To the Graduate Council:

I am submitting herewith a dissertation written by Mladen Bestvina entitled "Characterizing k-Dimensional Universal Menger Compacta." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

John J. Walsh, Major Professor

We have read this dissertation
and recommend its acceptance:

Accepted for the Council:

The Graduate School

CHARACTERIZING k-DIMENSIONAL UNIVERSAL MENGER COMPACTA

A Dissertation
Presented for the
Doctor of Philosophy
Degree
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Mladen Bestvina

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DEDICATION

To my parents, and to my wife Cynthia

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ABSTRACT

In the past few years H. Toruńczyk and R.D. Edwards-F. Quinn characterized infinite-dimensional manifolds modeled on Q or ℓ_2 and n-dimensional $(n \geq 5)$ manifolds modeled on Euclidean spaces, respectively. Briefly, if a space X satisfies the correct homotopy-theoretic local property, together with a certain "general positioning" property, then X is a manifold of the expected sort.

The main result of the thesis follows the same pattern: If a compact, (k-1)-connected, locally (k-1)-connected, k-dimensional metric space X has the "disjoint k-cells property" (i.e. any two maps $f, g: I^k \to X$ can be approximated by maps with disjoint images), then X is homeomorphic to the k-dimensional universal Menger space μ^k.

Using this result we answer in the affirmative the question whether different constructions of the universal k-dimensional compactum appearing in the literature yield the same space.

Also, we prove a few theorems strongly resembling the well-known facts about Q-manifolds. We mention only one example: the Z-set unknotting theorem. In particular, we show that μ^k is strongly homogeneous.

To prove these theorems, one needs to have a device for constructing homeomorphisms (with control) of μ^k onto itself. To that end, we use a "combinatorial approach": μ^k is represented as the intersection of a nice family of PL manifolds, each of which possesses a "handlebody structure" (where "handles" reflect the properties of μ^k). Using some standard PL-techniques we construct sequences of "handlebody

structures" with correct properties, which ultimately yield the desired homeomorphism.

The "philosophical" output can be summarized in one sentence. The universal k-dimensional Menger compactum μ^k and the Hilbert cube Q cannot be distinguished from the point of view of k-dimensional compacta.

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INTRODUCTION

One of the oldest tasks of topology is classification of spaces. The history of this question began in 1920 with the work of R.L. Moore [Mo] who characterized the unit interval $[0, 1]$ and the simple closed curve S^1 in terms of separation properties. Except for an excursion to 2-manifolds (S^2 was characterized by R.L. Moore, K. Kuratowski [Ku], L. Zippin [Zi], R.H. Bing [Bi_1] and others), this was followed by characterizing interesting one-dimensional continua, including the pseudo-arc [Bi_2] and the universal curve.

In [An_1] and [An_2] R.D. Anderson characterized the universal curve as the one dimensional locally connected continuum with no local cut points that has no non-empty open subset which embeds into the plane. He also proved that the universal curve is homogeneous.

The construction of the universal curve generalizes to give a universal k-dimensional Menger compactum. Several questions naturally arise:

1. Is there a way of characterizing the universal k-dimensional Menger compactum?
2. Do different constructions of the universal k-dimensional Menger compactum appearing in the literature (see [Mg], [Lf], [Pa]) yield the same space?
3. Is the universal k-dimensional Menger compactum homogeneous?

The universal k-dimensional Menger compactum possesses the attribute "universal" because every k-dimensional compactum embeds into it. In his Ph.D. thesis D. Wilson [Wi] established one more universal property of the universal curve: every locally connected continuum is the image of the universal curve under a monotone map. Keeping in mind the fact that the k-dimensional universal Menger compactum is (k-1)-connected (C^{k-1}) and locally (k-1)-connected (LC^{k-1}), we may wonder

4. Is every C^{k-1}, LC^{k-1} compactum the image of the k-dimensional universal Menger compactum under a UV^{k-1}-map?

The introduction of the disjoint k-cells property (DD^kP) was an important ingredient in the subsequent characterization theorems of Edwards-Quinn (Euclidean manifolds of dimension ≥ 5 [Ed], [Qu]) and Toruńczyk (Hilbert cube [To_1] and Hilbert space manifolds [To_2]). Looking at the characterization theorem for Q-manifolds (locally compact ANR that satisfies DD^kP for $k = 0,1,2,\ldots$) one is tempted to conjecture the following

<u>Characterization Theorem.</u> For a locally compact metric space X the following two statements are equivalent.

(1) X admits an open cover every element of which is homeomorphic to an open subset of the universal k-dimensional Menger compactum.

(2) X is LC^{k-1}, $\dim X = k$, and X satisfies DD^kP.

The ultimate goal of this treatise is to prove the above theorem. Along the way we answer in the affirmative the questions 1-4 listed above.

Comparing the two characterization theorems (for Q-manifolds and for Menger-space-manifolds) one realizes that there should be a strong parallel between Q-manifold theory and Menger-space-manifold theory.

Indeed, we prove a few theorems (e.g. The Z-set Unknotting Theorem) inspired by the corresponding theorems for Q-manifolds.

The treatise is organized as follows. In Chapter 1 we introduce the notion of a partition (a nice collection of submanifolds of a given manifold), which is the major tool in investigating the universal compacta. There we give a new construction of the k-dimensional universal compactum that appears to be more suitable for our purposes. We also define what we mean by a triangulated Menger-space-manifold. These are particularly convenient for a combinatorial analysis, since they can be represented as a nice intersection of PL manifolds ("stages"). In Chapter 2 we develop "PL techniques for k-dimensional universal compacta". This chapter consists of several sections, each of which describes a "move" we use later. Each move consists in improving a partition on a "stage".

Chapter 3 is devoted to a proof of The Z-set Unknotting Theorem. A special case of this theorem states that the universal k-dimensional Menger space is homogeneous.

In Chapter 4 we develop the decomposition theory of the universal k-dimensional Menger space. An important result is that every UV^{k-1}-surjection between Menger-space-manifolds can be approximated by a homeomorphism. This result shows that UV^{k-1}-surjections in the category of Menger-space-manifolds play the same role as cell-like mappings do in the category of Q-manifolds.

In Chapter 5 we prove The Resolution Theorem which answers in the affirmative the question 4. At the end of Chapter 5 we deduce the Characterization Theorem. One of its consequences is that all Menger-space-manifolds are "triangulable" in the sense of Chapter 1.

In the first five chapters we consider only the case of compact Menger-space-manifolds. With routine changes (open covers instead of epsilonics), the results are valid for non-compact Menger-space-manifolds. In Chapter 6 we outline parts of proofs for the non-compact case that are substantially different from the compact case.

DEFINITIONS AND NOTATION

All spaces in this treatise are separable and metric. By d we denote a metric on a space X compatible with the topology on it. If A is a subspace of X, and $x \in X$, by $d(x, A)$ we denote the number $\inf\{d(x, y): y \in A\}$, and we set $N_\varepsilon(A) = \{p \in X: d(p, A) < \varepsilon\}$ and $\operatorname{diam} A = \sup\{d(a, b): a, b \in A\}$. If $f,g: Y \to X$ are two maps, we use $(f, g) < \varepsilon$ to denote that $d(f(y), g(y)) < \varepsilon$ for all $y \in Y$. The identity map $X \to X$ is denoted by Id_X, or, more often, Id. We use I to denote the unit interval $[0, 1]$; E^m is the m-fold product of reals. A simplicial complex is a collection K of simplexes in some Euclidean space such that if $\sigma, \tau \in K$, $\sigma \cap \tau \neq \phi$ then $\sigma \cap \tau$ is a face of σ, and all faces of σ are in K. The underlying topological space is denoted by $|K|$. Abusing the notation, for a subcollection $L \subseteq K$, by $K - L$ we denote the simplicial complex consisting of all simplexes in $K - L$ and their faces. If K is a simplicial complex, βK is the barycentric subdivision obtained from K using geometric barycenters. Also, $\beta^2 K = \beta(\beta K)$.

All manifolds will be PL and, except in Chapter 6, compact, and all triangulations of manifolds will be combinatorial. We use superscripts to denote the dimension of a manifold. The boundary and the interior of a manifold M are denoted by ∂M and $\operatorname{Int} M$ respectively. We use $\operatorname{int} A$ and $\operatorname{Fr} A$ to denote the topological interior and the topological frontier of $A \subseteq X$, respectively.

Let P be a family of subsets of a space X, and let $A \subseteq X$. Then we set

$$\cup P = \cup\{p: p \in P\}, \quad \cap P = \cap\{p: p \in P\},$$

$$P|A = \{p \cap A: p \in P, p \cap A \neq \phi\},$$

$$St(A, P) = \cup\{p: p \in P, p \cap A \neq \phi\},$$

$$St\ P = \{St(p, P): p \in P\}, \text{ and}$$

$$\text{mesh } P = \sup\{\text{diam } p: p \in P\}.$$

We will use the phrase "$f: X \to Y$ induces an isomorphism of homotopy groups of dimension $\leq k - 1$". For $k = 0$ this phrase is vacuous, and for $k \geq 1$ it means that f induces a bijection between the components of X and Y, and that $f'_{\#}: \pi_i(C_X) \to \pi_i(C_Y)$ is an isomorphism for $i \leq k - 1$ and all choices of components $C_X \subseteq X$, $C_Y \subseteq Y$ with $f(C_X) \subseteq C_Y$, where $f': C_X \to C_Y$ is the restriction of f. A space X is $(k-1)$-connected (C^{k-1}) if $\pi_i(X) = 0$ for $i \leq k - 1$. A space X is locally $(k-1)$-connected (LC^{k-1}) if for every $x \in X$ and every neighborhood $U \ni x$ there exists a neighborhood $V \ni x$ with the property that every map $\alpha: \partial B^{i+1} \to V$ $(i = 0,1,\ldots,k-1)$ extends to $\tilde{\alpha}: B^{i+1} \to U$.

CHAPTER 1

PARTITIONS

1.1. Partitions on Compact PL-Manifolds (With Boundary)

In this section we introduce the notion of a partition on a manifold, adapting the definition in [Wa_1] to the case where the manifold has nonempty boundary. The example to keep in mind is the handlebody decomposition of a manifold with respect to a given (combinatorial) triangulation. We state and prove some basic properties of partitions. Finally, we introduce the notion of a k-partition $(k \geq 0)$, which is the Menger-space counterpart of the handlebody decomposition.

1.1.1. Definition. Let M^m be a compact PL-manifold with (possibly empty) boundary. A partition on M is a finite collection $P = \{p_\lambda : \lambda \in \Lambda\}$ of compact m-dimensional (PL) submanifolds of M so that (here $\Lambda^{(t)} = \{(\lambda(1), \lambda(2), \ldots, \lambda(t)) \in \Lambda^t | i \neq j \Rightarrow \lambda(i) \neq \lambda(j)\}$)

(1) $\cup P = M$,

(2) for any $t \geq 2$ and any $(\lambda(1), \ldots, \lambda(t)) \in \Lambda^{(t)}$ $p_{\lambda(1)} \cap \ldots \cap p_{\lambda(t)}$ is either ϕ or a $(m-t+1)$-dimensional submanifold of $\partial(p_{\lambda(1)} \cap \ldots \cap p_{\lambda(t-1)})$.

1.1.2. Example. Let L be a (combinatorial) triangulation of a manifold M^m, and let βL, $\beta^2 L$ be the first and the second barycentric subdivision respectively. A vertex v in L is the barycenter of some simplex $\sigma(v)$ in L. Set $\mathrm{Ind}(v) = \dim \sigma(v)$ (index of v), and define

$$P = \{St(v, \beta^2L) : v \text{ is a vertex in } \beta L\}$$

It is well known that the collection $\{St(v_i, \beta^2L), i \in I\}$ has non-empty intersection iff inclusion defines a linear order on $\{\sigma(v_i), i \in I\}$. In that case $\bigcap_{i\in I} St(v_i, \beta^2L)$ is the dual cell corresponding to the simplex in βL determined by $\{\sigma(v_i), i \in I\}$. This immediately implies that P is a partition on M.

1.1.3. Lemma. Let P be a partition on M, $q_1, q_2 \in P$, $q_1 \neq q_2$. Let $q = q_1 \cup q_2$. Then $Q = \{q\} \cup P - \{q_1, q_2\}$ is a partition on M.

Proof. (1) is clear. Also, q is a manifold. To prove (2), take $p_1, \ldots, p_t \in P - \{q_1, q_2\}$ mutually distinct, and observe that $p_1 \cap \ldots \cap p_t \cap q$ is the union of two $(m-t)$-manifolds $p_1 \cap \ldots \cap p_t \cap q_1$ and $p_1 \cap \ldots \cap p_2 \cap q_1$ whose intersection $p_1 \cap \ldots \cap p_t \cap q_1 \cap q_2$ is a submanifold of the boundary of both. Hence $p_1 \cap \ldots \cap p_t \cap q$ is an $(m-t)$-manifold. It is clear that $p_1 \cap \ldots \cap p_t \cap q \subseteq \partial(p_1 \cap \ldots \cap p_t)$ (since $p_1 \cap \ldots \cap p_t \cap q_i \subseteq \partial(p_1 \cap \ldots \cap p_t)$, $i = 1,2$). We need to show that $p_1 \cap \ldots \cap p_t \cap q_i \subseteq \partial(p_2 \cap \ldots p_t \cap q)$, $i = 1,2$. By assumption, $p_1 \cap \ldots \cap p_t \cap q_i \subseteq \partial(p_2 \cap \ldots \cap p_t \cap q_i) \subseteq$ $\subseteq \partial(p_2 \cap \ldots \cap p_t \cap q) \cup (p_2 \cap \ldots \cap p_t \cap q_1 \cap q_2)$. Since $p_1 \cap \ldots \cap p_t \cap q_i$ does not contain interior points of $p_2 \cap \ldots \cap p_t \cap q_1 \cap q_2$ (the intersection of these two sets is contained in the boundary of the latter one), and since $\partial(p_2 \cap \ldots \cap p_t \cap q)$ contains $\partial(p_2 \cap \ldots \cap p_1 \cap q_1 \cap q_2)$ (observe that

$\partial(p_2 \cap \dots \cap p_t \cap q) = \partial(p_2 \cap \dots \cap p_t \cap q_1) \cup \partial(p_2 \cap \dots \cap p_t \cap q_2)$ $- \operatorname{Int}(p_2 \cap \dots \cap p_t \cap q_1 \cap q_2))$, the desired containment follows.

<u>1.1.4. Definition.</u> Let $P = \{p_\lambda : \lambda \in \Lambda\}$, $Q = \{q_\nu : \nu \in N\}$ be two collections of sets. We say that Q is an <u>amalgamation</u> of P if there is a surjection $\phi\colon \Lambda \to N$ so that $q_\nu = \cup\{p_\lambda : \phi(\lambda) = \nu\}$.

<u>1.1.5. Corollary.</u> An amalgamation of a partition on a manifold is a partition.

<u>Proof</u>. Repeatedly use 1.1.3.

<u>1.1.6. Corollary.</u> If P is a partition on M, $p_1, \dots, p_t \in P$ mutually distinct elements with non-empty intersection, then $P_0 = \{p \cap p_1 \cap \dots \cap p_t : p \in P - \{p_1, \dots, p_t\},\ p \cap p_1 \cap \dots \cap p_t \neq \phi\}$ is a partition on $\cup P_0$.

<u>Proof.</u> The only non-trivial ingredient, that $\cup P_0$ is a manifold, follows from 1.1.5., since $\cup P_0 = (q_1 \cup \dots \cup q_1) \cap p_1 \cap \dots \cap p_t$, for some $q_1, \dots, q_t \in P - \{p_1, \dots, p_t\}$.

<u>1.1.7. Proposition.</u> If P is a partition, then any subcollection $Q \subseteq P$ is a partition on $\cup Q$.

<u>Proof.</u> Clear.

<u>1.1.8. Proposition.</u> Suppose M_1^m, M_2^m are two manifolds, $N^{m-1} = M_1 \cap M_2$ is a submanifold of both ∂M_1 and ∂M_2; P is a partition on M_1, Q is a partition on M_2. Further assume that K is a triangulation of N and that

(i) $\{q \cap N : q \in Q$ and $q \cap N \neq \phi\}$ is a partition on N ,

(ii) if $q \cap N \neq \phi$, then $q \cap N$ underlies a subcomplex of K ,

(iii) $\{p \cap N : p \in P$ and $p \cap N \neq \phi\} = \{St(v, \beta^2 K) : v$ is a vertex of $\beta K\}$.

Then $P \cup Q$ is a partition on $M_1 \cup M_2$.

Proof. The only nontrivial part of the proof is to check that $p_1 \cap \ldots \cap p_i \cap q_1 \cap q_2 \cap \ldots \cap q_j$ (if non-empty) is a submanifold of both $\partial(p_2 \cap \ldots \cap p_i \cap q_1 \cap \ldots \cap q_j)$ and $\partial(p_1 \cap \ldots \cap p_i \cap q_2 \cap \ldots \cap q_j)$ $(i,j \geq 1)$. However, this follows from

$$p_1 \cap \ldots \cap p_i \cap q_1 \cap \ldots \cap q_j =$$

$$= St(v_1, \beta^2 K) \cap \ldots \cap St(v_i, \beta^2 K) \cap (q_1 \cap N) \cap \ldots \cap (q_j \cap N),$$

where $p_\ell \cap N = St(v_\ell, \beta^2 K)$, $\ell = 1, \ldots, i$. (The expression on the right-hand side is the dual cell σ^* of the simplex σ in βK spanned by $v_1, \ldots, v_i$ in $(q_1 \cap N) \cap \ldots \cap (q_j \cap N)$ with respect to K [Hd] . Apply the fact that $\sigma^* \cap \partial N = (\sigma \cap \partial N)^*$ for a simplex σ in a triangulation of a manifold W , where $\sigma \cap \partial W$ is a face of σ , and that $\tau^* \subseteq \partial\sigma^*$, if $\tau \subseteq \partial\sigma$. If i or j is equal to 1, we have to apply 1.1.9.(c).)

1.1.9. Lemma. If P is a partition on M , and if $p_1, \ldots, p_t \in P$ are mutually distinct, then

(a) $\partial(p_1 \cap \ldots \cap p_t) \subseteq \partial(p_1 \cup \ldots \cup p_t)$

(b) $\mathrm{Int}(p_1 \cap \ldots \cap p_t) \subseteq \mathrm{Int}(p_1 \cup \ldots \cup p_t)$

(c) $p_1 \cap \ldots \cap p_t \cap \partial M \subseteq \partial(p_1 \cap \ldots \cap p_t)$.

Proof. We prove (a) and (b) by induction on t , starting non-trivially with $t = 2$.

$$\partial(p_1 \cup p_2) = \partial p_1 \cup \partial p_2 - \mathrm{Int}(p_1 \cap p_2) \supseteq \partial(p_1 \cap p_2)$$

$$\mathrm{Int}(p_1 \cup p_2) = \mathrm{Int} p_1 \cup \mathrm{Int} p_2 \cup \mathrm{Int}(p_1 \cap p_2) \supseteq \mathrm{Int}(p_1 \cap p_2)$$

Assuming (a) and (b) are true for $t - 1$, we prove it for $t \geq 3$. By 1.1.7 and 1.1.5, the collection $\{p_1, p_2, \ldots, p_{t-2}, p_{t-1} \cup p_t\}$ is a partition on $\bigcup_{i=1}^{t} p_i$. By inductive hypothesis we get

$$\begin{aligned}\partial(p_1 \cup \cdots \cup p_{t-1} \cup p_t) &\supseteq \partial(p_1 \cap \ldots \cap p_{t-2} \cap (p_{t-1} \cup p_t)) \\ &= \partial((p_1 \cap \ldots \cap p_{t-2} \cap p_{t-1}) \cup (p_1 \cap \ldots \cap p_{t-2} \cap p_t)) \\ &= \partial(p_1 \cap \ldots \cap p_{t-2} \cap p_{t-1}) \cup \partial(p_1 \cap \ldots \cap p_{t-2} \cap p_t) \\ &\quad - \mathrm{Int}(p_1 \cap \ldots \cap p_t) \supseteq \partial(p_1 \cap \ldots \cap p_t) ,\end{aligned}$$

and also

$$\begin{aligned}\mathrm{Int}(p_1 \cup \cdots \cup p_t) &\supseteq \mathrm{Int}(p_1 \cap \ldots \cap p_{t-2} \cap (p_{t-1} \cup p_t)) \\ &= \mathrm{Int}((p_1 \cap \ldots \cap p_{t-2} \cap p_{t-1}) \cup (p_1 \cap \ldots \cap p_{t-1} \cap p_2))\end{aligned}$$

$$= \mathrm{Int}(p_1 \cap \dots \cap p_{t-2} \cap p_{t-1}) \cup \mathrm{Int}(p_1 \cap \dots \cap p_{t-2} \cap p_t)$$

$$\cup \mathrm{Int}(p_1 \cap \dots \cap p_t) \supseteq \mathrm{Int}(p_1 \cap \dots \cap p_t) ,$$

which finishes the proof of (a) and (b). To prove (c), observe that

$$p_1 \cap \dots \cap p_t \cap \partial M \subseteq (p_1 \cap \dots \cap p_t) \cap \partial(p_1 \cup \dots \cup p_t) .$$

The latter set is contained (by (b)) in $\partial(p_1 \cap \dots \cap p_t)$.

<u>1.1.10. Lemma.</u> Let M_1^m, M_2^m be two manifolds, $N^{m-1} = M_1 \cap M_2$ a submanifold of both ∂M_1 and ∂M_2 . Assume P is a collection of subsets of $M_1 \cup M_2$ so that $P|M_1$, $P|M_2$, $P|N$ are partitions on M_1, M_2, N respectively. Then P is a partition on $M_1 \cup M_2$.

<u>Proof.</u> Let $p \in P$. Then $p = (p \cap M_1) \cup (p \cap M_2)$ is a union of two m-manifolds, whose intersection is $p \cap N$, an $(m - 1)$-manifold. Moreover, by 1.1.9(c) $p \cap N \subseteq \partial(p \cap M_i)$, $i = 1,2$, and hence p is an m-manifold. Since it is clear that $\cup P = M_1 \cup M_2$, we proceed by verifying 1.1.1.(2).

$$p_1 \cap \dots \cap p_t = [(p_1 \cap M_1) \cap \dots \cap (p_t \cap M_1)]$$

$$\cup [(p_1 \cap M_2) \cap \dots \cap (p_t \cap M_2)]$$

Hence $p_1 \cap \dots \cap p_t$ (if non-empty) is a union of two $(m-t+1)$-manifolds, whose intersection $(p_1 \cap N) \cap \dots \cap (p_t \cap N)$ is an (m-t)-manifold that is (by 1.1.9.(c)) contained in the boundary of both. Hence $p_1 \cap \dots \cap p_t$ is an (m-t+1)-manifold. Also,

$\partial(p_1 \cap \dots \cap p_{t-1}) = \partial((p_1 \cap M_1) \cap \dots \cap (p_{t-1} \cap M_1)) \cup$

$\cup\ \partial((p_1 \cap M_2) \cap \dots \cap (p_{t-1} \cap M_2)) - \mathrm{Int}((p_1 \cap N) \cap \dots \cap (p_{t-1} \cap N)) \supseteq$

$[(p_1 \cap M_1) \cap \dots \cap (p_t \cap M_1)] \cup [(p_1 \cap M_2) \cap \dots \cap (p_t \cap M_2)] =$

$p_1 \cap \dots \cap p_t$.

<u>1.1.11. Proposition.</u> Let M^m be a manifold, P a partition on M, and Q a collection of subsets of M so that for any $p_1, \dots, p_t \in P$ with non-empty intersection, $Q | p_1 \cap \dots \cap p_t$ is a partition on $p_1 \cap \dots \cap p_t$. Then Q is a partition on M .

<u>Proof.</u> By induction on card P . An application of 1.1.10 gives us a non-trivial start for card $P = 2$. So assume that $n =$ card P , and let $P = \{p_1, \dots, p_n\}$. Let $P' = \{p_1 \cup p_2, p_3, \dots, p_n\}$. By 1.1.5, P' is a partition on M . By 1.1.10, the hypothesis of 1.1.11 remain valid if we replace P by P' . Therefore, by inductive hypothesis, Q is a partition on M .

<u>1.1.12. Proposition.</u> Let K be a triangulation of a manifold M^m , and P a partition on M^m so that each $p \in P$ underlies a subcomplex of K . Let $\{v_i : i \in I\}$ be a collection of vertices of βK . Then the non-empty elements of $\{p - \mathrm{Int} \bigcup_{i \in I} St(v_i, \beta^2 K) : p \in P\}$ $\cup \{St(v_i\ \beta^2 K) : i \in I\}$ form a partition on M .

<u>Proof.</u> The proof follows by observing that the restriction of the handlebody decomposition corresponding to K on the sets of the form $p_1 \cap \dots \cap p_t$ is the handlebody decomposition of $p_1 \cap \dots \cap p_t$

corresponding to $K|p_1 \cap \cdots \cap p_t$, and applying 1.1.5. to handlebody decompositions obtained in this way.

The importance of the notion of a k-partition was recognized by J.J. Walsh [Wa$_3$] where he builds UV^{k-1}-surjections between spheres (see Appendix).

1.1.13. Definition. A partition $P = \{p_\lambda : \lambda \in \Lambda\}$ on M^m is a k-partition $(k \geq 0)$ provided conditions (3)-(5) listed below hold (in addition to (1) and (2) from 1.1.1.).

(3) For any t , $1 \leq t \leq k$, and any $(\lambda(1),\ldots, \lambda(t)) \in \Lambda^{(t)}$, $p_{\lambda(1)} \cap \ldots \cap p_{\lambda(t)}$ is either ϕ or $(k-t)$-connected.

(4) For any $(\lambda(1),\ldots, \lambda(k+2)) \in \Lambda^{(k+2)}$, $p_{\lambda(1)} \cap \ldots \cap p_{\lambda(k+2)} = \phi$

(5) For any $\Lambda_0 \subseteq \Lambda$ with $|\Lambda_0| \leq k + 1$, if $\lambda_1,\lambda_2 \in \Lambda_0$ imply that $p_{\lambda_1} \cap p_{\lambda_2} \neq \phi$, then $\cap \{p_\lambda : \lambda \in \Lambda_0\} \neq \phi$.

1.1.14. Remark. Condition (4) reflects k-dimensionality of the k-dimensional Menger space. Condition (5) is useful in inductive arguments (compare 1.1.15). The philosophy in (3) is to require all $(m-t+1)$-dimensional manifolds in sight be $(k-t)$-connected.

This property is preserved in the process of digging out k-holes, which ultimately yields the k-dimensional Menger space $(m \geq 2k + 1)$.

1.1.15. Proposition. If P is a k-partition on M and $p \in P$, then $St(p, P)$ is $(k-1)$-connected.

Proof. We prove by induction on i that if $p_1,\ldots,p_i$, $q_1,\ldots, q_j \in P - \{p\}$ are mutually distinct elements that meet p , then $p \cup ((p_1 \cup \ldots \cup p_i) \cap q_1 \cap \ldots \cap q_j)$ is $(k-j-1)$-connected $(0 \leq j \leq k-1)$. If $i = 1$, then both p and $p_1 \cap q_1 \cap \ldots \cap q_j$ are $(k-j-1)$-connected (by 1.1.13(3)). If $p_1 \cap q_1 \cap \ldots \cap q_j = \phi$, there is nothing to prove. If $p_1 \cap q_1 \cap \ldots \cap q_j \neq \phi$, 1.1.13.(5) implies that $p \cap p_1 \cap q_1 \cap \ldots \cap q_j \neq \phi$. By 1.1.13.(3), the latter intersection is $(k-j-2)$-connected, and hence $p \cup (p_1 \cap q_1 \cap \ldots \cap q_j)$ is $(k-j-1)$-connected (by Mayer-Vietoris theorem [Sp, p. 189] first $(k-j-1)$-homology groups are trivial. If $k-j-1 \geq 1$ by Van-Kampen [Sp., p. 151] this space is simply connected. The rest follows from Hurewicz theorem [Sp, p. 394]). For the inductive step, observe that $p \cup ((p_1 \cup \ldots \cup p_i) \cap q_1 \cap \ldots \cap q_j)$ can be represented as $A \cup B$, where

$A = p \cup ((p_1 \cup \ldots \cup p_{i-1}) \cap q_1 \cap \ldots \cap q_j)$, $B = p \cup (p_i \cap q_1 \cap \ldots \cap q_j)$.

By inductive hypothesis, A is $(k-j-1)$-connected, and so is B. Note that $A \cap B = p \cup ((p_1 \cup \ldots \cup p_{i-1}) \cap p_i \cap q_1 \cap \ldots \cap q_j)$ is $(k-j-2)$-connected, by inductive hypothesis. Consequently (Van Kampen, Meyer-Vietoris, Hurewicz), $A \cup B$ is $(k-j-1)$-connected. For $j = 0$ and $\{p_1, \ldots, p_i\} = \{p' \in P: p' \neq p, p' \cap p \neq \phi\}$ we get that $St(p, P)$ is $(k-1)$-connected.

<u>1.1.16. Definition.</u> Let P_i be a partition on M_i, $i = 1,2$. We say that a function $T: P_1 \to P_2$ <u>preserves intersections</u> if for any $p_1, \ldots, p_t \in P_1$ with $p_1 \cap \ldots \cap p_t \neq \phi$ it follows that $T(p_1) \cap \ldots \cap T(p_t) \neq \phi$. We say that T is a <u>one-to-one correspondence</u> if T is a bijection and both T and T^{-1} preserve intersections.

The next lemma will help us detect when certain submanifolds are highly connected.

<u>1.1.17. Lemma.</u> Let $\mathcal{U} = \{U_\lambda : \lambda \varepsilon \Lambda\}$, $\mathcal{V} = \{V_\lambda : \lambda \varepsilon \Lambda\}$ (same indexing set) be open covers of topological spaces X, Y respectively, and let $k \geq 1$ be an integer. Assume that for any t, $1 \leq t \leq k+1$, and any $(\lambda(1), \ldots, \lambda(t)) \varepsilon \Lambda^{(t)}$ the sets $U_{\lambda(1)} \cap \cdots \cap U_{\lambda(t)}$ and $V_{\lambda(1)} \cap \cdots \cap V_{\lambda(t)}$ are either both empty, or both non-empty and $(k-t)$-connected. Then X is $(k-1)$-connected iff Y is $(k-1)$-connected.

<u>Proof.</u> Assume that Y is $(k-1)$-connected, let $\ell \leq k-1$, and let $f: S^\ell \to X$ be a map. We need to show that f is null-homotopic. Let K be a triangulation of S^ℓ, and let $Q = \{St(v, \beta^2 K) : v$ is a vertex of $\beta K\}$ be the handlebody decomposition of S^ℓ with respect to K. We can pick K sufficiently small so that for any $q \varepsilon Q$ there is $\lambda(q) \varepsilon \Lambda$ with $f(q) \subseteq U_{\lambda(q)}$. Next, we construct a map $g: S^\ell \to Y$ so that $g(q) \subseteq V_{\lambda(q)}$, for all $q \varepsilon Q$. We define g successively on the intersections $q_{\nu(1)} \cap \cdots \cap q_{\nu(t)}$ of elements of Q, starting with maximal intersections $(t = \ell + 1 \leq k)$. In the inductive argument we require that $g(q_{\nu(1)} \cap \cdots \cap q_{\nu(t)}) \subseteq$ $\subseteq V_{\lambda(q_{\nu(1)})} \cap \cdots \cap V_{\lambda(q_{\nu(t)})}$. (The induction starts by defining g to be a constant map on maximal intersections. The inductive step consists of extending a map with $(k-t)$-connected target from the boundary of a $(\ell-t+1)$-cell onto the cell.) Since Y is ℓ-connected, g extends to $\tilde{g}: B^{\ell+1} \to Y$. Let L be a sufficiently small

triangulation of $B^{\ell+1}$, and P the handlebody decomposition of $B^{\ell+1}$ corresponding to L such that there is a function $\phi\colon P \to \Lambda$ with $\tilde{g}(p) \subseteq V_{\phi(p)}$, for all $p \in P$, and $f(p \cap S^{\ell}) \subseteq U_{\phi(p)}$, for all $p \in P$ with $p \cap S^{\ell} \neq \phi$. An argument as before produces a map $\tilde{f}\colon B^{\ell+1} \to X$ with $\tilde{f}(p) \subseteq U_{\phi(p)}$, for all $p \in P$. To finish the proof it remains to show that $\tilde{f}|S^{\ell} \simeq f$. Observe that $P|S^{\ell}$ gives a handlebody decomposition of S^{ℓ}, and for $p \cap S^{\ell} \in P|S^{\ell}$ both $f(P \cap S^{\ell})$ and $\tilde{f}(P \cap S^{\ell})$ are contained in a single element $U_{\phi(t)}$ of U. We now construct a homotopy $H\colon S^{\ell} \times I \to X$ between f and $\tilde{f}/S^{\ell}$ successively on $(p_{\rho(1)} \cap \dots \cap p_{\rho(t)} \cap S) \times I$ by "backward induction" on t, requiring that

$$H((p_{\rho(1)} \cap \dots \cap p_{\rho(t)} \cap S^{\ell}) \times I) \subseteq U_{\phi(p_{\rho(1)})} \cap \dots \cap U_{\phi(p_{\rho(t)})}$$

(the inductive step consists in extending a map with $(k-t)$-connected target from the boundary of a $(\ell-t+2)$-cell onto the cell).

<u>1.1.18. Corollary.</u> Suppose $T\colon P_1 \to P_2$ is a one-to-one correspondence between partitions P_1, P_2 on manifolds M_1, M_2 respectively. If both P_1, P_2 satisfy 1.1.13.(3), and if M_1 is $(k-1)$-connected, so is M_2.

<u>Proof.</u> Let K_i be a triangulation of M_i so that any $p \in P_i$ underlies a subcomplex of K_i $(i = 1,2)$. For $p \in P_1$, let U_p be the open second derived neighborhood of p with respect to K_1, and let V_p be the open second derived neighborhood of $T(p)$ with respect to K_2. Apply 1.1.17 to $U = \{U_p\colon p \in P_1\}$ and $V = \{V_p\colon p \in P_1\}$.

<u>1.1.19. Proposition.</u> Let P,Q be partitions on a manifold M^m so that if $q_1,\ldots, q_t \in Q$ are mutually distinct and have non-empty intersection, then $P|q_1 \cap \ldots \cap q_t$ is a partition $(t \geq 1)$. Then, symmetrically, if $p_1,\ldots, p_t \in P$ are mutually distinct and have non-empty intersection, it follows that $Q|p_1 \cap \ldots \cap p_t$ is a partition $(t \geq 1)$.

<u>Proof.</u> From hypotheses it follows that if $q_1 \cap \cdots \cap q_i \cap p_1 \cap \cdots \cap p_t \neq \phi$ $(q_1,\ldots, q_i \in Q$ mutually distinct), then it is a $(m-i-t+2)$-manifold. Moreover, $q_1 \cap \cdots \cap q_i \cap p_1 \cap \cdots \cap p_t \subseteq (q_2 \cap \cdots \cap q_i \cap p_1 \cap \cdots \cap p_t) \cap \partial(q_2 \cap \ldots \cap q_i) \subseteq \partial(q_2 \cap \ldots \cap q_i \cap p_1 \cap \ldots \cap p_t)$. (The last containment follows from 1.1.9.(c) applied to $P|q_2 \cap \ldots \cap q_i$).

1.2. The Standard Construction of the Universal k-Dimensional Menger Space μ^k and μ^k-Manifolds

In this section we define two classes $\mathcal{M}_0$ and $\mathcal{M}$ of spaces that "ought to" consist of the universal k-dimensional Menger space and (compact) manifolds modelled on it, respectively. In particular, all spaces in $\mathcal{M}_0$ ought to be homeomorphic to each other. At the end of Chapter 2 (2.8.2.) we show that this is indeed the case.

Let M_1^m be a compact manifold with a triangulation L_1, and assume $m \geq 2k + 1$. We produce a sequence $(M_1, L_1), (M_2, L_2), (M_3, L_3),\ldots$ of manifolds and triangulations inductively as follows. Suppose (M_i, L_i) has been constructed. Let

$$M_{i+1} = \cup\{St(v, \beta^2 L_i) : v \text{ is a vertex of } \beta L_i \text{ and } Ind(v) \geq m - k\}$$

(see 1.1.2.). By 1.1.5., M_{i+1} is a manifold, and let L_{i+1} be the triangulation of M_{i+1} obtained by restricting $\beta^2 L_i$ onto M_{i+1}.

1.2.1. Definition. The sequence $(M_1, L_1), (M_2, L_2), \ldots$ is called a μ^k-manifold defining sequence. Any space M that can be represented as $\bigcap_{i=1}^{\infty} M_i$ for some μ^k-manifold defining sequence $\{(M_i, L_i)\}_{i=1}^{\infty}$ is a triangulated μ^k-manifold obtained from M_1. The class of all triangulated μ^k-manifolds is denoted by $\mathcal{M}$. If M_1 is in addition (k-1)-connected, we say that the μ^k-manifold defining sequence $\{(M_i, L_i)\}_{i=1}^{\infty}$ is a μ^k-defining sequence. Any space μ^k that can be represented as $\bigcap_{i=1}^{\infty} M_i$ for some μ^k-defining sequence $\{(M_i, L_i)\}_{i=1}^{\infty}$ is a k-dimensional universal Menger space. The collection of all k-dimensional universal Menger spaces is denoted by $\mathcal{M}_0$.

We also define some terms that will be used in the sequel.

1.2.2. Definition. Let $\{(M_i^m, L_i)\}_{i=1}^{\infty}$ be a μ^k-manifold defining sequence. An (m-s)-submanifold N of M_i is convenient if N is (k-s-1)-connected and underlies a subcomplex of L_i. By C_i^s we denote the collection of all (m-s)-dimensional convenient submanifolds of M_i. A partition P on M_i is convenient if all elements of P

and all non-empty intersections of elements of P are convenient submanifolds of M_i) .

1.2.3. Remark. M_{i+1} is obtained from M_i by digging out a regular neighborhood of the (m-k-1)-skeleton of L_i , and thus keeping a regular neighborhood of the dual k-skeleton of L_i . In particular, the inclusion $M_{i+1} \subset M_i$ induces an isomorphism on $\pi_0, \pi_1, \ldots, \pi_{k-1}$ and an epimorphism on Π_k (by a general position argument). Similarly, if $N \in C_i^s$, then $N \cap M_{i+1} \in C_{i+1}^s$, and hence $N \supset N \cap M_{i+1} \supset N \cap N_{i+2} \supset \ldots$ is a μ^{k-s}-defining sequence. This fact allows us to use inductive arguments ("backward induction" on codimension). Although we never use it explicitly, it is only an exercise to show that for $k = 0$, $M_0 = M$ is a collection of Cantor sets. Note that each point in a triangulated μ^k-manifold has a neighborhood that belongs to M_0 . After we prove the uniqueness theorem (i.e. all spaces in M_0 are homeomorphic to each other), it will follow that any triangulated μ^k-manifold is locally homeomorphic to μ^k . The converse will follow from the characterization theorem.

1.3. A Combinatorial Characterization of μ^k .

Here we give two axioms and show that spaces having defining sequences satisfying these axioms are homeomorphic. Chapter 2 will be devoted to a proof that any μ^k-defining sequence satisfies the axioms.

An integer $k \geq 0$ is fixed. By a defining sequence we mean a sequence $M_1^m \supset M_2^m \supset M_3^m \supset \ldots$ of (compact, PL) manifolds so that M_{i+1} is a submanifold of M_i $(i \geq 1)$. On each M_i we distinguish a set C_i^s of non-empty submanifolds of codimension s $(s = 0,1,\ldots,k)$ subject to the following properties ((C1)-(C4))

(C1) If $N \varepsilon C_i^s$, then N is $(k-s-1)$-connected, and $N \cap M_j \varepsilon C_j^s$, for all $j \geq i$.

(C2) If N is a $(k-s-1)$-connected $(m-s)$-submanifold of M_i that admits a convenient partition, (the definition follows), then $N \varepsilon C_i^s$ $(s = 0,1,\ldots,k\ ;\ i \geq 1)$

(C3) $N \varepsilon C_i^k$ iff each component of N belongs to C_i^k. Furthermore, for each such N the number of components of $N \cap M_j$ goes to ∞ as $j \to \infty$

(C4) $M_1 \varepsilon C_1^0$

Note that by (C1) and (C4) all M_i's are $(k-1)$-connected.

We say that a partition P on a submanifold M_i is <u>convenient</u> if all elements and intersections of elements in P belong to C_i^s (for some s). Elements of C_i^s are <u>convenient manifolds</u>.

<u>1.3.1. Definition</u>. Let $M_1^m \supset M_2^m \supset \ldots$ be a defining sequence, and let $\{C_i^s : i \geq 1,\ 0 \leq s \leq k\}$ be a family of sets satisfying (C1)-(C4). The family $\{M_i, C_i^s\}_{i=1}^{\infty}$ is a μ^k-family if the following two axioms are satisfied (the metric on M_i is the restriction of some fixed metric on M_1).

<u>Axiom 1</u>. For any $i \geq 1$, any convenient k-partition $P = \{p_\lambda, \lambda \varepsilon \Lambda\}$ on M_i, and any $\varepsilon > 0$, there exist $j \geq i$ and a convenient k-partition Q on M_j so that

(a) mesh $Q < \varepsilon$

(b) $Q|p_{\lambda(1)} \cap \ldots \cap p_{\lambda(t)} \cap M_j$ is a $(k-t+1)$-partition on $p_{\lambda(1)} \cap \ldots \cap p_{\lambda(t)} \cap M_j$ for all t, $1 \le t \le k+1$, and all $(\lambda(1),\ldots,\lambda(t)) \in \Lambda^{(t)}$ with $p_{\lambda(1)} \cap \ldots \cap p_{\lambda(t)} \ne \phi$.

Axiom 2. Suppose the following is given: $i \ge 1$, $s \in \{0,1,\ldots,k\}$, $N \in C_i^s$, a submanifold N_0 of ∂N, with either $\dim N_0 = m - s - 1$ ($= \dim N$), or $N_0 = \phi$, partitions P_0 and Q_0 on N_0 so that P_0 is convenient and $Q_0|p_1 \cap \ldots \cap p_t$ is a convenient $(k-s-t)$-partition for all choices $p_1,\ldots, p_t \in P_0$ of mutually distinct elements having non-empty intersection. Also, given are a $(k-s)$-connected manifold M', a $(k-s)$-partition $\overline{Q}$ of M' and a function $T_0\colon Q_0 \to \overline{Q}$ with

(i) T_0 being injective, and

(ii) T_0 preserving intersections (see 1.1.15).

Then there exist $j \ge i$, a convenient $(k-s)$-partition Q on $N \cap M_j$, and a one-to-one correspondence $T\colon Q \to \overline{Q}$ such that

(a) $Q|N_0 \cap M_j = Q_0|N_0 \cap M_j$, and

(b) for any $q \in Q$ with $q \cap N_0 \ne \phi$

$$T(q) = T_0(q_0)$$

where $q_0 \in Q_0$ is the unique element with $q \cap N_0 = q_0 \cap M_j$.

The first axiom simply says that the sequence $\{M_i\}_{i=1}^{\infty}$ admits arbitrarily small k-partitions, while the second one asserts that the sequence $\{M_i\}_{i=1}^{\infty}$ is universal with respect to realizing abstract k-partitions.

The main theorem of this section, giving a combinatorial characterization of the universal k-dimensional Menger space, is the following.

<u>1.3.2. Theorem.</u> Suppose $\{M_i, C_i^s\}_{i=1}^{\infty}$ and $\{\hat{M}_i, \hat{C}_i^s\}$ are two μ^k-families. If $X = \bigcap_{i=1}^{\infty} M_i$, $\hat{X} = \bigcap_{i=1}^{\infty} M_i$, then $X \approx \hat{X}$.

Although the proof is lengthy (the rest of the section), it is simple-minded. We build fine k-partitions on $\{M_i\}$, $\{\hat{M}_i\}$ and one-to-one correspondences between them by a "backward induction" on strata (see 1.3.5.). The correspondences, in turn, give rise to a homeomorphism $f: X \to \hat{X}$ (see 1.3.3.).

<u>1.3.3. Lemma.</u> Suppose $i(1) \leq i(2) \leq i(3) \leq \ldots$ is a sequence of integers, P_r (resp., $\hat{P}_r$) is a convenient partition on $M_{i(r)}$ (resp., $\hat{M}_{i(r)}$) , $T_r\colon P_r \to \hat{P}_r$ is a one-to-one correspondence and the following holds.

$(i)_r$ If r is odd, then mesh $P_{r+1} < \dfrac{1}{2^{r+1}}$.

$(\hat{i})_r$ If r is even, then mesh $\hat{P}_{r+1} < \dfrac{1}{2^{r+1}}$.

$(ii)_r$ For any $p_r \varepsilon P_r$, $p_{r+1} \varepsilon P_{r+1}$

$p_r \cap p_{r+1} \neq \phi$ iff $T_r(p_r) \cap T_{r+1}(p_{r+1}) \neq \phi$.

$(iii)_r$ For any $p_r, p_r' \varepsilon P_r$, $p_{r+1}, p'_{r+n} \varepsilon P_{r+1}$ if

$p_r \cap p_{r+1} \neq \phi$, $p_{r+1} \cap p'_{r+1} \neq \phi$, $p'_{r+1} \cap p'_r \neq \phi$, then

$p_r \cap p'_r \neq \phi$.

$(\hat{iii})_r$ For any $\hat{p}_r, \hat{p}'_r \varepsilon \hat{p}_r$, $\hat{p}_{r+1}, \hat{p}_{r+1} \varepsilon \hat{P}_{r+1}$ if

$\hat{p}_r \cap \hat{p}_{r+1} \neq \phi$, $\hat{p}_{r+1} \cap \hat{p}'_{r+1} \neq \phi$, $\hat{p}'_{r+1} \cap \hat{p}'_r \neq \phi$,

then $\hat{p}_r \cap \hat{p}'_r \neq \phi$.

Then the following is true:

(*) For any $p_r \varepsilon P_r$, $p_{r+1} \varepsilon P_{r+1}$ with $p_r \cap p_{r+1} \neq \phi$

$$St(p_{r+1}, P_{r+1}) \subseteq \text{int } St(p_r, P_r)$$

(recall that int means topological interior, in this case with respect to $M_{i(r)}$) .

(**) For any $x \varepsilon X$, and any sequence $\{p_r \varepsilon P_r\}_{r=1}^{\infty}$ with $x \varepsilon p_r$, we have

$$\text{int } St(T_r(p_1), \hat{P}_r) \supseteq St(T_{r+1}(p_{r+1}), \hat{P}_{r+1}) ,$$

the intersection $\bigcap_{r=1}^{\infty} St(T_r(p_r), \hat{P}_r)$ is a single point, say $\hat{x}$, and the function $f: X \to \hat{X}$ defined by setting $f(x) = \hat{x}$ is well defined and a homeomorphism.

<u>Proof</u>. To prove (*), observe that if $x \notin \text{int } St(p_r, P_r)$, then there is $p'_r \varepsilon P_r$ with $x \varepsilon p'_r$ and $p_r \cap p'_r = \phi$. If, in addition, $x \varepsilon St(p_{r+1}, P_{r+1})$, then there is $p'_{r+1} \varepsilon P_{r+1}$ with $x \varepsilon p'_{r+1}$ and $p_{r+1} \varepsilon p'_{r+1} \neq \phi$. Then (iii) gives a contradiction, which proves (*).

If $\{p_r\}_{r=1}^{\infty}$ is a sequence as in (**), then (ii) gives $T_r(p_r) \cap T_{r+1}(p_{r+1}) \neq \phi$ $(r = 1,2,\ldots)$. The paragraph above (with "hats" where appropriate), shows that $\{St(T_r(p_r), \hat{P}_r)\}_{r=1}^{\infty}$ is a nested sequence of compacta. Note that (*) coupled with (i) implies that

$$\text{mesh } \hat{P}_r \leq 3 \cdot \text{mesh } \hat{P}_{r-1} \leq \frac{3}{2^{r-1}} ,$$

for $r > 2$ even. Hence for all $r > 2$, mesh $\hat{P}_r < \frac{3}{2^{r-1}}$ and therefore the intersection of the nested sequence $\{St(T_r(P_r), P_r)\}_{r=1}^{\infty}$ is a single point. If $\{p_r'\}_{r=1}^{\infty}$ is another sequence with $x \in p_r' \in P_r$, then $p_r \cap p_r' \neq \phi$, which implies that $T_r(p_r) \cap T_r(p_r') \neq \phi$. In particular, the point $x' = \bigcap_{i=1}^{\infty} St(T_r(p_r'), \hat{P}_r)$ is St P_r-close to x, $r = 1,2,\ldots$. This implies $\hat{x} = \hat{x}'$, and hence $f\colon X \to \hat{X}$ is well-defined. To show that f is continuous, observe that, by definition,

$$f(X \cap St(p_r, P_r)) \subseteq \hat{X} \cap St^2(T_r(p_r), \hat{P}_r) ,$$

and that by (*) $\{X \cap St(p_r, P_r)\}_{r=1}^{\infty}$ forms a neighborhood basis of $x = \bigcap_{r=1}^{\infty} p_r \in X$. Also, $\lim_{r\to\infty} \text{diam } St^2(T_r(p_r), \hat{P}_r) = 0$. We can repeat the argument to construct the map $\hat{f}\colon \hat{X} \to X$. Comparing the definitions, we observe that $\hat{f}f = Id_X$, $f\hat{f} = Id_{\hat{X}}$. (Specifically, $\hat{f}f(x)$ and x can be seen to be St P_r-close for $r = 1,2,\ldots$ using the fact that T_r is a one-to-one correspondence.)

Below, we inductively construct sequences $i(1) \leq i(2) \leq \ldots$, $\{P_r\}_{i=1}^{\infty}$ $\{P_i\}_{r=1}^{\infty}$, $\{T_r\colon P_r \to \hat{P}_r\}_{r=1}^{\infty}$ with properties (i)-($\hat{\text{iii}}$) from 1.3.3. and in addition

(iv) P_{r+1} and $\hat{P}_{r+1}$ are convenient k-partitions, $r = 0,1,\ldots$.

To start the inductive argument, we let $i(1) = 1$, $\hat{P}_1 = \{\hat{M}_1\}$, $\hat{P}_1 = \{\hat{M}_1\}$, $T_1\colon P_1 \to \hat{P}_1$ the only function between singletons (apply (C4)).

Assume that $r \geq 1$ and that $i(1) \leq \ldots \leq i(r)$, $P_1, \ldots, P_r$; $\hat{P}_1, \ldots, \hat{P}_r$; $T_1\colon P_1 \to \hat{P}_1, \ldots, T_r\colon P_r \to \hat{P}_r$ are constructed so as to satisfy $(i)_j$-$(iv)_j$, $j = 1, \ldots, r-1$. We show how to construct $i(r+1) \geq i(r)$, P_{r+1} , $\hat{P}_{r+1}$, $T_{r+1}\colon P_{i+1} \to \hat{P}_{r+1}$ so that $(i)_r$ - $(iv)_r$ hold, in the case when r is even. (If r is odd, interchange the roles of P_r and $\hat{P}_r$) .

Choose $\varepsilon > 0$, so that $\varepsilon < \frac{1}{2^{r+1}}$ and so that if $\hat{p}, \hat{p}' \in \hat{P}_r$, $\hat{p} \cap \hat{p}' = \phi$, then $N_\varepsilon(\hat{p}) \cap N_\varepsilon(\hat{p}') = \phi$. Using axiom 1, find $i \geq i(r)$ and a convenient k-partition P on M_i with mesh $\hat{P} < \varepsilon$ so that $\hat{P}|\hat{p}_1 \cap \ldots \cap \hat{p}_t \cap \hat{M}_i$ is a (k-t+1)-partition for all t , $1 \leq t \leq k+1$, and all choices of mutually distinct $\hat{p}_1, \ldots, \hat{p}_t \in \hat{P}_r$ having non-empty intersection. Observe that if $i(r+1) > i$, and if we set $\hat{P}_{r+1} = \hat{P}|\hat{M}_{i(r+1)}$, then $\hat{P}_{r+1}$ is a convenient k-partition on $\hat{M}_{i(r+1)}$, and $(\hat{i})_r$, $(i\hat{i}i)_r$ hold, by choice of ε . Moreover, $\hat{T}\colon \hat{P} \to \hat{P}_{r+1}$, $\hat{T}(\hat{p}) = \hat{p} \cap \hat{M}_{i(r+1)}$ is a one-to-one correspondence (here, apply (C1) and 1.1.18.). Therefore, it suffices to construct $i(r+1) > i(r)$, a convenient partition P_{r+1} on $M_{i(r+1)}$, and a one-to-one correspondence $T\colon P_{r+1} \to \hat{P}$ so that

$$\text{(!)} \quad \text{for any } p_r \in P_r\,,\ p_{r+1} \in P_{r+1}\,,$$
$$p_r \cap p_{r+1} \neq \phi \ \text{ iff } \ T_r(p_r) \cap T(p_r) \neq \phi$$

For there we can set $T_{r+1} = \hat{T} \circ T\colon P_{r+1} \to \hat{P}_{r+1}$. It is easy to see that $(ii)_r$ follows from (!) and (C1), $(iii)_r$ follows from $(ii)_r$ and $(i\hat{i}i)_r$, while $(i)_r$ is vacuous. Also, $(iv)_r$ follows from (C1) and 1.1.18.

For a partition Q and $j \geq 0$ define the <u>codimension j stratum</u> [Wa_1] by

$$S_j(Q) = \cup\{q_1 \cap \cdots \cap q_{j+1} : q_1, \ldots, q_{j+1} \in Q \text{ are mutually distinct}\}.$$

Let $\hat{R}_j = \hat{P}|S_j(\hat{P}_r)$. Since $S_j(\hat{P}_r)$ is not (in general) a manifold, $\hat{R}_j$ is not a partition. Note that $\hat{R}_j|\hat{p}_1 \cap \cdots \cap \hat{p}_{\ell+1} \cap \hat{M}_i$ ($= \hat{P}|\hat{p}_1 \cap \cdots \cap \hat{p}_{\ell+1} \cap \hat{M}_i$) is a $(k-\ell)$-partition ($\ell \geq j$) , for all choices of mutually distinct $\hat{p}_1, \ldots, \hat{p}_{\ell+1} \in \hat{P}_r$ having non-empty intersection. Denote this partition by $\hat{R}_j(\hat{p}_1, \ldots, \hat{p}_{\ell+1})$.

Consider the following statement $(j = 0,1,\ldots, k)$:

<u>(H_j)</u> There exist $i[j] \geq i$, a family R_j of subsets of $S_j(P_r) \cap M_{i[j]} = S_j(P_r|M_{i[j]})$, and a bijection $T^j: R_j \to \hat{R}_j$ so that:

(α_j) $R_j|p_1 \cap \cdots \cap p_{\ell+1} \cap M_{i[j]}$ is a convenient $(k-\ell)$-partition on $p_1 \cap \cdots \cap p_{\ell+1} \cap M_{i[j]}$ for any $\ell \geq j$ and any collection $p_1, \ldots, p_{\ell+1}$ of mutually distinct elements of P_r with non-empty intersection. This partition is denoted by $R_j(p_1, \ldots, p_{\ell+1})$;

(β_j) for any $\rho \in R_j$ and any mutually distinct $p_1, \ldots, p_{\ell+1} \in P_r$ $(\ell \geq j)$ $\rho \cap p_1 \cap \cdots \cap p_{\ell+1} \neq \phi$ iff $T^j(\rho) \cap T_r(p_1) \cap \cdots \cap T_r(p_{\ell+1}) \neq \phi$; and

(γ_j) $T^j(p_1, \ldots, p_{\ell+1}): R_j(p_1, \ldots, p_{\ell+1}) \to \hat{R}_j(T_r(p_1), \ldots, T_r(p_{\ell+1}))$ defined by

$\rho \cap p_1 \cap \ldots \cap p_{\ell+1} \mapsto T^j(\rho) \cap T_r(p_1) \cap \ldots \cap T_r(p_{\ell+1})$ is a one-to-one correspondence ($\ell \geq j$, and $p_1, \ldots, p_{\ell+1} \in P_r$ are mutually distinct and have non-empty intersection).

1.3.4. Lemma. (H_k) is a true statement.

Proof. Since P_r is a k-partition, $S_k(P_r)$ is a disjoint union of manifolds of the form $p_1 \cap \ldots \cap p_{k+1}$, where $p_1, \ldots, p_{k+1} \in P_r$ are mutually distinct. By assumption, $\hat{R}_k(T_r(p_1), \ldots, T_r(p_{k+1}))$ is a 0-partition. Let N be the number of elements in this partition. Also, by assumption, $p_1 \cap \ldots \cap p_{k+1} \in C^k_{i(r)}$, and hence by (C3) there is $i[k] \geq i(r)$ so that the number of components of $p_1 \cap \ldots \cap p_{k+1} \cap M_{i[k]}$ is $\geq N$. Let $R_k(p_1, \ldots, p_{k+1})$ be a 0-partition on $p_1 \cap \ldots \cap p_{k+1} \cap M_{i[k]}$ consisting of N elements obtained by "amalgamating" components of $p_1 \cap \ldots \cap p_{k+1} \cap M_{i[k]}$. By (C3), this is a convenient 0-partition. Let $T^k(p_1, \ldots, p_{k+1})$: $R_k(p_1, \ldots, p_{k+1}) \to \hat{R}_k(T_r(p_1), \ldots, T_r(p_{k+1}))$ be a bijection (necessarily a one-to-one correspondence), and set

$$R_k = \cup \{R_k(p_1, \ldots, p_{k+1}) : p_1, \ldots, p_{k+1} \in P_r \text{ are mutually distinct and have non-empty intersection}\}.$$

Define a bijection $T^k \colon R^k \to \hat{R}^k$ by setting $T^k(\rho) = T^k(p_1, \ldots, p_{k+1})(\rho)$, if $\rho \in R_k(p_1, \ldots, p_{k+1})$. One readily verifies (α_k)-(γ_k), and hence the proof of 1.3.4. is finished.

1.3.5. Lemma. If (H_{j+1}) is a true statement, then (H_j) is a true statement, $j = 0,1,\ldots, k-1$.

Proof. Fix mutually distinct elements $p_1,\ldots, p_{j+1} \in P_r$ with non-empty intersection. Let $N = p_1 \cap \ldots \cap p_{j+1} \cap M_{i[j+1]}$, and $N_0 = \cup \{p \cap p_1 \cap \ldots \cap p_{j+1} \cap M_{i[j+1]} : p \in P_r - \{p_1,\ldots, p_{j+1}\}\}$. By 1.1.6., N_0 is a submanifold of ∂N . Let P_0 be the partition on N_0 as defined in 1.1.6. Then P_0 is a convenient partition on N_0 . Let $Q_0 = R_{j+1}|N_0$ (observe that $N_0 \subseteq S_{j+1}(P_r/M_{i[j+1]})$). It is immediate from (α_{j+1}) that $Q_0|p_1^0 \cap \ldots \cap p_t^0$ is a convenient $(k-t-j)$-partition, for all mutually distinct $p_1^0,\ldots, p_t^0 \in P_0$ with non-empty intersection $(1 \leq t \leq k - j)$. By 1.1.11., Q_0 is a partition on N_0 . We define "hat versions" of N, N_0 by setting $\hat{N} = T_r(p_1) \cap \ldots \cap T_r(p_{j+1}) \cap \hat{M}_i$ and

$$\hat{N}_0 = \cup \{T_r(p) \cap T_r(p_1) \cap \ldots \cap T_r(p_{j+1}) \cap \hat{M}_i : \\ p \in P_r - \{p_1,\ldots, p_{j+1}\}\} .$$

The restriction of T^{j+1} defines a function from Q_0 into $\hat{R}_{j+1}|\hat{N}_0$ (by (β_{j+1}) it is well defined, and by (γ_{j+1}) it is a one-to-one correspondence). Composing with the "inclusion" map $\hat{R}_{j+1}|\hat{N}_0 \to \hat{R}_j|\hat{N}$, where $\hat{\rho} \cap N_0 \mapsto \hat{\rho} \cap \hat{N}$, we get an injective intersection preserving function $T_0 : Q_0 \to \hat{R}_j|\hat{N}$. By assumption, $\hat{R}_j|\hat{N}$ is a $(k-j)$-partition, and $\hat{N}$ is $(k-j-1)$-connected. By axiom 2, there is $i[j] \geq i[j+1]$ $(\geq i(r))$, a convenient $(k-j)$-partition Q

on $N \cap M_{i[j]}$ and a one-to-one correspondence $\overline{T}(p_1, \ldots, p_{j+1})$: $Q \to \hat{R}_j | \hat{N}$ such that $Q|N_0 \cap M_{i[j]} = Q_0|N_0 \cap M_{i[j]}$, and for any $q \in Q$ with $q \cap N_0 \neq \phi$ $\overline{T}(q) = T_0(q_0)$, where $q_0 \in Q_0$ is the unique element with $q \cap N_0 = q_0 \cap M_{i[j]}$.

We assume that this is done for all $(j+1)$-tuples $(p_1, \ldots, p_{j+1})$ of mutually distinct elements of P_r with non-empty intersection. Finally, we define

$(T^j)^{-1}(\hat{\rho}) = \cup \{\overline{T}(p_1, \ldots, p_{j+1})^{-1}(\hat{\rho} \cap T_r(p_1) \cap \ldots \cap T_r(p_{j+1}))$:
$p_1, \ldots, p_{j+1} \in P_r$ are mutually distinct and
$\hat{p} \cap T_r(p_1) \cap \ldots \cap T_r(p_{j+1}) \neq \phi\}$ and $R_j = \{(T^j)^{-1}(\hat{\rho}), \hat{p} \in \hat{R}_j\}$.

(α_j) is true for $\ell = j$ by construction $(R_j|p_1 \cap \ldots \cap p_{j+1} \cap M_{i[j]} = Q)$, and for $\ell > j$ by (α_{j+1}) $(R_j|p_1 \cap \ldots \cap p_{\ell+1} \cap M_{i[j]} = R_{j+1}|p_1 \cap \ldots \cap p_{\ell+1} \cap M_{i[j]}, \ell > j)$. Similarly, one establishes (β_{j+1}) and (γ_{j+1}). This concludes the proof of 1.3.5.

Repeatedly applying 1.3.5. (along with 1.3.4.), we conclude that (H_0) is a true statement. Let $i(r+1) = i[0]$, $P_{r+1} = R_0$, $T = T^0$. By (α_0) and 1.1.11., P_{r+1} is a partition and by (β_0) and (γ_0), T is a one-to-one correspondence.

<u>1.3.6. Lemma.</u> Let $q_1, \ldots, q_t \in P_{t+1}$ be mutually distinct elements with non-empty intersection. Then $P_r|q_1 \cap \ldots \cap q_t$ is a convenient partition. Moreover, the function
$F: P_r|q_1 \cap \ldots \cap q_t \to \hat{P}_r|T(q_1) \cap \ldots \cap T(q_t)$ is a one-to-one correspondence.

<u>Proof.</u> The first part follows from 1.1.19 (the convenience part is obvious from the definition). The second part is just a restatement of (γ_0).

Hence, by 1.1.18, $q_1 \cap \ldots \cap q_t$ is (k-t)-connected, and by (C2), $q_1 \cap \ldots \cap q_t$ is a convenient manifold.

To finish the proof of 1.3.2. it remains to observe that (β_0) implies (!) .

CHAPTER 2

BASIC MOVES

In this chapter, the main goal is to verify that any μ^k-defining sequence, together with the sets of convenient submanifolds (as defined in 1.2.), represents a μ^k-family. In particular, all spaces in M_0 (see 1.2.) are homeomorphic, and thus the universal k-dimensional Menger space μ^k is well-defined. For the reader's convenience, major steps (or "moves") of the proof form separate sections of the chapter. Later, we will apply same moves (sometimes with more control) to obtain refinements.

We start this chapter with some facts about LC^{k-1}-spaces and UV^{k-1}-maps.

2.1. On LC^{k-1}-Spaces and UV^{k-1}-Maps

In the next section we show that all triangulated μ^k-manifolds are LC^{k-1}. Although there is a rich source of information about LC^{k-1}-spaces in literature (e.g. [Bo], [Hu]), we failed to find statements we need. It is clear that in an LC^{k-1}-space (the reader is advised to think of μ^k) there are no homotopies (in general) between maps from k-dimensional spaces. We must allow "jumps over the k-holes" as well as homotopical moves. This gives rise to the notion of μ-homotopic maps, which allows us to state and prove theorems analogous to ones from Q-manifold theory.

As we shall see, UV^{k-1}-maps between LC^{k-1}-spaces play the role corresponding to cell-like maps between ANR's. A standard reference is [La].

2.1.1. Proposition. Let $f: X \to Y$ be a UV^{k-1}-surjection between metric compacta. Then X is UV^{k-1} iff Y is UV^{k-1}.

Sketch of proof. (For the readers well-versed in shape theory, this is a consequence of the Smale theorem in shape theory [Dy] and the fact that X is UV^{k-1} iff pro - $\pi_i(X) = 0$ for $i = 0,1,\ldots,k-1$.) Embed $X \subseteq M$, $Y \subseteq N$ where M, N are ANR's, and extend f to $F: M \to N$ (trimming back M if necessary). The main ingredient is the following approximate lifting property. Let $\mathcal{V}_k > \mathcal{V}_{k-1} > \ldots > \mathcal{V}_0$ be a sequence of locally finite covers of Y by open sets in N so that if $V \in \mathcal{V}_i$, then there is a $V' \in \mathcal{V}_{i-1}$ so that $St(V, \mathcal{V}_i) \subseteq V'$ and this inclusion, as well as $F^{-1}(St(V, \mathcal{V}_i)) \subseteq F^{-1}(V')$ induces null-homomorphisms of the homotopy groups in dimensions $\leq k-1$. Assume that K is a finite simplicial complex, $L \subseteq K$ a sub complex, $\dim(K - L) \leq k$, $\alpha: |L| \to M$, $\bar{\alpha}: |K| \to N$ maps so that $F\alpha$ and $\bar{\alpha}| |L|$ are $\mathcal{V}_0$-close, and if σ is a simplex of $K - L$, then $\bar{\alpha}(\sigma)$ is $\mathcal{V}_0$-small. Then there is a map $\tilde{\alpha}: |K| \to M$ so that $\tilde{\alpha}| |L| = \alpha$, and $F\tilde{\alpha}$ is $\mathcal{V}_k$-close to $\bar{\alpha}$ (compare with Lemma A in [La]). Moreover, any two $\mathcal{V}_0$-close maps $\beta_1, \beta_2: |K| \to N$ whose restrictions onto $|L|$ are $\mathcal{V}_0$-homotopic, are $\mathcal{V}_k$-homotopic.

(Lifts and homotopies are constructed successively on skeleta of K.)

2.1.2. Corollary. (i) If $f: X \to Y$, $g: Y \to Z$ are UV^{k-1}-surjections between metric compacta, so is $gf: X \to Z$. (ii) A UV^{k-1}-image of an LC^{k-1}-space is LC^{k-1}.

Proof. (i) $f\colon (gf)^{-1}(z) \to g^{-1}(z)$ is a UV^{k-1}-surjection, and $g^{-1}(z)$ is UV^{k-1}. (ii) See Theorem 16.11 of [Da].

2.1.3. Proposition. If A is a UV^{k-1}-compactum in an LC^{k-1}-compactum X, then A has property UV^{k-1} in X (i.e. for any neighborhood U of A there is a neighborhood V, $A \subseteq V \subseteq U$, so that inclusion $V \subseteq U$ induces null-homomorphisms of homotopy groups of dimension $\leq k - 1$).

Proof. Embed $X \subset Q$ (the Hilbert cube). Let $U_1 \supset U_2 \supset \dots$ be a nested sequence of neighborhoods of A in Q whose intersection is A, and so that inclusion $U_{n+1} \subseteq U_n$ induces null-homomorphisms of homotopy groups of dimension $\leq k - 1$. Let $\sum_i$ be a countable dense subset of the space of maps $f\colon (S^i, *) \to (X, *)$. For each element $f \in \sum_i$, let D_n^f be a countable dense subset of the space of maps $F\colon B^{i+1} \to U_n$ that extend f. We may assume that each $F \in D_n^f$ is an embedding when restricted to $B^{i+1} - S^i$. Let $P_n = \cup\{\mathrm{Im}F\colon F \in D_n^f,\ f \in \sum_0 \cup \dots \cup \sum_{k-1}\}$ and $P = \bigcup_{n=1}^{\infty} P_n$. Since X is LC^{k-1}, and $\dim(P - X) \leq k$ [H-W], there is a retraction (reindex if necessary) $r\colon P \cup X \to X$ [Hu]. We claim that if $r(P_{m-1}) \subseteq U_n$, then inclusion $U_m \cap X \subseteq U_n \cap X$ induces null-homomorphism of homotopy groups of dimension $\leq k - 1$. A representative $f\colon (S^i, *) \to (U_m \cap X, *)$ of an element of $\pi_i(U_m \cap X, *)$ can be chosen to be in $\sum_i$ (X is LC^{k-1}, [Hu]). Then f extends to $F\colon B^{i+1} \to U_{m-1}$, after adjusting $F \in D_{m-1}^f$. Hence $rF\colon B^{i+1} \to U_n \cap X$ extends f. This completes the proof of 2.1.3.

Using a modified "approximating lifting" argument (as in the proof of 2.1.1., only this time the covers will consist of open subsets of the target itself), plus the "nerve replacement trick" [Hu, p. 53] (replace the pair (X, A) by $(|N(\mathcal{U})| \cup A, A)$ for some cover $\mathcal{U}$ of $X - A$), we can prove the following:

<u>2.1.4. Proposition.</u> Suppose $\gamma: Y \to Y'$ is a UV^{k-1}-surjection between LC^{k-1}-compacta. Then for every $\varepsilon > 0$ there exists $\delta > 0$ so that the following holds. If A is a closed subset of a compact metric space X, $\dim(X - A) \leq k$, and if $f,g: A \to Y$ are two maps such that $(\gamma f, \gamma g) < \delta$ and f extends to $\tilde{f}: X \to Y$, then there is an extension of g $\tilde{g}: X \to Y$ with $(\gamma\tilde{f}, \gamma\tilde{g}) < \varepsilon$.

<u>2.1.5. Lemma.</u> For any compactum X there is a LC^{k-1}-space Z containing X so that $\dim(Z - X) \leq k$.

<u>Sketch of proof.</u> Embed X into the Hilbert cube $Q = \prod_{j=1}^{\infty} [-1, 1]_j$, and for $i \geq k$ let P_i be the infinite k-skeleton of $[-1, 1]^i \times \prod_{j>i} \{0\}_j$, so that $P_k \subseteq P_{k+1} \subseteq \ldots$. Then set $Z = X \cup \bigcup_{i \geq k} P_i$.

<u>2.1.6. Proposition.</u> Let $f, g: X \to Y$ be two maps between compacta, and assume that Y is LC^{k-1}. The following two statements are equivalent:

(i) For any embedding $X \to \tilde{X}$ of X into an LC^{k-1}-space with $\dim(\tilde{X} - X) \leq k$, and any extensions $\tilde{f}, \tilde{g}: U \to Y$ of f, g onto some neighborhood U of X in $\tilde{X}$, there is a neighborhood $V \subseteq U$ of X so that for any map $\alpha: P \to V$

from a polyhedron of dimension $\leq k-1$ we have

$f\alpha \simeq g\alpha$.

(ii) There exists an embedding $X \to \tilde{X}$ of X into an LC^{k-1}-space $\tilde{X}$, and there exist extensions $\tilde{f}, \tilde{g}\colon U \to Y$ of f,g onto a neighborhood U of X such that for some neighborhood $V \subseteq U$ of X, for any map $\alpha\colon P \to V$ from a polyhedron of dimension $\leq k-1$ it follows that

$\tilde{f}\alpha \simeq \tilde{g}\alpha$.

Proof. Implication (i) $\Rightarrow$ (ii) follows from 2.1.5. and the fact that f,g extend onto a neighborhood of X [Hu]. To show (ii) $\Rightarrow$ (i), let $(\tilde{X}', \tilde{f}', \tilde{g}', U')$ be another choice of an embedding $X \hookrightarrow \tilde{X}'$ and extensions $\tilde{f}', \tilde{g}'\colon U' \to Y$. Extending $\mathrm{Id}\colon X \to X \subseteq \tilde{X}(\tilde{X}')$ we can find neighborhoods W, V' of X in $\tilde{X}, \tilde{X}'$ respectively and maps $\phi\colon V' \to W$, $\psi\colon W \to U'$ so that for any maps $\alpha\colon P \to V'$, $\beta\colon P \to W$ of a $(k-1)$-dimensional polyhedron P we have $i\alpha \simeq \psi\phi\alpha$ ($i\colon V' \to U'$ inclusion, since $\tilde{X}'$ is LC^{k-1}, if V' is small enough, $i\alpha$ and $\psi\phi\alpha$ will be close maps [Hu]), and $\tilde{f}'\psi\beta \simeq \tilde{f}\beta$, $\tilde{g}'\psi\beta \simeq \tilde{g}\beta$ (again, if W is small enough, these two maps will be close). Finally, we have $\tilde{f}'\alpha \simeq \tilde{f}'\psi\phi\alpha \simeq \tilde{f}\phi\alpha \simeq \tilde{g}\phi\alpha \simeq \tilde{g}'\psi\phi\alpha \simeq \tilde{g}'\alpha$.

2.1.7. Definition. Two maps $f, g\colon X \to Y$ between compacta, where Y is LC^{k-1}, are μ-homotopic, $f \underset{\mu}{\simeq} g$, if (i) and (ii) from 2.1.6. hold.

The next proposition follows immediately from the definition and some elementary properties of LC^{k-1}-spaces (in (iii) use the lifting technique).

<u>2.1.8. Proposition.</u> (i) If Y is a LC^{k-1}-compactum, then there exists $\varepsilon > 0$ such that any two ε-close maps $f, g: X \to Y$ are μ-homotopic. (ii) If $f,g,h: X \to Y$ are maps into an LC^{k-1}-compactum Y, and if $f \underset{\mu}{\simeq} g$, $g \underset{\mu}{\simeq} h$, then $f \underset{\mu}{\simeq} h$. (iii) If $\gamma: Y \to Y'$ is a UV^{k-1}-surjection between LC^{k-1}-compacta, and $f,g: X \to Y$ maps from a compactum X with $\gamma f \underset{\mu}{\simeq} \gamma g$, then $f \underset{\mu}{\simeq} g$. (iv) ($\mu$-Homotopy Extension Theorem). If $A \subseteq X$ are compacta with $\dim X \leq k$, if $f,g: A \to Y$ are μ-homotopic maps into an LC^{k-1}-compactum Y, and if f extends onto $\tilde{f}: X \to Y$, then g extends onto $\tilde{g}: X \to Y$ with $\tilde{f} \underset{\mu}{\simeq} \tilde{g}$.

<u>Sketch of proof of (iv).</u> First, we realize (using (i) and 2.1.4. with $\gamma = \mathrm{Id}$) that it suffices to construct $\tilde{g}: X \to Y$ with $\tilde{g} \underset{\mu}{\simeq} \tilde{f}$ and $\tilde{g}|A$ δ-close to g (for sufficiently small $\delta > 0$). Second, this observation enables us to replace X by the nerve $|N(\mathcal{U})|$ of a fine cover $\mathcal{U}$ of X (of order $\leq k+1$), and $A \subseteq X$ by a subcomplex $|L| \subseteq |N(\mathcal{U})|$. Next, $f \underset{\mu}{\simeq} g$ leads to a homotopy $H_0: |L^{(k-1)}| \times I \to Y$ between $f||L^{(k-1)}|$ and $g||L^{(k-1)}|$, which in turn extends (simplex by simplex) to a map $H: |N(\mathcal{U})^{(k-1)}| \times I \cup |N(\mathcal{U})| \times \{0, 1\} \to Y$ with $H\, |N(\mathcal{U})| \times \{0\} = \tilde{f}$, $H||L| \times \{1\} = g$. Then set $\tilde{g}(x) = H(x, 1)$.

The main theorem of this section is an important attempt to reduce the resolution theorem (see section 5.1.) to the k-dimensional case. The author is indebted to J.J. Walsh for suggesting this approach.

2.1.9. Theorem. For any compactum X there exists a compactum $\tilde{X}$ with $\dim \tilde{X} \leq k$, and a UV^{k-1}-surjection $f: \tilde{X} \to X$.

Sketch of proof. Let $\{P_n, f_{nm}\}$ be an inverse sequence of finite polyhedra with limit X. Recursively on n define finite collections $\mathcal{U}_n = \{U_\alpha^{(n)}: \alpha \in A_n\}$, $\mathcal{V}_n = \{V_\alpha^{(n)}: \alpha \in A_n\}$ of sub-polyhedra of P_n whose interiors cover P_n with $V_\alpha^{(n)} \subseteq \operatorname{int} U_\alpha^{(n)}$, triangulation L_n of P_n and a simplicial approximation $g_{n-1,n}: |L_n| \to |L_{n-1}|$ of $f_{n-1,n}: P_n \to P_{n-1}$ so that

(a) $f_{n,m-1}\, g_{m-1,n}$ and f_{nm} are $\frac{1}{2^m}$-close $(m > n)$,

(b) for any $n \geq 1$, $\limsup_{m>n} \sup_{\alpha \in A_n} \operatorname{diam} f_{nm}(U_\alpha^{(m)}) = 0$,

(c) if $y_n \in P_n$, $g_{n-1,n}(y_n) = y_{n-1}$ $(n > 1)$, and if $y_m \notin U_\alpha^{(n)}$ for some $m \geq 1$, $\alpha \in A_m$, then for all $p > m$ it follows that $f_{mp}(y_p) \notin V_\alpha^{(m)}$

(d) if $x_n \in P_n$, $x_{n-1} \in P_{n-1}$, $f_{n-1,n}(x_n) = x_{n-1}$, $x_n \in U_\alpha^{(n)}$, $x_{n-1} \in U_\beta^{(n-1)}$, then $f_{n-1,n}(U_\alpha^{(n)}) \subseteq U_\beta^{(n-1)}$, and

(e) each $U_\alpha^{(n)} \in \mathcal{U}_n$ is contractible and underlies a subcomplex of L_n.

Then let $\bar{g}_{nm} = g_{nm} | |L_m^{(k)}|: |L_m^{(k)}| \to |L_n^{(k)}|$, and define $\tilde{X} = \varprojlim\{|L_n^{(k)}|, \bar{g}_{nm}\}$. Clearly, $\dim \tilde{X} \leq k$ [En]. Condition (a) implies that inclusions $|L_n^{(k)}| \to P_n$ "induce" a map $f: \tilde{X} \to X$ [Br]. Next, (b), (c) and (d) imply that if $x = (x_n) \in X$, then $f^{-1}(x) = \varprojlim\{U_\alpha^{(n)} \cap L_n^{(k)}, g_{nm}\}$ for any choice $U_\alpha^{(n)} \in \mathcal{U}_n$ with $x_n \in U_\alpha^{(n)}$. Finally, (e) asserts that each $f^{-1}(x)$ is non-empty and LC^{k-1}. For computational details the reader is referred to [Wa_2], where it is shown that if X has cohomological dimension less than or equal to k, then (with some additional control imposed) f is cell-like. Note that even if X is LC^{k-1}, $\tilde{X}$ may in general fail to be LC^{k-1}.

2.2. The Isotopy Move and Verification of Axiom 1.

Let (M_1, L_1), $(M_2, L_2), \ldots$ be a μ^k-manifold defining sequence (as described in 1.2.). In this section we present a move ("The Isotopy Move") that adjusts a given submanifold $N \subseteq M_i$ so that (after adjusting) $N \cap M_j$ is a subcomplex of L_j (for some $j > i$). Later, we will apply this move to finite collections of submanifolds (e.g. partitions).

For drawing conclusions, the description given below of the move is more important than any list of properties that can be preserved in the process of adjusting. The verification of axiom 1 is an easy application of this move.

The idea is to carefully choose starring points for various derived subdivisions in the construction of the μ^k-manifold defining sequence, and then (after going deep enough in the sequence to make sure that all desired submanifolds become subcomplexes) to specify an isotopy moving the prechosen starring points to barycenters.

2.2.1. Lemma. Assume σ is a simplex in a Euclidean space E^p, and $\sigma_1 < \sigma$ is one of its faces. For each face $\tau < \sigma$ choose a starring point $\hat{\tau} \in \operatorname{int} \tau$ according to the following rule (d is the standard metric on E^n),

(*) If $\tau_1 < \tau_2 < \sigma$ and $\tau_1 \cap \sigma_1 \neq \phi$, then $d(\hat{\tau}_1, \sigma_1) \leq d(\hat{\tau}_2, \sigma_1)$.

Let K be the simplicial complex obtained from σ by starring at the chosen starring points, and let $N = \cup\{\rho\colon \rho$ is a simplex in K, $\rho \cap \sigma_1 \neq \phi\}$ be the derived neighborhood of σ_1 [Hd]. Then N is a convex set.

Proof. Proceed by induction on $n = \dim \sigma$. By inductive hypothesis, $N \cap \tau$ is convex for any proper face $\tau \lneqq \sigma$ with $\tau \cap \sigma_1 \neq \phi$. Let N' be the convex closure of $\cup\{N \cap \tau\colon \tau \lneqq \sigma$, $\tau \cap \sigma_1 \neq \phi\}$. Clearly, $N' \cap \tau = N \cap \tau$ for any $\tau \lneqq \sigma$. Also, (*) implies that $\hat{\sigma} \notin \operatorname{int} N'$. By definition, N is the cone from $\hat{\sigma}$ over $\cup\{N \cap \tau : \tau \lneqq \sigma, \tau \cap \sigma_1 \neq \phi\}$. Since $\hat{\sigma} \notin \operatorname{int} N'$, it follows that N is the cone from $\hat{\sigma}$ over N' and hence convex.

We use 2.2.1. in the following setting: $\sigma_1 < \sigma$ are as in lemma, and $\tau = \tau_1 * \tau_2$ is a simplex in E^p with $\tau_1 \leq \sigma_1$, $\tau_2 \cap \sigma_1 = \phi$. Then 2.2.1. asserts that if we choose starring points carefully (i.e. so that (*) holds and so that $N \cap \tau_2 = \phi$), then $N \cap \tau$ is a regular neighborhood of τ_1 in τ (N as described in 2.2.1.).

We now proceed with a description of The Isotopy Move.

Let A be a finite collection of simplexes in M_i (M_i is assumed to be embedded in some Euclidean space E^p with all simplexes in L_i and A geometric). Let $\{c_1, c_2, \ldots, c_r\}$ be an ordering of simplexes in A and their faces in a non-decreasing way. Then c_1 is a vertex, and we choose a derived subdivision L_i' of L_i that contains c_1 as a vertex. Subdivide simplexes in A so that all simplexes τ in the subdivision have the form $\tau = \tau_1 * \tau_2$ where τ_1 is a simplex contained in the $(m\text{-}k\text{-}1)$-skeleton $L_i^{(m-k-1)}$ of L_i,

$\tau_2 \cap |L_i^{(m-k-1)}| = \phi$ and τ is contained in a simplex of L_i'. Observing that $L_i^{(m-k-1)}$ is a full subcomplex of L_i', we apply 2.2.1. to define a derived subdivision L_i'' of L_i' such that if N is the second derived neighborhood of $L_i^{(m-k-1)}$ with respect to L_i'', then $N \cap \tau$ is a regular neighborhood of $N \cap \tau_1$ for any $\tau = \tau_1 * \tau_2$ as above. From this, it follows that $N \cap \sigma$ is a regular neighborhood of $|L_i^{(m-k-1)}| \cap \sigma$ in σ for any $\sigma \varepsilon A$ (sketch of proof: let $\tilde{N}$ be the second derived neighborhood of $|L_i^{(m-k-1)}| \cap \sigma$ in σ with respect to a triangulation that subdivides everything in sight. Then construct a PL-homeomorphism
$(\sigma, \tilde{N}, |L_i^{(m-k-1)}| \cap \sigma) \cong (\sigma, N \cap \sigma, |L_i^{(m-k-1)}| \cap \sigma)$ successively on simplexes τ subdividing σ). Replace M_i by $M_{i+1}' = M_i - \mathrm{int}\, N$.
If c_1 is a vertex, proceed as above.

In the inductive step, we assume that we have constructed a manifold $M_{i+s}' \subseteq M_i$ and a triangulation $\overline{L}_{i+s}$ of M_{i+s}' that contains subdivisions of $c_1 \cap M_{i+s}', \ldots, c_s \cap M_{i+s}'$ (some of these are possibly empty). An argument in [Hd, Lemma 1.5.] shows that we can choose a derived neighborhood $\overline{L}_{i+s}'$ of $\overline{L}_{i+s}$ so that it contains a subdivision of $c_{s+1} \cap M_{i+s}'$. As above, choose a second derived $\overline{L}_{i+s}''$ so that for the second derived neighborhood N of $|\overline{L}_{i+s}^{(m-k-1)}|$ we have that $N \cap \sigma \cap M_{i+s}'$ is a regular neighborhood of $|\overline{L}_{i+s}^{(m-k-1)}| \cap \sigma \cap M_{i+s}'$ in $\sigma \cap M_{i+s}'$ for any $\sigma \varepsilon A$ (implicitely, $\sigma \cap M_{i+s}'$ is a manifold). Then let $M_{i+s+1}' = M_{i+s}' - \mathrm{int}\, N$, $\overline{L}_{i+s+1} = \overline{L}_{i+s}''|M_{i+s+1}'$.

After r steps we obtain a manifold $M_{i+r}' \subseteq M_i$. Let $\phi_t\colon M_i \to M_i$ be the PL-isotopy obtained by composing r PL-isotopies, the s^{th} isotopy ϕ_t^s taking starring points of $\phi_1^{s-1}(\overline{L}_{i+s}')$ and $\phi_1^{s-1}(\overline{L}_{i+s}'')$

vertices of βL_{i+s} and $\beta^2 L_{i+s}$ (ϕ_t^s is the identity on $M_i - \phi_1^{s-1}(M'_{i+s}))$.

Replacing $\sigma \in A$ by $\phi_1(\sigma) \cap M_{i+s}$ is The Isotopy Move. In practice, A will be the collection of simplexes of some triangulation of a submanifold $p \subseteq M_i$. Observe that if $\dim p \leq m - k - 1$, then $\phi_1(p) \cap M_{i+r+1} = \phi$ (since after r steps p becomes a part of the (m-k-1)-skeleton, which is thrown away). On the other hand, if $\dim p = m - t$, $t \leq k$, then inclusion $\phi_1(p) \cap M_{i+r} \to \phi_1(p)$ is a (k-t)-connected map (i.e. it induces isomorphisms of homotopy groups of dimension $\leq k - t - 1$ and an epimorphism in dimension $k - t$). This is a consequence of a general position argument, and the fact that $\phi_1(p) \cap M_{i+s+1}$ is obtained from $\phi_1(p) \cap M_{i+s}$ by throwing away a regular neighborhood of a (m-k-1)-complex. Also, observe that $\phi_t(\sigma) \subseteq \sigma$, for any $t \in [0, 1]$ and any simplex σ in L_i .

<u>2.2.2. Proposition.</u> Any μ^k-defining sequence $\{M_i\}_{i=1}^{\infty}$ together with collections C_i^s of convenient submanifolds as defined in 1.2. satisfies (C1)-(C4) and axiom 1.

<u>Proof.</u> To see that (C1)-(C4) are satisfied requires hardly more than a quick glance that we leave to the reader to make. Let P and $\varepsilon > 0$ be as in the axiom 1. Find $j > i$ such that mesh $L_j < \frac{\varepsilon}{6}$, and let Q' be the handlebody decomposition of M_j associated with L_j . Let $\phi_t\colon M_j \to M_j$ be a PL-isotopy as in The Isotopy Move such that $\phi_1(q_1 \cap \ldots \cap q_s) \cap M_\ell$ underlies a subcomplex of L_ℓ (for some $\ell \geq j$ and all $q_1, \ldots, q_s \in Q'$) . In particular, ϕ_1 moves points for less than $\frac{\varepsilon}{6}$. We set $Q = \phi_1(Q')|M_{\ell+1}$. Then

mesh $Q <$ mesh $Q' + 2 \cdot \frac{\varepsilon}{6} < \frac{\varepsilon}{2} + \frac{2\varepsilon}{6} < \varepsilon$. Also, Q is a convenient k-partition on $M_{\ell+1}$ (the undesirable non-empty intersections of handles in $\phi_1(Q')$ disappear after restricting to $M_{\ell+1}$ since their dimension is $\leq m - k - 1$, while the remaining intersections have the correct connectivity properties). The same argument, taken together with the fact that $\phi_t(p_1 \cap \ldots \cap p_s) \subseteq p_1 \cap \ldots \cap p_s$ for all $t \in [0, 1]$ and $p_1, \ldots, p_s \in P$ gives that $Q|p_1 \cap \ldots \cap p_s \cap M_{\ell+1}$ is a (k-s+1)-partition for all choices of mutually distinct $p_1, \ldots, p_s \in P$ having non-empty intersection.

2.3. Absorbing Maps and Basic Properties of μ_k-Manifolds

In this section we establish an "absorption property" of (triangulated) μ^k-manifolds, that leads to proofs of some "obvious" properties of μ^k-manifolds (LC^{k-1}, DD^kP). We also prove the Z-set approximation theorem (which follows "categorically" once μ^k-manifolds are known to be k-dimensional and LC^{k-1} and to satisfy the Disjoint k-Cells Property).

<u>The Absorption Move.</u> Let $\{M_i\}_{i=1}^{\infty}$ be a μ^k-manifold defining sequence. Suppose P is a polyhedron, $\dim P \leq k$, and $f_i: P \to M_i$ is a map. First approximate f_i by a general position map $f_i': P \to M_i$ whose image misses $|L_i^{(m-k-1)}|$ and define $f_{i+1}: P \to M_{i+1}$ as the composition $f_{i+1} = r f_i'$, where $r: M_i - |L_i^{(m-k-1)}| \to M_{i+1}$ is the canonical (strong) deformation retraction. Proceed recursively to produce a sequence $f_i, f_{i+1}, f_{i+2}, \ldots$ of maps. The important observation is that if $f_j(x) \in \sigma \in L_j$ then $f_{j+1}(x) \in St(\sigma, \beta^2 L_j)$,

and hence $f_n(x) \in St(\sigma, L_j)$, for all $n > j$. In particular, $\{f_j\}_{j=1}^{\infty}$ is a Cauchy sequence. Let $f = \lim_{n\to\infty} f_n$. Then $f\colon P \to M = \bigcap_{j=1}^{\infty} M_j$ is a map $(2 \cdot \text{mesh } L_i)$-close to f_i. Moreover, if $P_0 \subseteq P$ and $f_i(P_0) \subseteq M$, then $f|P_0 = f_i|P_0$.

2.3.1. Proposition. Any triangulated μ^k-manifold M satisfies the Disjoint k-Cells Property (DD^kP), i.e. any two maps $f, g\colon I^k \to M$ can be arbitrarily closely approximated by maps with disjoint images.

Proof. Let $\varepsilon > 0$, and choose $i > 0$ such that mesh $L_i < \varepsilon$. Approximate f, g by PL-embeddings $f_1, g_1\colon I^k \to M_i$ with disjoint images such that im f_1, im g_1 miss the (m-k-1)-skeleton of L_i and $(f,f_1) < \varepsilon$, $(g,g_1) < \varepsilon$. Let N_f, N_g be regular neighborhoods of im f_1, im g_1 respectively in M_i such that $N_f \cap N_g = \phi$. By The Isotopy Move (2.2.) there is $j > i$ and an isotopy $\phi_t\colon M_i \to M_i$ such that $\phi_1(N_f) \cap M_j$, $\phi_1(N_g) \cap M_j$ underlie subcomplexes of L_j. Moreover, following the description of The Isotopy Move, we choose derived subdivisions so that (in addition) various (m-k-1)-skeleta miss im f_1 $\cup$ im g_1 and so that various neighborhoods that are thrown away are small enough to miss im f_1 $\cup$ im g_1. Let $f_2 = \phi_1 f_1$, $g_2 = \phi_1 g_1$. Passing one stage deeper in the defining sequence if necessary, we may assume that $St(\phi_1(N_f) \cap M_j, L_j) \cap St(\phi_1(N_g) \cap M_j, L_j) = \phi$. This condition guarantees that if we replace f_2, g_2 by $f_3, g_3\colon I^k \to M$ as described (The Absorption Move), then im $f_3 \cap$ im $g_3 = \phi$. Finally, observe that $(f_3,f) < (f_3, f_2) + (f_2, f_1) + (f_1, f) < 2(\text{mesh } L_j) + \varepsilon + \varepsilon < 4\varepsilon$, and similarly $(g_3, g) < 4\varepsilon$.

<u>2.3.2. Proposition</u>. If $\{M_i\}_{i=1}^{\infty}$ is a μ^k-manifold defining sequence, $M = \bigcap_{i=1}^{\infty} M_i$, then inclusion induced homomorphism $i_\#: \pi_r(M) \to \pi_r(M_1)$ is an isomorphism for $r \leq k - 1$, and epimorphism for $r = k$.

<u>Proof.</u> a) $i_\#$ is an epimorphism, $r \leq k$. Let $f: (S^r, *) \to (M_1, *)$ be a map representing an element of $\pi_r(M_1)$ (the basepoint is chosen in $M \subseteq M_1$). Let $f': (S^r, *) \to (M, *)$ be obtained as described earlier. Then clearly f and f' are homotopic (rel $*$) as maps into M_1, since all the moves were either general position moves, or composing with a strong deformation retraction.

b) $i_\#$ is a monomorphism, $r \leq k - 1$. Let $f: (S^r, *) \to (M, *)$ be a map that extends to $F: B^{r+1} \to M_1$. Let $F': B^{r+1} \to M$ be a map obtained by absorbing F. Then $F'|S^r = f$, and hence f represents the trivial element in $\pi_r(M)$.

<u>2.3.3. Corollary.</u> A universal k-dimensional Menger space μ^k (i.e. any space belonging to $\mathcal{M}_0$, see 1.2.) is (k-1)-connected.

<u>2.3.4. Corollary.</u> Any triangulated μ^k-manifold is LC^{k-1}.

<u>Proof.</u> Any point in a triangulated μ^k-manifold has a closed neighborhood belonging to $\mathcal{M}_0$.

Just for the reference, let us note that

<u>2.3.5. Proposition.</u> Any triangulated μ^k-manifold M is k-dimensional.

<u>Proof.</u> The restriction to M of the canonical retraction of M_{i+1} onto the dual k-skeleton of L_i defines a small map $M \to$ k-polyhedron. By [H-W] $\dim M \leq k$. That $\dim M \geq k$ follows from 2.3.8. (as M contains many k-cells).

<u>2.3.6. Proposition.</u> Let X be a compact LC^{k-1}-space with $\dim X \leq k$. For a closed subset $A \subseteq X$ the following statements are equivalent.

(i) Any map $f: I^k \to X$ can be approximated arbitrarily closely by a map whose image misses A.

(ii) Any map $f: P \to X$ from a polyhedron of dimension $\leq k$ can be approximated by a map whose image misses A.

(iii) For any $\varepsilon > 0$ there is a map $g: X \to X - A$ ε-close to the identity map.

<u>Proof.</u> <u>(i) $\Rightarrow$ (ii).</u> Let K be a triangulation of P. Observe that (i) implies that the set of maps $\phi: P \to X$ with $\phi(\sigma) \cap A = \phi$ is open and dense in the space of all maps $P \to X$ (σ is a fixed simplex of K; here, we approximate $f|\sigma$ by $f': \sigma \to X - A$ and extend f' onto P using 2.1.4.). Consequently, the set of all maps $\phi: P \to X$ with $\phi(P) \cap A = \phi$ is dense in $\{f: P \to X\}$.

<u>(ii) $\Rightarrow$ (iii).</u> Let α be a small cover of X, and let β be a refinement of α such that every partial realization of a $\leq k$ dimensional simplicial complex relative to β extends to a full realization relative to α [Hu]. Let $p: X \to |N(\gamma)|$ be the projection map into the nerve of a cover γ that star-refines β and $\dim N(\gamma) \leq k$.

Define q on verticies V of $N(\gamma)$ so that $q(V) \varepsilon V$, and extend to $q: |N(\gamma)| \to X$. By (ii), q can be approximated by $q': |N(\gamma)| \to X - A$. Finally, $q'p: X \to X - A$ is the desired map.

(iii) ⇒ (i). Clear.

2.3.7. Definition. A closed subset A of an LC^{k-1} space X with $\dim X \leq k$ (in particular X can be a μ^k-manifold) is a Z-set if any one of (i)-(iii) from 2.3.6 are satisfied.

2.3.8. Theorem. (The Z-set approximation theorem). Let X be a LC^{k-1}-compactum with $\dim X \leq k$ that satisfies DD^kP. If B is any compactum with $\dim B \leq k$, then any map $f: B \to X$ can be approximated by a Z-embedding (i.e. by an embedding whose image is a Z-set). Moreover, if $B_0 \subseteq B$ is a closed subset and if $f|B_0$ is a Z-embedding, then the approximation can be chosen to agree with f on B_0.

Proof. First replace B by $B_0 \cup |N(\mathcal{U})|$ for some sufficiently small countable cover $\mathcal{U}$ of $B - B_0$ of order $\leq k - 1$. Let $K_1, K_2, \ldots$ be a sequence of triangulations of $|N(\mathcal{U})|$ with mesh $K_i \to 0$. For a simplex σ in one of these triangulations, let $F[\sigma]$ denote the set of maps $g: B_0 \cup |N(\mathcal{U})| \to X$ with $g|B_0 = f|B_0$ and $g(\sigma) \cap f(B_0) = \phi$. By 2.1.4. and assumption on $f(B_0)$, $F[\sigma]$ is open and dense in the space F of all maps $g: B_0 \cup |N(\mathcal{U})| \to X$ that agree with f on B_0. For disjoint simplexes σ_1, σ_2 of one of the triangulations K_i, let $F[\sigma_1, \sigma_2]$ denote the set of maps $g: B_0 \cup |N(\mathcal{U})| \to X$ that agree with f on B_0 and that satisfy $g(\sigma_1) \cap g(\sigma_2) = \phi$. By DD^kP and 2.1.4. $F[\sigma_1, \sigma_2]$ is open and dense in F. Let G be the intersection of (the countably many) sets of the form $F[\sigma]$ and $F[\sigma_1, \sigma_2]$. Observe that G consists of embeddings and that it is dense G_δ in F. Consequently, the space of ε-maps $B \to X$ that agree with f on B_0 is open and dense for every $\varepsilon > 0$. Applying Baire's theorem once again, we conclude that any map $f: B \to X$ can be approximated by an embedding (that agrees with $f|B_0$). To arrange that the image be a Z-set, we have to refine the proof as follows. Let $\{\alpha_1, \alpha_2, \ldots\}$ be a countable dense subset of the space of maps $\alpha: I^k \to X$. Using DD^kP, the fact that $f(B_0)$ is a Z-set, and the argument above, we approximate α_i by an embedding $\beta_i: I^k \to X$ so that $\operatorname{im} \beta_i \cap f(B_0) = \phi$ and, for $i \neq j$, $\operatorname{im} \beta_i \cap \operatorname{im} \beta_j = \phi$. (We do it recursively on i. Assuming $\beta_1^{i-1}, \ldots, \beta_{i-1}^{i-1}$ are embeddings approximating $\beta_1, \ldots, \beta_{i-1}$, choose embeddings $\beta_1^i, \ldots, \beta_{i-1}^i, \beta_i^i$ approximating $\beta_1^{i-1}, \ldots, \beta_{i-1}^{i-1}, \alpha_i$ having disjoint images that miss $f(B_0)$. Imposing sufficient control, the maps $\beta_i = \lim_{n>i} \beta_i^n$ are embeddings with the desired properties.) Thus $\{\beta_1, \beta_2, \ldots\}$ is a countable dense subset of $\{\alpha: I^k \to X\}$ and the β_i's are embeddings whose images miss $f(B_0)$. Observe that, by Definition 2.3.6.(i), if $Z \cap \beta_i(I^k) = \phi$, $i = 1,2,\ldots$ then Z is a Z-set. Following the first part of the argument, let $F[j, \sigma]$ be the set of maps $g: B_0 \cup |N(\mathcal{U})| \to X$ with $g|B_0 = f|B_0$ and $g(\sigma) \cap \beta_j(I^k) = \phi$. The fact that $\beta_j(I^k)$ is a Z-set implies that $F[j, \sigma]$ is open and dense in F. Let G' be the intersection of G and all sets of the form $F[j, \sigma]$. Then G' is dense G_δ in F, and consists of Z-embeddings. This amounts to the fact that the space $\Phi[j, \varepsilon]$ of ε-maps $h: B \to X$ with $h|B_0 = f|B_0$ and

$h(B) \cap \beta_j(I^k) = \phi$ is open and dense in the space Φ of all maps $g: B \to X$ with $g|B_0 = f|B_0$. A final Baire argument finishes the proof.

2.3.9. Proposition. Any one-point subset $\{x\}$ of a triangulated μ^k-manifold M is a Z-set.

Proof. As in the proof of 2.3.8., construct a family $\mathcal{B} = \{\beta_1, \beta_2, \ldots\}$ of embeddings $\beta_i: I^k \to M$ so that, for $i \neq j$, $\beta_i(I^k) \cap \beta_j(I^k) = \phi$ and so that $\mathcal{B}$ is dense in the space of all maps $I^k \to M$. If $x \in M - \bigcup_{i=1}^{\infty} \beta_i(I^k)$, then $\mathcal{B}$ detects that $\{x\}$ is a Z-set (cf. 2.3.6.(i)). If $x \in \beta_j(I^k)$, then the subcollection $\mathcal{B} - \{\beta_j\}$ detects that $\{x\}$ is a Z-set.

2.4. Building Partitions and Associated Maps

The reader should recall at this point that our goal is to prove that any μ^k-defining sequence (as defined in 1.2.) satisfies axiom 2 (see 1.3.). This axiom roughly says that we can build k-partitions on M_i's (with control). The strategy for verifying this axiom is to: (a) build a partition P on M_i that is in one-to-one correspondence with a given partition, and (b) perform "internal surgeries" in order to turn P into a k-partition. In this section we describe the first part of the strategy. The notion of associated maps (see [Wa_1] or 2.4.1.) appears to be an important refinement of the notion of intersection preserving functions. These maps come in handy for keeping the size of different moves under control. A helpful observation that simplifies the notation is that, by induction, it suffices to prove axiom 2 for $s = 0$.

2.4.1. Definition. Let P_1, P_2 be partitions on M_1, M_2 respectively, and assume that $T: P_1 \to P_2$ is a function that preserves intersections (1.1.15.). A map $\alpha: M_1 \to M_2$ is said to be associated to T if $\alpha(p) \subseteq T(p)$ for all $p \in P_1$.

2.4.2. Remark. If P_2 is a handlebody decomposition of a manifold, then any two associated maps are homotopic [Wa$_1$]. If P_2 is a k-partition, then any two associated maps are μ-homotopic.

2.4.3. Setting. $\{M_j\}_{j=1}^{\infty}$ is a μ^k-defining sequence, M_0 is a submanifold of ∂M_i , $\dim M_0 = m - 1$, $\dim M_j = m$ $(j = 1,2,\dots)$, Q_0 is a partition on M_0 such that all elements of Q_0 underlie a subcomplex of L_i , M' is a (k-1)-connected manifold, $\overline{Q}$ is a k-partition on M' , $T_0: Q_0 \to \overline{Q}$ is an injective function that preserves intersections.

2.4.4. Step 1. Define a map $\alpha_0: M_0 \cap M_{i[1]} \to M'$ associated to the function $Q_0|M_0 \cap M_{i[1]} \xrightarrow{I} Q_0 \xrightarrow{T_0} \overline{Q}$ $(I(q_0 \cap M_{i[1]}) = q_0$, $i[1] \geq i$ a sufficiently large index).

Recursively, for $t = k, k-1, \dots, 1, 0$ we define a map $\alpha_0^t: S_t(Q_0|M_0 \cap M_{i+k+1-t}) \to M'$ ($S_t(P)$ is the codimension t stratum, see 1.3.) requiring that $\alpha_0^t(q_1^0 \cap \dots \cap q_j^0 \cap M_{i+k+1-t}) \subseteq T_0(q_1^0) \cap \dots \cap T_0(q_j^0)$ for $j \geq t + 1$ and for all choices of mutually distinct $q_1^0, \dots, q_j^0 \in Q_0$ having non-empty intersection. For the inductive step, $\alpha_0^{t+1}|q_1^0 \cap \ldots \cap q_{t+1}^0 \cap S_{t+1}(Q_0|M_0 \cap M_{i+k-t})$ (with the $(k-t-1)$-connected set $T_0(q_1^0) \cap \ldots \cap T_0(q_{t+1}^0)$ as the range) extends to a regular neighborhood of the dual $(k-t)$-skeleton of the triangulation of $q_1^0 \cap \ldots \cap q_{t+1}^0 \cap M_{i+k-t}$ gotten by restricting L_{i+k-t}. This extension gives rise to $\alpha_0^t\colon q_1^0 \cap \ldots \cap q_{t+1}^0 \cap M_{i+k+1-t} \to M'$. By piecing these together we get $\alpha_0^t\colon S_t(Q_0|M_0 \cap M_{i+k+1-t}) \to M'$. Setting $\alpha_0 = \alpha_0^0$ and $i[1] = i + k + 1$ we get the desired map.

<u>2.4.5. Step 2.</u> Find a map $\alpha\colon M_{i[2]} \to M'$ (for some $i[2] \geq i[1]$) with $\alpha|M_0 \cap M_{i[2]} = \alpha_0|M_0 \cap M_{i[2]}$ and so that α, in addition, induces an epimorphism of k^{th} homotopy groups.

Let P be a polyhedron of dimension $\leq k$ and let $\beta\colon P \to M'$ be a map that induces epimorphism on π_k (e.g. P can be the k-skeleton of a triangulation of M'). Embed P into Int $M_{i[1]}$, and let R be a regular neighborhood of P. By The Isotopy Move (2.2.) there is an isotopy $\phi_t\colon M_{i[1]} \to M_{i[1]}$ and an index $j \geq i[1]$ such that $\phi_1(R) \cap M_j$ underlies a subcomplex of L_j. Observe that ϕ_t can be chosen to be identity on $\partial M_{i[1]} \supseteq M_0 \cap M_{i[1]}$. Also, if $\tilde{\beta}\colon R \to M'$ is an extension of β, then $\phi_1(R) \cap M_j \xrightarrow{\phi_1^{-1}} R \xrightarrow{\tilde{\beta}} M'$ induces an epimorphism on π_k. Now, $\alpha_0 \cup \tilde{\beta}\phi_1^{-1}|(M_0 \cap M_j) \cup (\phi_1(R) \cap M_j)$ as a map into a $(k-1)$-connected set M' extends onto a regular neighborhood of the dual k-skeleton of L_j. This map gives rise to $\alpha\colon M_{j+1} \to M'$. Set $i[2] = j + 1$.

<u>2.4.6. Step 3.</u> Redefine α so that, in addition, the following is true. There exists a collar $\phi\colon (M_0 \cap M_{i[2]}) \times I \to M_{i[2]}$ on $M_0 \cap M_{i[2]}$ such that

(i) for every $q_0 \in Q_0$ $\alpha(\phi((q_0 \cap M_{i[2]}) \times I)) \subseteq T_0(q_0)$, and

(ii) there exists $\varepsilon > 0$ such that if $A \subseteq M_{i[2]}$ is a set of diameter $< \varepsilon$, then $\alpha(A)$ is contained in a single element of $\overline{Q}$.

To get (i), attach a collar $(M_0 \cap M_{i[2]}) \times I$ on $M_{i[2]}$ and define $\alpha' = (M_0 \cap M_{i[2]}) \times I \cup M_{i[2]} \to M'$ by $\alpha'|(M_0 \cap M_{i[2]}) \times I = (\alpha|M_0 \cap M_{i[2]}) \circ \mathrm{pr}|M_0 \cap M_{i[2]}$. To get (ii), let $r\colon M' \to M'$ be a map extending a retraction of a regular neighborhood of the codimension 1 skeleton of a triangulation of M' (subdividing triangulations of the partition elements of $\overline{Q}$) onto this skeleton. Then replace α by $r\alpha'$. (The map r is chosen so that it is homotopic to $\mathrm{Id}\colon M' \to M'$, and so that $r(\sigma) \subseteq \sigma$ for all simplexes σ in the mentioned triangulation.)

<u>2.4.7. Step 4.</u> Define a partition Q on $M_{i[2]}$ that "extends" $Q_0|M_0 \cap M_{i[2]}$ and an injective function $T\colon Q \to \overline{Q}$ that "extends" T_0 (i.e. satisfying (a) and (b) from Axiom 2).

Let K be a subdivision of L_j such that mesh $K < \frac{\varepsilon}{3}$ and $\phi((q_0 \cap M_{i[2]}) \times I)$ underlies a subcomplex of K for any $q_0 \in Q_0$. Let K' be the subcomplex of K consisting of all simplexes in $M_{i[2]} - \phi((M_0 \cap M_{i[2]}) \times I)$ and their faces. Define $P = \{\phi((q_0 \cap M_{i[2]}) \times I)\colon q_0 \in Q_0\} \cup \{\mathrm{St}(v, \beta^2 K')\colon v \text{ is a vertex of } \beta K'\}$.

By 1.1.8., P is a partition on $M_{i[2]}$. We construct Q as an amalgamation of P, and hence Q is a partition (by 1.1.5.). For each $p \in P$ choose $\overline{q}(p) \in \overline{Q}$ with $\alpha(p) \subseteq \overline{q}(p)$. In addition, let $\overline{q}(\phi((q_0 \cap M_{i[2]}) \times I) = T_0(q_0)$ for $q_0 \in Q_0$. Define $q(\overline{q}) = \cup\{p \in P\colon \overline{q}(p) = \overline{q}\}$, and $T(q(\overline{q})) = \overline{q}$. Then $Q = \{q(\overline{q})\colon \overline{q} \in \overline{Q}\}$ and $T\colon Q \to \overline{Q}$ have the desired properties. Observe that T preserves intersections (if $x \in q_1 \cap \ldots \cap q_t$, then $\alpha(x) \in T(q_1) \cap \ldots \cap T(q_t))$.

<u>2.4.8. Step 5.</u> Perform some "internal surgeries" to replace the triple (Q, T, α) by a triple (Q', T', α') where T' is, in addition, a one-to-one correspondence.

T may fail to be a one-to-one correspondence for two reasons: (a) T may not be a surjection, (b) even if T is a surjection, T^{-1} may not preserve intersections. Using the following "move", we gradually increase $\operatorname{Im} T$, until eventually T becomes a surjection. If $k = 0$, T is already a surjection (since $\operatorname{Im} \alpha$ hits each component of M', and $\overline{Q}$ is a collection of pairwise disjoint manifolds). So assume $k \geq 1$. In particular, all elements of $\overline{Q}$ are connected. Let $\overline{q}, \overline{q}' \in \overline{Q}$ with $\overline{q} \notin \operatorname{Im} T$, $\overline{q}' \in \operatorname{Im} T$, $\overline{q} \cap q' \neq \phi$. Say $T(q') = \overline{q}'$ and specify $y \in \overline{q} \cap \overline{q}'$. Choose an m-ball $B \subset \operatorname{Int} q'$ and set $Q'' = \{q - \operatorname{Int} B\colon q \in Q\} \cup \{B\}$, and define $T''\colon Q'' \to \overline{Q}$ by requiring that $T''(q - \operatorname{Int} B) = T(q)$, and $T''(B) = \overline{q}$. Then T'' is an intersection preserving injection. Also, define $\alpha''\colon M_{i[2]} \to M'$ as follows.

$\alpha''|M_{i[2]} - \operatorname{Int} q' = \alpha|M_{i[2]} - \operatorname{Int} q'$, $\alpha''(B) = \{y\}$, and extend α'' onto $q' - \operatorname{Int} B$ so that $\alpha''(q' - \operatorname{Int} B) \subseteq \bar{q}'$ (the latter is possible by the Homotopy Extension Theorem [Hu]). Observe that $\alpha'' \simeq \alpha$. Repeated applications of this move produce (after omitting the primes) a triple (Q, T, α) such that T is a bijection.

For the reader's convenience, we list some important properties that (Q, T, α) satisfies at this point.

(A) $T: Q \to \bar{Q}$ is an intersection preserving bijection, and α is an associated map.

(B) $\alpha_{\#}: \pi_i(M_{i[2]}) \to \pi_i(M')$ is an isomorphism for $i \leq k - 1$, and an epimorphism for $i = k$.

(C) $Q|M_0 \cap M_{i[2]} = Q_0|M_0 \cap M_{i[2]}$, and for any $q \in Q$ with $q \cap M_0 \cap M_{i[2]} \neq \phi$ $T(q) = T_0(q_0)$, where $q_0 \in Q_0$ is the unique element with $q \cap M_0 \cap M_{i[2]} = q_0 \cap M_{i[2]}$.

Now, we want to turn T into a one-to-one correspondence. If $k = 0$, there is nothing to do; so assume $k \geq 1$. Suppose that $q_1, \ldots, q_t \in Q$ are mutually distinct elements with $q_1 \cap \ldots \cap q_t = \phi$ but $T(q_1) \cap \ldots \cap T(q_t) \neq \phi$. Let K be a triangulation of $M_{i[2]}$ having small mesh and refining $L_{i[2]}$, and let $\sigma = \langle v_1, \ldots, v_t \rangle$ be a (t-1)-simplex of K contained together with $\operatorname{St}(\sigma, K)$ in $\operatorname{Int} q_1$. We let $A_\ell = \operatorname{St}(v_\ell, \beta K)$, $\ell = 1, \ldots, t$. Then $A_1 \cap \ldots \cap A_t \neq \phi$, and if we set $q'_\ell = q_\ell \cup A_\ell$, $\ell = 2, \ldots, t$; $q'_1 = q_1 - \operatorname{Int}(A_2 \cup \ldots \cup A_t)$, $Q' = (Q - \{q_1, \ldots, q_t\}) \cup \{q'_1, \ldots, q'_t\}$, $T': Q' \to \bar{Q}$, $T'(q) = T(q)$ for $q \in Q$, $T'(q'_\ell) = T(q_\ell)$, $\ell = 1, \ldots, t$ then Q' is a partition, and T' is an intersection preserving bijection. If $y \in T(q_1) \cap \ldots \cap T(q_t)$, then the Homotopy Extension

Theorem [Hu] asserts that α is homotopic to a map $\alpha': M_{i[2]} \to M'$ with $\alpha' = \alpha$ on $M_{i[2]} - \operatorname{Int} q_1$, $\alpha'(A_1 \cup \ldots \cup A_t) = \{y\}$, $\alpha'(q_1) \subseteq T(q_1)$. A repeated application of this move produces (after omitting the primes) a triple (Q, T, α) satisfying (A)-(C) and, in addition,

(D) T is a one-to-one correspondence.

2.4.9. Remark. Instead of repeating this argument, we can do all reparations simultaneously, requiring that the collection $\{St(\sigma, K)\}$ of stars of simplexes used in the construction consists of mutually disjoint sets. Therefore, we can show that a triple (Q, T, α) satisfying (A)-(C) can be replaced by another triple satisfying (D) as well in such a way that the two associated maps are $\overline{Q}$-close.

2.5. Connecting Intersections

The second part of the strategy for proving axiom 2 consists in "connecting up" elements and intersections of elements of a partition Q. In this section we show how to increase connectivity of intersections, and in 2.7. we treat the hard part: increasing the connectivity of elements of Q. In practice, we intermingle the two constructions: first make all elements 0-connected, then all intersections of two elements 0-connected,..., then all intersections of k elements 0-connected, then all elements 1-connected,..., then all intersections of $(k-1)$-elements 1-connected, then all elements 2-connected,..., and finally all elements $(k-1)$-connected.

Let (Q, T, α) be a triple satisfying (A)-(D). Consider the following property $(-1 \leq \ell \leq k - 1,\ 1 \leq r \leq k - \ell)$

<u>$P(\ell, r)$:</u>

(i) For any $t \leq r$ and any $q_1, \ldots, q_t \in Q$ (mutually distinct), $q_1 \cap \ldots \cap q_t$ is either empty or ℓ-connected.

(ii) For any $t > r$ and any $q_1, \ldots, q_t \in Q$ (mutually distinct), $q_1 \cap \ldots \cap q_t$ is either empty or $\min(\ell - 1, k - t)$-connected.

Clearly, if (Q, T, α) satisfies $P(k - 1, 1)$ then Q is a k-partition (by 1.1.18.).

In this section we show how to replace (Q, T, α) that satisfies (A)-(D) and $P(\ell, r)$, $1 \leq r < k - \ell$, by a triple that satisfies (A)-(D) and $P(\ell, r + 1)$. Let $q_1, \ldots, q_{r+1} \in Q$ be mutually distinct elements with non-empty intersection. By $P(\ell, r)$, $q_1 \cap \ldots \cap q_{r+1}$ is $(\ell-1)$-connected, and we want to make this intersection ℓ-connected. Since $\pi_\ell(q_1 \cap \ldots \cap q_{r+1})$ is a finitely generated group, we can choose generators and maps $\gamma_1, \ldots, \gamma_p\colon S^\ell \to q_1 \cap \ldots \cap q_{r+1}$ so that every generator is freely homotopic to one of the γ_i's. Since $\dim(q_1 \cap \ldots \cap q_{r+1}) = m - r \geq 2\ell + 1$, a general position argument asserts that we may assume that $\gamma_1, \ldots, \gamma_p$ are PL-embeddings, that $\cup\, \gamma_i(S^\ell) \subseteq \mathrm{Int}(q_1 \cap \ldots \cap q_{r+1})$, and $\gamma_i(S^\ell) \cap \gamma_j(S^\ell) = \phi$ for $i \neq j$. Then $\cup\, \gamma_i(S^\ell) \subseteq \partial(q_1 \cap \ldots \cap q_r)$. By assumption, $q_1 \cap \ldots \cap q_r$ is ℓ-connected, and hence there exist maps $\tilde{\gamma}_1\colon B^{\ell+1} \to q_1 \cap \ldots \cap q_r$, $i = 1, \ldots, p$ so that $\tilde{\gamma}_i | \partial B^{\ell+1} = \gamma_i$. Moreover, since $2(\ell + 1) < m - r + 1 = \dim(q_1 \cap \ldots \cap q_r)$, we may assume that $\tilde{\gamma}_i$'s are PL-embeddings with mutually disjoint images, and that $\tilde{\gamma}_i(B^{\ell+1}) \cap \partial(q_1 \cap \ldots \cap q_r) = \gamma_i(S^\ell)$, $i = 1, \ldots, p$.

Let K be a triangulation of $M_{i[2]}$ having a small mesh and refining L_j so that each $\tilde{\gamma}_i(B^{\ell+1})$ underlies a subcomplex. Let O_i be the second derived neighborhood of $\tilde{\gamma}_i(B^{\ell+1})$ in $M_{i[2]}$ with respect to K. Define

$$q'_{r+1} = q_{r+1} \cup \left(\bigcup_{i=1}^{p} O_i \right)$$

$$q'_{\delta} = q_{\delta} - \mathrm{Int}\left(\bigcup_{i=1}^{p} O_i \right), \ \delta = 1, \ldots, r .$$

By 1.1.12. and 1.1.15., $Q' = (Q - \{q_1, \ldots, q_{r+1}\}) \cup \{q'_1, \ldots, q'_{r+1}\}$ is a partition on $M_{i[2]}$. If $i(1) < \ldots < i(a) \leq r$, then

$q_{i(1)} \cap \ldots \cap q_{i(a)} = (q_{i(1)} \cap \ldots \cap q_{i(a)}) - \mathrm{Int}\left(\bigcup_{\beta=1}^{a} O_{i(\beta)} \right) =$

$(q_{i(1)} \cap \ldots \cap q_{i(a)}) - \mathrm{Int}(\text{reg. neighborhood of } \bigcup_{\beta=1}^{a} \tilde{\gamma}_{i(\beta)} (B^{\ell+1}))$.

The latter set is $(\ell-1)$-connected by a general position argument $(\ell + (\ell + 1) < m - a + 1 = \dim(q_{i(1)} \cap \ldots \cap q_{i(a)}))$. If $q(1), \ldots, q(v) \in Q - \{q_1, \ldots, q_{r+1}\}$, then

$q(1) \cap \ldots \cap q(v) \cap q'_{i(1)} \cap \ldots \cap q'_{i(a)} =$

$= q(1) \cap \ldots \cap q(v) \cap q_{i(1)} \cap \ldots \cap q_{i(a)}$, and

$q(1) \cap \ldots \cap q(v) \cap q'_{i(1)} \cap \ldots \cap q'_{i(a)} \cap q'_{r+1} =$

$= q(1) \cap \ldots \cap q(v) \cap q_{i(1)} \cap \ldots \cap q_{i(a)} \cap q_{r+1}$, and hence these intersections are correctly connected. To show that $q'_{i(1)} \cap \ldots \cap q'_{i(a)} \cap q'_{r+1}$ is correctly connected, observe that this intersection can be represented as $A \cup B$, where

$A = (q_{i(1)} \cap \ldots \cap q_{i(a)} \cap q_{r+1}) - \mathrm{Int}\left(\bigcup_{i=1}^{p} O_i \right) =$

$(q_{i(1)} \cap \ldots \cap q_{i(a)} \cap q_{r+1}) - \mathrm{Int}(\text{reg. neighborhood of } (\cup \tilde{\gamma}_i(B^{\ell+1}))$

$\cap q_{i(1)} \cap \ldots \cap q_{i(a)} \cap q_{r+1})$, $B = q_{i(1)} \cap \cdots \cap q_{i(a)} \cap (\bigcup_{i=1}^{p} \partial O_i)$. By construction, $\tilde{\gamma}_i(B^{p+1})$ hits $q_{i(1)} \cap \ldots \cap q_{i(a)} \cap q_{r+1}$ in $\gamma_i(S^\ell)$, and therefore (by a general position argument) A is $(\ell-1)$-connected. B can be represented as a disjoint union $B_1 \cup \ldots \cup B_p$, where $B_i = q_{i(1)} \cap \ldots \cap q_{i(a)} \cap \partial O_i$, $i = 1,\ldots, p$. If $a < r$, then $\tilde{\gamma}_i(B^{\ell+1}) \subseteq \partial(q_{i(1)} \cap \ldots \cap q_{i(a)})$, and hence B_i is a $(m-a)$-ball. Also, $A \cap B_i = q_{i(1)} \cap \ldots \cap q_{i(a)} \cap q_{r+1} \cap \partial O_i$ which is a $(m-a-1)$-ball (a regular neighborhood of a cell in the boundary of a manifold N is a cell whose boundary hits ∂N in a cell [Hd]) . By a Van-Kampen and Mayer-Vietoris argument [Sp] $A \cup B_1 \cup \ldots \cup B_p$ is $(\ell-1)$-connected. Let $a = r$. Then $\tilde{\gamma}_i(B^{\ell+1})$ hits $\partial(q_1 \cap \ldots \cap q_r)$ regularly (i.e. in $\tilde{\gamma}_i(S^\ell)$) . Hence $B_i = q_1 \cap \ldots \cap q_r \cap \partial O_i$ is a copy of $\partial B^{m-r-\ell} \times B^{\ell+1}$, which is ℓ-connected $(m - r - \ell - 2 \geq \ell)$, and $A \cap B_i = q_1 \cap \ldots \cap q_r \cap q_{r+1} \cap \partial O_i$ is a copy of $\partial B^{m-r-\ell} \times \partial B^{\ell+1}$, which is $(\ell-1)$-connected. Again, a Van-Kampen and Mayer-Vietoris argument detects that $A \cup B$ is $(\ell-1)$-connected. To show that $A \cup B$ is ℓ-connected, let $f: (S^\ell, *) \to (A \cup B, *)$ be a map (base point in $q_1 \cap \ldots \cap q_r \cap q_{r+1}$ is chosen away from O_i's) . By general positioning $(\ell + (m - r - \ell) < m)$ we may assume that $\mathrm{Im}\, f$ misses the "cocore" $B^{m-r-\ell} \times 0$ of B_i , $i = 1,\ldots, p$. Consequently, f is homotopic to a map $f': (S^\ell, *) \to (A \cup B, *)$ with $\mathrm{Im}\, f' \subseteq A \subseteq q_1 \cap \ldots \cap q_r \cap q_{r+1}$. Since in $\pi_\ell(q_1 \cap \ldots \cap q_r \cap q_{r+1})$ we can represent $[f'] = \sum_{i=1}^{p} n_i[f_i]$ ($[f_i]$'s are chosen generators), it suffices to show that if f' is homotopic to f_i (i.e. to γ_i) in $q_1 \cap \ldots \cap q_r \cap q_{r+1}$, then f' is null-homotopic in $A \cup B$. The general position argument shows that f' is then homotopic to a a "parallel copy" of γ_i, namely to a map $\gamma_i^!: S^\ell \to A \cap B_i \cong \partial B^{m-r-\ell} \times \partial B^{\ell+1}$ of the form {constant} × Id . (Take a homotopy H in $q_1 \cap \ldots \cap q_r \cap q_{r+1}$ between f' and $\gamma_i^!$, general position it with respect to $\mathrm{Im}(\gamma_j)$ and then slide it into A.) Finally, $\gamma_i^!$ is null-homotopic in $\partial B^{m-r-\ell} \times B^{\ell+1} = B_i \subseteq A \cup B_i$. Define $T': Q' \to \overline{Q}$ $T'(q) = T(q)$ for $q \in Q - \{q_1, \ldots, q_{r+1}\}$, $T'(q_i^!) = T(q_i)$. It remains to construct an associated map $\alpha': M_{i[2]} \to M'$.

We construct a homotopy $H: M_{i[2]} \times I \to M'$ such that

(i) $H(x, 0) = \alpha(x)$, for all $x \in M_{i[2]}$;

(ii) $H(x, t) = \alpha(x)$, for all $x \in M_{i[2]} - (q_1 \cup \ldots \cup q_{r+1})$;

(iii) $H(q_j \times I) \subseteq T(q_j)$, $j = 1, \ldots, r+1$;

(iv) $H(O_i \times \{1\}) \subseteq T(q_1) \cap \ldots \cap T(q_r) \cap T(q_{r+1})$.

Then, if we set $\alpha'(x) = H(x, 1)$, the triple (Q', T', α') clearly satisfies (A)-(D) . Repeating this argument, we produce a triple that satisfies (A)-(D) and $P(\ell, r+1)$.

The homotopy H itself can be constructed as follows. First, since O_i is the second derived neighborhood of $\tilde{\gamma}_i(B^{\ell+1})$ with respect to K , there exists a strong deformation retraction $G_i: O_i \times I \to O_i$ with $G_i(x, 0) = x$, $r_i = G_i(-, 1): O_i \to \tilde{\gamma}_i(B^{\ell+1})$ a retraction, and $G_i((\sigma \cap O_i) \times I) \subseteq \sigma \cap O_i$ for any simplex $\sigma \in \beta K$ [Hd] . Next, $\alpha \cdot \tilde{\gamma}_1: (B^{\ell+1}, \partial B^{\ell+1}) \to$ $(T(q_1) \cap \ldots \cap T(q_r), T(q_1) \cap \ldots \cap T(q_r) \cap T(q_{r+1}))$ represents a trivial element in $\pi_{\ell+1}(T(q_1) \cap \ldots \cap T(q_r), T(q_1) \cap \ldots \cap T(q_r) \cap T(q_{r+1}))$ (the latter pair is $(\ell+1)$-connected, since $T(q_1) \cap \ldots \cap T(q_r)$ is $k - r(\geq \ell + 1)$ connected, and $T(q_1) \cap \ldots \cap T(q_{r+1})$ is ℓ-connected). Hence, there is a homotopy $F_i: (B^{\ell+1}, \partial B^{\ell+1}) \times I \to (T(q_1) \cap \ldots \cap T(q_r),$

$T(q_1) \cap \ldots \cap T(q_{r+1}))$ with $F_i(x, 0) = \alpha \cdot \tilde{\gamma}_i(x)$, $F_i(x, 1) \in T(q_1) \cap \ldots \cap T(q_{r+1})$, for all x . Define $H|O_i \times I: O_i \times I \to T(q_1) \cap \ldots \cap T(q_r)$ by setting

$$H(x, \tau) = \begin{cases} \alpha \cdot G_i(x, 2\tau) & , 0 \leq \tau \leq \frac{1}{2} \\ F_i(\tilde{\gamma}_i^{-1} \cdot r_i(x), 2\tau - 1) & , \frac{1}{2} \leq \tau \leq 1 \end{cases}$$

$(i = 1, \ldots, p)$.

We need to extend H over $M_{i[2]} \times I$ preserving (iii). To that end, we invoke the Homotopy Extension Theorem, and define H by "backward induction", each time extending H on the set of the form $(q^1 \cap \ldots \cap q^t) \times I$, under the assumption that M is already defined on $(q^1 \cap \ldots \cap q^t) \times \{0\} \cup C(q^1, \ldots, q^t) \times I$, where $C(q^1, \ldots, q^t) \subseteq q' \cap \ldots \cap q^t$ is the "frontier" $\cup\{q^1 \cap \ldots \cap q^t \cap q\colon q \in Q - \{q^1, \ldots, q^t\}\}$.

<u>2.5.1. Remark.</u> Property (iii) also shows that α' is $\overline{Q}$-close to α . Fixing all intersections of $r + 1$ elements simultaneously, rather than one at a time, we can arrange that the triple (Q, T', α') satisfying $P(\ell, r + 1)$ as promissed, has the property that α' and α are $\overline{Q}$-close (even $\overline{Q}$-homotopic).

2.6. Correct Ordering

Before giving a strategy for "connecting up" elements of Q, we state and prove a lemma we need in the sequel.

2.6.1. Definition. Let M^m be a compact PL-manifold, $M_0^{m-1} \subseteq \partial M$ a submanifold, and $\ell \geq 0$. We say that (M, M_0) allows correct ℓ-ordering if the following is true. Suppose M is a submanifold of a manifold N^m with $\mathrm{Fr}M = M_0$ (frontier with respect to N), and assume that $\overline{N - M}$ is a manifold and that P is a partition on $\overline{N - M}$. Then for every $\varepsilon > 0$ there exists an ordered collection $A = \{A_1, A_2, \ldots, A_r\}$ of submanifolds of M so that

(a) $P \cup A$ is a partition on N,

(b) A_i is ℓ-connected, $i = 1, \ldots, r$,

(c) $(\overline{N - M} \cup A_1 \cup \ldots \cup A_{i-1}) \cap A_i$ is a non-empty $(\ell-1)$-connected set, and

(d) $\operatorname{diam} A_i < \varepsilon$, $i = 1, \ldots, r$.

Clearly, if (M, M_0) allows correct ℓ-ordering, then (M, M_0) is ℓ-connected. The purpose of this section is to show that certain "nice" manifold pairs allow correct ℓ-ordering.

2.6.2. Lemma. $(B^n \times I^r, \partial B^n \times I^r)$ allows correct $(n-1)$-ordering $(n \geq 1, r \geq 0)$.

Proof. Let $B^n \times I^r \subset N^{n+r}$, $(B^n \times I^r) \cap \overline{(N - (B^n \times I^r))} = \partial B^n \times I^r$, let P be a partition on $\overline{N - (B^n \times I^r)}$, and let $\varepsilon > 0$. Let $B_0^n \subseteq \operatorname{Int} B^n$ be a PL n-ball of small size, and choose a triangulation L of I^r of small mesh. Then $H = \{St(r, \beta^2 L): v \in \beta L\}$ is a handlebody decomposition of I^r. We can choose B_0^n and L small enough so that $\operatorname{diam}(B_0^n \times h) < \varepsilon$, for all $h \in H$. Let K be a triangulation of N so that: (i) all elements of P and all sets of the form $B_0^n \times h$, $B^n \times h$ and $(\partial B_0^n) \times h$ underlie subcomplexes of K $(h \in H)$ and (ii) $\operatorname{mesh}(K|B^n \times I^n) < \frac{\varepsilon}{3}$. Since $\overline{(B^n - B_0^n)} \times I^r \searrow (\partial B^n) \times I^r$, there exists a subdivision K' of $K|\overline{(B^n - B_0^n)} \times I^r$ so that K' collapses to $K'|\partial B^n \times I^r$ [Hd]. Following the collapse we can order simplexes of K' as $\sigma_1, \sigma_2, \ldots, \sigma_s, \sigma_{s+1}, \sigma_{s+2}, \ldots, \sigma_{s+2t}$ where $\sigma_1, \ldots, \sigma_s \subseteq \partial B^n \times I^r$, $\sigma_{s+1}, \ldots, \sigma_{s+2t} \not\subseteq \partial B^n \times I^r$, $\sigma_1 \cup \sigma_2 \cup \ldots \cup \sigma_{s+2i-1} \cup \sigma_{s+2i} \searrow \sigma_1 \cup \ldots \cup \sigma_{s+2i-2}$ and σ_{s+2i} is the free face of σ_{s+2i-1}. Let $B_i = St(v_i\ \beta^2 K')$, where v_i is the barycenter of σ_i. Then $\mathcal{B} = \{B_i: i = 1, \ldots, s + 2t\}$ is the handlebody decomposition of $\overline{(B^n - B_0^n)} \times I^r$ associated with K'.

<u>Claim.</u> The sequence $B_1, B_2, \ldots, B_{s+2t}, B_0^n \times h_1, B_0^n \times h_2, \ldots, B_0^n \times h_q$ satisfies (a)-(d). (Here, $h_1, h_2, \ldots, h_q$ is an arbitrary ordering of H.)

Properties (b) and (d) are clear: all elements in the sequence are balls of size $< \varepsilon$. Part (a) follows from 1.1.8.. To prove (c), we consider 4 cases.

<u>Case 1.</u> We show that $B_i \cap ((\partial B^n \times I^r) \cup B_1 \cup \ldots \cup B_{i-1})$ is contractible, if $i \leq s$. Observe that the collection $\mathcal{Q}$ consisting of the non-empty sets in $\{B_i \cap (\partial B^i \times I^r)\} \cup \{B_i \cap B_j: j < i\}$ is a partition on $B_i \cap ((\partial B^n \times I^r) \cup B_1 \cup \ldots \cup B_{i-1})$ (compare

1.1.5., 1.1.7., and 1.1.6.). Furthermore, all elements and all non-empty intersections of elements of Q are cells ("the dual cells", [Hd]). Also, notice that $B_{j(1)} \cap \ldots \cap B_{j(v)} \cap B_i \neq \phi \Rightarrow B_{j(1)} \cap \ldots \cap B_{j(v)} \cap B_i \cap (\partial B^n \times I^r) \neq \phi$. Consequently, the nerve $|N(Q)|$ has a cone structure with $B_i \cap (\partial B^n \times I^r)$ playing the role of the cone point. Therefore, by 1.1.17. (applied to open regular neighborhoods of elements of Q and open regular neighborhoods of elements of the cover $\{|St(v, \beta N(Q))|: v$ is a vertex in $N(Q)\}$ of $|N(Q)|$) , we conclude that $\cup\{q \in Q\} = B_i \cap ((\partial B^n \times I^r) \cup B_1 \cup \ldots \cup B_{i-1})$ is ℓ-connected, for all ℓ (i.e. contractible).

<u>Case 2.</u> We show that $B_{s+2i} \cap ((\partial B^n \times I^r) \cup B_1 \cup \ldots \cup B_{s+2i-1})$ is contractible, $i \leq t$. Again the collection Q consisting of non-empty sets in $\{B_{s+2i} \cap B_j : j < s + 2i\}$ is a partition on $B_{s+2i} \cap ((\partial B^n \times I^r) \cup B_1 \cup \ldots \cup B_{s+2i-1})$, all elements of Q and all non-empty intersections of elements of Q are cells. It is easy to see that $N(Q)$ has a cone structure, with $B_{s+2i} \cap B_{s+2i-1}$ playing the role of the cone point, and hence, as in Case 1, $\cup Q$ is contractible.

<u>Case 3.</u> We show that $B_{s+2i-1} \cap ((\partial B^n \times I^r) \cup B_1 \cup \ldots \cup B_{s+2i-2})$ is contractible, $i \leq t$. Consider the collection $Q = \{B_{s+2i-1} \cap B_j : j < s + 2i - 1\}$. Q is a partition on $B_{s+2i-1} \cap ((\partial B^n \times I^r) \cup B_1 \cup \ldots \cup B_{s+2i-2})$, and all elements and intersections are cells. Here, $N(Q)$ is isomorphic to $\beta K' | \overline{\partial\sigma_{s+2i-1} - \sigma_{s+2i}}$, and hence contractible.

Case 4. We show that $(B_0^n \times h_i) \cap \overline{((B^n - B_0^n)} \times I^r \cup B_0^n$ $(h_1 \cup \ldots \cup h_{i-1}))$ is (n-2)-connected, $i = 1,\ldots, q$. First observe that this intersection is equal to $(\partial B_0^n \times h_i) \cup B_0^n \times$ $((h_1 \cap h_i) \cup (h_2 \cap h_i) \cup \ldots \cup (h_{i-1} \cap h_i))$. If $i = 1$, then $\partial B_0^n \times h_1 \approx S^{n-1} \times I^r$ is (n-2)-connected. For $i > 1$, the proof follows from the next lemma.

2.6.3. Lemma. For $1 \leq i(1) < i(2) < \ldots < i(\alpha) \leq q$ and $D \subseteq \{1,\ldots, q\} - \{i(1),\ldots, i(\alpha)\}$ the set $(\partial B_0^n \times (h_{i(1)} \cap \ldots \cap h_{i(\alpha)})) \cup (B_0^n \times (\bigcup_{\delta\varepsilon D} (H_{i(1)} \cap \ldots \cap h_{i(\alpha)} \cap h_\delta)))$ is (n-2)-connected.

Proof. We proceed by induction on card D . If $D = \phi$, it is clear. Let card $D = d > 0$, and fix $\delta' \varepsilon D$. By inductive hypothesis, the following two sets are (n-2)-connected:

$X_1 = (\partial B_0^n \times (h_{i(1)} \cap \ldots \cap h_{i(\alpha)}))$
$\cup (B_0^n \times (\bigcup_{\delta\varepsilon D-\{\delta'\}} (h_{i(1)} \cap \ldots \cap h_{i(\alpha)} \cap h_\delta)))$,
$X_0 = (\partial B_0^n \times (h_{i(1)} \cap \ldots \cap h_{i(\alpha)} \cap h_{\delta'}))$
$\cup (B_0^n \times (\bigcup_{\delta\varepsilon D-\{\delta'\}} (h_{i(1)} \cap \ldots \cap h_{i(\alpha)} \cap h_{\delta'} \cap h_\delta))$. Set $X_2 =$ $B_0^n \times (h_{i(1)} \cap \ldots \cap h_{i(\alpha)} \cap h_{\delta'})$. Then X_2 is contractible, $X_1 \cap X_2 = X_0$, and $X_0 = \phi$ only if $X_2 = \phi$. By Van Kampen and Mayer-Vietoris $X = X_1 \cup X_2$ is (n-2)-connected. It remains to observe that $X = (\partial B_0^n \times (h_{i(1)} \cap \ldots \cap h_{i(\alpha)}))$
$\cup (B_0^n \times (\bigcup_{\delta\varepsilon D} (h_{i(1)} \cap \ldots \cap h_{i(\alpha)} \cap h_\delta)))$.

2.6.4. Lemma. $(B^n \times S^q \times I^r , \partial B^n \times S^q \times I^r)$ allows correct (n-1)-ordering $(n \geq 1,\ r,q \geq 0)$.

Proof. By induction on q. If $q = 0$, apply 2.6.2. twice (to each component of $B^n \times S^0 \times I^r$). So assume $q > 0$, $B^n \times S^q \times I^r \subseteq N^{n+q+r}$, P is a partition on $\overline{N - (B^n \times S^q \times I^r)}$, $\overline{N - (B^n \times S^q \times I^r)} \cap B^n \times S^q \times I^r = \partial B^n \times S^q \times I^r$, and $\varepsilon > 0$. Represent S^q as $B^q_+ \cup S^{q-1} \times I \cup B^q_-$, where $S^{q-1} \times I$ is a bicollar on the equatorial sphere S^{q-1}, and B^q_+, B^q_- are closures of complementary domains of the bicollar. Applying the inductive hypothesis on $(B^n \times (S^{q-1} \times I) \times I^r, \partial B^n \times (S^{q-1} \times I) \times I^r)$, where $B^n \times (S^{q-1} \times I) \times I^r$ is regarded as a subset of $N' = \overline{N - (B^n \times B^q_+ \times I^r \cup B^n \times B^q \times I^r)}$ we get a collection $A = \{A_1, \ldots, A_u\}$ of subsets of $B^n \times S^{q-1} \times I \times I^r$ so that

(i) each A_i is $(n-1)$-connected and ε-small,

(ii) $A_i \cap ((\partial B^n \times S^{q-1} \times I \times I^r) \cup A_1 \cup \ldots \cup A_{i-1})$ is a non-empty $(n-2)$-connected set,

(iii) $P \cup A$ is a partition on N'.

Two applications of 2.6.2. give collections $B = \{B_1, \ldots, B_v\}$, $C = \{C_1, \ldots, C_w\}$ such that

(iv) each B_i, C_i is $(n-1)$-connected and ε-small,

(v) $B_i \cap ((\partial B^n \times B^q_+ \times I^r) \cup B_1 \cup \ldots \cup B_{r-1})$ and $C_i \cap ((\partial B^n \times B^q_- \times I^r) \cup C_1 \cup \ldots \cup C_{i-1})$ are non-empty $(n-2)$-connected sets,

(vi) $P \cup A \cup B \cup C$ is a partition on N.

Finally, $A_1, A_2, \ldots, A_u, B_1, \ldots, B_v, C_1, \ldots, C_w$ is the desired collection.

2.7. Increasing the Connectivity of Partition Elements

In this section we start with a triple (Q, T, α) satisfying (A)-(D) (see 2.4.) and $P(\ell - 1, k - \ell + 1)$, $0 \leq \ell \leq k - 1$ (see 2.5.). We transform this triple (performing "internal surgeries") to a triple (Q', T', α') that satisfies (A)-(D) and $P(\ell, 1)$. In the process of doing that, we will have to go deeper in the defining sequence. The goal is to make elements of Q ℓ-connected, without losing connectivity properties of intersections; specifically the intersection of t (≥ 2) elements is $\min(\ell - 1, k - t)$-connected.

<u>2.7.1. Step 1.</u> For each $q_\delta \in Q$ choose $\gamma_1^\delta, \ldots, \gamma_{p(\delta)}^\delta : S^\ell \to q_\delta$ so that for some (any) basepoint $* \in q_\delta$ all elements of some set of generators of $\pi_\ell(q_\delta)$ are freely homotopic to at least one of the chosen maps. Also, we may assume that γ_i^δ's are PL-embeddings with mutually disjoint images which are contained in $\operatorname{Int} q_\delta$.

<u>2.7.2. Step 2.</u> Choose extensions $\tilde{\gamma}_1^\delta, \ldots, \tilde{\gamma}_{p(q)}^\delta : B^{\ell+1} \to M_{i[2]}$. (If $M_{i[2]}$ is (k-1)-connected, as in axiom 2, we need not struggle. In general, we know that α induces a monomorphism on π_ℓ , and $\alpha \cdot \gamma_i^\delta$ is null-homotopic in $T(q_\delta) \subseteq M'$.) A general position argument shows that we may assume that all chosen maps are embeddings into $\operatorname{Int} M_{i[2]}$ with mutually disjoint images.

<u>2.7.3. Step 3.</u> Here we show how to choose extensions $\tilde{\gamma}_i^\delta$, $i = 1, \ldots, p(\delta)$, $q_\delta \in Q$ so that the map $\alpha \cdot \tilde{\gamma}_i^\delta : B^{\ell+1} \to M'$ is homotopic (rel $\partial B^{\ell+1}$) to a map whose image is contained in $T(q_\delta)$. For notational simplicity, we omit indexes, starting with $\gamma: S^\ell \to q$

and an extension $\tilde{\gamma}: B^{\ell+1} \to M_{i[2]}$. Let $G: B^{\ell+1} \to T(q)$ be a map with $G|\partial B^{\ell+1} = \alpha\gamma$. We choose $\tilde{\gamma}(0)$ and $\alpha\tilde{\gamma}(0)$ to be basepoints (inducing an epimorphism on homotopy groups is a property that does not depend on the choice of basepoints). Define $P: (B^{\ell+1}, \partial B^{\ell+1}) \to (M', \alpha\tilde{\gamma}(0))$ by (we view $B^{\ell+1}$ as $I \times S^{\ell}/\{0\} \times S^{\ell}$)

$$P(r, \phi) = \begin{cases} G(2r, \phi) & , 0 \le r \le \frac{1}{2} \\ \alpha\,\tilde{\gamma}(2 - 2r, \phi) & , \frac{1}{2} \le r \le 1 . \end{cases}$$

Then P represents an element of $\pi_{\ell+1}(M', \alpha\tilde{\gamma}(0))$. Since $\alpha: (M_{i[2]}, \tilde{\gamma}(0)) \to (M', \alpha\tilde{\gamma}(0))$ induces an epimorphism on $\pi_{\ell+1}$, there is a map $S: (B^{\ell+1}, \partial B^{\ell+1}) \to (M_{i[2]}, \tilde{\gamma}(0))$ such that $\alpha S \simeq P$ (rel $B^{\ell+1}$) . Define $H: B^{\ell+1} \to M_{i[2]}$ as

$$H(r, \phi) = \begin{cases} S(2r, \phi) & , 0 \le r \le \frac{1}{2} \\ \tilde{\gamma}(2r - 1, \phi) & , \frac{1}{2} \le r \le 1 . \end{cases}$$

A straightforward calculation shows that $H|\partial B^{\ell+1} = \gamma$ and $\alpha H \simeq G: B^{\ell+1} \to M'$ (rel $\partial B^{\ell+1}$). In the sequel we assume that this is done with all $\tilde{\gamma}_i^{\delta}$'s and that, in addition to what was said about $\tilde{\gamma}_i^{\delta}$'s in step 2, $\alpha\,\tilde{\gamma}_i^{\delta}$ is homotopic (rel$\partial B^{\ell+1}$) to a map $\chi_i^{\delta}: B^{\ell+1} \to T(q_{\delta})$.

<u>2.7.4. Step 4.</u> The idea consists of adding a regular neighborhood of $\tilde{\gamma}_i^{\delta}(B^{\ell+1})$ to q_{δ} (and subtracting its interior from other

elements of Q) . Unfortunately, if we do that we lose an important property: T will not be an intersection preserving function ($\tilde{\gamma}_i^\delta(B^{\ell+1})$ may intersect elements of Q that q_δ does not hit). Thus, the following "move" is forced upon us. We subtract from other elements of Q a slightly bigger regular neighborhood, and we take care of the "annulus" region between the two later.

Choose the following.

1. A small closed regular neighborhood N_i^δ of $\tilde{\gamma}_i^\delta(B^{\ell+1})$ so that the chosen neighborhoods are mutually disjoint (for different choices of i, δ). Also, in order to make N_i^δ in "general position" with elements of Q (to be able to assert that the collection defined below is a partition that satisfies $P(\ell - 1, k - \ell + 1)$) , we require that N_i^δ be obtained as the second derived neighborhood with respect to a triangulation of $M_{i[2]}$ that subdivides everything in sight.

2. A "parallel copy" of $\gamma_i^\delta(S^\ell)$ in ∂N_i^δ , i.e. an embedding $\bar{\gamma}_i^\delta\colon S^\ell \to \partial N_i^\delta \cap q_\delta$. We also assume that $\bar{\gamma}_i^\delta$ and γ_i^δ are homotopic in $N_i^\delta \cap q_\delta$.

3. A "reimbedded copy" of $\tilde{\gamma}_i^\delta(B^{\ell+1})$, i.e. an embedding $\hat{\gamma}_i^\delta\colon B^{\ell+1} \to \operatorname{Int} N_i^\delta$ "close" to $\tilde{\gamma}_i^\delta$ so that $\hat{\gamma}_i^\delta|S^\ell$ and $\tilde{\gamma}_i^\delta|S^\ell$ are homotopic in $q_\delta \cap N_i^\delta$. (Most of the time we can choose $\hat{\gamma}_i^\delta = \tilde{\gamma}_i^\delta$, only in the proof of the σ-Z-set shrinking theorem will we have to choose a different embedding.)

4. An annulus $A_i^\delta \approx S^\ell \times I$ whose boundary components are $\bar{\gamma}_i^\delta(S^\ell)$ and $\hat{\gamma}_i^\delta(S^\ell)$ with $A_i^\delta \subseteq q_\delta \cap N_i^\delta$, $A_i^\delta \cap \partial N_i = \bar{\gamma}_i^\delta(S^\ell)$ and $A_i^\delta \cap \hat{\gamma}_i^\delta(B^{\ell+1}) = \hat{\gamma}_i^\delta(S^\ell)$ (e.g. embed a homotopy between $\bar{\gamma}_i^\delta$ and $\hat{\gamma}_i^\delta|S^\ell$).

5. A second derived neighborhood $\hat{N}_i^\delta$ of $\hat{\gamma}_i^\delta(B^{\ell+1}) \cup A_i^\delta$ in N_i^δ with respect to a triangulation that subdivides everything. In particular, $\hat{N}_i^\delta$ looks like a $(\ell+1)$-handle attached to ∂N_i^δ, and $F_i^\delta = \overline{N_i^\delta - \hat{N}_i} \cong B^{\ell+1} \times S^{m-\ell-2} \times I$. We represent $\hat{N}_i^\delta$ as the union $\hat{N}_i^\delta(A) \cup \hat{N}_i^\delta(C)$ of the second derived neighborhood $\hat{N}_i^\delta(A)$ of A_i^δ and the second derived neighborhood $\hat{N}_i^\delta(C)$ of the "core" $\hat{\gamma}_i^\delta(B^{\ell+1})$.

6. "Levels of a product structure", i.e. maps $\lambda_i^\delta \colon M_{i[2]} \to I$ such that $\lambda_i^\delta(x) = 0$ for $x \notin N_i^\delta$, $\lambda_i^\delta(x) = 1$ for $x \in \hat{N}_i^\delta(C)$.

7. A map $r\colon M_{i[2]} \to M_{i[2]}$ homotopic to Id such that $r(x) = x$ if $\lambda_i^\delta(x) = 0$ for all choices (i, δ); $r(x) \in \hat{\gamma}_i^\delta(B^{\ell+1})$ if $\lambda_i^\delta(x) \geq \frac{1}{2}$; $r(\hat{N}_i^\delta(A)) \subseteq q_\delta$; and $r(x) \in \hat{\gamma}_i^\delta(S^\ell)$ if $x \in \hat{N}_i^\delta(A)$ with $\lambda_i^\delta(x) \geq \frac{1}{2}$.

Let $\overline{M} = M_{i[2]} - \mathrm{Int}(\bigcup_{i,\delta} N_i^\delta)$. By 1.1.12., $Q|\overline{M}$ is a partition. Observe that if an intersection of elements from Q is non-empty, then it has dimension $\geq m - k > k$. Consequently, if N_i^δ's are chosen sufficiently small, the function $I\colon Q|\overline{M} \to Q$, $I(q \cap \overline{M}) = q$ is a one-to-one correspondence. Moreover, a general position argument $(N_i^\delta \cap q_{\delta(1)} \cap \ldots \cap q_{\delta(j)}$ is a regular neighborhood of the polyhedron $\tilde{\gamma}_i^\delta(B^{\ell+1}) \cap q_{\delta(1)} \cap \ldots \cap q_{\delta(j)}$ of dimension $\leq k)$ reveals that $Q|\overline{M}$ satisfies $P(\ell - 1, k - \ell + 1)$.

Attach "handles" $\hat{N}_i^\delta$ to elements $q_\delta \cap \overline{M}$ of $Q|\overline{M}$ to obtain the partition $Q(0)$ on $\overline{M} \cup \bigcup_{i,\delta} \hat{N}_i^\delta$ (i.e. $q_\delta(0) = (q_\delta \cap \overline{M}) \cup \bigcup_i N_i^\delta)$.

It is then clear that $Q(0)$ is a partition that still satisfies

$P(\ell - 1, k - \ell + 1)$. A little computation reveals that all elements of $Q(0)$ are ℓ-connected and hence $Q(0)$ satisfies $P(\ell, 1)$. (Show that "parallel copies" $\overline{\gamma}_i^\delta$, $i = 1, \ldots, p(\delta)$, "generate" $\pi_\ell(q_\delta \cap \overline{M})$. Attaching $\hat{N}_i^\delta$'s , we "kill" all of the generators.)

The problem we are faced with is that $Q(0)$ is not a partition on $M_{i[2]}$: it does not cover sets $F_i^\delta = \overline{N_i^\delta - \hat{N}_i^\delta}$ ("fringe regions"). We define a map $\alpha(0): M_{i[2]} \to M'$ whose restriction on $M(0) = \cup\{q \in Q(0)\}$ is an associated map to the one-to-one correspondence $T(0): Q(0) \to \overline{Q}$, $T(0)(q_\delta(0)) = T(q_\delta)$. We use this map as a guide to "fill in" fringe regions with partition elements.

2.7.5. Step 5. (The definition of $\alpha(0): M_{i[2]} \to M'$.) We set

$$\alpha(0)(x) = \begin{cases} \alpha r(x) & , \text{ if } \lambda_i^\delta(x) \le \frac{1}{2} \text{ for all } i, \delta \\ \\ H_i^\delta((\hat{\gamma}_i^\delta)^{-1} r(x) , 2\lambda_i^\delta(x) - 1) & , \text{ if } \lambda_i^\delta(x) \ge \frac{1}{2} \end{cases}$$

where $H_i^\delta: B^{\ell+1} \times I \to M'$ is a homotopy with $H_i^\delta(b, 0) = \alpha\hat{\gamma}_i^\delta(b)$, $H_i^\delta(b, 1) \in T(q_\delta)$ for $b \in B^{\ell+1}$ and $H_i^\delta(b,t) = \alpha\hat{\gamma}_i^\delta(b)$ for $b \in \partial B^{\ell+1}$, $t \in I$. ($\tilde{\gamma}_i^\delta$ was chosen so that such a homotopy exists, $\hat{\gamma}_i^\delta$ is "close" to $\tilde{\gamma}_i^\delta$.). A straightforward verification shows that $\alpha(0)(q_\delta(0)) \subseteq T(0)(q_\delta(0))$. Also, if we set

$$f_t(x) = \begin{cases} \alpha r(x) , \text{ if } \lambda_i^\delta(x) \le \frac{1}{2} \text{ for all } i, \delta \\ \\ H_i^\delta((\hat{\gamma}_i^\delta)^{-1} r(x) , \min\{t, 2\lambda_i^\delta(x) - 1\}), \text{ if } \lambda_i^\delta(x) \ge \frac{1}{2} \end{cases}$$

then $\{f_t\}_{t\varepsilon I}$ is a homotopy between $f_0 = \alpha\, r$ and $f_1 = \alpha(0)$. Since $r \simeq \mathrm{Id}$, we conclude that $\alpha(0) \simeq \alpha$.

2.7.6. Step 6. As in 2.4.6. adjust $\alpha(0)$ so that there is $\varepsilon > 0$ with the property that if $A \subseteq M_{i[2]}$ is a set of diameter $< \varepsilon$, then $\alpha(0)(A)$ is contained in a single element of $\overline{Q}$. Choose an ordering $\omega(1), \omega(2),\ldots, \omega(\nu)$ of the set $\Omega = \{(i, \delta): i = 1,\ldots, p(\delta)\ , q_\delta \varepsilon Q\}$. For each $\omega \varepsilon \Omega$ let $A_\omega = \{A_\omega[1],\ldots, A_\omega[s[w]]\}$ be a family of submanifolds of $F_i^\delta = F_\omega$ as guaranteed by 2.6.4., i.e.

(a) $Q(0) \cup A_{\omega(1)} \cup \ldots \cup A_{\omega(v)}$ is a partition $(v = 1,\ldots, \nu)$,

(b) $A_\omega[j]$ is ℓ-connected and diam $A_\omega[j] < \varepsilon$ $(j = 1,\ldots, s[\omega];\ \omega \varepsilon \Omega)$,

(c) $(\overline{M} \cup A_\omega[1] \cup \ldots \cup A_\omega[j-1]) \cap A_\omega[j]$ is a non-empty $(\ell-1)$-connected set $(j = 1,\ldots, s[\omega];\ \omega \varepsilon \Omega)$

(Observe that $(F_\omega, \partial F_\omega) \cong (B^{\ell+2} \times S^{m-\ell-2}, \partial B^{\ell+2} \times S^{m-\ell-2})$ and hence allows a correct $(\ell+1)$-ordering.) An application of The Isotopy Move (2.2.) produces an isotopy $\phi_t\colon M_{i[2]} \to M_{i[2]}$ and $i[3] \geq i[2]$ so that $\phi_1(q_\delta(0)) \cap M_{i[3]}$, $\phi_1(A_W[j]) \cap M_{i[3]}$ underlie subcomplexes of $L_{i[3]}$. Also, all connectivity properties are preserved. We replace $\alpha(0)$ by $\alpha(0) \cdot \phi_1^{-1}|M_{i[3]}$. In order to simplify the notation, we proceed as if $\phi_t = \mathrm{Id}$, $0 \leq t \leq 1$.

It remains to make the $A_\omega[j]$'s "join the crowd". This is done recursively, using the preferred ordering of $A_\omega[j]$'s. We show how to do it for the first $\omega = \omega(1)$. Others are treated in exactly the same way. We inductively define a sequence $Q(0), Q(1),\ldots, Q(j),\ldots, Q(\bar{s})$

$(s = s(\omega(1))$ of partitions so that the following is true. (To simplify the notation, we let $A[j] = A_{\omega(1)}[j]$.)

$H(j)_1$: $Q(j)$ is a partition on $(\overline{M} \cup A[1] \cup \ldots \cup A[j]) \cap M_{i\{j\}}$, for some (large enough) $i\{j\}$,

$H(j)_2$: $Q(j) \cup \{A[j+1] \cap M_{i\{j\}}, \ldots, A[s] \cap M_{i\{j\}}\}$ is a partition;

$H(j)_3$: $T(j): Q(j) \to \overline{Q}$ is a one-to-one correspondence,

$H(j)_4$: $\alpha(j)_{\#}: \pi_i(M_{i\{j\}}) \to \pi_i(M')$ is an isomorphism for $i \leq k-1$, and an epimorphism for $i = k$.

$H(j)_5$: $\alpha(j) | \cup Q(j)$ is a map associated to $T(j)$,

$H(j)_6$: $Q(j)$ satisfies $P(\ell, 1)$,

$H(j)_7$: $\alpha(j) = \alpha(0)$ on the complement of $A[1] \cup \ldots \cup A[j]$.

The induction starts with the triple $(Q(0), T(0), \alpha(0))$. After s steps we produce a triple $(Q(\bar{s}), T(\bar{s}), \alpha(\bar{s}))$ which satisfies $P(\ell, 1)$ and also covers the first "fringe region" (or what is left of it after going deeper in the defining sequence). Repeating this argument as many times as there are fringe regions, we construct the sought after triple that satisfies (A)-(D) and $P(\ell, 1)$ (see 2.4. and 2.5.).

The rest of the section describes the inductive step. We assume the triple $(Q(j-1), T(j-1), \alpha(j-1))$ satisfying $H(j-1)_1$-$H(j-1)_7$ is constructed, and proceed with the construction of $(Q(j), T(j), \alpha(j))$.

By assumption, $\alpha(j-1)(A[j]) \subseteq \overline{q}$ for some $\overline{q} \varepsilon \overline{Q}$ (see $H(j-1)_7$, (b) and the choice of $\varepsilon > 0$) .

Let $S = St(T(j-1)^{-1}(\overline{q}), Q(j-1))$. Observe that the obviously defined partition on S (a subcollection of $Q(j-1)$) is by $H(j-1)_3$ in one-to-one correspondence with the suitable subcollection of $\overline{Q}$ that covers $St(\overline{q}, \overline{Q})$, which is (k-1)-connected ($\overline{Q}$ is a k-partition, see 1.1.15.). Consequently, by $H(j-1)_6$ and 1.1.18., S is ℓ-connected.

Note that $S \cap A[j] \cap M_{i\{j-1\}} = (\overline{M} \cup A[1] \cup \ldots \cup A[j-1]) \cap A[j] \cap M_{i\{j-1\}}$. (The inclusion $\subseteq$ is clear. If x belongs to the right-hand side, and $x \varepsilon q \varepsilon Q(j-1)$, then $\alpha(j-1)(x) \varepsilon T(j-1)(q) \cap \overline{q}$ (by $H(j-1)_7$) , which implies $q \cap T(j-1)^{-1} (\overline{q}) \neq \phi$, and thus $x \varepsilon q \subseteq S$.) In particular, $S \cap A[j] \cap M_{i\{j-1\}}$ is (ℓ-1)-connected (by (c), and the fact that going deeper in the defining sequence does not lead to any loss of connectivity properties). Consequently, $S \cup (A[j] \cap M_{i\{j-1\}})$ is ℓ-connected (by Van Kampen and Mayer-Vietoris).

We would like to add $A[j] \cap M_{i\{j-1\}}$ to $T(j-1)^{-1}(\overline{q})$ thus obtaining $Q(j)$. The problem is that $T(j-1)^{-1}(q) \cup (A[j] \cap M_{i\{j-1\}})$ need not be ℓ-connected. To remedy that, we add to it "a thickened copy of the dual k-skeleton" of $S \cup (A[j] \cap M_{i\{j-1\}})$. We will automatically get desired connectivity properties of the intersections. In view of further refinements, we only list properties of the "dual k-skeleton" to be used.

Let P be a polyhedron of dimension $\leq k$ in $S \cup (A[j] \cap M_{i\{j-1\}})$ with the following properties (we let p denote $T(j-1)^{-1}(q) \cup (A[j] \cap M_{i\{j-1\}})$.

(a) $P \subseteq Int(S \cup (A[j] \cap M_{i\{j-1\}}))$,

(b) $P \cup p$ is ℓ-connected,

(c) if $q_1, \ldots, q_s \in Q(j-1) - \{T(j-1)^{-1}(q)\}$ are mutually distinct elements that intersect p with $q_1 \cap \ldots \cap q_s \neq \phi$ then $P \cap q_1 \cap \ldots \cap q_s$ is non-empty and $\min(\ell - 1, k - s)$-connected. If $s = 1$, this intersection is non-empty and ℓ-connected,

(d) if $q_1, \ldots, q_s \in Q(j-1) - \{T(j-1)^{-1}(q)\}$ are mutually distinct elements such that $p \cap q_1 \cap \ldots \cap q_s \neq \phi$, then $P \cap p \cap q_1 \cap \ldots \cap q_s \neq \phi$ and the pair $(p \cap q_1 \cap \ldots \cap q_s , P \cap p \cap q_1 \cap \ldots \cap q_s)$ is $(k-s)$-connected.

Then let N be the second derived neighborhood of P with respect to a triangulation of $M_{i\{j-1\}}$ having small mesh and subdividing everything. Define $\hat{p} = p \cup N$, $\hat{q} = q - \text{Int } N$ for $q \in Q(j - 1)$, $q \neq T(j - 1)^{-1}(\overline{q})$. Let $\hat{Q} = \{\hat{q} : q \in Q(j-1) - \{T(j - 1)^{-1}(\overline{q})\}\} \cup \{\hat{p}\}$. It should be clear that $\hat{Q}$ is a partition (cf. 1.1.12.). By (b), $\hat{p}$ is ℓ-connected, and a general position argument $(1 + k + \ell < m)$ asserts that all other elements of $\hat{Q}$ are ℓ-connected. If $\hat{q}_1, \ldots, \hat{q}_s \in \hat{Q}$ are mutually distinct, and $\hat{q}_i \neq \hat{p}$, $i = 1, \ldots, s$, then $\hat{q}_1 \cap \ldots \cap \hat{q}_s = (q_1 \cap \ldots \cap q_s) - \text{Int } N$, and again a general position argument $(1 + k + (k - s) < m - s + 1)$ asserts that this intersection, if non-empty, is $\min(\ell - 1, k - s)$-connected $(s \geq 2)$. Assume that $\hat{p} \cap \hat{q}_1 \cap \ldots \cap \hat{q}_s \neq \phi$. This intersection can be represented as $A \cup B$, where $A = (p \cap q_1 \cap \ldots \cap q_s) - \text{Int } N$ and $B = q_1 \cap \ldots \cap q_s \cap \partial N$. Note that

$A \cap B = p \cap q_1 \cap \ldots \cap q_s \cap \partial N$. Assume, in addition, that $s \leq k$. By 1.1.13.(5) we have $p \cap q_1 \cap \ldots \cap q_s \neq \phi$, and thus by (d) the pair $(p \cap q_1 \cap \ldots \cap q_s, P \cap p \cap q_1 \cap \ldots \cap q_s)$ is (k-s)-connected. Consequently, $(p \cap q_1 \cap \ldots \cap q_s , p \cap q_1 \cap \ldots \cap q_s \cap N)$ is (k-s)-connected, and (by general positioning with P) so is $(p \cap q_1 \cap \ldots \cap q_s, p \cap q_1 \cap \ldots \cap q_s \cap \partial N)$. Since the pair $(p \cap q_1 \cap \ldots \cap q_s , p \cap q_1 \cap \ldots \cap q_s - \operatorname{Int} N)$ is (k-s)-connected by general position, it follows that $(A, A \cap B)$ is (k-s)-connected. By (i) and general positioning, B is $\min(\ell - 1, k - s)$-connected. Since the inclusion induced homomorphism $H_q(A \cap B) \to H_q(A)$ is an isomorphism for $q < k - s$ and an epimorphism for $q = k - s$, from the Mayer-Vietoris sequence for (A, B) we get that $H_q(A \cup B) = 0$ for $q \leq \min(\ell - 1, k - s)$. Also, $A \cup B$ is simply-connected if $\min(\ell - 1, k - s) \geq 1$. If $A_1,\ldots, A_u$ are the components of A then, using Van Kampen's theorem, we show by induction that $B \cup A_1 \cup \ldots \cup A_w$ is simply-connected, $w = 1,\ldots, n$. The inductive step consists in observing that $(A_w, A_w \cap B)$ is 1-connected.) Thus, $\hat{Q}$ has correct connectivity properties. Observe that it may happen that $\hat{p} \cap \hat{q}_1 \cap \ldots \cap \hat{q}_s \neq \phi$ even if $p \cap q_1 \cap \ldots \cap q_s = \phi$. However, this happens only if $q_1 \cap \ldots \cap q_s \neq \phi$, and thus, by (a) and 1.1.13.(5), if $s = k + 1$. By The Isotopy Move, we can find $i\{j\} \geq i\{j - 1\}$ and an isotopy (as in 2.2.) that takes all elements of $\hat{Q}$ to submanifolds of $M_{i\{j-1\}}$ which intersected with $M_{i\{j\}-1}$ underlie subcomplexes of $L_{i\{j\}-1}$. Going one stage deeper, to $M_{i\{j\}}$, all undesirable intersections, having dimension $\leq m - k - 1$, disappear. This partition is the sought after partitoin $Q(j)$. The function

$\hat{T}: \hat{Q} \to \overline{Q}$ is easily defined by $\hat{T}(\hat{p}) = \overline{q}$, $\hat{T}(\hat{q}) = T(j - 1)(q)$, and transferred by the isotopy to $T(j): Q(j) \to \overline{Q}$. A straightforward argument shows that $T(j)$ is a one-to-one correspondence, and $H(j)_1$-$H(j)_3$ and $H(j)_6$ are satisfied. It remains to define $\alpha(j)$. We repeat the argument from steps 1 and 2 in 2.4. To make sure that $H(j)_4$ is satisfied, choose a polyhedron P_0 in $M_{i\{j-1\}}$ (e.g. the k-skeleton of L) such that $\alpha(j - 1)|P_0: P_0 \to M'$ induces an isomorphism of homotopy groups of dimension $\leq k - 1$ and epimorphism in dimension k; and that $P_0 \hookrightarrow M_{i\{j-1\}}$ induces isomorphisms of homotopy groups of dimension $\leq k-1$. We require one more property for P:

(e) $P \cap P_0 = \phi$

(e.g. if we let P be a slightly shrunken copy of the dual k-skeleton of some fine triangulation of $S \cup (A[j] \cap M_{i\{j-1\}})$, it will satisfy (a)-(e)). We then choose N so that $N \cap P_0 = \phi$. We can also choose a regular neighborhood N_0 of P_0 so that $N \cap N_0 = \phi$. By The Isotopy Move 2.2., we may assume that N and N_0 underlie subcomplexes of $L_{i\{j\}}$. Then we define $\alpha(j)$ on $(M_{i\{j\}} - \mathrm{Int}(S \cup A[j])) \cup N_0$ as $\alpha(j - 1)$ and then extend to the rest of $M_{i\{j\}}$ as in 2.2.4. and 2.4.5. (going deeper in the defining sequence, if necessary). Then $H(j)_7$ is clear, and $H(j)_5$ is built into the construction of $\alpha(j)$. Also, $H(j)_4$ follows from the choice of N_0, and the fact that $N_0 \cap M_{i\{j\}} \to M_{i\{j\}}$ induces an isomorphism of homotopy groups of dimension $\leq k - 1$ (for simplicity, we assume that the isotopy in The Isotopy Move is identity).

This finishes the inductive argument.

2.8. Some Easy Consequences

In this section we present theorems whose conclusions include the stipulation that there are homeomorphisms between various triangulated μ^k-manifolds. The proofs consist of a careful analysis of the construction of k-partitions given in 2.2.-2.7.

First, let us note that we have achieved the goal set in 1.3.

<u>2.8.1. Proposition.</u> If $(M_1, L_1), (M_2, L_2), \ldots$ is a μ^k-defining sequence (as in 1.2.1.), and C_i^s the class of convenient manifolds (as in 1.2.2.), $s = 0, \ldots, k$; $i = 1, 2, \ldots,$ then $\{M_i, C_i^s\}_{i=1}^{\infty}$ is a μ^k-family (as in 1.3.1.).

<u>Proof.</u> By 2.2.2., (C1)-(C4) and Axiom 1 are satisfied. Verifying Axiom 2, we may assume that $s = 0$. First note that Q_0 is a partition on N_0 consisting of submanifolds that underly subcomplexes of L_i. In 2.4. we have shown how to "extend" Q_0 onto N, i.e. to build a partition Q on $N \cap M_j$ ($j \geq i$ large enough) and a one-to-one correspondence $T: Q \to \overline{Q}$ satisfying (a) and (b). Using the move described in 2.7., we adjust Q so that all elements of Q are connected. Then we perform a move described in 2.5. to make all non-empty intersections of $\leq k$ elements of Q connected. We proceed by applying the move from 2.7. to make elements of Q simply-connected, and performing a move from 2.5. to make all non-empty intersections of $\leq$ (k-1)-elements of Q simply-connected. This process is continued until the desired connectivity properties are obtained. Some of these steps require going deeper in the defining

sequence. Observe that we never lose properties already achieved (in particular (a) and (b) from Axiom 2). After finitely many steps, we have a convenient k-partition Q in our hands, and the proof is finished.

2.8.2. Corollary. All spaces in M_0 (see 1.2.) are pairwise homeomorphic.

Proof. This follows from 2.8.1 and 1.3.2.

From now on, we can talk about the space μ^k without ambiguity.

2.8.3. Definition. A (compact) metric space X is a (compact) μ^k-manifold if each point $x \in X$ has a neighborhood homeomorphic to an open subset of μ^k.

2.8.4. Corollary. Any triangulated μ^k-manifold (as in 1.2.1.) is a μ^k-manifold.

The converse is a consequence of the Characterization Theorem. For further reference, we list properties that μ^k-manifolds possess.

2.8.5. Proposition. Any μ^k-manifold is a k-dimensional LC^{k-1}-space satisfying DD^kP.

Proof. It is clear that any μ^k-manifold X is LC^{k-1}. Being a countable union of k-dimensional closed subsets, it is k-dimensional [H-W].

Let $f, g\colon I^k \to X$ be two maps. Find a triangulation K of I^k such that if $\sigma, \tau \in K$ then either $f(\sigma) \cap g(\tau) = \phi$ or

$f(\sigma) \cup g(\tau)$ is contained in an open set $U(\sigma, \tau)$ which is homeomorphic to an open subset of μ^k. Let $(\sigma_1, \tau_1), (\sigma_2, \tau_2), \ldots, (\sigma_n, \tau_n)$ be an ordering of all pairs (σ, τ) of k-simplexes in K, and let $\varepsilon > 0$. We define a sequence $(f_1, g_1), \ldots, (f_n, g_n)$ of pairs of maps $I^k \to X$ satisfying $(i,j = 1, \ldots, n)$

$(1)_i$ $(f_i, f) < \frac{i}{n} \varepsilon$, $(g_i, g) < \frac{i}{n} \varepsilon$,

$(2)_i$ $f_i(\sigma_j) \cap g_i(\tau_j) = \phi$ if either $j \leq 1$ or $f(\sigma_j) \cap g(\tau_j) = \phi$

$(3)_i$ $f_i(\sigma_j) \cup g_i(\tau_j) \subseteq U(\sigma_j, \tau_j)$ if $f(\sigma_j) \cap g(\tau_j) = \phi$.

Then f_n, g_n are desired approximations. Since obtaining (f_1, g_1) from (f, g) is easier than obtaining (f_i, g_i) from (f_{i-1}, g_{i-1}), we proceed with the inductive step. Let $\eta > 0$ be a small enough number such that if $(f_i, f_{i-1}) < \eta$, $(g_i, g_{i-1}) < \eta$ then $(1)_{i-1}$-$(3)_{i-1}$ hold. By 2.1.4. we can find $\delta > 0$ such that if f', g' approximate $f_{i-1}|\sigma_i$, $g_{i-1}|\tau_i$ respectively, then f', g' extend to f_i, g_i with $(f_i, f_{i-1}) < \eta$, $(g_i, g_{i-1}) < \eta$. If $f_{i-1}(\sigma_i) \cap g_{i-1}(\tau_i) = \phi$, let $f_i = f_{i-1}$, $g_i = g_{i-1}$. If $f_{i-1}(\sigma_i) \cap g_{i-1}(\tau_i) \neq \phi$, then by $(2)_{i-1}$ $f(\sigma_i)$ $g(\tau_i) \neq \phi$ and hence $f(\sigma_i) \cup g(\tau_i) \subseteq U(\sigma_i, \tau_i)$ and, by $(3)_{i-1}$, $f_{i-1}(\sigma_i) \cup g_{i-1}(\tau_i)$ _ $U(\sigma_i, \tau_i)$. Applying DD^kP on $U(\sigma_i, \tau_i)$, we approximate $f_{i-1}|\sigma_i$ and $g_{i-1}|\tau_i$ by f', g' such that $(f_{i-1}|\sigma_i, f') < \delta$, $(g_{i-1}|\tau_i, g') < \delta$ and $f'(\sigma_i) \cap g'(\tau_i) = \phi$. Then we let f_i, g_i be extensions of f', g' with $(f_i, f_{i-1}) < \eta$, $(g_i, g_{i-1}) < \eta$.

It is well-known that: 1. any map $f: M \to N$ between Q-manifolds that induces isomorphisms on homotopy groups is a homotopy equivalence [Wh], and 2. any simple homotopy equivalence $f: M \to N$ between

Q-manifolds is homotopic to a homeomorphism [Ch] . In the theory of μ^k-manifolds there are no simple-homotopy obstructions, and we can prove the counterparts of both statements at once.

2.8.6. Theorem. Let $f\colon M_1 \to M_2$ be a map between triangulated μ^k-manifolds that induces isomorphisms of homotopy groups of dimension $\leq k - 1$. Then f is μ-homotopic to a homeomorphism.

Proof. Without loss of generality, we assume that $k \geq 1$ and that M_1, M_2 are connected. Let $\{M_1(n)\}_{n=1}^{\infty}$ and $\{M_2(n)\}_{n=1}^{\infty}$ be sequences of PL-manifolds as in the definition 1.2.1. of triangulated μ^k-manifolds.

Step 1. Find $n' \geq 1$, $i \geq 1$, a map $F\colon M_1(i) \to M_2(n)$ and a k-partition $\overline{Q}$ on $M_2(n)$ such that: 1. If two maps $\alpha, \beta\colon X \to M_2(n')$ are St $\overline{Q}$-close, then they are homotopic as maps into $M_2(1)$; 2. The map F induces isomorphisms of homotopy groups of dimension $\leq k - 1$ and an epimorphism in dimension k ; 3. The maps f and $F|M_1$ are homotopic as maps into $M_2(n)$.

It is clear how to find $n' \geq 1$ and $\overline{Q}$. Extend $f\colon M_1 \to M_2 \subseteq M_2(n)$ to a map $F'\colon M_1(i) \to M_2(n)$ for some i ($M_2(n)$ is an ANR) . Observe that F' induces isomorphisms of homotopy groups of dimension $\leq k - 1$. Finally, we find $F\colon M_1(i) \to M_2(n)$ that is μ-homotopic to F' and such that $F'_{\#}\colon \pi_k(M_1(i)) \to \pi_k(M_2(n))$ is an epimorphism using the following.

2.8.7. Lemma. If $F'\colon X \to Y$ induces an isomorphism of homotopy groups of dimension $\le k-1$, if Y is LC^{k-1}, and if $\alpha\colon P \to Y$ is a map of a polyhedron of dimension $\le k$, then there is a map $\beta\colon P \to X$ such that $F'\beta \simeq_{\mu} \alpha$.

Proof. We produce a map β such that $F'\beta \,|\, |K^{(k-1)}| \simeq \alpha \,|\, |K^{(k-1)}|$, for some fixed triangulation K of P. By induction, we build maps $\beta_i\colon |K^{(i)}| \to X$ $(i = 0,1,\dots, k-1)$ and homotopies $H_i\colon |K^{(i)}| \times I \to Y$ such that $H_i(x, 0) = \alpha(x)$, $H_i(x, 1) = F'\beta_i(x)$ for $x \in |K^{(i)}|$. For $i = 0$, let $\beta_0(x)$ be a point in the component of X that corresponds to the component of Y containing $\alpha(x)$ (F' induces a bijection between the components of X and Y), and let $H_i(x, t)$ be a path between $\alpha(x)$ and $F'\beta_0(x)$. Inductively, assume that β_{i-1}, H_{i-1} have been constructed $(i \le k-1)$. Let σ be an i-simplex of K. Then $\alpha \cup H_{i-1}\colon \sigma \times \{0\} \cup (\partial\sigma) \times I \to Y$ extends to $H\colon \sigma \times I \to Y$. Hence $\beta_{i-1}|\partial\sigma\colon \partial\sigma \to X$ is null-homotopic, since $F'\beta_{i-1}|\partial\sigma = H_{i-1}|(\partial\sigma) \times \{1\} = H|(\partial\sigma) \times \{1\}$ is null-homotopic in Y and F' induces a monomorphism on π_{i-1}. Hence, there is an extension $\beta'\colon \sigma \to X$ of $\beta_{i-1}|\partial\sigma$. As in 2.7.3., we can pick an extension β' such that $F'\beta' \simeq H|\sigma \times \{1\}$ (rel $\partial\sigma$) (using the information that $F'_{\#}\colon \pi_i(X) \to \pi_i(Y)$ is an epimorphism). By The Homotopy Extension Theorem [Hu], find $H'\colon \sigma \times I \to Y$ such that $H'|\sigma \times 0 = \alpha|\sigma$, $H'|(\partial\sigma) \times I = H_{i-1}|(\partial\sigma) \times I$ and $H'(x, 1) = F'\beta'(x)$, for $x \in \sigma$. Finally, set $\beta_i|\sigma = \beta'$, $H_i|\sigma \times I = H'$. We extend β_{k-1} onto k-simplexes of K using the fact that $\beta_{k-1}|\partial\sigma\colon \partial\sigma \to X$ is null-homotopic (since $F'\beta_{k-1}|\partial\sigma \simeq \alpha|\partial\sigma$ is null-homotopic, and F' induces a monomorphism on π_{k-1}).

Returning back to Step 1 of the proof of 2.8.6., we specify a polyhedron P of dimension $\leq k$ and a map $\gamma: P \to M_2(n)$ such that $\gamma_\#: \pi_i(P) \to \pi_i(M_2(n))$ is an isomorphism for $i \leq k - 1$, and an epimorphism for $i = k$ (e.g., P can be the k-skeleton of a triangulation of $M_2(n)$, and γ can be inclusion). By 2.8.7. there is a map $\delta: P \to M_1(i)$ such that $F'\beta' \underset{\mu}{\simeq} \gamma$. By adjusting, we can assume that δ is a PL embedding. Then $F'|\delta(P)$ and $\gamma\delta^{-1}$ are μ-homotopic maps, and The μ-Homotopy Extension Theorem 2.1.8.(iv) yields a map $F: M_1(i) \to M_2(n)$ μ-homotopic to F' and agreeing with $\gamma\delta^{-1}$ on $\delta(P)$. In particular, F induces an epimorphism of k^{th} homotopy groups.

<u>Step 2.</u> Using F as a guide, build a partition Q on $M_1(i)$ that is in one-to-one correspondence with $\overline{Q}$, as in steps 3, 4, and 5 of 2.4. (there, the guiding map was denoted by α). Note that the triple (Q, T, α) obtained at the end has the property that α is homotopic to the map F.

<u>Step 3.</u> We proceed, as in 2.5. and 2.7., to "connect up" elements and intersections of elements of Q. It is of utmost importance that we maintain the following property of associated maps α: Maps $\alpha|M_1: M_1 \to M_2(n) \subseteq M_2(1)$ and $f: M_1 \to M_2 \subseteq M_2(1)$ are μ-homotopic. This property is clearly preserved if we replace α by a homotopic map (as in steps 3, 4, 5 of 2.4., 2.5., and step 5 of 2.7.). Furthermore, it is preserved if we replace α by $\alpha|M_1(j)$, for some $j \geq i$ (as we do very often, in particular in Step 6 of 2.7.), or if we replace α by a map that is St $\overline{Q}$-close to α (as in Step 6 of 2.7.), by choice of n and $\overline{Q}$.

Consequently, we can assume that there is a triple (Q, T, α) where $\alpha\colon M_1(j) \to M_2(n)$ is a map with $\alpha|M_1 \simeq_\mu f\colon M_1 \to M_2(1)$, Q is a k-partition on $M_1(j)$, $T\colon Q \to \overline{Q}$ is a one-to-one correspondence, and α is a map associated to T.

Step 4. Using Axioms 1 and 2 on elements and intersections of elements of partitions Q and $\overline{Q}$, build sequences $\{P_r\}$ and $\{\hat{P}_r\}$ of k-partitions with properties as in 1.3.3. (The inductive construction is described in details in 1.3. The only difference is that the induction begins by setting $P_0 = Q$, $\hat{P}_0 = Q$, $T_0 = T$. Let $h\colon M_1 \to M_2$ be the homeomorphism as in the conclusion of 1.3.3. (there, it was named f). Observe that, by definition, h and f are St $\overline{Q}$-close, and therefore h and f are μ-homotopic as maps $M_1 \to M_2(1)$. It remains to observe that h and f are μ-homotopic as maps $M_1 \to M_2$ by "absorbing maps", since we can absorb the guaranteed homotopy in $M_2(1)$ to get the required homotopy in M_2 (see 2.3.).

2.8.8. Corollary. If M_1, M_2 are two triangulated μ^k-manifolds obtained from triangulated PL-manifolds M_1^{PL} and M_2^{PL} respectively, and if there is a map $f\colon M_1^{PL} \to M_2^{PL}$ that induces an isomorphism of homotopy groups of dimension $\leq k - 1$, then $M_1 \approx M_2$.

Proof. By 2.8.6. it suffices to produce a map $g\colon M_1 \to M_2$ such that g and $f|M_1$ are μ-homotopic as maps into M_2^{PL}. This can be done by "absorbing" $f|M_1\colon M_1 \to M_2^{PL}$. (Although in 2.3. we described absorbings of maps defined on k-dimensional polyhedra, it should be clear how to do the same if the domain is a k-dimensional compactum, by approximately factoring the map through the nerve of some fine cover.)

<u>2.8.9. Corollary.</u> The topological type of a triangulated μ^k-manifold M does not depend on the choice of the triangulation of the PL-manifold that yields M.

<u>2.8.10. Corollary.</u> Any triangulated μ^k-manifold can be obtained from a (2k+1)-dimensional PL-manifold.

<u>Proof.</u> Let $\{M_i\}_{i=1}^{\infty}$ be a defining sequence for a μ^k-manifold M, as in 1.2.1., let P be the k-skeleton of a triangulation of M_1 and let N be a regular neighborhood of P PL-embedded into E^{2k+1}. Then $\dim N = 2k + 1$ and $N \xrightarrow{r} P \xrightarrow{i} M$ is a map that induces an isomorphism of homotopy groups of dimension $\leq k - 1$ ($r: N \to P$ is the canonical retraction, $i: P \to M$ is inclusion). If M' is the μ^k-manifold obtained from N, then by 2.8.8. $M' \approx M$.

<u>2.8.11. Corollary.</u> Any compact (k-1)-connected triangulated μ^k-manifold is homeomorphic to μ^k.

CHAPTER 3

THE Z-SET UNKNOTTING THEOREM

In this chapter we prove several versions of "The Z-set Unknotting Theorem" inspired by the corresponding theorem in Q-manifold theory. Roughly, it states that a homeomorphism between two Z-sets in a triangulated μ^k-manifold M can be extended to a homeomorphism of M, provided necessary μ-homotopy conditions are met. Using techniques developed in Chapter 2, we will construct a homeomorphism of M that only approximately extends a given homeomorphism of Z-sets. The sought after homeomorphism will be obtained as the limit of such homeomorphisms.

As a first application, we prove that triangulated μ^k-manifolds are (strongly) homogeneous.

3.1. The Z-set Unknotting Theorem

Our first Z-set unknotting theorem is the following:

<u>3.1.1. Theorem.</u> Let M be a triangulated μ^k-manifold, $f: M \to Y$ a UV^{k-1}-surjection, $Z \subseteq M$ a Z-set (see 2.3.7.) and U a neighborhood of $f(Z)$ in Y. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds.

If $Z' \subseteq M$ is a Z-set, and if $h: Z \to Z'$ is a homeomorphism with $(fh, f) < \delta$, then h extends to a homeomorphism $\tilde{h}: M \to M$ of M such that $\tilde{h}$ = Identity on $M - f^{-1}(U)$ and $(f\tilde{h}, f) < \varepsilon$.

In most applications f will be the identity map. The major step in proving 3.1.1. consists of showing that h "approximately extends" to a homeomorphism of M.

3.1.2. Lemma. Under assumptions of 3.1.1., for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\eta > 0$ the following holds.

If $Z' \subseteq M$ is a Z-set, and if $h: Z \to Z'$ is a homeomorphism with $(fh, f) < \delta$, then there is a homeomorphism $\tilde{h}: M \to M$ such that $\tilde{h} =$ Identity on $M - f^{-1}(U)$, $(f\tilde{h}, f) < \varepsilon$ and $(\tilde{h}|Z, h) < \eta$.

Proof. Since Y is LC^{k-1} (as a UV^{k-1}-image of an LC^{k-1}-space), for any $\varepsilon > 0$ there is $\delta > 0$ such that any map $g: S^i \to Y$ with δ-small image extends to a map $\tilde{g}: B^{i+1} \to Y$ with ε-small image, $i = 0,1,\ldots, k-1$ [Hu]. In that case we write $\delta < \varepsilon(LC^{k-1})$.

Choose positive numbers $A_{k-1}, \xi_{k-1}, \eta_{k-1}$ such that $A_{k-1} < \varepsilon$, $\eta_{k-1} < \eta$, $N(f(Z), \xi_{k-1}) \subseteq U$ (by $N(B, \xi)$ we denote the ξ-neighborhood of B). Then choose positive numbers $\varepsilon'_{k-2}, \varepsilon_{k-2}, A'_{k-2}, \varepsilon'_{k-1}, \varepsilon_{k-1}, A_{k-1}, \xi_{k-1}, \ldots, \varepsilon'_0, \varepsilon_0, A_0, \xi_0, \varepsilon'_{-1}, \varepsilon_{-1}, A_{-1}, \xi_{-1}, \bar{\varepsilon}, \bar{\eta}$ (in this order) so that

(1) $\xi_\ell + (k + 2)\bar{\varepsilon} + \varepsilon'_\ell < \xi_{\ell+1}$,

(2) $A_\ell + (k + 1)\bar{\varepsilon} < \varepsilon_\ell \; (LC^{k-1})$,

(3) $A_\ell + \varepsilon_\ell + (k + 1)\bar{\varepsilon} < \varepsilon'_\ell \; (LC^{k-1})$,

(4) $A_\ell + (k + 2)\bar{\varepsilon} + \varepsilon'_\ell < A_{\ell+1}$,

(5) $\eta_\ell + k\bar{\eta} < \eta_{\ell+1} \qquad (-1 \leq \ell \leq k - 2)$,

(6) if $A \subseteq M$ is an $\bar{\eta}$-small set, then $f(A)$ is $\bar{\varepsilon}$-small,

(7) $3\bar{\varepsilon} < \min(A_{-1}, \xi_{-1})$, and

(8) $3\bar{\eta} < \eta_{-1}$.

Name stage M_n in the defining sequence for M that admits a k-partition $\bar{Q}$ with mesh $\bar{Q} < \bar{\eta}$. Let W_n denote the union of all elements of $\bar{Q}$ that hit $f^{-1}f(Z)$. For $i \geq n$, we set $W_i = W_n \cap M_i$, and $W = W_n \cap M$. Since W is a (closed) neighborhood of $f^{-1}f(Z)$, there is $\varepsilon' > 0$ such that $f^{-1}(N(f(Z), \varepsilon')) \subseteq W$ and such that any two ε'-close maps into Y are μ-homotopic (2.1.8.(i)) . Finally, choose $\delta > 0$ such that (see 2.1.4.) if $A \subseteq X$ is a closed subset of a compactum of dimension $\leq k$, if $\alpha, \beta: A \to M$ are maps such that $f\alpha$ and $f\beta$ are δ-close, and if β extends to $\tilde{\beta}: X \to M$, then it follows that α extends to $\tilde{\alpha}: X \to M$ with $(f\tilde{\alpha}, f\tilde{\beta}) < \varepsilon'$.

Let $Z' \subseteq M$ be a Z-set, and $h: Z \to Z'$ a homeomorphism with $(fh, f) < \delta$.

By the choice of δ , the map $\alpha: \overline{M - W} \cup Z \to M$ defined as Id on $\overline{M - W}$ and as h on Z extends to $\tilde{\alpha}: M \to M$ with $(f\tilde{\alpha}, f) < \varepsilon'$. It follows that $f\tilde{\alpha} \simeq \mu f$, which implies that $f\tilde{\alpha}$, and hence $\tilde{\alpha}$, induces an isomorphism of homotopy groups of dimension $\leq k - 1$. Requiring that $\tilde{\alpha}$ be identity on a suitable subset of M , we can arrange that $\tilde{\alpha}: M \to M \subseteq M_n$ induces an epimorphism of k^{th} homotopy groups. Since M_n is an ANR , there is $i \geq n$ and an extension $\tilde{\alpha}: M_i \to M_n$. We can also assume that $\tilde{\alpha}|M_i - W_i$ = Inclusion. As in step 3 of 2.4., we can adjust $\tilde{\alpha}$ to get a map $\tilde{\alpha}': M_i \to M_n$ with the additional property that the image of a small set is contained in a single element of $\bar{Q}$. Going deeper in the sequence, we can assume that any (3 mesh L_i)-small set has this property. As in step 4 of 2.4., we can define a partition Q which is an amalgamation of the handlebody decomposition associated with L_i, and an injective, intersection preserving function $T: Q \to \overline{Q}$. Note that T is the "identity map" off of W_i. If we require, in addition, that $\tilde{\alpha}$ be identity on a point from the interior of each $\overline{q} \in \overline{Q}$, and if $\tilde{\alpha}'$ is close enough to $\tilde{\alpha}$, then T will be a bijection. As in step 5 of 2.4., we can replace the triple $(Q, T, \tilde{\alpha}')$ by a triple $(Q_{-1}, T_{-1}, \alpha_{-1})$, where T_{-1} is, in addition, a one-to-one correspondence. By 2.4.9., $(\alpha_{-1}, \tilde{\alpha}) < 2\overline{\eta}$. Also, performing The Isotopy Move (2.2), we can assume that each element in Q_{-1} underlies a subcomplex of $L_{i(-1)}$ (for some $i(-1) \geq i$). Slightly adjusting as in 2.3., we may assume that $\alpha_{-1}(M) \subseteq M$. After this move, we will have $(\alpha_{-1}, \tilde{\alpha}) < 3\overline{\eta}$. Therefore, by (6), (7), (8), the following claim is true for $\ell = -1$:

(C_ℓ): There exist $i(\ell) \geq i$ and a triple $(Q_\ell, T_\ell, \alpha_\ell)$ with the following properties.

- $(a)_\ell$ Q_ℓ is a partition on $M_{i(\ell)}$ such that each element of Q_ℓ underlies a subcomplex of $L_{i(\ell)}$,
- $(b)_\ell$ Q_ℓ satisfies $P(\ell, 1)$ (see 2.5.),
- $(c)_\ell$ $T_\ell: Q_\ell \to \overline{Q}$ is a one-to-one correspondence,
- $(d)_\ell$ $\alpha_\ell: M_{i(\ell)} \to M_n$ is a map associated with T_ℓ, and $\alpha_\ell(M) \subseteq M$,
- $(e)_\ell$ α_ℓ = Inclusion on each $q \in Q_\ell$ with $f(q \cap M) \cap N(f(Z), \xi_\ell) = \phi$,
- $(f)_\ell$ $(f\alpha_\ell|M, f) < A_\ell$, and
- $(g)_\ell$ $(\alpha_\ell|Z, h) < \eta_\ell$.

<u>Inductive Step: $(C_\ell) \Rightarrow (C_{\ell+1})$, $-1 \leq \ell \leq k - 2$.</u> First, using techniques in 2.5. replace $(Q_\ell, T_\ell, \alpha_\ell)$ by $(Q'_\ell, T'_\ell, \alpha'_\ell)$ that satisfies $P(\ell, k - \ell)$ (if $\ell \geq 0$; otherwise there is nothing to do). Also, by 2.2. (The Isotopy Move), we can assume that Q' consists of elements that underlie subcomplexes and that $\alpha'_\ell(M) \subseteq M$. Then

(e)' α'_ℓ = inclusion on each $q \in Q'_\ell$ with $f(q \cap M) \cap N(f(Z), \xi_\ell + k\bar{\varepsilon}) = \phi$,

(f)' $(f\alpha'|M, f) < A_\ell + k \cdot \bar{\varepsilon}$,

(g)' $(\alpha'_\ell|Z, h) < \eta_\ell + k \cdot \bar{\eta}$

(since $(\alpha_\ell, \alpha'_\ell) < k \cdot \bar{\eta})$.

We continue by applying methods of 2.7. to make elements of Q'_ℓ $(\ell + 1)$-connected. We will, however, impose some control on choices of "cores" used in 2.7. Roughly, we require that cores and the associated fringe regions be disjoint from Z (here we use the fact that Z is a Z-set), and that homotopies used to redefine the associated map be small measured in Y . Let $\gamma: S^{\ell+1} \to q$ be one of the "generators" as in step 1 of 2.7. Let $\gamma': S^{\ell+1} \to q \cap M$ be a map obtained by absorbing γ , as in 2.3. (Note that γ' determines the same "generator" as γ .) Since $\alpha'_\ell\gamma'(S^{\ell+1})$ is contained in a single element of $\bar{Q}$, it is $\bar{\eta}$-small, and hence $f\alpha'_\ell\gamma'(S^{\ell+1})$ is $\bar{\varepsilon}$-small by (6) . By (f)' it follows that $f\gamma'(S^{\ell+1})$ is $A_\ell + (k + 1)\bar{\varepsilon}$-small. Then (2) implies that $f\gamma'$ bounds a ε_ℓ-small $(\ell + 2)$-cell in Y , which can be approximately lifted to M to yield a map $\tilde{\gamma}': B^{\ell+2} \to M$ that extends γ' and whose image under f is ε_ℓ-small in Y . By (f)' it follows that $f\alpha'_\ell\tilde{\gamma}'(B^{\ell+2})$ is

$A_\ell + k \cdot \overline{\varepsilon} + \varepsilon_\ell$-small in Y. Hence, the map $S^{\ell+2} \to M$ defined on the upper hemisphere as $\alpha_\ell'\tilde{\gamma}'$ and on the lower hemisphere as a null-homotopy $B^{\ell+2} \to T_\ell'(q) \cap M$ of $\alpha_\ell'\gamma'$ (as in step 3 of 2.7.) is $A_\ell + (k+1)\overline{\varepsilon} + \varepsilon_\ell$ -small in Y. Consequently, (3) implies that the homotopy $H\colon B^{\ell+1} \times I \to M$ (as in step 5 of 2.7., indexes are omitted) can be chosen to be ε_ℓ'-small (measured in Y). Next, we can go deeper in the defining sequence to make sure that subsequent Isotopy Moves are small, and then approximate γ' by a PL embedding that gives us the core. Also, choose small regular neighborhoods of the core and define $\alpha_\ell(0)\colon M_j \to M_n$ as in step 5 of 2.7. Proceed as in step 6 of 2.7. and adjust to get $\alpha_{\ell+1}\colon M_{i(\ell+1)} \to M_n$ with $\alpha_{\ell+1}(M) \subseteq M$, and a triple $(Q_{\ell+1}, T_{\ell+1}, \alpha_{\ell+1})$ satisfying $P(\ell+1, 1)$. Then $(a)_{\ell+1}$-$(d)_{\ell+1}$ are clear; $(g)_{\ell+1}$ follows from $(g)'$, (5), and the fact that $\alpha_{\ell+1} = \alpha_\ell'$ on Z; $(f)_{\ell+1}$ follows from $(f)'$, (4), and the fact that in the fringe region move $f\alpha_{\ell+1}$ and $f\alpha_\ell'$ differ only for the size of the homotopy H (measured in Y), plus $2\overline{\varepsilon}$ which we may lose doing Isotopy Moves. Similarly, (1) and $(e)'$ imply $(e)_{\ell+1}$.

Consequently, (C_{k-1}) is a true statement. Then Q_{k-1} is a k-partition and $\alpha_{k-1}\colon M_{i(k-1)} \to M_n$ is a map associated to a one-to-one correspondence $T_{k-1}\colon Q_{k-1} \to \overline{Q}$. Moreover, by the choice of A_{k-1}, ξ_{k-1}, η_{k-1}, we get that $(f\alpha_{k-1}, f) < \varepsilon$, $\alpha_{k-1} = \mathrm{Id}$ off of $f^{-1}(U)$, and $(\alpha_{k-1}|Z, h) < \eta$. Again going deeper in the sequence (choosing $j \geq i(k-1)$ and restricting α_{k-1} onto M_j) to make sure that our next move is small, we proceed as in 1.3.3. to get a homeomorphism $\tilde{h}: M \to M$ that also satisfies $(f\tilde{h}, f) < \varepsilon$, $\tilde{h} = \mathrm{Id}$ off of $f^{-1}(U)$ and $(\tilde{h}|Z, h) < \eta$.

Proof of 3.1.1. First we consider the case when $f = \mathrm{Id}: M \to M$. For a Z-set $Z \subset M$, a neighborhood U of Z and numbers $\varepsilon, \delta > 0$ by $H(Z, U, \varepsilon; \delta)$ denote the following statement. If $h: Z \to Z'$ is a homeomorphism between Z-sets in M that is δ-close to the identity and if $\eta > 0$, then there is a homeomorphism $\tilde{h}: M \to M$ such that $\tilde{h} = \mathrm{Id}$ on $M - U$, $\tilde{h}$ is ε-close to the identity, and $(\tilde{h}\, Z, h) < \eta$.

Let Z, U, ε be given. By 3.1.2. there exists $\delta > 0$ such that $H(Z, U, \frac{\varepsilon}{4}; \delta)$. Suppose $h: Z \to Z'$ is a homeomorphism between Z-sets that is δ-close to the identity. Setting $\varepsilon_0 = \varepsilon_0' = \frac{\varepsilon}{4}$, inductively, for $n = 1,2,3,\ldots$, choose (in this order) open neighborhoods V_n and V_n' of Z and Z' respectively, positive numbers η_n', ε_n' , a homeomorphism $h_n': M \to M$, positive numbers ε_n, η_n , and a homeomorphism $h_n: M \to M$ such that the following is true.

$(1)_n$ $V_n' \subseteq N(Z', \frac{1}{2^{n-1}}) \cap V_{n-1}'$, $n \geq 2$ and $V_1' \subseteq U$,

$(2)_n$ $H(h_{n-1} \ldots h_1(Z'), h_{n-1} \ldots h_1(V_n'), \varepsilon_{n-1}; \eta_n')$; $n \geq 2$,

$(3)_n$ $h_n' = \mathrm{Id}$ on $M - h_{n-1}' \ldots h_1'(V_{n-1})$, $(h_n', \mathrm{Id}) < \varepsilon_{n-1}'$,
$h_n'|h_{n-1}' \ldots h_1'(Z)$ and
$h_{n-1} h_{n-2} \ldots h_1 h\, h_1'^{-1} \ldots h_{n-1}'^{-1}|h_{n-1}' \ldots h_1'(Z)$ are
η_n'-close; $n \geq 1$ $(h_0 = h_0' = \mathrm{Id}$, $V_0 = U)$,

$(4)_n$ $V_n \subseteq N(Z, \frac{1}{2^n}) \cap V_{n-1}$, $n \geq 2$; $V_1 \subseteq U$

$(5)_n$ $H(h_n' \ldots h_1'(Z), h_n' \ldots h_1'(V_n)$, $\varepsilon_n'; \eta_n)$, $n \geq 1$,

$(6)_n$ $h_n = \mathrm{Id}$ on $M - h_{n-1} \cdots h_1(V_n')$, $(h_n, \mathrm{Id}) < \varepsilon_{n-1}$, and $h_n/h_{n-1} \cdots h_1(Z')$ is η_n-close to $h_n' \cdots h_1' h^{-1} h_1^{-1} \cdots h_{n-1}^{-1} | h_{n-1} \cdots h_1(Z')$, $n \geq 1$.

(The existence of h_n , $n \geq 2$, and h_n , $n \geq 1$, is guaranteed by $(5)_{n-1}$ and $(2)_n$ respectively. The existence of h_1' follows from the choice of ε_0' and δ.)

Note that we choose ε_n, ε_n' after homeomorphisms h_{n-1}, h_{n-1}' are already chosen. We can arrange (by requiring $\varepsilon_n < \frac{1}{2^n}$) that $\tilde{h} = \lim_{i\to\infty} h_i \cdots h_1$ and $\tilde{h}' = \lim_{i\to\infty} h_i' \cdots h_1'$ exist, and from [An_3] it follows that we can also arrange that $\tilde{h}$, $\tilde{h}'$ are homeomorphisms. Also, if ε_n's and ε_n''s are sufficiently small, we have $(\tilde{h}, \mathrm{Id}) < \frac{\varepsilon}{2}$, $(\tilde{h}', \mathrm{Id}) < \frac{\varepsilon}{2}$. Then $(3)_n$ implies that $h_n' h_{n-1}' \cdots h_1'$ and $h_{n-1} h_{n-2} \cdots h_1 h$ are η_n'-close when restricted to Z . Requiring that $\eta_n' \to 0$, when $n \to \infty$, and passing to the limit, we get $\tilde{h}' = \tilde{h} h$. Hence, $\tilde{h}'^{-1} \tilde{h}: M \to M$ is a homeomorphism ε-close to the identity, extending h . By $(3)_n$, $(6)_n$, $(1)_n$ and $(4)_n$ it follows that both $\tilde{h}$ and $\tilde{h}'$, and consequently $\tilde{h}'^{-1} \tilde{h}$, are fixed on $M - U$.

For the general case, when $f: M \to Y$ is a UV^{k-1}-surjection, first apply 3.1.2. to get a homeomorphism $\hat{h}: M \to M$ fixed on $M - f^{-1}(U)$ and close to Id measured in Y , such that $\hat{h}|Z$ and h are close. Then use the argument just presented (with $f = \mathrm{Id}$), to get a small homeomorphism $\bar{h}: M \to M$ fixed on $M - f^{-1}(U)$ extending $\hat{h} h^{-1}: Z' \to \hat{h}(Z)$. Then $\bar{h}^{-1} \hat{h}$ is the desired homeomorphism.

We want to discuss two more versions of the Z-set unknotting theorem: The ε - δ version (3.1.3.), and the μ-homtopy version (3.1.4.). We give only sketches of proofs, since the details are the same as in the proof of 3.1.2.

<u>3.1.3. Theorem.</u> Let $f: M \to Y$ be a UV^{k-1}-surjection from a triangulated μ^k-manifold. Then for any $\varepsilon > 0$ there exists $\delta > 0$ with the following property.

If $h: Z_1 \to Z_2$ is a homeomorphism between two Z-sets in M with $(fh, f) < \delta$, then there exists a homeomorphism $\tilde{h}: M \to M$ with $\tilde{h}|Z_1 = h$ and $(f\tilde{h}, f) < \varepsilon$.

<u>Sketch of proof.</u> Using 2.1.4., if $\delta > 0$ is sufficiently saml1 we can define the "guide map" $\alpha: M \to M$ extending h such that $f\alpha$ and f are close. Now we proceed as in the proof of 3.1.2. to "turn α into a homeomorphism" $\hat{h}: M \to M$ such that $f\hat{h}$ is close to f and $\hat{h}|Z_1$ is close to h . Then apply 3.1.1. (with $f = \mathrm{Id}$) to obtain a small homeomorphism $\bar{h}: M \to M$ extending $\hat{h}\, h^{-1}: Z_2 \to \hat{h}(Z_1)$. Finally, $\bar{h}^{-1}\, \hat{h}$ is the desired homeomorphism.

<u>3.1.4. Theorem.</u> Let Z_1, Z_2 be two Z-sets in a triangulated μ^k-manifold M , and let $h: Z_1 \to Z_2$ be a homeomorphism. Denote by i_j the inclusion map $Z_j \to M$ $(j = 1, 2)$. If $i_1 \simeq_\mu i_2 h$, then h extends to a homeomorphism $\tilde{h}: M \to M$.

<u>Sketch of proof.</u> By the μ-Homotopy Extension Theorem 2.1.8.(iv) we can find a "guide map" $\alpha: M \to M$ extending h (and inducing an isomorphism of homotopy groups of dimension $\leq$ k-1 and an epimorphism of

k^{th} homotopy groups). Then we proceed by turning α into a homeomorphism. We need only make sure that "gross moves" (as in 2.7.) are performed away from Z_1 (i.e. the "cores", together with regular neighborhoods are chosen away from Z_1). The constructed homeomorphism $\hat{h}: M \to M$ has the property that $\hat{h}/Z_1$ is close to h. To finish the proof, apply 3.1.1. to get a homeomorphism $\bar{h}: M \to M$ extending $\hat{h}\, h^{-1}: Z_2 \to \hat{h}(Z_1)$. Then $\bar{h}^{-1}\, \hat{h}: M \to M$ extends h.

<u>3.1.5. Corollary.</u> Any homeomorphism $h: Z_1 \to Z_2$ between Z-sets in μ^k extends to a homeomorphism $\bar{h}: \mu^k \to \mu^k$.

<u>Proof.</u> Since μ^k is C^{k-1}, it follows that $i_1 \simeq_\mu i_2 h$.

<u>3.1.6. Remark.</u> For $k = 0$ this result is well-known (Z-sets in the Cantor set are nowhere dense closed subsets). A closed subset of μ^1 is a Z-set iff it does not locally separate. The result in 3.1.5 for $k = 1$ also appears in [M-O-T].

3.2. Homogeneity of μ^k

Although there are no hard proofs in this section (all results are consequences of The Z-set Unknotting Theorem), we devote the entire section to discussing homogeneity, since it was the author's main goal in undertaking the project of understanding Menger spaces. The results of this section answer (in the affirmative) questions about homogeneity of Menger space.

A space X is <u>homogeneous</u> if for any $x,y \in X$ there is a homeomorphism $h: X \to X$ with $h(x) = y$.

3.2.1. Theorem. Any connected triangulated μ^k-manifold $(k \geq 1)$ is homogeneous.

Proof. Follows directly from 3.1.4. and the fact that points are Z-sets (2.3.9.).

A space X is *strongly locally homogeneous* if for any $x \in X$ and any neighborhood U of x there exists a neighborhood V of x such that if $y \in V$ then there is a homeomorphism $h: X \to X$ with $h(x) = y$ and $h = \mathrm{Id}$ on $X - U$.

3.2.2. Theorem. Any triangulated μ^k-manifold is strongly locally homogeneous.

Proof. This follows from 3.1.1.

Using strong local homogeneity, via the standard argument consisting of constructing a homeomorphism as the limit of homeomorphisms h_n throwing n points of D_1 into D_2 we prove the following.

3.2.3. Corollary. If D_1, D_2 are countable, dense subsets of a triangulated μ^k-manifold M, then for any $\varepsilon > 0$ there is a homeomorphism $h: M \to M$ with $h(D_1) = D_2$ and $(h, \mathrm{Id}) < \varepsilon$.

CHAPTER 4

THE DECOMPOSITION THEORY OF MENGER MANIFOLDS

In this chapter we carry out a program similar to that used by R.D. Edwards in the setting of Euclidean manifolds to prove "half" of the Characterization Theorem. We show that if G is a UV^{k-1}-decomposition of a triangulated μ^k-manifold M, and if M/G is k-dimensional and satisfies DD^kP, then G is shrinkable.

In the first section we discuss the simplest case when the non-degeneracy set N_G is contained in a Z-set in M. Here we need not assume that M/G satisfies DD^kP since it follows from the hypothesis that N_G is contained in a Z-set.

The hard work is done in 4.2., where we consider the case where N_G is a countable union of Z-sets ("The σ-Z-set Shrinking Theorem"). The proof consists in a careful analysis of "moves" described in Chapter 2.

Finally, in 4.3. we prove the theorem in general.

The general case follows "categorically" from The Z-set Unknotting Theorem and "The σ-Z-set Shrinking Theorem".

We will show in the next chapter (The Resolution Theorem 5.1.8.) that every LC^{k-1}-compactum Y admits a compact triangulated μ^k-manifold M and a UV^{k-1}-surjection $f: M \to Y$. Embed M as a Z-set into μ^k, and consider the upper semicontinuous decomposition G of μ^k given by the fibers of f and singletons. If Y is $> k$ dimensional, then $\mu^k/G \supseteq Y$ is also $> k$ dimensional, and hence

$\mu^k/G \not\approx \mu^k$. In the next section we show that if $\dim Y \leq k$, then $\mu^k/G \approx \mu^k$.

4.1. The Z-set Shrinking Theorem

Both results in this section follow from The Z-set Unknotting Theorem. The first theorem corresponds to the "Closed 0-dimensional Shrinking Theorem" for Euclidean manifolds, while the second one corresponds to The 1-LCC Shrinking Theorem.

For a surjection $f: X \to Y$ between compacta we set $N_f = \{x \in X: f^{-1}f(x) \neq \{x\}\}$. We say that f is a <u>near-homeomorphism</u> if f can be arbitrarily closely approximated by a homeomorphism. Clearly, any near-homeomorphism between LC^{k-1}-compacta must be a UV^{k-1}-surjection. We say that a surjection $f: X \to Y$ is a strong near-homeomorphism if for any open set $U \subseteq Y$ containing $\overline{f(N_f)}$ and any $\varepsilon > 0$ there is a homeomorphism $h: X \to Y$ such that $(h, f) < \varepsilon$ and $h = f$ on $f^{-1}(Y - U)$. The following "Shrinking Criterion" due to R.H. Bing is well-known.

<u>4.1.1. Proposition.</u> A surjection $f: X \to Y$ between compacta is a strong near-homeomorphism if and only if for any $\varepsilon > 0$, any $\eta > 0$, and any neighborhood U of $\overline{N_f}$ there is a homeomorphism $h: X \to X$ such that $(fh, f) < \varepsilon$, $\operatorname{diam} h(f^{-1}(y)) < \eta$ for every $y \in Y$, and $h = \mathrm{Id}$ on $X - U$.

The question we want to address is when is a given UV^{k-1}-surjection $f: M \to Y$ from a triangulated μ^k-manifold a (strong) near-homeomorphism.

4.1.2. Theorem. If $f: M \to Y$ is a UV^{k-1}-surjection from a triangulated μ^k-manifold such that $\overline{N_f}$ is a Z-set in M, and if $\dim Y = k$, then f is a strong near-homeomorphism.

Proof. Let $\varepsilon > 0$, $\eta > 0$, $U \supseteq \overline{N_f}$ as in 4.1.1. By 3.1.1. we can choose $\delta > 0$ such that if $h: \overline{N_f} \to Z$ is a homeomorphism onto a Z-set $Z \subset M$ with $(fh, f) < \delta$, then h extends to a homeomorphism $\tilde{h}: M \to M$ with $(f\tilde{h}, f) < \varepsilon$ and $\tilde{h} = \mathrm{Id}$ on $M - U$. (Note that $U = f^{-1}(f(U))$, and $f(U)$ is an open neighborhood of $f(\overline{N_f})$.) Hence, to finish the proof we need to construct a Z-embedding $h: \overline{N_f} \to M$ such that $(fh, f) < \delta$ and $\operatorname{diam} h(f^{-1}(y)) < \eta$, $y \in Y$. By 2.3.8. it suffices to find a map $\alpha: \overline{N_f} \to M$ such that $(f\alpha, f) < \delta$ and $\alpha(f^{-1}(y)) = \text{point}$, $y \in Y$. We can take $\alpha = gf|\overline{N_f}$, where $g: N \to M$ is an approximate lift of f, i.e. a map with $(fg, \mathrm{Id}_{Y}) < \delta$ (which exists since $\dim Y = k$, and f is a UV^{k-1}-surjection).

4.1.3. Theorem. Suppose $f: M \to Y$ is a UV^{k-1}-surjection from a triangulated μ^k-manifold M. If $\dim Y = k$ and if $f(\overline{N_f})$ is a Z-set in Y (see 2.3.7.), then f is a strong near-homeomorphism.

Proof. Let $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ be a countable dense subset of the space of maps $I^k \to M$, such that α_i is a Z-embedding, $i = 1,2,\ldots$ and $\alpha_i(I^k) \cap \alpha_j(I^k) = \phi$ for $i \neq j$ (see the proof of 2.3.8.). Choose inductively for $n = 1,2,\ldots$ (in this order) $\varepsilon_n > 0$, a neighborhood U_n of $f(\overline{N_f})$ in Y, and a homeomorphism $h_n: M \to M$ such that

$(1)_n$ $(f, fh_n) < \varepsilon_n$,

$(2)_n$ $h_n = \mathrm{Id}$ on $M - f^{-1}(U_n)$,

$(3)_n$ $h_n(h_{n-1} \dots h_1(\alpha_n(I^k))) \cap \overline{N}_f = \phi$, and

$(4)_n$ if $i < n$, then $h_{n-1} \dots h_1(\alpha_i(I^k)) \cap f^{-1}(U_n) = \phi$.

Observe that $(3)_{n-1}$, $(4)_i$ and $(2)_i$ $(i < n)$ imply that $h_{n-1} \dots h_1(\alpha_i(I^k)) \cap \overline{N}_f = \phi$ for $i < n$, and hence we can choose U_n so that $(4)_n$ holds. Let W be a triangulated μ^k-manifold obtained by choosing a deep enough stage M_i and a fine k-partition Q on M_i , and let $W_i = \cup\{q \in Q : q \subseteq f^{-1}(U_n)\}$ and $W = W_i \cap M$. If Q is sufficiently fine, $N_f \subset \operatorname{int} W$. Note that $f|W: W \to f(W)$ is a UV^{k-1}-surjection and $W_0 = (\partial W_i) \cap M$ is a Z-set in W . Since $f(\overline{N}_f)$ is a Z-set in Y , we can approximate $f\alpha_n: I^k \to Y$ by a map $\beta_n: I^k \to Y$ whose image misses $f(\overline{N}_f)$ such that $f\alpha_n = \beta_n$ on $(f\alpha_n)^{-1}(Y - f(W))$. Then $f^{-1}\beta_n: I^k \to M$ is a well-defined map whose image misses $\overline{N}_f$. By 2.3.8. we can approximate $f^{-1}\beta_n$ by a Z-embedding $\gamma_n: I^k \to M$ such that $\gamma_n = \alpha_n$ on $(f\alpha_n)^{-1}\overline{(Y - f(W))}$. Applying 3.1.1. to W and the homeomorphism $h = \mathrm{Id} \cup \gamma_n^{-1}\alpha_n^{-1}: W_0 \cup (\alpha_n(I^k) \cap W) \to W_0 \cup (\gamma_n(I^k) \cap W)$ (between Z-sets in W) which satisfies $(fh, f) < \delta$ for a small $\delta > 0$ we get a homeomorphism $\tilde{h}: W \to W$ extending h . We can extend $\tilde{h}$ onto $M - W$ by the Identity map to get $h_n: M \to M$ satisfying $(1)_n$-$(4)_n$. If $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, then the sequence $\{f_n = fh_n h_{n-1} \dots h_1\}_{n=1}^{\infty}$ forms a Cauchy sequence of maps $M \to Y$. Let $f' = \lim_{n\to\infty} f_n$. Then f' , as the limit of UV^{k-1}-surjections, is a UV^{k-1}-surjection that approximates f (if $\sum_{n=1}^{\infty} \varepsilon_n$ is small). Notice that if $U_1 \supseteq U_2 \supseteq \dots$ and $\bigcap_{n=1}^{\infty} U_n = f(\overline{N}_f)$, then $f'(\overline{N}_{f'}) \subseteq f(\overline{N}_f)$ by $(2)_n$. Also, $(2)_n$ and $(4)_n$ imply that $f'^{-1}(f(N_f))$ is contained in the complement of

$\bigcup_{n=1}^{\infty} \alpha_n(I^k)$. Consequently, $\bar{N}_{f'}$ is a Z-set in M . By 4.1.2. f' is a strong near-homeomorphism. Since f' approximates f and $f' = f$ on $M - f^{-1}(U_1)$ (and U_1 can be chosen arbitrarily small), we conclude that f is a strong near-homeomorphism.

4.2. The σ-Z-Set Shrinking Theorem

In this section we make a final analysis of the "moves" described in Chapter 2 to prove The σ-Z-Set Shrinking Theorem 4.2.2. The remaining part of the program consists of "categorical" consequences of the theorems already proved.

4.2.1. Definition. A subset $\Sigma \subseteq M$ of a triangulated μ^k-manifold M is a σ-Z-set if it is a countable union of Z-sets.

4.2.2. Theorem. Let $f\colon M \to Y$ be a UV^{k-1}-surjection from a triangulated μ^k-manifold onto a space Y with $\dim Y = k$. If N_f is a σ-Z-set in M such that

(*) if $i \geq 1$ and if $N \subseteq M_i$ is a convenient submanifold of M_i of codimension s $(s = 0,1,\ldots,k)$, then $N_f \cap N$ is a σ-Z-set in $N \cap M$ (which is a copy of μ^{k-s}) .

Then f is a strong near-homeomorphism.

The condition (*) has a technical nature. From the main theorem it follows that 4.2.2. is true without assuming (*) .

Proof. Fix positive numbers $A_{k-1}, \xi_{k-1}, \eta_{k-1}$. We need to construct a homeomorphism $h\colon M \to M$ such that $(fh, f) < A_{k-1}$,

$h = \mathrm{Id}$ off of $f^{-1}(N(\overline{f(N_f)}, \xi_{k-1}))$ and $\mathrm{diam}\, h(f^{-1}(y)) < \eta_{k-1}$ for all $y \in Y$.

Inductively on $\ell = k-2, k-3, \ldots, -1$ choose (in this order) positive numbers $\varepsilon_\ell', \varepsilon_\ell, A_\ell, \xi_\ell$ and then choose $\overline{\varepsilon}, \overline{\eta} > 0$ such that

(1) $\xi_\ell + (k+2)\overline{\varepsilon} + \varepsilon_\ell' < \xi_{\ell+1}$,

(2) $A_\ell + (k+1)\overline{\varepsilon} < \varepsilon_\ell(LC^{k-1})$,

(3) $A_\ell + \varepsilon_\ell + (k+1)\overline{\varepsilon} < \varepsilon_\ell'\,(LC^{k-1})$,

(4) $A_\ell + (k+2)\overline{\varepsilon} + \varepsilon_\ell' < A_{\ell+1}$,

(5) $\eta_\ell + (k+2)\overline{\eta} < \eta_{\ell+1}$ $\quad(-1 \le \ell \le k-2)$,

(6) if $A \subseteq M$ is an $\overline{\eta}$-small set, then $f(A)$ is $\overline{\varepsilon}$-small,

(7) $3\overline{\varepsilon} < \min(A_{-1}, \xi_{-1})$, and

(8) $3\,\overline{\eta} < \eta_{-1}$.

As in the proof of 3.1.2. find a deep enough stage M_n in the defining sequence for M that admits a k-partition $\overline{Q}$ with mesh $\overline{Q} < \overline{\eta}$. Let W_n denote the union of all elements of $\overline{Q}$ that hit $\overline{N}_f$. For $i \geq n$ we set $W_i = W_n \cap M_i$ and $W = W_n \cap M$. Applying lifting arguments as in [La] to the UV^{k-1}-surjection $f|W: W \to f(W)$ we obtain a map $g: f(W) \to W$ such that $fg: f(W) \to f(W)$ is close to Id. We set $\alpha = gf|W: W \to W$. Observe that we can arrange (choosing g carefully) that $\alpha = \mathrm{Id}$ on $\mathrm{Fr}W$. Thus α extends to $\tilde{\alpha}: M \to M$ setting $\tilde{\alpha} = \mathrm{Id}$ on $\overline{M - W}$. Note that $f\tilde{\alpha}: M \to Y$ is close to f and hence $\tilde{\alpha}$ induces an isomorphism of homotopy groups of dimension $\leq k-1$. Requiring that $\tilde{\alpha}$ be identity on a suitable (closed) subset A of M, we can arrange that $\tilde{\alpha}: M \to M \subseteq M_n$ induces an epimorphism of k^{th} homotopy groups. (Since N_f is a σ-Z-set, we can assume that $A \cap N_f = \phi$. Then choose g to be an extension of the partial lift $f^{-1}|: A \to M$.)

Extending $\tilde{\alpha}$ to a map into M_n, we obtain a map $\alpha': M_i \to M_n$ for some $i \geq n$. We also assume that α' = Inclusion on $\overline{M_i - W_i}$.

Adjusting α', we get a partition Q' on M_i and an intersection preserving function $T': Q' \to \overline{Q}$ such that α' is a map associated to T'. As in the proof of 3.1.2., requiring that α be identity on a point in the interior of each $\overline{q} \in \overline{Q}$, we can arrange that T' is a bijection. As in Step 5 of 2.4. we replace the triple (Q', T', α') by a triple $(Q_{-1}, T_{-1}, \alpha_{-1})$ where T_{-1} is, in addition, a one-to-one correspondence. The reader can easily verify, using (6), (7), and (8), that the triple $(Q_{-1}, T_{-1}, \alpha_{-1})$ after adjusting α_{-1} so that $\alpha_{-1}(M) \subseteq M$ satisfies the following statement for $\ell = -1$. (To get $(a)_{-1}$ we may have to perform The Isotopy Move and go deeper in the defining sequence.)

(S_ℓ): There exist $i(\ell) \geq i$ and a triple $(Q_\ell, T_\ell, \alpha_\ell)$ with the following properties.

$(a)_\ell$ Q_ℓ is a partition on $M_{i(\ell)}$ such that each element of Q_ℓ underlies a subcomplex of $L_{i(\ell)}$,

$(b)_\ell$ Q_ℓ satisfies $P(\ell, 1)$ (see 2.5.),

$(c)_\ell$ T_ℓ: $Q_\ell \to \overline{Q}$ is a one-to-one correspondence,

$(d)_\ell$ $\alpha_\ell: M_{i(\ell)} \to M_n$ is a map associated with T_ℓ, and $\alpha_\ell(M) \subseteq M$,

$(e)_\ell$ α_ℓ = Inclusion on each $q \in Q_\ell$ with $f(q \cap M) \cap N(\overline{f(N_f)}, \xi_\ell) = \phi$,

$(f)_\ell$ $(f\alpha_\ell|M, f) < A_\ell$, and

$(g)_\ell$ $\operatorname{diam} \alpha_\ell(f^{-1}(y)) < \eta_\ell$ for any $y \in Y$.

<u>Inductive step: $(S_\ell) \Rightarrow (S_{\ell+1})$, $-1 \leq \ell \leq k - 2$.</u> Using techniques developed in 2.5. replace $(Q_\ell, T_\ell, \alpha_\ell)$ by $(Q'_\ell, T'_\ell, \alpha'_\ell)$ that satisfies $P(\ell, k - \ell)$ (again, if $\ell = -1$ there is nothing to do). Performing The Isotopy Move and "absorbing" we may assume that Q'_ℓ consists of elements that underlie subcomplexes (of some stage $i' \geq i(\ell)$) and that $\alpha'_\ell(M) \subseteq M$. Then we have:

(e)' α'_ℓ = inclusion on each $q \in Q'_\ell$ with

$f(q \cap M) \cap N(\overline{f(N_f)}, \xi_\ell + k\overline{\varepsilon}) = \phi$,

(f)' $(f\alpha'_\ell|M, f) < A_\ell + k\overline{\varepsilon}$, and

(g)' $\operatorname{diam} \alpha'_\ell(f^{-1}(y)) < \eta_\ell + k\overline{\eta}$ for any $y \in Y$.

(Compare with the proof of 3.1.2.).

<u>Choosing the cores.</u> Let $\gamma: S^{\ell+1} \to M$ be a "generator" of $q \in Q'_\ell$. There are two restrictions on the choice of the "core" $\tilde{\gamma}: B^{\ell+2} \to M$ (see 2.7., steps 3-5): 1. It has to be "small" measured in Y , and 2. it must be away from a big part of the σ-Z-set N_f so that the redefined map $\alpha(0): M_{i'} \to M_n$ (as in Step 5 of 2.7.) does not stretch the fibers of f too much. Our first choice is similar to the one as in the proof of 3.1.2. We pick an extension $\tilde{\gamma}_1: B^{\ell+2} \to M$ of γ such that $\operatorname{Image}(f\tilde{\gamma}_1)$ is ε_ℓ-small in Y . We also pick a homotopy (rel $S^{\ell+1}$) $\tilde{H}: B^{\ell+2} \times I \to M$ between $\alpha'_\ell\tilde{\gamma}_1$ and a map into a single element of $\overline{Q}$, such that $\operatorname{Image}(fH)$ is ε'_ℓ-small. Next, we pass to a deeper stage such that all subsequent moves are sufficiently small. Our final choice γ of the core will

be close to $\tilde{\gamma}_1$, and so the condition 1 will be satisfied. Since N_f is a σ-Z-set, we may assume that Image($\tilde{\gamma}_1$) misses N_f . Approximate $\tilde{\gamma}_1$ by a PL-embedding $\tilde{\gamma}$, and let N be a regular neighborhood of Image($\tilde{\gamma}$) . (We follow the notation from 2.7., Step 4.) Performing The Isotopy Move, we can assume that N becomes a convenient submanifold when intersected with a deeper stage M_j . Absorbing, we approximate $\tilde{\gamma}$ by a map $\tilde{\gamma}_2$: $B^{\ell+2} \to M$. (Note that $N \cap M$ is a neighborhood of Image($\tilde{\gamma}_2$).) Again, we assume that Image($\tilde{\gamma}_2$) $\cap N_f = \phi$. Using the upper semicontinuity of the decomposition G of M induced by f , we can find a nested sequence $U_0 \supset U_1 \supset \ldots \supset U_p$ of small (in particular $\overline{U}_0 \subset \mathrm{int}(N \cap M)$) saturated neighborhoods of Image($\tilde{\gamma}_2$), where p is a fairly large number to be specified later (but depending only on data at hand). In view of later requirements, we describe a specific construction of the nested sequence. In the process we will change the core $\tilde{\gamma}$ p times. Also, we prefer to think of U_s's as being submanifolds of some stage of the defining sequence. Since we are dealing with several stages at the same time, we summarize: α_ℓ' is a map defined on M_i', and we wish to transform α_ℓ' (as in 2.7., step 5) to a map $\alpha(0)$: $M_i' \to M_n$. Various regular neighborhoods to be defined (including N that is already defined) are regular neighborhoods of polyhedra in M_i' , and underlie subcomplexes of L_j when intersected with a deep enough stage M_j . (Also, $M_j \cap N \hookrightarrow N$ induces an isomorphism of homotopy groups of dimension $\leq k - 1$) .

Say $N \cap M_{j(0)}$ is convenient. To get U_0 , approximate $\tilde{\gamma}_2$ by a PL-embedding $\hat{\gamma}_2$: $B^{\ell+2} \to M_{j(0)}$, and let U_0 be a regular neighborhood of $\mathrm{Im}(\hat{\gamma}_2)$ in $M_{j(0)}$ (and hence in $M_{i'}$). Performing The Isotopy Move, we assume that $U_0 \cap M_{j(1)}$ is convenient. Note that $U_0 \subseteq \mathrm{int}(N \cap M_{j(0)})$, since the isotopy keeps $N \cap M_{j(0)}$ invariant.

Absorbing $\hat{\gamma}_2$, we find a map $\tilde{\gamma}_3\colon B^{\ell+2} \to M$ with $\mathrm{Im}(\tilde{\gamma}_3) \subseteq \mathrm{int}\, U_0$. Using the fact that N_f is a σ-Z-set, we can assume that $\mathrm{Im}(\tilde{\gamma}_3) \cap N_f = \phi$. By upper semicontinuity, we can find a neighborhood V of $\mathrm{Im}(\tilde{\gamma}_3)$ such that no fiber of f hits both V and $\overline{M_{j(1)} - U_0}$. Go deeper in the defining sequence so that the size of the preferred triangulation $L_{j'}$ of the current stage $M_{j'}$ is small compared to the shortest distance between $\overline{V}$ and $\overline{M_{j(1)} - U_0}$. Then approximate $\tilde{\gamma}_3$ by a PL-embedding $\hat{\gamma}_3\colon B^{\ell+2} \to M_{j'}$ and let U_1 be a regular neighborhood of $\mathrm{Im}(\hat{\gamma}_3)$ in $M_{j'}$. Performing The Isotopy Move in $M_{j'}$ we assume that, in addition, that $U_1 \cap M_{j(2)}$ underlies a subcomplex of $L_{j(2)}$.

In this way we obtain a PL-core $\hat{\gamma}\colon B^{\ell+2} \to M_{i'}$ and sequences $U_0 \supset U_1 \supset \dots \supset U_p$ and $j(0) \le j(1) \le \dots \le j(p)$ such that

(1) each U_s is a regular neighborhood of $\mathrm{Im}(\hat{\gamma})$ in $M_{i'}$,

(2) $U_s \subseteq M_{j(s)}$, and

(3) no fiber of f hits both U_s and $\overline{M_{i'} - U_{s-1}}$ $(1 \le s \le p)$.

<u>Defining $\lambda\colon M_{i'} \to [0, 1]$</u> (see 2.7., Step 6). First choose a "parallel copy" $\overline{\gamma}\colon S^{\ell+1} \to \partial N \cap q$ of the generator, and an anulus $A \approx S^{\ell+1} \times I$ whose boundary components are $\overline{\gamma}(S^{\ell+1})$ and $\gamma(S^{\ell+1})$ as in 2.7., Step 4. We also set $\hat{N}(C) = U_p$. From Urysohn's Lemma [Du] it follows that there is a map $\lambda\colon M_{i'} \to [0, 1]$ such that

$\lambda(x) = 0$ if $x \notin N$, $0 \leq \lambda(x) \leq \frac{1}{2}$ if $x \notin U_0$, $\frac{1}{2} + \frac{s}{2p} \leq \lambda(x) \leq \frac{1}{2} + \frac{s+1}{2p}$ if $x \in U_s - U_{s+1}$ $(0 \leq s \leq p-1)$, and $\lambda(x) = 1$ if $x \in U_p = \hat{N}(C)$.

<u>Defining $r: M_{i'} \to M_{i'}$</u> (see 2.7., Step 4). We may assume that second derived neighborhoods $\hat{N}(C)$ of $Im(\hat{\gamma})$ and $\hat{N}(A)$ of A are chosen with respect to the same triangulation. We appeal several times to The Homotopy Extension Theorem. On $\{x \in \hat{N}(A):\lambda(x) \geq \frac{1}{2}\}$ we define r as the restriction of a retraction from $\hat{N}(A)$ onto $\hat{\gamma}(S^{\ell+1})$. We set r = Identity on $\{x \in M_{i'}:\lambda(x) = 0\}$ and extend r onto $\hat{N}(A)$ so that $r(\hat{N}(A)) \subseteq q$. (This is possible by The Homotopy Extension Theorem). Next, we extend r onto $\{x \in M_{i'}:\lambda(x) \geq \frac{1}{2}\}$ so that $r(x) \in C = Im(\hat{\gamma})$ whenever $\lambda(x) \geq \frac{1}{2}$. The key is that whenever we define x on a set, it is always homotopic to inclusion into $M_{i'}$. Finally, we extend r onto a map $r: M_{i'} \to M_{i'}$. Before we proceed, let us make a crucial remark. If we use "$\varepsilon - \delta$" version of The Homotopy Extension Theorem (if G is a compact ANR and $\varepsilon > 0$, then there is $\delta > 0$ such that if $f,g: A \to G$ are δ-close maps, and if f extends onto $\tilde{f}: X \to G$, then it follows that g extends onto $\tilde{g}: X \to G$ which is ε-close to f), and if we choose N sufficiently small, then r can be chosen to be arbitrarily close to $Id: M_{i'} \to M_{i'}$.

We proceed as in Step 5 of 2.7. to define $\alpha(0): M_{i'} \to M_n$. Let $H: B^{\ell+2} \times I \to M_n$ to be a homotopy rel $S^{\ell+1}$ such that $H(b, 0) = \alpha\hat{\gamma}(b)$, $H(b, 1) \in T(q)$ for $b \in B^{\ell+1}$. We can obtain H by adjusting the corresponding homotopy $\tilde{H}$ for $\tilde{\gamma}_1: B^{\ell+2} \to M$.

<u>Claim.</u> If N is sufficiently small, if r is sufficiently close to $\mathrm{Id}: M_{i'} \to M_{i'}$, and if $p \geq 1$ is a sufficiently large number (depending only on α_ℓ' and $\tilde{H}$) then $\alpha(0)(f^{-1}(y))$ has size $< \eta_\ell + k\overline{\eta}$, for any $y \in Y$.

Indeed, this is straightforward from the definition of $\alpha(0)$ (see 2.7. Step 5). Homotopy H acts in two "directions". There is no danger in the core direction, since if N is sufficiently small all fibers of f inside N are small. Choosing p large (using uniform continuity of H which determines epsilonics of H) we can arrange that H does not stretch the fibers of f in the I direction either.

<u>Setting up the fringe region</u>. As in Step 6 of 2.7. we can adjust $\alpha(0)$ and partition the fringe region (with a preferred ordering of the partition elements). To be able to extend the partition of the complement onto the fringe region we have to require that all elements of the partition be sufficiently small so that their image by (adjusted) $\alpha(0)$ is contained in a single element of Q. Since we are just about to perform The Isotopy Move we require, in addition, that no element of the partition of the fringe region hits both U_s and $M_{i'} - U_{s-1}$ $(s = 1,\ldots, p)$.

<u>The Isotopy Move.</u> We want to find an isotopy $\phi_t: M_{i'} \to M_{i'}$ with $\phi_0 = \mathrm{Id}$ such that ϕ_1 takes partition elements in the fringe region to convenient submanifolds (after intersecting with a deeper stage). We will also replace $\alpha(0)$ by $\alpha(0)\phi_1$. Since i' was chosen large, The Isotopy Move is small in the core direction. The

only danger is to stretch partition elements across (in I-direction of the homotopy). Recall at this point that neighborhoods U_s were chosen so that $U_s \cap$ deeper stage is a convenient manifold. Moreover, if we look at the sequence $U_s = U_s \cap M_{i'} = U_s \cap M_{j(s)} \supseteq U_s \cap M_{j(s)+1} \supseteq \cdots \supseteq U_s \cap M_{\text{deep stage}}$ we see the following. Each element is obtained from the previous one by taking away a <u>regular neighborhood</u> of a (m-k-1)-dimensional polyhedron. It follows that we can perform an isotopy move $\psi_t \colon M_{i'} \to M_{i'}$ <u>keeping</u> U_s's <u>invariant</u> (and hence not stretching the partition elements), so that ψ_1(partition element) $\cap$ (some deeper stage) is still a (k-1)-connected manifold. The same applies to non-empty intersections of partition elements. Consequently, we can pick a deep enough stage so that The Isotopy Move as described in 2.2. is small (even in I-direction), and hence does not stretch. The composition of the two isotopies gives us the required isotopy ϕ_t . Observe that $\alpha(0)$ does not stretch the fibers of f to a size $\geq \eta_{\ell+1}$ (by (g)' and (5)) .

<u>Extending the partition</u> (Step 6 of 2.7.). In this step we use assumption (*) . As in 2.7., step 6, let $A[j]$ be a submanifold of some stage M_r . The partition is already defined on a submanifold of M_r and the map $\alpha(j-1) \colon M_r \to M_n$ tells us that we want to "amalgamate" $A[j]$ with $T(j-1)^{-1}(\overline{q})$ (notation as in 2.7. Step 6). Denoting $p = A[j] \cup T(j-1)^{-1}(\overline{q})$ we find a polyhedron P satisfying (a)-(e) as in Step 6 of 2.7. Note that the redefined map $\alpha(j)$ is mesh $\overline{Q}$-close to $\alpha(j)$. It follows that only those fibers of f that are mapped by $\alpha(j-1)$ to a set of size $\geq \eta_{\ell+1} - \overline{\eta}$ are in danger

of getting stretched to a size $\geq \eta_\ell$. Note that the union of such fibers forms a Z-set in M , and also (by (*)) in each copy of μ^{k-s} of the form $M \cap q_1 \cap \ldots \cap q_{s+1}$. We will readjust P so that it does not intersect the "dangerous" Z-set. We also have to take into account The Isotopy Move which may move P to hit the Z-set. However, we will prevent that from happening by finding a polyhedron P (maybe in some even deeper stage) such that the distance between P and the Z-set is large compared to the size of the preferred triangulation of that stage.

We start with a "shrunken" copy P_1 of the dual k-skeleton of the triangulation L_v restricted to $St(T(j-1)^{-1}(\bar{q}), Q(j-1)) \cup A[j]$. A careful "absorbing" as in 2.3. gives us a map $g\colon P_1 \to M$ such that $g(x) \in \sigma \cap M$ for every $x \in P_1 \cap \sigma$ (σ is a simplex of L_v) . Using (*) and 2.1.4. we inductively on skeleta define a map $g'\colon P_1 \to M$ that approximates g and whose image misses the "dangerous" Z-set. As a part of inductive hypotheses we require that $g'(x) \in \sigma \cap M$ whenever $x \in P_1 \cap \sigma$ (σ is a simplex of L_v) . Next, go deeper in the defining sequence to find an index $v' \geq v$ such that mesh $L_{v'}$ is small compared to the shortest distance between $Im(g')$ and the "dangerous" Z-set. Finally, approximate g' by a PL-embedding $g''\colon P_1 \to M_{v'}$ such that $P = g''(P_1)$ has required properties. Using such P and a small enough regular neighborhood of P (as in Step 6 of 2.7.) we realize that the redefined map $\alpha(j)$ does not stretch the fibers of f to a size $\geq \eta_{\ell+1}$.

This concludes the inductive argument. We learned that the statement (S_{k-1}) is valid. It gives us a (convenient) k-partition

Q_{k-1} , a one-to-one correspondence $T_{k-1}: Q_{k-1} \to \overline{Q}$ and a map $\alpha_{k-1}: M_{i(k-1)} \to M_n$ associated with T_{k-1} . Also, $(e)_{k-1}$, $(f)_{k-1}$, and $(g)_{k-1}$ hold. The triple $(Q_{k-1}, T_{k-1}, \alpha_{k-1})$ gives rise to a homeomorphism $h: M \to M$ (as at the end of the proof of 3.1.2.) with properties that correspond to $(e)_{k-1}$, $(f)_{k-1}$, $(g)_{k-1}$; namely

(E) $h(x) = x$ if $x \in M$ with $f(x) \notin N(\overline{f(N_f)}, \xi_{k-1})$,

(F) $(fh, f) < A_{k-1}$, and

(G) diam $h(f^{-1}(y)) < \eta_{k-1}$ for any $y \in Y$.

By 4.1.1., f is a strong near-homeomorphism.

4.3. The Main Shrinking Theorem

In this section we use results of 4.1 and 4.2. to prove a more general shrinking theorem. Rather than assuming that the nondegeneracy set of the map is "small", we hypothesize that the decomposition space is "nice". The strategy of proof follows Edwards' program for Euclidean manifolds, and consists of successive "improvements" of the given map. The ultimate improvement is a homeomorphism approximating the given map.

<u>4.3.1. Theorem.</u> Let M be a compact triangulated μ^k-manifold, and $f: M \to Y$ a UV^{k-1}-surjection. If $\dim Y = k$ and if Y satisfies DD^kP, then f is a strong near-homeomorphism.

<u>Proof.</u> Let U be a neighborhood of $\overline{f(N_f)}$ in Y , and let $\varepsilon > 0$. We will find a homeomorphism $h: M \to Y$ such that $h = f$ on $f^{-1}(Y - U)$ and $(f, h) < \varepsilon$.

Step 1. As in 2.3.8. we find a countable dense subset $\{\alpha_1, \alpha_2, \ldots\}$ of $\{\alpha: I^k \to Y\}$ such that each α_i is an embedding, and $\alpha_i(I^k) \cap \alpha_j(I^k) = \phi$ whenever $i \neq j$. In this step we approximate f by a UV^{k-1}-surjection $f': M \to Y$ such that f' is 1-1 over $\bigcup_{i=1}^{\infty} \alpha_i(I^k)$. We obtain f' as the limit of UV^{k-1}-surjections $f = f_0, f_1, \ldots$ which we construct inductively. The relevant controls include some obvious ones:

$(1)_i$ $(f_i, f_{i-1}) < \dfrac{\varepsilon}{3 \cdot 2^i}$,

$(2)_i$ $f_i | f_i^{-1}(C_i): f_i^{-1}(C_i) \to C_i$ is a homeomorphism, where $C_i = \alpha_i(I^k)$,

$(3)_i$ $f_i = f_j$ on $f_j^{-1}(C_j)$, $j < i$.

These three conditions guarantee that $f' = \lim_{i\to\infty} f_i$ is a UV^{k-1}-surjection that is $\frac{\varepsilon}{3}$-close to f. The control that prevents points from $M - f_i^{-1}(C_i)$ slipping (in the limit process) into C_i is more cumbersome. After f_i has been defined, we choose a sequence $U_i^{i+1} \supseteq U_i^{i+2} \supseteq \ldots$ of neighborhoods of $f_i^{-1}(C_i)$ that intersects down to $f_i^{-1}(C_i)$. We impose a condition that guarantees that the image of a point in $M - U_j^i$ stays away from C_j:

$(4)_i$ $(f_i, f_{i-1}) < \frac{1}{3} \cdot \mathrm{dist}(f_{i-1}(M - U_j^i), C_j)$, $j < i$

(here $\mathrm{dist}(A, B) = \inf\{d(a, b): a \in A, b \in B\}$)

The inductive argument goes as follows. Assume that f_{i-1} is constructed. Conditions $(1)_i$ and $(4)_i$ give us $\delta > 0$ such that we need to get $(f_i, f_{i-1}) < \delta$. Also, to get $(3)_i$ we need to have $f_i = f_{i-1}$ on $f_{i-1}^{-1}(C_1) \cup \ldots \cup f_{i-1}^{-1}(C_{i-1})$. Let G be the decomposition of M induced over C_i (i.e. G consists of the fibers $f^{-1}(y)$, $y \in C_i$ and singletons). We get a commutative diagram

$$\begin{tikzcd} & M \arrow[dl, "\pi"'] \arrow[dd, "f_{i-1}"] \\ M/G \arrow[dr, "P"'] & \\ & Y \end{tikzcd}$$

where $\pi: M \to M/G$ is the decomposition map, and $P: M/G \to Y$ is the induced map. Note that $P|P^{-1}(C_i): P^{-1}(C_i) \to C_i$ is a homeomorphism, and that $P^{-1}(C_i) \subseteq M/G$ is a Z-set (a straightforward argument). Also note that $\pi(N_\pi) \subseteq P^{-1}(C_i)$, and hence by 4.1.3. π is a (strong) near-homeomorphism. Let $H: M \to M/G$ be a homeomorphism approximating π such that $H = \pi$ on $f_{i-1}^{-1}(C_1) \cup \dots \cup f_{i-1}^{-1}(C_{i-1})$. Finally, let $f_i = P\,H$. If H is close enough to π, we have $(f_i, f_{i-1}) < \delta$ which gives $(1)_i$ and $(4)_i$. It is clear from the construction that $(2)_i$ and $(3)_i$ are satisfied. To get the "strong" part of the theorem, we need to know that $f' = f$ on $f^{-1}(Y - U)$. We get this requiring in the inductive process that

$(5)_i$ $\quad f_i = f_{i-1}$ on $f_{i-1}^{-1}(Y - U)$.

<u>Step 2.</u> We $\frac{\varepsilon}{3}$-approximate f' by a UV^{k-1}-surjection $f'': M \to Y$ such that $N_{f''}$ is a σ-Z-set in M that satisfies the technical condition (*) from 4.2.2. (The σ-Z-set Shrinking Theorem). Once this is done, we appeal to 4.2.2. concluding that f'' can be $\frac{\varepsilon}{3}$-approximated by a homeomorphism, which in turn is an ε-approximation to f.

We start by carefully choosing a countable family $C = \{\beta_1, \beta_2, \ldots\}$ of mappings from cells (of various dimensions $\leq k$) into M such that each β_i is an embedding, $\mathrm{Im}(\beta_i) \cap \mathrm{Im}(\beta_j) = \phi$ when $i \neq j$ and, most important of all, any σ-compact subset N of $M - \bigcup_{i=1}^{\infty} \mathrm{Im}\, \beta_i$ is a σ-Z-set that satisfies (*) from 4.2.2. We will make sure that if $M' \subset M$ is a copy of μ^{k-s} obtained by intersecting a convenient submanifold of one of the stages M_i with M $(0 \leq s \leq k)$, then C contains a countable dense family of embedded (k-s)-cells in M'.

Say $\{M(1, s), M(2, s), \ldots\}$ is the collection of all copies of μ^{k-s} in M that come from (countably many) convenient submanifolds of dimension $m - s$ (in various stages) and let $M(0, 0) = M$. For each pair (i, s) fix a disjoint union $X(i, s)$ of countably many (k-s)-cells and let X denote the disjoint union $\bigcup_{i,s} X(i, s)$. Let $\alpha: X \to M$ be a map such that $\alpha|X(i, s)$ gives a countable dense family of maps $I^{k-s} \to M(i, s)$. We topologize $T = \{\beta: X \to M: \beta(X(i, s)) \subseteq M(i, s)$, for all $i, s\}$ defining a neighborhood basis for $\beta: X \to M$. Fix a map $\varepsilon: X \to (0, \infty)$ and let $N(\beta; \varepsilon) = \{\beta' \in T: d(\beta'(x), \beta(x)) < \varepsilon(x)\}$. Varying ε, we get a basis of neighborhoods for β. Clearly, T is not a first countable space, but a routine argument shows that it is a Baire space. Cover X by countably many cells by covering $X(i, s)$ by countably many (k-s) cells (all i, s) so that if $x, y \in X$, $x \neq y$, then there are disjoint cells σ, τ in the chosen collection such that $x \in \sigma$, $y \in \tau$.

Let $T(\sigma, \tau) = \{\beta \in T: \beta(\sigma) \cap \beta(\tau) = \phi\}$. An argument similar to the one given in the proof of 2.3.1. shows that $T(\sigma, \tau)$ is an open and dense subset of T. Consequently, the intersection T_0 of all sets of the form $T(\sigma, \tau)$ for chosen cells σ, τ is a dense G_δ subset of T. Observe that T_0 consists of 1-1 maps, and hence any map $\beta \in T$ can be approximated by a map $\beta' \in T_0$. In particular, we can replace α by $\alpha' \in T_0$ requiring that each $\alpha' | X(i,s)$ be also a countable dense family of maps $I^{k-s} \to M(i, s)$. (This will happen if $\varepsilon: X \to (0, \infty)$ is small enough.) The map α' gives the desired family C .

Continuing the proof of 4.3.1. we recall our goal: To approximate $f': M \to Y$ by a map $f'': M \to Y$ such that $N_{f''}$ is contained in the complement of $\bigcup_{i=1}^{\infty} \operatorname{Im} \beta_i$. Again we obtain f'' as the limit of maps $f_i': M \to Y$ of the form $f_i' = f'h_i \dots h_2h_1$, where $h_j: M \to M$ is a homeomorphism, $j = 1,2,\dots$. The relevant control is

$(1)_i$ h_i is fixed on $h_{i-1} \dots h_1(\operatorname{Im}\beta_1 \cup \dots \operatorname{Im}\beta_{i-1})$,

$(2)_i$ $h_i h_{i-1} \dots h_1(\operatorname{Im}\beta_i) \subseteq f'^{-1}(C)$, for some cell C over which f' is 1-1, and

$(3)_i$ $(fh_i, f) < \dfrac{\varepsilon}{3 \cdot 2^{i+1}}$.

Also, for each i we fix a decreasing sequence $U_i^{i+1} \supset U_i^{i+2} \supset \dots$ of neighborhoods of $\operatorname{Im} \beta_i$ whose intersection is $\operatorname{Im} \beta_i$, and we require that

$(4)_i$ $(f'h_i \dots h_1, f'h_{i-1} \dots h_1)$

$$< \frac{1}{3} \cdot \operatorname{dist}(f'h_{i-1} \dots h_1(M - U_j^i), f'h_{i-1} \dots h_1(\operatorname{Im}\beta_j) \quad \text{for } j < i .$$

The reader can verify that $f'' = \lim_{i\to\infty} f_i'$ satisfies the required properties.

The inductive step. Assuming that $h_1, \ldots, h_{i-1}$ have been constructed, we show how to find h_i. There is $\delta > 0$ such that if $(f'h_i, f') < \delta$, then h_i satisfies $(3)_i$ and $(4)_i$. The map $f'h_{i-1} \ldots h_1\beta_i\colon I^{k-s} \to Y$ can be approximated by a map $\gamma\colon I^{k-s} \to C \subseteq Y$, where C is one of the cells over which f' is one-to-one. We apply 3.1.1. to get a homeomorphism $h_i\colon M \to M$ (small measured in Y) such that $h_i(h_{i-1} \ldots h_1\beta_i(I^{k-s}))$ is contained in $f'^{-1}(C)$. (Note that both $h_{i-1} \ldots h_1\beta_i(I^{k-s})$ and $f'^{-1}(C)$ are Z-sets.) Also, h_i can be chosen to be fixed outside a neighborhood of $f'h_{i-1} \ldots h_1\beta_i(I^{k-s})$, which gives $(1)_i$. If we require also

$$(5)_i \quad h_i = \mathrm{Id} \quad \text{on} \quad f'^{-1}(Y - U)$$

we realize that $f'' = f$ on $f'^{-1}(Y - U)$, which implies that f is a strong near-homeomorphism.

The following result can be proved directly, using the partition argument.

4.3.2. Corollary. A map $f\colon M \to N$ between triangulated μ^k-manifolds is a near-homeomorphism if and only if f is a UV^{k-1}-surjection.

CHAPTER 5

THE CHARACTERIZATION THEOREM

In the first section of this chapter we prove The Resolution Theorem which establishes a link between arbitrary LC^{k-1}-spaces and μ^k-manifolds. This theorem, coupled with The Main Shrinking Theorem, yields The Characterization Theorem.

5.1. The Resolution Theorem

In what follows, Y denotes an arbitrary locally (k-1)-connected compact metric space. A resolution of Y is a pair (M, f), where M is a triangulated μ^k-manifold and $f: M \to Y$ is a UV^{k-1}-surjection. The goal of this section is to show that Y has a resolution.

<u>5.1.1. Remark.</u> Using partition techniques we could prove the uniqueness theorem; namely, if (M_1, f_1) and (M_2, f_2) are resolutions of Y, then for every $\varepsilon > 0$ there is a homeomorphism $h: M_1 \to M_2$ such that $(f_2h, f_1) < \varepsilon$. Without any work, from 2.8.6. it follows that $M_1 \approx M_2$.

Before outlining the strategy, we introduce a device of improving a resolving map.

<u>5.1.2. Proposition.</u> Let $f: M \to Y$ be a resolution of Y, and let $Z \subset M$ be a Z-set. Let $g: Z \to Y$ be any map with $g \underset{\mu}{\simeq} f|Z$ (see 2.1.7.). Then there is a resolution $f': M \to Y$ such that $f'|Z = g$.

Proof. We note that this is a "categorical" consequence of the Z-set unknotting theorem. By 3.1.3. there is a sequence $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \dots > 0$ such that if $h: Z_1 \to Z_2$ is a homeomorphism between Z-sets in M with $(fh, f|Z_1) < \varepsilon_i$, then h extends to a homeomorphism $\tilde{h}: M \to M$ with $(f\tilde{h}, f) < \varepsilon_{i-1}$ $(i = 2,3,\dots)$. We also assume that $\varepsilon_i < \frac{1}{2^i}$, for all i. Denote by $j: Z \to M$ the inclusion map. Let $\alpha_0: Z \to M$ be a Z-embedding such that $(f\alpha_0, g) < \frac{\varepsilon_2}{2}$ (since f is UV^{k-1}, we can lift g approximately, and then approximate the lift by a Z-embedding). We can also assume, by 2.1.8.(i), that $f\alpha_0 \underset{\mu}{\simeq} g$. By assumption $g \underset{\mu}{\simeq} fj$, and hence by 2.1.8.(ii) $f\alpha_0 \underset{\mu}{\simeq} fj$. Consequently, by 2.1.8.(iii), $\alpha_0 \underset{\mu}{\simeq} j$. Invoking 3.1.4., we produce a homeomorphism $h_0: M \to M$ such that $h_0|Z = \alpha_0$. Let $\alpha_i: Z \to M$ be a Z-embedding with $(f\alpha_i, g) < \frac{\varepsilon_{i+2}}{2}$, $i \geq 1$. The choice of ε_i's produces a sequence $\{h_i: M \to M\}$ of homeomorphisms such that $h_i|\alpha_{i-1}(Z) = \alpha_i\alpha_{i-1}^{-1}|\alpha_{i-1}(Z)$, and $(fh_i, f) < \varepsilon_i$. (Note that $(f\alpha_i, f\alpha_{i-1}) < \varepsilon_{i+1}$.) Let $f_i = fh_i h_{i-1} \dots h_1 h_0$. The reader can verify that $(f_i, f_{i-1}) < \varepsilon_i < \frac{1}{2^i}$, $i \geq 1$, and hence $f' = \lim_{i\to\infty} f_i$ is a well defined UV^{k-1}-surjection. Also, $(f_i|Z, g) < \frac{\varepsilon_{i+2}}{2}$ which, taking the limit, implies that $f'|Z = g$.

Clearly, a resolving map $f: M \to Y$ induces an isomorphism of homotopy groups of dimension $\leq k - 1$. We start by producing a finite polyhedron P and a map $f: P \to Y$ that has the same property.

5.1.3. Proposition. For every compact LC^{k-1}-space Y there exist a finite polyhedron P with dimension $\leq k$ and a map $f: P \to Y$ such that $f_\#: \pi_i(P) \to \pi_i(Y)$ is an isomorphism for $i \leq k - 1$.

Proof. For $k = 0$ there is nothing to prove, and for $k = 1$ we can use a finite set P whose cardinality is equal to the number of components of Y. So we assume that $k \geq 2$ and that Y is connected.

Choose a sequence $\varepsilon_{k-1} > \varepsilon_{k-2} > \dots > \varepsilon_0$ of positive numbers such that every map $\alpha: S^{i+1} \to Y$ whose image is $4\varepsilon_i$-small extends to a map $\tilde{\alpha}: B^{i+1} \to Y$ with ε_{i+1}-small image $(i = 0,1,\dots, k-2)$. Also require that if $\alpha, \beta: (S^j, *) \to (Y, *)$ are ε_{k-1}-close maps, then $\alpha \simeq \beta$ (rel $*$). We can certainly choose $\varepsilon_{k-1} > 0$ such that there is a small homotopy between α and β. During this homotopy the basepoint describes a small loop which is trivial for $k \geq 2$.

Let $\mathcal{V}$ be a finite cover of Y such that any partial realization of a $\leq k$ dimensional simplicial complex into $\mathrm{St}\,\mathcal{V}$ extends to a full realization such that all simplexes are mapped to sets of size $< \varepsilon_0$. Choose a partition of unity $\{\psi_V : V \in \mathcal{V}\}$ subordinate to $\mathcal{V}$ such that $\psi_V(x_V) = 1$ for some $x_V \in V$. Then $\{\psi_V\}$ defines a map $\psi: Y \to |N(\mathcal{V})| = P_0$. For a vertex V of $N(\mathcal{V})$ define $\phi(V) = x_V$. Then ϕ is a partial realization of $N(\mathcal{V})$ into $\mathrm{St}\,\mathcal{V}$, so it extends to a full realization $\phi: |N(\mathcal{V})^{(k)}| \to Y$ as promised by choice of $\mathcal{V}$. From now on, we will view $P_0 = |N(\mathcal{V})|$ and $P_0^{(k)} = |N(\mathcal{V})^{(k)}|$ as CW-complexes. Using The Cellular Approximation Theorem, find a cellular approximation $h^0: P_0^{(k)} \to P_0$ to $\psi\phi: P_0^{(k)} \to P_0$. Since $\psi\phi | P_0^{(0)} =$ inclusion we have $h^0 | P_0^{(p)} =$ inclusion.

Setting $f_0 = \phi$, $g_0 = \psi$ and choosing $\mathcal{V}$ sufficiently small, we realize that the following statement is true for $i = 0$.

(S_i) There exist a finite CW-complex P_i and maps $f_i\colon P_i^{(k)} \to Y$, $g_i\colon Y \to P_i$, and $h^i\colon P_i^{(k)} \to P_i$, such that

$(1)_i$ $g_i f_i | P_i^{(0)} =$ inclusion,

$(2)_i$ h_i is a cellular approximation to $g_i f_i$ such that $(f_i h_i, f_i) < \varepsilon_i$,

$(3)_i$ if e is a cell of $P_i^{(k)}$, then $\operatorname{diam} f_i(e) < \varepsilon_i$,

$(4)_i$ there is a homotopy $H_i\colon P_i^{(i)} \times I \to P_i^{(k)}$ between inclusion and h_i such that $d(f_i H(x,t), f_i(x)) < \varepsilon_i$ for $x \in P_i^{(i)}$, $t \in I$,

$(5)_i$ if $\alpha\colon W \to Y$ is a map of a CW-complex W with $\dim W \leq k$, then there is a cellular approximation $\beta\colon W \to P_i^{(k)}$ to $g_i\alpha$ such that $(f_i\beta, \alpha) < \varepsilon_i$, and

$(6)_i$ P_i is obtained from P_{i-1} by attaching $(i+1)$-cells $(i > 0)$.

<u>Inductive Step $((S_j)\colon 0 \leq j \leq i) \Rightarrow (S_{i+1})$, $i = 0,1,\ldots,k-2$.</u>

Let e be a $(i+1)$-cell of P_i. We attach a $(i+2)$-cell E to P_i using the attaching map $\gamma\colon \partial(B^{i+1} \times I) = B^{i+1} \times \{0\} \cup B^{i+1} \times \{1\} \cup \partial B^{i+1} \times I \to P_i^{(i+1)}$. On $B^{i+1} \times \{0\}$ the map γ is defined as the characteristic map χ of e. On $\partial B^{i+1} \times I$ we define $\gamma(x,t) = H_i(\chi(x), t)$. Then $\gamma|\partial B^{i+1} \times \{1\}$ can be extended to $\gamma'\colon B^{i+1} \times \{1\} \to P_i$ using a homotopy between h^i and $g_i f_i$, and the map $g_i f_i \chi$. Then let $\gamma|B^{i+1} \times \{1\}$ be a cellular approximation to this extension. The following algorithm of getting cellular approximations guarantees that $f_i\gamma(B^{i+1} \times \{1\})$ will be a small set. Since

$B^{i+1} \times \{1\}$ is compact, $\gamma'(B^{i+1} \times \{1\})$ hits only finitely many cells of P_i. By $(6)_0$-$(6)_i$ cells of P_i of dimension $> i + 1$ are simplexes, and we can homotope γ' off of the interior of such simplexes, one at a time. The important observation is that γ' and $\gamma|B^{i+1} \times \{1\}$ will be close, as well as homotopic. It follows that $f_i\gamma(\partial B^{i+1} \times I)$ is a $4\varepsilon_i$-small set. Hence we can extend f_i onto E to get f_i' such that $f_i'(E)$ is ε_{i+1}-small in Y. Extend h^i onto E by approximating $g_i f_i'|E$. Attaching a $(i+2)$-cell for each $(i+1)$-cell in P_i, extending f_i and h^i as described, and setting $g_{i+1} = g_i$, we obtain P_{i+1}, f_{i+1}, h^{i+1}, and g_{i+1}. Finally, define $H_{i+1}\colon P_{i+1}^{(i+1)} \times I \to P_i^{(k)}$ on $P_{i+1}^{(i)}$ as H_i and on $e \times I$ using the attached cell E. (Specifically, $H_{i+1}(x,t) = \tilde{\gamma}(\chi^{-1}(x), t)$, where $\tilde{\gamma}$ is the characterastic map for t.) The reader can easily check $(1)_{i+1}$ - $(6)_{i+1}$.

The verification that the map $f_{k-1}\colon P_{i-1}^{(k)} \to Y$ promissed by (S_{k-1}) induces an isomorphism of homotopy groups of dimension $\leq k - 1$ is straightforward. Clearly the CW-complex $P_{k-1}^{(k)}$ can be built using PL attaching maps so that $P_{k-1}^{(k)}$ is a polyhedron.

<u>5.1.4. Corollary.</u> There exist a triangulated μ^k-manifold N and a map $f\colon N \to Y$ such that $f_\#\colon \pi_i(N) \to \pi_i(Y)$ is an isomorphism for $i \leq k - 1$.

<u>Proof.</u> Model N on the regular neighborhood of the polyhedron $P \subset E^{2k+1}$ as in 5.1.3.

The map we want to put our hands on is constructed in the next proposition. The overall strategy resembles West's [We] (in resolving

ANR's by Q-manifolds). However, there are several important differences. We do not encounter simple-homotopy obstruction, which allows us to produce the resolution map inductively, improving the connectivity of the fibers. On the other hand, our universal space μ^k (or a μ^k-manifold) contains only k-dimensional spaces, which causes some minor technical difficulties in dealing with arbitrary LC^{k-1}-compacta. The critical fact that allows us to successfully approach this problem comes from 2.1.9.

<u>5.1.5. Proposition.</u> Let $\alpha: X \to Y$ be a UV^{k-1}-surjection, such that $\dim X \leq k$ (see 2.1.9.). Then there is a triangulated μ^k-manifold N, an embedding $X \subseteq N$ and a map $r: N \to Y$ extending α such that r induces an isomorphism of homotopy groups of dimension $\leq k - 1$.

The reader is advised to think of α as being the identity map (i.e. $\dim Y \leq k$).

<u>Proof.</u> Let $\mathcal{U}$ be a small finite cover of X of order $\leq k + 1$.

<u>Step 1.</u> Let $p: X \to |N(\mathcal{U})|$ be a "projection" map. Using the fact that Y is LC^{k-1}, build a map $g: |N(\mathcal{U})| \to Y$ such that gp and α are close. In particular, we can arrange that $gp \underset{\mu}{\simeq} \alpha$ (see 2.1.8.).

<u>Step 2.</u> Let $f: N \to Y$ be a map from a triangulated μ^k-manifold as in 5.1.4. We build a map $\beta: |N(\mathcal{U})| \to N$ such that $f\beta \underset{\mu}{\simeq} g$ (or, equivalently, $f\beta| |N(\mathcal{U})^{(k-1)}| \simeq g| |N(\mathcal{U})^{(k-1)}|$). To that end, we

construct by induction on i a homotopy $H_i\colon |N(\mathcal{U})^{(i)}| \times I \to Y$ and $g\,|\,|N(\mathcal{U})^{(i)}|$ $(i = 0,1,\ldots,k-1)$. We get H_{i+1} as an extension of H_i. Let σ be a $(i+1)$-simplex of $N(\mathcal{U})$. Then $H_i \cup g$ is defined on $\partial\sigma \times I \cup \sigma \times \{1\}$, and β_i is defined on $\partial\sigma$. Observe that $f \circ \beta_i|\partial\sigma$ is null homotopic in Y. Since f induces a monomorphism of i^{th} homotopy groups, it follows that $\beta_i|\partial\sigma$ is null-homotopic in N. It remains to show that there is an extension $\beta_{i+1}\colon \sigma \to N$ of β_i such that the map $H_i \cup f\beta_{i+1}|\partial(\sigma \times I)\colon \partial(\sigma \times I) \to Y$ is null-homotopic. Using the fact that f induces an epimorphism of $(i+1)^{st}$ homotopy groups, we can pick any extension $\beta'_{i+1}\colon \sigma \to N$ of β_i, and then redefine β'_{i+1} (similarly to what was done in 2.7.3.) to obtain the desired map β_{i+1}. To get H_{i+1}, we extend $H_i \cup f\beta_{i+1}|\partial(\sigma \times I)$ onto $\sigma \times I$.

Repeating the first part of the argument (using that $f_\#\colon \pi_{k-1}(N) \to \pi_{k-1}(Y)$ is a monomorphism), we can define $\beta = \beta_k$, extending H_{k-1}. This is the sought after map.

Step 3. By 2.3.8. we can approximate $\beta p\colon X \to N$ by a Z-embedding $j\colon X \to N$. Since $f|j(X)\colon j(X) \to Y$ and $\alpha j^{-1}\colon j(X) \to Y$ are μ-homotopic mpas $(fj \simeq_\mu f\beta p \simeq_\mu gp \simeq_\mu \alpha)$, The μ-Homotopy Extension Theorem 2.1.8.(iv) asserts that f is μ-homotopic to a map that extends $\alpha|j(X)$.

The next two propositions are critical ingredients in the inductive construction of a resolving map.

<u>5.1.6. Proposition.</u> Suppose $M_1, M_2, M_0 = M_1 \cap M_2$ are triangulated μ^k-manifolds, and M_0 is a Z-set in both M_1 and M_2. Then $M_1 \cup M_2$ admits a resolution.

<u>Proof.</u> Let $\gamma_i: P_i \to M_i$ $(i = 0,1,2)$ be a map from a finite polyhedron inducing an isomorphism of homotopy groups of dimension $\leq k - 1$. As in step 2 of 5.1.5. we can find maps $j_1: P_0 \to P_1$, $j_2: P_0 \to P_1$ such that $\gamma_i j_i \underset{\mu}{\simeq} \gamma_0$. Replacing P_i by $P_i \times I^{2k+1}$ $(i = 1,2)$ we can assume that j_i is an embedding, and hence $P_0 \subseteq P_i$. Let $P = P_1 \cup (P_0 \times I) \cup P_2$ be obtained by identifying the top and the bottom level of $P_0 \times I$ with $P_0 \subseteq P_1$ and $P_0 \subseteq P_2$ respectively. We may assume that P is a subpolyhedron of some Euclidean space E^m (m large). Let $\overline{M}_1, \overline{M}_2$ be small regular neighborhoods of P_1, P_2 in E^m, and let $\overline{M}_0$ be a small regular neighborhood of $P_0 \times I$ minus $\operatorname{int}(M_1 \cup M_2)$. Denote by $r_i: \overline{M}_i \to P_i$ the retraction $(i = 1,2)$, and let M_i' be the μ^k-manifold obtained from $\overline{M}_i$ (the initial triangulation of $\overline{M}_i$ subdivides $\overline{M}_i \cap \overline{M}_0$). The map $M_i' \hookrightarrow \overline{M}_i \xrightarrow{r_i} P_i \xrightarrow{\gamma_i} M_i$ induces an isomorphism of homotopy groups of dimension $\leq k - 1$. (see 2.3.2.). By 2.8.6. there is a homeomorphism $h_i: M_i' \to M$ μ-homotopic to $\gamma_i r_i | M_i'$. Let M_0' be the μ^k-manifold obtained from $\overline{M}_0$. Similarly, there is a map $r_0: \overline{M}_0 \to P_0 \times I$ inducing an isomorphism of homotopy groups of dimension $\leq k - 1$ and extending r_i's. Then the map $M_0' \hookrightarrow \overline{M}_0 \xrightarrow{r} P_0 \times I \xrightarrow{pr} P_0 \xrightarrow{\gamma_0} M_0$ induces an isomorphism of homotopy groups of dimension $\leq k - 1$. Let $h_0: M_0' \to M_0$ be a homeomorphism μ-homotopic to this map. By M' denote the μ^k-manifold obtained from $\overline{M}_1 \cup \overline{M}_0 \cup \overline{M}_2$. Clearly,

$M' = M_1' \cup M_0' \cup M_2'$. To show that there is a UV^{k-1}-surjection $M' \to M_1 \cup M_2$ it suffices to find a UV^{k-1}-surjection $h_i': M_i' \to M_i$ extending $h_0|M_i' \cap M_0'$. (Then $h = h_1' \cup h_0 \cup h_2': M' \to M_1 \cup M_2$ is the desired map). Note that $M_i' \cap M_0'$ is a Z-set in M_i' , and hence, in order to apply 5.1.2., we need to know that $h_0|M_i' \cap M_0' \simeq_\mu h_i|M_i' \cap M_0'$ as maps into M_i . They are actually μ-homotopic as maps into M_0 . Indeed,

$$h_0|M_i' \cap M_0' \simeq_\mu (M_i' \cap M_0' \to \overline{M}_i' \cap \overline{M}_0 \xrightarrow{r_0|} P_0 \xrightarrow{\gamma_0} M_0)$$

$$= (M_i' \cap M_0' \hookrightarrow \overline{M}_i \cap \overline{M}_0 \xrightarrow{r_i} P_0 \xrightarrow{\gamma_i} M_0) \simeq_\mu h_i|M_i' \cap M_0' .$$

We say that Y is i-resolvable $(0 \le i \le k-1)$, if there is a triangulated μ^k-manifold M and a UV^i-surjection $f: M \to Y$.

<u>5.1.7. Proposition.</u> Suppose A, B, and $A \cap B$ are LC^{k-1}-spaces such that A,B are (k-1)-resolvable, and $A \cap B$ is i-resolvable. Then $A \cup B$ is min(i+1, k-1)-resolvable.

<u>Proof.</u> Let $\alpha: N_A \to A$ and $\beta: N_B \to B$ be UV^{k-1}-surjections, and let $\gamma: N \to A \cap B$ be a UV^i-surjection (N, N_A, N_B are triangulated μ^k-manifolds). Since α, β are UV^{k-1} , we can approximately lift γ to $\gamma_A: N \to N_A$, $\gamma_B: N \to N_B$. We may also assume that γ_A, γ_B, are Z-embeddings, and $\alpha\gamma_A \simeq_\mu i\gamma$, $\beta\gamma_B \simeq_\mu j\gamma$ ($i: A \cap B \to A$, $j: A \cap B \to B$ are inclusions). By 5.1.2., there are UV^{k-1}-surjections $\alpha': N_A \to A$, $\beta': N_B \to B$ such that $\alpha'\gamma_A = \gamma$, $\beta'\gamma_B = \gamma$. Finally, $\alpha' \cup \beta'$: $N_A \cup N_B \to A \cup B$ ($\gamma_A(N)$ is identified with $\gamma_B(N)$) is a

UV^s-surjection, $s = \min(i+1, k-1)$. (The union of two UV^{k-1}-compacta with UV^i-intersection is UV^s) . By 5.1.6., $N_A \cup N_B$ is $(k-1)$-resolvable, and thus $A \cup B$ is s-resolvable.

We are now ready to prove the resolution theorem.

5.1.8. Theorem. For any LC^{k-1}-compactum Y there is a compact triangulated μ^k-manifold M and a UV^{k-1}-surjection $f: M \to Y$.

Proof. Let $\alpha: X \to Y$ be a UV^{k-1}-surjection with $\dim X \leq k$ (as in 2.1.9), and let $r: N \to Y$ be a map as in 5.1.5. Fix Z-embeddings $h_0: N \to N$, $h_1: N \to N$ with disjoint images and denote $N_0 = \mathrm{Im}(h_0)$, $N_1 = \mathrm{Im}(h_1)$. We can assume that h_0 and h_1 are μ-homotopic to $\mathrm{Id}: N \to N$. The reader should think of N_0 and N_1 as the top and the bottom levels of a mapping cylinder structure defined on N . In the category of μ^k-manifolds we remedy the lack of stability (which is helpful in the theory of Q-manifolds) by changing the definition of the mapping cylinder. We also need the "collapse" $c: N \to N_1$. We require that c be a UV^{k-1}-surjection, and that $c|N_1 = \mathrm{Id}$ and $c|N_0 = h_1 h_0^{-1}$. Such a map can be constructed using 5.1.2. applied to the resolving map $h_1: N \to N_1$ and the Z-set $N_0 \cup N_1 \subset N$.

Let $(N', N_0', N_1', h_0', h_1', c')$ be a duplicate of $(N, N_0, N_1, h_0, h_1, c)$ with $h: N \to N'$ the identification homeomorphism. (In particular, $hh_0 = h_0'h$ etc.). We now follow [We] to define a space that resolves Y . Then, by induction on i , we show that this space is i-resolvable $(i = 0,1,\ldots,k-1)$. First define a space T which is obtained from the disjoint sum $N \cup N'$ identifying $z \in N_1 \subset N$ with $z' \in N_0' \subset N'$ provided $z' \in h_0'(X)$ and $r(h_1^{-1}(z)) = \alpha(h^{-1}h'^{-1}(z'))$; and also identifying $z \in N_0 \subset N$ with $z' \in N_1' \subset N'$ provided $z \in h_0(X)$ and $r(h^{-1}h'^{-1}(z')) = \alpha(h_0^{-1}(z))$. By $p: N \cup N' \to T$ denote the decomposition map, and let $\phi: T \to Y$ be defined as:

1. $\phi p|N: N \to Y$ is the composition $N \xrightarrow{c} N_1 \xrightarrow{h_1^{-1}} N \xrightarrow{r} Y$
2. $\phi p|N': N' \to Y$ is the composition
$$N' \xrightarrow{c'} N_1' \xrightarrow{h_1^{-1}} N' \xrightarrow{h^{-1}} N \xrightarrow{r} Y .$$

It is straightforward that ϕ is well-defined. For $y \in Y$, let $g_y = p(c^{-1}h_1\ \alpha^{-1}(y) \cup c'^{-1}h_1'\ h\alpha^{-1}(y)) \subseteq Y$. The reader can check that $\phi(g_y) = \{y\}$, for all $y \in Y$, and that the collection G consisting of $\{g_y: y \in Y\}$ and singletons is an upper semi-continuous decomposition of T . Consequently, ϕ induces a map $\tilde{\phi}: T/G \to Y$. Notice that $\tilde{\phi}^{-1}(y)$ is homeomorphic to the image under a UV^{k-1}-map of the wedge of two copies of $c^{-1}(h_1\ r^{-1}(y))/A$, where $A = h_1\ r^{-1}(y) \cup c^{-1}(h_1\ \alpha^{-1}(y))$. It follows that $\tilde{\phi}$ is a UV^{k-1}-surjection. To set off the inductive argument, we represent T/G as $T_1 \cup T_2$ where $T_1 = \pi p(N) \cup \pi p(N_0')$, $T_2 = \pi p(N') \cup \pi_p(N_0)$ ($\pi: T \to T/G$ is the decomposition map). Note that $T_1 \cap T_2 = \pi p(N_0) \cup \pi p(N_0')$. If we denote by $\tilde{N}$ the adjunction space $N \cup_\alpha Y$, then $T_1 \cap T_2$ is the union of two copies of $\tilde{N}$ along a subset homeomorphic to Y . We will prove that both T_1 and T_2 are $(k-1)$-resolvable. Once having that, we start an inductive argument. Assume Y is i-resolvable, $i < k - 1$. Since $\tilde{N}$ is $(k-1)$-resolvable, (by definition), it follows from 5.1.7. that $T_1 \cap T_2$ is (i+1)-resolvable. Again, by 5.1.7., $T_1 \cup T_2$ is min(i+2, k-1)-resolvable, and so is Y as the image of $T_1 \cup T_2$ under a UV^{k-1}-map.

Hence, to finish the argument, we need to show that T_1 (and T_2) is (k-1)-resolvable. Note that the "mapping cylinder" $p(N) \cup p(N_0') \subseteq T$ resolves T_1 (via π). The map $p: N \to p(N) \cup p(N_0')$ is one-to-one over the complement of $p(N_0')$. Let N'' be the union of two copies of N sewn together along $N_1 \subset N$. We claim that N'' (which is (k-1)-resolvable by 5.1.6.) resolves $p(N) \cup p(N_0')$. On the first copy of N in N'' define f as p. It remains to define a UV^{k-1}-map $g: N \to p(N_0')$ that agrees with f on the common copy of N_1, i.e. if we identify $p(N_0')$ with $\tilde{N} = N \cup_\alpha Y$ we need to find a UV^{k-1}-map $g: N \to \tilde{N}$ such that $g|N_1 = q\ r\ h_1^{-1}$, where $q: N \cup Y \to \tilde{N}$ is the decomposition map. There is an obvious UV^{k-1}-map $N \to \tilde{N}$ (restrictions of q). Note that $qr\ h_1^{-1}: N_1 \to \tilde{N}$ induces an isomorphism of homotopy groups of dimension $\leq k-1$ (r does by assumption, h_1^{-1} is a homeomorphism, and $q|Y: Y \xrightarrow{\subseteq} N$ from the equality $\alpha = ri$, $i: X \to N$ is the inclusion map). Let $\psi: N_1 \to N$ be an approximate lift of $qr\ h_1^{-1}$ with respect to $q|N: N \to \tilde{N}$. Then $\psi_\#: \pi_j(N_1) \to \pi_j(N)$ is an isomorphism for $j \leq k-1$. By 2.8.6., there is a homeomorphism $\tilde{\psi}: N_1 \to N$ μ-homotopic to ψ. Then $q\tilde{\psi}: N_1 \to \tilde{N}$ is a UV^{k-1}-surjection μ-homotopic to $qr\ h_1^{-1}: N_1 \to \tilde{N}$. We now apply 5.1.2. to the map $q\tilde{\psi}c: N \to \tilde{N}$ (which is UV^{k-1}) and the Z-set $N_1 \subset N$, to get another UV^{k-1}-surjection $g: N \to \tilde{N}$ that agrees on N_1 with $qr\ h_1^{-1}$.

5.2. The Characterization Theorem

Finally, as a dessert, we can derive the characterization theorem.

5.2.1. Theorem. Suppose X is a compact LC^{k-1}-space with $\dim X = k$ that satisfies the disjoint k-cells property. Then X is homeomorphic to a triangulated μ^k-manifold.

Proof. By 5.1.8. (The Resolution Theorem) there exist a compact triangulated μ^k-manifold M and a UV^{k-1}-surjection $f: M \to X$. Applying 4.3.1. (The Main Shrinking Theorem), we find out that $X \approx M$.

5.2.2. Corollary. (The Triangulation Theorem). Any compact μ^k-manifold (i.e. a space every point of which has a neighborhood homeomorphic to an open subset of μ^k) is homeomorphic to a triangulated μ^k-manifold.

Proof. Follows directly from 5.2.1. and 2.8.5.

5.2.3. Corollary. A compact metric space X is homeomorphic to μ^k if and only if it is (k-1)-connected, locally (k-1)-connected, k-dimensional, and satisfies the disjoint k-cells property.

Proof. By 5.2.1. X is a triangulated μ^k-manifold. It follows that $X \approx \mu^k$ by 2.8.11.

There are several constructions of universal k-dimensional spaces appearing in the literature. The best known one is due to K. Menger [Mg] (although he consider only the case $k = 1$). Start with the cube $Q_0 \subset E^m$, $m \geq 2k + 1$, and subdivide it into 3^m congruent cubes. Keep the cubes adjacent to the k-skeleton of Q_0 and delete others to get Q_1 . Repeat the process on each cube that is left over. Successively obtain a sequence $Q_0 \supset Q_1 \supset Q_2 \supset \ldots$, and let $M_k^m = \bigcap_{i=1}^{\infty} Q_i$. A routine argument shows that M_k^m $(m \geq 2k + 1)$ satisfies conditions of 5.2.3. and consequently $M_k^m \approx \mu^k$.

Replacing cubes by simplexes in the above description, we get the Lefschetz construction of the universal k-dimensional space [Lf] . Again 5.2.3. implies that this space is homeomorphic to μ^k .

The construction of Pasynkov's [Pa] is somewhat different. Let $\{U_i\}_{i=1}^{\infty}$ be a basis of open sets of S^k , so that each U_i is an open cell with the property that $\operatorname{diam}(U_i) \to 0$. The universal space P_k can be obtained as the inverse limit of spaces X_i with bonding maps $f_i\colon X_{i+1} \to X_i$ constructed inductively. We set $X_0 = S^k$, and we obtain X_{i+1} from X_i by "bubbling over U_i" , i.e. X_{i+1} is the quotient space gotten from the disjoint union $X_i \cup X_i$ identifying the two copies of $x \in X_i$ precisely when $f_{0i}(x) \notin U_i$, where $f_{0i}\colon X_i \to X_0$ is the bonding map. The map $f_i\colon X_{i+1} \to X_i$ is induced by the obvious map $X_i \cup X_i \to X_i$. A helpful observation is that X_i can be identified with a subset of X_{i+1} in which case the bonding maps become retractions. It is easy to check that P_k satisfies DD^kP , that $\dim P_k = k$, and that P_k is a locally connected continuum. If we are careful with the choice of U_i's (e.g. we can choose "round" open balls) then P_k will also be LC^{k-1} and C^{k-1} , and thus homeomorphic to μ^k . Indeed, in that case we can argue inductively on the number of bubbles that X_{i+1} is obtained from X_i by adding a (k-1)-connected set along a (k-s)-connected set. (Use inductive hypotheses on ∂U_i). In particular, $(f_{0i}^{-1}(U_i), f_{0i}^{-1}(\partial U_i))$ is a (k-1)-connected pair and hence any map $\phi: S^j \to X_{i+1}$ $(j < k)$ can be homotoped (by a small homotopy) to a map into X_i. If $\phi: S^j \to P_k$ is a small map, then the sequence $f_0\phi, f_1\phi, f_2\phi, \ldots$ converges to ϕ ($f_i: P_k \to X_i$ is the bonding map). Inserting a (small) homotopy between $f_i\phi$ and $f_{i+1}\phi$, we get that ϕ is homotopic (via a small homotopy) to $f_r\phi$, which is a small map into X_r, and thus null-homotopic. It follows that P_k is LC^{k-1} and that each X_i is (k-1)-connected. In particular, P_k is C^{k-1}, and the claim follows.

CHAPTER 6

NONCOMPACT MENGER MANIFOLDS

In this chapter we generalize work done in the first five chapters to the noncompact case. The proofs usually consist of repeating the arguments for the compact case, with the standard changes (using covers rather than epsilonics). We will give relevant definitions and major statements leading to a proof of The Characterization Theorem, only pointing out places in the proof of the corresponding statements for the compact case that do not obviously generalize to the noncompact case.

A partition on a (noncompact, but separable) PL m-manifold M is a locally finite collection P of compact PL m-submanifolds of M that satisfies (1) and (2) from 1.1.1. The partition P is a k-partition on M if properties (3)-(5) from 1.1.13. hold.

Let M_1^m be a PL manifold with a (combinatorial) triangulation L_1, $m \geq 2k+1$. Produce a sequence $(M_1, L_1), (M_2, L_2), \ldots$ as in 1.2. and let $M = \bigcap_{i=1}^{\infty} M_i$. Then M is a triangulated μ^k-manifold. As in 2.3. we see that any μ^k-manifold (i.e. a space every point of which has a neighborhood homeomorphic to an open subset of μ^k) is k-dimensional, LC^{k-1}, and satisfies DD^kP.

A closed subset $A \subseteq M$ is a Z-set if any map $f\colon I^k \to M$ can be arbitrarily closely approximated by a map whose image misses A. Equivalently, A is a Z-set in M if for any open cover $\mathcal{U}$ of M there is a map $f\colon M \to M - A$ such that f is $\mathcal{U}$-close to the identity.

<u>The Z-set Approximation Theorem.</u> If B is a locally compact space with $\dim B \leq k$, then any proper map $f\colon B \to M$ can be approximated (up to a given open cover of M) by a closed Z-embedding. Moreover, if $B_0 \subseteq B$ is a closed subset of B, and if $f|B_0$ is a closed Z-embedding, then the approximation can be chosen to agree with f on B_0.

If A is a closed subset of a locally compact space X with $\dim X \leq k$, if Y is a locally compact LC^{k-1}-space, and if $f,g\colon A \to Y$ are proper maps, we say that f and g are properly μ-homotopic, $f \simeq_{\mu} g$, provided there are a neighborhood U of A and extensions $\tilde{f}, \tilde{g}$ of f, g respectively such that for every proper map $\alpha\colon P \to U$ from a polyhedron P with $\dim P \leq k - 1$ the maps $\tilde{f}\alpha$ and $\tilde{g}\alpha$ are properly homotopic in Y.

We have seen earlier that if there is a map $f\colon M_1 \to M_2$ between compact μ^k-manifolds that induces an isomorphism on homotopy groups of dimension $\leq k$, then $M_1 \approx M_2$. In the noncompact case we have to add that M_1 and M_2 have "matching ends". We say that a proper map $f\colon X \to Y$ between locally compact spaces induces an epimorphism of j^{th} homotopy groups of ends $(j \geq 0)$ if for every compactum $C \subseteq Y$ there exists a compactum $K \subseteq Y$ such that for each point $x \in X - f^{-1}(K)$ and every map $\alpha\colon (S^j, *) \to (Y - K, f(x))$ there exist a map $\tilde{\alpha}\colon (S^j, *) \to (X - f^{-1}(C), x)$ and a homotopy $f\tilde{\alpha} \simeq \alpha$ (rel $*$) in $Y - C$. Similarly, we say that $f\colon X \to Y$ induces a monomorphism of j^{th} homotopy groups of ends if for every compactum $C \subseteq Y$ there exists a compactum $K \subseteq Y$ such that for every map $\tilde{\alpha}\colon S^j \to X - f^{-1}(K)$

with the property that $f\tilde{\alpha}$ is null-homotopic in $Y - K$ it follows that $\tilde{\alpha}$ is null-homotopic in $X - f^{-1}(C)$.

If $f,g\colon M_1 \to M_1$ are properly μ-homotopic maps then f induces an epimorphism (monomorphism) of j^{th} homotopy groups of ends $(0 \leq j \leq k - 1)$ iff g does.

<u>Theorem.</u> Let $f\colon M_1 \to M_2$ be a proper map between locally compact triangulated μ^k-manifolds that induces an isomorphism of homotopy groups of dimension $\leq k - 1$ and an isomorphism (i.e. both monomorphism and epimorphism) of homotopy groups of ends of dimension $\leq k - 1$.

Then f is properly μ-homotopic to a homeomorphism.

<u>Sketch of proof.</u> Let $M_1 = \bigcap_{n=1}^{\infty} M_1(n)$, $M_2 = \bigcap_{n=1}^{\infty} M_2(n)$. As before, we can find a small k-partition $\overline{Q}$ on $M_2(n)$, for some $n \geq 1$. We can extend f (viewing it as a map into $M_2(n)$) onto a neighborhood of M_1 . This neighborhood may not contain an element of the defining sequence for M_1 , and hence we have to work with sequences of stages, rather than with stages themselves. We can choose a countable increasing collection $M_1^1 \subset M_1^2 \subset M_1^3 \subset \ldots \subset M_1^s \subset \ldots$ of compacta in $M_1(1)$ such that the union of their interiors is $M_1(1)$, and then look at the sequence $M_1(n_1) \supseteq M_1(n_2) \supseteq M_1(n_3) \supseteq \ldots \supseteq M_1(n_s) \supseteq \ldots$ of stages such that the extension of f is defined on $M_1(n_1, n_2, \ldots) = (M_1(n_1) \cap M_1^1) \cup (M_1(n_2) \cap M_1^2) \cup (M_1(n_3) \cap M_1^3) \cup \ldots$. By trimming back, we may assume, that $M_1(n_s) \cap M_1^s$ is a submanifold of $M_1(n_s)$ and underlies a subcomplex

of L_{n_s} . As in 2.4. f induces a triple (Q, T, α) , where Q is a partition on $M_1(n_1, n_2, \ldots)$. Proceed by turning Q into a k-partition. In the process we will have to "pass to a deeper stage" several times (i.e. replace $M_1(n_1, n_2, \ldots)$ by $M_1(n_1', n_2', \ldots)$) . The move described in 2.5. (connecting intersection) has local nature, and is adopted here without changes. The real danger is in the move described in 2.7. (increasing the connectivity of partition elements). We will have to choose infinitely many "cores" to kill a locally finite collection of generators. If infinitely many of them hit the same compactum, we will not have enough room to choose pairwise disjoint regular neighborhoods of the cores. However, this will not happen since our hypotheses about f guarantee that cores can be chosen near ∞ if the generators are near ∞ .

<u>The Z-set Unknotting Theorem.</u> All versions of The Z-set Unknotting Theorem for the compact case listed in 3.1. (3.1.1., 3.1.3., 3.1.4.) generalize to the noncompact case after replacing numbers $\varepsilon, \delta > 0$ by open covers, and using the notion of properly μ-homotopic maps rather than μ-homotopic. (We first prove The Proper μ-Homotopy Extension Theorem).

The σ-Z-set Shrinking Theorem (4.2.2.) also generalizes, leading to a proof of The Main Shrinking Theorem: Let M be a (locally compact) triangulated μ^k-manifold and $f: M \to Y$ a proper UV^{k-1}-surjection. If $\dim Y = k$, and if Y satisfies DD^kP , then f is a strong near-homeomorphism.

Finally, The Resolution Theorem (every locally compact LC^{k-1}-space Y admits a triangulated μ^k-manifold and a proper UV^{k-1}-surjection $f: M \to Y$) can be proved using the strategy for compact case. The fact that every locally compact space Y admits a locally compact space X and a UV^{k-1}-surjection $f: X \to Y$ with $\dim X \le k$ follows from 2.1.7. by taking an embedding $Y \to Q$, and restricting the map $f: X \to Q$ (as in 2.1.7.) onto $f^{-1}(Y)$. The analogue of 5.1.3. states that every locally compact LC^{k-1}-space admits a locally finite polyhedron P with $\dim P \le k$, and a proper map $f: P \to Y$ that induces an isomorphism of homotopy groups of dimension $\le k - 1$, and of homotopy groups of ends of dimension $\le k - 1$.

We close this chapter by stating The Characterization Theorem and The Triangulation Theorem.

<u>The Characterization Theorem.</u> Suppose X is a locally compact LC^{k-1}-space with $\dim X = k$ that satisfies the disjoint k-cells property. Then X is homeomorphic to a triangulated μ^k-manifold.

<u>The Triangulation Theorem.</u> Any μ^k-manifold (i.e. a space every point of which has a neighborhood homeomorphic to an open subset of μ^k) is homeomorphic to a triangulated μ^k-manifold.

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VITA

Mladen Bestvina was born in Osijek, Yugoslavia, on December 1, 1959. He graduated from Osijek High School in Osijek, Yugoslavia, in 1978. He holds a Bachelor of Science degree in Mathematics from the University of Zagreb at Zagreb, Yugoslavia (1982). In 1982, he began graduate study at The University of Tennessee, Knoxville, where he was awarded the Doctor of Philosophy degree in Mathematics in August 1984.