The G_2-Hitchin Component of Triangle Groups: Dimension and Integer Points

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To my family and many friends whom I never would have made it this far without.

To my parents and my twin brother, who supported my decision to attend graduate school and have never questioned my thirst for knowledge.

To Dr. Anna Lawson, who was my constant confidant and companion throughout this program, and her family, who cheered me on just as much as they had for Anna.

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Abstract

The image of $\mathrm{PSL}(2,\mathbb{R})$ under the irreducible representation into $\mathrm{PSL}(7,\mathbb{R})$ is contained in the split real form $G_2^{4,3}$ of the exceptional Lie group G_2. This irreducible representation therefore gives a representation ρ of a hyperbolic triangle group $\Gamma(p,q,r)$ into $G_2^{4,3}$, and the *Hitchin component* of the representation variety $\mathrm{Hom}(\Gamma(p,q,r),G_2^{4,3})$ is the component of $\mathrm{Hom}(\Gamma(p,q,r),G_2^{4,3})$ containing ρ.

This thesis is in two parts: (i) we give a simple, elementary proof of a formula for the dimension of this Hitchin component, this formula having been obtained earlier in [Alessandrini et al.], [3], as part of a wider investigation using Higgs bundle techniques, and (ii) we prove the existence of an infinite sequence of integer points on the G_2-Hitchin component of the (2,4,6)-triangle group.

One reason for studying hyperbolic triangle groups is that they contain surface groups as subgroups of finite index. Integer representations in Hitchin components then often provide examples of surface groups represented as elusive *thin matrix groups*, see [Sarnak] [16], [Long and Reid] [12], and [Kontorovich et al.] [11].

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Chapter 1

Concepts underlying the main results

In this section, definitions are given of mathematical terms used in the statements and proofs of the two main theorems.

1.1 Definitions pertaining to Hitchin components

1.1.1 Triangle groups

Despite the different ways to view triangle groups geometrically, topologically, or algebraically, the algebraic description (and matrix representations) of these groups $\Gamma(p, q, r)$, also often denoted as $T(p, q, r)$ or $\triangle(p, q, r)$, will be used the most often here. However, to obtain the group representation and matrix representations of the generators, the geometric interpretation of triangle groups as rotational symmetries of a tiling of a unit 2-sphere, Euclidean plane, or hyperbolic plane by geodesic triangles with angles π/p, π/q, and π/r is utilized.

The topological interpretation of these triangle groups gives motivation for studying triangle groups, as $\Gamma(p, q, r)$ is the orbifold fundamental group of a sphere with cone points of order p, q, and r. Hyperbolic triangle groups contain fundamental groups of compact surfaces (surface groups) as subgroups of finite index, and part of the motivation for studying hyperbolic triangle groups is to provide more information about these fundamental groups.

For a triangle ABC as in Figure 1.1, not necessarily Euclidean, let ϕ_1, ϕ_2, ϕ_3 be reflections in the sides BC, CA, AB, respectively. Then, $a = \phi_2 \circ \phi_3$ is the counter-clockwise rotation through 2α about the vertex A. Similarly, $b = \phi_3 \circ \phi_1$ is the rotation through 2β about B, and $c = \phi_1 \circ \phi_2$ is the rotation through 2γ about C. The case where

$$\alpha = \frac{\pi}{p}, \quad \beta = \frac{\pi}{q}, \quad \gamma = \frac{\pi}{r}$$

for positive integers p, q, $r \geq 2$, is of particular interest as then the rotations a, b, c have finite orders p, q, r respectively. Also, note that since a reflection always has order 2,

$$abc = (\phi_2\phi_3)(\phi_3\phi_1)(\phi_1\phi_2) = 1.$$

The angle sum $\alpha + \beta + \gamma$ can be greater than, equal to, or less than π corresponding to triangles on the unit 2-sphere, the Euclidean plane, or the hyperbolic plane. Triangles are *spherical* on the unit 2-sphere, *Euclidean* on the Euclidean plane, and *hyperbolic* on the hyperbolic plane.

Let Γ be the group of isometries generated by the reflections a, b, c. These isometries, applied to the triangle ABC, generate a *tiling* of the unit 2-sphere, Euclidean plane, or hyperbolic plane by triangles all congruent to the original triangle.

For example, for $(p, q, r) = (2, 3, 5)$, the subdivision of the icosahedral (or dodecahedral) tiling of the sphere is as shown in figure 1.2. For $(p, q, r) = (3, 3, 3)$, the usual tiling of the (Euclidean) plane by equilateral triangles is obtained, and for $(p, q, r) = (2, 3, 7)$, figure 1.3 is obtained for the subdivision of the tiling of the hyperbolic plane by regular heptagons.

The (p, q, r)-triangle group $\Gamma(p, q, r)$ is the subgroup of Γ consisting of all orientation-preserving isometries of the upper half plane $\mathbb{H}^2$ that map the tiling into itself, i.e. $\Gamma(p, q, r)$ is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^2)$.

As is noted in [14], the product of the three rotations going around the triangle is always $-I$; however, this is not an issue as 2×2 matrices representing isometries are only defined up to sign. $\Gamma(p, q, r)$ is generated by the ρ_i, and the relations between the ρ_i above form a complete set of relations for the group, which can be verified with the Seifert-van Kampen

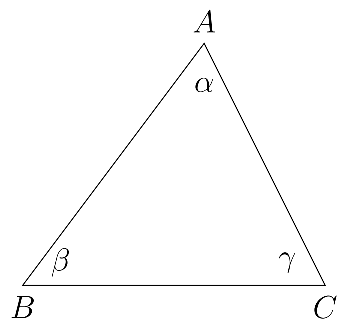

Figure 1.1: Triangle ABC, not necessarily Euclidean

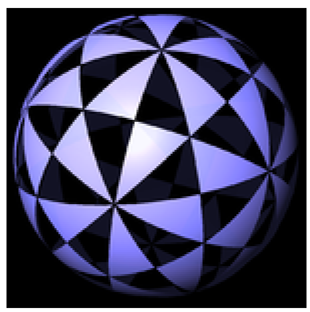

Figure 1.2: Tiling of the 2-sphere by triangles with angles $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{5}$.

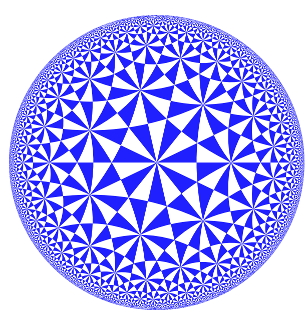

Figure 1.3: Tiling of the hyperbolic plane by triangles with angles $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{7}$.

theorem, given below:

$$\Gamma(p,q,r) = \langle\, a,\ b,\ c\ :\ a^p = b^q = c^r = abc = 1\,\rangle.$$

If the space (the unit 2-sphere, Euclidean plane, or hyperbolic plane) is factored out by the action of $\Gamma(p,q,r)$, the result is a 2-sphere with three cone points whose orders are p, q, r. The quotient map is an orbifold covering map, and since the covering space is simply connected, the orbifold fundamental group of this orbifold is naturally isomorphic to the group $\Gamma(p,q,r)$.

1.1.2 Linear groups and their projectivizations

Definition 1.1. *Let $\mathbb{F}$ be a field, and let n be a positive integer. The* ***general linear group*** $\mathrm{GL}(n,\mathbb{F})$ *is the group of nonsingular $n \times n$ matrices with entries in $\mathbb{F}$, under matrix mulitplication. The* ***special linear group*** $\mathrm{SL}(n,\mathbb{F})$ *is the subgroup of* $\mathrm{GL}(n,\mathbb{F})$ *consisting of matrices of determinant 1.*

Definition 1.2. *The* ***projective (general) linear group*** *is the quotient group*

$$\mathrm{PGL}(n,F) = \mathrm{GL}(n,\mathbb{F})/\,\mathrm{Z},$$

where $\mathrm{GL}(n,\mathbb{F})$ *is the general linear group of $n \times n$ matrices with entries in $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and* Z *is the subgroup of* $\mathrm{GL}(n,\mathbb{F})$ *of all nonzero scalar transformations of $\mathbb{F}^n$, i.e. the center of the general linear group.*

Definition 1.3. *The* ***projective special linear group*** *is the quotient group*

$$\mathrm{PSL}(n,\mathbb{F}) = \mathrm{SL}(n,\mathbb{F})/\,\mathrm{SZ},$$

where $\mathrm{SL}(n,\mathbb{F})$ *is the special linear group of $n \times n$ matrices with entries in $\mathbb{F}$ and* SZ *is the subgroup of* $\mathrm{SL}(n,\mathbb{F})$ *of scalar transformations with determinant 1, i.e.* SZ *is the center of* $\mathrm{SL}(n,\mathbb{F})$.

For $n > 2$ odd, as the only scalar transformation with determinant 1 in $\mathrm{SL}(n, \mathbb{R})$ is I, the center of $\mathrm{SL}(n, \mathbb{R})$ is trivial. So, for $n > 2$ odd, $\mathrm{PSL}(n, \mathbb{R}) = \mathrm{SL}(n, \mathbb{R})$.

1.1.3 Linear representations of groups

Definition 1.4. *A **linear representation** of a group G is a homomorphism of G into the multiplicative group $\mathrm{GL}(n, \mathbb{F})$ of non-singular $n \times n$ matrices over $\mathbb{F}$, for some positive integer n and some field $\mathbb{F}$.*

There now follows an extended example central to this thesis, namely the *holonomy representation* of a hyperbolic triangle group into $\mathrm{PSL}(2, \mathbb{R})$.

An upper half-plane model may be used for the construction of the hyperbolic triangles as this model is conformal, i.e. angle measurements are preserved. After constructing a triangle in $\mathbb{H}^2$ with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$, and $\frac{\pi}{r}$ as shown in Figure A.1, one can obtain the matrix representations for a, b, c and the reflections ϕ_1, ϕ_2, and ϕ_3 in the sides of the triangle.

Without loss of generality, one may assume that the smaller semicircle has a radius of 1 since dilation is an isometry in the hyperbolic plane. The center of the smaller semi-circle is at $\cos(\pi/p)$ on the real axis, and the center of the larger semicircle is at $-r_2 \cos(\pi/q)$, where r_2 is its radius. r_2 can be calculated in terms of the angles using elementary trigonometry of overlapping circles, as r_2 is a solution to the quadratic equation:

$$2r_2c_3 - \left(1 + r_2^2 - (c_1 + r_2c_2)^2\right) = 0,$$

which, taking the positive root, gives

$$r_2 = \frac{-c_1c_2 - c_3 + \sqrt{-1 + c_1^2 + c_2^2 + 2c_1c_2c_3 + c_3^2}}{-1 + c_2^2},$$

where $c_1 = \cos\left(\frac{\pi}{p}\right)$, $c_2 = \cos\left(\frac{\pi}{q}\right)$, and $c_3 = \cos\left(\frac{\pi}{r}\right)$ for simplicity. Now that r_2 is in terms of the angles, 2×2 matrix representations can be found for the reflections in the sides of the triangle.

A reflection in the hyperbolic plane is (Euclidean) inversion in a geodesic, and in the special case where the geodesic is the semicircle of unit radius centered at 0, this is $z \mapsto \frac{1}{\bar{z}}$. A reflection is orientation reversing, which is why complex conjugation is present in this expression. A matrix can be used to represent this as with Möbius transformations:

$$\frac{(0)\overline{z}+1}{(1)\overline{z}+0} \quad \to \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If the geodesic is a semicircle of radius r centered at 0, the reflection is

$$z \mapsto \frac{r^2}{\overline{z}} \quad \to \quad \begin{pmatrix} 0 & r^2 \\ 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & r \\ 1/r & 0 \end{pmatrix},$$

where for convenience scale the matrix so that its determinant is -1.

An elementary result in the geometry of the hyperbolic plane $\mathbb{H}^2$ is that the orientation reversing isometries $\mathbb{H}^2$ are precisely those given by the transformations

$$z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d} \text{ with } a,\, b,\, c,\, d \in \mathbb{R}, \quad ad - bc = -1\,,$$

and that orientation preserving isometries are precisely those given by transformations

$$z \mapsto \frac{az+b}{cz+d} \text{ with } a,\, b,\, c,\, d \in \mathbb{R}, \quad ad - bc = 1\,,$$

corresponding to the fact that, as in Euclidean geometry, each orientation preserving isometry can be expressed as the product of two orientation reversing isometries.

Moreover, the value of a fraction $\frac{az+b}{cz+d}$ is unchanged by reversing signs of all a, b, c, d, so orientation preserving isometries of $\mathbb{H}^2$ are represented by elements of the projective special linear group $\mathrm{PSL}(2,\mathbb{R})$. Indeed, this correspondence gives rise to an isomorphism between the group $\mathrm{Isom}^+(\mathbb{H}^2)$ of orientation preserving isometries of the hyperbolic plane with the projective special linear group $\mathrm{PSL}(2,\mathbb{R})$.

For the inversion in a semicircle of radius r and center at $(k, 0)$ on the real axis, the previous matrix is conjugated by the isometry $z \mapsto z + k$ that moves the origin to the center $(k, 0)$:

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & r \\ 1/r & 0 \end{pmatrix} \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} k/r & -k^2/r + r \\ 1/r & -k/r \end{pmatrix}.$$

Let ϕ_1 be the reflection across the y-axis. Let ϕ_2 be the reflection about the side of the triangle overlapping the semicircle of radius 1, and let ϕ_3 be the reflection about the side of the triangle overlapping the semicircle of radius r_2. Then:

$$\phi_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -c_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c_1 & 1 - c_1^2 \\ 1 & -c_1 \end{pmatrix},$$

$$\phi_3 = \begin{pmatrix} 1 & -r_2 c_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & r_2 \\ 1/r_2 & 0 \end{pmatrix} \begin{pmatrix} 1 & r_2 c_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -c_2 & r_2(1 - c_2^2) \\ 1/r_2 & c_2 \end{pmatrix}.$$

Now, the generators of the triangle group $a = \phi_3 \circ \phi_1$, $b = \phi_1 \circ \phi_2$, and $c = \phi_2 \circ \phi_3$ can be found in terms of 2×2 (invertible) matrices.

1.1.4 Irreducibility of a representation

Definition 1.5. *Let F be a subfield of an algebraically closed field K, and let Γ be a subgroup of $\mathrm{GL}(n, F)$. The action of Γ on F^n is **irreducible** if there does not exist a proper, non-trivial subspace of F^n invariant under Γ, and the action is **absolutely irreducible** if there does not exist a non-trivial subspace of K^n invariant under Γ, regarded as a subgroup of $\mathrm{M}(n, K)$.*

*Let $\phi : \Gamma \to \mathrm{GL}(n, F)$ be a linear representation of a group Γ. The representation ϕ is **irreducible** if the action $\phi(\Gamma)$ on F^n is irreducible, and ϕ is **absolutely irreducible** if the action of $\phi(\Gamma)$ on F^n is absolutely irreducible.*

1.1.5 The irreducible representation of $\mathrm{PSL}(2,\mathbb{R})$ in $\mathrm{PSL}(n,\mathbb{R})$

From the representation theory of $\mathrm{SL}(2,\mathbb{R})$, there exists for $n \geq 3$ an irreducible representation ρ_n, unique up to conjugacy, of $\mathrm{SL}(2,\mathbb{R})$ in $\mathrm{SL}(n,\mathbb{R})$. Let V be the real vector space of homogeneous polynomials of degree $n-1$ in two variables x, y. Then V has dimension n over $\mathbb{R}$, with basis $\mathcal{B} = \{x^{n-1},\, x^{n-2}y,\, \ldots,\, xy^{n-2},\, y^{n-1}\}$, and we define a linear action of $\mathrm{SL}(2,\mathbb{R})$ on V as follows. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \;, \qquad \begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{pmatrix}.$$

Then for $f(x,y) \in V$ we define a linear map $T_A : V \to V$ by

$$T_A(f(x,y)) \;=\; f(x',\, y') \;,$$

and take $\rho_n(A)$ to be the transpose of the matrix of T_A with respect to the basis $\mathcal{B}$. The reason for taking the transpose is to ensure that ρ_n is a group homomorphism, *i.e.* that $\rho_n(AB)$ is equal to $\rho_n(A)\rho_n(B)$ rather than $\rho_n(B)\rho_n(A)$.

Let $I_n \in \mathrm{SL}(n,\mathbb{R})$ denote the identity $n \times n$ matrix. It is readily checked that

$$\rho_n(-I_2) \;=\; \begin{cases} -I_n\,, & (n \;\text{ even}) \\ I_n\,, & (n \;\text{ odd}) \end{cases}$$

and so ρ_n induces a representation $\mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R})$, which (with mild abuse of notation) we also denote ρ_n, and which makes the following diagram commute:

$$\begin{tikzcd} \mathrm{SL}(2,\mathbb{R}) \arrow[r, "\rho_n"] \arrow[d] & \mathrm{SL}(n,\mathbb{R}) \arrow[d] \\ \mathrm{PSL}(2,\mathbb{R}) \arrow[r, "\rho_n"] & \mathrm{PSL}(n,\mathbb{R}) \end{tikzcd}$$

In the above diagram, the vertical homomorphisms are the natural quotient maps.

Since $\mathrm{PSL}(2,\mathbb{R})$ is a simple group, the representation $\rho_n : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R})$ is faithful, and its restriction to a hyperbolic triangle group, regarded as a subgroup of $\mathrm{PSL}(2,\mathbb{R})$, is a faithful representation of that group.

Having to deal with $\mathrm{PSL}(n,\mathbb{R})$ rather than with $\mathrm{SL}(n,\mathbb{R})$ is in general a mild technical obstacle, but in this thesis we will be concerned solely with the case $n = 7$, whereby this obstacle vanishes on account of 7 being odd.

1.1.6 Lie group

Definition 1.6. *A* ***real Lie group*** *is a group that is also a finite-dimensional real smooth manifold, in which the group operations of multiplication and inversion are smooth maps.*

Here we will be dealing exclusively with *matrix* or *linear* Lie groups, *i.e.* Lie groups that are subgroups of $\mathrm{GL}(n,\mathbb{C})$ for some n. The *closed subgroups theorem* asserts that a subgroup G of $\mathrm{GL}(n,\mathbb{C})$ is a Lie group if G is topologically closed in $\mathrm{GL}(n,\mathbb{C})$.

1.1.7 Representation variety and character variety

Let Γ be a finitely presented group, with presentation consisting of generators $\gamma_1, \ldots, \gamma_r$ and relators $r_1, \ldots, r_s$, and let $G \subset \mathrm{GL}(n,\mathbb{R})$ be a real Lie group.

If $\rho : \Gamma \to \mathrm{GL}(n,\mathbb{R})$ is a representation, then for each $1 \leq i \leq s$, we have $\rho(r_i) = I$, I being the identity $n \times n$ matrix, and this determines a system of polynomial equations in matrix entries of the matrices $\rho(\gamma_j)$, for which the representation ρ is essentially a solution. The set of all representations from Γ to $\mathrm{GL}(n,\mathbb{R})$ is therefore the set of solutions to this system of polynomial equations, and is thus a real affine algebraic variety called the *representation variety* of Γ in $\mathrm{GL}(n,\mathbb{R})$.

If instead ρ is a representation of Γ into a proper subgroup G of $\mathrm{GL}(n,\mathbb{R})$, the above system of polynomial equations is augmented by extra equations corresponding to the constraint of ρ being in G, and the set of all representations of Γ into G is the set of solutions to this augmented system.

Definition 1.7. *For G a subgroup of $\mathrm{GL}(n,\mathbb{R})$, we denote the representation variety of Γ in G by $\mathrm{Hom}(\Gamma, G)$.*

Taking Γ, G as above, often it is convenient to consider that representations ρ, ρ' of Γ into G are *equivalent* if $\rho' = \tau \circ \rho$ for some inner automorphism τ of G. The set of equivalence classes is then the *character variety* $\chi(\Gamma, G)$. Describing the topology of $\chi(\Gamma, G)$ in general is technical, although in the cases under construction here, $\mathrm{Hom}(\Gamma, G)$, $\chi(\Gamma, G)$ will be product spaces $\mathbb{R}^s$, $\mathbb{R}^t$, respectively, with t the result of subtracting the dimension of G from s. For details on the topology of $\chi(\Gamma, G)$, we refer the reader to [1].

1.1.8 Hitchin component of the representation variety of a triangle group

The next definition is of fundamental importance to this document.

Definition 1.8. *Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a finitely presented group, and let $\rho_n : \mathrm{PSL}(2, \mathbb{R}) \to \mathrm{PSL}(n, \mathbb{R})$ be the irreducible representation given by the action on homogeneous polynomials in two variables. Suppose that $\rho_n(\Gamma)$ is contained in a Lie group $G \subset \mathrm{PSL}(n, \mathbb{R})$. The **Hitchin component** of $\mathrm{Hom}(\Gamma, G)$ is the component of $\mathrm{Hom}(\Gamma, G)$ containing ρ_n.*

1.2 The exceptional Lie group G_2

1.2.1 The Lie algebra of a Lie group, the exponential map and the action of Ad of G on its Lie algebra

We give a brief overview of concepts relating to Lie algebras that are needed for Theorem 2.1. Background material may be found in many texts, for example [9] or [10].

Let G be a Lie group with tangent bundle TG, and for $g \in G$, let $L_g : G \to G$ be the left multiplication operator $x \mapsto gx$ ($x \in G$). The differential $dL_g : TG \mapsto TG$ has two-sided inverse $dL_{g^{-1}}$, so maps the tangent space at the identity element of G isomorphically to the tangent space at $g \in G$.

Let v be a tangent vector to G at the identity element. The set

$$X_v = \{dL_g(v) : g \in G\} \subset TG$$

is a vector field on G, invariant under all dL_g and thus termed a *left invariant vector field on G*.

The *Lie algebra* of G is $\mathfrak{g} = \{X_g : g \in G\}$, and since the correspondence $v \leftrightarrow X_v$ is bijective we can equally consider $\mathfrak{g}$ as the tangent space to G at the identity element.

In addition to this vector space structure, the bracket operation $[X_g, X_h] = X_g X_h - X_h X_g$ on vector fields provides $\mathfrak{g}$ with a *bracket operation* $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ on the Lie algebra satisfying the three conditions

1. $[\,,\,]$ is bilinear;
2. $[\,,\,]$ is skew-symmetric;
3. The *Jacobi identity*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

holds for all $X, Y, Z \in \mathfrak{g}$.

Here we are treating a vector field on the smooth manifold G as a differential operator, specifically a derivation, acting on smooth functions on G. Given vector fields X_g, X_h on G and a smooth function f on G, the bracket $[X_g, X_h]$ is the derivation given by

$$[X_g, X_h](f) = X_g(X_h(f)) - X_h(X_g(f)).$$

In the case where $G \subset \mathrm{GL}(n, \mathbb{C})$ is a matrix Lie group, tangent vectors to G at the identity can naturally be identified with (not necessarily invertible) $n \times n$ matrices, and the bracket operation is then given by $[X, Y] = XY - YX$. As an example, let us compute the Lie algebra of the 3-dimensional Lie group $\mathfrak{sl}(2, \mathbb{R})$ of $G = \mathrm{SL}(2, \mathbb{R})$. We have 1-paramater subgroups

$$\left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}, \quad \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} : t \in \mathbb{R} \right\}, \quad \left\{ \begin{pmatrix} 1+t & 0 \\ 0 & \frac{1}{1+t} \end{pmatrix} : t \neq 1 \right\}.$$

of $\mathrm{SL}(2,\mathbb{R})$, and evaluating derivatives with respect to t at $t = 0$, we obtain the three linearly independent Lie algebra elements

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

X, Y, Z form a vector space basis for $\mathfrak{sl}(2,\mathbb{R})$, which is seen to consist precisely of 2×2 matrices over $\mathbb{R}$ with zero trace. Brackets are easily computed; for example,

$$[X, Y] = XY - YX = Z.$$

The Lie group G (or at least a neighborhood of the identity) can be recovered from its Lie algebra by applying the exponential map:

$$\exp : \mathfrak{g} \to G \quad \exp(A) = e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

For $g \in G$, let ψ_g be the inner automorphism $h \mapsto ghg^{-1}$ of G. The differential $d\psi_g : TG \to TG$ maps the tangent space at the identity of G to itself by an automorphism which we denote Ad_g. This gives us an action Ad of the Lie group G on its Lie algebra $\mathfrak{g}$:

$$\mathrm{Ad} : G \to \mathrm{Aut}(G), \quad g \mapsto \mathrm{Ad}_g,$$

called the *adjoint representation* of G.

In the case where G is a matrix Lie groups, the automorphism Ad_g is simply conjugation by the matrix g:

$$\mathrm{Ad}_g(A) = gAg^{-1}, (A \in \mathfrak{g}).$$

In the proof of Theorem 2.1, the following identity will be used:

Proposition 1.9.

$$\exp(\mathrm{Ad}_g(A)) = \psi_g(\exp(A))$$

This identity may be verified informally by conjugating all terms in the exponential series for the matrix A by g.

1.2.2 Summary of the classification of simple Lie groups over $\mathbb{C}$

Finite dimensional complex simple Lie algebras were completely classified by Wilhelm Killing and Élie Cartan around 1900. There are four infinite families, one of which is the family $\mathfrak{sl}(n, \mathbb{C})$, $n = 1, 2, 3, \ldots$ and exactly five so-called *exceptional Lie algebras*; these five being the Lie algebras of the Lie groups G_2, F_4, E_6, E_7, E_8 of complex dimensions 14, 52, 78, 133, 248, respectively, [2, 8]. G_2 and F_4 are sometimes referred in older documents as E_2 and E_4, respectively.

1.2.3 k-forms on $\mathbb{C}^n$ or $\mathbb{R}^n$

Definition 1.10. *A* k***-form on an*** $\mathbb{F}$***-vector space*** V *(for* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$*) is an element of the* $\mathbb{F}$*-vector space*

$$\bigwedge^k(V^*) = \underbrace{V^* \wedge \cdots \wedge V^*}_{k}.$$

The k-form can be considered an object with k entries filled with k vectors from V; the form then produces a scalar in $\mathbb{F}$. The k-form is k-multilinear and also *alternating*, i.e. if a couple of vectors are switched, then the resulting scalar is multiplied by -1.

Let $\{e_i^* : 1 \leq i \leq 7\}$ be the basis for V^* dual to the standard basis $\{e_i\}$ for V, i.e. $e_i^*(e_j) = \delta_{ij}$. The $\mathbb{F}$-vector space $\bigwedge^3(V^*)$ has dimension $\binom{7}{3} = 35$, with basis $\{e_i^* \wedge e_j^* \wedge e_k^* : i < j < k\}$. The group of nonsingular matrices $\mathrm{GL}(7, \mathbb{F})$ acts on $\bigwedge^3(V^*)$ as follows:

$$\mathrm{GL}(7, \mathbb{F}) \times \bigwedge^3(V^*) \to \bigwedge^3(V^*) \quad (A, \Omega) \mapsto A \cdot \Omega = A^*(\Omega),$$

where the pullback $A^*(\Omega)$ is given by

$$A^*(X \wedge Y \wedge Z) = A^{-1}(X) \wedge A^{-1}(Y) \wedge A^{-1}(Z).$$

The isotropy group of a form is its stabilizer under this action.

A 3-form Ω on V is *generic* if its orbit under the action of $\mathrm{GL}(n, \mathbb{F})$ is open in $\bigwedge^3(V^*)$. Note that it is impossible for a 3-form on $\mathbb{F}^n$ to be generic for $n > 8$ as the dimension $\binom{n}{3}$ of $\bigwedge^3(\mathbb{F}^n)$ should not exceed n^2, the dimension of $\mathrm{GL}(n, \mathbb{F})$.

In a paper published in 1900, F. Engel proved:

Theorem 1.11. *There is just one orbit of generic 3-forms in $\bigwedge^3(\mathbb{C}^7)^*$. A representative of the orbit is*

$$\Omega_0 = -2e_1^* \wedge e_2^* \wedge e_5^* - 2e_1^* \wedge e_3^* \wedge e_6^* - 2e_1^* \wedge e_4^* \wedge e_7^* - 4e_2^* \wedge e_3^* \wedge e_4^* + 4e_5^* \wedge e_6^* \wedge e_7^*.$$

Let $e_i^* \wedge e_j^* \wedge e_k^*$ be denoted by e^{ijk}, where e_i^* is the associated dual basis element in $V^* = (\mathbb{F}^7)^*$. Then, the 3-form Ω_0 can be rewritten as

$$\Omega_0 = -2e^{125} - 2e^{136} - 2e^{147} - 4e^{234} + 4e^{567}.$$

In 1907, Engel's student Walter Reichel defended his Ph.D. thesis, which contained this result:

Theorem 1.12. *There are precisely two orbits of generic 3-forms in $\bigwedge^3(\mathbb{R}^7)^*$. Representatives are*

1. *Ω_0 interpreted as being over $\mathbb{R}$, and*

2. *$\Omega_1 = e^{147} + e^{257} + e^{367} + e^{123} - e^{156} + e^{246} - e^{345}$.*

1.2.4 The Cayley-Dickson process

The Cayley-Dickson process is a construction that mimics the way in which the complex numbers can be constructed as pairs of real numbers. Given an $\mathbb{R}$-algebra A of dimension n, equipped with a *conjugation* $^* : A \to A$, application of the process produces an $\mathbb{R}$-algebra of dimension $2n$, also equipped with a conjugation, with underlying set the Cartesian product $A \times A$.

The construction of the algebra $A \times A$ from A is as follows:

$$1_{A\times A} = (1_A, 0_A)$$
$$-(a,\, b) = (-a, -b)\,(a,\, b \in A)$$
$$(a,\, b)^* = (a^*, -b)\,(a,\, b \in A)$$
$$(a,\, b) + (c,\, d) = (a + c,\, b + d)\,(a,\, b,\, c,\, d \in A)$$
$$(a,\, b)(c,\, d) = (ac - d^*b,\, da + bc^*)\,(a,\, b,\, c,\, d \in A)$$

Inductive application of this construction, starting with the real numbers, yields the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, the octonions $\mathbb{O}$ and thereafter increasingly exotic algebras containing zero divisors. If instead we start with the so-called *split complex numbers*, a variation on the complex numbers where $i^2 = +1$ in place of the familiar $i^2 = -1$, we obtain the *split quaternions* and then the *split octonions* $\mathbb{O}_{4,4}$.

The split complex numbers can be defined as the quotient $\mathbb{R}[x]/(x^2 - 1)$, a variation on $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$.

The relevance of the algebras $\mathbb{O}$ and $\mathbb{O}_{4,4}$ will be made apparent in the next section.

The algebra $\mathbb{O}_{4,4}$ is not isomorphic to the "true" or "standard" octonions $\mathbb{O}$, as it has nonzero elements of norm 0 that are not invertible. Multiplication in $\mathbb{O}$ and $\mathbb{O}_{4,4}$ is not associative, but it is *alternative*, i.e. if any two of X, Y, or Z are equal, then $(XY)Z = X(YZ)$.

$\mathbb{R}$ is the completely ordered field, and the 2-dimensional $\mathbb{R}$-algebra $\mathbb{C}$ is not ordered but an algebraically complete field. The 4-dimensional $\mathbb{R}$-algebra $\mathbb{H}$ is non-commutative but is still an associative normed division algebra. The 8-dimensional $\mathbb{R}$-algebra $\mathbb{O}$ is not associative but is an alternative normed division algebra, [4].

1.2.5 The exceptional Lie group G_2

As stated above in 1.11, there is a unique orbit of generic 3-forms in the space $\bigwedge^3(\mathbb{C}^7)^*$, a representative being

$$\Omega_0 = -2e^{125} - 2e^{136} - 2e^{147} - 4e^{234} + 4e^{567}.$$

In the absence of a canonical representative of this orbit, we take Ω_3 as an arbitrary choice, and make the following definition.

Definition 1.13. *G_2 is the isotropy group of Ω_0 within $\mathrm{SL}(7, \mathbb{C})$, i.e.*

$$G_2 = \{A \in \mathrm{SL}(7, \mathbb{C}) : A.\Omega_0 = \Omega_0\}.$$

If instead of Ω_0 we chose a different representative of its orbit, the isotropy group would be in the same conjugacy class of G_2 as defined above; therefore, there is a sense in which we are only defining G_2 up to conjugacy.

Recall also that there are precisely two orbits of generic 3-forms in $\bigwedge^3(\mathbb{R}^7)^*$:

1. Ω_0 as being interpreted over $\mathbb{R}$, and
2. $\Omega_1 = e^{147} + e^{257} + e^{367} + e^{123} - e^{156} + e^{246} - e^{345}$.

Theorem 1.14. *The respective isotropy groups in $\mathrm{SL}(7, \mathbb{R})$ of Ω_0, Ω_1 are the noncompact and compact real forms of G_2, [8].*

Proofs of these theorems are given in [8] and are quite lengthy. The standard notation for the noncompact real form of G_2 is $G_2^{4,3}$, reflecting the fact that it respects a bilinear form of signature (4, 3), so is a subgroup of SO(4, 3). The compact real form is denoted G_2^c and is a subgroup of SO(7).

The real forms $G_2^{4,3}$, G_2^c are the automorphism groups of $\mathbb{O}_{4,4}$, $\mathbb{O}$, respectively. The complex Lie group G_2 also has a characterization as the automorphism group of the complexification of the octonion algebra $\mathbb{O}$.

1.2.6 The Lie algebra of G_2

$\mathfrak{g}_2$ is the simple (complex) Lie algebra over G_2, and if $\mathbb{C}$ is replaced with $\mathbb{R}$ in the above definition, the result is the (noncompact) split real form $G_2^{4,3}$ of G_2, whose Lie algebra $\mathfrak{g}_2^{4,3}$ is the (noncompact) real form of $\mathfrak{g}_2$. There is also a compact real form (whose complexification is also $\mathfrak{g}_2$), which can be described using cross products, [8].

Let $\mathbb{F}$ denote either of the fields $\mathbb{C}$, $\mathbb{R}$. Recall the vector cross product in $\mathbb{F}^3$:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Denote the above 3×3 matrix as ℓ_x, which represents the map $y \mapsto x \times y$.

Representing elements of $\mathbb{F}^3$ by column vectors, one can represent $\mathfrak{g}_2$ or $\mathfrak{g}_2^{4,3}$ (for $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, respectively) as the following set of 7×7 matrices, as seen in [8]:

$$\left\{ M_{(a,x,y)} = \begin{pmatrix} 0 & -2y^t & -2x^t \\ x & a & \ell_y \\ y & \ell_x & -a^t \end{pmatrix} : a \in \mathfrak{sl}_3(\mathbb{F}),\ x, y \in \mathbb{F}^3 \right\}.$$

1.3 Integer points on the Hitchin component of the (2,4,6)-triangle group

1.3.1 Visualization of the Hitchin component

By Theorem 2.1, the G_2-Hitchin component $\mathcal{H}_{G_2}(\Gamma(2,4,6))$ of the (2,4,6)-triangle group (in its character variety form) has dimension 2. An exact formulation similar to the process of [13] was found, but an extra "filter" was needed to ensure that the Newton's method process converged (numerically) to a representation preserving a generic 3-form.

The exact version of $\mathcal{H}_{G_2}(\Gamma(2,4,6))$ used (which is on file) has generators a and b of orders 4 and 6, respectively with ab of order 2. The parameters chosen are $u = \mathrm{tr}(aaab)+1$ and $v = \mathrm{tr}(aabb)+1$, adding 1 to make the computations more digestible. The matrix entries lie in a quadratic extension $\mathbb{Q}(u,v)(\alpha)$ of $\mathbb{Q}(u,v)$, where

$$\alpha = \sqrt{2(-72u^2+72u^3-16u^4-u^5+v(-192u+64u^2+20u^3)+v^2(-192-32u+2u^2)-32v^3)}.$$

The calculation for α is a lengthy process involving the convergence of Newton's method and is a similar process to that of finding α in [7] but in $\mathrm{SL}(7,\mathbb{R})$ with $u = \mathrm{tr}(aaab)+1$ and $v = \mathrm{tr}(aabb)+1$. Since there are two generators, there are $49\cdot 2 = 98$ unknown matrix entries; each relator gives 49 equations, so from the relators we have $49\cdot 3 = 147$ equations. Then there are $35\cdot 2 = 70$ equations for preserving the 3-form, and a further 2 equations for steering towards the desired values of u, v. In total, this is 219 equations in 98 unknowns, *i.e.* a Jacobian of size 219×98. Experimentation revealed that $\mathrm{tr}(ab^2a^{-2}b^{-1}) = \frac{1}{4}(-4-2u^2+4u+2uv+\alpha)$. Since $\mathrm{tr}(aaab) = u-1$, $\mathrm{tr}(aabb) = v-1$, and $\mathrm{tr}(ab^2a^{-2}b^{-1}) = \frac{1}{4}(-4-2u^2+4u+2uv+\alpha)$, it makes sense to search for integer values of u and v such that α is also an integer, as all the traces of words in a and b must be integral for use of Proposition 4.1.

Proposition 1.15. *A necessary condition for a representation in the Hitchin component of the triangle group $\Gamma(2,4,6)$ to be written over the integers is that u, v should be integers and that α should be an even integer.*

Proof. Let the generators of $\rho(\Gamma(2,4,6))$ be a, b as given in *B*.1 in the Appendix, with a, b, ab of orders 4, 6, 2, respectively. Let us suppose that ρ is written over the integers. Then, clearly traces of all words in a, b are integers. From B.1, we have

$$\mathrm{tr}(a^3b) = u-1, \quad \mathrm{tr}(a^2b^2) = v-1, \quad \mathrm{tr}(ab^2a^2b^5) = \frac{1}{4}(-4-2u^2+4u+2uv+\alpha).$$

From the first two of these traces u, v are integers, and from the third trace, $4(-1+u)+2u(-u+v)+\alpha$ is an integer multiple of 4. ∎

The base representation of $\mathcal{H}_{G_2}(\Gamma(2,4,6))$ ρ_0 is at $u = 72$ and $v = 378$, and the tautological representation is given in B.1.

After finding some integer values of u and v for which α is also an integer, Mathematica was used to run through many values of u and v and print out pairs for which $\alpha \in \mathbb{Z}$. Since α must be real, one can exclude cases where $\alpha^2 < 0$. We kept the range for u between 70 and 500 and the range for v between 350 and 10,000, starting with 70 and 350 respectively as the base representation ρ_0 is at $u = 72$, $v = 378$. The values of u and v that were obtained with these parameters can be found in Section B.2 in the Appendix.

Using Mathematica by taking subsets of six points from Section B.2, we wanted to see whether the $u-$ and $v-$coordinates fit a polynomial of smaller degree than 5 and assume each subset contains $(72,\ 378)$. Using this method, two candidates were found:

- $\{(72,\ 378),\ (76,\ 418),\ (78,\ 446),\ (84,\ 482),\ (108,\ 810),\ (172,\ 1978)\}$ with the polynomials $u = 68 + \frac{4x}{3} + \frac{14x^2}{3} - \frac{7x^3}{3} + \frac{x^4}{3}$ and $v = 570 - \frac{1366x}{3} + 359x^2 - \frac{320x^3}{3} + 11x^4$,
- $\{(72,\ 378),\ (76,\ 418),\ (88,\ 550),\ (108,\ 810),\ (136,\ 1258),\ (172,\ 1978)\}$ with the polynomial $u = 76 - 8x + 4x^2$ and $v = 418 - 82x + 45x^2 - 4x^3 + x^4$.

The first candidate has non-integer rational coefficients, which is not helpful. However, the second one has all integer coefficients. One can check whether the second candidate is a good one by substituting $u = 76 - 8x + 4x^2$ and $v = 418 - 82x + 45x^2 - 4x^3 + x^4$ into α^2:

$$\alpha^2 = 64(-1 + x)^2(16 - 2x + x^2)^2(19 - 2x + x^2)^2.$$

Let $t = x - 1$. Then,

$$\begin{aligned} u &= 4(19 - 2x + x^2) = 4(18 + t^2), \\ v &= (x^2 - 2x + 19)(x^2 - 2x + 22) = (t^2 + 18)(t^2 + 21), \\ \text{and } \alpha &= \sqrt{64t^2(15 + t^2)^2(18 + t^2)^2} = 8t(15 + t^2)(18 + t^2). \end{aligned}$$

This substitution gives a sequence of points of the G_2 Hitchin component which, as shown by Theorem 2.2, can be conjugated so as to be over $\mathbb{Z}$ for any integer t, where $t = 0$ gives the base representation ρ_0 at $u = 72$ and $v = 378$. Let (ρ_n) denote the sequence of representations in $\mathcal{H}_{G_2}(\Gamma(2,4,6))$, where

$$\rho_n \text{ is the representation at } (u,\, v) = (4(18+t^2),\, (t^2+18)(t^2+21))\,, \quad n = t \in \mathbb{Z}_{\geq 0.} \quad (1.1)$$

Figure 1.4 is an illustration of the curve $\alpha = 0$ in the (u, v)-plane with the points ρ_n for $n = t = 0, 1, \ldots 6$ indicated. The Hitchin component is homeomorphic to $\mathbb{R}^2$, [3], and can be visualized as a horn shape with two sheets, corresponding to α and its negative, glued along the curve $\alpha = 0$.

1.3.2 Algebras over a field

Definition 1.16. *A* ***central simple algebra*** *over a field K is a finite-dimensional associative K-algebra which is simple and for which its center is exactly K.*

Theorem 1.17. *(Skolem-Noether, as in [15]) Let $K \subset B \subset A$, where B is a simple subring of the central simple K-algebra A. Then, every K-isomorphism φ of B onto a subalgebra $\tilde{B}$ of A extends to an inner automorphism of A, that is, there exists an invertible $a \in A$ such that*

$$\varphi(b) = aba^{-1}, \quad b \in B.$$

Definition 1.18. *An element b of a commutative ring B is* ***integral*** *over a subring A of B if b is a root of a monic polynomial over A. The* ***integral closure*** *of A in a set B is the set of elements in B that are integral over A.*

The following proposition is Proposition 2.2 (c - e) in [6], where K is an algebraically closed field and Γ is a multiplicative submonoid of $\mathrm{M}(n, K)$ that acts irreducibly on K^n, so that the Burnside Lemma furnishes a K-basis $s_1, \ldots, s_{n^2}$ in Γ of $\mathrm{M}(n, K)$, and the dual basis $t_1, \ldots, t_{n^2}$:

$$\mathrm{tr}(t_i s_j) = \delta_{ij}, \quad (i, j = 1, \ldots, n^2).$$

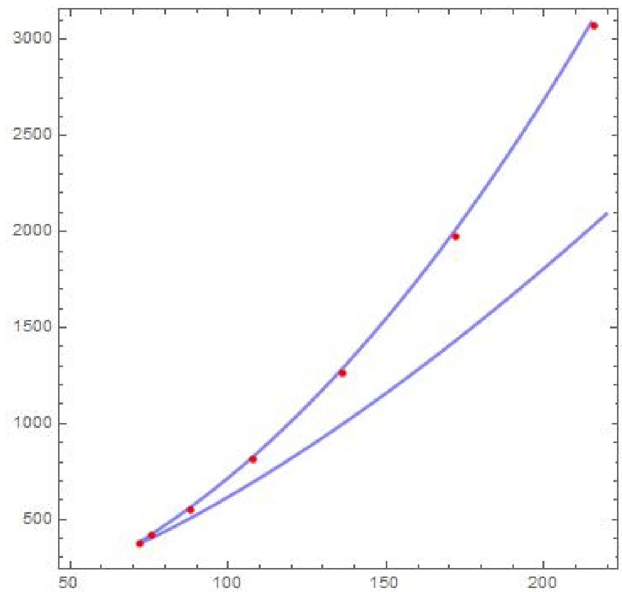

Figure 1.4: The curve $\alpha = 0$ in the (u,v)-plane with points ρ_n at $n = t = 0, 1, \ldots, 6$.

Proposition 1.19.

(i) $F\Gamma$ *is a central simple* F*-algebra of dimension* n^2 *and hence isomorphic to* $\mathrm{M}(r, D)$ *where* D *is a division algebra of dimension* s^2 *over the center* F *with* $rs = n$*. The integer* s *is called the Schur index of the representation of* Γ *on* K^n*.*

(ii) There is an extension E *of degree* s *of* F *in* K*, isomorphic to a maximal subfield of* D*, such that* $E\Gamma(\cong E \otimes_F F\Gamma)$ *is isomorphic to the* E*-algebra* $\mathrm{M}(n, E)$*.*

(iii) $E\Gamma$ *is conjugate in* $\mathrm{M}(n, K)$ *to* $\mathrm{M}(n, E)$*.*

The following are Corollaries 2.3 and 2.5, respectively, in [6] and are used in the non-constructive proof of 4.1.

Corollary 1.20. *Let* A *be a Noetherian subring of* K *with field of fractions* F*. Assume that the integral closure of* A *in any finite extension of* F *is a finitely generated* A*-module. Assume further that* $\mathrm{tr}(\Gamma)$ *is integral over* A*, i.e. that each element of* $\mathrm{tr}(\Gamma)$ *is so, and that* $\mathrm{tr}(\Gamma) \subset L$ *for some finitely generated field extension* L *of* F*. Then,* $A\Gamma$ *is a finitely generated* A*-module.*

Corollary 1.21. *In the proposition above, suppose that* F *is a finite extension of the prime field of* K*. Then there is a finite extension* L *of* F *in* K *such that, if* B *is the integral closure of* A *in* L*, then* $s\Gamma s^{-1} \subset \mathrm{M}(n, B)$ *for some* $s \in \mathrm{GL}(n, K)$*.*

1.3.3 Burnside Γ-basis for $\mathrm{M}(n, \mathbb{F})$

Definition 1.22. *Let* F *be a (not necessarily algebraically closed) field, and let* Γ *be a multiplicative group of matrices in* $\mathrm{M}(n, F)$*. A* ***Burnside*** Γ***-basis*** *for* $\mathrm{M}(n, F)$ *is a subset of* Γ *that is a basis for the* n^2*-dimensional* F*-vector space* $\mathrm{M}(n, F)$*.*

Lemma 1.23. *(Burnside Lemma) Suppose that* Γ *acts irreducibly on* F^n*.*

(a) Γ *contains a basis* $s_1, \ldots, s_{n^2}$ *of* $\mathrm{M}(n, F)$*. Let* $t_1, \ldots, t_{n^2}$ *be the dual basis relative to the trace form:* $\mathrm{tr}(t_i s_j) = \delta_{ij}$ $(1 \leq i, j \leq n^2)$*.*

(b) For any $s \in \mathrm{M}(n, F)$*,* $s = \sum_i \mathrm{tr}(s s_i) t_i$*. Hence,* $\Gamma \subset \sum_i \mathrm{tr}(\Gamma) t_i$*.*

Chapter 2

Statement of main results

2.1 The Dimension of $\mathcal{H}_{G_2}$ for $\Gamma(p, q, r)$

Theorem 2.1. *Let $S = \{n \in \mathbb{N} : n \geq 2\}$, and let $f : S \to \mathbb{N}$ be defined by:*

$$f(n) = \begin{cases} 8 & (n = 2) \\ 10 & (n = 3, 4, 5) \\ 12 & (n \geq 6) \end{cases} .$$

The dimension of the Hitchin component $\mathcal{H}_{G_2}$ of $\Gamma(p, q, r)$ in $G_2^{4,3}$ is $f(p) + f(q) + f(r) - 14$, and that of the corresponding component of the character variety is $f(p) + f(q) + f(r) - 28$.

Remark. A proof of 2.1 is given in Alessandrini et al., [3], as part of a wider investigation using Higgs bundles techniques. Here, we provide an alternate proof that is elementary in nature.

2.2 Integer points in $\mathcal{H}_{G_2}(\Gamma(2, 4, 6))$

The following theorem makes reference to the sequence of representations (ρ_n) for $n \in \mathbb{Z}_{\geq 0}$ as defined in Equation 1.1 for matrices a, b in the tautological representation given in B.1 in the Appendix. However, we substitute the parameters u, v in terms of t and work with the tautological representation given in Equation 4.1.

Theorem 2.2. *The representations ρ_n for $n \in \mathbb{Z}_{\geq 0}$ are pairwise non-conjugate and are integral, i.e. each can be conjugated to a representation in* $\mathrm{SL}(7, \mathbb{Z})$.

Chapter 3

Proof of Theorem 2.1

The proof given here is partly an adaptation of the proof of the main theorem in [14]; the part of the argument concerning $G_2^{4,3}$ and its Lie algebra is however new.

Let α, β, γ be the images of the generators a, b, c of $\Gamma(p,q,r)$ in $G_2^{4,3}$. Observe that under a continuous deformation, these images can only move within their conjugacy classes in $G_2^{4,3}$ since they are of finite order. Writing $[\alpha]$, $[\beta]$, $[\gamma]$ for the respective conjugacy classes of the representation from $\mathrm{PSL}(2,\mathbb{R})$ to $G_2^{4,3}$, there is a smooth map

$$\Phi : [\alpha] \times [\beta] \times [\gamma] \to G_2^{4,3}, \quad \Phi : (\alpha', \beta', \gamma') \mapsto \alpha'\beta'\gamma',$$

which by irreducibility of ρ_0 is shown to be a submersion close to (α, β, γ) in the proof of [14]. Thus, it follows from the defining relation $abc = 1$ that

$$\dim \mathcal{H}_{G_2} = \dim[\alpha] + \dim[\beta] + \dim[\gamma] - \dim G_2^{4,3} = \dim[\alpha] + \dim[\beta] + \dim[\gamma] - 14.$$

The goal is to compute the subalgebra of the Lie algebra $\mathfrak{g}_2$ consisting of elements that commute with Ad_g of a given generator g. The dimension of the conjugacy class of a generator g is equal to $14 - \dim(C_{\mathfrak{g}_2}(g))$, where $C_{\mathfrak{g}_2}(g)$ is the set of elements $\xi \in \mathfrak{g}_2$ such that $\mathrm{Ad}_g(\xi) = \xi$.

Here is the explicit version of $\mathfrak{g}_2$ is given by [8]:

$$\mathfrak{g}_2 = \left\{ \begin{bmatrix} 0 & -2y^t & -2x^t \\ x & a & \ell_y \\ y & \ell_x & -a^t \end{bmatrix} : a \in \mathfrak{sl}_3(\mathbb{C}),\, x, y \in \mathbb{C}^3 \right\},\ \text{where } \ell_x = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

Hence, an arbitrary element ξ of g_2 is of the form:

$$\xi = \begin{bmatrix} 0 & -2y_1 & -2y_2 & -2y_3 & -2x_1 & -2x_2 & -2x_3 \\ x_1 & a_{11} & a_{12} & a_{13} & 0 & -y_3 & y_2 \\ x_2 & a_{21} & a_{22} & a_{23} & y_3 & 0 & -y_1 \\ x_3 & a_{31} & a_{32} & -a_{11}-a_{22} & -y_2 & y_1 & 0 \\ y_1 & 0 & -x_3 & x_2 & -a_{11} & -a_{21} & -a_{31} \\ y_2 & x_3 & 0 & -x_1 & -a_{12} & -a_{22} & -a_{32} \\ y_3 & -x_2 & x_1 & 0 & -a_{13} & -a_{23} & a_{33} \end{bmatrix},$$

where x_i, y_i, $a_{ij} \in \mathbb{C}$ for $i, j = 1,\ 2,\ 3$. Note that $a_{33} = -a_{11} - a_{22}$ since the matrix $a \in \mathfrak{sl}(3, \mathbb{C})$ has zero trace.

The following facts are from basic Lie theory:

(i) There exists a neighborhood of $0 \in \mathfrak{g}_2$ for which the exponential map

$$\exp : \mathfrak{g}_2 \to G_2 \quad \exp(A) = e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

is a diffeomorphism to a neighborhood of the identity element of G_2;

(ii) As seen in Proposition 1.9, $\exp(\mathrm{Ad}_g\, \xi) = g \exp(\xi) g^{-1}$, for $g \in G_2$ and $\xi \in \mathfrak{g}_2$, where $\mathrm{Ad} : G_2 \to \mathrm{Aut}(\mathfrak{g}_2)$ is the adjoint map.

Identity (ii) can be checked by merely expanding the right-hand side. Hence, the dimension of $C_{G_2}(\gamma)$ is equal to the dimension of the subspace

$$W = \{\xi \in \mathfrak{g}_2 : \exp(\mathrm{Ad}_\gamma\, \xi) = \exp(\xi)\} = \{\xi \in \mathfrak{g}_2 : \mathrm{Ad}_\gamma\, \xi = \xi\}$$

of the Lie algebra $\mathfrak{g}_2$. The dimension of the real vector space W depends only on the conjugacy class in $\mathrm{PSL}(2,\mathbb{R})$ of an elliptic generator g, say of order k. For convenience, since the eigenvalues of g are $e^{i\pi/k}$ and $e^{-i\pi/k}$, one can divert attention to $\mathrm{PSL}(2,\mathbb{C})$. Take g to be the diagonal 2×2 matrix with these diagonal entries without loss of generality, and the image of g under the irreducible representation from $\mathrm{PSL}(2,\mathbb{C})$ to $\mathrm{PSL}(2,\mathbb{C})$ is the 7×7 matrix below:

$$\begin{bmatrix} e^{i\pi/k} & 0 \\ 0 & e^{-i\pi/k} \end{bmatrix} \to \begin{bmatrix} e^{-6i\pi/k} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-4i\pi/k} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-2i\pi/k} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{2i\pi/k} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{4i\pi/k} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{6i\pi/k} \end{bmatrix}$$

We are essentially computing the complex dimension of the complexification of W, which is sufficient as the (complex) dimension of G_2 is equal to the (real) dimension of its split real form $G_2^{4,3}$.

By (i) above, one can recover an element of the Lie group G_2 from an element of its Lie algebra $\mathfrak{g}_2$. However, as G_2 is only defined up to conjugacy in $\mathrm{SL}(7,\mathbb{C})$, one must ensure that the triangle group generators have been conjugated so that they fix the same "standard" generic 3-form (up to scalar multiple) as that fixed by the version of G_2 given in [8]:

$$\Omega_0 = -2e^{125} - 2e^{136} - 2e^{147} - 4e^{234} + 4e^{567}.$$

Unfortunately, the 3-form fixed by the image of $\mathrm{PSL}(2,\mathbb{C})$ under the irreducible representation into $\mathrm{PSL}(7,\mathbb{C})$ (as an action on homogeneous polynomials) is

$$\Omega = -15e^{345} + 6e^{246} - 3e^{156} - 3e^{237} + e^{147}.$$

The forms Ω_0 and Ω are different, but they are thankfully in the same orbit under the action of $\mathrm{GL}(7,\mathbb{R})$. So, once the irreducible representation from $\mathrm{PSL}(2,\mathbb{C})$ to $\mathrm{PSL}(7,\mathbb{C})$

is taken, the conjugation of the image of g must be taken to be in G_2. Experimentation revealed that the matrix

$$M = \frac{1}{6 \cdot 5^{1/3}} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & -30 & 0 & 0 & 0 \end{bmatrix}$$

takes the form Ω to Ω_0. Hence, conjugating the image of g under this irreducible representation by M gives:

$$g_7 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2\pi i/k} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{4\pi i/k} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-6\pi i/k} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-2\pi i/k} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-4\pi i/k} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{6\pi i/k} \end{bmatrix}.$$

All that remains is to get conditions on the 14 parameters of an arbitrary element ξ of $\mathfrak{g}_2$ as explicitly defined in [8] so that $g_7\xi = \xi g_7$, i.e. ξ is in the centralizer of g_7. $g_7\xi - \xi g_7 = 0$ gives the following conditions :

$$\begin{aligned} 2y_1(e^{2\pi i/k} - 1) &= 0 & a_{12}(e^{-2\pi i/k} - e^{-4\pi i/k}) &= 0 \\ 2y_2(e^{4\pi i/k} - 1) &= 0 & a_{13}(e^{2\pi i/k} - e^{-6\pi i/k}) &= 0 \\ 2y_3(e^{-6\pi i/k} - 1) &= 0 & a_{21}(e^{2\pi i/k} - e^{4\pi i/k}) &= 0 \\ 2x_1(e^{-2\pi i/k} - 1) &= 0 & a_{23}(e^{4\pi i/k} - e^{-6\pi i/k}) &= 0 \\ 2x_2(e^{-4\pi i/k} - 1) &= 0 & a_{31}(e^{2\pi i/k} - e^{-6\pi i/k}) &= 0 \\ 2x_3(e^{6\pi i/k} - 1) &= 0 & a_{32}(e^{-4\pi i/k} - e^{-6\pi i/k}) &= 0 \end{aligned}$$

In every possible case below, $y_1 = x_1 = a_{12} = a_{21} = 0$. Otherwise $k = 1$, a contradiction since $k \geq 2$.

1. If $k = 2$, then $y_3 = x_3 = a_{23} = a_{32} = 0$. So, there are 6 free variables, and the dimension of the centralizer of g_7 in G_2 is 6. The dimension of the conjugacy class of g_7 in G_2 is 14 - 6 = 8.

2. If $k = 3$, then $y_2 = x_2 = a_{13} = a_{31} = a_{23} = a_{32} = 0$. So, there are 4 free variables, and the dimension of the centralizer of g_7 in G_2 is 4. The dimension of the conjugacy class of g_7 in G_2 is 14 - 4 = 10.

3. If $k = 4$, then $y_2 = x_2 = y_3 = x_3 = a_{23} = a_{32} = 0$. So, there are 4 free variables, and the dimension of the centralizer of g_7 in G_2 is 4. The dimension of the conjugacy class of g_7 in G_2 is 14 - 4 = 10.

4. If $k = 5$, then $y_2 = x_2 = y_3 = x_3 = a_{13} = a_{31} = 0$. So, there are 4 free variables, and the dimension of the centralizer of g_7 in G_2 is 4. The dimension of the conjugacy class of g_7 in G_2 is 14 - 4 = 10.

5. For any $k \geq 6$, $y_2 = x_2 = y_3 = x_3 = a_{12} = a_{21} = a_{13} = a_{31} = 0$. So, there are only 2 free variables, a_{11} and a_{22}, and the dimension of the centralizer of g_7 in G_2 is 2. The dimension of the conjugacy class of g_7 in G_2 is 14 - 2 = 12.

Thus, the images α, β, γ of the generators a, b, c of the triangle group $\Gamma(p,q,r)$ in G_2 are of sizes $f(p)$, $f(q)$, and $f(r)$, respectively. The dimension of the Hitchin component $\mathcal{H}_{G_2}$ of $\Gamma(p,q,r)$ in G_2 is $f(p)+f(q)+f(r)-14$. Since the matrix used to conjugate Ω to obtain Ω_0 is over the real numbers and the (complex) dimension of G_2 is equal to the (real) dimension of $G_2^{4,3}$, the dimensions of the centralizers of g_7 in $G_2^{4,3}$ will agree with their respective cases in G_2.

The fact that $f(3)$, $f(4)$, and $f(5)$ are all equal seems to be a coincidence since the computations for these values, as seen above, appear unrelated.

Representations ρ_1, ρ_2 are equivalent if $\rho_2 = \tau_g \circ \rho_1$ where $\tau_g(x) = gxg^{-1}$ for x, $g \in G_2$.

Given a representation ρ in the Hitchin component, suppose that there are elements g_1, $g_2 \in G_2$ such that $\tau_{g_1} \circ \rho = \tau_{g_2} \circ \rho$. Then, $\tau_{g_2^{-1}g_1} \circ \rho = \rho$, and since ρ is irreducible, by Schur's

Lemma, $g_2^{-1}g_1$ is a scalar multiple of the identity. However, since each element of G_2 has determinant 1, $g_2^{-1}g_1 = 1$.

The map that assigns to each representation its equivalence class is locally a projection of a product to one of its factors, the other factor being a space of representations of dimension 14. So, passing from the representation variety to the character variety results in another reduction in dimension of 14. $\square$

Chapter 4

Preliminary Results and Proof of Theorem 2.2

The proof of this theorem requires some preliminary results, which are provided and proven here, and contains some moderately heavy computations in Mathematica. The output data is too onerous for inclusion in this (or any) article, as it deals with large matrices whose entries are quotients of polynomials, often of large degree; However, this method is fairly elementary, and as it is described here in full, the interested reader familiar with computer algebra systems will be able to carry out these or similar computations.

4.1 Results needed for the proof of Theorem 2.2

The following result is a special case of Corollary 1.21 from [6]:

Proposition 4.1. *Let Γ be a subgroup of $\mathrm{SL}(n, \mathbb{Q})$ satisfying:*

(i) The action of Γ on $\mathbb{Q}^n$ is absolutely irreducible;

(ii) $\chi(\Gamma) \subset \mathbb{Z}$, i.e. each element of Γ has integer trace.

Then there exists $\tau \in \mathrm{GL}(n, \mathbb{Q})$ with $\tau\Gamma\tau^{-1} \subset \mathrm{SL}(n, \mathbb{Z})$.

Proof. Let $\mathbb{Z}\Gamma$ denote the set of all finite $\mathbb{Z}$-linear combinations of elements of Γ. Since Γ is absolutely irreducible over $\mathbb{Q}^n$, Γ acts irreducibly over $\mathbb{C}^n$. Since $\chi(\Gamma) \subset \mathbb{Z}$, $\chi(\Gamma)$ is (naturally) integral over $\mathbb{Z}$. Hence, by Corollary 1.20, $\mathbb{Z}\Gamma$ is a $\mathbb{Z}$-order of $\mathrm{M}(n, \mathbb{Q})$.

The order $\mathbb{Z}\Gamma$ is contained in a maximal order $\mathcal{O}$ of $\mathrm{M}(n,\mathbb{Q})$; from Theorem 21.6 in [15], such a maximal order is of the form $\{t \in \mathrm{M}(n,\mathbb{Q}) \mid t(\Lambda) \subset \Lambda\}$ for some full lattice Λ in $\mathbb{Q}^n$. Since Λ is isomorphic (as a $\mathbb{Z}$-module) to the free module $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$, so $\mathcal{O} \cong \mathrm{M}(n,\mathbb{Z})$.

Apply the Skolem-Noether theorem to obtain $t \in \mathrm{GL}(n,\mathbb{Q})$ with $\phi_t(\mathcal{O}) \subset \mathrm{M}(n,\mathbb{Z})$, with ϕ_t being the inner automorphism $x \mapsto txt^{-1}$ of $\mathrm{M}(n,\mathbb{Q})$.

$$\begin{tikzcd}
\mathrm{M}(n,\mathbb{Q}) \arrow[r, dashed, "\phi_t"] & \mathrm{M}(n,\mathbb{Q}) \\
\mathcal{O} \arrow[u, hook] \arrow[r, "\cong"] & \mathrm{M}(n,\mathbb{Z}) \arrow[u, hook]
\end{tikzcd}$$

$\square$

The generators a, b of the tautological representation of the (2,4,6)-triangle group (after conjugation and replacing parameters u, v in terms of t) have entries in the field of fractions $\mathbb{Q}(t)$ of the polynomial ring $\mathbb{Z}[t]$. So, take F to be this field and $\Gamma \subset \mathrm{M}(7,F)$ to be the multiplicative group generated by a, b. With the aid of Mathematica, it is relatively simple to find a Burnside Γ-basis for $\mathrm{M}(7,F)$, and the following basis will be used. Here I denotes the identity 7×7 matrix, A and B denote the inverses of a and b, respectively.

$$\begin{aligned}
&\{I,\ Ab,\ (Ab)^2,\ (Ab)^3,\ aB,\ (aB)^2,\ (aB)^3,\ bA,\ (bA)^2,\ (bA)^3,\ Ba,\ (Ba)^2,\\
&(Ba)^3,\ aaB,\ (aaB)^2,\ aBB,\ (aBB)^2,\ AAb,\ (AAb)^2,\ Abb,\ (Abb)^2, bbA,\\
&(bbA)^2,\ bAA,\ (bAA)^2,\ BBa,\ (BBa)^2,\ Baa,\ (Baa)^2,\ aba,\ (aba)^2,\ aBa,\\
&(aBa)^2,\ AbA,\ (AbA)^2,\ ABA,\ bab,\ (bab)^2,\ bAb,\ (bAb)^2,\ BaB,\ (BaB)^2,\\
&abAB,\ aBAb,\ abbABB,\ ABBabb\,\}
\end{aligned}$$

Verifying that this basis, denoted $\mathcal{B} = \{s_1, \ldots, s_{49}\}$, is indeed a Burnside Γ-basis is easy to do in Mathematica as the most time-consuming portion is typing up the data itself. Here, we have $s_1 = I$, but one can always take the first basis element to be the identity matrix by multiplying all basis elements by s_1^{-1} if necessary.

From the classical result Lemma 1.23 by Burnside, such a basis exists if Γ is absolutely irreducible, but the converse is much easier and is of immediate use.

Proposition 4.2. *Let F be a (not necessarily algebraically closed) field, and let Γ be a multiplicative group of matrices in $\mathrm{M}(n, F)$. If $\mathrm{M}(n, F)$ admits a Burnside Γ-basis, then Γ is absolutely irreducible in $\mathrm{M}(n, F)$.*

Proof. Suppose that the invariant subspace of Γ is a non-trivial, proper subspace W of $\mathrm{M}(n, K)$, where K is the algebraic closure of the field F. Choose a basis for this subspace W and extend it to a basis of $\mathrm{M}(n, K)$.

With respect to this basis, the matrix for every element of Γ, regarded as a group of linear transformations of $\mathrm{M}(n, K)$, has a block of zeros in the bottom left-hand corner, and so Γ is conjugate in $\mathrm{GL}(n, K)$ to a group Γ' of matrices of this form.

Clearly, the matrix whose bottom left-hand corner entry is 1 and all other entries 0 is not in the K-span of Γ', so the subspace of $\mathrm{M}(n, K)$ consisting of all K-linear combinations of elements of Γ' has dimension strictly less than n^2. The dimension is equal to that of the subspace of $\mathrm{M}(n, K)$ consisting of all K-linear combinations of elements of Γ, a contradiction. Thus, if there is a Burnside Γ-basis, then the invariant subspace of Γ is the entire space $\mathrm{M}(n, K)$. So Γ is absolutely irreducible in $\mathrm{M}(n, F)$. $\square$

Proposition 4.3. *Let Γ_n, $n \in \mathbb{N}$, be the multiplicative subgroup of $\mathrm{M}(7, \mathbb{Q})$ obtained by setting $t = n$ in $\Gamma = \langle a, b \rangle$. Then, Γ_n is absolutely irreducible in $\mathrm{M}(7, \mathbb{Q})$ for all $n \geq 1$.*

Proof. Let T be the 49×49 transition matrix from the basis $\mathcal{B}$ to the standard basis for $\mathrm{M}(7, \mathbb{Q}(t))$ consisting of elementary matrices E_{ij} whose (i, j)-entry is 1 and all other entries are 0. The jth column of T consists of the 49 entires of the jth element of the basis $\mathcal{B}$. From Proposition 4.2, it is sufficient to check that the determinant of T remains nonzero when substituting $t = 1, 2, 3, \ldots$ Mathematica quickly calculates that the determinant is actually a polynomial of degree 161 of the form:

$$Ct^5(15 + t^2)^{22}(17 + t^2)(18 + t^2)(20 + t^2) \cdot f(t) \cdot g(t),$$

where $C = 4398046511104$ and f, g are irreducible polynomials of degrees 26, 36, respectively. Therefore, the substitution $t = 0$ produces a non-basis for $\mathrm{M}(7, \mathbb{Q})$ but for all integers $n \geq 1$, the substitution $t = n$ produces a Burnside Γ-basis for $\mathrm{M}(7, \mathbb{Q})$. $\square$

4.2 Proof of Theorem 2.2

Experimentation has revealed that conjugating the tautological representation in B.1 by the diagonal matrix with diagonal entries $\{1, u^{-1}, 1, 1, u^{-1}, 1, u^{-1}\}$ makes for easier computation.

Replacing the parameters in the tautological representation B.1 with $u = 4(18 + t^2)$, $v = (t^2 + 18)(t^2 + 21)$ gives:

$$a = \begin{bmatrix} 1 & 0 & -4 & 2(18+t^2) & 0 & 0 & 4(18+t^2) \\ 0 & 0 & 2 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}, b = \begin{bmatrix} 0 & 0 & -4 & 1 & \frac{p_1}{25+t^2} & \frac{q_1}{2(16+t^2)(25+t^2)} & 0 \\ 0 & 1 & 2 & 0 & \dfrac{p_2}{2(25+t^2)} & \dfrac{q_2}{4(16+t^2)(25+t^2)} & 0 \\ 0 & 0 & -1 & 0 & \dfrac{p_3}{2(25+t^2)} & \dfrac{q_3}{4(25+t^2)} & 0 \\ 0 & 0 & -2 & 0 & \dfrac{p_4}{25+t^2} & \dfrac{q_4}{2(25t^2)} & 0 \\ 0 & 0 & 1 & 0 & \dfrac{p_5}{25+t^2} & \dfrac{q_5}{2(16+t^2)(25+t^2)} & 1 \\ 1 & 0 & -2 & 0 & \dfrac{p_6}{25+t^2} & \dfrac{q_6}{25+t^2} & 0 \\ 0 & 0 & 1 & 0 & \dfrac{p_7}{2(25+t^2)} & \dfrac{q_7}{4(16+t^2)(25+t^2)} & 0 \end{bmatrix}, \quad (4.1)$$

where
$$\begin{aligned} p_1 &= 420 - 295t + 60t^2 - 34t^3 + 2t^4 - t^5, & q_1 &= -480 - 3400t - 28t^2 - 693t^3 - 46t^5 - t^7, \\ p_2 &= 780 + 40t + 97t^2 + 2t^3 + 3t^4, & q_2 &= 14480 + 1420t + 2552t^2 + 169t^3 + 151t^4 + 5t^5 + 3t^6, \\ p_3 &= (16 + t^2)(-15 + 2t - t^2), & q_3 &= -140 + 17t - 25t^2 + t^3 - t^4, \\ p_4 &= -240 - 31t^2 - t^4, & q_4 &= -190 - 15t - 29t^2 - t^3 - t^4, \\ p_5 &= -80 + 20t - 5t^2 + t^3, & q_5 &= -280 + 240t - 50t^2 + 31t^3 - 2t^4 + t^5, \\ p_6 &= 660 - 25t + 55t^2 - t^3 + t^4, & q_6 &= 55 - 20t + 4t^2 - t^3, \\ p_7 &= 190 + 27t^2 + t^4, & q_7 &= (16 + t + t^2)(140 + 25t^2 + t^4). \end{aligned}$$

For $t \in \mathbb{Z}$, the entries of the generator a are already integers. For a particular integer t, by focusing on columns 5 and 6 of b, one can conjugate the representation in several steps so that all matrix entries are integers. This was achieved for the first twelve representations in the sequence, but a pattern is not yet apparent. So, we resort to a non-constructive proof of Theorem 2.2. The proof involves some technical details requiring a number of definitions, but broadly it can be broken down into two steps, which are adapted from [5], [14], together with a smaller step. Here is a summarized description of these steps:

1. Let a, b be the generating matrices with entries in the ring of rational functions $\mathbb{Q}(t)$, given after the statement of Theorem 2.2. The first step is to reduce the problem to showing that (i) the group G of matrices generated by a, b is absolutely irreducible in $\mathrm{SL}(7, \mathbb{Q})$ and (ii) each element of G has integer trace. This is the content of Proposition 4.1.

2. The second step involves showing that hypotheses (i) and (ii) of Proposition 4.1 are satisfied. For (i), a *Burnside basis* was constructed for the 49-dimensional vector space of 7×7 matrices over $\mathbb{Q}$ consisting of elements of G, and for (ii), a computational device used by Ballas and Long in [5] was utilized here. The underlying idea is:

   (a) Note that the set of primes dividing denominators of elements of G is $\mathcal{D} = \{2, 16+t^2, 25+t^2\}$. Therefore, the primes in this set $\mathcal{D}$ are the only primes that can divide denominators of traces of elements of G.

   (b) The method of [5] is to use the Burnside basis to construct a single 49×49 matrix M over $\mathbb{Q}(t)$ where it is guaranteed that all primes dividing denominators of traces of elements of G occur as factors of denominators of entries of M.

   (c) This new set of primes $\mathcal{D}'$, consisting of the primes occurring in denominators of entries of M, depends on the choice of Burnside basis, but a Burnside basis was produced for which $\mathcal{D}'$ did not contain either of the primes $16 + t^2$, $25 + t^2$.

3. The set $\mathcal{D}'$ did contain the prime 2, but G was conjugated so that all denominators occurring in the conjugated generators were odd. Since traces are invariant under conjugation, this concluded the proof that all traces in G are integral.

Since $\mathcal{B}$ is a Burnside Γ-basis, by Proposition 4.2, the group generated by a, b is absolutely irreducible for $F = \mathbb{Q}(t)$, and we hope that the specializations of Γ obtained by setting $t = 0, 1, 2, \ldots$ form a sequence of absolutely irreducible multiplicative subgroups of $\mathrm{M}(7, \mathbb{Q})$. As seen in Proposition 4.3, the sequence of these subgroups is in fact absolutely irreducible with the exception of $t = 0$. Hence, condition (i) of Proposition 4.1 is confirmed for all representations ρ_n for $n \geq 1$. Thankfully, an integral representation conjugate to ρ_0 was found *ad hoc*:

$$a_0 = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 2 & 16 \\ 0 & 0 & 0 & -1 & 0 & 4 & -1 \\ 0 & 0 & 0 & -1 & -1 & -7 & 1 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -1 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & -1 \end{bmatrix}, \quad b_0 = \begin{bmatrix} 0 & 0 & 0 & 7 & 0 & -11 & 2 \\ 0 & 1 & 0 & -7 & -1 & 21 & 0 \\ 0 & 0 & 0 & -8 & 0 & 14 & -1 \\ 0 & 0 & 0 & -7 & 1 & 9 & -1 \\ 0 & 0 & 1 & -6 & -1 & 10 & 1 \\ 1 & 0 & 0 & -5 & 2 & 6 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & 0 \end{bmatrix}.$$

Condition (ii) of Proposition 4.1 still needs to be checked if it holds for all ρ_n, $n \in \mathbb{N}$. It is sufficient to show that traces of all words in the matrices a, b are in the polynomial ring $\mathbb{Z}[t]$, as then substituting $t \in \mathbb{Z}$ will give integer trace.

It is easy to check that the matrices in the Burnside Γ-basis $\mathcal{B}$ have integral trace, and in general the traces of elements of Γ are certainly in the ring $R = \mathbb{Z}\left[t, \frac{1}{2}, \frac{1}{16+t^2}, \frac{1}{25+t^2}\right]$ as 2, $16+t^2$, and $25+t^2$ are precisely the primes occurring in the denominators of matrix entries of a, b.

Traces are unaffected by conjugation, so if one can conjugate the representation so that the matrix entries lie in a different subring S of the field of rational functions $\mathbb{Q}(t)$, the traces must lie in $R \cap S$. The idea is to show that the representation can be conjugated so as to be over a ring S distinct from R. This method was used to expunge the prime 2 from denominators, but for dealing with the primes $16+t^2$ and $25+t^2$, we follow a strategy in [5] which, despite not being algorithmic, is often effective. All of the computational details will not be described here as they are too onerous, but the process is described in sufficient detail for one to repeat the computation.

The first step is to find, by inspection, a collection of 49 words in the matrices a, b that constitute a vector space basis $\{s_1, \ldots, s_{49}\}$ for the 49-dimensional $\mathbb{Q}(t)$-vector space $\mathrm{M}(7, \mathbb{Q}(t))$. Use the Burnside Γ-basis $\mathcal{B} = \{s_1, \ldots, s_{49}\}$.

The trace form $\langle s, t\rangle = \mathrm{tr}(st)$ is non-degenerate, so there exists a basis $\{t_1, \ldots, t_{49}\}$ dual to the s_i with respect to this form, i.e. $\mathrm{tr}(s_i t_j) = \delta_{ij}$, by the Burnside Lemma, i.e. Lemma 1.23.

Here are a couple of immediate observations:

(i) $\mathrm{tr}(t_i) = \mathrm{tr}(s_1 t_i) = \begin{cases} 1 & (i = 1) \\ 0 & (i \neq 1) \end{cases}$,

(ii) Let $M = \sum_{i=1}^{49} \mathrm{tr}(s_i) t_i$. Then, $\langle s_j, M \rangle = \sum_{i=1}^{49} \mathrm{tr}(s_i) \langle s_j, t_i \rangle = \mathrm{tr}(s_j) = \langle s_j, I \rangle$. As the trace form is non-degenerate, one can deduce that $M = I$, and so,

$$I = \sum_{i=1}^{49} \mathrm{tr}(s_i) t_i.$$

The group Γ acts on the algebra $\mathrm{M}(7, \mathbb{Q}(t))$ by left multiplication, and one can use the dual basis $\{t_i\}$ to express this as a representation Φ of Γ into $\mathrm{SL}(49, \mathbb{Q}(t))$: for each $g \in G$, write:

$$g.t_i = \sum_{j=1}^{49} \alpha_{ij}(g) t_j, \qquad \Phi(g) = (\alpha_{ij}(g)).$$

Taking traces and from the first observation (i) above:

$$\mathrm{tr}(g.t_i) = \sum_{i=1}^{49} \alpha_{ij}(g) \mathrm{tr}(t_j) = \alpha_{i1}(g).$$

Multiply both sides of the second observation (ii) above on the left by g to obtain:

$$\mathrm{tr}(g) = \sum_{i=1}^{49} \mathrm{tr}(s_i) \mathrm{tr}(g.t_i) = \sum_{i=1}^{49} \mathrm{tr}(s_i) \alpha_{i1}(g),$$

and recall that $\mathrm{tr}(s_i) \in \mathbb{Z}[t]$ $(1 \leq i \leq 49)$.

So, the trace of the matrix g is now equated with a sum of products of polynomials with matrix entries of $\Phi(g)$. If $\mathrm{tr}(g)$ does have a nontrivial denominator, then any prime dividing that denominator must appear as a factor of the denominator of some matrix entry of $\Phi(g)$ and in turn as a factor of the denominator of some matrix entry of the generators $\Phi(a)$, $\Phi(b)$.

This computation was carried out in Mathematica with a relatively simple program to write as it only involves elementary linear algebraic concepts. However, it was slow to execute due to the complexity of the expressions involved. The primes appearing in denominators of the original subgroup Γ of $\mathrm{SL}(7, \mathbb{Q}(t))$ are 2, $16+t^2$, $25+t^2$, and the latter two of these primes do no appear in denominators of $\Phi(a)$, $\Phi(b)$.

Therefore, the traces of elements of Γ have been shown to lie in the ring $\mathbb{Z}\left[t, \frac{1}{2}\right]$, and denominators that are powers of 2 are still to be contended with. However, for $t \in \mathbb{Z}$ (the case we are interested in), this will be resolved if one can show that the original representation can always be conjugated so that all denominators are odd.

The final stage of the proof of Theorem 2.2 is to conjugate the generating matrices a, b so that substitution of an integer for the parameter t results in matrices where all denominators are odd. It seems that there is not a conjugating matrix that will achieve this for all integers, so partition the integers into four subsets:

$$S_1 = \{4k+2\}, \quad S_2 = \{8k+4\}, \quad S_3 = \{8k\}, \quad S_4 = \{2k+1\} \quad (k \in \mathbb{Z}).$$

Case (i): Suppose $t \in S_1$. Then, $t = 4k+2$ for some integer k. By substituting $t = 4k+2$, the simplified form of all entries in b have odd denominators of the form $29+16k+16k^2$ or $145+196k+260k^2+128k^3+64k^4$. The numerators are described in B.2.

$$b = \begin{bmatrix} 0 & 0 & -2 & 1 & -\frac{b_{15}}{29+16k+16k^2} & -\frac{b_{16}}{145+196k+260k^2+128k^3+64k^4} & 0 \\ 0 & 1 & 2 & 0 & \frac{b_{25}}{29+16k+16k^2} & \frac{b_{26}}{145+196k+260k^2+128k^3+64k^4} & 0 \\ 0 & 0 & -1 & 0 & -\frac{b_{35}}{29+16k+16k^2} & -\frac{b_{36}}{29+16k+16k^2} & 0 \\ 0 & 0 & -1 & 0 & -\frac{b_{45}}{29+16k+16k^2} & -\frac{b_{46}}{29+16k+16k^2} & 0 \\ 0 & 0 & 1 & 0 & \frac{b_{55}}{29+16k+16k^2} & \frac{b_{56}}{145+196k+260k^2+128k^3+64k^4} & 1 \\ 1 & 0 & -1 & 0 & \frac{b_{65}}{29+16k+16k^2} & \frac{b_{66}}{29+16k+16k^2} & 0 \\ 0 & 0 & 1 & 0 & \frac{b_{75}}{29+16k+16k^2} & \frac{b_{76}}{145+196k+260k^2+128k^3+64k^4} & 0 \end{bmatrix}.$$

Case (ii): Suppose $t \in S_2$. Then, $t = 8k+4$ for some integer k. Unfortunately, substituting $t = 8k+4$ alone will not give odd denominators, so conjugation is required. The numerators for columns 5 and 6 are described in B.3.

$$b = \begin{bmatrix} 0 & 0 & -2 & 1 & -\frac{b_{15}}{41+64k+64k^2} & -\frac{b_{16}}{41+146k+274k^2+256k^3+128k^4} & 0 \\ 0 & 1 & 2 & 0 & \frac{b_{25}}{41+64k+64k^2} & \frac{b_{26}}{2(41+146k+274k^2+256k^3+128k^4)} & 0 \\ 0 & 0 & -1 & 0 & -\frac{b_{35}}{41+64k+64k^2} & -\frac{b_{36}}{41+64k+64k^2} & 0 \\ 0 & 0 & -1 & 0 & -\frac{b_{45}}{41+64k+64k^2} & -\frac{b_{46}}{41+64k+64k^2} & 0 \\ 0 & 0 & 1 & 0 & \frac{b_{55}}{41+64k+64k^2} & \frac{b_{56}}{4(41+146k+274k^2+256k^3+128k^4)} & 1 \\ 1 & 0 & -1 & 0 & \frac{b_{65}}{41+64k+64k^2} & -\frac{b_{66}}{41+64k+64k^2} & 0 \\ 0 & 0 & 1 & 0 & \frac{b_{75}}{41+64k+64k^2} & \frac{b_{76}}{4(41+146k+274k^2+256k^3+128k^4)} & 0 \end{bmatrix}.$$

Conjugating a and b simultaneously by the diagonal matrix γ_1 with diagonal entries $\{1,4,1,1,4,1,4\}$ will multiply rows 2, 5, and 7 by 4 while dividing columns 2, 5, and 7 by 4 and give the following representation for b. The representation for $\gamma_1 a \gamma_1^{-1}$ and the numerators for entries in columns 5 and 6 in $\gamma_1 b \gamma_1^{-1}$ are described in B.4 and B.5, respectively.

$$\gamma_1 b \gamma_1^{-1} = \begin{bmatrix} 0 & 0 & -2 & 1 & -\frac{c_{15}}{41+64k+64k^2} & -\frac{c_{16}}{41+146k+274k^2+256k^3+128k^4} & 0 \\ 0 & 1 & 8 & 0 & \frac{c_{25}}{41+64k+64k^2} & \frac{c_{26}}{41+146k+274k^2+256k^3+128k^4} & 0 \\ 0 & 0 & -1 & 0 & -\frac{c_{35}}{41+64k+64k^2} & -\frac{c_{36}}{41+64k+64k^2} & 0 \\ 0 & 0 & -1 & 0 & -\frac{c_{45}}{41+64k+64k^2} & -\frac{c_{46}}{41+64k+64k^2} & 0 \\ 0 & 0 & 4 & 0 & \frac{c_{55}}{41+64k+64k^2} & \frac{c_{56}}{41+146k+274k^2+256k^3+128k^4} & 1 \\ 1 & 0 & -1 & 0 & \frac{c_{65}}{41+64k+64k^2} & -\frac{c_{66}}{41+64k+64k^2} & 0 \\ 0 & 0 & 4 & 0 & \frac{c_{75}}{41+64k+64k^2} & \frac{c_{76}}{41+146k+274k^2+256k^3+128k^4} & 0 \end{bmatrix}.$$

Hence, b is conjugate to a matrix with all odd denominators when $t \equiv 4 \pmod 8$.

Case (iii): Suppose $t \in S_3$. Then, $t = 8k$ for some integer k. Substituting $t = 8k$ gives a similar issue to the above case $t \in S_2$. The numerators for entries in columns 5 and 6 are described in B.6.

$$b = \begin{bmatrix} 0 & 0 & -2 & 1 & -\frac{b_{15}}{25+64k^2} & -\frac{b_{16}}{25+164k^2+256k^4} & 0 \\ 0 & 1 & 2 & 0 & \frac{b_{25}}{25+64k^2} & \frac{b_{26}}{2(25+164k^2+256k^4)} & 0 \\ 0 & 0 & -1 & 0 & -\frac{b_{35}}{25+64k^2} & \frac{b_{36}}{25+64k^2} & 0 \\ 0 & 0 & -1 & 0 & -\frac{b_{45}}{25+64k^2} & -\frac{b_{46}}{25+64k^2} & 0 \\ 0 & 0 & 1 & 0 & \frac{b_{55}}{25+64k^2} & \frac{b_{56}}{2(25+164k^2+256k^4)} & 1 \\ 1 & 0 & -1 & 0 & \frac{b_{65}}{25+64k^2} & \frac{b_{66}}{25+64k^2} & 0 \\ 0 & 0 & 1 & 0 & \frac{b_{75}}{25+64k^2} & \frac{b_{76}}{25+164k^2+256k^4} & 0 \end{bmatrix}.$$

Conjugating a and b simultaneously by the diagonal matrix γ_1 with diagonal entries $\{1, 2, 1, 1, 2, 1, 2\}$ will multiply rows 2, 5, and 7 by 2 while dividing columns 2, 5, and 7 by 2 and gives the following representation. The representation for $\gamma_1 a \gamma_1^{-1}$ and the numerators for entries in columns 5 and 6 in $\gamma_1 b \gamma_1^{-1}$ are described in B.7 and B.8, respectively.

$$\gamma_1 b \gamma_1^{-1} = \begin{bmatrix} 0 & 0 & -2 & 1 & \frac{c_{15}}{25+64k^2} & -\frac{c_{16}}{25+164k^2+256k^4} & 0 \\ 0 & 1 & 2 & 0 & \frac{c_{25}}{25+64k^2} & \frac{c_{26}}{25+164k^2+256k^4} & 0 \\ 0 & 0 & -1 & 0 & -\frac{c_{35}}{25+64k^2} & \frac{c_{36}}{25+64k^2} & 0 \\ 0 & 0 & -1 & 0 & -\frac{c_{45}}{25+64k^2} & -\frac{c_{46}}{25+64k^2} & 0 \\ 0 & 0 & 1 & 0 & \frac{c_{55}}{25+64k^2} & \frac{c_{56}}{2(25+164k^2+256k^4)} & 1 \\ 1 & 0 & -1 & 0 & \frac{c_{65}}{25+64k^2} & \frac{c_{66}}{25+64k^2} & 0 \\ 0 & 0 & 1 & 0 & \frac{c_{75}}{25+64k^2} & \frac{c_{76}}{25+164k^2+256k^4} & 0 \end{bmatrix}.$$

Hence, b is conjugate to a matrix with all odd denominators when $t \equiv 0 \pmod 8$, and so, b is conjugate to a matrix with all odd denominators when t is even.

Case (iv): Suppose $t \in S_4$, i.e. t is odd. Then, $t = 2k+1$ for some integer k. Substituting $t = 2k+1$ gives a more complicated case. The numerators of columns 5 and 6 of b are described in B.9.

$$b = \begin{bmatrix} 0 & 0 & -2 & 1 & \frac{b_{15}}{13-2k+2k^2} & \frac{b_{16}}{221-86k+94k^2-16k^3+8k^4} & 0 \\ 0 & 1 & 2 & 0 & \frac{b25}{2(13-2k+2k^2)} & \frac{b_{26}}{221-86k+94k^2-16k^3+8k^4} & 0 \\ 0 & 0 & -1 & 0 & -\frac{b_{35}}{2(13-2k+2k^2)} & \frac{b_{36}}{13-2k+2k^2} & 0 \\ 0 & 0 & -1 & 0 & \frac{b_{45}}{13-2k+2k^2} & \frac{b_{46}}{13-2k+2k^2} & 0 \\ 0 & 0 & 1 & 0 & \frac{b_{55}}{13-2k+2k^2} & \frac{b_{56}}{221-86k+94k^2-16k^3+8k^4} & 1 \\ 1 & 0 & -1 & 0 & \frac{b_{65}}{2(13-2k+2k^2)} & \frac{b_{66}}{13-2k+2k^2} & 0 \\ 0 & 0 & 1 & 0 & \frac{b_{75}}{2(13-2k+2k^2)} & \frac{b_{76}}{221-86k+94k^2-16k^3+8k^4} & 0 \end{bmatrix}.$$

Conjugating a and b simultaneously by the matrix γ_1,

$$\gamma_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

gives us the following representation. The representation for $\gamma_1 a \gamma_1^{-1}$ and the numerators for entries in columns 5 and 6 in $\gamma_1 b \gamma_1^{-1}$ are described in B.10 and B.11, respectively.

$$\gamma_1 b \gamma_1^{-1} = \begin{bmatrix} 0 & 0 & -4 & 1 & \frac{c_{15}}{13-2k+2k^2} & \frac{c_{16}}{221-86k+94k^2-16k^3+8k^4} & 2 \\ 0 & 1 & 3 & 0 & \frac{c_{25}}{13-2k+2k^2} & \frac{c_{26}}{221-86k+94k^2-16k^3+8k^4} & -2 \\ 0 & 0 & 0 & 0 & \frac{c_{35}}{13-2k+2k^2} & \frac{c_{36}}{(13-2k+2k^2)(17-4k+4k^2}) & 0 \\ 0 & 0 & -2 & 0 & -\frac{c_{45}}{13-2k+2k^2} & \frac{c_{46}}{13-2k+2k^2} & 1 \\ 0 & 0 & 1 & 0 & \frac{c_{55}}{13-2k+2k^2} & \frac{c_{56}}{221-86k+94k^2-16k^3+8k^4} & 0 \\ 1 & 0 & -2 & 0 & \frac{c_{65}}{13-2k+2k^2} & \frac{c_{66}}{13-2k+2k^2} & 1 \\ 0 & 0 & 2 & 0 & \frac{c_{75}}{13-2k+2k^2} & \frac{c_{76}}{221-86k+94k^2-16k^3+8k^4} & -1 \end{bmatrix}.$$

Hence, b is conjugate to a matrix with all odd denominators for any integer t, and so, all elements of the ring S, the ring containing $\chi(\Gamma)$, are conjugate to elements with odd denominators. However, S is contained in the ring $\mathbb{Z}[t, \frac{1}{2}]$. Thus, $\chi(\Gamma) \subset \mathbb{Z}$ for Γ the (4, 6, 2)-triangle group, and by Proposition 4.1, there exists $\tau \in \mathrm{GL}(n, \mathbb{Q})$ with $\tau \Gamma \tau^{-1} \subset \mathrm{SL}(n, \mathbb{Z})$, i.e. the representations of the (4, 6, 2)-triangle group are conjugate to matrices with integer entries. $\square$

Chapter 5

Further enquires

5.1 Mysterious Parabolas

In each of these examples, the Hitchin component is two-dimensional, and the occurrence of these parabolas was discovered through experimentation. We currently do not understand why these parabolas occur. For all examples of Hitchin components of dimension two below, choosing $u = \mathrm{tr}(b^{-1}a)$ and $v = \mathrm{tr}(a^{-1}b)$ worked.

Example 5.1.1. For finding the Hitchin component of the representation of the (3,4,4)-triangle group in $\mathrm{SL}(3, \mathbb{R})$, the dot plot of (u, v) given in Figure 5.1 such that $\alpha^2 = u^2v^2 - 4(u^3 + v^3) - 4(u^2 + v^2) + 14uv + 4(u + v) - 7$ is a positive integer revealed the presence of an apparently infinite set of points lying on a parabola, which has equations $u = 3n^2 - n + 2$ and $v = 3n^2 + n + 2$ for integers $n \geq 0$.

Example 5.1.2. A similar dot plot was obtained in Figure 5.2 for the Hitchin component of the representation of the (3,3,4)-triangle group in $\mathrm{SL}(3, \mathbb{R})$with two parabolas. The inner parabola has equations $u = 4n^2 + 3n + 2$ and $v = 4n^2 + 5n + 3$, and the outer parabola on the dot plot (u, v) has equations $u = n^2 + 2$ and $v = (n + 1)^2 + 2$ for integers $n \geq 0$.

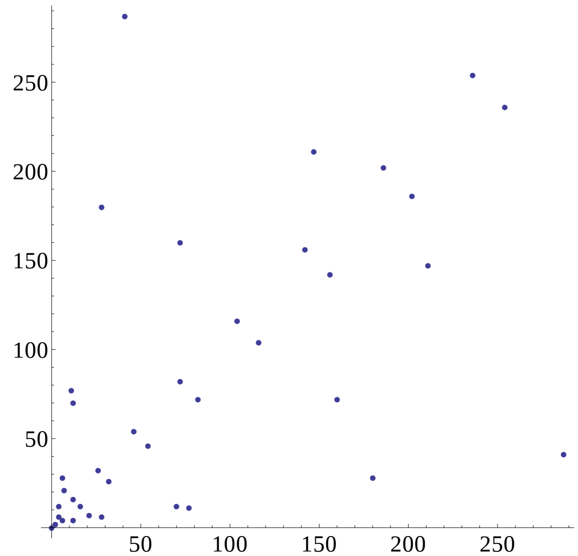

Figure 5.1: Dot plot of (u, v) for (3,4,4)-triangle group in $\mathrm{SL}(3, \mathbb{R})$

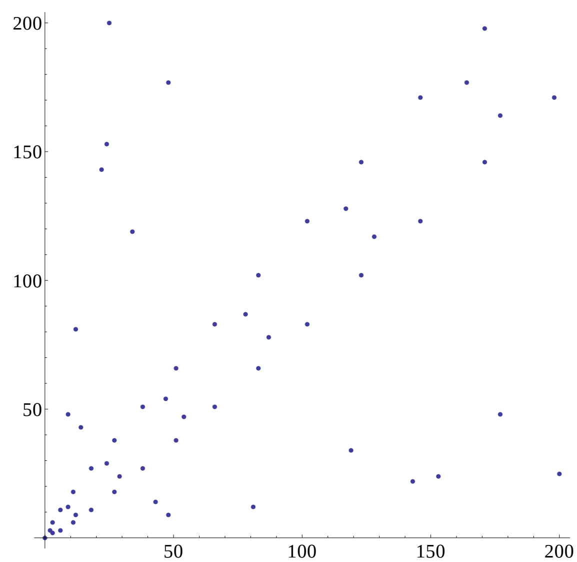

Figure 5.2: Dot plot of (u, v) for (3,3,4)-triangle group in $\mathrm{SL}(3, \mathbb{R})$

Example 5.1.3. A similar dot plot was obtained in Figure 5.3 for the Hitchin component of the representation of the (3,3,4)-triangle group in $\mathrm{SL}(4, \mathbb{R})$ with a parabola, which has equations $u = 4n^2 - n + 5$ and $v = 4n^2 + n + 5$ for integers $n \geq 0$.

Example 5.1.4. A similar dot plot was obtained in Figure 5.4 for the Hitchin component of the representation of the (3,3,4)-triangle group in $\mathrm{SL}(5, \mathbb{R})$, with two parabolas. The inner parabola has equations $u = 12n^2 + 9n + 3$ and $v = 12n^2 + 15n + 6$, and the outer parabola on the dot plot (u, v) has equations $u = n^2$ and $v = (n+1)^2$ for integers $n \geq 0$.

In B.1 in the Appendix, the parameters set for finding this particular set of solutions included requiring the basepoint (72, 378) to be part of the solution set. However, we did not look into other solutions not containing this point. Such a solution might look like one of the mysterious parabolas previously mentioned.

5.2 Infinite Sequence of ρ_n for $n \equiv 0 \pmod 4$

An infinite sequence of exact integer representations of the $(2, 4, 6)$ triangle group into G_2 has been discovered for the case t is a multiple of 4, but integer versions of representations for t not divisible by 4 are pending.

For this sequence, use the generators c, d of $\Gamma(2, 4, 6)$ where $c^2 = d^6 = (cd)^4 = 1$.

$$c = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

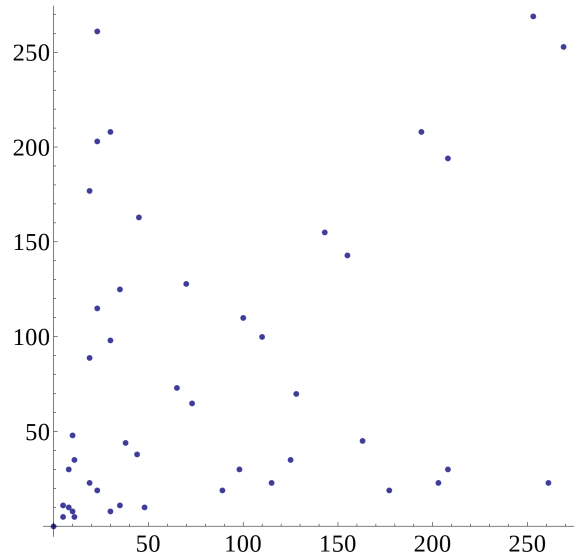

Figure 5.3: Dot plot of (u, v) for (3,3,4)-triangle group in $\mathrm{SL}(4, \mathbb{R})$

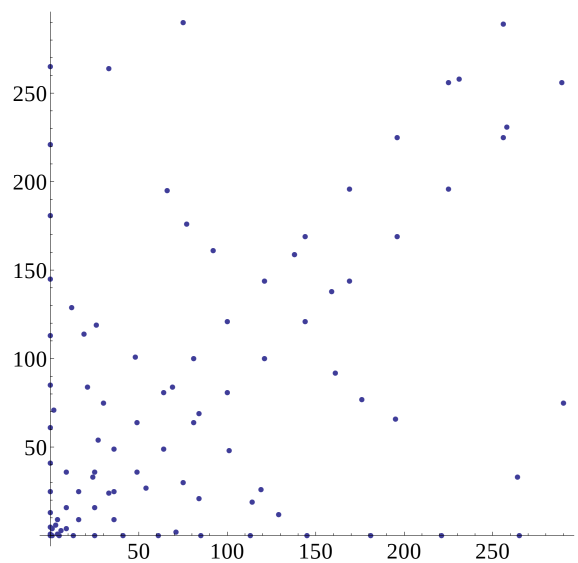

Figure 5.4: Dot plot of (u, v) for (3,3,4)-triangle group in $\mathrm{SL}(5, \mathbb{R})$

$$d = \begin{bmatrix} -7-4k & 1-4k & 77+4k & 80+8k & 19+4k & -4k & 16 \\ 4+3k & k & -63+8k & -65+8k & -16 & k & -8 \\ -7-2k & -1-2k & 98-44k+4k^2 & 102-44k+4k^2 & 25-8k & -1-2k & 12-4k \\ 16+5k & 1+3k & -170+54k-4k^2 & -174+54k-4k^2 & -43+8k & 1+3k & -22+4k \\ -38-8k & -4k & 288-72k+4k^2 & 287-72k+4k^2 & 72-8k & -4k & 40-4k \\ 8+5k & 2-k & -112+28k & -115+32k & -29+4k & 1-k & -8 \\ -9-4k-3k^2 & -k-k^2 & 71-k-2k^2 & 71-k-2k^2 & 18+2k & -k-k^2 & 9+2k \end{bmatrix}$$

The original generators (the tautological representation in terms of u, v) are $a = cd^{-1}$ and $b = d$.

Using Mathematica, one can confirm that the orders of a, b, and ab are 4, 6, and 2, respectively. Recall that the original parameters are $u = \operatorname{tr}(aaab) + 1$, $v = \operatorname{tr}(aabb) + 1$ and that the coordinates of the sequence of integer points were $u = 4(18+t^2)$, $v = (18+t^2)(25+t^2)$.

Calculating u and v based on the matrices a and b above, $u = \operatorname{tr}(aaab)+1 = 72+64k^2 = 4(18+(4k)^2)$ and $v = 378 + 624k^2 + 256k^4 = (18+(4k)^2)(21+(4k)^2)$. Substituting $t = 4k$, the parameters become $u = 4(18+t^2)$ and $v = (18+t^2)(21+t^2)$. Hence, there is an infinite sequence of representations ρ_n with integer entries for $n = t \equiv 0 \pmod 4$.

The generators $a = cd^{-1}$ and $b = d$ are written explicitly in terms of n in Section B.1.3.

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Appendix

Appendix A

Figures

The following is a drawn construction of the hyperbolic (p, q, r)-triangle in the upper-half plane model.

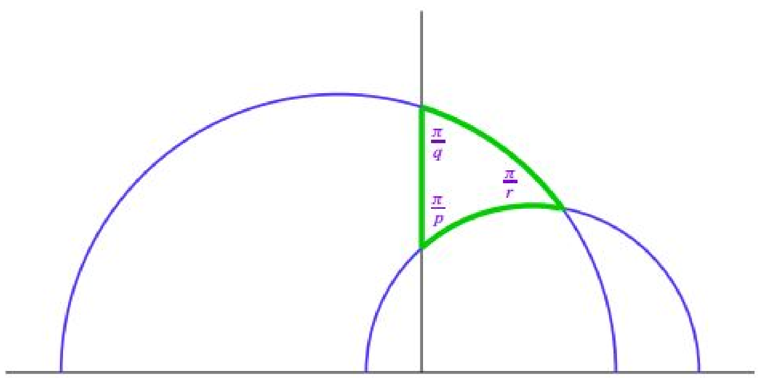

Figure A.1: The (p, q, r)-triangle

Appendix B

Summary of Equations

B.1 Matrix Representations

B.1.1 Tautological Representation for the Hitchin component of $\Gamma(2,4,6)$

By means of a *tautological representation*, we can express the entire Hitchin component by means of a single matrix for each generator. The matrix entries are algebraic expressions in the parameters u, v of the variety, and assigning the values to u, v together with a sign of α determines a specific representation in the variety. The choice of values for u, v should give a point in the correct region of the (u, v)-plane as specified in Figure 1.4. Recall that in Figure 1.4 the Hitchin component consists of two sheets glued along the curve $\alpha = 0$, and the sign of α determines the particular sheet where the representation lies.

Here matrices with entries in the field $\mathbb{Q}(u, v)(\alpha)$ are given for the triangle group $\Gamma(2,4,6)$, a, b generators of orders 4, 6 respectively, and their product ab has order 2. The *base representation* ρ_0 is the composition of the holomony representation in $\mathrm{PSL}(2,\mathbb{R})$ (given by the hyperbolic structure) with the irreducible representation of $\mathrm{PSL}(2,\mathbb{R})$ in $\mathrm{PSL}(7,\mathbb{R}) = \mathrm{SL}(7,\mathbb{R})$ and has coordinates $(u, v) = (72, 378)$, $\alpha = 0$.

The base representation of $\mathcal{H}_{G_2}(\Gamma(2,4,6))$ ρ_0 is at $u = 72$ and $v = 378$, and the tautological representation (in parameters u and v) is

$$a = \begin{bmatrix} 1 & 0 & -4 & \frac{1}{2}u & 0 & 0 & 1 \\ 0 & 0 & 2u & -u & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & u & -u & 0 & 0 & -1 \\ 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & u & 0 & 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 & 0 & -4 & 1 & \frac{\eta_{15}}{2u\delta_1} & \frac{\eta_{16}}{2\delta_1} & 0 \\ 0 & 1 & 2u & 0 & \frac{\eta_{25}}{2\delta_1} & \frac{\eta_{26}}{2\delta_1} & 0 \\ 0 & 0 & -1 & 0 & -\frac{\eta_{35}}{\delta_1} & -\frac{\eta_{36}}{\delta_1} & 0 \\ 0 & 0 & -2 & 0 & -\frac{\eta_{45}}{\delta_2} & \frac{\eta_{46}}{\delta_2} & 0 \\ 0 & 0 & u & 0 & \frac{\eta_{55}}{2\delta_1} & \frac{\eta_{56}}{\delta_1} & 1 \\ 1 & 0 & -2 & 0 & \frac{\eta_{65}}{2u\delta_1} & \frac{\eta_{66}}{2\delta_1} & 0 \\ 0 & 0 & u & 0 & \frac{\eta_{75}}{\delta_1} & -\frac{\eta_{76}}{\delta_1} & 0 \end{bmatrix}, \quad \text{(B.1)}$$

where $\delta_1 = 8u^5 + u^6 - 32u^3(-6+v) - 256uv(6+v) - 256v^2(6+v) - 8u^4(14+3v) + 32u^2v(34+5v)$,

$$\delta_2 = 8u^3 + u^4 + 32u(6+v) + 32v(6+v) - 16u^2(7+v),$$

$$\begin{aligned} \eta_{15} = {} & -8u^6 - u^7 + 16u^5(10+v) - 64v(6+v)\alpha + 2u^4(-168 - 48v + 2v^2 + \alpha) \\ & + 32u^2(-144v - 36v^2 - 2v^3 + 7\alpha) + u^3(32v^2 - 28\alpha + v(320 - 2\alpha)) \\ & + 16u(6+v)(32v^2 - 4\alpha + v(96+\alpha)), \end{aligned}$$

$$\begin{aligned} \eta_{16} = {} & 1484 + 6(-1+u)^6 - 6(-1+u)^5(-16+v) - 4406(-1+v) - 7728(-1+v)^2 \\ & - 2864(-1+v)^3 - 256(-1+v)^4 - 74(-1+u)^4(-7+3v) \\ & + 4(-1+u)^3(-124 - 263v + 24v^2) + 2(-1+u)^2(-711 - 910v + 400v^2 + 8v^3) \\ & + 2(-1+u)(136 + 1249v + 1552v^2 + 112v^3) \\ & + (-4+u)(-16u2 + u^3 + 16u(3+v) - 8v(6+v))\alpha, \end{aligned}$$

$$\begin{aligned} \eta_{25} = {} & u[-4u^5 + 2u^4(22+v) - 8u^2(-36 + 44v + 3v^2) + u^3(-192 + 32v + \alpha) \\ & - 4u(22\alpha + 3v(-128+\alpha)) + 32(2v^3 + 6\alpha + 3v(-24+\alpha))], \end{aligned}$$

$$\eta_{26} = u[-2u^6 + 2u^5 v + 16(-6+v)v\alpha + u^4(-224 + 48v + \alpha) - 4u^3(-264 - 64v + 10v^2 + \alpha)$$
$$- 4u^2(104v^2 + 3v(32+\alpha) + 8(36+\alpha)) + 16u(24v^2 + 14v^3 + 6\alpha + v(-72 + 5\alpha))],$$
$$\eta_{35} = 2(-6 + 2u - v)(4u^3 + 96v + 16v^2 - 2u^2(6+v) - 8\alpha + u(-32v + \alpha)),$$
$$\eta_{36} = 4(-6 + 2u - v)(-14u^3 + u^4 + 4u^2(6+v) + 2v\alpha - 2u(-12v + 2v^2 + \alpha)),$$
$$\eta_{45} = 8(6 - 2u + v)^2,$$
$$\eta_{46} = -16u^2 - 8u^3 + u^4 - 2(6+v)(-16v + \alpha) + 4u(12 - 4v - v^2 + \alpha),$$
$$\eta_{55} = u[4u^5 - 2u^4(-26+v) + 8u^2(300 + 48v + 5v^2) + u^3(-736 - 112v + \alpha)$$
$$- 4u(576 - 80v^2 + 22\alpha + 3v(-128+\alpha)) + 32(-48v^2 - 4v^3 + 6\alpha + 3v(-48+\alpha))],$$
$$\eta_{56} = u[2u^5 + u^6 - 4u^4(62 + 3v) - 16v(6+v)\alpha - 4u^3(-264 - 78v + v^2 + \alpha)$$
$$+ 32uv(-36 - 6v + 2v^2 + \alpha) + 2u^2(-64v^2 + 6(-96+\alpha) + v(-96+\alpha))],$$
$$\eta_{65} = 8u^6 + u^7 - 8u^5(22 + 3v) - 64v(6+v)\alpha + u^4(576 + 32v - 2\alpha)$$
$$- 16u^3(36 - 88v - 9v^2 + \alpha) - 64u(6+v)(-12v + 2v^2 + \alpha)$$
$$+ 32u^2(-24v^2 + 7\alpha + v(-144+\alpha)),$$
$$\eta_{66} = -6u^6 + 2u^5(-34+v) + 512v^2(6+v) - 8u^3(348 + 40v + 5v^2) + u^4(960 + 160v - \alpha)$$
$$- 32u(-64v^2 - 4v^3 + 6\alpha + 3v(-80+\alpha)) + 4u^2(576 - 160v^2 + 22\alpha + v(-928 + 3\alpha)),$$
$$\eta_{75} = 24u^5 + u^6 - 256v^2(6+v) - 32u^4(12+v) + 48u^3(26 + 3v) + 32uv(-84 - 8v + v^2)$$
$$+ 64u^2(-18 + 17v + v^2),$$
and $\eta_{76} = 2u[-14u^2 + u^3 - 4(-6+v)v + 4u(6+v)][-4u^2 + 2u(6+v) + \alpha]$.

B.1.2 Details of the last step in the proof of Theorem 2.2

In the last step of the proof of Theorem 2.2, it was shown how to conjugate the representation ρ_n so that all denominators are odd. It was necessary to partition the integers into subsets S_1, S_2, S_3, S_4:

$$S_1 = \{4k+2\}, \quad S_2 = \{8k+4\}, \quad S_3 = \{8k\}, \quad S_4 = \{2k+1\} \quad (k \in \mathbb{Z})$$

and find conjugations applicable to each subset. Here, we specify the numerators of the fractions involved in these conjugations.

Representation of b for $t \in S_1$

$$b = \begin{bmatrix} 0 & 0 & -2 & 1 & -\frac{b_{15}}{29+16k+16k^2} & -\frac{b_{16}}{145+196k+260k^2+128k^3+64k^4} & 0 \\ 0 & 1 & 2 & 0 & \frac{b_{25}}{29+16k+16k^2} & \frac{b_{26}}{145+196k+260k^2+128k^3+64k^4} & 0 \\ 0 & 0 & -1 & 0 & -\frac{b_{35}}{29+16k+16k^2} & -\frac{b_{36}}{29+16k+16k^2} & 0 \\ 0 & 0 & -1 & 0 & -\frac{b_{45}}{29+16k+16k^2} & -\frac{b_{46}}{29+16k+16k^2} & 0 \\ 0 & 0 & 1 & 0 & \frac{b_{55}}{29+16k+16k^2} & \frac{b_{56}}{145+196k+260k^2+128k^3+64k^4} & 1 \\ 1 & 0 & -1 & 0 & \frac{b_{65}}{29+16k+16k^2} & \frac{b_{66}}{29+16k+16k^2} & 0 \\ 0 & 0 & 1 & 0 & \frac{b_{75}}{29+16k+16k^2} & \frac{b_{76}}{145+196k+260k^2+128k^3+64k^4} & 0 \end{bmatrix}, \tag{B.2}$$

where $b_{15} = 101 + 958k + 1408k^2 + 1856k^3 + 1024k^4 + 512k^5$,

$b_{16} = 1817 + 7978k + 17076k^2 + 24744k^3 + 23680k^4 + 16640k^5 + 7168k^6 + 2048k^7$,

$b_{25} = 8(82 + 137k + 181k^2 + 104k^3 + 48k^4)$,

$b_{26} = 4(989 + 2433k + 4155k^2 + 4114k^3 + 3048k^4 + 1312k^5 + 384k^6)$,

$b_{35} = 2(5 + 4k + 4k^2)(15 + 8k + 16k^2)$,

$b_{36} = 107 + 206k + 344k^2 + 224k^3 + 128k^4$,

$b_{45} = 2(95 + 156k + 220k^2 + 128k^3 + 64k^4)$,

$b_{46} = 2(90 + 175k + 236k^2 + 144k^3 + 64k^4)$,

$b_{55} = 4(-13 + 12k + 4k^2 + 16k^3)$,

$b_{56} = 62 + 428k + 672k^2 + 880k^3 + 512k^4 + 256k^5$,

$b_{65} = 62 + 428k + 672k^2 + 880k^3 + 512k^4 + 256k^5$,

$b_{66} = 23 - 64k - 32k^2 - 64k^3$,

$b_{75} = 157 + 280k + 408k^2 + 256k^3 + 128k^4$,

$b_{76} = 4(176 + 523k + 997k^2 + 1106k^3 + 888k^4 + 416k^5 + 128k^6)$.

Representations when $t \in S_2$

$$b = \begin{bmatrix} 0 & 0 & -2 & 1 & -\frac{b_{15}}{41+64k+64k^2} & -\frac{b_{16}}{41+146k+274k^2+256k^3+128k^4} & 0 \\ 0 & 1 & 2 & 0 & \frac{b_{25}}{41+64k+64k^2} & \frac{b_{26}}{2(41+146k+274k^2+256k^3+128k^4)} & 0 \\ 0 & 0 & -1 & 0 & -\frac{b_{35}}{41+64k+64k^2} & -\frac{b_{36}}{41+64k+64k^2} & 0 \\ 0 & 0 & -1 & 0 & -\frac{b_{45}}{41+64k+64k^2} & -\frac{b_{46}}{41+64k+64k^2} & 0 \\ 0 & 0 & 1 & 0 & \frac{b_{55}}{41+64k+64k^2} & \frac{b_{56}}{4(41+146k+274k^2+256k^3+128k^4)} & 1 \\ 1 & 0 & -1 & 0 & \frac{b_{65}}{41+64k+64k^2} & -\frac{b_{66}}{41+64k+64k^2} & 0 \\ 0 & 0 & 1 & 0 & \frac{b_{75}}{41+64k+64k^2} & \frac{b_{76}}{4(41+146k+274k^2+256k^3+128k^4)} & 0 \end{bmatrix}. \tag{B.3}$$

where $b_{15} = 4(311 + 2215k + 6368k^2 + 10368k^3 + 9216k^4 + 4096k^5)$,

$b_{16} = 1912 + 15555k + 59288k^2 + 136104k^3 + 202240k^4 + 195584k^5 + 114688k^6 + 32768k^7$,

$b_{25} = 2(847 + 3360k + 6544k^2 + 6400k^3 + 3072k^4)$,

$b_{26} = 3996 + 23359k + 67592k^2 + 115600k^3 + 124288k^4 + 78848k^5 + 24576k^6$,

$b_{35} = 16(1 + 2k + 2k^2)(23 + 48k + 64k^2)$,

$b_{36} = 4(83 + 391k + 872k^2 + 960k^3 + 512k^4)$,

$b_{45} = 16(31 + 126k + 254k^2 + 256k^3 + 128k^4)$,

$b_{46} = 517 + 2204k + 4384k^2 + 4352k^3 + 2048k^4$,

$b_{55} = 16(-1 + 14k + 28k^2 + 32k^3)$,

$b_{56} = 297 + 2096k + 6160k^2 + 10176k^3 + 9216k^4 + 4096k^5$,

$b_{65} = 4(204 + 623k + 1112k^2 + 960k^3 + 512k^4)$,

$b_{66} = 25 + 288k + 512k^2 + 512k^3$,

$b_{75} = 439 + 1888k + 3936k^2 + 4096k^3 + 2048k^4$,

$b_{76} = 1791 + 11790k + 37024k^2 + 67872k^3 + 77056k^4 + 51200k^5 + 16384k^6$.

Conjugating a and b by the diagonal matrix γ_1 with diagonal entries $\{1,4,1,1,4,1,4\}$ gives

$$\gamma_1 a \gamma_1^{-1} = \begin{bmatrix} 1 & 0 & -2 & 4(17+32k+32k^2) & 0 & 0 & 17+32k+32k^2 \\ 0 & 0 & 8 & -8 & 1 & 0 & -1 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & -8 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 & -1 \end{bmatrix}, \tag{B.4}$$

$$\gamma_1 b \gamma_1^{-1} = \begin{bmatrix} 0 & 0 & -2 & 1 & -\frac{c_{15}}{41+64k+64k^2} & -\frac{c_{16}}{41+146k+274k^2+256k^3+128k^4} & 0 \\ 0 & 1 & 8 & 0 & \frac{c_{25}}{41+64k+64k^2} & \frac{c_{26}}{41+146k+274k^2+256k^3+128k^4} & 0 \\ 0 & 0 & -1 & 0 & -\frac{c_{35}}{41+64k+64k^2} & -\frac{c_{36}}{41+64k+64k^2} & 0 \\ 0 & 0 & -1 & 0 & -\frac{c_{45}}{41+64k+64k^2} & -\frac{c_{46}}{41+64k+64k^2} & 0 \\ 0 & 0 & 4 & 0 & \frac{c_{55}}{41+64k+64k^2} & \frac{c_{56}}{41+146k+274k^2+256k^3+128k^4} & 1 \\ 1 & 0 & -1 & 0 & \frac{c_{65}}{41+64k+64k^2} & -\frac{c_{66}}{41+64k+64k^2} & 0 \\ 0 & 0 & 4 & 0 & \frac{c_{75}}{41+64k+64k^2} & \frac{c_{76}}{41+146k+274k^2+256k^3+128k^4} & 0 \end{bmatrix}, \tag{B.5}$$

where $c_{15} = 311 + 2215k + 6368k^2 + 10368k^3 + 9216k^4 + 4096k^5$,

$c_{16} = 1912 + 15555k + 59288k^2 + 136104k^3 + 202240k^4 + 195584k^5 + 114688k^6 + 32768k^7$,

$c_{25} = 2(847 + 3360k + 6544k^2 + 6400k^3 + 3072k^4)$,

$c_{26} = 2(3996 + 23359k + 67592k^2 + 115600k^3 + 124288k^4 + 78848k^5 + 24576k^6)$,

$c_{35} = 4(1 + 2k + 2k^2)(23 + 48k + 64k^2)$,

$c_{36} = 4(83 + 391k + 872k^2 + 960k^3 + 512k^4)$,

$c_{45} = 4(31 + 126k + 254k^2 + 256k^3 + 128k^4)$,

$c_{46} = 517 + 2204k + 4384k^2 + 4352k^3 + 2048k^4$,

$c_{55} = 16(-1 + 14k + 28k^2 + 32k^3)$,

$c_{56} = 297 + 2096k + 6160k^2 + 10176k^3 + 9216k^4 + 4096k^5$,

$c_{65} = 204 + 623k + 1112k^2 + 960k^3 + 512k^4$,

$c_{66} = 25 + 288k + 512k^2 + 512k^3$,

$c_{75} = 439 + 1888k + 3936k^2 + 4096k^3 + 2048k^4$,

$c_{76} = 1791 + 11790k + 37024k^2 + 67872k^3 + 77056k^4 + 51200k^5 + 16384k^6$.

Representations when $t \in S_3$

$$b = \begin{bmatrix} 0 & 0 & -2 & 1 & -\frac{b_{15}}{25+64k^2} & -\frac{b_{16}}{25+164k^2+256k^4} & 0 \\ 0 & 1 & 2 & 0 & \frac{b_{25}}{25+64k^2} & \frac{b_{26}}{2(25+164k^2+256k^4)} & 0 \\ 0 & 0 & -1 & 0 & -\frac{b_{35}}{25+64k^2} & \frac{b_{36}}{25+64k^2} & 0 \\ 0 & 0 & -1 & 0 & -\frac{b_{45}}{25+64k^2} & -\frac{b_{46}}{25+64k^2} & 0 \\ 0 & 0 & 1 & 0 & \frac{b_{55}}{25+64k^2} & \frac{b_{56}}{2(25+164k^2+256k^4)} & 1 \\ 1 & 0 & -1 & 0 & \frac{b_{65}}{25+64k^2} & \frac{b_{66}}{25+64k^2} & 0 \\ 0 & 0 & 1 & 0 & \frac{b_{75}}{25+64k^2} & \frac{b_{76}}{25+164k^2+256k^4} & 0 \end{bmatrix}, \tag{B.6}$$

where $b_{15} = 2(-105 + 590k - 960k^2 + 4352k^3 - 2048k^4 + 8192k^5)$,

$b_{16} = 15 + 850k + 56k^2 + 11088k^3 + 47104k^4 + 65536k^7$,

$b_{25} = 390 + 160k + 3104k^2 + 512k^3 + 6144k^4$,

$b_{26} = 905 + 710k + 10208k^2 + 5408k^3 + 38656k^4 + 10240k^5 + 49152k^6$,

$b_{35} = 8(1 + 4k^2)(15 - 16k + 64k^2)$,

$b_{36} = -70 + 68k - 800k^2 + 256k^3 - 2048k^4$,

$b_{45} = 8(15 + 124k^2 + 256k^4)$,

$b_{46} = 95 + 60k + 928k^2 + 256k^3 + 2048k^4$,

$b_{55} = 16(-5 + 10k - 20k^2 + 32k^3)$,

$b_{56} = -35 + 240k - 400k^2 + 1984k^3 - 1024k^4 + 4096k^5$,

$b_{65} = 2(165 - 50k + 880k^2 - 128k^3 + 1024k^4)$,

$b_{66} = 55 - 160k + 256k^2 - 512k^3$,

$b_{75} = 95 + 864k^2 + 2048k^4$,

$b_{76} = 70 + 35k + 1080k^2 + 400k^3 + 5248k^4 + 1024k^5 + 8192k^6$.

Conjugating a and b simultaneously by the diagonal matrix γ_1 with diagonal entries $\{1,2,1,1,2,1,2\}$ gives:

$$\gamma_1 a \gamma_1^{-1} = \begin{bmatrix} 1 & 0 & -2 & 4(9+32k^2) & 0 & 0 & 18+64k^2 \\ 0 & 0 & 4 & -4 & 1 & 0 & -1 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -4 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & -1 \end{bmatrix}, \tag{B.7}$$

$$\gamma_1 b \gamma_1^{-1} = \begin{bmatrix} 0 & 0 & -2 & 1 & \frac{c_{15}}{25+64k^2} & -\frac{c_{16}}{25+164k^2+256k^4} & 0 \\ 0 & 1 & 2 & 0 & \frac{c_{25}}{25+64k^2} & \frac{c_{26}}{25+164k^2+256k^4} & 0 \\ 0 & 0 & -1 & 0 & -\frac{c_{35}}{25+64k^2} & \frac{c_{36}}{25+64k^2} & 0 \\ 0 & 0 & -1 & 0 & -\frac{c_{45}}{25+64k^2} & -\frac{c_{46}}{25+64k^2} & 0 \\ 0 & 0 & 1 & 0 & \frac{c_{55}}{25+64k^2} & \frac{c_{56}}{2(25+164k^2+256k^4)} & 1 \\ 1 & 0 & -1 & 0 & \frac{c_{65}}{25+64k^2} & \frac{c_{66}}{25+64k^2} & 0 \\ 0 & 0 & 1 & 0 & \frac{c_{75}}{25+64k^2} & \frac{c_{76}}{25+164k^2+256k^4} & 0 \end{bmatrix}, \tag{B.8}$$

where $c_{15} = -105 + 590k - 960k^2 + 4352k^3 - 2048k^4 + 8192k^5$,

$c_{16} = 15 + 850y + 56y^2 + 11088y^3 + 47104y^4 + 65536y^7$,

$c_{25} = 390 + 160k + 3104k^2 + 512k^3 + 6144k^4$,

$c_{26} = 905 + 710k + 10208k^2 + 5408k^3 + 38656k^4 + 10240k^5 + 49152k^6$,

$c_{35} = 4(1+4k^2)(15 - 16k + 64k^2)$,

$c_{36} = -70 + 68k - 800k^2 + 256k^3 - 2048k^4$,

$c_{45} = 4(15 + 124k^2 + 256k^4)$,

$c_{46} = 95 + 60k + 928k^2 + 256k^3 + 2048k^4$,

$c_{55} = 16(-5 + 10k - 20k^2 + 32k^3)$,

$c_{56} = -35 + 240k - 400k^2 + 1984k^3 - 1024k^4 + 4096k^5$,

$c_{65} = 165 - 50k + 880k^2 - 128k^3 + 1024k^4$,

$c_{66} = 55 - 160k + 256k^2 - 512k^3$,

$c_{75} = 95 + 864k^2 + 2048k^4$,

$c_{76} = 2(70 + 35k + 1080k^2 + 400k^3 + 5248k^4 + 1024k^5 + 8192k^6)$.

Representations when $t \in S_4$

$$b = \begin{bmatrix} 0 & 0 & -2 & 1 & \frac{b_{15}}{13-2k+2k^2} & \frac{b_{16}}{221-86k+94k^2-16k^3+8k^4} & 0 \\ 0 & 1 & 2 & 0 & \frac{b25}{2(13-2k+2k^2)} & \frac{b_{26}}{221-86k+94k^2-16k^3+8k^4} & 0 \\ 0 & 0 & -1 & 0 & -\frac{b_{35}}{2(13-2k+2k^2)} & \frac{b_{36}}{13-2k+2k^2} & 0 \\ 0 & 0 & -1 & 0 & \frac{b_{45}}{13-2k+2k^2} & \frac{b_{46}}{13-2k+2k^2} & 0 \\ 0 & 0 & 1 & 0 & \frac{b_{55}}{13-2k+2k^2} & \frac{b_{56}}{221-86k+94k^2-16k^3+8k^4} & 1 \\ 1 & 0 & -1 & 0 & \frac{b_{65}}{2(13-2k+2k^2)} & \frac{b_{66}}{13-2k+2k^2} & 0 \\ 0 & 0 & 1 & 0 & \frac{b_{75}}{2(13-2k+2k^2)} & \frac{b_{76}}{221-86k+94k^2-16k^3+8k^4} & 0 \end{bmatrix}. \tag{B.9}$$

where $b_{15} = 203 - 265k + 184k^2 - 104k^3 + 28k^4 - 8k^5$,

$b_{16} = 908 - 2830k + 2532k^2 - 2376k^3 + 1060k^4 - 536k^5 + 112k^6 - 32k^7$,

$b_{25} = 419 - 160k + 218k^2 - 40k^3 + 24k^4$,

$b_{26} = 3898 - 1887k + 2946k^2 - 890k^3 + 684k^4 - 104k^5 + 48k^6$,

$b_{35} = (9 - 4k + 2k^2)(17 - 4k + 4k^2)$,

$b_{36} = -46 + 37k - 34k^2 + 10k^3 - 4k^4$,

$b_{45} = -68 + 33k - 37k^2 + 8k^3 - 4k^4$,

$b_{46} = -51 + 22k - 32k^2 + 6k^3 - 4k^4$,

$b_{55} = -53 + 33k - 16k^2 + 4k^3$,

$b_{56} = 2(-151 + 223k - 165k^2 + 98k^3 - 28k^4 + 8k^5)$,

$b_{65} = 371 - 142k + 128k^2 - 20k^3 + 8k^4$,

$b_{66} = 40 - 31k + 14k^2 - 4k^3$,

$b_{75} = 109 - 58k + 66k^2 - 16k^3 + 8k^4$,

$b_{76} = 664 - 515k + 716k^2 - 298k^3 + 204k^4 - 40k^5 + 16k^6$.

Conjugating a and b simultaneously by the matrix γ_1,

$$\gamma_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

gives us the following representations:

$$\gamma_1 a \gamma_1^{-1} = \begin{bmatrix} 1 & 0 & -4 & 38-8k+8k^2 & 0 & 0 & 8(5-k+k^2) \\ 0 & 1 & 3 & -1 & 1 & 0 & -3 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & -2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & -2 & 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 & 0 & 0 & -3 \end{bmatrix}, \tag{B.10}$$

$$\gamma_1 b \gamma_1^{-1} = \begin{bmatrix} 0 & 0 & -4 & 1 & \frac{c_{15}}{13-2k+2k^2} & \frac{c_{16}}{221-86k+94k^2-16k^3+8k^4} & 2 \\ 0 & 1 & 3 & 0 & \frac{c_{25}}{13-2k+2k^2} & \frac{c_{26}}{221-86k+94k^2-16k^3+8k^4} & -2 \\ 0 & 0 & 0 & 0 & \frac{c_{35}}{13-2k+2k^2} & \frac{c_{36}}{(13-2k+2k^2)(17-4k+4k^2}) & 0 \\ 0 & 0 & -2 & 0 & -\frac{c_{45}}{13-2k+2k^2} & \frac{c_{46}}{13-2k+2k^2} & 1 \\ 0 & 0 & 1 & 0 & \frac{c_{55}}{13-2k+2k^2} & \frac{c_{56}}{221-86k+94k^2-16k^3+8k^4} & 0 \\ 1 & 0 & -2 & 0 & \frac{c_{65}}{13-2k+2k^2} & \frac{c_{66}}{13-2k+2k^2} & 1 \\ 0 & 0 & 2 & 0 & \frac{c_{75}}{13-2k+2k^2} & \frac{c_{76}}{221-86k+94k^2-16k^3+8k^4} & -1 \end{bmatrix}, \tag{B.11}$$

where $c_{15} = 2(203 - 265k + 184k^2 - 104k^3 + 28k^4 - 8k^5)$,

$c_{16} = 908 - 2830k + 2532k^2 - 2376k^3 + 1060k^4 - 536k^5 + 112k^6 - 32k^7$,

$c_{25} = 264 - 109k + 142k^2 - 28k^3 + 16k^4$,

$c_{26} = 2281 - 1201k + 1831k^2 - 594k^3 + 444k^4 - 72k^5 + 32k^6$,

$c_{35} = -22 + 23k - 10k^2 + 4k^3$,

$c_{36} = -59 + 149k - 97k^2 + 78k^3 - 20k^4 + 8k^5$,

$c_{45} = 2(-68 + 33k - 37k^2 + 8k^3 - 4k^4)$,

$c_{46} = -51 + 22k - 32k^2 + 6k^3 - 4k^4$,

$c_{55} = -53 + 33k - 16k^2 + 4k^3$,

$c_{56} = -151 + 223k - 165k^2 + 98k^3 - 28k^4 + 8k^5$,

$c_{65} = 371 - 142k + 128k^2 - 20k^3 + 8k^4$,

$c_{66} = 40 - 31k + 14k^2 - 4k^3$,

$c_{75} = 109 - 58k + 66k^2 - 16k^3 + 8k^4$,

$c_{76} = 664 - 515k + 716k^2 - 298k^3 + 204k^4 - 40k^5 + 16k^6$.

B.1.3 Explicit representations for $n \equiv 0 \pmod 4$

As mentioned above, eventually an explicit formula was found for representations over the integers for $n \equiv 0 \pmod 4$. Explicit formulae for the other congruence classes modulo 4 are pending.

$$
b = d = \begin{bmatrix}
-7-n & 1-n & 77+n & 80+2n & 19+n & -n & 16 \\
4+\frac{3n}{4} & \frac{n}{4} & -63+2n & -65+2n & -16 & \frac{n}{4} & -8 \\
-7-\frac{n}{2} & -1-\frac{n}{2} & 98-11n+\frac{n^2}{4} & 102-11n+\frac{n^2}{4} & 25-2n & -1-\frac{n}{2} & 12-n \\
16+\frac{5n}{4} & 1+\frac{3n}{4} & -170+\frac{27n}{2}-\frac{n^2}{4} & -174+\frac{27n}{2}-\frac{n^2}{4} & -43+2n & 1+\frac{3n}{4} & -22+n \\
-38-2n & -n & 288-18n+\frac{n^2}{4} & 287-18n+\frac{n^2}{4} & 72-2n & -n & 40-n \\
8+\frac{5n}{4} & 2-\frac{n}{4} & -112+7n & -115+8n & -29+n & 1-\frac{n}{4} & -8 \\
-9-n-\frac{3n^2}{16} & -\frac{n}{4}-\frac{n^2}{16} & 71-\frac{n}{4}-\frac{n^2}{8} & 71-\frac{n}{4}-\frac{n^2}{8} & 18+\frac{n}{2} & -\frac{n}{4}-\frac{n^2}{16} & 9+\frac{n}{2}
\end{bmatrix}
$$

$$
a = cd^{-1} = \left[\begin{matrix}
-6+\frac{n}{2}-\frac{n^2}{4} & 26-\frac{3n}{2}+\frac{5n^2}{4} & -242+\frac{39n}{2}-16n^2+\frac{n^3}{2} \\
-2-\frac{n}{4} & -4+\frac{5n}{4} & 17-\frac{33n}{2}+\frac{n^2}{2} \\
4+\frac{n}{2} & 8-\frac{7n}{2} & -38+41n-\frac{5n^2}{4} \\
-4-\frac{n}{4} & 1+\frac{9n}{4} & -34-23n+\frac{3n^2}{4} \\
-1-n & -33+6n & 257-79n+\frac{9n^2}{4} \\
-11-\frac{n}{4}-\frac{n^2}{4} & 17+\frac{9n}{4}+\frac{5n^2}{4} & -191-30n-\frac{29n^2}{2}+\frac{n^3}{2} \\
\frac{n}{4}-\frac{n^2}{16} & 9-\frac{3n}{2}+\frac{5n^2}{16} & -71+\frac{81n}{4}-\frac{9n^2}{2}+\frac{n^3}{8}
\end{matrix}\right. \quad \cdots
$$

$$
\cdots \quad \left.\begin{matrix}
-249+\frac{41n}{2}-16n^2+\frac{n^3}{2} & -63+3n-4n^2 & 7-\frac{n}{2}+\frac{n^2}{4} & 34+2n^2 \\
15-\frac{33n}{2}+\frac{n^2}{2} & 4-4n & 2+\frac{n}{4} & -2+2n \\
-34+41n-\frac{5n^2}{4} & -9+10n & -4-\frac{n}{2} & 4-5n \\
-38-23n+\frac{3n^2}{4} & -9-6n & 4+\frac{n}{4} & 4+3n \\
256-79n+\frac{9n^2}{4} & 64-18n & 1+n & -28+9n \\
-204-29n-\frac{29n^2}{2}+\frac{n^3}{2} & -51-9n-4n^2 & 12+\frac{n}{4}+\frac{n^2}{4} & 28+6n+2n^2 \\
-71+\frac{81n}{4}-\frac{9n^2}{2}+\frac{n^3}{8} & -18+\frac{9n}{2}-n^2 & -\frac{n}{4}+\frac{n^2}{16} & 9-2n+\frac{n^2}{2}
\end{matrix}\right]
$$

It is worth noting that the generator of order 2 is a matrix with constant entries:

$$ab = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

B.2 Details of the search for a candidate sequence of integer points

First Mathematica was made to run through a large set of integer values of the parameters u, v in a search for pairs (u, v) for which α is an integer. Specifically, this was fun for $70 \leq u \leq 500$ and $350 \leq v \leq 10000$, avoiding pairs outside the permitted region in the (u, v)-plane. The output is given below; from these, a search was made for a sequence of a few pairs that could be extrapolated to give an infinite sequence of pairs with $\alpha \in \mathbb{Z}$. As mentioned earlier, this search was only conducted for sequences starting with the base representation $(u, v) = (72, 378)$. It is possible that there are other sequences yielding integer representations that do not contain the base representation.

$$\begin{aligned}
&\{(u, v) : u, v \in \mathbb{Z}, \alpha \in \mathbb{N} \cup \{0\}\} = \\
&\quad \{(72, 378), (76, 418), (78, 446), (84, 482), (88, 506), (88, 550), (90, 525), (96, 630), \\
&\quad (100, 650), (102, 639), (108, 810), (114, 783), (126, 917), (136, 1190), (136, 1258), \\
&\quad (140, 1050), (142, 1331), (144, 1206), (150, 1170), (154, 1217), (156, 1287), (162, 1661), \\
&\quad (164, 1386), (168, 1386), (172, 1978), (174, 1566), (174, 1827), (176, 1617), (180, 1575), \\
&\quad (186, 1628), (186, 2015), (192, 2070), (194, 2517), (196, 2450), (196, 2482), (202, 2323), \\
&\quad (214, 2011), (214, 2018), (216, 3078), (220, 2930), (228, 3458), (234, 2808), (238, 2485), \\
&\quad (242, 2433), (250, 2585), (250, 3644), (252, 2738), (268, 3410), (268, 4690), (270, 3630), \\
&\quad (270, 4125), (278, 3153), (280, 4970), (282, 3276), (282, 4311), (284, 3282), (292, 4106), \\
&\quad (294, 3402), (318, 6093), (328, 6970), (330, 4895), (330, 5544), (334, 4342), (334, 5702), \\
&\quad (336, 6633), (340, 4610), (342, 7245), (364, 5138), (364, 6266), (388, 9506), (396, 5490), \\
&\quad (396, 8210), (404, 6402), (414, 9063), (420, 6138), (424, 5837), (432, 5994), (432, 9153), \\
&\quad (444, 6327), (446, 7737)\}.
\end{aligned}$$

Vita

Hannah Downs was born in Chattanooga, Tennessee to Roger and Marsha Downs alongside her twin brother Zachary Downs. She grew a special interest in mathematics after skipping eighth-grade math and being introduced to a pilot program that reduced Algebra I and Algebra II down to a single year. After graduating as Salutatorian from Bledsoe County High School, she enrolled as a first generational college student at Tennessee Technological University, where she graduated summa cum laude with a B.S. in Mathematics and summa cum laude with an M.S. in Mathematics. She began her career at the University of Tennessee, Knoxville in August 2017 to complete her Ph.D. in Mathematics under Dr. Morwen Thistlethwaite. After graduation, she will become a lecturer for the Mathematics department at UTK and continue to work in academia.