\;=\left(\hat{p}_x^2\;+\hat{p}_y^2\right)=1-\hat{p}_z^2=1-\mu^2\,, \qquad (2.111)$$

lead to the relatively simple form

$$\vec{\mathcal{B}}_0=\frac{ie^2}{m}\frac{\beta}{(1-\beta^2\cos^2\theta)}$$

$$\times\left\{-\mu\,\hat{z}+\hat{e}_-\,\hat{e}_+\cdot\hat{p}\;+\;\hat{e}_+\,\hat{e}_-\cdot\hat{p}\;+\;\hat{p}\left(1-\frac{m}{E}\right)\right\}. \qquad (2.112)$$

In (2.98) for $\mathcal{M}(+,+)$ one needs $\hat{e}_+\cdot\vec{\mathcal{B}}(+,+)$, $\hat{z}\cdot\vec{\mathcal{B}}(+,+)$ and $\hat{e}_-\cdot\vec{\mathcal{B}}(+,+)$. Using (2.112) and the relations (2.106) and (2.107), one has

$$\hat{e}_+\cdot\vec{\mathcal{B}}_0=\frac{-ie^2}{m}\;\frac{\beta}{1-\beta^2\mu^2}\;\hat{p}\cdot\hat{e}_+\,, \qquad (2.13)$$

$$\hat{z}\cdot\vec{\mathcal{B}}_0=\frac{-ie^2}{m}\;\frac{\beta}{1-\beta^2\mu^2}\;\hat{p}\cdot\hat{z} \qquad (2.114)$$

and

$$\hat{e}_-\cdot\vec{\mathcal{B}}_0=\frac{-ie^2}{m}\;\frac{\beta}{1-\beta^2\mu^2}\;\hat{p}\cdot\hat{e}_-\,. \qquad (2.115)$$

With the explicit $Y_{1,m}(\theta,\phi)$,

$$Y_{1,\pm1}(\Omega)=\mp\sqrt{\frac{3}{8\pi}}\;sin\theta\;e^{\pm i\phi} \qquad (2.116)$$

and

$$Y_{1,0}(\Omega) = \sqrt{\frac{3}{4\pi}}\, cos\theta\ , \tag{2.117}$$

we find that

$$\hat{p}\cdot\hat{e}_{\pm} = \frac{\mp 1}{\sqrt{2}}(\hat{p}_x \pm i\hat{p}_y) = \frac{\mp 1}{\sqrt{2}}\, sin\theta\ e^{\pm i\phi} = \sqrt{\frac{4\pi}{3}}\, Y_{1,\pm 1}(\Omega) \tag{2.118}$$

and

$$\hat{p}\cdot\hat{z} = cos\theta \equiv \mu = \sqrt{\frac{4\pi}{3}}\, Y_{1,0}(\Omega)\ . \tag{2.119}$$

From (2.113)-(2.115) (2.118)-(2.119) we find

$$\hat{e}_{+}\cdot\vec{\mathcal{B}}_0 = -\sqrt{\frac{4\pi}{3}}\ \frac{ie^2}{m}\ \frac{\beta}{1-\beta^2\mu^2}\, Y_{1,1}(\Omega)\ , \tag{2.120}$$

$$\hat{z}\cdot\vec{\mathcal{B}}_0 = -\sqrt{\frac{4\pi}{3}}\ \frac{ie^2}{m}\ \frac{\beta}{1-\beta^2\mu^2}\, Y_{1,0}(\Omega) \tag{2.121}$$

and

$$\hat{e}_{-}\cdot\vec{\mathcal{B}}_0 = -\sqrt{\frac{4\pi}{3}}\ \frac{ie^2}{m}\ \frac{\beta}{1-\beta^2\mu^2}\, Y_{1,-1}(\Omega)\ . \tag{2.122}$$

Using (2.120)-(2.122) in (2.98) for $\mathcal{M}(+,+;\hat{z})$, we have

$$\mathcal{M}_0 \equiv \mathcal{M}(+,+;\hat{z}) = -\sqrt{\frac{2}{(2\pi)^3}}\int d\vec{p}\ \ \frac{m}{E}\left(-\sqrt{\frac{4\pi}{3}}\frac{ie^2\beta}{E(1-\beta^2\mu^2)}\right)$$

$$\times\Big\{\mathcal{C}^{l\otimes 1}_{-1,1}(J)\,\Phi_{l,-1}(\vec{p})Y_{1,1}(\Omega) + \ \mathcal{C}^{l\otimes 1}_{0,0}(J)\,\Phi_{l,0}(\vec{p})\,Y_{1,0}(\Omega)$$

$$+\ \mathcal{C}^{l\otimes 1}_{1,-1}(J)\,\Phi_{l,+1}(\vec{p})\,Y_{1,-1}(\Omega)\Big\}\ . \tag{2.123}$$

Now expanding the denominator using

$$\frac{1}{x^2-\mu^2} = \frac{1}{x}\sum_{\substack{j=0\\ \text{even}}}^{\infty} (2j+1)\, Q_j(x)\, P_j(\mu)\,, \tag{2.124}$$

we may write

$$\frac{1}{1-\beta^2\mu^2} = \sum_{\substack{j=0\\ \text{even}}}^{\infty} \sqrt{4\pi(2j+1)}\,\beta^{-1}\, Q_j(\beta^{-1})\, Y_{j,0}(\Omega)\,. \tag{2.125}$$

Separating the angular dependence, and using (2.123) and (2.125) and the identity $E^{-2} = \frac{1-\beta^2}{m^2}$, we obtain

$$\begin{aligned}\mathcal{M}_0 = \frac{ie^2}{\pi m}\sqrt{\frac{4\pi}{3}}\int dp\, p^2\ \phi_l(p)\,\left(1-\beta^2\right)\\ \times\sum_{\substack{j=0\\ \text{even}}}^{\infty} \sqrt{2j+1}\, Q_j(\beta^{-1})\int\ d\Omega\ Y^*_{j,0}\ \Big\{ \mathcal{C}^{l\otimes 1}_{-1,1}(J)\, Y_{l,-1}(\Omega)\ Y_{1,1}(\Omega)\\ + \mathcal{C}^{l\otimes 1}_{0,0}(J)\, Y_{l,0}(\Omega)\, Y_{1,0}(\Omega) + \ \mathcal{C}^{l\otimes 1}_{1,-1}(J)\, Y_{l,+1}(|Om)\, Y_{1,-1}(\Omega)\Big\}\,.\end{aligned} \tag{2.126}$$

To evaluate the angular integrals we use the Gaunt formula[49]

$$\begin{aligned}\int d\Omega\ Y^*_{l_3,m_3}(\Omega)\ Y_{l_2,m_2}(\Omega)\ Y_{l_1,m_1}(\Omega) =\\ \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l_3+1)}}\ \langle l_1, m_1; l_2, m_2|l_3, m_3\rangle\ \langle l_1, 0; l_2, 0|l_3, 0\rangle\,.\end{aligned} \tag{2.127}$$

Applying this to our case, we have $l_3 = j$ and $m_3 = 0$, and we define $l \equiv l_2$ and $l' = l_1$, which gives

$$\int d\Omega\ Y^*_{j,0}(\Omega)\ Y_{l,m}(\Omega)\ Y_{l',m'}(\Omega) =$$

$$\sqrt{\frac{(2l'+1)(2l+1)}{4\pi(2j+1)}}\ \langle l,m;l',m'|j,0\rangle\ \langle l,0;l',0|j,0\rangle\ . \tag{2.129}$$

By using properties of the Clebsch-Gordan coefficients we can infer several useful properties of these integrals. Note first that[49]

$$\langle l,m;l',m'|j,j_z\rangle = (-1)^{l+l'-j}\langle l,-m;l',-m'|j,-j_z\rangle\ , \tag{2.130}$$

which when applied to the $\langle l,0;l',0|j,0\rangle$ in (2.129) enforces the requirement that

$$l+l'-j = \begin{cases} odd, & \text{integral vanishes;} \\ even, & \text{integral may be nonzero.} \end{cases} \tag{2.131}$$

In our case $l' = 1$ in (2.126), and $j = even$. These together imply the following;

$$l = \begin{cases} even, & \mathcal{M}_0 \text{ vanishes;} \\ odd, & \mathcal{M}_0 \text{ may be nonzero.} \end{cases} \tag{2.132}$$

This result agrees with expectation that $\Gamma_{\gamma\gamma} = 0$ for $l = even$ in the spin-triplet case, since this implies $C = -1$, which cannot couple to $\gamma\gamma$ states. Also for $l = odd$ all the integrals in (2.126) may be nonzero, with one simplification given by (2.130),

$$\int d\Omega\ Y^*_{j,0}(\Omega)\ Y_{l,m}(\Omega)\ Y_{l',m'}(\Omega) = \int d\Omega\ Y^*_{j,0}(\Omega)\ Y_{l,-m}(\Omega)\ Y_{l',-m'}\ . \tag{2.133}$$

Since here $l' = 1 = s$ for all three integrals in (2.126) we may rewrite (2.129) using our definition (2.99) for $C^{l\otimes 1}_{m,s_z}$ as

$$\int d\Omega\ Y^*_{j,0}(\Omega)\ Y_{l,m}(\Omega)\ Y_{l',m'}(\Omega) =$$

$$\sqrt{\frac{(2l'+1)(2l+1)}{4\pi(2j+1)}}\, \mathcal{C}^{l\otimes 1}_{m,m'}(j)\, \mathcal{C}^{l\otimes 1}_{0,0}(j)\,. \tag{2.134}$$

Substituting the results (2.134) into (2.126), and using (2.133) gives

$$\mathcal{M}_0 = \frac{ie^2}{\pi m}\sqrt{2l+1}\int dp\, p^2\, \phi_l(p)\left(1-\beta^2\right)$$

$$\times \sum_{\substack{j=0\\ \text{even}}}^{\infty} Q_j(\beta^{-1})\, \mathcal{C}^{l\otimes 1}_{0,0}(j)\Big\{\mathcal{C}^{l\otimes 1}_{1,-1}(j)\Big[\mathcal{C}^{l\otimes 1}_{-1,1}(J) + \mathcal{C}^{l\otimes 1}_{1,-1}(J)\Big]$$

$$+\, \mathcal{C}^{l\otimes 1}_{0,0}(j)\, \mathcal{C}^{l\otimes 1}_{0,0}(J)\Big\}\,. \tag{2.135}$$

Since our summation index j is even, the quantum number l is odd, and since $\mathcal{C}^{l\otimes 1}_{m,m'}(j)$ vanishes unless $j \in \{l-1, l, l+1\}$, it follows that j takes on only the two values

$$j = l-1 \qquad \text{and} \qquad j = l+1\,. \tag{2.136}$$

(2.135) and (2.136) give the following expression for $\mathcal{M}_o$

$$\mathcal{M}_0 = \frac{ie^2}{\pi m}\sqrt{2l+1}\int dp\, p^2\, \phi_l(p)\left(1-\beta^2\right)$$

$$\times\Big\{\mathcal{T}_{l-1}(J)\, Q_{l-1}(\beta^{-1}) + \mathcal{T}_{l+1}\, Q_{l+1}(\beta^{-1})\Big\}\,, \tag{2.137}$$

where we have defined

$$\mathcal{T}_j(J) \equiv \mathcal{C}^{l\otimes 1}_{0,0}(j)\Big\{\mathcal{C}^{l\otimes 1}_{1,-1}(j)$$

$$\times\Big[\mathcal{C}^{l\otimes 1}_{1,-1}(J) + \mathcal{C}^{l\otimes 1}_{-1,1}(J)\Big] + \mathcal{C}^{l\otimes 1}_{0,0}(j)\, \mathcal{C}^{l\otimes 1}_{0,0}(J)\Big\}\,. \tag{2.138}$$

We now require $\mathcal{M}_0$ for the three possible values of the total angular momentum J for given l. Actually we already know the result for one case, $J = l$; $\mathcal{M}_0$ must vanish because there is no helicity zero state for two photons. This provides a check of our general result (2.137). From **Table 2.1**, we see that

$$\mathcal{C}^{l\otimes 1}_{1,-1}(J=l) = -\mathcal{C}^{l\otimes 1}_{-1,1}(J=l)\,, \tag{2.139}$$

and

$$\mathcal{C}^{l\otimes 1}_{0,0}(J=l) = 0\,, \tag{2.140}$$

which when substituted into (2.138) give

$$\mathcal{T}_j(J=l) = 0 \qquad \forall\, j \in \{l-1, l+1\}\,, \tag{2.141}$$

and therefore

$$\Gamma^{J++}_{\gamma\gamma}(\lambda=0; J=l) = 0\,, \tag{2.142}$$

as required. For the two nontrivial cases $J = l \pm 1$ we have from **Table 2.1**

$$\mathcal{C}^{l\otimes 1}_{1,-1}(J=l\pm 1) = \mathcal{C}^{l\otimes 1}_{-1,1}(J=l\pm 1)\,; \tag{2.143}$$

we may therefore rewrite $\mathcal{T}_j(J)$ for these cases as

$$\mathcal{T}_j(J=l\pm 1) \equiv \mathcal{C}^{l\otimes 1}_{0,0}(j)\Big\{2\mathcal{C}^{l\otimes 1}_{1,-1}(j)\mathcal{C}^{l\otimes 1}_{1,-1}(J) \;+\; \mathcal{C}^{l\otimes 1}_{0,0}(j)\,\mathcal{C}^{l\otimes 1}_{0,0}(J)\Big\}\,. \tag{2.144}$$

The values of the Clebsch-Gordan coefficients required are given in **Table 2.1**;

$$\mathcal{C}^{l\otimes 1}_{0,0}(l-1) = -\sqrt{\frac{l}{2l+1}}\,, \tag{2.145}$$

Table 2.1. Clebsch-Gordan coefficients for $l \otimes 1$ [48].

$\mathcal{C}^{l\otimes 1}_{m,s_z}(J)$	$s_z = +1$	$s_z = 0$	$s_z = -1$
$J = l+1$	$\sqrt{\frac{(l+M)(l+M+1)}{(2l+1)(2l+2)}}$	$\sqrt{\frac{(l-M+1)(l+M+1)}{(2l+1)(l+1)}}$	$\sqrt{\frac{(l-M)(l-M+1)}{(2l+1)(2l+2)}}$
$J = l$	$-\sqrt{\frac{(l+M)(l-M+1)}{2l(l+1)}}$	$\frac{M}{\sqrt{l(l+1)}}$	$\sqrt{\frac{(l-M)(l+M+1)}{2l(l+1)}}$
$J = l-1$	$\sqrt{\frac{(l-M)(l-M+1)}{2l(2l+1)}}$	$-\sqrt{\frac{(l-M)(l+M)}{l(2l+1)}}$	$\sqrt{\frac{(l+M+1)(l+M)}{2l(2l+1)}}$

$$\mathcal{C}^{l\otimes 1}_{0,0}(l+1) = \sqrt{\frac{l+1}{2l+1}}\,, \tag{2.146}$$

$$\mathcal{C}^{l\otimes 1}_{1,-1}(l-1) = -\sqrt{\frac{l+1}{2(2l+1)}} \tag{2.147}$$

and

$$\mathcal{C}^{l\otimes 1}_{1,-1}(l+1) = -\sqrt{\frac{l}{2(2l+1)}}\,. \tag{2.148}$$

After substituting these into (2.144), we see that

$$\mathcal{T}_{l-1}(J = l-1) = -\sqrt{\frac{l}{2l+1}}\,, \tag{2.149}$$

$$\mathcal{T}_{l+1}(J = l-1) = 0\,, \tag{2.150}$$

$$\mathcal{T}_{l-1}(J = l+1) = 0 \tag{2.151}$$

and

$$\mathcal{T}_{l+1}(J = l+1) = \sqrt{\frac{l+1}{2l+1}}\,, \tag{2.152}$$

These give for $\mathcal{M}_0$

$$\mathcal{M}_0(J=l+1) = \frac{ie^2}{\pi m}\sqrt{l+1}\int dp\; p^2\;\phi_l(p)\left(1-\beta^2\right)Q_{l+1}(\beta^{-1})$$

$$= \frac{ie^2}{\pi m}\sqrt{J}\int dp\; p^2\;\phi_l(p)\left(1-\beta^2\right)Q_J(\beta^{-1}) \tag{2.153}$$

and

$$\mathcal{M}_0(J=l-1) = \frac{-ie^2}{\pi m}\sqrt{l}\int dp\; p^2\;\phi_l(p)\left(1-\beta^2\right)Q_{l-1}(\beta^{-1})$$

$$= \frac{-ie^2}{\pi m}\sqrt{J+1}\int dp\; p^2\;\phi_l(p)\left(1-\beta^2\right)Q_J(\beta^{-1})\,. \tag{2.154}$$

To obtain the widths from (2.153) and (2.154), we make use of (2.97), which leads to the relatively simple forms

$$\Gamma^{J++}_{\gamma\gamma}(\lambda=0;J=l+1) = \frac{J}{2J+1}\frac{2\alpha^2}{\pi m^2}$$

$$\times\left|\int dp\; p^2\;\phi_l(p)\left(1-\beta^2\right)Q_J(\beta^{-1})\right|^2 \tag{2.155}$$

and

$$\Gamma^{J++}_{\gamma\gamma}(\lambda=0;J=l-1) = \frac{J+1}{2J+1}\frac{2\alpha^2}{\pi m^2}$$

$$\times\left|\int dp\; p^2\;\phi_l(p)\left(1-\beta^2\right)Q_J(\beta^{-1})\right|^2. \tag{2.156}$$

The helicity two ($\lambda = 2$) case

This involves $\vec{\mathcal{B}}(+,-)$. Thus we use

$$\vec{\epsilon}_1 = \vec{\epsilon}_1(\lambda_1 = +) = \hat{e}_+ \qquad \vec{\epsilon}_2 = \vec{\epsilon}_2(\lambda_2 = -) = \hat{e}_+ \, , \tag{2.157}$$

which together with (2.107) lead to

$$\vec{\epsilon}_1^{\,*} = \vec{\epsilon}_2^{\,*} = -\hat{e}_- \, . \tag{2.158}$$

Using this expression for the polarization vectors in (2.41) and the relations (2.106) and (2.107), we obtain

$$\vec{\mathcal{B}}_2 \equiv \vec{\mathcal{B}}(+,-;\hat{z}) = \frac{2ie^2}{m} \frac{\beta}{(1-\beta^2\mu^2)} \, \hat{p}\cdot\hat{e}_- \left\{ \hat{e}_- - \left(1 - \frac{m}{E}\right) \hat{p}\, \hat{p}\cdot\hat{e}_- \right\} . \tag{2.159}$$

For $\mathcal{M}(+,-)$ in (2.101) one requires $\hat{e}_\pm \cdot \vec{\mathcal{B}}_2$ and $\hat{z}\cdot\vec{\mathcal{B}}_2$; using (2.159) and the relations (2.106)-(2.107), one has

$$\hat{e}_+ \cdot \vec{\mathcal{B}}_2 = \frac{-ie^2}{m} \frac{\beta}{1-\beta^2\mu^2} \hat{p}\cdot\hat{e}_- \left\{ 1 + \frac{m}{E} + \left(1 - \frac{m}{E}\right) \mu^2 \right\} , \tag{2.160}$$

$$\hat{z}\cdot\vec{\mathcal{B}}_2 = \frac{-ie^2}{m} \left(1 - \frac{m}{E}\right) \mu \left(\hat{p}\cdot\hat{e}_-\right)^2 \tag{2.161}$$

and

$$\hat{e}_- \cdot \vec{\mathcal{B}}_2 = \frac{-ie^2}{m} \left(1 - \frac{m}{E}\right) \left(\hat{p}\cdot\hat{e}_-\right)^3 . \tag{2.162}$$

Using explicit $Y_{3,m}(\theta,\phi)$ spherical harmonics

$$Y_{3,-1}(\Omega) = \frac{1}{4}\sqrt{\frac{21}{4\pi}} \; sin\theta \; \left(5\, cos^2\theta - 1\right) e^{-i\phi} \, , \tag{2.163}$$

$$Y_{3,-2}(\Omega) = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\ sin^2\theta\ cos\theta\ e^{-2i\phi}\ , \tag{2.164}$$

and

$$Y_{3,-3}(\Omega) = \frac{1}{4}\sqrt{\frac{35}{4\pi}}\ sin^3\theta\ e^{-3i\phi}\ , \tag{2.165}$$

one can rewrite (2.160)-(2.162) as

$$\hat{e}_+ \cdot \vec{\mathcal{B}}_2 = \frac{-ie^2}{m}\frac{\beta}{1-\beta^2\mu^2}\,\frac{2}{5}\sqrt{\frac{4\pi}{3}}$$

$$\times\left\{\left(3+2\frac{m}{E}\right)Y_{1,-1}(\Omega)\ +\ \sqrt{\frac{2}{7}}\left(1-\frac{m}{E}\right)Y_{3,-1}(\Omega)\right\}\ , \tag{2.166}$$

$$\hat{z} \cdot \vec{\mathcal{B}}_2 = \frac{-ie^2}{m}\frac{\beta}{1-\beta^2\mu^2}\ \sqrt{\frac{4\pi}{3}}\left(1-\frac{m}{E}\right)\sqrt{\frac{8}{35}}\ Y_{3,-2}(\Omega) \tag{2.167}$$

and

$$\hat{e}_- \cdot \vec{\mathcal{B}}_2 = \frac{-ie^2}{m}\frac{\beta}{1-\beta^2\mu^2}\sqrt{\frac{4\pi}{3}}\left(1-\frac{m}{E}\right)\sqrt{\frac{24}{35}}\ Y_{3,-3}(\Omega)\ . \tag{2.168}$$

Now we are prepared to evaluate $\mathcal{M}(+,-;\hat{z})$. Using (2.166)-(2.167) in (2.101), we have

$$\mathcal{M}(+,-;\hat{z}) = \frac{-\sqrt{2}}{\sqrt{(2\pi)^3}}\int d\vec{p}\ \frac{m}{E}\left(\frac{-ie^2}{m}\frac{\beta}{1-\beta^2\mu^2}\right)\sqrt{\frac{4\pi}{3}}$$

$$\times\left\{\frac{2}{5}\mathcal{C}^{l\otimes 1}_{1,1}(J)\left[\left(3+2\frac{m}{E}\right)Y_{1,-1}(\Omega)+\sqrt{\frac{2}{7}}\left(1-\frac{m}{E}\right)Y_{3,-1}(\Omega)\right]\Phi_{l,1}(\vec{p})\right.$$

$$+\sqrt{\frac{8}{35}}\left(1-\frac{m}{E}\right)\mathcal{C}^{l\otimes 1}_{2,0}(J)\,Y_{3,-2}(\Omega)\,\Phi_{l,2}(\vec{p})$$

$$\left.+\sqrt{\frac{24}{35}}\left(1-\frac{m}{E}\right)\mathcal{C}^{l\otimes 1}_{3,-1}(J)\,Y_{3,-3}(\Omega)\,\Phi_{l,3}(\vec{p})\right\}\ . \tag{2.169}$$

Employing (2.125) and (2.169) as before gives

$$\mathcal{M}(+,-;\hat{z}) = \frac{ie^2}{\pi}\sqrt{\frac{4\pi}{3}}\int dp\; p^2\;\phi_l(p)\,\frac{1}{E}\sum_{\substack{j=0\\ \text{even}}}^{\infty}\sqrt{2j+1}\,Q_j(\beta^{-1})$$

$$\times\int d\Omega\, Y^*_{j,0}(\Omega)\left\{\frac{2}{5}\left(3+2\frac{m}{E}\right)\mathcal{C}^{l\otimes 1}_{1,\bar{1}}(J)\,Y_{l,1}(\Omega)\,Y_{1,-1}(\Omega)\right.$$

$$+\left[\sqrt{\frac{8}{175}}\,\mathcal{C}^{l\otimes 1}_{1,\bar{1}}(J)\,Y_{l,1}(\Omega)\,Y_{3,-1}(\Omega)\;+\;\sqrt{\frac{8}{35}}\,\mathcal{C}^{l\otimes 1}_{2,\bar{0}}(J)\,Y_{l,2}(\Omega)\,Y_{3,-2}(\Omega)\right.$$

$$\left.\left.+\sqrt{\frac{24}{35}}\,\mathcal{C}^{l\otimes 1}_{3,-1}(J)\,Y_{l,3}(\Omega)\,Y_{3,-3}(\Omega)\right]\right\}\,. \quad (2.170)$$

According to the arguments that led to (2.131) one has

$$l = \begin{cases} even, & \mathcal{M}(+,-;\hat{z}) \text{ vanishes;} \\ odd, & \mathcal{M}(+,-;\hat{z}) \text{ survives,} \end{cases} \quad (2.171)$$

which agrees with our expectation that $\Gamma_{\gamma\gamma} = 0$ for an $l = even$ spin-singlet state, since this implies $C = -1$ which cannot couple to $\gamma\gamma$ states. These arguments also imply that for $l = odd$ none of the angular integrals in (2.170) vanish, so they must all be evaluated. Using (2.127) and (2.128), we find two types of integrals; one with $l' = 1$ which was previously discussed (2.134), and a new case with $l' = 3$, which is

$$\int d\Omega\; Y^*_{j,0}(\Omega)\; Y_{l,m}(\Omega)\; Y_{3,m'}(\Omega) = \sqrt{\frac{7(2l+1)}{4\pi(2j+1)}}\;\mathcal{C}^{l\otimes 3}_{m,m'}(j)\,\mathcal{C}^{l\otimes 3}_{0,0}(j)\,. \quad (2.172)$$

Here we have defined the Clebsch-Gordan coefficients of the $j = l\otimes l' = l\otimes 3$ by

$$\mathcal{C}^{l\otimes 3}_{l_z,l'_z}(j) \equiv \langle l, l_z; 3, l'_z | j, j_z\rangle\,. \quad (2.173)$$

After performing the angular integrals we have from (2.170)

$$\mathcal{M}_2 = \frac{ie^2}{\pi}\sqrt{2l+1}\int dp\, p^2\, \phi_l(p)\, \frac{1}{E}$$

$$\Bigg\{\frac{2}{5}\Big(3+2\,\frac{m}{E}\Big)\mathcal{C}^{l\otimes 1}_{1,1}(J)\sum_{\substack{j=0\\ \text{even}}}^{\infty} Q_j(\beta^{-1})\,\mathcal{C}^{l\otimes 1}_{1,-1}(j)\,\mathcal{C}^{l\otimes 1}_{0,0}(j)$$

$$+\sqrt{\frac{8}{75}}\Big(1-\frac{m}{E}\Big)\sum_{\substack{j=0\\ \text{even}}}^{\infty} Q_j(\beta^{-1})\,\mathcal{C}^{l\otimes 3}_{0,0}(j)\,\Big[\mathcal{C}^{l\otimes 1}_{1,1}(J)\,\mathcal{C}^{l\otimes 3}_{1,-1}(j)$$

$$+\sqrt{5}\,\mathcal{C}^{l\otimes 1}_{2,0}(J)\,\mathcal{C}^{l\otimes 3}_{2,-2}(j) \;+\; \sqrt{15}\,\mathcal{C}^{l\otimes 1}_{3,-1}(J)\,\mathcal{C}^{l\otimes 3}_{3,-3}(j)\Big]\Bigg\}$$

$$= \frac{ie^2}{\pi}\sqrt{2l+1}\int dp\, p^2\, \phi_l(p)\, \frac{1}{E}$$

$$\times\Bigg\{\frac{2}{5}\Big(3+2\,\frac{m}{E}\Big)\Big[\mathcal{F}'_{l-1}(J)\,Q_{l-1}(\beta^{-1}) \;+\; \mathcal{F}'_{l+1}(J)\,Q_{l+1}(\beta^{-1})\Big]$$

$$+\sqrt{\frac{8}{75}}\,\Big(1-\frac{m}{E}\Big)\Big[\Big(1-\delta^l_1\Big)\,\Big(\mathcal{G}'_{l-3}(J)\,Q_{l-3}(\beta^{-1}) + \mathcal{G}'_{l-1}(J)\,Q_{l-1}(\beta^{-1})\Big)$$

$$+\,\mathcal{G}'_{l+1}(J)\,Q_{l+1}(\beta^{-1}) \;+\; \mathcal{G}'_{l+3}(J)\,Q_{l+3}(\beta^{-1})\Big]\Bigg\}\,, \qquad (2.174)$$

where we have defined

$$\mathcal{F}'_j(J) \equiv \mathcal{C}^{l\otimes 1}_{1,1}(J)\,\mathcal{C}^{l\otimes 1}_{1,-1}(j)\,\mathcal{C}^{l\otimes 1}_{0,0}(j) \qquad (2.175)$$

and

$$\mathcal{G}'_j(J) \equiv \mathcal{C}^{l\otimes 3}_{0,0}(j)\,\Big[\mathcal{C}^{l\otimes 1}_{1,1}(J)\,\mathcal{C}^{l\otimes 3}_{1,-1}(j)$$

$$+\sqrt{5}\,\mathcal{C}^{l\otimes 1}_{2,0}(J)\,\mathcal{C}^{l\otimes 3}_{2,-2}(j) \;+\; \sqrt{15}\,\mathcal{C}^{l\otimes 1}_{3,-1}(J)\,\mathcal{C}^{l\otimes 3}_{3,-3}(j)\Big]\,. \qquad (2.176)$$

The factor of $(1-\delta_1^l)$ in (2.174) derives from the composition of $j = l \otimes 3$, which implies that the angular momentum j can only assume the values $j \in \{|l-3|, \cdots, l+3\}$; in our case the orbital angular momentum l must be an odd positive integer, so for $l \geq 3$ one can remove the absolute value signs and simply sum over $j \in \{l-3, l-1, l+1, l+3\}$ (recall that $j = even$). However $l = 1$ is a special case, since only $j \in \{2, 4\}$ is allowed. In other words, of the four generally allowed values of j, for $l = 1$ j can only take on the two values $(l+1,\ l+3)$. To incorporate this special case in our general formulas we inserted a factor of $(1-\delta_1^l)$ for the other two values after removing the absolute value signs. In the first sum, however, which involves an integral over three spherical harmonics with $j = l \otimes 1$, the range of j values is $j \in \{|l-1|, \cdots, l+1\}$; since $j =$ *even* and $l \geq 1$, one can remove the absolute value signs there without introducing spurious j values.

We now have to determine 18 constants to evaluate (2.175) and (2.176) (six for each of the three possible values of J). For this we use the Clebsch-Gordan coefficients given in **Table 2.1** and **Table 2.2**, which contain $\mathcal{C}^{l \otimes 1}_{m,m'}(j)$ and $\mathcal{C}^{l \otimes 3}_{m,-m}(j)$, respectively. Before evaluating these constants, note first that **Table 2.2** implies $\mathcal{C}^{l \otimes 3}_{0,0}(l-3) = \mathcal{C}^{l \otimes 3}_{0,0}(l-1) = 0$ for $l = 1$, and since $\mathcal{G}'_j(J)$ is proportional to $\mathcal{C}^{l \otimes 3}_{0,0}(j)$, then $\forall J$ and for $l = 1$, $\mathcal{G}'_{l-3}(J) = \mathcal{G}'_{l-1}(J) = 0$. This actually implies that the $(1-\delta_1^l)$ in (2.174) is superfluous. So, we may rewrite

Table 2.2. Clebsch-Gordan coefficients for $l \otimes 3$[50].

Note, since l is odd, $\mathcal{C}^{l\otimes 3}_{m,-m}(j) = \mathcal{C}^{l\otimes 3}_{-m,m}(j)$ by (2.130).

$\mathcal{C}^{l\otimes 3}_{m,-m}(j)$	$m=0$	$m=1$
$j=l+3$	$\sqrt{\frac{5(l+1)(l+2)(l+3)}{2(2l+1)(2l+3)(2l+5)}}$	$\sqrt{\frac{15l(l+2)(l+3)}{8(2l+1)(2l+3)(2l+5)}}$
$j=l+1$	$-\sqrt{\frac{3l(l+1)(l+2)}{2(2l-1)(2l+1)(2l+5)}}$	$-(l-5)\sqrt{\frac{l+2}{8(2l-1)(2l+1)(2l+5)}}$
$j=l-1$	$\sqrt{\frac{3l(l-1)(l+1)}{2(2l-3)(2l+1)(2l+3)}}$	$-(l+6)\sqrt{\frac{l-1}{8(2l-3)(2l+1)(2l+3)}}$
$j=l-3$	$-\sqrt{\frac{5(l-2)(l-1)l}{2(2l-3)(2l-1)(2l+1)}}$	$\sqrt{\frac{15(l-2)(l-1)(l+1)}{8(2l-3)(2l-1)(2l+1)}}$
$\mathcal{C}^{l\otimes 3}_{m,-m}(j)$	$m=2$	$m=3$
$j=l+3$	$\sqrt{\frac{3(l-1)l(l+3)}{4(2l+1)(2l+3)(2l+5)}}$	$\sqrt{\frac{(l-2)(l-1)l}{8(2l+1)(2l+3)(2l+5)}}$
$j=l+1$	$(l+4)\sqrt{\frac{5(l-1)}{4(2l-1)(2l+1)(2l+5)}}$	$\sqrt{\frac{15(l-2)(l-1)(l+3)}{8(2l-1)(2l+1)(2l+5)}}$
$j=l-1$	$-(l-3)\sqrt{\frac{5(l+2)}{4(2l-3)(2l+1)(2l+3)}}$	$\sqrt{\frac{15(l-2)(l+2)(l+3)}{8(2l-3)(2l+1)(2l+3)}}$
$j=l-3$	$-\sqrt{\frac{3(l-2)(l+1)(l+2)}{4(2l-3)(2l-1)(2l+1)}}$	$\sqrt{\frac{(l+1)(l+2)(l+3)}{8(2l-3)(2l-1)(2l+1)}}$

(2.174) as

$$\mathcal{M}_2 = \frac{ie^2}{\pi}\sqrt{2l+1}\int dp\; p^2\; \phi_l(p)\;\frac{1}{E}$$

$$\times\Bigg\{\frac{2}{5}\Big(3+2\,\frac{m}{E}\Big)\Big[\mathcal{F}'_{l-1}(J)\,Q_{l-1}(\beta^{-1}) \;+\; \mathcal{F}'_{l+1}(J)\,Q_{l+1}(\beta^{-1})\Big]$$

$$+\sqrt{\frac{8}{75}}\,\Big(1-\frac{m}{E}\Big)\Big[\,\mathcal{G}'_{l-3}(J)\,Q_{l-3}(\beta^{-1}) \;+\; \mathcal{G}'_{l-1}(J)\;Q_{l-1}(\beta^{-1})$$

$$+\,\mathcal{G}'_{l+1}(J)\;Q_{l+1}(\beta^{-1}) \;+\; \mathcal{G}'_{l+3}(J)\;Q_{l+3}(\beta^{-1})\Big]\Bigg\}\,. \quad (2.177)$$

Note that by using $e^2 = 4\pi\alpha$ and the identity

$$\frac{2}{5}\Big(3+2\,\frac{m}{E}\Big) = 2-\frac{4}{5}\Big(1-\frac{m}{E}\Big)\,, \quad (2.178)$$

we may rewrite $\mathcal{M}_2$ (2.177) in the rather simpler form

$$\mathcal{M}_2 = i4\alpha\sqrt{2l+1}\int dp\; p^2\; \phi_l(p)\;\frac{1}{E}$$

$$\times\Bigg\{\mathcal{F}_{l-1}(J)\,Q_{l-1}(\beta^{-1}) \;+\; \mathcal{F}_{l+1}(J)\,Q_{l+1}(\beta^{-1})$$

$$+\Big(1-\frac{m}{E}\Big)\Big[\,\mathcal{G}_{l-3}(J)\,Q_{l-3}(\beta^{-1}) \;+\; \mathcal{G}_{l-1}(J)\;Q_{l-1}(\beta^{-1})$$

$$+\,\mathcal{G}_{l+1}(J)\;Q_{l+1}(\beta^{-1}) \;+\; \mathcal{G}_{l+3}(J)\;Q_{l+3}(\beta^{-1})\Big]\Bigg\}\,, \quad (2.179)$$

where the new constants are defined by

$$\mathcal{F}_j(J) \equiv 2\mathcal{F}'_j(J)\,, \quad (2.180)$$

$$\mathcal{G}_{l\pm1}(J) \equiv \sqrt{\frac{8}{75}}\,\mathcal{G}'_{l\pm1}(J) \;-\; \frac{4}{5}\,\mathcal{F}'_{l\pm1} \quad (2.181)$$

and

$$\mathcal{G}_{l\pm3}(J) \equiv \sqrt{\frac{8}{75}}\, \mathcal{G}'_{l\pm3}(J)\,. \tag{2.182}$$

We will now evaluate the helicity two amplitude in (2.179) for the triplet states for each of the three values of the total angular momentum J, by evaluating the six constants given in (2.180) and (2.182) for each case. This uses **Table 2.1** and **Table 2.2**.

<u>The case $J = l + 1$</u>

For this case we find that the various $\mathcal{F}_j(J)$ and $\mathcal{G}_j(J)$ constants in (2.179) are as follows;

$$\mathcal{F}_{l-1}(l+1) = \frac{-1}{2l+1}\sqrt{\frac{l(l+2)(l+3)}{2l+1}}\,, \tag{2.183}$$

$$\mathcal{F}_{l+1}(l+1) = \frac{1}{2l+1}\sqrt{\frac{l(l+2)(l+3)}{2l+1}}\,, \tag{2.184}$$

$$\mathcal{G}_{l-3}(l+1) = 0\,, \tag{2.185}$$

$$\mathcal{G}_{l-1}(l+1) = \frac{(l+1)}{(2l+3)(2l+1)}\sqrt{\frac{l(l+2)(l+3)}{2l+1}}\,, \tag{2.186}$$

$$\mathcal{G}_{l+1}(l+1) = \frac{-2(l+1)}{(2l+5)(2l+1)} \sqrt{\frac{l(l+2)(l+3)}{2l+1}} \tag{2.187}$$

and finally

$$\mathcal{G}_{l+3}(l+1) = \frac{(l+1)}{(2l+5)(2l+3)} \sqrt{\frac{l(l+2)(l+3)}{2l+1}}\,. \tag{2.188}$$

On substituting these expressions in (2.179) for $\mathcal{M}_2$, and using the latter in (2.100), we finally obtain the helicity two positronium partial width $\Gamma_{\gamma\gamma}^{\lambda=2}$ for $J = l+1$;

$$\Gamma_{\gamma\gamma}^{\lambda=2}(J = l+1, s=1) = \frac{2\alpha^2}{\pi m^2} \frac{l(l+2)(l+3)}{(2l+3)(2l+1)^2}$$

$$\times \left| \int_0^\infty dp\, p^2\, \phi_l(p) \Bigg\{ \sqrt{1-\beta^2} \Big[Q_{l+1}(\beta^{-1}) \;-\; Q_{l-1}(\beta^{-1}) \Big] \right.$$

$$+ \frac{l+1}{(2l+3)(2l+5)} \Big(\sqrt{1-\beta^2} \;-\; (1-\beta^2) \Big) \Big[(2l+5) Q_{l-1}(\beta^{-1})$$

$$\left. -\, 2(2l+3) Q_{l+1}(\beta^{-1}) \;+\; (2l+1) Q_{l+3}(\beta^{-1}) \Big] \Bigg\} \right|^2 . \tag{2.189}$$

<u>*The case $J = l$*</u>

Proceeding similarly, we find that the constants in this case are given by

$$\mathcal{F}_{l-1}(l) = \frac{1}{2l+1} \sqrt{(l+2)(l-1)}\,, \tag{2.190}$$

$$\mathcal{F}_{l+1}(l) = \frac{-1}{2l+1} \sqrt{(l+2)(l-1)} \tag{2.192}$$

and

$$\mathcal{G}_{l-3}(l) = \mathcal{G}_{l-1}(l) = \mathcal{G}_{l+1}(l) = \mathcal{G}_{l+3}(l) = 0\,, \tag{2.192}$$

which when substituted as for $J = l+1$ yield the positronium $\Gamma_{\gamma\gamma}^{\lambda=2}$ for this state

$$\Gamma_{\gamma\gamma}^{\lambda=2}(J=l, s=1) = \frac{2\alpha^2}{\pi m^2} \frac{(l-1)(l+2)}{(2l+1)^2}$$
$$\times \left| \int_0^\infty dp\, p^2\, \phi_l(p) \sqrt{1-\beta^2} \left\{ Q_{l+1}(\beta^{-1}) \;-\; Q_{l-1}(\beta^{-1}) \right\} \right|^2 . \tag{2.193}$$

Note that this $\lambda = 2$ result, combined with the absence of a helicity zero contribution for this state, is consistent with the Landau-Yang theorem, since if $J = l = 1$ we have $\Gamma_{\gamma\gamma} = 0$.

<u>The case $J = l-1$</u>

For this case we find

$$\mathcal{F}_{l-1}(l-1) = \frac{-1}{2l+1} \sqrt{\frac{(l-2)(l-1)(l+1)}{2l+1}}\,, \tag{2.194}$$

$$\mathcal{F}_{l-1}(l-1) = \frac{1}{2l+1} \sqrt{\frac{(l-2)(l-1)(l+1)}{2l+1}}\,, \tag{2.195}$$

$$\mathcal{G}_{l-3}(l-1) = \frac{-l}{(2l-3)(2l-1)} \sqrt{\frac{(l-2)(l-1)(l+1)}{2l+1}}\,, \tag{2.196}$$

$$\mathcal{G}_{l-1}(l-1) = \frac{2l}{(2l-3)(2l+1)} \sqrt{\frac{(l-2)(l-1)(l+1)}{2l+1}}\,, \tag{2.197}$$

$$\mathcal{G}_{l+1}(l-1) = \frac{-l}{(2l-1)(2l+1)} \sqrt{\frac{(l-2)(l-1)(l+1)}{2l+1}} \tag{2.198}$$

and finally

$$\mathcal{G}_{l+3}(l-1) = 0\,, \tag{2.199}$$

which give

$$\begin{aligned}\Gamma^{\lambda=2}_{\gamma\gamma}(J=l-1, s=1) &= \frac{2\alpha^2}{\pi m^2}\frac{(l-2)(l-1)(l+1)}{(2l-1)(2l+1)^2} \\ &\times \Bigg| \int_0^\infty dp\, p^2\, \phi_l(p) \Bigg\{ \sqrt{1-\beta^2}\Big[Q_{l+1}(\beta^{-1}) - Q_{l-1}(\beta^{-1})\Big] \\ &- \frac{l}{(2l-3)(2l-1)}\Big(\sqrt{1-\beta^2} - (1-\beta^2)\Big)\Big[(2l+1)Q_{l-3}(\beta^{-1}) \\ &- 2(2l-1)Q_{l-1}(\beta^{-1}) + (2l-3)Q_{l+1}(\beta^{-1})\Big]\Bigg\}\Bigg|^2 . \end{aligned} \tag{2.200}$$

This is consistent with the expected $\Gamma^{\lambda=2}_{\gamma\gamma}(0^{++}) = 0$ result, and together with (2.156) reproduce the relativistic result of Bergström, Hulth and Snellman (1.52) if we set in their formula our free process approximation

$\sqrt{\vec{p}^2 + m_q{}^2} = E = \omega = M/2$ and after transforming (2.200) according to (1.31).

Thus far we have considered $\Gamma_{\gamma\gamma}$ relativistically for positronium states; in the next chapter we shall apply these results to quarkonium, as described in section **1.4**, and we will see how our formulas reproduce the relativistic special cases treated by Bergström, Hulth and Snellman (cited in section **1.6**). First however we will evaluate the nonrelativistic limit of our triplet $\Gamma_{\gamma\gamma}$ results for general J^P given explicit positronium wavefunctions.

The nonrelativistic limit

We will now derive the nonrelativistic limit of the triplet $\Gamma_{\gamma\gamma}$ for general angular quantum numbers. We keep the leading term in the expansion of $Q_J(\beta^{-1})$ in powers of β, given in (2.74), and substitute $p \approx m\beta$ and $\sqrt{1-\beta^2} \approx 1 - \frac{1}{2}\beta^2$, and retain the lowest order contribution to $\Gamma_{\gamma\gamma}$. Since the total $\Gamma_{\gamma\gamma}$ for the triplet states is the sum of helicity zero and helicity two contributions, it is possible that their lowest order contributions may occur at different orders in β. One then neglects the higher power contribution, and says that the $\gamma\gamma$ width is zero for that helicity in the nonrelativistic limit.

Starting with the top of the triplet, $J = l + 1$, and noting that $Q_J(\beta^{-1}) \propto \beta^{J+1}$ in this limit, the helicity zero contribution (2.155) is order β^{l+2}. However in view of the expansion (2.74), the nonrelativistic helicity two contribution comes from the first $Q_{l-1}(\beta^{-1})$ term in (2.189), and is order β^l. Therefore in the nonrelativistic limit this state decays only to $\lambda = 2$ two-photon final states. (One can show explicitly that the lowest order contribution to $\mathcal{M}(\lambda_1, \lambda_2)$ is always order β^l). This is consistent with previous nonrelativistic calculations which find no $\lambda = 0$ contribution in the $J^{PC} = 2^{++}$ P-wave. As noted previously, this is supported by experiment (the $\lambda = 0$ contribution to $\Gamma_{\gamma\gamma}(a_2(1320))$ is < 0.082 at 95% $c.l.$[51]). To derive the general nonrelativistic limit, we use (2.74) and find

$$\Gamma^{NR}_{\gamma\gamma}(J = l+1, s = 1) = \frac{2\alpha^2}{\pi m^2} \frac{l(l+2)(l+3)}{(2l+3)(2l+1)^2}$$

$$\times \left| \int_0^\infty dp\, p^2\, \phi_l(p)\, \frac{2^{l-1}(l-1)!^2}{(2l-1)!} \beta^l \right|^2 . \qquad (2.201)$$

Now, using (2.76) and the identity (2.77), one finds the simple form

$$\Gamma^{NR}_{\gamma\gamma}(s = 1; J = l+1) = \frac{(l+2)(l+3)}{l(2l+3)}\, \hat{\Gamma}\, , \qquad (2.202)$$

where $\hat{\Gamma}$ is the spin singlet nonrelativistic positronium $\Gamma_{\gamma\gamma}$ given in (2.78),

$$\hat{\Gamma} = \frac{\alpha^2}{m^{2l+2}} \left|\psi^{(l)}(0)\right|^2 . \qquad (2.203)$$

Using the explicit $\hat{\Gamma}$ derived from hydrogenic wavefunctions (2.80), we find the positronium $\Gamma_{\gamma\gamma}$ for the top of the multiplet ($J = l + 1$) in closed form;

$$\Gamma_{\gamma\gamma}(e^+e^-(s = 1; J = l + 1)) =$$

$$\frac{(l+2)(l+3)}{2l(l+1)(2l+3)} \frac{\binom{n+l}{2l+1}}{\binom{2l+1}{l}} n^{-(2l+4)} \cdot m\alpha^{2l+5} . \qquad (2.204)$$

This reproduces the 2^{++} formula of Alekseev and Tumanov (1.49) for the special case of $l = 1$.

Now, considering $J = l = odd$, we have only a $\lambda = 2$ two-photon state. Proceeding as previously, we obtain

$$\Gamma^{NR}_{\gamma\gamma}(s = l; J = 1) = \frac{(l-1)(l+2)}{l^2} \hat{\Gamma} . \qquad (2.205)$$

Again using (2.80) we find the positronium $\Gamma_{\gamma\gamma}$ for the middle of the multiplet ($J = l$) in closed form;

$$\Gamma_{\gamma\gamma}(e^+e^-(s = 1; J = l)) = \frac{(l-1)(l+2)}{2l^2(l+1)} \frac{\binom{n+l}{2l+1}}{\binom{2l+1}{l}} n^{-(2l+4)} \cdot m\alpha^{2l+5} . \qquad (2.206)$$

Finally, for the bottom of the multiplet ($J = l - 1$), we see that for $\lambda = 0$ the leading contribution in (2.156) is order β^l (2.156). However, for the helicity two amplitude careful analysis of (2.200) is required, since there are actually two terms that give leading contributions at this order; the usual $Q_{l-1}(\beta^{-1})$ term, and a new contribution coming from $Q_{l-3}(\beta^{-1})$.

If one does the calculation keeping only the lowest order contribution in the integrand (the leading contribution to the invariant amplitude $\mathcal{M}_{fi}$), one finds that the term coming from $Q_{l-3}(\beta^{-1})$ actually arises from the β^3 terms of $\vec{\mathcal{B}}$ in (2.41). Thus one would arrive at a different and erroneous results by truncating the amplitude before angular integration, specifically by keeping only the terms multiplied by β in $\vec{\mathcal{B}}$! Again, using (2.200) and (2.156) as before, but retaining the two new terms in (2.200), one finds

$$\Gamma^{NR}_{\gamma\gamma}(s=1; J=l-1) = \Bigg\{ \underbrace{\frac{(2l+1)^2}{l(2l-1)}}_{\lambda=0}$$

$$+ \underbrace{\frac{(l-2)(l-1)(l+1)}{l^2(2l-1)} \cdot \Bigg(\overbrace{1}^{\beta \text{ term}} + \overbrace{\frac{l(2l+1)}{2(l-1)(l-2)}}^{\beta^3 \text{ term}} \Bigg)^2 \cdot \left(1-\delta^l_1\right)}_{\lambda=2} \Bigg\} \cdot \hat{\Gamma}$$

$$= \Bigg\{ \underbrace{\frac{(2l+1)^2}{l(2l-1)}}_{\lambda=0} + \underbrace{\frac{4(l+1)\left(l^2 - \frac{5}{4}l + 1\right)^2}{l^2(l-2)(l-1)(2l-1)} \cdot \left(1-\delta^l_1\right)}_{\lambda=2} \Bigg\} \cdot \hat{\Gamma}\,. \qquad (2.207)$$

Again care must be exercised in imposing the multiplier $\left(1-\delta^l_1\right)$, since a cursory inspection of the relativistic $\Gamma_{\gamma\gamma}(J=l-1)$ in (2.200) suggests the required result of no $\lambda=0$ contribution for the $J=0$ case (for $l=1$), but one divides by $l-1$ in this case in the amplitude and obtains the result (2.207), which would diverge for $l=1$ had we not included the multiplier $\left(1-\delta^l_1\right)$.

To obtain the closed-form positronium $\Gamma_{\gamma\gamma}$ for this case, we again use (2.80) and find

$$\Gamma_{\gamma\gamma}(e^+e^-(s=1;J=l-1)) = \Bigg\{ \underbrace{\frac{(2l+1)^2}{2l(l+1)(2l-1)}}_{\lambda=0}$$

$$+ \underbrace{\frac{2\left(l^2-\frac{5}{4}l+1\right)^2}{l^2(l-2)(l-1)(2l-1)} \cdot \left(1-\delta_1^l\right)}_{\lambda=2} \Bigg\} \cdot \frac{\binom{n+l}{2l+1}}{\binom{2l+1}{l}}\, n^{-(2l+4)} \cdot m\alpha^{2l+5} \;, \quad (2.208)$$

which reproduces the positronium $\Gamma_{\gamma\gamma}(0^{++})$ (1.48) of Tumanov and Alekseev for the special case of $l = 1$, and together with (2.204) and (2.206) reproduce the well known $\Gamma_{\gamma\gamma}(0^{++}) : \Gamma_{\gamma\gamma}(1^{++}) : \Gamma_{\gamma\gamma}(2^{++})$ ratios of $15/4 : 0 : 1$.

CHAPTER 3

COMPARISON WITH DATA

3.1 Summary of quarkonium $\Gamma_{\gamma\gamma}$ formulas

We will now present numerical predictions using our formulas. First we shall summarize our results for the $\Gamma_{\gamma\gamma}(J^P)$ of $C = +1$ mesons. Nonrelativistically, we have[52]

$$\Gamma^{NR}_{\gamma\gamma}(l^-) = 3 \cdot \mathcal{F}_q \cdot \{\underbrace{1}_{\lambda=0} + \underbrace{0}_{\lambda=2}\} \cdot \hat{\Gamma}_q \,, \tag{3.1}$$

$$\Gamma^{NR}_{\gamma\gamma}((l+1)^+) = 3 \cdot \mathcal{F}_q \cdot \left\{\underbrace{0}_{\lambda=0} + \underbrace{\frac{(l+2)(l+3)}{l(2l+3)}}_{\lambda=2}\right\} \cdot \hat{\Gamma}_q \,, \tag{3.2}$$

$$\Gamma^{NR}_{\gamma\gamma}(l^+) = 3 \cdot \mathcal{F}_q \cdot \left\{\underbrace{0}_{\lambda=0} + \underbrace{\frac{(l-1)(l+2)}{l^2}}_{\lambda=2}\right\} \cdot \hat{\Gamma}_q \,, \tag{3.3}$$

$$\Gamma^{NR}_{\gamma\gamma}((l-1)^+) = 3 \cdot \mathcal{F}_q \cdot \left\{\underbrace{\frac{(2l+1)^2}{l(2l-1)}}_{\lambda=0} + \underbrace{\frac{4(l+1)\left(l^2 - \frac{5}{4}l + 1\right)^2}{l^2(l-2)(l-1)(2l-1)} \cdot (1 - \delta^l_1)}_{\lambda=2}\right\} \cdot \hat{\Gamma}_q \,, \tag{3.4}$$

where $\hat{\Gamma}_q$ is obtained from $\hat{\Gamma}$, the positronium two photon nonrelativistic width in terms of the l^{th} derivative of the radial wavefunction at contact given in

(2.70), by changing $m \to m_q$, and is

$$\hat{\Gamma}_q = \frac{\alpha^2}{{m_q}^{2(l+1)}} \cdot \left|\psi^{(l)}(0)\right|^2 . \tag{3.5}$$

The relativistic results for general J^P may be written in a unified form with the dependence on λ and J^P incorporated in a single function, $\mathcal{Q}^{(\lambda)}_{J^P}(\beta)$[47,53];

$$\Gamma_{\gamma\gamma}(J^P) = 3 \cdot \mathcal{F}_q \cdot \frac{2\alpha}{\pi {m_q}^2} \sum_{\lambda=0,2} \left| \int_0^\infty dp\; p^2\; \phi_l(p) \cdot \mathcal{Q}^{(\lambda)}_{J^P}(\beta) \right|^2 \tag{3.6}$$

This function for all allowed quantum numbers is

$$\mathcal{Q}^{(0)}_{l^-}(\beta) = \frac{1-\beta^2}{\beta}\; Q_l(\beta^{-1}) , \tag{3.6}$$

$$\mathcal{Q}^{(0)}_{(l+1)^+}(\beta) = \sqrt{\frac{l+1}{2l-1}}\; (1-\beta^2)\; Q_{l+1}(\beta^{-1}) , \tag{3.8}$$

$$\mathcal{Q}^{(0)}_{(l-1)^+}(\beta) = \sqrt{\frac{l}{2l-1}}\; (1-\beta^2)\; Q_{l-1}(\beta^{-1}) , \tag{3.9}$$

$$\begin{aligned} \mathcal{Q}^{(2)}_{(l+1)^+}(\beta) = \sqrt{\frac{l(l+2)(l+3)}{(2l+3)(2l+1)^2}} \times \Bigg\{ & \sqrt{1-\beta^2}\Big[Q_{l+1}(\beta^{-1}) \;-\; Q_{l-1}(\beta^{-1})\Big] \\ & + \frac{l+1}{(2l+3)(2l+5)}\Big(\sqrt{1-\beta^2} \;-\; (1-\beta^2)\Big)\; \Big[(2l+5)Q_{l-1}(\beta^{-1}) \\ & - 2(2l+3)Q_{l+1}(\beta^{-1}) \;+\; (2l+1)Q_{l+3}(\beta^{-1})\Big]\Bigg\} , \end{aligned} \tag{3.10}$$

$$Q_{l+}^{(2)}(\beta) = \sqrt{\frac{(l-1)(l+2)}{(2l+1)^2}}\sqrt{1-\beta^2}\Big\{ Q_{l+1}(\beta^{-1}) - Q_{l-1}(\beta^{-1})\Big\} \qquad (3.11)$$

and

$$\begin{aligned} Q_{(l-1)+}^{(2)}(\beta) = \sqrt{\frac{(l-2)(l-1)(l+1)}{(2l-1)(2l+1)^2}} \Bigg\{ & \sqrt{1-\beta^2}\Big[Q_{l+1}(\beta^{-1}) \ - \ Q_{l-1}(\beta^{-1})\Big] \\ & - \frac{l}{(2l-3)(2l-1)}\Big(\sqrt{1-\beta^2} \ - \ (1-\beta^2)\Big) \Big[(2l+1)Q_{l-3}(\beta^{-1}) \\ & - 2(2l-1)Q_{l-1}(\beta^{-1}) \ + \ (2l-3)Q_{l+1}(\beta^{-1})\Big]\Bigg\} . \end{aligned} \qquad (3.12)$$

3.2 Phenomenological treatment of phase space

In chapter 1 we referred to a previous relativistic $\Gamma_{\gamma\gamma}(0^{-+})$ due to Bergström, Snellman and Tengstrand, later derived independently by Hayne and Isgur; we also noted that the latter multiply the rate by an *ad hoc* overall factor of $(M_R/M_{ref})^3$ to impose an expected approximate M_R^3 mass dependence;

$$\Gamma(\pi^0 \to \gamma\gamma) = \Gamma(\pi^0 \to \gamma\gamma)\Big|_{\substack{\text{quark}\\ \text{model}}} \cdot \left(\frac{M_R}{M_{ref}}\right)^3 . \qquad (3.13)$$

Multiplication of the rate by $(M_R/M_{ref})^3$ is primarily motivated by comparison with experiment. The $\gamma\gamma$ widths of the light pseudoscalars π^0, η and η' seen in e^+e^- experiments[15] are known to increase rapidly with M_R;

$$\Gamma_{\gamma\gamma}(\pi^0) = 7.29 \pm 0.19 \text{ eV}, \qquad (3.14)$$

$$\Gamma_{\gamma\gamma}(\eta) = 0.524 \pm 0.031 \text{ keV}, \tag{3.15}$$

$$\Gamma_{\gamma\gamma}(\eta') = 4.25 \pm 0.19 \text{ keV} . \tag{3.16}$$

In contrast, direct evaluation of the theoretical decay rate (2.73) for $J^P = 0^-$ would incorrectly give comparable $\gamma\gamma$ widths for these resonances, because this formula makes no direct reference to the physical resonance mass M_R.

Note that the observed $\gamma\gamma$ widths are approximately proportional to M_R^3 ($M_R \approx 135$ MeV, 547 MeV and 958 MeV for the π^0, η and η', respectively[19]),

$$\Gamma_{\gamma\gamma}(\pi^0)/M_{\pi^0}^3 = 2.96 \pm 0.08 \text{ keV GeV}^{-3} \tag{3.17}$$

$$\Gamma_{\gamma\gamma}(\eta)/M_{\eta}^3 = 3.17 \pm 0.19 \text{ keV GeV}^{-3} \tag{3.18}$$

$$\Gamma_{\gamma\gamma}(\eta')/M_{\eta'}^3 = 4.84 \pm 0.22 \text{ keV GeV}^{-3} . \tag{3.19}$$

This approximate M_R^3 dependence can be motivated by consideration of a simple pseudoscalar-$\gamma\gamma$ effective lagrangian,

$$\mathcal{L}_I = \frac{1}{2} g \phi F^{\mu\nu} \tilde{F}_{\mu\nu} , \tag{3.20}$$

which leads to a 0^{-+} $\gamma\gamma$ width proportional to M_R^3,

$$\Gamma_{\gamma\gamma} = \frac{1}{64\pi} \cdot g^2 M_R^3 . \tag{3.21}$$

The problem with the quark model calculation is that it does not incorporate the physical mass of the initial state, which must be restored as a phenomenological input. In effect we treat the quark model calculation as a determination of g in (3.20) (for 0^{-+}), and then impose the overall M_R^3 dependence found in (3.21). There is clearly a need for an improved calculation of these rates at the quark level which incorporates the physical meson mass *ab initio*.

Our prescription for restoring the physical meson mass, suggested by Hayne and Isgur[32], is to "correct" the M_R dependence of the quark model $\Gamma_{\gamma\gamma}$ calculation by introducing an overall multiplicative factor, as in (3.13). This requires specification of a "reference" mass M_{ref}, which is a characteristic mass scale in the $q\bar{q}$ model wavefunction. Hayne and Isgur originally suggested setting M_{ref} equal to the mass actually calculated in the quark model. This has the advantage of eliminating this $(M_R/M_{ref})^p$ correction as models become more accurate, but the disadvantage that adding a constant to $V(r)$ changes M_{ref} and hence the predicted decay rate. A choice for M_{ref} which does not suffer this unphysical feature is the expected rest plus kinetic energy; this was used by Godfrey and Isgur[7], in the relativistic form $\sqrt{\vec{p}^{\,2}+m^2}$. There is clearly considerable arbitrariness in the choice of M_{ref}, and hence a corresponding uncertainty in the overall scale of decay rates. The relative rates are predicted more reliably, as M_{ref} cancels in a ratio of rates.

For the $\gamma\gamma$ decays of $l=1$ $q\bar{q}$ states and the $l=2$, $J^{PC}=2^{-+}$ state we also expect an M_R^3 dependence in $\Gamma_{\gamma\gamma}$, since the effective lagrangians

$$\mathcal{L}_I^{(2^{++})} = \frac{1}{2} g f_{\mu\nu} F^{\mu\alpha} F^{\nu}_{\alpha} \,, \tag{3.22}$$

$$\mathcal{L}_I^{(0^{++})} = \frac{1}{2} g \phi F_{\mu\nu} F^{\mu\nu} \tag{3.23}$$

and

$$\mathcal{L}_I^{(2^{-+})} = \frac{1}{2} g f_{\mu\nu} F^{\mu\alpha} \tilde{F}^{\nu}_{\alpha} \tag{3.24}$$

have the same number of derivatives. For higher-spin initial mesons, more derivatives will be required to construct an effective lagrangian, which will lead to a higher power of M_R. For example, in the 4^{-+} case two additional derivatives are required, so we incorporate an $(M_R/M_{ref})^7$ dependence in $\Gamma(4^{-+} \to \gamma\gamma)$. The higher-spin $\gamma\gamma$ partial widths are evidently much more sensitive to the choice of the reference mass M_{ref}, and their absolute scale is correspondingly less well determined.

3.3 The $\pi_2(1670)$ $\gamma\gamma$ width

For our numerical estimate of the $\pi_2(1670)$ decay rate (assuming it to be $q\bar{q}$) we take M_{ref} to be the rms relativistic kinetic energy of the quarkonium state,

$$M_{ref} = 2\langle\psi|(\vec{p}^{\,2} + {m_q}^2)|\psi\rangle^{\frac{1}{2}} \,, \tag{3.25}$$

in terms of which our $\gamma\gamma$ partial width is

$$\Gamma(\pi_2 \to \gamma\gamma) = \frac{\alpha^2}{3\pi {m_q}^2}$$

$$\times \left| \int_0^\infty dp\ p^2\ \phi(p) \Big(\frac{1-\beta^2}{\beta}\Big)\ Q_2(\beta^{-1}) \right|^2 \cdot \left(\frac{M_R}{M_{ref}}\right)^3 , \qquad (3.26)$$

where the Legendre function is $Q_2(x) = \frac{1}{4}(3x^2 - 1)\ln(\frac{x+1}{x-1}) - \frac{3}{2}x$. We evaluate this decay rate numerically using wavefunctions satisfying the Schrödinger equation with a Coulomb-plus-linear potential;

$$V_{q\bar{q}}(r) = -\frac{4}{3}\frac{\alpha_s}{r} + ar , \qquad (3.27)$$

which is known to be very realistic for heavy quarks, and gives surprisingly good results for light quark systems. We use a "standard" light-quark parameter set $\alpha_s = 0.6$, $a = 0.18\ \mathrm{GeV}^2$ and $m_q = 0.33\ \mathrm{GeV}$; the wavefunction integral is found to be

$$\int_0^\infty dp\ p^2\ \phi(p)\Big(\frac{1-\beta^2}{\beta}\Big)Q_2(\beta^{-1}) \ = \ 2.617 \times 10^{-2}\ \mathrm{GeV}^{\frac{3}{2}} \qquad (3.28)$$

and the rms relativistic kinetic energy (3.25) is

$$M_{ref} \ = \ 1.164\ \mathrm{GeV}\ . \qquad (3.29)$$

Combining these and taking $M_R = 1.67$ GeV, we find a $\gamma\gamma$ partial width of

$$\Gamma(\pi_2(1670) \to \gamma\gamma) = 0.105\ \mathrm{keV}\ , \qquad (3.30)$$

which is approximately an order of magnitude smaller than the experimental value given in (1.60).

In view of the discrepancy between our result (3.30) and experiment, it is important to investigate the stability of this $\Gamma_{\gamma\gamma}$ with respect to changes in the parameters α_s, a, m_q and M_{ref}. We find that the $\gamma\gamma$ width is rather insensitive to α_s and a but depends strongly on m_q and M_{ref}; for small changes from the "standard" parameters quoted above we find

$$\frac{\delta\Gamma_{\gamma\gamma}}{\Gamma_{\gamma\gamma}} = 0.06 \cdot \frac{\delta\alpha_s}{\alpha_s} + 0.32 \cdot \frac{\delta a}{a} - 2.61 \cdot \frac{\delta m_q}{m_q} \,, \tag{3.31}$$

so that the strongest dependence by far is on m_q. (M_{ref} is implicitly a function of α_s, a and m_q here, as we have set it equal to the rms relativistic kinetic energy.) If we vary the definition of M_{ref} with fixed α_s, a and m_q, we find

$$\frac{\delta\Gamma_{\gamma\gamma}}{\Gamma_{\gamma\gamma}} = -3 \cdot \frac{\delta M_{ref}}{M_{ref}} \,, \tag{3.32}$$

so the scale of partial widths also depends strongly on the M_{ref} assumed.

The sensitivity of decay widths to m_q was previously noted by Hayne and Isgur[32] in a study of weak and electromagnetic quarkonium decays in a relativistic quark model. They concluded that a better numerical fit to the data followed from an "extended" parameter set with a light quark mass of $m_q = 220$ MeV. As we have no particular reason to prefer the conventional value of $m_q = 330$ MeV, we also quote results for this lighter m_q. This gives

$$\Gamma(\pi_2(1670) \to \gamma\gamma) = 0.275 \text{ keV} \,, \tag{3.33}$$

which motivates our final estimate of

$$\Gamma(\pi_2(1670) \to \gamma\gamma) = 0.1 - 0.3\,\text{keV}\,. \tag{3.34}$$

This is somewhat smaller than the experimental result of ≈ 1 keV, but in view of the remaining freedom in the choice of M_{ref} and the rather *ad hoc* incorporation of M_R in $\Gamma_{\gamma\gamma}$, we do not believe that (3.34) argues against a $q\bar{q}$ assignment for the $\pi_2(1670)$. The appropriate test of this assignment is a search for the $I = 0$ partner $\eta_2(\approx 1700)$, which should have a $\gamma\gamma$ width of about $2 - 3$ keV $\left(\frac{25}{9} \cdot \Gamma_{\gamma\gamma}(\pi_2)\right)$. If the discrepancy between (3.34) and experiment is confirmed, this will indicate the importance of developing more realistic techniques for the calculation of $\gamma\gamma$ widths. A method in which the approximate M_R^3 dependence arises naturally in $\Gamma_{\gamma\gamma}$ would be a particularly useful improvement in technique.

As our estimate (3.34) for the $\gamma\gamma$ width of the $\pi_2(1670)$ differs considerably from the previous numerical estimate of Anderson, Austern and Cahn (1.61), it is instructive to compare our decay formulas to understand the origin of this discrepancy. One important difference is their use of the nonrelativistic "contact" approximation in evaluating the decay amplitude. This assumes that it is a good approximation to keep only the leading term in a series expansion of the amplitude in (3.26) in powers of p/m_q, which for $J = 2$ requires by use of (2.74)

$$\frac{\mathcal{I}_0}{\mathcal{I}} = \frac{\int_0^\infty dp\; p^2\;\phi(p) \cdot \left\{\frac{2p^2}{15{m_q}^2}\right\}}{\int_0^\infty dp\; p^2\;\phi(p) \cdot \left\{\left(\frac{1-\beta^2}{\beta}\right)Q_2(\beta^{-1})\right\}} \approx 1\,. \tag{3.35}$$

For light-quark parameters $\alpha_s = 0.6$, $a = 0.18\ \mathrm{GeV}^2$ and $m_q = 0.33\ \mathrm{GeV}$, we find that this ratio is actually

$$\frac{\mathcal{I}_0}{\mathcal{I}} = \frac{\int_0^\infty dp\ p^2\ \phi(p) \cdot \left\{\frac{2p^2}{15 m_q{}^2}\right\}}{\int_0^\infty dp\ p^2\ \phi(p) \cdot \left\{\left(\frac{1-\beta^2}{\beta}\right) Q_2(\beta^{-1})\right\}} = 7.996\ , \tag{3.36}$$

so the contact approximation is a poor one for light 2^{-+} decays. Similar conclusions have been reported for other light $q\bar{q}$ states[13,30]. Note however that (3.36) implies that the contact approximation should *overestimate* the $\gamma\gamma$ decay rate by about a factor of 64, whereas Anderson, Austern and Cahn underestimate it by two to three orders of magnitude (assuming the $\pi_2(1670)$ is indeed the ${}^1D_2$ $q\bar{q}$ state). The compensating factor is their identification of the quark mass m_q with half the resonance mass, $m_q = M_R/2 \approx 835\ \mathrm{MeV}$, instead of the usual constituent mass $m_q \approx 330\ \mathrm{MeV}$. Since the nonrelativistic decay formula (1.57) scales as $m_q{}^6$, replacing 330 MeV by 835 MeV reduces their $\Gamma_{\gamma\gamma}$ by a factor of $(835/330)^6 = 262$. The combination of these two large factors scales their decay rate down from our relativistic result by an overall factor of ≈ 4. Our phenomenological $(M_R/M_{ref})^3$ multiplier in (3.26) increases the predicted $\Gamma_{\gamma\gamma}$ of the $\pi_2(1670)$ by about a factor of 3. This together with the ≈ 4 above accounts for the order-of-magnitude difference between our result of 0.11 keV for $m_q = 0.33\ \mathrm{GeV}$ in a Coulomb-plus-linear potential and their result of 6 eV in the very similar Cornell model. (They also give results for several less well motivated potentials, which leads to their quoted range of 1.2 eV (power-law)−9.4 eV (logarithmic).)

3.4 $\Gamma_{\gamma\gamma}$ for other light $q\bar{q}$ states

As we have derived relativistic $\Gamma_{\gamma\gamma}$ formulas for arbitrary total J, it may be of interest to present numerical results for other higher-spin singlet states as well as for radial excitations. For levels without well-established experimental candidates we use the mass estimates of Godfrey and Isgur[7], which are indicated by question marks. For light quarks we use the standard parameters $\alpha_s = 0.6$, $a = 0.18\,\mathrm{GeV}^2$, and quote a range which follows from $m_q = 330\,\mathrm{MeV} \to 220\,\mathrm{MeV}$, as we find that the sensitive dependence of $\Gamma_{\gamma\gamma}$ on m_q generalizes to all light quarks. We give results for $I=1$ 0^{-+}, 2^{-+} and 4^{-+} singlet states in **Table 3.1**, and for the P-wave triplet (0^{++} and 2^{++}) in **Table 3.2**, and for their first radial excitation with assumed mass of 1800 GeV in **Table 3.3**.

Table 3.2 indicates that the ratio of the $\Gamma_{\gamma\gamma}$ of the P-wave 0^{++} to the 2^{++} is reduced from 15/4 nonrelativistically (1.50) to ≈ 1.8 by relativistic corrections. This is quite similar to the result of Li, Close, and Barnes, who found it to be ≈ 2 at order β^2[26]. However one sees in this table that the ratio of $\lambda = 0$ and $\lambda = 2$ contributions to $\Gamma_{\gamma\gamma}(2^{++})$ (which is expected to be zero nonrelativistically for this case, as well as for any $J^{PC} = (l+1)^{++}$ by (3.2)), does not experience important relativistic effects, since it becomes $\approx 0.02 \rightarrow 0.03$. The calculation of Li *et al* finds this relativistic ratio to be ≈ 0.05[26]. Here a general conclusion may be reached as we find that $\Gamma_{\gamma\gamma}^{\lambda=0}(^3P_2)/\Gamma_{\gamma\gamma}^{tot}(^3P_2) \leq 0.047$ for any nodeless $q\bar{q}$ wavefunction $\phi(p)$ (see Appendix B).

Table 3.3 also shows a small $\lambda = 0$ contribution to the total $\Gamma_{\gamma\gamma}$ of the radially excited $^3P_2'(1800?)$. The same is true for the $^3F_4$ $l = 3$ state, as indicated in **Table 3.4**. Although only $\lambda = 0$ is allowed for the bottom of the

Table 3.1 Estimated and observed I=1 quarkonium $\gamma\gamma$ widths for the range $m_q = 330$ MeV $\rightarrow$ 220 MeV.

0^{-+}	2^{-+}	4^{-+}
$\pi^0(135)$ $3.4 \rightarrow 6.4$ eV expt.[15] 7.29 ± 0.19 eV		
$\pi(1300)$ $0.43 \rightarrow 0.49$ keV	$\pi_2(1670)$ $0.11 \rightarrow 0.27$ keV expt.[19] 1.35 ± 0.26 keV	
$\pi(3S)(1880?)$ $0.74 \rightarrow 1.00$ keV	$\pi_2(2D)(2130?)$ $0.10 \rightarrow 0.16$ keV	$\pi_4(1G)(2330?)$ $0.21 \rightarrow 1.6$ keV

Table 3.2 Estimated and observed P-wave quarkonium $\Gamma_{\gamma\gamma}$ for the range $m_q = 330$ MeV $\rightarrow$ 220 MeV.

^3P_J $\frac{(u\bar{u}+d\bar{d})}{\sqrt{2}}$	Theory	Experiment
$f_0(1250)$	$3.25 \rightarrow 6.46$ keV	$4.3 \pm 0.8 \pm 0.6$[28] 8.0 ± 1.67 keV[29]
$f_2(1270)$	$\underbrace{(0.04 \rightarrow 0.11)}_{\lambda=0} + \underbrace{(1.71 \rightarrow 3.93)}_{\lambda=2}$ $= 1.75 \rightarrow 4.04$ keV	2.8 ± 0.4 keV[19] 3.01 ± 0.12 keV[54]
$\lambda = 0/\lambda = 2$	$2.2 \rightarrow 2.9\%$	$< 8.2\%$ (95% $c.l.$)[51]
f_0/f_2	$1.86 \rightarrow 1.60$	

Table 3.3 Predictions for radially excited P-wave quarkonium $\Gamma_{\gamma\gamma}$ for the range $m_q = 330$ MeV $\rightarrow$ 220 MeV.

$^3P_J'$	Theory
$^3P_0'(1800?)$	$2.16 \rightarrow 2.32$ keV
$^3P_2'(1800?)$	$\underbrace{(0.08 \rightarrow 0.16)}_{\lambda=0} + \underbrace{(1.53 \rightarrow 2.44)}_{\lambda=2} = 1.61 \rightarrow 2.60$ keV
$\lambda = 0/\lambda = 2$	$5.2\% \rightarrow 6.6\%$
$^3P_0'/^3P_2'$	$1.34 \rightarrow 1.12$

Table 3.4 Estimated $\Gamma_{\gamma\gamma}$ for $l = 3$ light quarkonia

for the range $m_q = 330$ MeV $\to$ 220 MeV.

$^3F_J(u\bar{u} + d\bar{d})/\sqrt{2}$	Theory
$^3F_4(2050)$	$(\underbrace{0.03 \to 0.20}_{\lambda=0}) + (\underbrace{0.33 \to 1.56}_{\lambda=2}) = 0.36 \to 1.76$ keV $\lambda = 0 \Rightarrow \quad \approx 8\% \to \approx 11.5\%$
$^3F_3(2050?)$	$0.50 \to 2.49$ keV only $\lambda = 0$
$^3F_2(2050?)$	$(\underbrace{0.63 \to 2.62}_{\lambda=0}) + (\underbrace{1.85 \to 8.49}_{\lambda=2}) = 2.48 \to 11.11$ keV $\lambda = 0 \Rightarrow \quad \approx 25\% \to \approx 23\%$

multiplet $l = 1$ $^3P_0$ state, the $^3F_2$ state at the bottom of the $l = 3$ multiplet couples with comparable strength to both $\lambda = 0$ and $\lambda = 2$ $\gamma\gamma$ states.

Nonrelativistically their ratio (3.4) is $294/919 \approx 32\%$ for the $\lambda = 0/total$ contribution to this $^3F_2$ 2^{++} decay, which should be compared to the relativistic results (in **Table 3.4**) which finds that this ratio is $\approx 25\%$. This indicates again that the nonrelativistic helicity selection rules in (3.1)-(3.4) are accurate when compared to the relativistic calculations, to within the accuracies expected in quark model calculations. Thus, because of the simplicity of these nonrelativistic formulas, they are very useful as approximate helicity selection rules. However, as we have seen for $l = 0, 1$ and 2 states, for the overall scale of $\Gamma_{\gamma\gamma}$ widths the relativistic formulas are certainly required. (The multiplet ratios are not so sensitive to relativistic corrections, $\Gamma_{\gamma\gamma}(4^{++}) : \Gamma_{\gamma\gamma}(3^{++}) : \Gamma_{\gamma\gamma}(2^{++})$ is $1 : 1 : 919/100$ nonrelativistically (3.2)-(3.4), and becomes $1 : \approx 1.4 : \approx 6.5$ relativistically, as shown in **Table 3.4**).

We do not find significant suppression of light-$q\bar{q}$ $\gamma\gamma$ widths with increasing l, which is consistent with the observation of the $a_2(1320)$ and the $\pi_2(1670)$ with comparable $\gamma\gamma$ widths. Higher-l states such as the $l = 3$ $4^{++}/3^{++}/2^{++}$ multiplet at ≈ 2050 MeV and the $l = 4$ $\pi_4(2330?)$ should therefore be observable in $\gamma\gamma$ collisions. We emphasize that the difference between the very small $\lambda = 0$ contribution to the $l = 1$ 2^{++} state and the comparable $\lambda = 0$ and $\lambda = 2$ contributions in the $l = 3$ 2^{++} $q\bar{q}$ decay should be a characteristic signature of the level of orbital excitation of a given 2^{++} state.

We also do not find significant $\Gamma_{\gamma\gamma}$ suppression with radial excitation in any of the $q\bar{q}$ systems considered here (see **Table 3.1** and **Table 3.3**). This suggests that the $\pi(1300)$, radially excited light η states, radially excited 2^3P_2 $q\bar{q}$ states at ≈ 1800 MeV, and a radial $\pi_2(2130?)$ may all be observable in $\gamma\gamma$. As no radial excitations have been reported in $\gamma\gamma$ collisions to date, these states may provide particularly sensitive tests of the model calculations presented here.

3.5 $\Gamma_{\gamma\gamma}$ for $c\bar{c}$, $b\bar{b}$ and $t\bar{t}$ states

We present our numerical results for $c\bar{c}$ states in **Table 3.5** and **Table 3.6**, and for $b\bar{b}$ in **Table 3.7** and **Table 3.8**. The $c\bar{c}$ and $b\bar{b}$ parameter sets used are $(\alpha_s, a, m_Q) = (0.4, 0.18\ \mathrm{GeV}^2, 1.4\ \mathrm{GeV})$ and $(0.15, 0.18\ \mathrm{GeV}^2, 4.5\ \mathrm{GeV})$ respectively; these are typical of values found in fits to the spectrum of states. For the ${}^1G_4$ $c\bar{c}$ and $b\bar{b}$ states, which are not discussed by Godfrey and Isgur, we use an approximate mass extrapolated from their $l < 4$ results. We caution the reader that the 4^{-+} rates incorporate an $(M_R/M_{ref})^7$ factor, so the M_{ref} ambiguity discussed previously is quite large, and these rates are at best order-of-magnitude estimates. Once more we find that the overall scales for $c\bar{c}$ states are rather insensitive to α_s and a but depend strongly on m_c. For example, changing m_c from 1.4 GeV to 1.6 GeV changes $\Gamma_{\gamma\gamma}(\eta_c)$ from 4.8 keV to 3.4 keV and $\Gamma_{\gamma\gamma}(\chi_{c2})$ from 0.56 keV to 0.34 keV. The ratios of these rates however are not affected so strongly (for example the ratio $\Gamma_{\gamma\gamma}(\eta_c)/\Gamma_{\gamma\gamma}(\chi_{c2}) \approx 10$ for a broad range of m_c).

Notice that our results for $c\bar{c}$ and $b\bar{b}$ states are consistent with the expectation of smaller relativistic effects for heavier quark systems. For example, the $\lambda = 0$ contribution to $\Gamma_{\gamma\gamma}(\chi_{c2})$ is predicted to be 0.005 relativistically, which is much closer to the null contribution found nonrelativistically than the 0.02−0.03 predicted for light quarks. Furthermore, the ratio $\Gamma_{\gamma\gamma}(0^{++})/\Gamma_{\gamma\gamma}(2^{++})$ is ≈ 2.8

Table 3.5 Estimated and observed singlet $c\bar{c}$ $\gamma\gamma$ widths.

0^{-+}	2^{-+}	4^{-+}
$\eta_c(2980)$ 4.8 keV expt.[19] $6.6^{+2.4}_{-2.1}$ keV		
$\eta_c(3590)$ 3.7 keV	$\eta_c(1D)(3840?)$ eV 20. eV	
$\eta_c(2S)(4060?)$ 3.3 keV	$\eta_c(2D)(4210?)$ 35. eV	$\eta_c(1G)(4350?)$ 0.92 eV

Table 3.6 Estimates and measurements of $\Gamma_{\gamma\gamma}$ for $l = 1$ $c\bar{c}$ states

^3P_J $c\bar{c}$	Theory	Experiment
$\chi_{c2}(3555)$	0.56 keV	$0.251 \pm 0.069^{+0.014}_{-0.017}$ keV[55]
		< 1.0 keV (95% *c.l.*)[19,56]
	only $\approx 0.5\% \lambda = 0$	$3.8 \pm 1.7 \pm 1.2$ keV[57]
$\chi_{c0}(3415)$	1.56 keV	4.0 ± 2.8 keV[19]
χ_{c0}/χ_{c2}	2.79	

Table 3.7 Estimated singlet $b\bar{b}$ $\gamma\gamma$ widths.

0^{-+}	2^{-+}	4^{-+}
$\eta_b(1S)(9400?)$ 0.17 keV		
$\eta_b(2S)(9980?)$ 0.13 keV	$\eta_b(1D)(10150?)$ 33. meV	
$\eta_b(3S)(10340?)$ 0.11 keV	$\eta_b(2D)(10450?)$ 69. meV	$\eta_b(1G)(10500?)$ 59. μeV

Table 3.8 Estimated P-wave $b\bar{b}$ $\gamma\gamma$

widths and ratio

^3P_J $b\bar{b}$	Theory
$\chi_{b2}(9900)$	0.0037 keV
$\chi_{b0}(9900)$	0.0013 keV
χ_{b0}/χ_{b2}	3.514

for $c\bar{c}$ and ≈ 3.5 for $b\bar{b}$ systems, much closer to the nonrelativistic expectation of 15/4 than the ≈ 1.8 expected for light quarks.

A qualitative conclusion regarding $c\bar{c}$ and $b\bar{b}$ systems is that $\Gamma_{\gamma\gamma}$ is quickly suppressed by orbital excitation. This orbital suppression grows more pronounced as the quark mass increases, so that each step of $\Delta l = 2$ results in a decrease in $\Gamma_{\gamma\gamma}$ by about two orders of magnitude for $c\bar{c}$ and about three orders of magnitude for $b\bar{b}$. As the current statistical accuracy in resolving $c\bar{c}$ states in $\gamma\gamma$ collisions is at the keV level, **Table 3.2** and **Table 3.3** show that it is clearly unrealistic to expect to observe $l \geq 2$ singlet $c\bar{c}$ and $b\bar{b}$ states at e^+e^- machines. Radial excitation however does not significantly decrease $\Gamma_{\gamma\gamma}$ for these states, as is evident in **Table 3.5** and **Table 3.7**. As an example from the $s = 1$ states, we find

$$\Gamma_{\gamma\gamma}(\chi'_{c2}) = 0.64\,\mathrm{keV}\;. \tag{3.37}$$

Since our calculations are presumably more reliable for heavier quark systems, radial excitations such as η'_c are high-priority targets, as no radial excitations have yet been observed in $\gamma\gamma$ collisions.

For $t\bar{t}$ systems we consider only the lowest-lying state, the pseudoscalar η_t. Assuming a mass of $m_t = 140$ GeV, and taking $\alpha_s = 0.1$ we expect

$$\Gamma_{\gamma\gamma}(\eta_t) \approx 5\,\mathrm{keV}\;. \tag{3.38}$$

Again the $t\bar{t}$ orbital excitations are expected to have very small $\Gamma_{\gamma\gamma}$.

CHAPTER 4

SUMMARY AND CONCLUSIONS

We have derived relativistic and nonrelativistic results for the $\gamma\gamma$ partial widths of positronium and quarkonium states with general angular quantum numbers (J^{PC}). We find that

1. Our relativistic $\Gamma_{\gamma\gamma}$ formulas agree with previously reported relativistic calculations of the 0^{-+} and 0^{++} widths as special cases (Note the latter was based on a different technique).

2. Our nonrelativistic formulas lead to closed form results for the $\Gamma_{\gamma\gamma}$ of any positronium bound state, and agree as special cases with the results previously derived by many authors for 0^{-+}, 0^{++} and 2^{++}, and the recently derived $\Gamma_{\gamma\gamma}$ for 2^{-+} due to Anderson, Austern and Cahn.

3. For the 2^{-+} case, we find large relativistic corrections to the result of Anderson, Austern and Cahn, and our relativistic $\Gamma_{\gamma\gamma}(2^{-+})$ is numerically much closer to the observed $\pi_2(1670) \to \gamma\gamma$ partial width.

4. We find that the light 2^{-+}, $l = 3$ 2^{++}, 3^{++} and 4^{++}, $l = 4$ 4^{-+} $q\bar{q}$ states, and their radial excitations should all be observable in $\gamma\gamma$ collisions, as we

find no significant radial or orbital suppression for the $\Gamma_{\gamma\gamma}$ of light quark systems.

5. For heavier quark systems such as $c\bar{c}$ and $b\bar{b}$, we find that orbital excitation beyond $l = 1$ quickly reduces the $\gamma\gamma$ width below the current experimental accuracy in resolving $c\bar{c}$ states in $\gamma\gamma$ collisions. However we find that moderate radial excitation does not significantly decrease the $\gamma\gamma$ couplings of these systems.

6. The nonrelativistic formulas provide helicity selection rules accurate to $\approx$ 10% for $\gamma\gamma$ decays of $q\bar{q}$ systems. The accuracy (compared to the relativistic results) increases with increasing quark mass, as expected. For example, the vanishing nonrelativistic $\lambda = 0$ contribution for the $l = 1$ 2^{++} state receives small relativistic corrections to become $\approx 2\% - 3\%$ for light quarks, and these effects are smaller for the χ_{c2}, which has a $\approx 0.5\%$ $\lambda = 0$ contribution. For any nodeless wavefunction we find that the $\lambda = 0$ contribution to $\Gamma_{\gamma\gamma}(^3P_2)$ is at most 4.7%.

7. The ratio $\Gamma_{\gamma\gamma}(0^{++})/\Gamma_{\gamma\gamma}(2^{++})$, which is 15/4 nonrelativistically, is suppressed by relativistic corrections to ≈ 1.8 for light quarks. Again the departure from the nonrelativistic value (15/4) decreases with increasing quark mass.

8. We find new results for the light $l = 3$ multiplet, $\Gamma_{\gamma\gamma}(4^{++}) : \Gamma_{\gamma\gamma}(3^{++}) : \Gamma_{\gamma\gamma}(2^{++}) = 1 : \approx 1.4 : \approx 6.5$ relativistically. (Nonrelativistically, it is $1 : 1 : 919/100$.) The helicity structure of the $\gamma\gamma$ decays of this multiplet is nontrivial; the top of the multiplet (4^{++}) has a small $\lambda = 0$ contribution, the middle 3^{++} state is only $\lambda = 2$ ($\lambda = 0$ is not allowed for odd^{++} states), and the bottom 2^{++} state has comparable $\lambda = 0$ and $\lambda = 2$ contributions. This 2^{++} is particularly interesting, as its distinctive helicity structure allows one to distinguish between $l = 1$ and $l = 3$ states, since the $l = 1$ 2^{++} has a very small $\lambda = 0$ contribution.

These results suggest several interesting possibilities for experimental searches. Since the $l = 2$ $\pi_2(1670)$ has been observed in $\gamma\gamma$ events, we suggest that the next target might be the $l = 3$ ${}^3F_J(2050)$ multiplet, which should be observable in $\gamma\gamma$ according to our results; the bottom of this multiplet, the 2^{++}, is especially interesting as it is expected to have a nontrivial helicity structure. We also suggest the radial excitations as experimental targets, since no radial excitations have yet been identified in $\gamma\gamma$ collisions. As examples, the $\eta(1295)$, $\eta(1440)$, $f_2(1810)$ and $\eta_c(3590)$ should all be observable in $\gamma\gamma$ collisions with $\gamma\gamma$ widths near or above 1 keV.

Our formulas for $\Gamma_{\gamma\gamma}(q\bar{q})$ complete a program of theoretical calculations of two-photon couplings of fermion-antifermion bound states at $\mathcal{O}(\alpha)$ which has spanned ≈ 50 years. Considerable interesting work remains to be done for the relativistic $q\bar{q}$ decays, however; since we incorporated the dependence of $\Gamma_{\gamma\gamma}$ on the mass of the decaying resonance as a phenomenological input, a method which incorporates such a dependence naturally is an obvious next improvement in $\Gamma_{\gamma\gamma}(q\bar{q})$ calculations. A very closely related topic is investigated in single tag experiments (in which one of the scattered leptons in e^+e^- is detected, so that one may investigate $Q^2 \neq 0$ couplings). These experiments will accumulate enough data in future to allow the detailed study of $\gamma\gamma^*$ couplings to mesons, and it will be very useful to have relativistic $\Gamma_{\gamma\gamma^*}(q\bar{q})$ results available for comparison with experiment.

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APPENDICES

APPENDIX A

One may demonstrate that the term in (2.67), $\sum \equiv \sum_{\lambda_1,\lambda_2} \left|\hat{k}\cdot(\vec{\epsilon}_1^{\,*} \times \vec{\epsilon}_2^{\,*})\right|^2 =$ 2 for an arbitrary direction $\hat{k}$ by expanding the cross product $\vec{\epsilon}_1^{\,*} \times \vec{\epsilon}_2^{\,*}$ using the Levi-Civita tensor, ϵ^{lmn},

$$\vec{A} \times \vec{B} = \epsilon^{lmn} A_m B_n \, \hat{x}_l \, , \tag{A.1}$$

where repeated indices are summed over, and by using the identity

$$\sum_{\lambda_i} \epsilon_i^{*l} \epsilon_i^{*m} = \delta^{lm} - \frac{k_i^l k_i^m}{k_i^2} \, , \tag{A.2}$$

and relations between the Levi-Civita tensors, and $\vec{k}_1 = -\vec{k}_2$. A less involved route would be to rewrite the sum $\sum_{\lambda_1,\lambda_2} \rightarrow \sum_{\lambda}$, where λ is the helicity of the $\gamma\gamma$ state, and to use the corresponding $\gamma\gamma$ helicity states in (1.38)-(1.40), as in the derivation of (2.92) and (2.96). Now, since for J^{-+} only $\lambda = 0$ is allowed, we may use (1.38) to write

$$\begin{aligned}\sum &= \left|\hat{k}\cdot(\vec{\epsilon}_1^{\,*} \times \vec{\epsilon}_2^{\,*})\big|_{\lambda=0}\right|^2 \\ &= \left|\frac{1}{\sqrt{2}}\{\vec{\epsilon}_1^{\,*}(+) \times \vec{\epsilon}_2^{\,*}(+) - \vec{\epsilon}_1^{\,*}(-) \times \vec{\epsilon}_2^{\,*}(-)\}\right|^2 . \end{aligned} \tag{A.3}$$

Since we may set $\hat{k} = \hat{z}$ without loss of generality, we may use (2.103), (2.104), (2.106) and (2.107) to obtain $\sum = 2$.

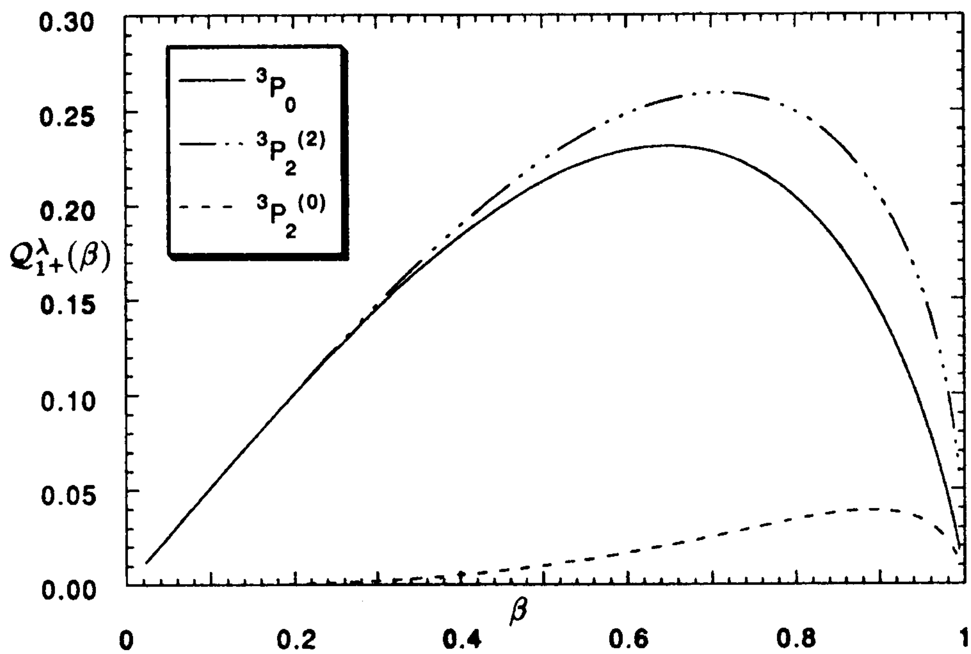

Fig 3. Weight functions: $Q_{0^+}(\beta)$ and $Q_{2^+}^{\lambda=0,2}(\beta)$. (Note that $Q_{0^+}(\beta)$ is multiplied by $\sqrt{4/15}$ for comparison with $Q_{2^+}(\beta)$.)

Here we display two light $q\bar{q}$ wavefunctions $\phi(p)$ and the weight functions $Q^{\lambda}_{J^P}(\beta)$ for P-wave quarkonia. **Fig 3** shows the weight functions for $l = 1$; note the $\lambda = 0$ and $\lambda = 2$ components of the 2^{++} amplitude. **Fig 4** and **Fig 5** show the scaled wavefunctions $\tilde{\phi}(p) \equiv p^2\phi(p)dp/d\beta$ for the $l = 1$ $q\bar{q}$ ground state and first radial excitation, respectively. Inspection of the ratio $Q_{2^+}^{\lambda=0}(\beta)/Q_{2^+}^{\lambda=2}(\beta)$ leads to the general conclusion that for any nodeless wavefunction $\phi(p)$ one has

$$\frac{\Gamma^{\lambda=0}_{\gamma\gamma}(^3P_2(q\bar{q}))}{\Gamma^{tot}_{\gamma\gamma}(^3P_2(q\bar{q}))} \leq 4.7\% \, . \qquad (B.1)$$

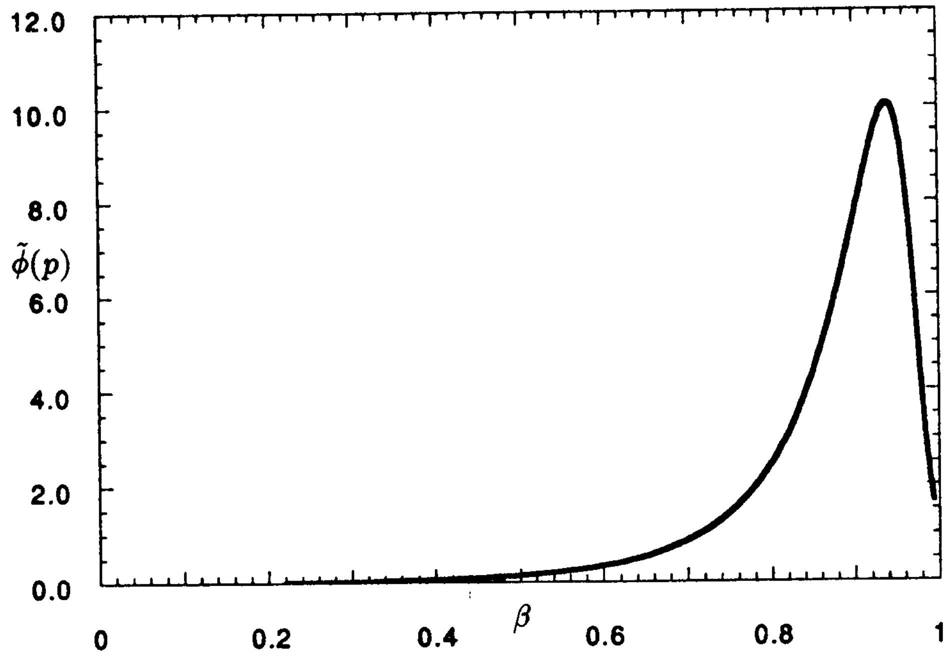

Fig 4. Scaled radial wavefunction for light $l = 1$ $q\bar{q}$ state.

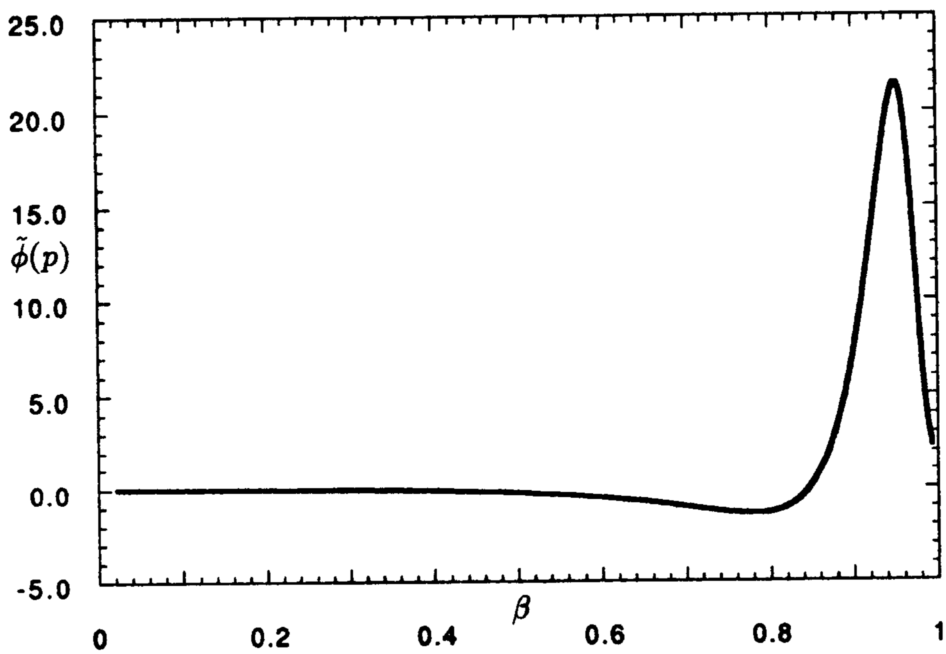

Fig 5. Scaled 1^{st} radially excited wavefunction; light $l = 1$ $q\bar{q}$.

VITA

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