

Parametric Instability Investigation and Stability Based Design for Transmission Systems Containing Face-gear Drives

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ABSTRACT

The objective of this dissertation is to provide a novel design methodology for face-gear transmissions based on system stability - a dynamics viewpoint. The structural dynamics models of transverse and torsional vibrations are developed for face-gear drives with spur pinions to investigate the parametric instability behavior in great depth. The unique face-gear meshing kinematics and the fluctuation of mesh stiffness due to a non-unity contact-ratio are considered in these models. Since the system is periodically time-varying, Floquet theory is utilized to solve the Mathieu-Hill system equations and determine the system stability numerically. To avoid complex numerical computations, Treglod's approximation is employed to calculate face-gear contact-ratio.

For transverse vibration, the model of face-gear with one spur pinion and in-plane symmetric centrifugal stress field is investigated first, and next the face-gear meshing with multiple pinions is explored, finally, the one pinion case is recalculated by taking into account the in-plane asymmetric stress field resulting from in-plane driving force. The results show that the system stability depends on rotation speed, geometrical dimension and mesh load. In stability based design, the system stability is one design constraint; the other constraint is input power. The power level determines the maximum stress at pinion tooth root and the in-plane driving force on face-gear body. Based on parametric instability investigations, the macroscopic design methodology of the face-gear body is explored by considering the input power and stability constraints. Moreover, the relationship of system stability to spatial configuration of input pinions is also explored for the multiple pinion case.

For torsional vibration, the system stability is investigated numerically with respect to rotation speed, rotational inertia, mesh stiffness, and characteristics of transmission shafts. Furthermore, a perturbation method is applied to the stability boundary tracing for design purposes. The stability results provide the necessary information for vibration suppressions. Hereinto, the effect of the system inertia distribution on the system stability is explored to develop passive vibration suppression methods and to find an optimal design with least weight; the damping and stiffness of shafts can also be adjusted individually so as to achieve passive and active vibration controls.

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Chapter 1

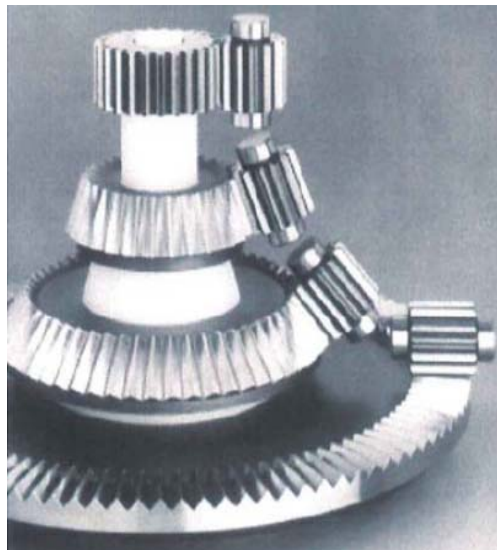
INTRODUCTION

1.1 Background

Face-gear drive is an important mechanical component which transfers rotation and power between intersecting shafts. It has existed for a long time as an alternative to bevel gears, but the applications of face-gear are limited due to its complex tooth geometry and relatively high cost. Face-gear drives were mostly used for low power applications until helicopter industry showed research interests in them about two decades ago [1]. For helicopter transmission applications, the basic face-gear drive consists of a face-gear and an involute spur pinion. Some examples of face-gear drive with spur pinion are displayed in Figure 1.1.

Different from bevel gears, there are no strict installation requirements for face-gear drives since the spur pinion can move along the axial direction freely and also has favorable misalignment tolerances in other directions. The meshing of spur pinion and face-gear does not yield any axial forces, which saves large thrust bearings. This component is always required for bevel gears. The weight benefits are obtained from the low sensitivity of face-gear to pinion misalignment by reducing bearing support stiffness requirements. Moreover, the low misalignment sensitivity of face-gear drives produces

less transmission error, also less vibration and noise, than in other types of gear drives. Face-gear drive has constant rotation velocities theoretically due to its conjugated generation by a shaper. In addition, the axial freedom of spur pinion makes face-gear drive more favorable than bevel one in split-torque transmission designs [2].



(a) Face-gear / spur pinion with different shaft angles [3-4].



(b) A test face-gear drive in RDS-21 program [5].

Fig. 1. 1: Examples of face-gear drive meshing with spur pinion.

Litvin et al. [6] summarized the theoretical advantages of face-gear drives: (1) lower misalignment sensitivity, (2) reduced level of noise owing to small transmission errors (3) more favorable load transfer between adjacent tooth pairs; and (4) no accurate axial location requirement for the spur pinion.

After twenty years development under the joint Army/NASA Advanced Rotorcraft Transmission (ART) program [1], helicopter main transmissions containing face-gear drive were finally implemented. The highlight is the Boeing 3400 HP Apache Block III Improved Drive System, based on a split-torque face-gear system, which can provide a 20% more power throughput than the previous Apache Longbow Drive System [7-8]. At the same time, Sikorsky Aircraft has also been developing and testing its own helicopter face-gear transmission system [9]. As one of key technologies for the U.S. Army Rotorcraft Drive Systems for the 21st Century (RDS-21) Program, the further advancements of face-gear technology are in development [5, 8-9]. However, the dynamics consequences of face-gear drives have not been fully investigated. Thus, this dissertation is to advance the current state-of-the-art.

1.2 Helicopter Main Transmission

Helicopter main transmission transfers power from engines to rotors with reduced speed and enhanced torque. Transmission technology determines, to a great extent, helicopter operation performance, noise and vibration, maintenance requirement, service life, and manufacturing cost. As reported in previous literatures, transmission systems account for about 12~20% of helicopter cost, 30% of maintenance expense, 10~15% of helicopter weight, and 16% of the mechanically related malfunctions that often result in

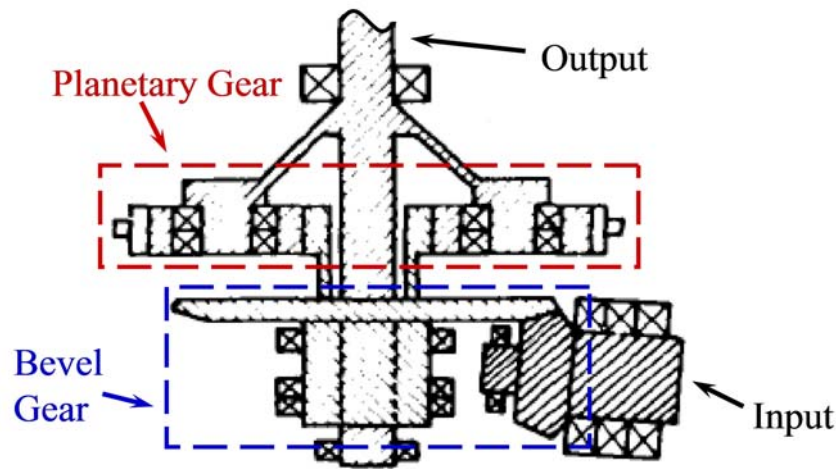
the loss of helicopter [10-11]. Both ART program and RDS-21 program aim at the transmission systems with higher power-to-weight ratio, lower noise level, longer service life and economic cost [1, 5, 8-9].

The typical main gearboxes applied in helicopter transmission can be identified as bevel gear / planetary gear type, bevel gear / cylindrical gear type, face-gear / planetary gear type, and face-gear / face-gear type according to their components; or single power path design and split power path design according to their drivetrain configuration.

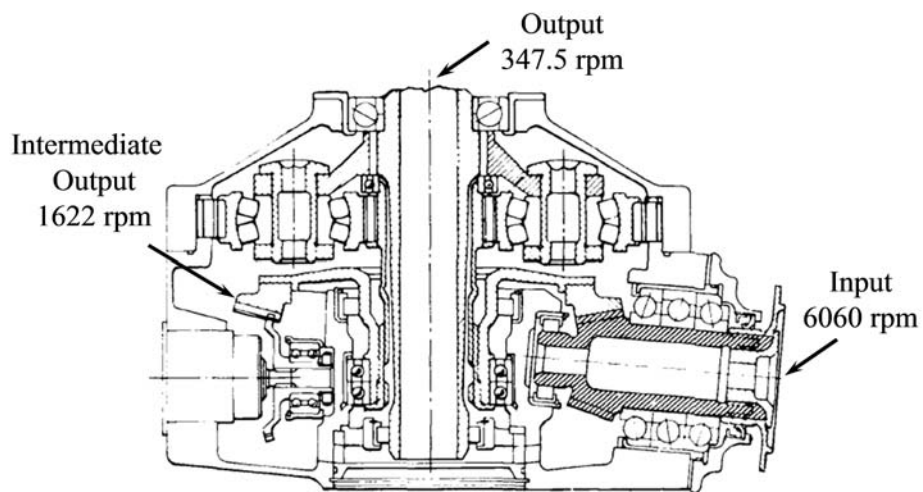
In order to reduce the weight of drive system, helicopter main gearbox demands fewer stages, more parallel power paths and larger reduction ratios at the final stage. The volume of gear is proportional to the square of gear radius, while their torque-carrying capacity is proportional to lower order of gear radius. So, the gear sizes can be reduced substantially when the input torque is split equally into multiple paths. Also, the gearbox weight can be reduced by arranging a larger reduction ratio at the final stage because of the lower torques at preceding stages [2, 12]. Face-gear drive possesses high reduction ratio capacity and the unique advantage in split-torque design; hence it is the most favorable candidate for helicopter transmission applications now.

1.2.1 Bevel gear / planetary gear type helicopter main gearbox

The helicopter main gearboxes, consisting of bevel gear angle turn stage and planetary gear deceleration output stage, had been applied to Bell OH-58 Kiowa, Boeing CH-47 Chinook, and Sikorsky UH-60 Black Hawk before 1980s [13].



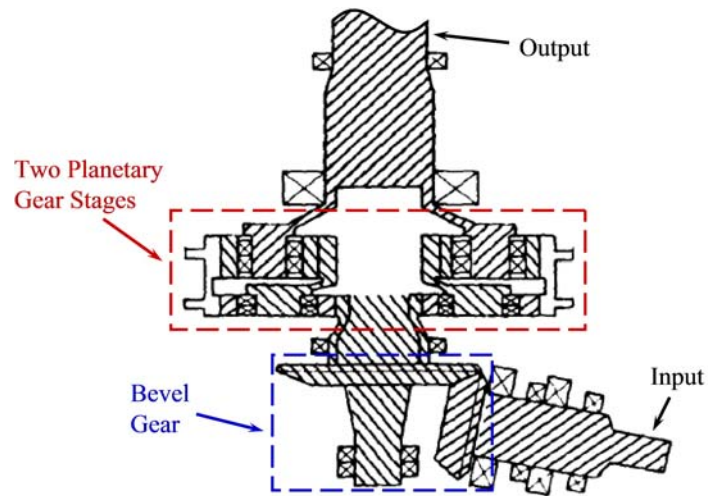
(a) Schematic



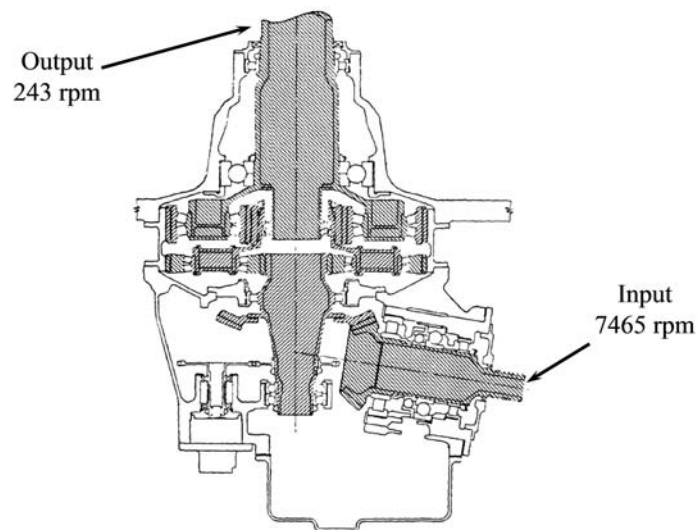
(b) Detail

Fig. 1. 2: OH-58 main gearbox [13].

The OH-58 main gearbox comprises an input bevel gear stage serving as angle turning function and a planetary gear stage as reduction output, shown in Figure 1.2. It is a single main rotor transmission with only one power input path, usually used in light helicopters. This gearbox provides a speed reduction from 6060 rpm to 347.5 rpm.



(a) Schematic



(b) Detail

Fig. 1. 3: CH-47 forward gearbox [13].

The CH-47 transmissions are a typical tandem rotor design. Its forward gearbox is very similar to OH-58, displayed in Figure 1.3. The differences are: CH-47 contains two planetary gear stages in series for a higher reduction ratio while OH-58 has only one; OH-58 output shaft goes downward through the bevel gear while CH-47 does not.

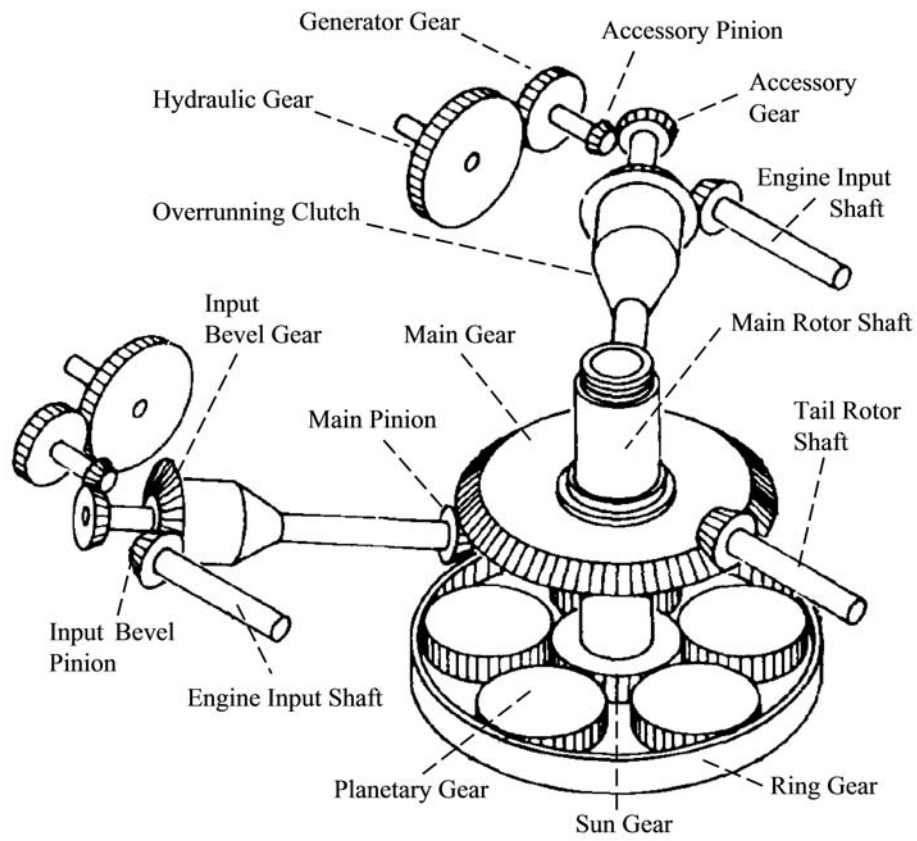


Fig. 1. 4: Schematic of UH-60 Black Hawk main gearbox [13].

The schematic of UH-60 main gearbox is illustrated in Figure 1.4. This gearbox consists of one main module, two engine input modules, and two accessory modules. The power is transmitted into the main module from two engines via two separated input modules. The power path changes the direction through bevel gear drives, and then, a combining bevel gear is used to transfer power to main planetary gear system and tail rotor gearbox. The planetary gear system contains five planet pinions with a reduction ratio of 4.67:1 [13]. The power is transmitted from the central sun gear to the main rotor shaft via the five planet pinions and the planet carrier successively [14].

1.2.2 Bevel gear / cylindrical gear type helicopter main gearbox

The split-torque helicopter main gearbox with cylindrical gears at the final stage is a promising alternative for conventional planetary gears. The reduction ratio of the final stage can reach up to 14:1 under the split-torque configuration with two parallel paths, while the maximum reduction ratio is about 7:1 for the usual planetary gear sets with 3 to 18 parallel power paths [12]. In split-torque design, both power splitting and large reduction ratio at the final stage contribute to weight savings. The typical applications are the proposed split-torque upgrade gearbox for Sikorsky UH-60A Black Hawk helicopter and the main gearbox of Boeing-Skiersky RAH-66 Comanche helicopter, shown in Figure 1.5 and Figure 1.6 respectively.

The UH-60A upgrade gearbox transfer power from two engines to the main rotor via one bevel gear angle turning stage, one bevel gear torque splitting stage, and one spur gear torque recombination stage. The power from each engine is divided into two parallel paths via the bevel gear torque splitting stage [15].

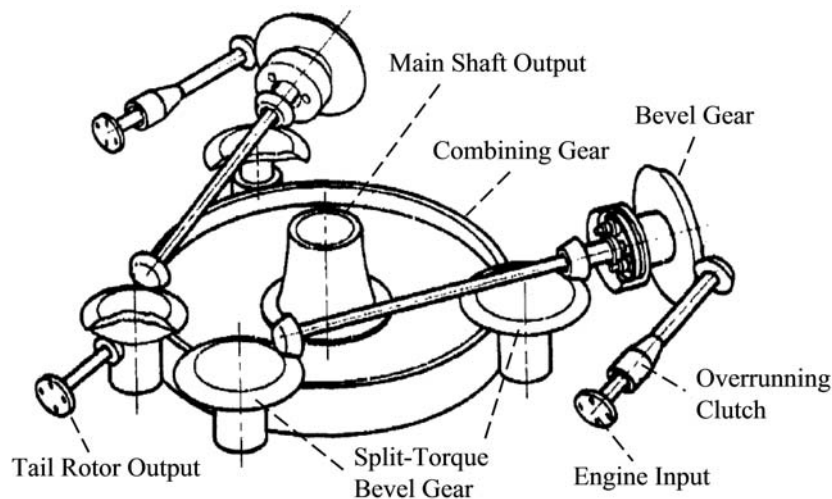
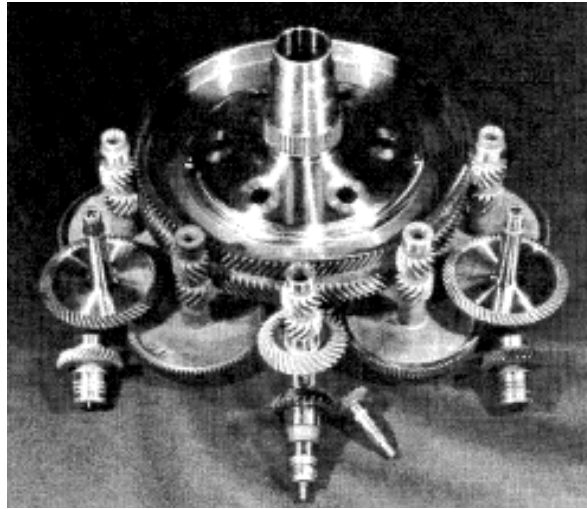
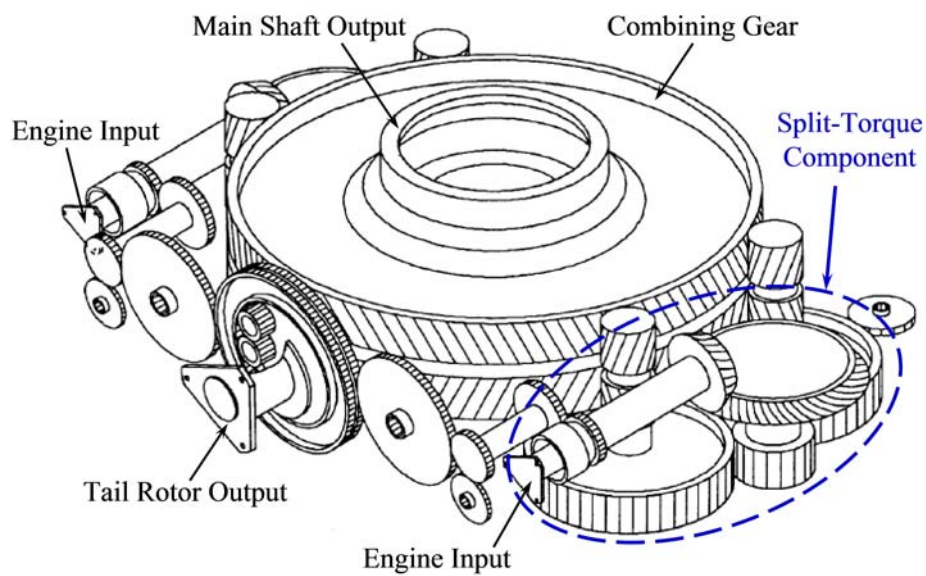


Fig. 1. 5: Split-torque upgrade gearbox compatible with the UH-60A helicopter [15].



(a) Layout of Comanche helicopter main gearbox [12]



(b) A proposed design by G. White [16]

Fig. 1. 6: Split-path helicopter transmission for two engines.

RAH-66 Comanche helicopter is also equipped with a split-torque main rotor gearbox for two engines. A dual-helical combining gear is used to cancel axial thrust. Different from UH-60A upgrade gearbox, RAH-66 employs cylindrical spur gears to split

torque. The detail of the spur gear stage for split-torque design is illustrated in Figure 1.7. The input pinion meshes with two parallel shaft gears simultaneously to split the power into two paths evenly.

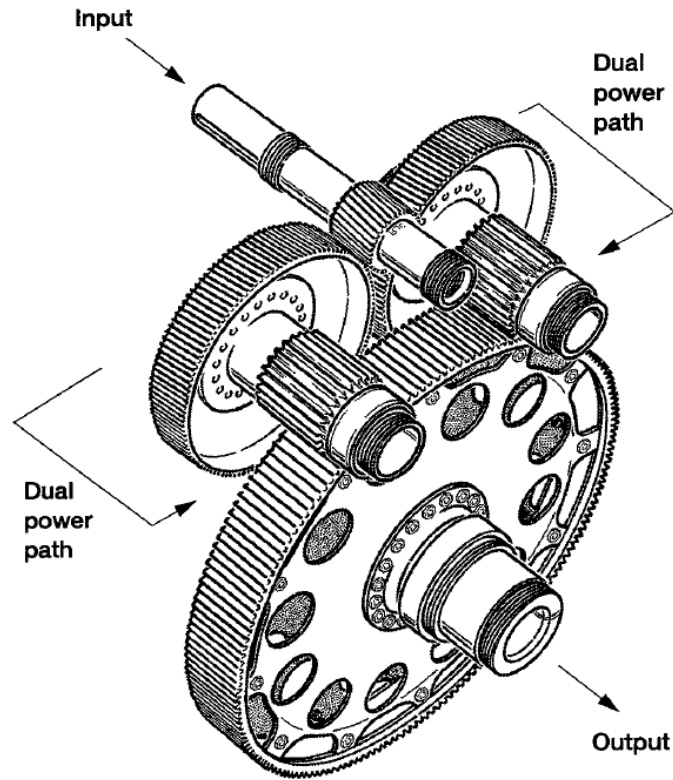


Fig. 1. 7: Example of split-torque design based on cylindrical gears [12].

1.2.3 Face-gear / planetary gear type helicopter main gearbox

The typical example of face-gear / planetary gear helicopter main gearbox is a split-torque / planetary hybrid configuration proposed by McDonnell Douglas Helicopter Company for the ART program [1]. See Figure 1.8.

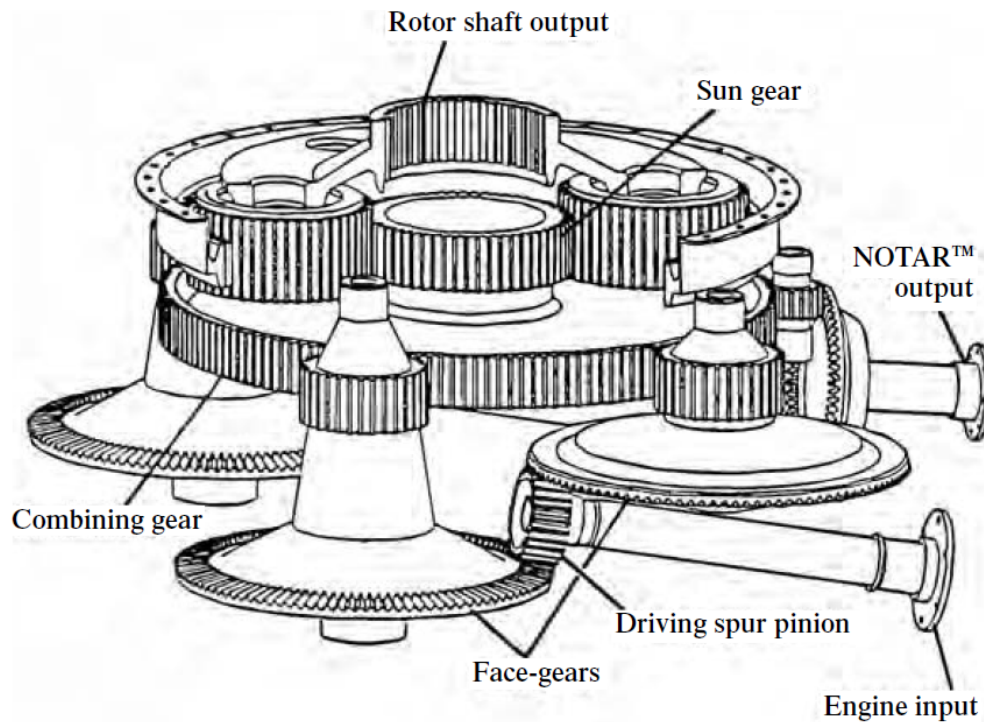


Fig. 1. 8: Example of face-gear split-torque / planetary hybrid transmission [1-2].

This conceptual baseline was derived directly from the AH-64A helicopter [1]. Each input pinion splits the torque into two face-gears and changes the drivetrain direction simultaneously. Next, a spur gear stage is used to recombine all torques and provides the input power for the final planetary stage and the tail rotor. This design employed face-gear/spur-pinion drives, instead of spiral bevel gears, serving as the drivetrain angle turning stage because of the favorable installation misalignment tolerance and no axial positioning requirement. The compare between bevel and face-gear drives in split-torque designs is illustrated in Figure 1.9. The bevel pinion has no axial freedom, while the spur pinion has.

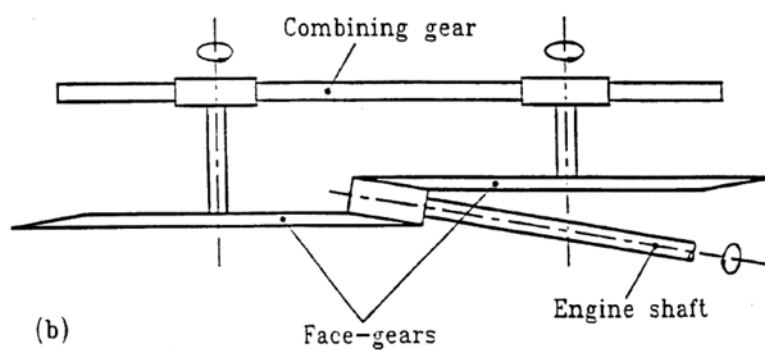
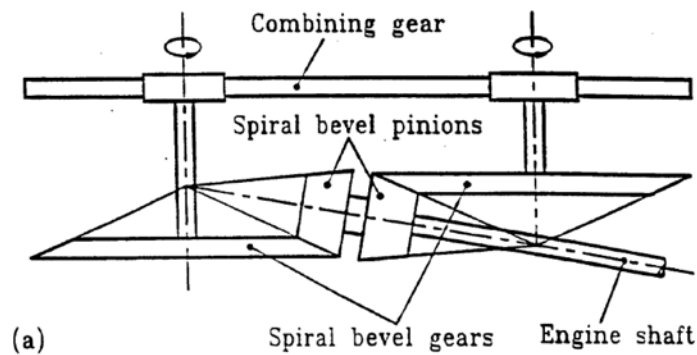


Fig. 1. 9: Examples of split-torque design based on bevel and face-gear drives [2].

In addition, the latest Apache Block III Improved Drive System also utilizes face-gear to split the torque and planetary gear to drive the main rotor system. The split-torque face-gear system consists of two input tapered pinions, upper face-gear, lower face-gear, two tapered idlers, and one tapered idler for subsidiary power outputs, shown in Figure 1.10. The torque on each input pinion is split between idlers and pinions via lower face-gear, and then, the upper face-gear recombines all torques for planetary gear stage input. The Boeing 3400 HP Apache Block III Improved Drive System containing this split-torque face-gear system can provide a 20% more power throughput than the previous Apache Longbow Drive System [7-8].

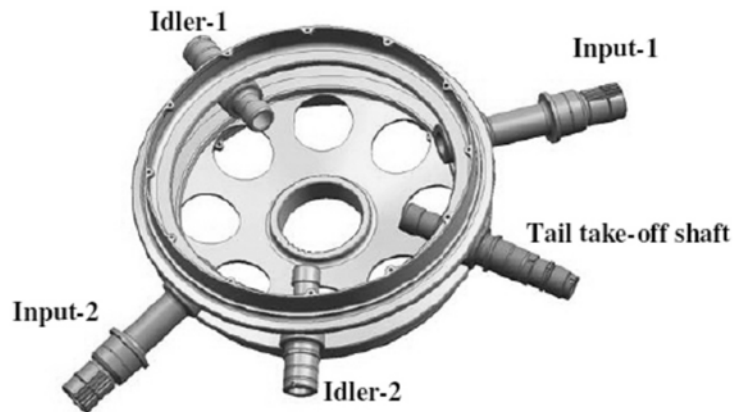


Fig. 1. 10: Face-gear system in the Apache Block III Improved Drive System [4, 7].

1.2.4 Face-gear / face-gear type helicopter main gearbox

As one of key technologies for the RDS-21 program, Sikorsky has been working on a novel transmission system with two face-gear stages in series [9]. Figure 1.11 shows the 3D CAD model of this drivetrain system.

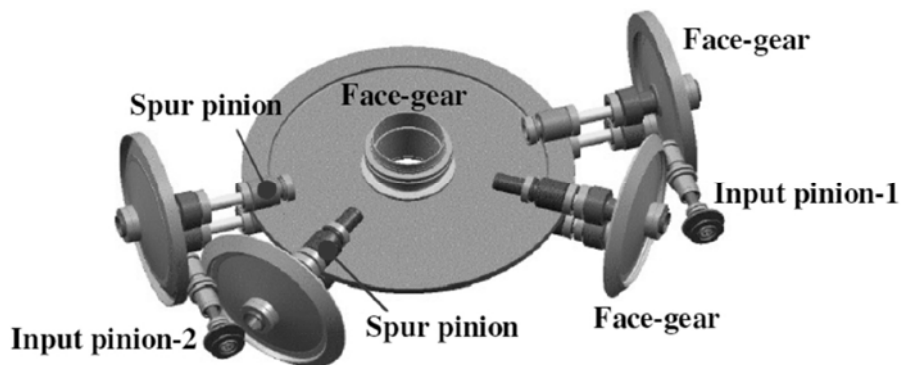


Fig. 1. 11: 3D CAD model of Sikorsky RDS-21 drivetrain system [4, 9].

The RDS-21 drivetrain system is designed for a dual engine configuration and contains two face-gear reduction stages. The power is transmitted to the first stage via the

input pinion that splits torque into two face-gears; the second stage uses a face-gear to recombine all torques and drive the main rotor. These two stages change the power direction continuously. A trade study on different reduction ratio arrangements between two stages was conducted for a 5100 HP RDS-21 demonstrator gearbox by Sikorsky. The results indicated that the design with the second stage gear reduction ratio of 10.04 and the total gear reduction ratio of 85.3 achieves the maximum power-to-weight ratio [9].

1.3 Development of Face-gear Research

The basic theory and process technology for face-gear appeared as early as the 1930s [3]. The research on face-gear progressed slowly until its possible application in helicopter transmission was proposed in the early 1990s. Litvin et al. [17] developed the detailed analytical geometry of face-gear drives with spur involute pinion via simulating the meshing of the face-gear and its shaper. In order to reduce the stresses and transmission errors, some face-gear designs with modified tooth profiles were presented in [18-19]. The stress expectation for these modified versions was also validated based on the FEA models of tooth pairs. Furthermore, Litvin et al. first introduced the analytical derivation for the face-gear generation by a grinding worm in [19-20], and the stress analyses on this type of face-gear were conducted by FEA tools too. The grinding technics helps harden tooth surface. The dynamic instability of face-gear drives with spur pinion, induced by meshing kinematics, was investigated based on a model of spinning disk / moving load in [21].

The application of helical pinion reduces contact stress and increases contact ratio. The theory of face-gear drives with a helical pinion was established by Litvin and

coworkers in [22-23]. They investigated two types of helical pinion: one is screw involute helicoid, the other is determined as the envelope to a skew parabolic rack-cutter; the generation of the conjugated face-gear was accomplished by shaper and worm respectively. The research on worm face-gear drives with conical and cylindrical worms was presented in [24]. In addition, Litvin et al. [2] and He et al. [9] explored offset face-gear drives, namely, the input and output shafts are crossed but not intersected.

Handschuh et al. [25] conducted experimental evaluations of face-gear at NASA Lewis facility with the focus of failure modes and load capacity tests. Lewicki et al. [5, 26] and Heath et al. [8] explored face-gear fatigue characteristics. The instantaneous load sharing on all teeth in mesh and the corresponding experimental validation were investigated for face-gear in [27]. To simplify the complexity in face-gear design, Guingand et al. [28] provided two simple analytical formulae based on statistical methods to calculate the inner and outer radii of face-gear tooth respectively, which saves the intensive computations required for plotting Litvin's radius factor chart [17]. FEA techniques were developed for face-gear drives to perform a 3-D tooth contact analysis in [29], to simulate the meshing with profile modified pinion or misalignment in [30-31], and to analyze the bending stress based on different parameters in [32]. Finally, some lightweight split-torque helicopter transmission designs based on face-gear drives were proposed, reasoned for feasibility, tested under high-speed high-load conditions, and adjusted to achieve evenly torque splitting [6, 33-34].

The tooth contact mode is an important factor to determine the operation characteristics of face-gear drives. When the pinion is an identical copy of the shaper that generates the face-gear, the meshing tooth pairs are in line contact mode; when the pinion

has less number of teeth than the shaper, they are in point contact mode. The point contact mode has advantage in misalignment tolerance, while the line contact mode gives better performances in contact ratio, maximum load, contact pressure, and transmission error [17, 35]. To optimize the line contact mode, a slight pinion tooth profile modification (crowning) was proposed to reduce the sensitivity to misalignments in [35]. Therefore, the line contact mode is chosen as the object of this study in the following chapters.

1.4 Dynamics of Gearbox

In the past half century, dynamic characteristics of gear/gearbox have been intensively explored by lots of scholars. Ozguven and Houser [36] presented an excellent summary on the mathematical models of gear dynamics published before 1987 and classified them according to the flexible elements that were considered in models. These models evolved from treating all components as rigid to incorporating as many as flexible degrees-of-freedom. Currently, they are being further developed based on newly gear geometry and material characteristics.

Utagawa and Harada [37-38] conducted both experimental and theoretical studies on the dynamic loads over the spur gear teeth with pressure angle and pitch errors respectively under high speed conditions. In their theoretical analyses, an equivalent linear spring-mass model was employed. Kasuba and Evans [39] proposed a concept of variable-variable mesh stiffness for the dynamic load calculation by considering the effects of transmitted load, tooth profile modifications and errors, hub torsional deformation, position of contact areas, and so on. Kubo [40] researched the rotational

vibration of helical gears with manufacturing and alignment errors based on the rigid disk / flexible teeth model. This type of model is widely used in gear dynamic analyses and a typical example, including mesh stiffness, mesh damping and the excitation due to gear errors, for one pair of gears is shown in Figure 1.12. Furthermore, the gear dynamics models with nonlinearities, such as gear errors and backlash, were also developed in the similar way [41-42].

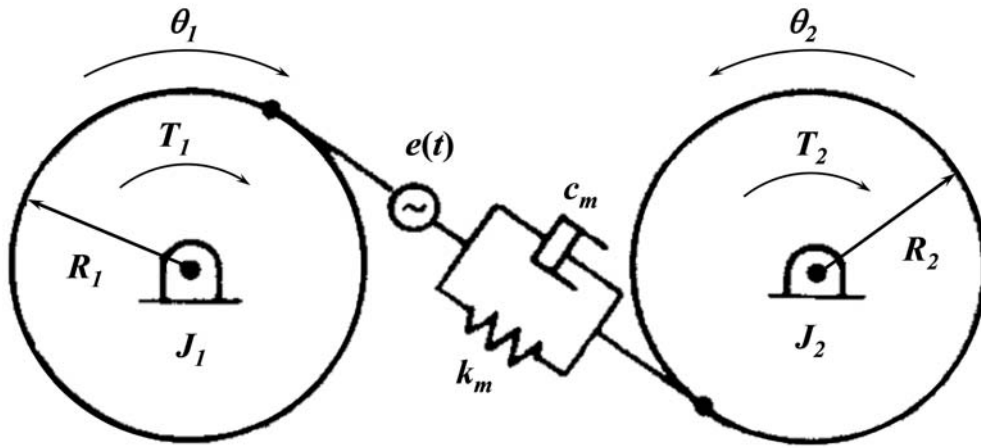


Fig. 1. 12: An example of gear dynamics model [41].

The dynamic characteristics of shafts and bearings are able to affect the performance of gear systems and can also be adjusted to improve the performance. Kubo and Kiyono [43] incorporated shaft and bearing stiffnesses into the vibration model of gears with tooth errors. Bahgat et al. [44] considered the mass of shafts besides their stiffness when calculating the instantaneous dynamic loads on spur gear teeth. However, the mass of shaft is usually much less than gear, so the accuracy of the models without shaft mass is enough in most cases. By taking into account the characteristics of shafts and bearings, both linear and nonlinear models were developed to analyze system

stability, dynamic tooth load, forced response, profile modification, backlash effect, and so on in [45-47]. Via the multi-scale method, the parametric instability was explored for a two-stage gear system with time-varying meshing stiffness resulting from a non-unity contact ratio in [48]. Additionally, a dual gearbox system, connected by a flexible shaft, was analyzed in [49]. Nonlinear backlash, time-varying meshing stiffness and dynamic characteristics of coupling shaft were considered, and the periodic steady-state response was also computed by using a harmonic balance and numerical arc-length continuation approach.

By considering coupled torsional-lateral motions of shafts, the gear model becomes rotor model with the unique gear characteristics essentially. Gearbox was modeled as the rotor system coupled by gear mesh loads and also analyzed for stability and response under different types of mesh assumptions in [50-52]. When helical or bevel gears are used, the axial thrust bearings are needed. Consequently, the coupled axial-torsional-lateral dynamic models were utilized to research such geared rotor systems [53-56].

Moreover, some investigators consider the whole gear as rigid by neglecting the flexibility of mesh tooth pairs. This assumption makes the gear dynamics close to pure torsional vibration problems. Iida et al. [57-58] focused on dynamics of shafts by using meshing force to connect gears instead of stiffness. The continuous system model of shafts was also incorporated into gear systems in order to contain shaft inertia [49, 59]. In addition, the plastic deformation of gear teeth under heavy loads was investigated experimentally and theoretically to minimize the dynamic load [60].

1.5 Objectives of Dissertation

As cited above, the theory of face-gear design, generation and stress analysis have been well established; the feasibility and durability were also validated by experiments. However, the papers about the dynamic models of face-gear drives are scarce and incomprehensive. The objective of this dissertation is to develop the structural dynamics models of transverse and torsional vibrations for face-gear drives with spur pinions to fully investigate the parametric instability behavior. Subsequently, the design guidelines for face-gear transmissions are proposed based on system stability - a dynamics viewpoint.

The unique face-gear meshing kinematics and the fluctuation of mesh stiffness due to a non-unity contact-ratio kinematics are considered in these models. Since the system is periodically time-varying, Floquet theory is utilized to solve the Mathieu-Hill system equations and determine the system stability numerically. To avoid complex numerical computations, Treglod's approximation is employed to calculate face-gear contact-ratio.

For transverse vibration, the stability of a face-gear meshing with one spur pinion and subjected to an in-plane symmetric centrifugal stress field is investigated first, and next the face-gear meshing with multiple pinions is explored, finally, the one-pinion case is recalculated by incorporating in-plane asymmetric stress field resulting from the in-plane tangential driving force. The system stability is analyzed with respect to various rotation speeds, geometric dimensions and mesh loads.

The transverse stability based face-gear design comprises two main constraints, one is system stability; the other is input power. This power determines the maximum

stress at pinion tooth root and the in-plane tangential driving force on face-gear body. All parameters that are able to affect system stability can be designed to avoid instability and satisfy with pinion tooth stress restriction. Based on parametric instability investigations, the macroscopic dimensions of face-gear body and the input power level will be designed for specific operation speed ranges. Moreover, the multiple pinion case will also explore the relationships of system stability to spatial configuration of input paths.

For torsional vibration, the system stability is investigated based on mesh stiffness, system inertia distribution, and characteristics of input and output shafts. It is possible to develop passive and active vibration suppression strategies via adjusting the damping and stiffness of shafts or the mass moment of inertia distribution among pinion, face-gear and load. Also, a perturbation technique is employed to obtain the approximate stability boundary analytically for design purposes.

Chapter 2

MATHEMATICAL METHODS FOR STABILITY ANALYSIS

2.1 System Introduction

This dissertation is to establish structural dynamics models for transmission systems containing face-gear drives, incorporating the unique face-gear meshing kinematics and fluctuation of mesh load resulting from non-unity contact-ratio kinematics. Due to the kinematically induced variation of the face-gear/pinion contact locus during gear rotation, this system has periodically time-varying dynamical behavior. Both meshing kinematics and mesh load fluctuations may excite dynamic instability. This chapter is to introduce the characteristics of this type of systems and the computation methods for the system stability criteria.

First of all, let us consider a general periodically linear time-varying system without force input in mechanical engineering. The equations-of-motion are given as

$$\mathbf{M}(t)\ddot{q}(t) + \mathbf{C}(t)\dot{q}(t) + \mathbf{K}(t)q(t) = 0 \quad (2-1)$$

and

$$\mathbf{M}(t + T_0) = \mathbf{M}(t), \quad \mathbf{C}(t + T_0) = \mathbf{C}(t), \quad \mathbf{K}(t + T_0) = \mathbf{K}(t) \quad (2-2)$$

where $\mathbf{M}(t)$, $\mathbf{C}(t)$, $\mathbf{K}(t)$ are respectively generalized system mass, damping, stiffness matrices with a period of T_0 , $q(t)$ is n -dimensional generalized coordinators, and “ $\cdot$ ”

represents differentiation with respect to time t . The Eq. (2-1) is classified as *Hill equation* in general or as *Mathieu equation* when the time-varying coefficient matrices are sinusoidal. The corresponding state-space form is

$$\dot{\mathbf{X}} = \mathbf{A}(t)\mathbf{X} \quad \text{with} \quad \mathbf{X} = \begin{bmatrix} q(t)^T & \dot{q}(t)^T \end{bmatrix}^T \quad (2-3)$$

and

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}(t)^{-1}\mathbf{K}(t) & -\mathbf{M}(t)^{-1}\mathbf{C}(t) \end{bmatrix}_{2n \times 2n} \quad \text{and} \quad \mathbf{A}(t + T_0) = \mathbf{A}(t) \quad (2-4)$$

Here, $\mathbf{X}$ is a column vector including $2n$ system state variables; $\mathbf{A}(t)$ is a $2n \times 2n$ periodic system matrix with the period of T_0 ; $\mathbf{0}$ and $\mathbf{I}$ are $n \times n$ zero and identity matrix respectively.

2.2 Stability Analysis Methods

To predict and determine the system dynamical behavior, the stability analyses must be implemented for Hill equations, either in Eq. (2-1) form or in Eq. (2-3) form. Fortunately, previous scholars have successfully developed a lot of mathematical methods. The most common ones include Floquet's method, Bolotin's approach, Lyapunov theory, and perturbation technique [61].

The essence of Floquet's method is to examine the stability of the state transition matrix that maps an initial state to the state after one period [62-63]. The state transition matrix can be formed from Wronskian matrix, but the numerical integrals require time-consuming computations especially for high-dimensional systems. An approximate strategy that treats the state transition matrix as stepwise constant or discrete was utilized to evaluate the transition matrix [62-65], and its convergence and stability characteristics were proved to be the same as the original system theoretically by Hsu [66]. Bauchau and

Nikishkov [67-68] conducted an implicit Floquet analysis which examines only the dominant eigenvalues of the transition matrix.

Bolotin [69] proposed a method for stability boundary tracing: two types of solutions with periods of T_0 and $2T_0$ are given as Fourier series respectively; the areas surrounded by two solutions with identical period are unstable and by two solutions with different periods are stable. This method was extended to find combination resonance boundaries in [70-71]. Jang and Jeong [72], and Pei [73] explored rotating system with gyroscopic effect by Bolotin's approach. Pei also found the unstable areas may be enlarged for gyroscopic systems via comparing with the results from Floquet's method.

Lyapunov theory was employed to determine the stability of time-varying system at a viewpoint of energy variation [74-77]. Hsu [78-79] developed a perturbation technique to determine the system combination resonance. By comparing the stability results, it is revealed that Hsu's technique works much faster than Floquet method due to no numerical computations, but Hsu's technique cannot catch all instabilities since only resonance items are considered [80]. In addition, Xu and Gasch [81] presented a harmonic balance method to transfer the periodically time-varying system matrix into a constant hyper-dimensional one. The eigenvalues of the hyper-dimensional system matrix governs the stability of the original system.

This dissertation employs Floquet theory and Hsu's method for stability analyses, and they are respectively explained below in detail.

2.2.1 Floquet Theory and Approximate Algorithm

According to Floquet theory, the stability of the periodically linear time-varying system, such as Eq. (2-3) and Eq. (2-4), can be represented by the stability status over only one period. The system state is transferred from one time point to another through a transition matrix, $\Phi(t, t_0)$, and it is expressed as

$$\mathbf{X}(t) = \Phi(t, t_0) \mathbf{X}(t_0) \quad (2-5)$$

Thus, when the initial state is identity, the state after one period, T_0 , is

$$\mathbf{X}(T_0) = \Phi(T_0, 0) \mathbf{X}(0) = \Phi(T_0, 0) \quad \text{when} \quad \mathbf{X}(0) = \mathbf{I}_{2n} \quad (2-6)$$

Here, $\mathbf{I}_{2n}$ is a $2n \times 2n$ identity matrix. In general, the state after k -integral periods is [62-63]

$$\mathbf{X}(t + kT_0) = \Phi(T_0, 0)^k \mathbf{X}(t) \quad (2-7)$$

The state transition matrix, $\Phi(T_0, 0)$, can be obtained by numerically integrating Eq. (2-3) from 0 to T_0 with identity initial conditions. This transition matrix is called *Floquet Transition Matrix* (FTM), and its eigenvalues, λ_i , are Floquet multipliers which govern the overall stability characteristics of the system. The stability criterion is given

$$\frac{\ln \lambda_i}{T_0} = \alpha_i + \mathbf{i}\beta_i \quad \text{and} \quad \begin{cases} \alpha_i < 0 & \text{stable} \\ \alpha_i \geq 0 & \text{unstable} \end{cases} \quad (2-8)$$

This method can catch all instabilities but pays back intensive computations due to numerical integral. As mentioned above, an approximate strategy that treats the system matrix as stepwise constant is used to evaluate FTM in this dissertation [62-65]. By dividing one period into Q time sub-intervals with initial values $[0, t_1, \dots, t_{Q-1}]$, the system matrix $\mathbf{A}(t)$ is assumed as constant in each sub-interval and evaluated at the middle time point of the sub-interval. Thus, the solutions for each sub-interval are obtained

$$\begin{aligned}
\mathbf{X}(t_1) &= e^{\mathbf{A}(\frac{t_1}{2}) \times t_1} \mathbf{X}(0) \\
\mathbf{X}(t_2) &= e^{\mathbf{A}(\frac{t_2+t_1}{2}) \times (t_2-t_1)} \mathbf{X}(t_1) \\
&\vdots \\
\mathbf{X}(T_0) &= e^{\mathbf{A}(\frac{T_0+t_{Q-1}}{2}) \times (T_0-t_{Q-1})} \mathbf{X}(t_{Q-1})
\end{aligned} \tag{2-9}$$

By using the solution of the last sub-interval as the initial state for the next sub-interval, cascading all equations in Eq. (2-9) yields

$$\mathbf{X}(T_0) = \prod_{p=1}^Q e^{\mathbf{A}\left(\frac{t_p+t_{p-1}}{2}\right)(t_p-t_{p-1})} \mathbf{X}(0) \quad \text{with} \quad \begin{cases} t_0 = 0 \\ t_Q = T_0 \end{cases} \tag{2-10}$$

and the approximate FTM is obtained

$$\Phi(T) = \prod_{p=1}^Q e^{\mathbf{A}\left(\frac{t_p+t_{p-1}}{2}\right)(t_p-t_{p-1})} \tag{2-11}$$

Here, the computational accuracy and efficiency can be easily controlled and balanced by adjusting the number of sub-intervals, Q , within one period.

2.2.2 Hsu's Method and Combination Resonance

Compared with the numerical integral based Floquet method, perturbation techniques provide more favorable computation efficiency because they are based on analytical approximations. A general and simple perturbation algorithm, developed by Hsu [78-79], is employed here to trace the stability boundary approximately. This method is explained below: Eq. (2-1) is transformed as a standard form through normalization and diagonalization processes

$$\ddot{q}(t) + \varepsilon \mathbf{P}(t) \dot{q}(t) + [\mathbf{U}^{(0)} + \varepsilon \mathbf{U}(t)] q(t) = 0 \tag{2-12}$$

where ε is a small real number. $\mathbf{U}^{(0)}$ is a diagonal matrix with positive real numbers (square of natural frequencies) on its diagonal line and can be expressed as

$$\mathbf{U}^{(0)} = \begin{bmatrix} \omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \ddots & \\ & & & \omega_n^2 \end{bmatrix} \quad \text{with} \quad \omega_1^2 \leq \omega_2^2 \leq \dots \leq \omega_n^2 \quad (2-13)$$

Here, $\omega_1, \omega_2, \dots, \omega_n$ are system natural frequencies. $\mathbf{P}(t)$ and $\mathbf{U}(t)$ are periodically time-varying and expanded as Fourier series

$$\begin{aligned} \mathbf{U}(t) &= \sum_{s=1}^S \left(\mathbf{D}^{(s)} \cos s\omega t + \mathbf{E}^{(s)} \sin s\omega t \right) \\ \mathbf{P}(t) &= \mathbf{F}^{(0)} + \sum_{s=1}^S \left(\mathbf{F}^{(s)} \cos s\omega t + \mathbf{G}^{(s)} \sin s\omega t \right) \end{aligned} \quad (2-14)$$

Substituting Eq. (2-13) and Eq. (2-14) into Eq. (2-12) yields,

$$\begin{aligned} \ddot{\mathbf{q}}(t) + \mathbf{U}^{(0)}\mathbf{q}(t) &= -\varepsilon \left[\mathbf{F}^{(0)} + \sum_{s=1}^S \left(\mathbf{F}^{(s)} \cos s\omega t + \mathbf{G}^{(s)} \sin s\omega t \right) \right] \dot{\mathbf{q}}(t) \\ &\quad - \varepsilon \left[\sum_{s=1}^S \left(\mathbf{D}^{(s)} \cos s\omega t + \mathbf{E}^{(s)} \sin s\omega t \right) \right] \mathbf{q}(t) \end{aligned} \quad (2-15)$$

and its component form,

$$\begin{aligned} \ddot{q}_i(t) + \omega_i^2 q_i(t) &= -\varepsilon \left[\sum_{j=1}^n f_{ij}^{(0)} \dot{q}_j(t) + \sum_{s=1}^S \sum_{j=1}^n \left(f_{ij}^{(s)} \cos s\omega t + g_{ij}^{(s)} \sin s\omega t \right) \dot{q}_j(t) \right] \\ &\quad - \varepsilon \sum_{s=1}^S \sum_{j=1}^n \left(d_{ij}^{(s)} \cos s\omega t + e_{ij}^{(s)} \sin s\omega t \right) q_j(t), \quad i = 1, 2, \dots, n \end{aligned} \quad (2-16)$$

where lowercase letter represents the element of its corresponding capital letter matrix.

The first order form of Eq. (2-16) can be written as

$$\begin{aligned}
\dot{q}_i(t) &= y_i(t) \\
\dot{y}_i(t) + \omega_i^2 q_i(t) &= -\varepsilon \left[\sum_{j=1}^n f_{ij}^{(0)} y_j(t) + \sum_{s=1}^S \sum_{j=1}^n \left(f_{ij}^{(s)} \cos s\omega t + g_{ij}^{(s)} \sin s\omega t \right) y_j(t) \right] \\
&\quad - \varepsilon \sum_{s=1}^S \sum_{j=1}^n \left(d_{ij}^{(s)} \cos s\omega t + e_{ij}^{(s)} \sin s\omega t \right) q_j(t)
\end{aligned} \tag{2-17}$$

One possible perturbation solution is taken as the form of

$$\begin{aligned}
q_i(t) &= A_i(t) \cos \omega_i t + B_i(t) \sin \omega_i t + \sum_{r=1}^{\infty} \varepsilon^r q_i^{(r)}(t) \\
y_i(t) &= \omega_i [-A_i(t) \sin \omega_i t + B_i(t) \cos \omega_i t] + \sum_{r=1}^{\infty} \varepsilon^r \dot{q}_i^{(r)}(t)
\end{aligned} \tag{2-18}$$

On the right hand side of Eq. (2-18), the items involving A_i and B_i are the variational part and the rest is the perturbation part. Replacing the y and q in Eq. (2-17) by the expressions in Eq. (2-18) and truncating to the first order of ε , the resulting equations are

$$\begin{aligned}
\dot{A}_i \cos \omega_i t + \dot{B}_i \sin \omega_i t &= 0 \\
-\omega_i \dot{A}_i \sin \omega_i t + \omega_i \dot{B}_i \cos \omega_i t + \varepsilon (\ddot{q}_i^{(1)} + \omega_i^2 q_i^{(1)}) &= \\
&\quad - \frac{\varepsilon}{2} \sum_{s=1}^S \sum_{j=1}^n \left[H_1^{(s)} \cos(\omega_j t + s\omega t) + H_2^{(s)} \cos(\omega_j t - s\omega t) \right. \\
&\quad \left. + H_3^{(s)} \sin(\omega_j t + s\omega t) + H_4^{(s)} \sin(\omega_j t - s\omega t) \right] \\
&\quad - \varepsilon \sum_{j=1}^n f_{ij}^{(0)} \omega_j (B_j \cos \omega_j t - A_j \sin \omega_j t)
\end{aligned} \tag{2-19}$$

where

$$\begin{aligned}
H_1^{(s)} &= d_{ij}^{(s)} A_j - e_{ij}^{(s)} B_j + f_{ij}^{(s)} \omega_j B_j + g_{ij}^{(s)} \omega_j A_j \\
H_2^{(s)} &= d_{ij}^{(s)} A_j + e_{ij}^{(s)} B_j + f_{ij}^{(s)} \omega_j B_j - g_{ij}^{(s)} \omega_j A_j \\
H_3^{(s)} &= d_{ij}^{(s)} B_j + e_{ij}^{(s)} A_j - f_{ij}^{(s)} \omega_j A_j + g_{ij}^{(s)} \omega_j B_j \\
H_4^{(s)} &= d_{ij}^{(s)} B_j - e_{ij}^{(s)} A_j - f_{ij}^{(s)} \omega_j A_j - g_{ij}^{(s)} \omega_j B_j
\end{aligned} \tag{2-20}$$

For the second equation in Eq. (2-19), all possible unstable items on the right hand side are collected with the variational part and the rest is associated with the

perturbation part. By treating the A_i and B_i as constant, the perturbation equations are always stable since $\omega_i > 0$. Thus, the variational equations govern the system stability.

When the excitation frequency of the coefficient matrices in Eq. (2-12) is equal or close to some combinations of the system natural frequencies, the parametric resonance occurs. First of all, the case of the excitation frequency close to some summation combinations of two natural frequencies is analyzed below,

$$\omega = \frac{1}{s}(\omega_k + \omega_j) + \varepsilon\lambda \quad (2-21)$$

Here, λ is a finite real number and $\varepsilon\lambda$ is a small quantity since ε is small. The ω_k and ω_j items are included in the variational equations as

$$\begin{aligned} \dot{A}_k \cos \omega_k t + \dot{B}_k \sin \omega_k t &= 0 \\ -\dot{A}_k \sin \omega_k t + \dot{B}_k \cos \omega_k t &= -\frac{\varepsilon}{2\omega_k} \left[H_2^{(s)} \cos(\omega_k t + \varepsilon\lambda st) - H_4^{(s)} \sin(\omega_k t + \varepsilon\lambda st) \right] \\ &\quad - \mathcal{F}_{kk}^{(0)} (B_k \cos \omega_k t - A_k \sin \omega_k t) \\ \dot{A}_j \cos \omega_j t + \dot{B}_j \sin \omega_j t &= 0 \\ -\dot{A}_j \sin \omega_j t + \dot{B}_j \cos \omega_j t &= -\frac{\varepsilon}{2\omega_j} \left[H_2^{*(s)} \cos(\omega_j t + \varepsilon\lambda st) - H_4^{*(s)} \sin(\omega_j t + \varepsilon\lambda st) \right] \\ &\quad - \mathcal{F}_{jj}^{(0)} (B_j \cos \omega_j t - A_j \sin \omega_j t) \end{aligned} \quad (2-22)$$

where $H_2^{*(s)}$ and $H_4^{*(s)}$ are the $H_2^{(s)}$ and $H_4^{(s)}$ with the subscripts k and j exchanged respectively. After decoupling A_k , B_k , A_j , and B_j from Eq. (2-22), all resulting equations are integrated with respect to $\omega_k t$ and $\omega_j t$ over $[0, 2\pi]$ and then substituted by their integral average. The variational equations become

$$\begin{aligned}
\dot{A}_k &= -\mathcal{F}_{kk}^{(0)} \frac{A_k}{2} - \frac{\varepsilon}{4\omega_k} \left[H_2^{(s)} \sin \varepsilon \lambda s t + H_4^{(s)} \cos \varepsilon \lambda s t \right] \\
\dot{B}_k &= -\mathcal{F}_{kk}^{(0)} \frac{B_k}{2} - \frac{\varepsilon}{4\omega_k} \left[H_2^{(s)} \cos \varepsilon \lambda s t - H_4^{(s)} \sin \varepsilon \lambda s t \right] \\
\dot{A}_j &= -\mathcal{F}_{jj}^{(0)} \frac{A_j}{2} - \frac{\varepsilon}{4\omega_j} \left[H_2^{*(s)} \sin \varepsilon \lambda s t + H_4^{*(s)} \cos \varepsilon \lambda s t \right] \\
\dot{B}_j &= -\mathcal{F}_{jj}^{(0)} \frac{B_j}{2} - \frac{\varepsilon}{4\omega_j} \left[H_2^{*(s)} \cos \varepsilon \lambda s t - H_4^{*(s)} \sin \varepsilon \lambda s t \right]
\end{aligned} \tag{2-23}$$

According to Eq. (2-20), the expressions of $H_2^{*(s)}$, $H_4^{*(s)}$, $H_2^{(s)}$, and $H_4^{(s)}$ are substituted into Eq. (2-23), and then the equations are further simplified based on the following transformation,

$$\begin{aligned}
X_1 &= A_k + \mathbf{i}B_k, & Y_1 &= A_j + \mathbf{i}B_j \\
X_2 &= A_k - \mathbf{i}B_k, & Y_2 &= A_j - \mathbf{i}B_j
\end{aligned} \tag{2-24}$$

where $\mathbf{i}^2 = -1$. The results are

$$\begin{aligned}
\dot{X}_1 &= -\frac{1}{2} \mathcal{F}_{kk}^{(0)} X_1 - \frac{\varepsilon}{4\omega_k} \left[\mathbf{i} \left(d_{kj}^{(s)} - g_{kj}^{(s)} \omega_j \right) - \left(e_{kj}^{(s)} + f_{kj}^{(s)} \omega_j \right) \right] e^{-\mathbf{i} \varepsilon \lambda s t} Y_2 \\
\dot{X}_2 &= -\frac{1}{2} \mathcal{F}_{kk}^{(0)} X_2 - \frac{\varepsilon}{4\omega_k} \left[-\mathbf{i} \left(d_{kj}^{(s)} - g_{kj}^{(s)} \omega_j \right) - \left(e_{kj}^{(s)} + f_{kj}^{(s)} \omega_j \right) \right] e^{\mathbf{i} \varepsilon \lambda s t} Y_1 \\
\dot{Y}_1 &= -\frac{1}{2} \mathcal{F}_{jj}^{(0)} Y_1 - \frac{\varepsilon}{4\omega_j} \left[\mathbf{i} \left(d_{jk}^{(s)} - g_{jk}^{(s)} \omega_k \right) - \left(e_{jk}^{(s)} + f_{jk}^{(s)} \omega_k \right) \right] e^{-\mathbf{i} \varepsilon \lambda s t} X_2 \\
\dot{Y}_2 &= -\frac{1}{2} \mathcal{F}_{jj}^{(0)} Y_2 - \frac{\varepsilon}{4\omega_j} \left[-\mathbf{i} \left(d_{jk}^{(s)} - g_{jk}^{(s)} \omega_k \right) - \left(e_{jk}^{(s)} + f_{jk}^{(s)} \omega_k \right) \right] e^{\mathbf{i} \varepsilon \lambda s t} X_1
\end{aligned} \tag{2-25}$$

The system stability is determined by Eq. (2-25) which are slightly coupled first order equations. Therefore, the stability criterion for the excitation frequency close to the summation of k^{th} and j^{th} natural frequencies, $(\omega_k + \omega_j)/s$, is

$$\left[\frac{\alpha + (\alpha^2 + \beta^2)^{1/2}}{2} \right]^{1/2} \Rightarrow \begin{cases} < f_{kk}^{(0)} + f_{jj}^{(0)} & \text{stable} \\ \geq f_{kk}^{(0)} + f_{jj}^{(0)} & \text{unstable} \end{cases} \tag{2-26}$$

with

$$\alpha = (f_{kk}^{(0)} + f_{jj}^{(0)})^2 - 4(f_{jj}^{(0)} f_{kk}^{(0)} + s^2 \lambda^2) + \frac{1}{\omega_k \omega_j} \left[(d_{kj}^{(s)} - g_{kj}^{(s)} \omega_j)(d_{jk}^{(s)} - g_{jk}^{(s)} \omega_k) \right. \\ \left. + (e_{kj}^{(s)} + f_{kj}^{(s)} \omega_j)(e_{jk}^{(s)} + f_{jk}^{(s)} \omega_k) \right] \quad (2-27)$$

$$\beta = 4s\lambda(f_{jj}^{(0)} - f_{kk}^{(0)}) + \frac{1}{\omega_k \omega_j} \left[(e_{kj}^{(s)} + f_{kj}^{(s)} \omega_j)(d_{jk}^{(s)} - g_{jk}^{(s)} \omega_k) \right. \\ \left. - (d_{kj}^{(s)} - g_{kj}^{(s)} \omega_j)(e_{jk}^{(s)} + f_{jk}^{(s)} \omega_k) \right]$$

Based on the same approach, the stability criterion for the excitation frequency close to the subtraction of k^{th} and j^{th} ($j > k$) natural frequencies, $(\omega_j - \omega_k)/s$, is similar to the summation case expect different α and β ,

$$\alpha = (f_{kk}^{(0)} + f_{jj}^{(0)})^2 - 4(f_{jj}^{(0)} f_{kk}^{(0)} + s^2 \lambda^2) - \frac{1}{\omega_k \omega_j} \left[(d_{kj}^{(s)} - g_{kj}^{(s)} \omega_j)(d_{jk}^{(s)} + g_{jk}^{(s)} \omega_k) \right. \\ \left. + (e_{kj}^{(s)} + f_{kj}^{(s)} \omega_j)(e_{jk}^{(s)} - f_{jk}^{(s)} \omega_k) \right] \quad (2-28)$$

$$\beta = 4s\lambda(f_{jj}^{(0)} - f_{kk}^{(0)}) + \frac{1}{\omega_k \omega_j} \left[(d_{kj}^{(s)} - g_{kj}^{(s)} \omega_j)(e_{jk}^{(s)} - f_{jk}^{(s)} \omega_k) \right. \\ \left. - (e_{kj}^{(s)} + f_{kj}^{(s)} \omega_j)(d_{jk}^{(s)} + g_{jk}^{(s)} \omega_k) \right]$$

When the case of multiple identical natural frequencies appears, all corresponding items must be included in the analysis if the identical natural frequency is involved in combination resonances.

2.3 Summary

This chapter gives the equations-of-motion and the corresponding state-space form for a general periodically linear time-varying system. The equations are classified as *Mathieu-Hill equation*.

According to Floquet theory, the state transition matrix that maps an initial state to the next state after one period determines the stability of the state mapping, also the stability of the system. This transition matrix is *Floquet Transition Matrix* and an intensive numerical integral is required to calculate FTM. To increase the computation

efficiency, a validated approximation strategy is employed to evaluate FTM by treating the system matrix as a stepwise constant function.

As an analytical method, the perturbation technique is used to predict the system stability boundary approximately. The high-accuracy results can be achieved when the system damping and the time-varying components of the system stiffness are much smaller than the constant component of the system stiffness.

Chapter 3

GEOMETRY AND MESHING KINEMATICS OF FACE-GEAR DRIVES

3.1 Overview

The face-gear is generated by shaper or worm and the necessary mathematical theory has been well established. This chapter is to introduce the mathematical descriptions, developed by Litvin et al. [2, 17, 82-83], for the analytical geometry of face-gear drives and the meshing simulation. The face-gear drive, discussed in this dissertation, comprises one (multiple) spur involute pinion and one face-gear generated by a shaper that is identical to the pinion. Therefore, all parameters of the shaper are the same as those of the pinion. The face-gear generation by a shaper is illustrated in Figure 3.1.

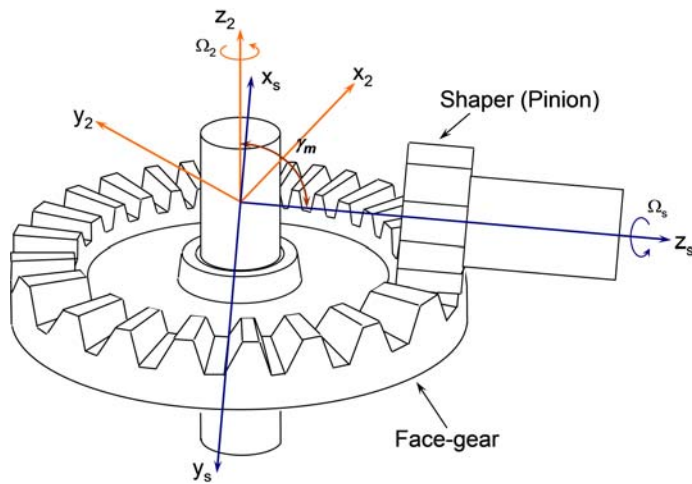


Fig. 3. 1: Diagram of face-gear generation by a shaper.

3.2 Shaper (Pinion) Tooth Surface

The shaper (pinion) is a standard spur involute gear and the projection of its tooth profile on x_s - y_s plane is shown in Figure 3.2. The angle corresponding to half of the circular width of the space between two adjacent teeth, θ_{0s} , is determined by [82],

$$\theta_{0s} = \frac{\pi}{2N_s} - \text{inv } \alpha_c \quad (3-1)$$

where N_s is number of shaper teeth, α_c is shaper pressure angle and “inv” represents the equation of involute curve.

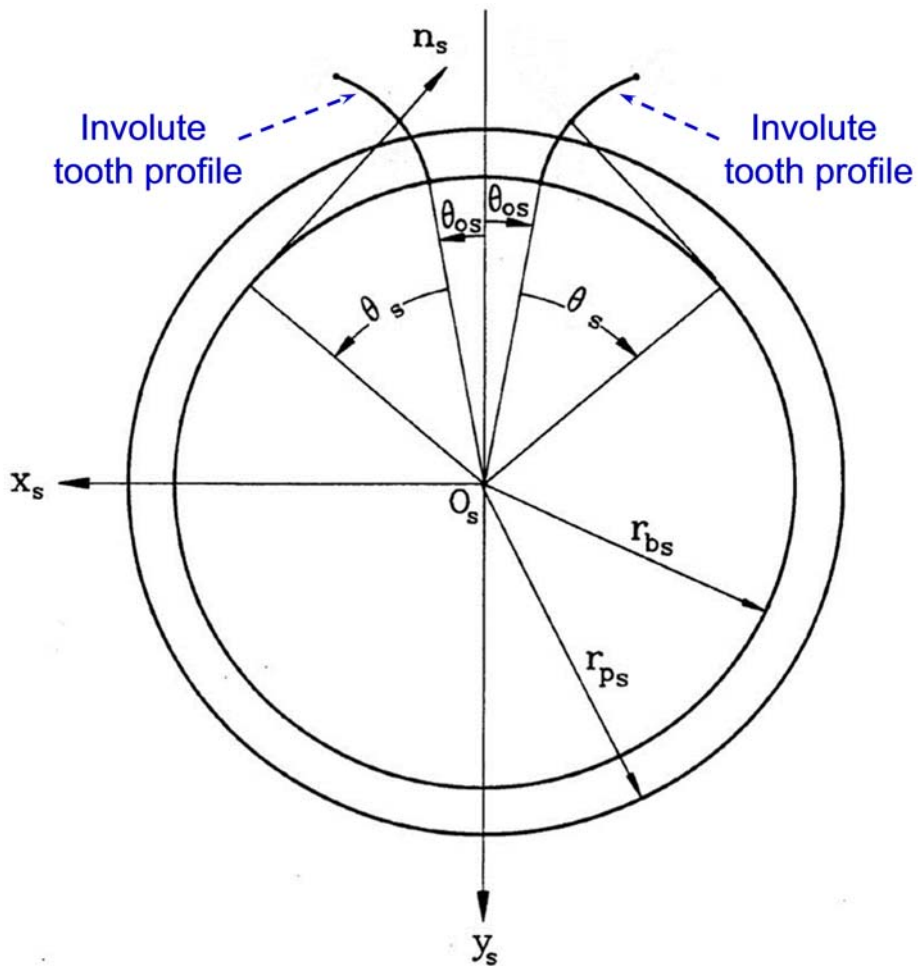


Fig. 3. 2: Shaper (pinion) with involute tooth profile [2, 82].

The position vector of a point on shaper tooth surface is expressed in shaper's body fixed coordinate $\boldsymbol{I}_s(x_s, y_s, z_s)$ as [82]

$$\mathbf{r}_s(u_s, \theta_s) = \begin{bmatrix} x_s \\ y_s \\ z_s \end{bmatrix} = \begin{bmatrix} r_{bs}[\sin(\theta_{0s} + \theta_s) - \theta_s \cos(\theta_{0s} + \theta_s)] \\ -r_{bs}[\cos(\theta_{0s} + \theta_s) + \theta_s \sin(\theta_{0s} + \theta_s)] \\ u_s \end{bmatrix} \quad (3-2)$$

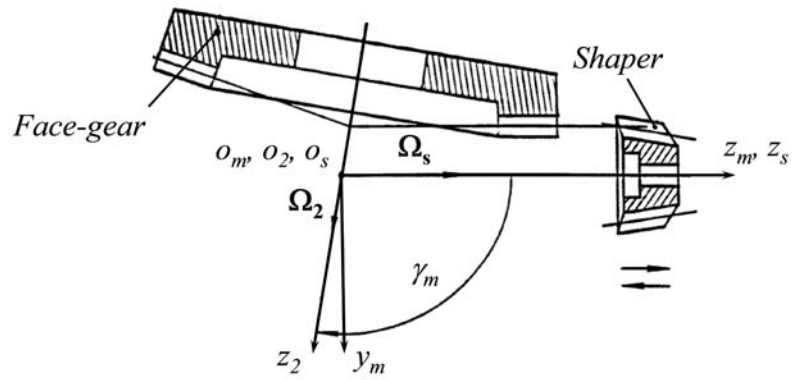
Here, (u_s, θ_s) are the Gaussian coordinates of the involute shaper surface, and r_{bs} is the radius of shaper base circle. Furthermore, the unit normal to the shaper tooth surface is also obtained based on the position vector [82],

$$\mathbf{n}_s = \frac{\frac{\partial \mathbf{r}_s}{\partial \theta_s} \times \frac{\partial \mathbf{r}_s}{\partial u_s}}{\left| \frac{\partial \mathbf{r}_s}{\partial \theta_s} \times \frac{\partial \mathbf{r}_s}{\partial u_s} \right|} = - \begin{bmatrix} \cos(\theta_{0s} + \theta_s) \\ \sin(\theta_{0s} + \theta_s) \\ 0 \end{bmatrix} \quad (3-3)$$

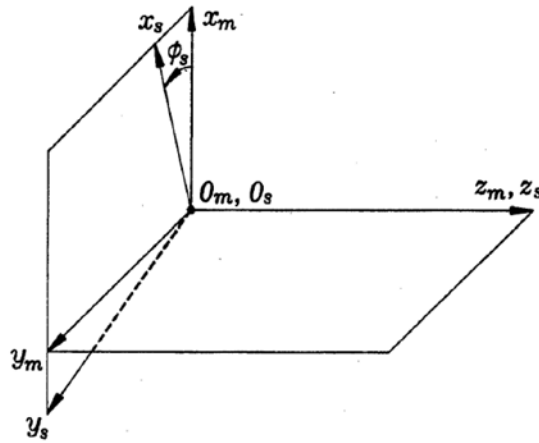
3.3 Coordinate Transformation

The shaper's body fixed coordinate $\boldsymbol{I}_s(x_s, y_s, z_s)$ is transformed to the face-gear's body fixed coordinate $\boldsymbol{I}_2(x_2, y_2, z_2)$ through two inertial coordinate systems: one is $\boldsymbol{I}_m(x_m, y_m, z_m)$, aligning with the initial shaper position; the other is $\boldsymbol{I}_p(x_p, y_p, z_p)$, aligning with the initial face-gear position. The coordinates for the face-gear generation by a shaper is sketched in Figure 3.3(a); the shaper coordinate $\boldsymbol{I}_s$ and its corresponding inertial coordinate $\boldsymbol{I}_m$ are shown in Figure 3.3(b); the face-gear coordinate $\boldsymbol{I}_2$ and its corresponding inertial coordinate $\boldsymbol{I}_p$ are shown in Figure 3.3(c);

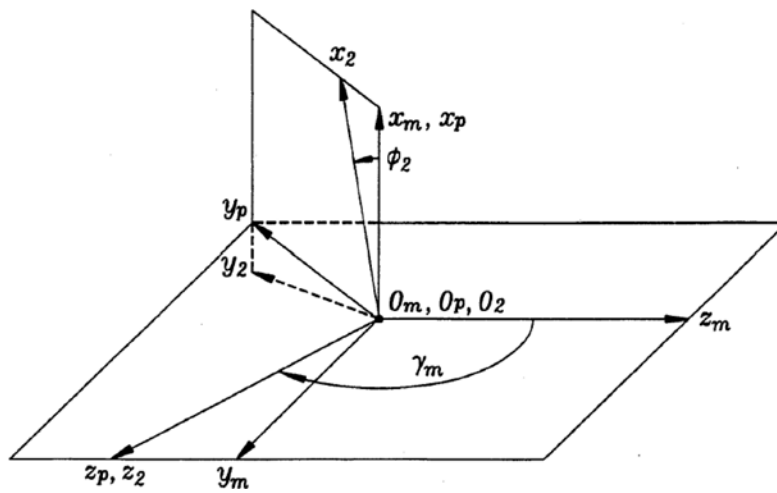
This procedure comprises three coordinate rotations: firstly, $\boldsymbol{I}_s$ rotates clockwise to inertial coordinate $\boldsymbol{I}_m$ along z_m axis; next, $\boldsymbol{I}_m$ rotates clockwise to $\boldsymbol{I}_p$ along x_m axis; and finally, $\boldsymbol{I}_p$ rotates count-clockwise to $\boldsymbol{I}_2$ along z_p axis.



(a) Sketch of the face-gear generation by a shaper



(b) Coordinates for shaper



(c) Coordinates for face-gear

Fig. 3. 3 Coordinate systems for face-gear and shaper [2, 82].

The transform matrix from $\mathbf{F}_s$ to $\mathbf{F}_2$ is the product of three coordinate rotation matrices in order [82],

$$\begin{aligned}\mathbf{M}_{2s} &= \mathbf{M}_{2p}(\phi_2)\mathbf{M}_{pm}(\gamma_m)\mathbf{M}_{ms}(\phi_s) \\ &= \begin{bmatrix} \cos\phi_2 \cos\phi_s + \cos\gamma_m \sin\phi_2 \sin\phi_s & -\cos\phi_2 \sin\phi_s + \cos\gamma_m \sin\phi_2 \cos\phi_s & -\sin\gamma_m \sin\phi_2 \\ -\sin\phi_2 \cos\phi_s + \cos\gamma_m \cos\phi_2 \sin\phi_s & \sin\phi_2 \sin\phi_s + \cos\gamma_m \cos\phi_2 \cos\phi_s & -\sin\gamma_m \cos\phi_2 \\ \sin\gamma_m \sin\phi_s & \sin\gamma_m \cos\phi_s & \cos\gamma_m \end{bmatrix}\end{aligned}\quad (3-4)$$

and

$$\mathbf{r}_2(u_s, \theta_s, \phi_s) = \mathbf{M}_{2s}(\phi_s)\mathbf{r}_s(u_s, \theta_s) \quad \text{with} \quad \phi_2 = \phi_s \frac{N_s}{N_2} \quad (3-5)$$

where γ_m is the shaft angle of face-gear drive ($\gamma_m \geq 90^\circ$), $\mathbf{r}_2$ is the position vector in $\mathbf{F}_2$, N_2 is the number of face-gear teeth, and ϕ_s , ϕ_2 are rotation angles of shaper and face-gear respectively.

3.4 Equation of Meshing

The equation of meshing is derived from the condition that the relative velocity of a contact point on tooth surface is vertical to its normal [82-83]. The relative velocity of shaper to face-gear is expressed in $\mathbf{F}_s$ as [82]

$$\begin{aligned}\mathbf{v}_s^{(s2)} &= \mathbf{v}_s^{(s)} - \mathbf{v}_s^{(2)} = (\boldsymbol{\Omega}_s^{(s)} - \boldsymbol{\Omega}_s^{(2)}) \times \mathbf{r}_s \\ &= \begin{bmatrix} v_{xs}^{(s2)} \\ v_{ys}^{(s2)} \\ v_{zs}^{(s2)} \end{bmatrix} = \boldsymbol{\Omega}_s^{(s)} \begin{bmatrix} -y_s(1 - m_{2s} \cos\gamma_m) - z_s m_{2s} \sin\gamma_m \cos\phi_s \\ x_s(1 - m_{2s} \cos\gamma_m) + z_s m_{2s} \sin\gamma_m \sin\phi_s \\ m_{2s} \sin\gamma_m (x_s \cos\phi_s - y_s \sin\phi_s) \end{bmatrix}\end{aligned}\quad (3-6)$$

where gear ratio is $m_{2s} = N_s/N_2$, $\mathbf{v}_s^{(s)}$ and $\mathbf{v}_s^{(2)}$ are the velocity vectors of the shaper and the face-gear in $\mathbf{F}_s$ respectively, $\boldsymbol{\Omega}_s^{(s)}$ is shaper rotation speed, and $\boldsymbol{\Omega}_s^{(s)}$ and $\boldsymbol{\Omega}_s^{(2)}$ are the

angular velocity vectors of the shaper and the face-gear in $\mathbf{I}_s$ respectively. Combining with Eq. (3-3), the equation of meshing is given as [82]

$$f(u_s, \theta_s, \phi_s) = \mathbf{n}_s \bullet \mathbf{v}_s^{(s2)} = r_{bs}(1 - m_{2s} \cos \gamma_m) - u_s m_{2s} \sin \gamma_m \cos(\phi_s + \theta_{0s} + \theta_s) = 0 \quad (3-7)$$

3.5 Face-gear Tooth Surface Generation

Since face-gear tooth is generated by meshing with a shaper, the tooth surface must satisfy with the equation of meshing, Eq. (3-7). The mathematical procedures for forming the face-gear tooth surface are: 1) define the position vector of the shaper tooth surface; 2) solve the equation of meshing to locate tooth contact areas based on (u_s, θ_s, ϕ_s) parameters scan; 3) express the contact areas in face-gear body fixed coordinate system via coordinate transform matrix Eq. (3-4). For each ϕ_s , the values of (u_s, θ_s) define an instantaneous contact line between the teeth of face-gear and shaper (pinion). The contact lines constitute the face-gear tooth surface. The flowchart of face-gear generation is illustrated in Figure 3.4.

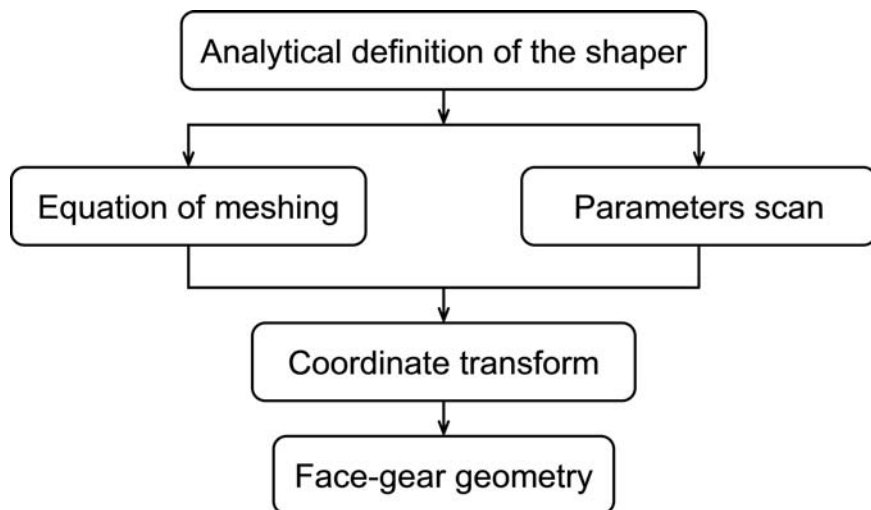


Fig. 3. 4: Flowchart of face-gear generation [27].

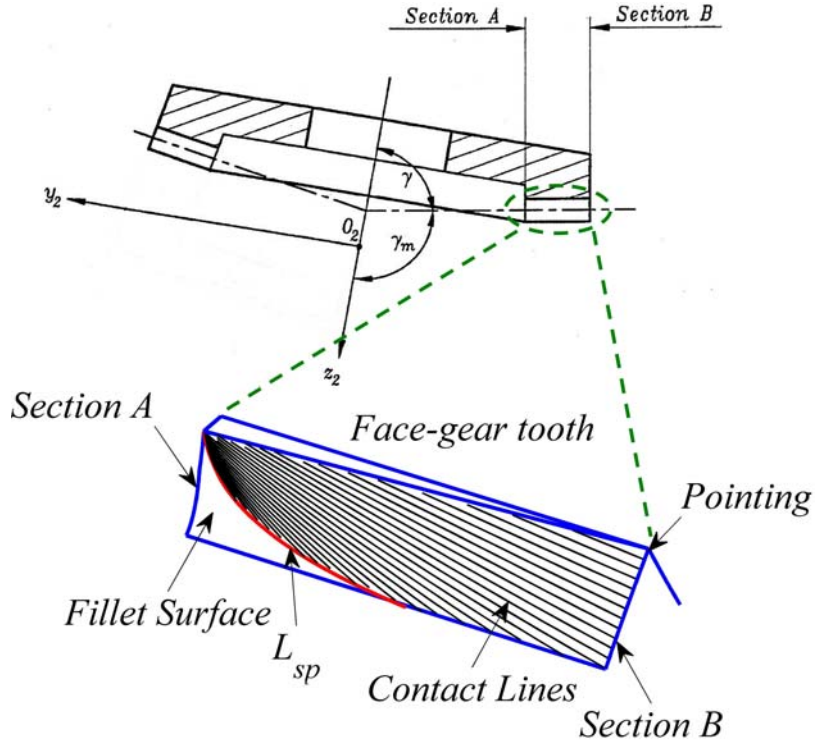


Fig. 3. 5: Face-gear tooth and contact lines [2].

Figure 3.5 shows a sample generated face-gear tooth and contact lines. Each contact line represents the location where face-gear and shaper (pinion) teeth interact with each other in an instant. The mesh cycle starts from *section B* at the outer edge of face-gear tooth and ends in *section A* at the inner edge. It is revealed in Figure 3.5 that contact lines continuously change their length and location in one mesh cycle. This variation indicates the mesh load moves its effective position periodically during rotations, which is possible to excite dynamic instability.

The cross section of a face-gear drive is shown in Figure 3.6 to illustrate the face-gear tooth radius calculation. The inner edge of face-gear tooth (*Section A*) is limited by nonundercutting condition. It avoids the undercutting on the fillet surface that is under the boundary curve L_{sp} in Figure 3.5.

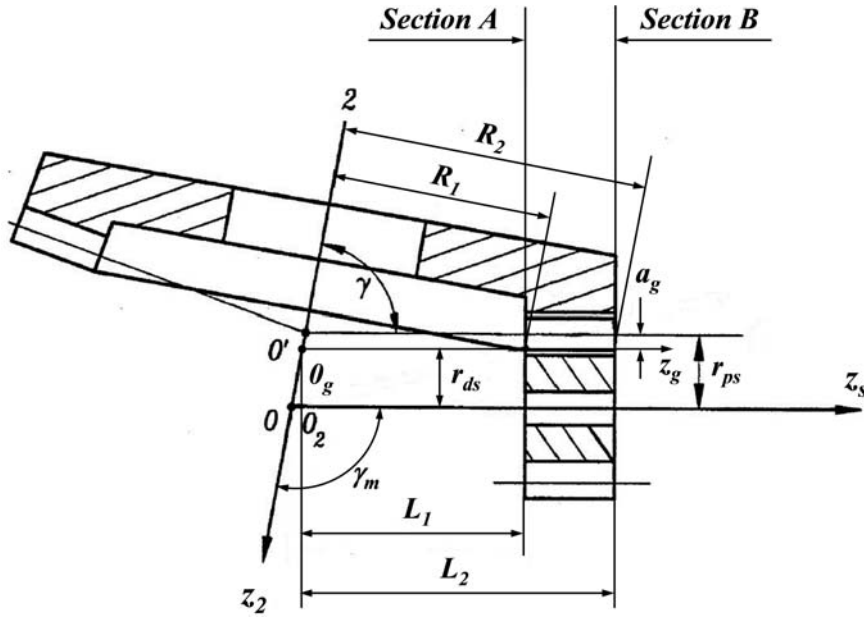


Fig. 3. 6: Inner and outer limiting ends of face-gear tooth [2, 82].

The equations for nonundercutting condition are given [82].

$$\Delta_1^2 + \Delta_2^2 + \Delta_3^2 = 0 \quad (3-8)$$

where

$$\Delta_1 = \begin{vmatrix} \frac{\partial x_s}{\partial u_s} & \frac{\partial x_s}{\partial \theta_s} & v_{xs}^{(s2)} \\ \frac{\partial y_s}{\partial u_s} & \frac{\partial y_s}{\partial \theta_s} & v_{ys}^{(s2)} \\ \frac{\partial f}{\partial u_s} & \frac{\partial f}{\partial \theta_s} & \frac{\partial f}{\partial \phi_s} \Omega_s^{(s)} \end{vmatrix} = 0 \quad (3-9)$$

$$\Delta_2 = \begin{vmatrix} \frac{\partial x_s}{\partial u_s} & \frac{\partial x_s}{\partial \theta_s} & v_{xs}^{(s2)} \\ \frac{\partial z_s}{\partial u_s} & \frac{\partial z_s}{\partial \theta_s} & v_{zs}^{(s2)} \\ \frac{\partial f}{\partial u_s} & \frac{\partial f}{\partial \theta_s} & \frac{\partial f}{\partial \phi_s} \Omega_s^{(s)} \end{vmatrix} = 0 \quad (3-10)$$

$$\Delta_3 = \begin{vmatrix} \frac{\partial y_s}{\partial u_s} & \frac{\partial y_s}{\partial \theta_s} & v_{ys}^{(s2)} \\ \frac{\partial z_s}{\partial u_s} & \frac{\partial z_s}{\partial \theta_s} & v_{zs}^{(s2)} \\ \frac{\partial f}{\partial u_s} & \frac{\partial f}{\partial \theta_s} & \frac{\partial f}{\partial \phi_s} \Omega_s^{(s)} \end{vmatrix} = 0 \quad (3-11)$$

Here, f is the equation of meshing in Eq. (3-7), (x_s, y_s, z_s) is the position vector of the shaper tooth surface in Eq. (3-2), and $(v_{xs}^{(s2)}, v_{ys}^{(s2)}, v_{zs}^{(s2)})$ is the relative velocity vector in Eq. (3-6).

The critical point of undercutting locates on the intersection of the shaper addendum and the boundary curve L_{sp} on the face-gear tooth. The polar angle of the shaper addendum in Gaussian coordinates, denoted by θ_s^* , is [82]

$$\theta_s^* = \frac{\sqrt{r_{as}^2 - r_{bs}^2}}{r_{bs}} \quad (3-12)$$

with the radius of the shaper addendum circle, r_{as} . The superscript asterisk indicates the parameters for the critical point of undercutting. The inner limiting end of the face-gear tooth is found by following procedures: 1) input θ_s^* into any equation of Eq. (3-9), Eq. (3-10), and Eq. (3-11) to solve the parameter ϕ_s^* ; 2) substitute θ_s^* and ϕ_s^* into the equation of meshing Eq. (3-7) to determine the parameter u_s^* ; 3) locate the coordinates (x_s^*, y_s^*, z_s^*) of the critical point of undercutting by Eq. (3-2) and parameters (u_s^*, θ_s^*) . The inner limiting value, L_l , is obtained by [17]

$$L_l = z_s^* - \frac{r_{ds}}{\tan \gamma} \quad \text{and} \quad \gamma \leq 90^\circ \quad (3-13)$$

where r_{ds} is the radius of the shaper dedendum and $\gamma_m + \gamma = 180^\circ$.

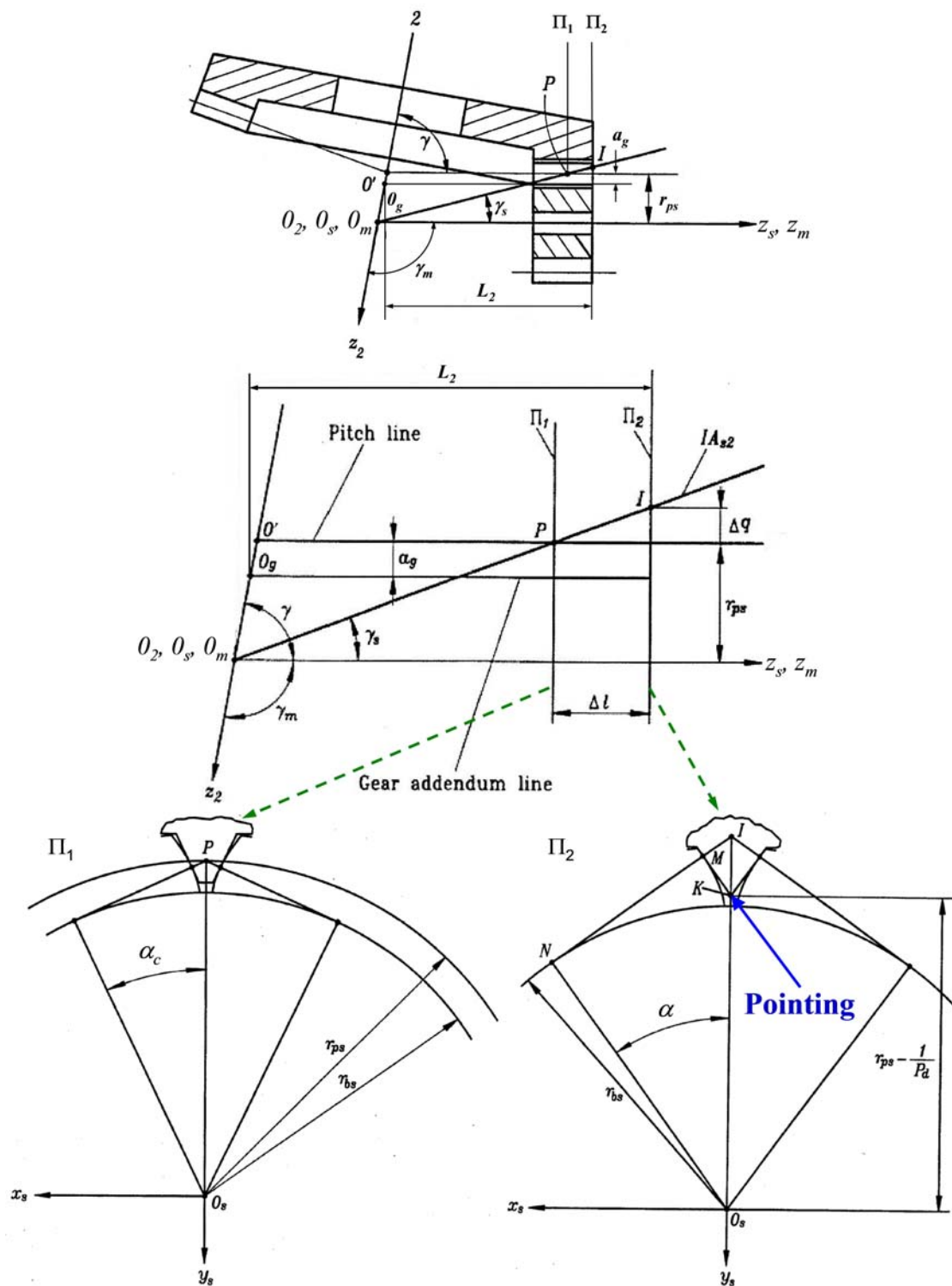


Fig. 3. 7: Schematic of pointing position and cross sections of planes Π_1 and Π_2 [2, 82].

The outer edge of face-gear tooth (*Section B* in Figure 3.6) is limited by pointing condition. Pointing means the planes of both tooth sides meet at a point where the tooth thickness equals to zero (marked in Figure 3.5). The Π_2 plane containing pointing position is shown in Figure 3.7. The outer limiting value, L_2 , is given by [82]

$$L_2 = \frac{N_s}{2P_d} \left(\frac{1}{\tan \gamma_s} - \frac{1}{\tan \gamma} \right) + \frac{a_g}{\tan \gamma} + \frac{N_s}{2P_d \tan \gamma_s} \left(\frac{\cos \alpha_c - \cos \alpha}{\cos \alpha} \right) \quad (3-14)$$

where P_d is diametral pitch, a_g is the shaper tooth addendum, and α , γ_s are solved via Eq. (3-15) and Eq. (3-16) respectively [82]

$$\alpha - \frac{(N_s - 2) \sin \alpha}{N_s \cos \alpha_c} = \theta_{0s} \quad (3-15)$$

$$\cot \gamma_s = \frac{1 + m_{2s} \cos \gamma}{m_{2s} \sin \gamma} \quad (3-16)$$

Consequently, the corresponding inner and outer limiting radii of face-gear tooth, measured on the plane of the face-gear body disk, are respectively (Figure 3.6)

$$R_1 = L_1 \sin \gamma \quad (3-17)$$

$$R_2 = L_2 \sin \gamma \quad (3-18)$$

3.6 Contact Point and Contact Centroid

The tooth size is much smaller than the face-gear dimensions, so does the length of contact line. This dissertation assumes the mesh load that distributes along the tooth surface as a lumped one acting on the midpoint of each contact line in turn. This midpoint is called “*Contact Point*” and marked in Figure 3.8 with “x” symbol.

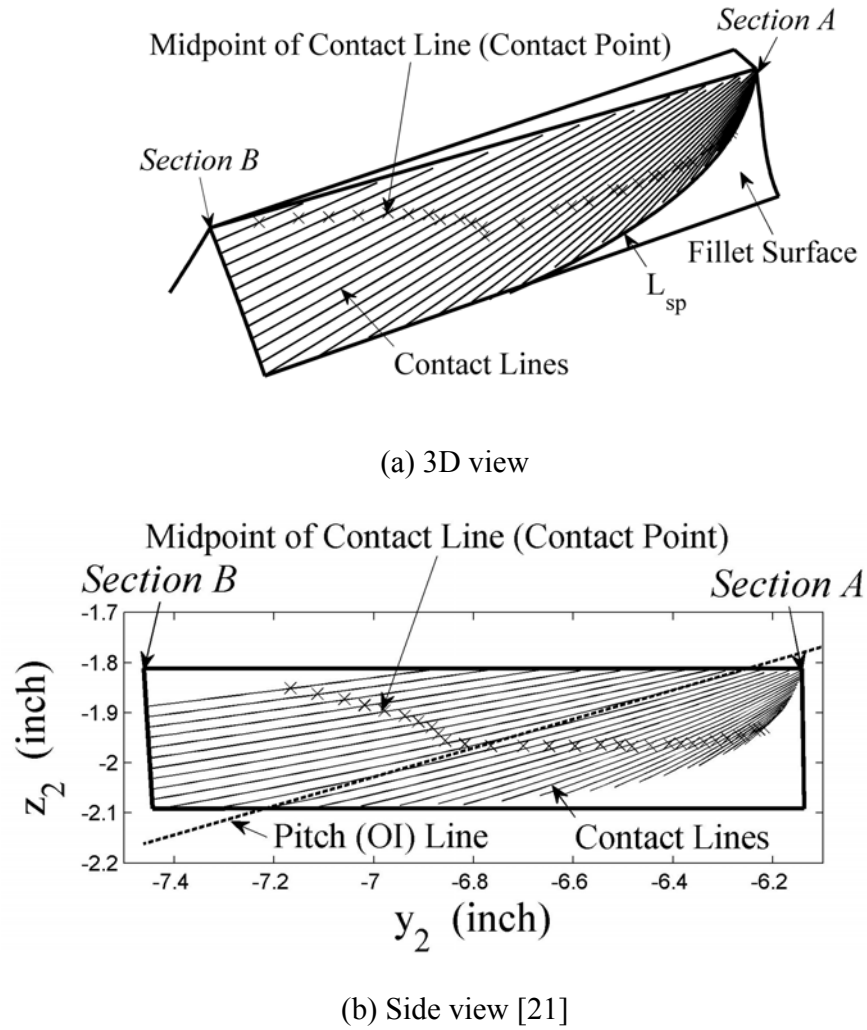


Fig. 3. 8: Contact points on face-gear tooth.

The contact point moves from the outer end of the face-gear tooth to the inner end periodically as gear rotations. The total time of one engaged tooth pair running from initial contact to out of meshing, T_e , is

$$T_e = cr \times T_0 \quad (3-19)$$

and

$$T_0 = \frac{2\pi}{\omega_m} \quad \text{and} \quad \omega_m = \Omega_2 N_2 \quad (3-20)$$

where cr is contact-ratio, T_0 is mesh period, ω_m is mesh frequency and Ω_2 is face-gear rotation speed. The radial and circumferential positions of contact points, r_i and θ_i , for four successive meshing tooth pairs on an example face-gear drive are plotted in the space fixed coordinate frame overlaying the face-gear surface, shown in Figure 3.9. For clockwise rotation direction from the top view of face-gear, both radial and circumferential positions of the contact point continuously vary from their maximum to minimum during one mesh cycle, and then the meshing tooth pair runs out of contact.

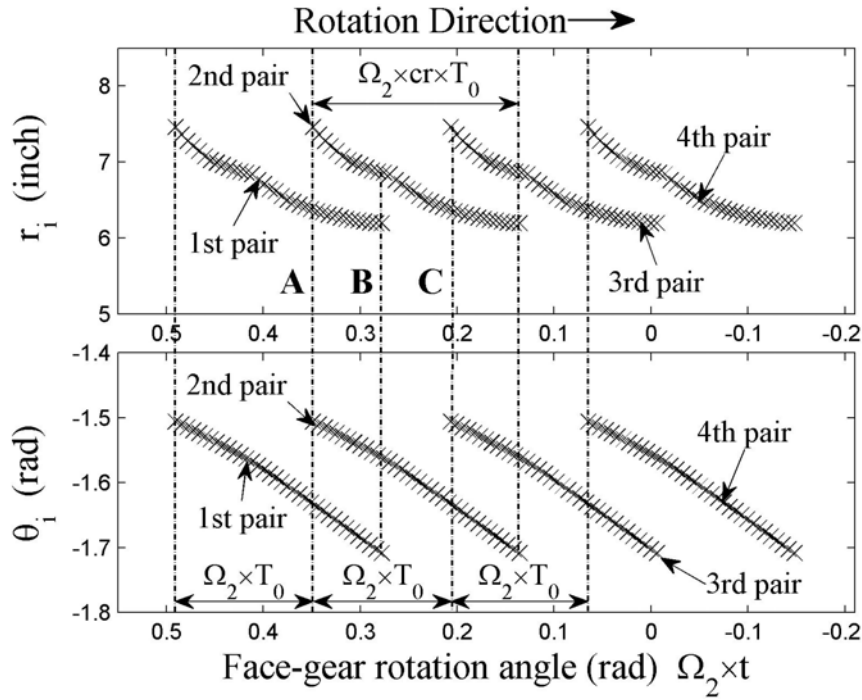


Fig. 3. 9: An example of effective radial and circumferential positions of contact points and phase lags between different meshing tooth pairs, for $cr = 1.5$.

For the contact-ratio greater than one, multiple pairs of teeth may stay in mesh simultaneously but with different phase lags. For example, a 1.7 contact-ratio means one pair of teeth are always in contact and 70% of the mesh time two pairs are in contact [21].

The contact-ratio in Figure 3.9 is 1.5. It indicates that the first and second pairs stay in mesh simultaneously for half mesh period, from time **A** to time **B**. Afterwards, the first pair goes out of contact and the second pair keeps in engagement alone for the rest of the mesh period until the third pair begins to mesh, from time **B** to time **C**. The cycle repeats in the same pattern as gear rotations. For a non-unity contact-ratio, the number of meshing pairs fluctuates during one period, which causes the variation of the total mesh load. Moreover, figure 3.9 also illustrates the phase difference between adjacent meshing pairs is $\Omega_2 \times T_0$.

For the transverse vibration analysis, one-point-contact is required in order to incorporate the effects of the pinion bearing on the face-gear dynamics. Therefore, a simplified strategy is employed to combine all contact points in mesh by an equivalent contact point, called “*Contact Centroid*” and illustrated in Figure 3.10.

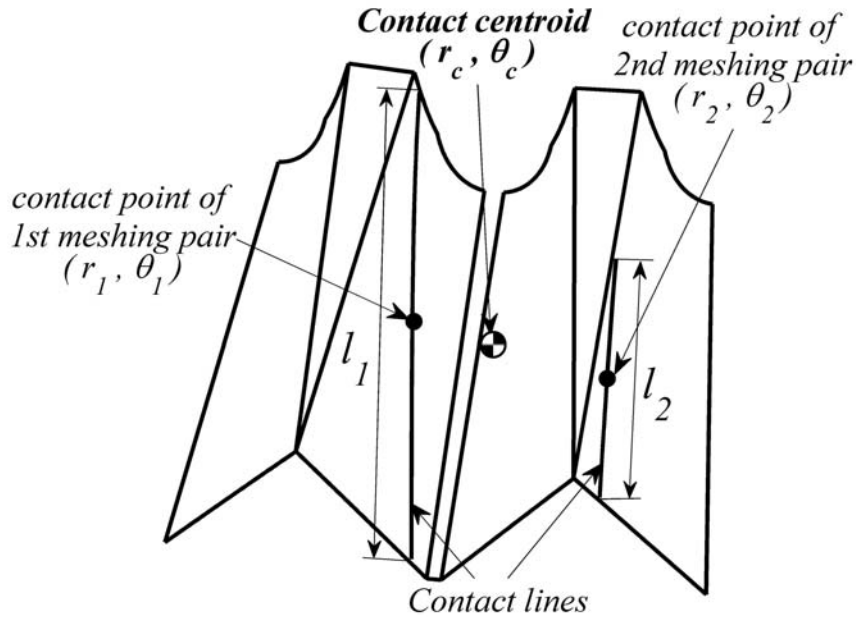


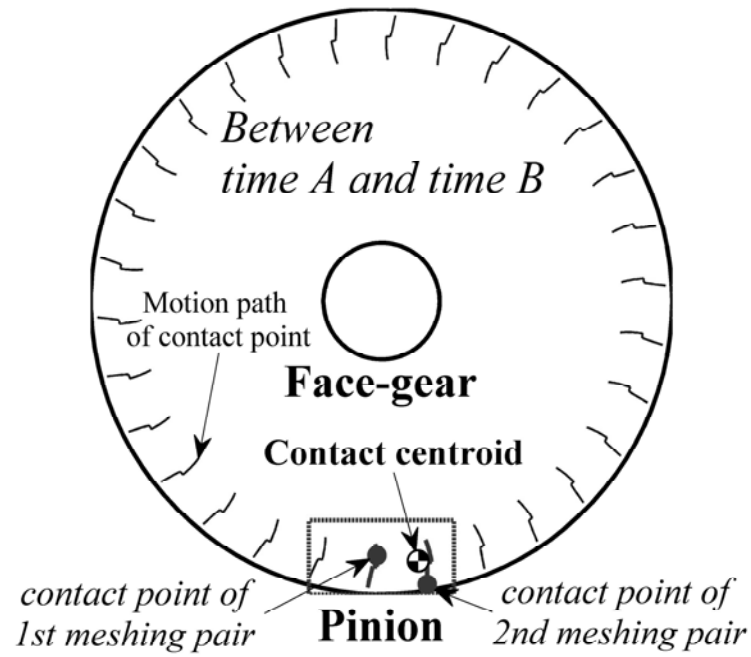
Fig. 3. 10: Contact centroid, for $1 < cr < 2$.

The contact centroid position (r_c, θ_c) is calculated in space fixed frame (r, θ) by the following equations:

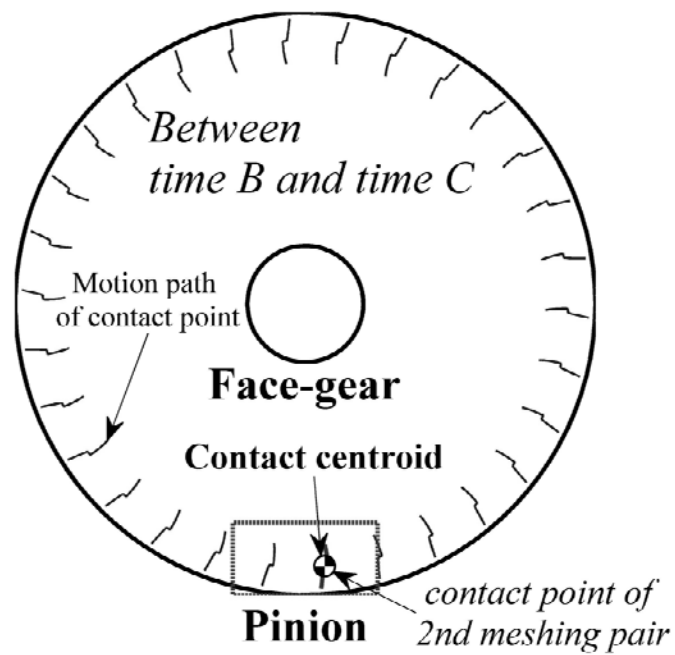
$$\begin{aligned} r_c(t) &= \frac{\sum_{j=1}^{\text{ceil}(cr)} [r_j(t)l_j(t)]}{\sum_{j=1}^{\text{ceil}(cr)} [l_j(t)]} \\ \theta_c(t) &= \frac{\sum_{j=1}^{\text{ceil}(cr)} [\theta_j(t)l_j(t)]}{\sum_{j=1}^{\text{ceil}(cr)} [l_j(t)]} \end{aligned} \quad (3-21)$$

where (r_j, θ_j) is the contact point position of the j^{th} meshing pair, l_j is the corresponding contact line length, and “ceil(•)” is the function that rounds a number upwards toward its nearest integer. Here, the contact line length is used as a weight parameter to evaluate the contact centroid position. Both contact point position and contact line length depend on gear rotation angles, also on time.

Figure 3.11 illustrates the positions of contact points and corresponding contact centroid on face-gear plane for 1.5 contact-ratio. When the face-gear rotation angle (time) is between angle **A** and angle **B** indicated in Figure 3.9, two meshing pairs are in contact simultaneously and the contact centroid locates between two contact points (see Figure 3.11(a)). When the face-gear rotation angle is between angle **B** and angle **C**, only one meshing pair engages and the contact centroid overlaps with this contact point (see Figure 3.11(b)). The positions of contact points and corresponding contact centroid during two adjacent mesh periods for a sample face-gear drive are plotted in Figure 3.12. The radial and circumferential positions are shown in Figure 3.12(a) and Figure 3.12 (b) respectively.

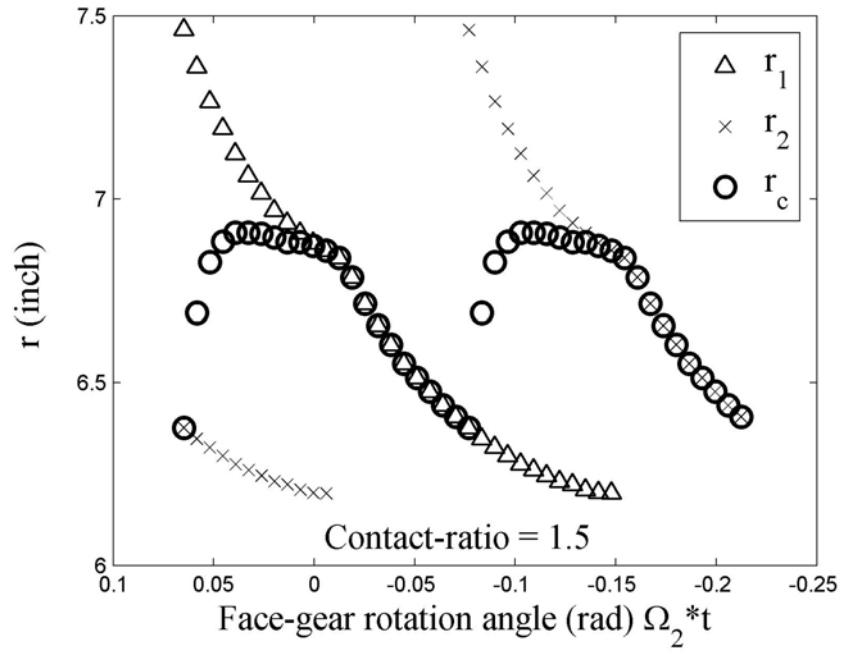


(a) Between time **A** and time **B**

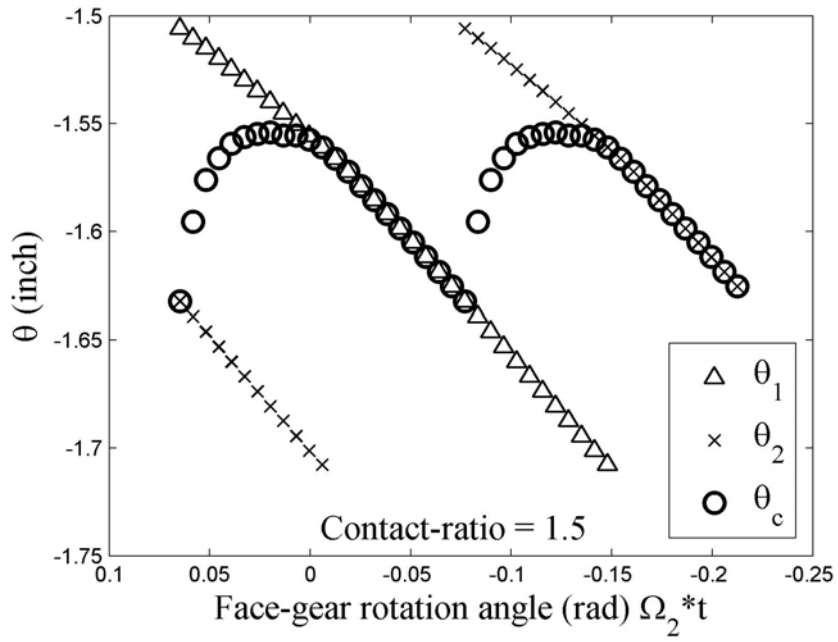


(b) Between time **B** and time **C**

Fig. 3. 11: Sketch of contact points and corresponding contact centroid, for $cr = 1.5$.



(a) Radial position



(b) Circumferential position

Fig. 3. 12: Positions of contact points and corresponding contact centroid, for $cr = 1.5$.

3.7 Approximate Movement Path of Contact Points

To easily expand the meshing kinematics as Fourier series for perturbation analyses, a simplification strategy is employed in the torsional vibration investigation. This strategy is to approximate the movement path of contact points as a skew line for each meshing tooth pair individually, and then superpose the mesh loads from all meshing pairs but with their phase lags, shown in Figure 3.13. It should be noted that the slope of the circumferential variation approximates to one. This indicates the contact point rotates with the face-gear at almost the same speed so that its circumferential variation relative to the face-gear body can be neglected.

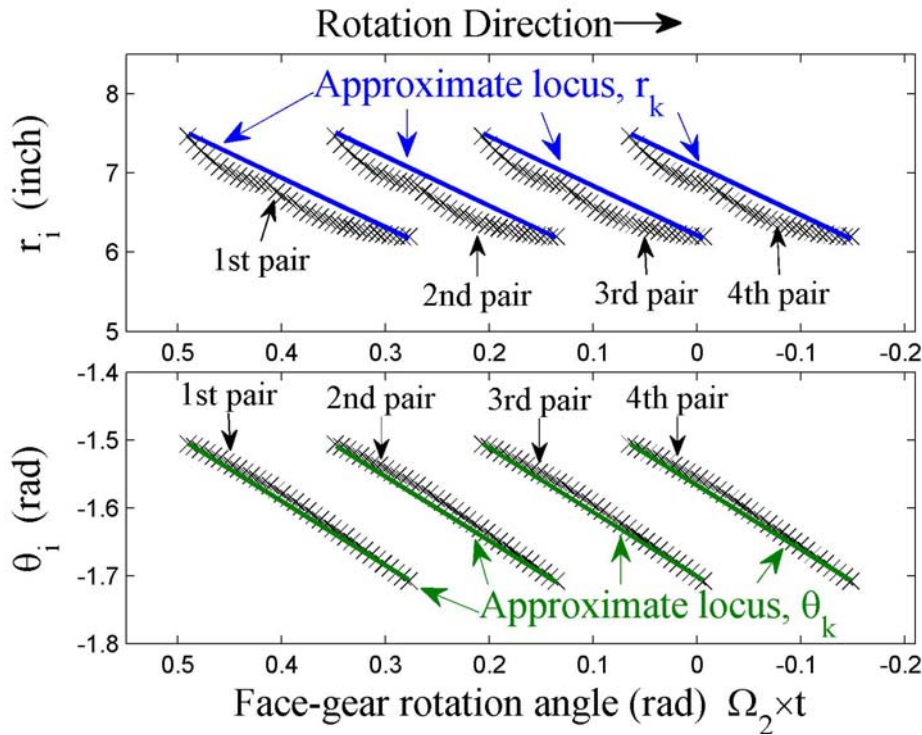


Fig. 3. 13: Approximate loci for the position of contact points, for $cr = 1.5$ [21].

Based on this simplification strategy, the approximate radial and circumferential positions of contact point for a meshing tooth pair, (r_k, θ_k) , are expressed in space fixed frame (r, θ) as:

$$\begin{aligned} r_k(t) &= \begin{cases} R_2 + \frac{(R_1 - R_2)}{crT_0}t & 0 \leq t \leq crT_0 \\ no\ contact & crT_0 < t \leq T_0 \text{ceil}(cr) \end{cases} \\ \theta_k(t) &= \begin{cases} \Omega_2 t + \theta_r & 0 \leq t \leq crT_0 \\ no\ contact & crT_0 < t \leq T_0 \text{ceil}(cr) \end{cases} \end{aligned} \quad (3-22)$$

where θ_r is initial angle. Without loss of generality, the initial angle is set as zero for simplicity.

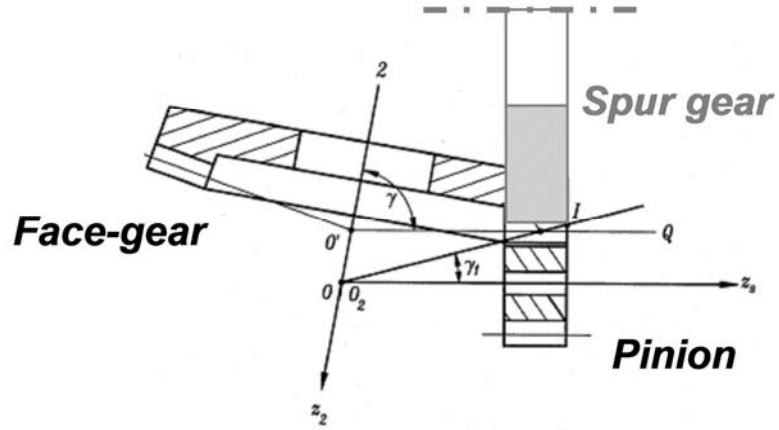
3.8 Contact-Ratio Approximation

Contact-ratio is an important parameter for gear dynamics investigations, but no exact close-form formula exists for spur pinion/face-gear drive. Since the numerical methods, based on differential geometry and theory of gearing, are time-consuming [17, 82], this dissertation employs so-called Tregold's approximation [84] for face-gear contact-ratio calculation.

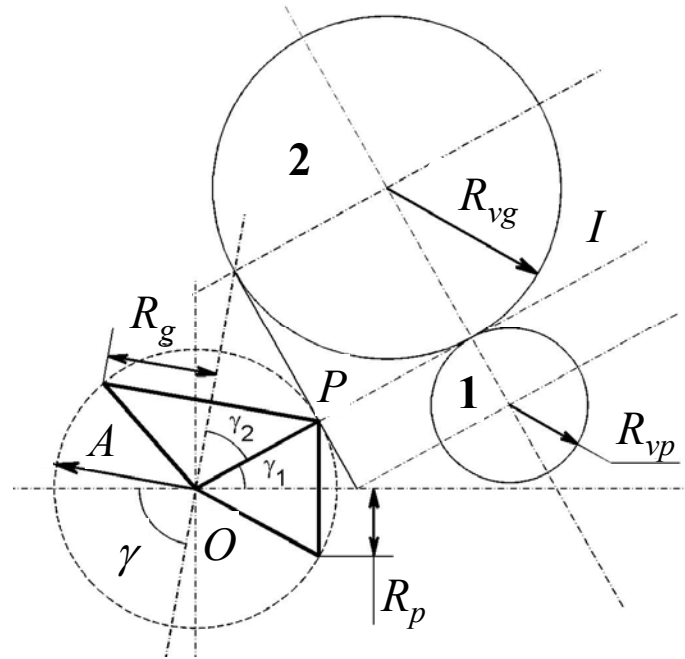
The idea is to image a spur gear instead of the face-gear to mesh with the spur pinion, which transforms the face-gear drive into a formative spur gear drive. This method is a common practice used in bevel gearing and is illustrated in Figure 3.14. The apex semiangles of pinion and face-gear pitch cones, γ_1 and γ_2 , are given in [82]:

$$\gamma_1 = \text{arc cot} \left(\frac{m_{12} + \cos \gamma}{\sin \gamma} \right) \quad \text{and} \quad \gamma_2 = \text{arc cot} \left(\frac{1 + m_{12} \cos \gamma}{m_{12} \sin \gamma} \right) \quad (3-23)$$

where the gear ratio m_{12} equals to N_2/N_1 , N_1 is the number of pinion teeth and N_2 is the number of face-gear teeth.



(a) Formative spur gear [82, 84]



(b) Mapping from face-gear drives to spur gear drives [21, 84]

Fig. 3. 14: Tregold's approximation.

The number of teeth for the formative spur gear drives, N_{v1} and N_{v2} , are calculated by [84]:

$$\begin{aligned} N_{v1} &= N_1 / \cos \gamma_1 \\ N_{v2} &= N_2 / \cos \gamma_2 \end{aligned} \quad (3-24)$$

The contact-ratio of face-gear drive is approximated by the formative spur gear contact-ratio. The analytical formula for the contact-ratio of spur gears is [82]:

$$cr = \frac{N_{v1}(\tan \alpha_{va1} - \tan \alpha_c) + N_{v2}(\tan \alpha_{va2} - \tan \alpha_c)}{2\pi} \quad (3-25)$$

$$\cos \alpha_{vai} = r_{vbi} / r_{vai} \quad (i = 1, 2) \quad (3-26)$$

where α_c , α_{vai} , r_{vbi} , r_{vai} are respectively the pressure angle, the addendum pressure angle, the base circle radius, and the addendum circle radius of the formative spur gears. Here, r_{vai} is obtained in the usual way by adding tooth addendum to the pitch radius.

3.9 Summary

This chapter introduces the analytical geometry of face-gear drives, developed by Litvin et al. The face gear tooth surface is generated through a conjugated meshing with the shaper. The contact lines between face-gear and shaper form the face-gear tooth surface. Essentially the tooth generation is to find the mathematical expression of the contact lines and the limiting value of the tooth. Based on the known parameters of the shaper, the position vectors of contact lines on shaper tooth surface are solved via the equation of meshing. Next, a coordinate transformation is applied to obtain the position vectors of contact lines on face-gear tooth surface. Through the parameter scan, the contact lines within one mesh cycle constitute the face-gear tooth surface. The inner and

outer ends of the face-gear tooth are respectively limited by nonundercutting and pointing conditions. Since the spur pinion is an identical copy of the shaper, all parameters of the shaper are the same as those of the pinion.

The contact point is defined as the midpoint of each contact line. The movement path of the contact points can be directly obtained through the mathematical expression of the contact lines. For the transverse vibration of face-gear, the concept of contact centroid is utilized to collect the effects of multiple contact points that are in mesh at the same time, when the contact-ratio is greater than one. For the torsional vibration, the movement path of the contact points is approximated as a skew line for each meshing tooth pair. The mesh loads from multiple meshing tooth pairs are superposed after calculating each mesh load from a single pair individually.

Finally, Tregold's approximation is employed to calculate face-gear contact-ratio. This method transforms a spur pinion/face-gear drive to a formative spur gear drive via assuming a spur gear instead of the face-gear to mesh with the spur pinion. Since no exact closed-form formula exists for the face-gear contact-ratio calculation, this approximation provides an efficient method to save the complex numerical computations based on the face-gear differential geometry.

Chapter 4

STRUCTURAL DYNAMICS MODEL AND MACROSCOPIC DESIGN FOR FACE-GEAR DRIVES WITH A SPUR PINION

4.1 Overview

This chapter is to establish a structural dynamics model for transverse vibration of face-gear drives with a spur pinion and provide information for the macroscopic design of face-gear body based on stability and stress constraints. The face-gear is modeled as a spinning annular disk according to Kirchhoff plate theory and the mesh load is modeled as a unit with the prescribed unique face-gear kinematics. This mesh load unit comprises pinion mass, pinion bearing stiffness, and pinion bearing damping. The out-of-plane movement of the unit couples with the transverse vibration of the spinning disk.

The research on the transverse vibration of spinning disk has a long history owing to its wide applications in circular saw, computer storage device, vehicle brake, and gear transmission. The natural frequencies of transverse vibration were investigated theoretically and experimentally for annular disks with pre-stresses resulting from rotational centrifugal force by [85-86]. Adams [87] and Renshaw [88] derived the critical speed for floppy disks on elastic foundation and in rigid enclosure respectively. Pei and Tan [89] conducted a parametric study on the lateral instability of a disk under

periodically varying angular speeds. The stability and dynamic response analyses of the spinning disk subjected to stationary transverse loads, stiffness, damping, mass, force or their combination, were intensively implemented in [90-93]. Moreover, Chen and Bogy [94-95] took into account pitching loads, making the disk vertical plane tilting, besides transverse loads. Extra single degree-of-freedom was added to the continuous disk system for coupling an oscillating load in [96-97]. The response of spinning disk to a moving force was calculated by [98-99] and the stability of spinning or stationary disk under rotating loads was analyzed by [100-103]. Pei et al. [104] developed a general model to study the parametric instability induced by magnetic head reciprocation in hard disk drive, which is similar to the face-gear model proposed in this dissertation. In addition, the spinning disk flutter was widely investigated by [105-109]. Manzione and Hayfeh [110-111] explored the instability mechanisms of rotating disk under stationary load based on nonlinear plate theory. Also, the control strategies were developed to suppress the vibration of spinning disk systems with various instabilities in [112-113].

4.2 Structural Dynamics Model of Face-gear Drives with a Spur Pinion

To simplify the face-gear model and the stability analysis, major assumptions made in this dissertation for the structural dynamics models of transverse vibration of face-gear drive are as follow:

- (1) The spur pinion is an identical copy of the shaper generating the face-gear. So, the pinion has a line contact mode with the face-gear and all parameters of the pinion are the same as those of the shaper.
- (2) The face-gear bearing, pinion body and gear teeth are rigid.

- (3) The face-gear body is an elastic annular disk. Its inner edge is clamped and outer edge is free.
- (4) The mesh load has prescribed in-plane movements and is made up of the pinion mass and the pinion bearing.
- (5) The whole system is well manufactured and installed without any misalignments.

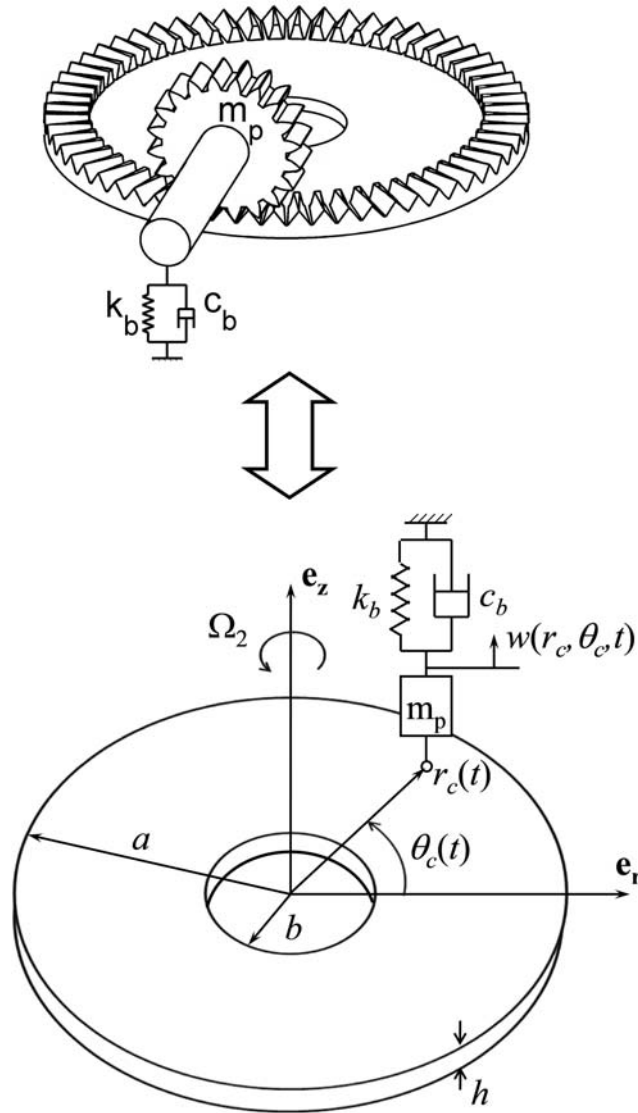


Fig. 4. 1: Face-gear structural dynamics model: spinning elastic disk and a concentrated meshing load unit that moves as the unique face-gear meshing kinematics.

The structural dynamics model of the face-gear drive with a spur pinion is illustrated in Figure 4.1. Based on assumptions above and the geometrical analyses in chapter 3, the face-gear is modeled as isotropic, homogeneous, elastic annular disk with uniform thickness based on Kirchhoff linear plate theory. Its inner boundary is rigidly clamped while the outer one is free. The mesh loads of all tooth pairs in mesh are accumulated and modeled as an equivalent concentrated load unit contacting the face-gear surface vertically at the contact centroid. The in-plane movements of the contact centroid result from the unique face-gear meshing kinematics described in section 3.6. This concentrated load unit comprises pinion mass, pinion bearing stiffness, and pinion bearing damping. Since the pinion is rigid, the out-of-plane displacement and velocity of the pinion bearing stiffness and damping are the same as those of the disk at the contact centroid.

The face-gear/pinion model is classified as a spinning disk / moving load system. Some similar systems, for example hard disk/magnetic head system [104], have been investigated. Following their work [85-104, 112-113], the governing equation-of-motion for the face-gear model is expressed in space fixed polar coordinates (r, θ) as:

$$\begin{aligned} \rho h \left(\frac{\partial^2 w}{\partial t^2} + 2\Omega_2 \frac{\partial^2 w}{\partial t \partial \theta} + \Omega_2^2 \frac{\partial^2 w}{\partial \theta^2} \right) + \xi D \nabla^4 \left(\frac{\partial w}{\partial t} + \Omega_2 \frac{\partial w}{\partial \theta} \right) + D \nabla^4 w \\ - h \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \sigma_r \frac{\partial w}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\sigma_\theta \frac{\partial w}{\partial \theta} \right) \right] + \frac{\delta(r - r_c(t)) \delta(\theta - \theta_c(t))}{r} (m_p \ddot{w} + c_b \dot{w} + k_b w) = 0 \end{aligned} \quad (4-1)$$

where $w(r, \theta, t)$ is the face-gear disk transverse deflection and $\nabla^4 = \nabla^2 \nabla^2$ is the biharmonic operator. Here ρ is material density of face-gear, h is face-gear disk thickness, ξ is the viscous loss factor, $D = Eh^3/12(1-\nu^2)$ is plate bending stiffness with Young's Modulus E and Poisson's ratio ν . Also, m_p is pinion mass, k_b is pinion bearing stiffness, and c_b is

pinion bearing damping. Finally, Dirac delta function, $\delta(\bullet)$, is used to indicate that the concentrated mesh load unit acts on the contact centroid, $(r_c(t), \theta_c(t))$, defined in Eq. (3-21).

The centrifugal membrane pre-stress due to rotation is a symmetric stress field, given by [113-114]

$$\begin{aligned}\sigma_r &= \rho a^2 \Omega_2^2 \left[b_0 \left(\frac{a}{r} \right)^2 + b_1 - \frac{3+\nu}{8} \left(\frac{r}{a} \right)^2 \right] \\ \sigma_\theta &= \rho a^2 \Omega_2^2 \left[-b_0 \left(\frac{a}{r} \right)^2 + b_1 - \frac{1+3\nu}{8} \left(\frac{r}{a} \right)^2 \right]\end{aligned}\quad (4-2)$$

with

$$\begin{aligned}b_0 &= \frac{(1-\nu)(b/a)^2[(3+\nu) - (1+\nu)(b/a)^2]}{8[(1+\nu) + (1-\nu)(b/a)^2]} \\ b_1 &= \frac{(1+\nu)[(3+\nu) + (1-\nu)(b/a)^4]}{8[(1+\nu) + (1-\nu)(b/a)^2]}\end{aligned}\quad (4-3)$$

here, a and b are the outer and the inner radii of the disk respectively. The clamped boundary condition at $r=b$ and the free boundary condition at $r=a$ are [86, 113]

$$\begin{aligned}w|_{r=b} &= 0 \\ \partial w / \partial t|_{r=b} &= 0 \\ \left[\frac{\partial^2 w}{\partial t^2} + \nu \left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \right]_{r=a} &= 0 \\ \left[\frac{\partial}{\partial r} (\nabla^2 w) + \frac{1-\nu}{r^2} \frac{\partial^2}{\partial \theta^2} \left(\frac{\partial w}{\partial r} - \frac{w}{r} \right) \right]_{r=a} &= 0\end{aligned}\quad (4-4)$$

4.3 System Discretization

The eigenfunctions of the simpler related problem can be used as admissible functions for the original problem [115]. The stationary disk problem is the associated

simpler problem for the face-gear model in Eq. (4-1) - Eq. (4-4). As the stationary disk eigenfunction, Bessel functions have been utilized as the radial mode function to solve various problems containing annular disk [89, 93, 97-98, 100-101, 104, 116-119]. Hence the solution of Eq. (4-1) is assumed to be in the form:

$$w(r, \theta, t) = \sum_{m=0}^M \sum_{n=-N}^N R_{mn}(r) e^{in\theta} \eta_{mn}(t) \quad (4-5)$$

with

$$R_{mn}(r) = A_{mn} J_n(\beta_{mn} r) + B_{mn} Y_n(\beta_{mn} r) + C_{mn} I_n(\beta_{mn} r) + D_{mn} K_n(\beta_{mn} r) \quad (4-6)$$

where m and n are respectively the number of nodal circle and nodal diameter of vibration mode (m, n) , $\eta_{mn}(t)$ is the corresponding generalized coordinate and $i^2 = -1$; J_n , Y_n are Bessel functions and I_n , K_n are modified Bessel functions of order n . Their constant coefficients, A_{mn} , B_{mn} , C_{mn} , D_{mn} and parameter, β_{mn} , are determined by boundary conditions, Eq. (4-4), and the orthonormality condition below

$$\int_b^a [R_{kn}(r) R_{jn}(r)] r dr = \delta_{k,j} = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases} \quad (4-7)$$

By applying Galerkin's method, the system equation-of-motion can be partially decoupled, via substituting the assumed solution Eq. (4-5) into Eq. (4-1), multiplying the resulting equation by $R_{mn}(r) e^{-in\theta}$, and then integrating over the domain $\{b \leq r \leq a, 0 \leq \theta \leq 2\pi\}$, as

$$\begin{aligned} & \ddot{\eta}_n + (\xi \mathbf{B}_n + i 2n \Omega_2 \mathbf{I}_\beta) \dot{\eta}_n + (\mathbf{B}_n + i \xi n \Omega_2 \mathbf{B}_n + \mathbf{L}_n - n^2 \Omega_2^2 \mathbf{I}_\beta) \eta_n \\ & + \sum_{q=-N}^N \frac{e^{i(-n+q)\theta_c(t)}}{2\pi \rho h} \mathbf{H}_{n,q}(r_c(t)) (m_p \ddot{\eta}_q + c_b \dot{\eta}_q + k_b \eta_q) = 0 \end{aligned} \quad (4-8)$$

with

$$\left\{ \begin{array}{l} \mathbf{\beta}_n = \text{diag}[\beta_{mn}^4 D / \rho h] \\ \mathbf{I}_\beta \text{ is an identity matrix with the same size of } \mathbf{\beta}_n \\ \mathbf{L}_n = \text{full}[k_{m1,m2}^n] \\ \text{with } k_{m1,m2}^n = \frac{1}{\rho} \int_b^a \left(\sigma_r R'_{m1,n}(r) R'_{m2,n}(r) + \sigma_\theta \frac{n^2}{r^2} R_{m1,n}(r) R_{m2,n}(r) \right) r dr \\ \mathbf{H}_{n,q}(r_c(t)) = \text{full}[h_{m1,m2}^{nq}(r_c(t))] \\ \text{with } h_{m1,m2}^{nq}(r_c(t)) = R_{m1,n}(r_c(t)) R_{m2,q}(r_c(t)) \end{array} \right. \quad (4-9)$$

where $\eta_n = [\cdots \eta_{mn}(t) \cdots]^T$ for $(m = 0, 1, \dots, M)$, “ $\cdot$ ” represents differentiation with respect to time t , “ $\text{diag}[\bullet]$ ” represents diagonal matrix or diagonal block matrix, and “ $\text{full}[\bullet]$ ” represents full matrix.

By collecting all general coordinates, $q(t) = [\cdots \eta_n^T \cdots]^T$ for $(n = -N, \dots, -1, 0, 1, \dots, N)$, the resulting equation-of-motion are obtained as:

$$[\mathbf{M}_0 + \mathbf{M}(t)]\ddot{q}(t) + [\mathbf{C}_0 + \mathbf{C}(t)]\dot{q}(t) + [\mathbf{K}_0 + \mathbf{K}(t)]q(t) = 0 \quad (4-10)$$

with

$$\left\{ \begin{array}{l} \mathbf{M}_0 = \text{diag}[\mathbf{I}_\beta] \\ \mathbf{C}_0 = \text{diag}[\xi \mathbf{\beta}_n + \mathbf{i} 2n \Omega_2 \mathbf{I}_\beta] \\ \mathbf{K}_0 = \text{diag}[\mathbf{\beta}_n + \mathbf{L}_n + \mathbf{i} \xi n \Omega_2 \mathbf{\beta}_n - n^2 \Omega_2^2 \mathbf{I}_\beta] \\ \mathbf{M}(t) = \text{full}[e^{\mathbf{i}(-n+q)\theta_c(t)} \mathbf{H}_{n,q}(r_c(t))] m_p / 2\pi \rho h \\ \mathbf{C}(t) = \text{full}[e^{\mathbf{i}(-n+q)\theta_c(t)} \mathbf{H}_{n,q}(r_c(t))] c_b / 2\pi \rho h \\ \mathbf{K}(t) = \text{full}[e^{\mathbf{i}(-n+q)\theta_c(t)} \mathbf{H}_{n,q}(r_c(t))] k_b / 2\pi \rho h \end{array} \right. \quad (4-11)$$

In order to use Floquet theory to determine the stability of the periodically time-varying system described by Eq. (4-10), it is first recast into the standard state-space form as:

$$\dot{\mathbf{X}} = \mathbf{A}(t)\mathbf{X} \quad \text{with} \quad \mathbf{X} = [q(t)^T \quad \dot{q}(t)^T]^T \quad (4-12)$$

and

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -[\mathbf{M}_0 + \mathbf{M}(t)]^{-1}[\mathbf{K}_0 + \mathbf{K}(t)] & -[\mathbf{M}_0 + \mathbf{M}(t)]^{-1}[\mathbf{C}_0 + \mathbf{C}(t)] \end{bmatrix} \quad (4-13)$$

The rest is to determine the stability by following the method given in section 2.2.1.

4.4 Pinion Parameters

Pinion parameters are rescaled to obtain a better understanding in the physical meaning of the effects of pinion parameters on the system stability.

Firstly, the stiffness of face-gear body is chosen as the reference quantity to scale the pinion bearing stiffness. In the face-gear model, this body stiffness value can be numerically evaluated via dividing a transverse force by the disk steady-state displacement produced by the force. This process is summarized below as:

$$\begin{aligned} 0 &= \mathbf{A}_d(\Omega_2)\mathbf{X} + \mathbf{B}_d F_d \\ y_d &= \mathbf{C}_d \mathbf{X} \end{aligned} \quad (4-14)$$

where F_d is the transverse force acting on the face-gear disk and

$$\begin{aligned} \mathbf{A}_d(\Omega_2) &= \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}_0^{-1}\mathbf{K}_0(\Omega_2) & -\mathbf{M}_0^{-1}\mathbf{C}_0(\Omega_2) \end{bmatrix} \\ \mathbf{B}_d &= \frac{1}{2\pi\rho h} [\dots 0 \dots, \dots R_{mn}(r_d)e^{-in\theta_d} \dots]^T \\ \mathbf{C}_d &= [\dots R_{mn}(r_d)e^{in\theta_d} \dots, \dots 0 \dots] \end{aligned} \quad (4-15)$$

By solving Eq. (4-14), the face-gear body stiffness, k_d , is obtained as

$$k_d = k_d(\Omega_2, r, h) = \frac{F_d}{y_d} = -\frac{1}{\mathbf{C}_d \mathbf{A}_d^{-1}(\Omega_2) \mathbf{B}_d} \quad (4-16)$$

The stiffness of face-gear body depends on rotation speed (due to the centrifugal stress field), effective position, and thickness. Here, the reference is the average value of

the body stiffness at the face-gear tooth center over the entire speed range. The nondimensional (N.D.) bearing stiffness, μ , is defined as the ratio of pinions bearing stiffness to the reference

$$\mu = \frac{k_b}{\text{avg}[k_d(\Omega_2, \frac{R_1 + R_2}{2}, h)]} \quad (4-17)$$

where “avg[•]” represents the average operator over the entire rotation speed range, and R_1 and R_2 are given respectively in Eq. (3-17) and Eq. (3-18). The proportional bearing damping, ζ , is the ratio of pinion bearing damping to the pinion bearing stiffness,

$$\zeta = \frac{c_b}{k_b} \quad (4-18)$$

The N.D. pinion mass, τ , is the ratio of real pinion mass to the nominal pinion mass which is calculated by treating pinion as a solid cylinder with the pinion's dimensions,

$$\tau = \frac{m_p}{\rho \pi r_p^2 (L_2 - L_1)} \quad (4-19)$$

Here, r_p is the pitch circle radius of the pinion, and L_1 and L_2 are given respectively in Eq. (3-13) and Eq. (3-14).

4.5 Bending Stress of Pinion Tooth

The maximum bending stress occurs at the root of spur pinion tooth. This stress is closely related to the pinion geometry, input power, and rotation speed. For simplification, the stress value is estimated by treating the tooth as a cantilever beam with a distributed load on the pitch circle [21, 120], illustrated in Figure 4.2.

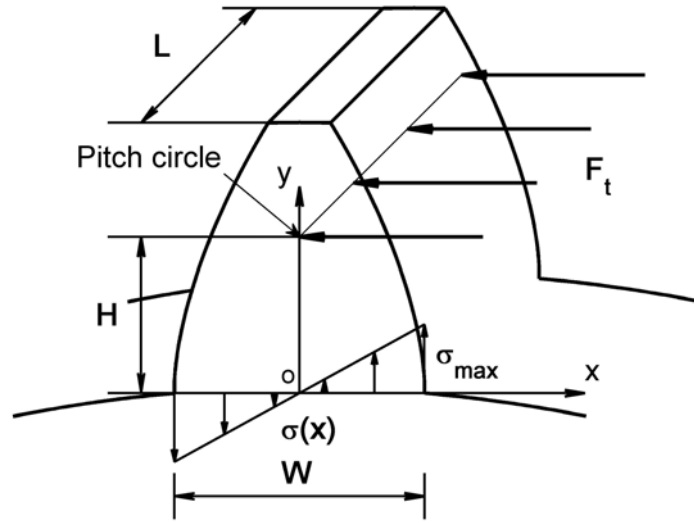


Fig. 4. 2: Pinion tooth geometry and root stress [21].

The maximum normal stress, σ_{max} , at the tooth root is calculated approximately by the following formula:

$$\sigma_{max} = \frac{6F_t H}{LW^2} \quad (4-20)$$

with

$$F_t = \frac{P}{\Omega_1 r_p} \quad \text{and} \quad L = L_2 - L_1 \quad (4-21)$$

where H is dedendum, L is tooth face width, W is tooth root thickness, F_t is tangential load on pitch circle, P is input power, and Ω_1 is the pinion rotation speed. The maximum bending stress can not be greater than the yield strength, σ_Y , with some safety factor, SF , namely

$$\sigma_{max} < \frac{\sigma_Y}{SF} \quad (4-22)$$

4.6 Stability Results

Two sample face-gear drives are employed to demonstrate the stability analysis method developed above. The parameters of the face-gear drives are given in Table 4.1; the corresponding parameters in the face-gear model are shown in Table 4.2; and the material characteristics are listed in Table 4.3.

Furthermore, the convergence tests are carried out by examining the largest real part of the system eigenvalues as the increases of modes (M and N). It reveals that $M=1$ and $N=3$ can yield good results for the models of the two sample face-gear drives. The parametric instability is numerically determined for various mesh load parameters and rotation speeds.

Table 4. 1: Face-gear drive parameters

Parameter	Value	
	<i>Case I</i>	<i>Case II</i>
Pinion number of teeth, N_1	26	31
Gear ratio, m_{12}	5	
Module, m_d	3.175 (mm) [$P_d = 8$ (1/in)]	
Shaft angle, γ	90°	
Pressure angle, α_c	25°	

Table 4. 2: Face-gear model parameters

Parameter	Value	
	<i>Case I</i>	<i>Case II</i>
Face-gear body thickness, h	$0.03a$	$0.05a$
Outer radius of disk, a	$R_2 \sin(\gamma)$	
Inner radius of disk, b	50 (mm)	
Input speed range, Ω_I	0 ~ 5000 (rpm)	
Mode number parameter, M	1	
Mode number parameter, N	3	

Table 4. 3: Material characteristics

Parameter	Value	
	<i>Case I</i>	<i>Case II</i>
Gear density, ρ	7840 (kg/m ³)	
Yield strength, σ_Y	439.9 (Mpa)	
Young's modulus, E	200 (Gpa)	
Poisson's ratio, ν	0.3	
Viscous loss factor, ζ	5e-006 (sec)	

First of all, the approximate contact-ratio of face-gear drives is calculated by Tregold's approximation described in section 3.8. The contact-ratio of face-gear drives with perpendicular intersecting shaft angle is shown in Figure 4.3.

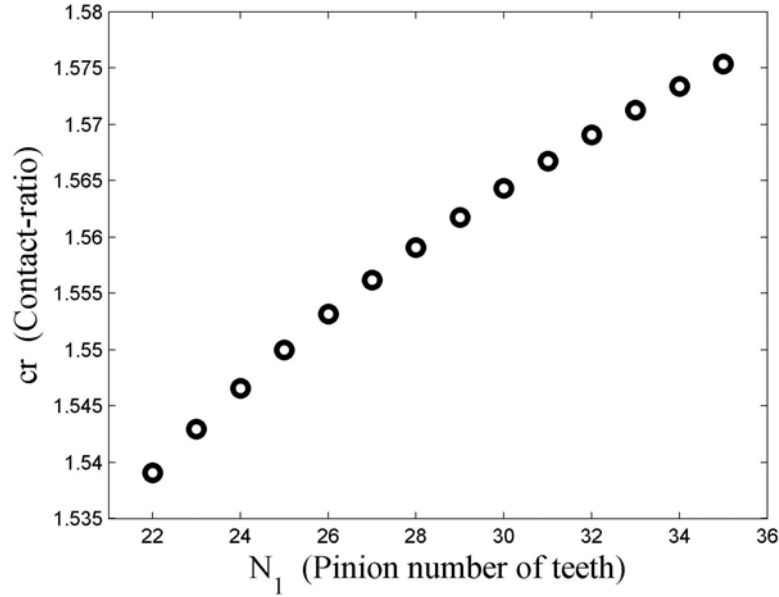


Fig. 4. 3: Approximate contact-ratio of face-gear drives, $m_{12} = 5$, $\gamma=90^\circ$ and $P_d = 8$ (1/in).

The system stability results for two different face-gear drives are displayed below. For *Case I*, the number of pinion teeth is 26 and thickness is 3 percent of face-gear tooth outer radius, shown in Figure 4.4; for *Case II*, the number of pinion teeth is 31 and thickness is 5 percent of face-gear tooth outer radius, shown in Figure 4.5. The black region indicates the parametric instability of transverse vibration of face-gear drive in this dissertation.

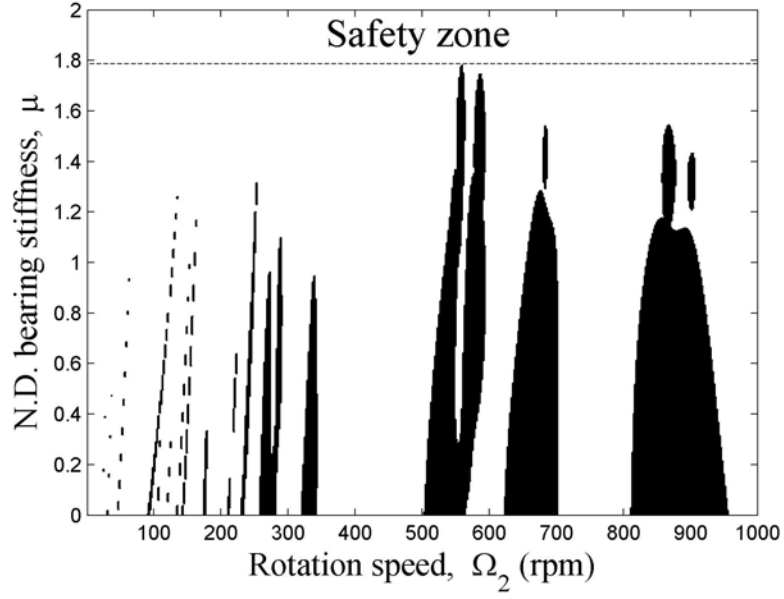
The effects of pinion bearing stiffness on system stabilities are explored in Figure 4.4(a) and Figure 4.5(a). The vertical axis is N.D. bearing stiffness and the horizontal

axis is face-gear rotation speed. It is indicated that the system is unstable for some lower bearing stiffness levels but stable for other levels. The safety zone of the system is also marked.

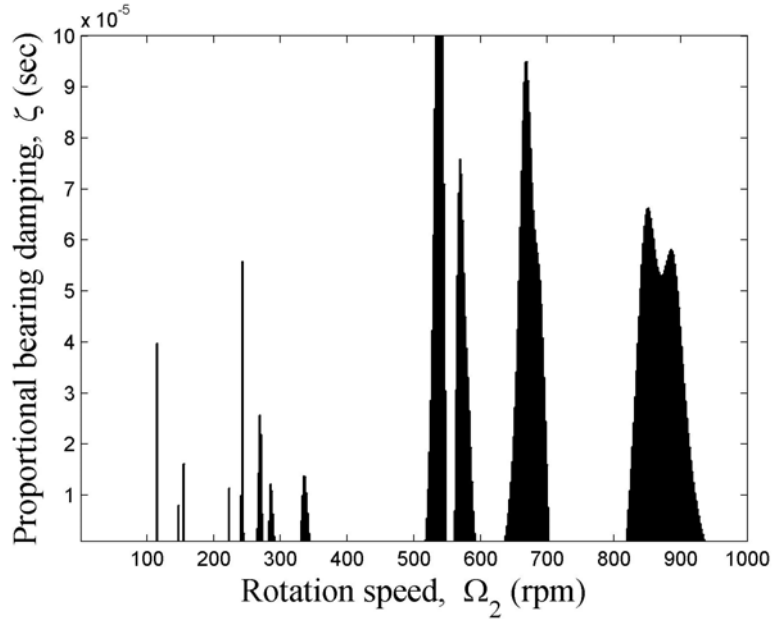
Figure 4.4(b) and Figure 4.5(b) shows the relationship between pinion bearing damping and system instabilities. The vertical axis is proportional pinion bearing damping and the horizontal axis is face-gear rotation speed.

The stability results based on pinion mass are drawn in Figure 4.4(c) and Figure 4.5(c). The vertical axis is N.D. pinion mass and the horizontal axis is face-gear rotation speed.

Parametric instabilities can be clearly observed from Figures 4.4-4.5 and depend on rotation speed, pinion mass, bearing damping, and bearing stiffness.

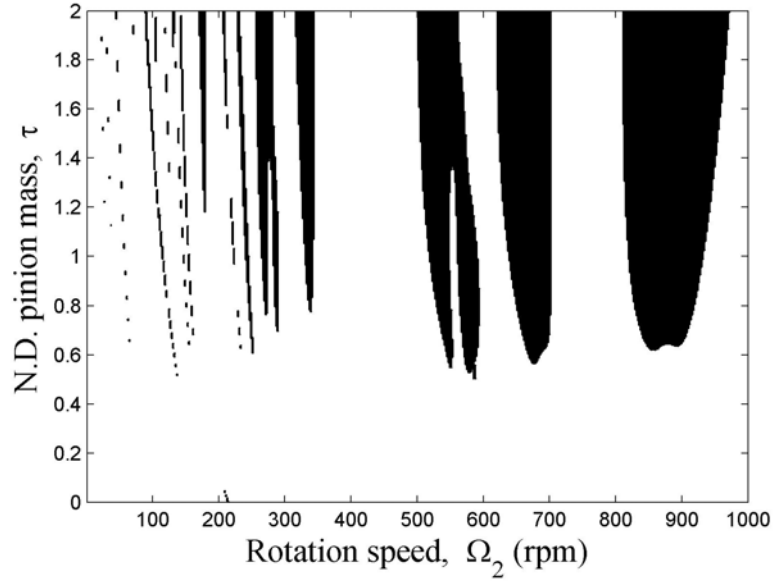


(a) Instabilities due to pinion bearing stiffness and rotation speed, $\tau = 1$, $\zeta = 3 \times 10^{-6}$ sec.



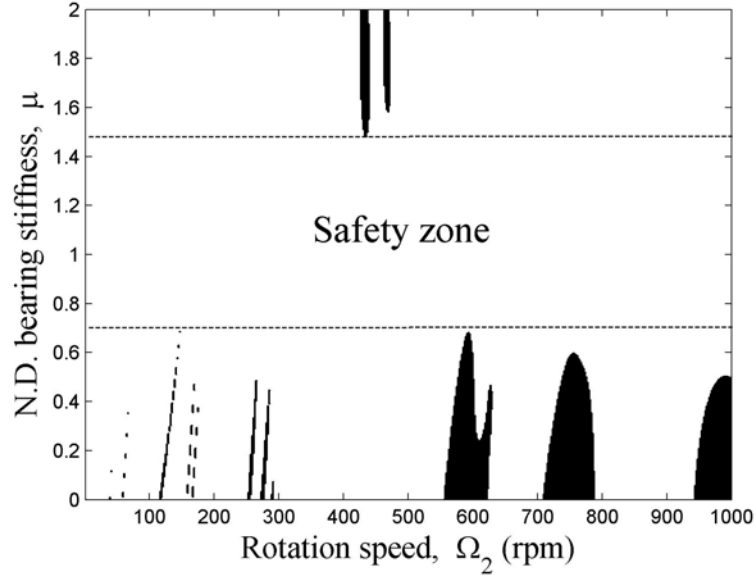
(b) Instabilities due to pinion bearing damping and rotation speed, $\mu = 0.6$, $\tau = 1$.

Fig. 4. 4: Face-gear/pinion system parametric instability, *Case I*: $N_1 = 26$, $N_2 = 130$, $a = 231.94$ mm, $h = 0.03a$, $cr = 1.5532$.

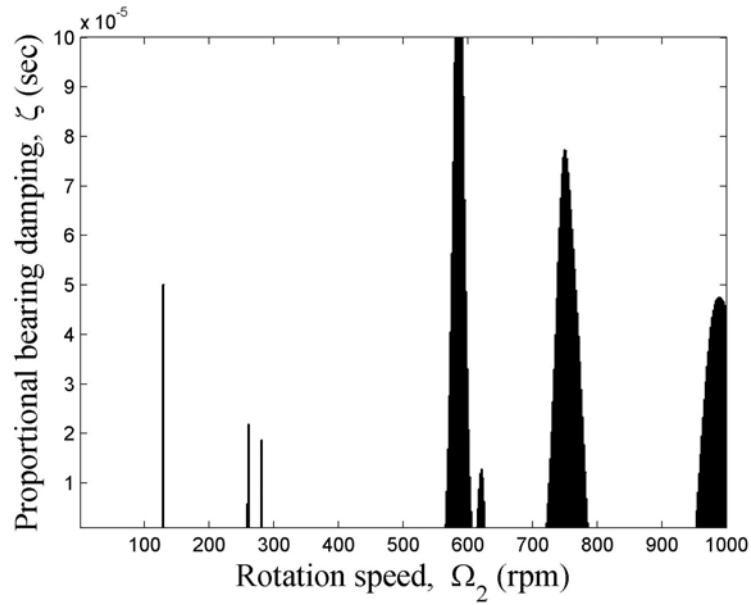


(c) Instabilities due to pinion mass and rotation speed, $\mu = 0.6$, $\zeta = 3 \times 10^{-6}$ sec.

Fig. 4. 4: Continued

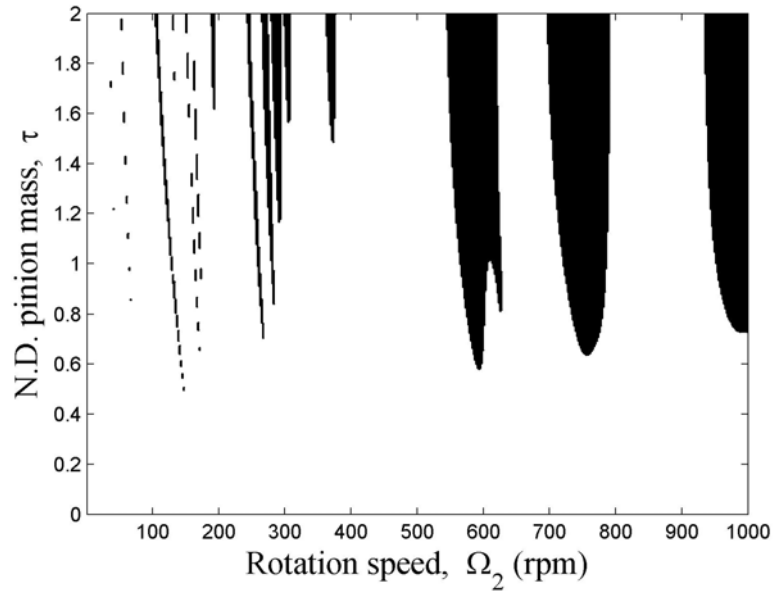


(a) Instabilities due to pinion bearing stiffness and rotation speed, $\tau = 1$, $\zeta = 1 \times 10^{-6}$ sec.



(b) Instabilities due to pinion bearing damping and rotation speed, $\mu = 0.25$, $\tau = 1$.

Fig. 4. 5: Face-gear/pinion system parametric instability, *Case II*: $N_1 = 31$, $N_2 = 155$, $a = 274.54$ mm, $h = 0.05a$, $cr = 1.5668$.



(c) Instabilities due to pinion mass and rotation speed, $\mu = 0.25$, $\zeta = 1 \times 10^{-6}$ sec.

Fig. 4. 5: Continued

4.7 Stability Based Face-Gear Macroscopic Design

In this chapter, the macroscopic design for face-gear body is discussed. The stability based design includes two constraints: maximum bending stress and system stability. The main processes are summarized in Figure 4.6.

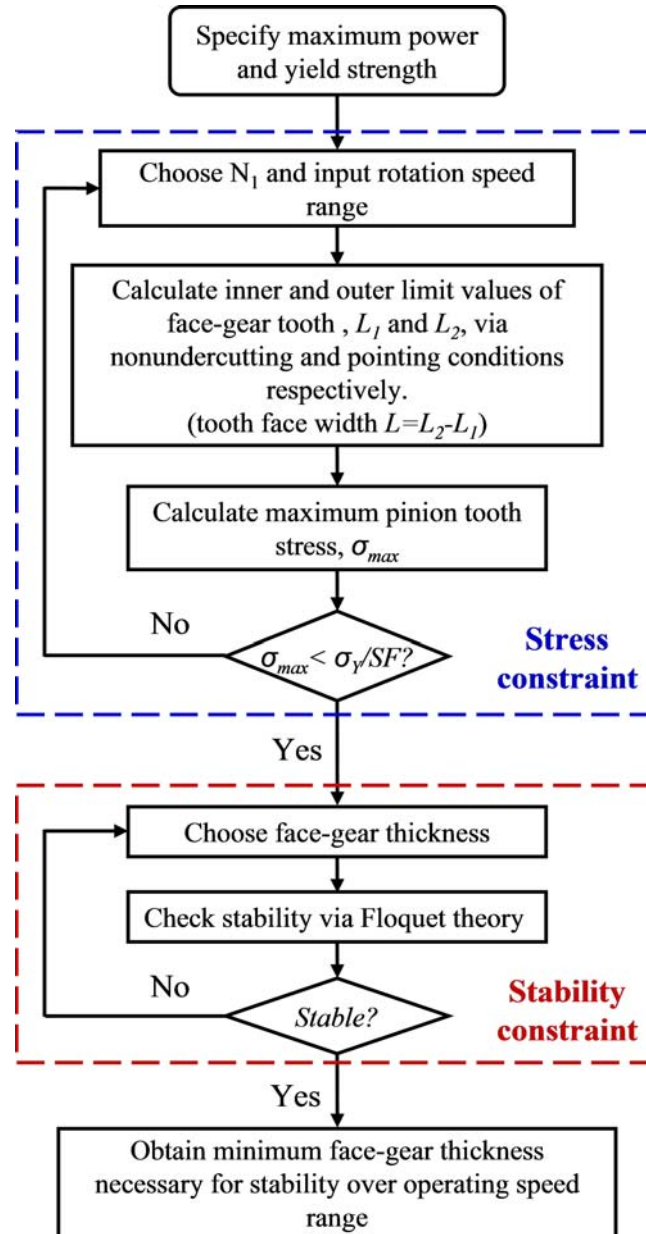


Fig. 4. 6: Flowchart of the stability based face-gear design [21].

The maximum bending stress at the root of pinion teeth must be less than yield strength with some safety factor. The stress constraint is able to affect face-gear size because the face-gear and its pinion have the same tooth face width. The system stability is closely related to face-gear body thickness. Based on the dynamics viewpoint, the minimum thickness necessary for sustaining stability over entire operating speed range is obtained. Both constraints can help decide face-gear dimensional limits in macroscopic designs, and also determine the face-gear weight. Here, a prototypical 600HP class face-gear drive is studied. Two designs are explored in this analysis: one is based on constant pinion diametral pitch; the other is based on constant pinion pitch radius. The design requirements are listed in Table 4.4, and other parameters are given in Table 4.1-4.3 if no other special assignments.

Table 4. 4: Design requirements

Parameter	Value
Maximum input power, P	600 (hp)@ $\Omega_1=5000$ (rpm)
Pinion number of teeth, N_I	22 ~ 35
Stress safety factor, SF	1.5

4.7.1 Design with constant diametral pitch

The face-gear tooth face width is determined by pointing and nonundercutting conditions and it is calculated by the second equation of Eq. (4-21). In this design, the pinion and the face-gear have the same tooth face width, and the bending stress must be lower than the yield strength with a safety factor of 1.5. The tooth face width and the

maximum stress at the root of the pinion tooth are respectively plotted in Figure 4.7 and Figure 4.8 for different numbers of pinion teeth under a constant diametral pitch.

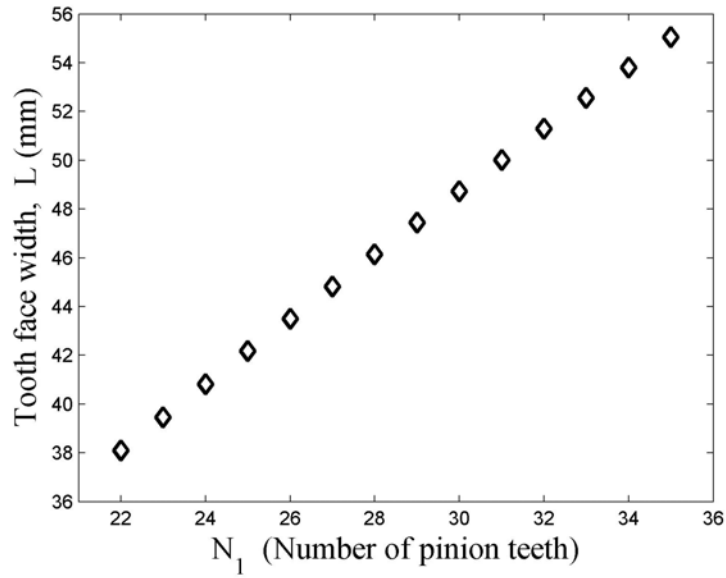


Fig. 4. 7: Tooth face width for different numbers of pinion teeth, $P_d = 8$ (1/in).

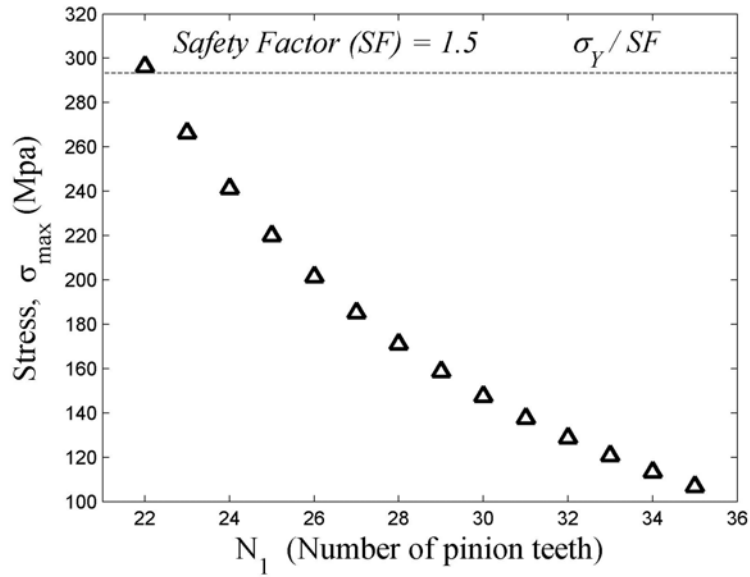
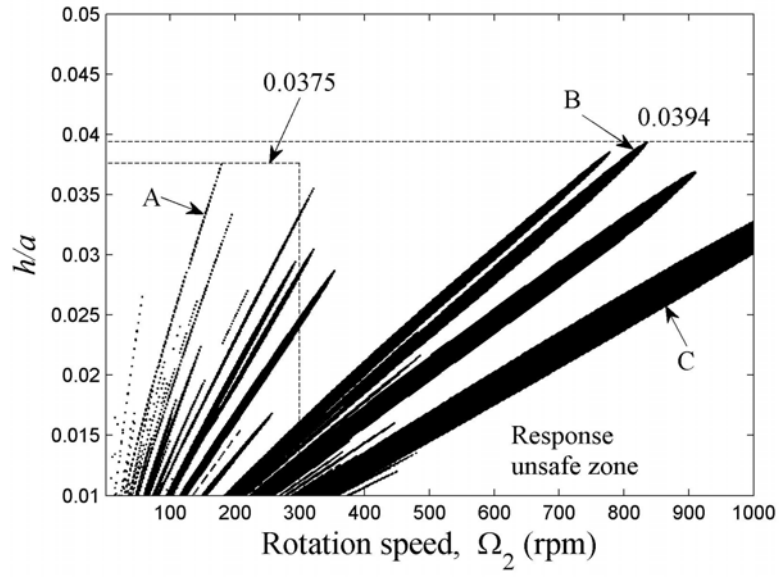


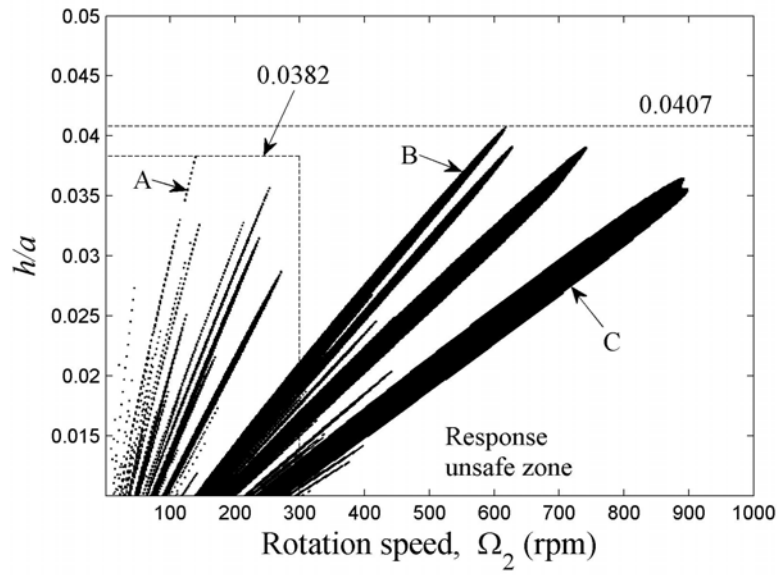
Fig. 4. 8: Maximum stresses at the root of pinion tooth for different numbers of pinion teeth, and 600 Hp @ 5000 RPM, $P_d = 8$ (1/in).

The gear dimensions here are based on the constant gear ratio and constant diametral pitch assumptions. Consequently, the gears with higher number of teeth bear lower tooth bending stresses but in return have bigger sizes [21]. The area below the dash line in Figure 4.8 is stress safe.

The object of this design is a 5:1 reduction ratio face-gear drive. Under the same diametral pitch, three cases for different choices of number of pinion teeth are analyzed. The face-gear outer radius increases with the number of teeth. The parametric instability zones, with respect to face-gear body thickness and rotation speed, for each of these cases are displayed in Figure. 4.9. The vertical axis is the ratio of face-gear body thickness to outer radius and the horizontal axis is face-gear rotation speed.

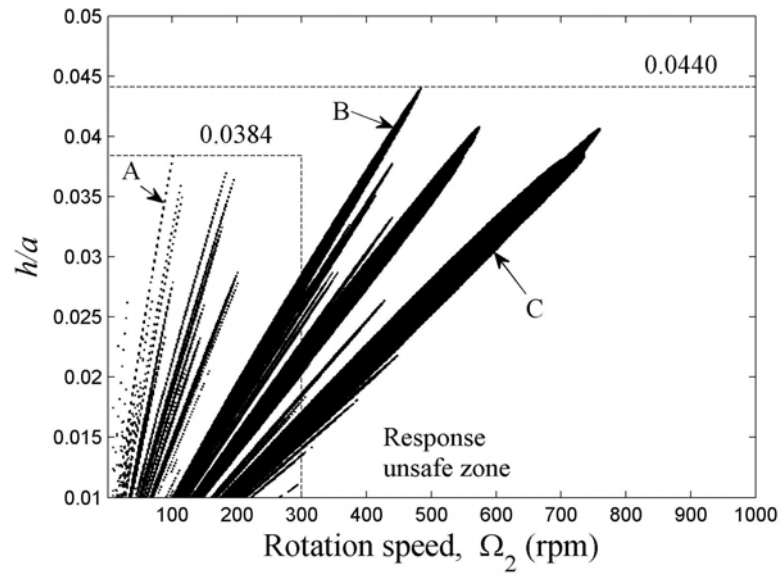


(a) $N_1 = 25, N_2 = 125, a = 223.4 \text{ mm}, r_p = 39.69 \text{ mm}, cr = 1.55$



(b) $N_1 = 28, N_2 = 140, a = 249 \text{ mm}, r_p = 44.45 \text{ mm}, cr = 1.5591$

Fig. 4. 9: Face-gear/pinion system parametric instability based on face-gear body thickness and rotation speed, $\tau = 1, \mu = 0.8, \zeta = 1 \times 10^{-5} \text{ sec}, P_d = 8 \text{ (1/in)}$.



(c) $N_1 = 32$, $N_2 = 160$, $a = 283$ mm, $r_p = 50.8$ mm, $cr = 1.5691$

Fig. 4. 9: Continued

The minimum face-gear body thickness ratio (h/a) necessary for sustaining the system stability over the entire operating speed region is determined from the stability plots, called “*critical thickness ratio*”. Their values for different operation speed regions can be read directly from Figure 4.9(a)-(c). For example, $h/a \approx 0.0375$ for the operating speed [0~300] rpm, which is decided by the unstable lobe A and $h/a \approx 0.0394$ for [0~1000] rpm, which is decided by the unstable lobe B, as shown in Figure 4.9(a).

It should be noted that the critical thickness ratio is chosen above unstable lobe C rather than below it because lower speed range is the primary operating zone in many applications; the other reason is the areas below the unstable lobe C have small thickness so that the displacement response of the face-gear may reach an unacceptable high level. Hence, the area at bottom is called “*response unsafe zone*” and also marked in Figure 4.9.

Once the critical thickness ratio is given, the minimum allowable thickness, h_{cr} , is obtained and called “*critical thickness*”. The annular cylinder volume formula is used to evaluate a nominal volume of the face-gear body, $V_f = \pi(a^2 - b^2)h_{cr}$. Consequently, the product of the nominal volume and the material density estimates the corresponding nominal weight.

The design based on the system stability and stress constraints are explored for different tooth counts, two operation speed regions, a 5:1 reduction ratio, and constant diametral pitch $P_d = 8$ (1/in). The results are shown in Figure 4.10. It reveals that the design requirements for weight and stress conflict with each other because the lower stress level designs tend to be less stable and therefore require larger values of face-gear body thickness to sustain stability [21]. Also, this trade off design depends on the desired operating speed range.

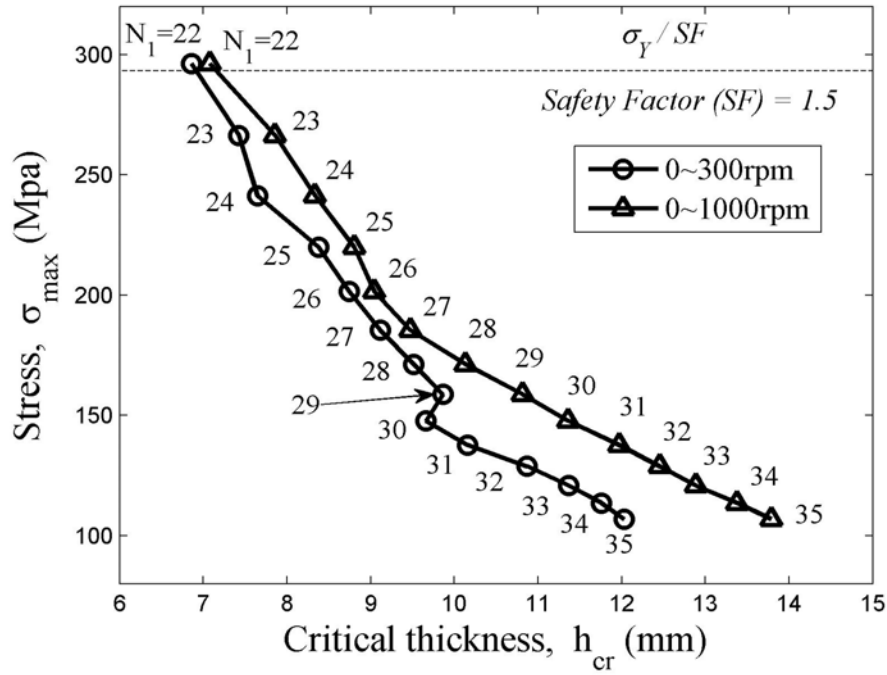


Fig. 4. 10: Maximum pinion tooth root stress for 5:1 reduction design VS face-gear critical thickness for various pinion tooth counts and operating speeds, $P_d = 8$ (1/in).

4.7.2 Design with constant pinion pitch radius

For the designs with constant pinion pitch radius, the tooth face width and the maximum stresses at the root of the pinion tooth are respectively plotted in Figure 4.11 and Figure 4.12 for different numbers of pinion teeth. The face width is inversely proportional to the number of pinion teeth. The stress increases as the pinion tooth counts growth, because larger tooth count leads to bigger diametral pitch under a constant pitch radius, which reduces the tooth root thickness and also enhances the stress level. However, the increase of the stress is small. Both relationships of face width / tooth count and stress / tooth count for constant pinion pitch radius are contrary to those in the design with constant diametral pitch (Figure 4.7 and Figure 4.8).

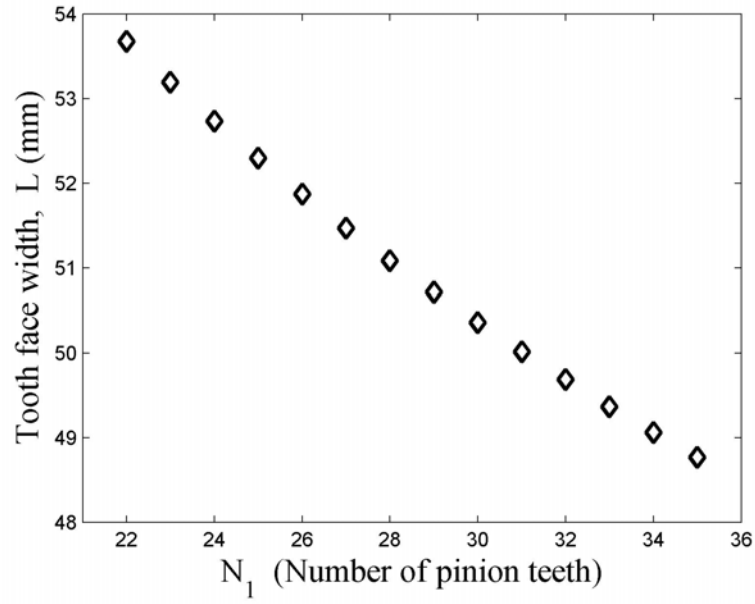


Fig. 4. 11: Pinion tooth face width for different numbers of pinion teeth, $r_p = 1.9375$ inch
or $r_p = 49.2$ mm.

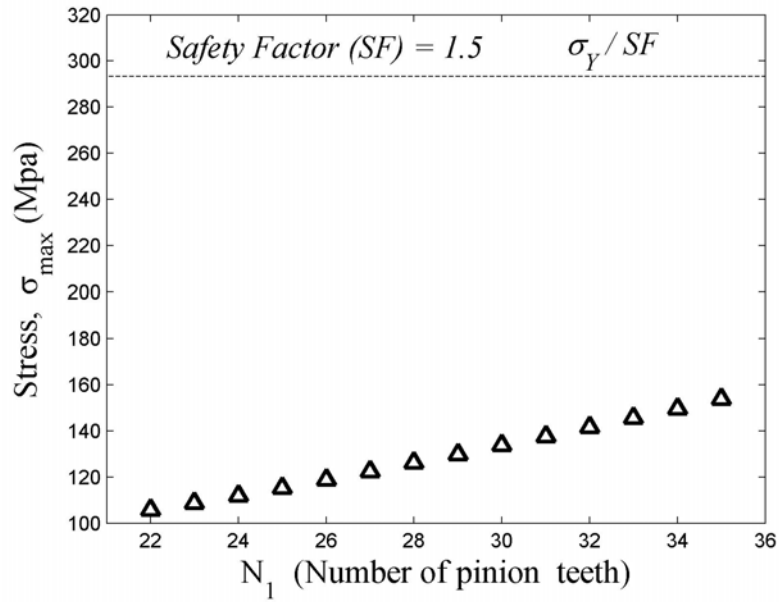
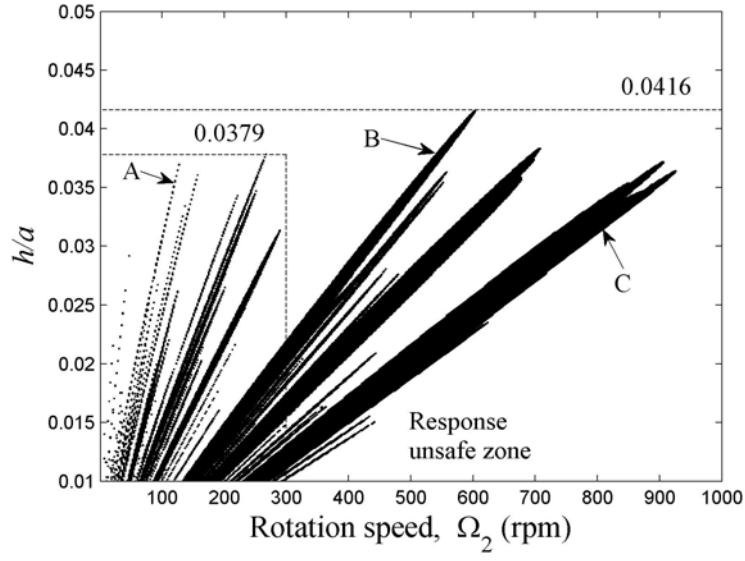
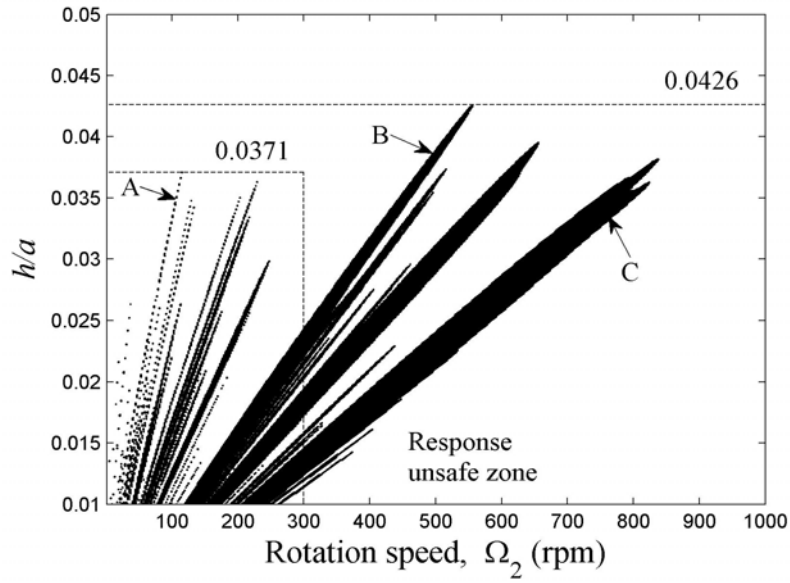


Fig. 4. 12: Maximum stresses at the root of the pinion tooth for different numbers of
pinion teeth, and 600 Hp @ 5000 RPM, $r_p = 1.9375$ inch or $r_p = 49.2$ mm.

Under the same design requirements, a 5:1 reduction ratio face-gear drives with constant pinion pitch radius are also analyzed. The three cases in the constant diametral pitch design above are re-evaluated for the constant pinion pitch radius design, shown in Figure 4.13(a)-(c).

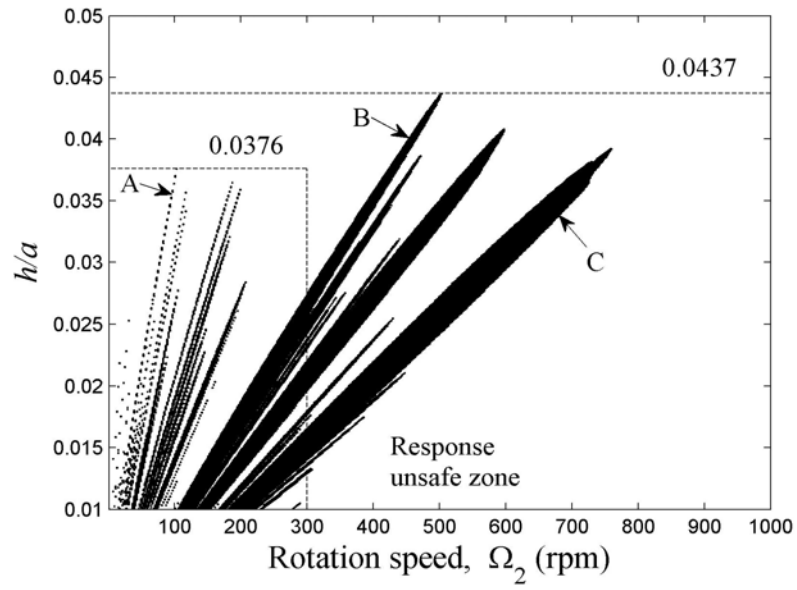


(a) $N_1 = 25, N_2 = 125, a = 277 \text{ mm}, P_d = 6.45 \text{ (1/in)}, cr = 1.55$



(b) $N_1 = 28, N_2 = 140, a = 275.7 \text{ mm}, P_d = 7.23 \text{ (1/in)}, cr = 1.5591$

Fig. 4. 13: Face-gear/pinion system parametric instability based on face-gear body thickness and rotation speed, $\tau = 1, \mu = 0.8, \zeta = 1 \times 10^{-5} \text{ sec}, \rho_p = 7840 \text{ kg/m}^3, r_p = 1.9375$ inch or $r_p = 49.2 \text{ mm}$.



(c) $N_1 = 32$, $N_2 = 160$, $a = 274.2$ mm, $P_d = 8.26$ (1/in), $cr = 1.5691$

Fig. 4. 13: Continued

The results indicate that the distribution of the parametric instability zones is similar among these cases with different tooth counts. The critical thickness ratio values for the operating speed ranges [0~300] rpm and [0~1000] rpm are also marked in Figure 4.13(a)-(c).

Figure 4.14 summarizes the results for many different tooth counts for a 5:1 reduction ratio design with a constant pinion pitch radius $r_p = 1.9375$ inch or $r_p = 49.2$ mm. The maximum bending stress at the pinion tooth root and the critical thickness are displayed for different numbers of pinion teeth. This figure shows that the critical thickness value varies a little for different pinion tooth counts, so the stress is the main constraint to be considered. In this case study, the constant pinion pitch radius has a good value to keep all designs away from the yield strength. Hence, the pinion pitch radius is the key parameter in this design.

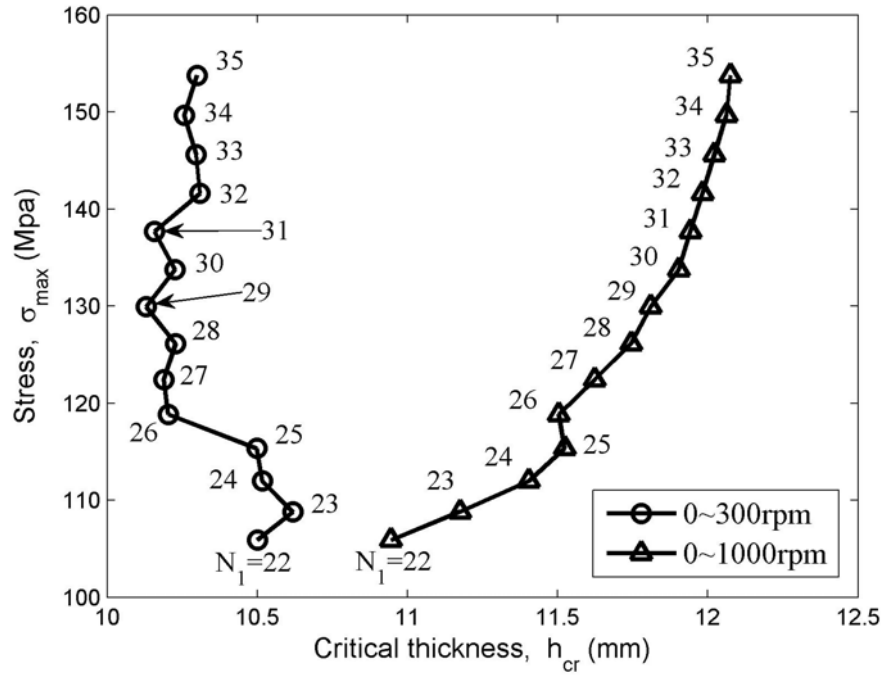


Fig. 4. 14: Maximum pinion tooth root stress for 5:1 reduction design VS face-gear critical thickness for various pinion tooth counts and operating speeds, with constant pinion pitch radius $r_p = 1.9375$ inch or $r_p = 49.2$ mm.

4.8 Summary

This chapter establishes a structural dynamics model for the face-gear drive with one spur pinion. The face-gear transverse vibrations and the unique meshing kinematics are considered in this model. A concept of contact centroid is proposed to taking into account the effects of both meshing kinematics and non-unity contact-ratio kinematics, and the contact line length serves as the weight function that scales the contributions resulting from different meshing tooth pairs at each moment of rotation angle. The face-

gear contact-ratio calculated by Tregold's approximation is plotted for various numbers of pinion teeth. Finally, the parametric studies are conducted on pinion parameters. The results clearly show that the system instability depends on pinion mass, pinion bearing stiffness, pinion bearing damping, and rotation speed.

The stability based design for face-gear macroscopic dimensions is explored based on tooth stress and system stability constraints. The face-gear and pinion have the same tooth face width in designs. The stress level at the root of pinion tooth is inversely proportional to the face width which is determined by the inner and outer limits of the face-gear tooth. That is to say nearly all face-gear drive parameters can affect the face width. In the meantime, the macroscopic dimensions and tooth limiting radii of the face-gear are closely related to the system stability. Two designs are explored respectively based on constant diametral pitch and constant pinion pitch radius. Under the constant diametral pitch, the design requirements for weight and stress conflict with each other because the lower stress level designs tend to be less stable and therefore require larger values of face-gear body thickness to sustain stability. Also, this trade off design depends on the desired operating speed range. For a constant pinion pitch radius, the critical thicknesses for the system stability are almost the same for various tooth counts. So, the key is to design the pinion pitch radius to satisfy with the stress requirement.

Chapter 5

PARAMETRIC STABILITY ANALYSIS AND SPATIAL CONFIGURATION DESIGN FOR FACE-GEAR DRIVES WITH MULTIPLE SPUR PINIONS

5.1 Overview

A lot of helicopters equip with two or more engines (see section 1.2), so more than one input pinion is required to work with multiple power input paths in the helicopter transmission systems. As a key design merit to reduce overall transmission weight, the torque splitting technique is also implemented through the face-gear drive with multiple pinions. Figure 1.10 and Figure 1.11 display two examples of face-gear/multi-pinions drive and the sketch of one face-gear meshing with two pinions is shown in Figure 5.1. The stability study on this type of drives is important in order to lower noise, improve operation reliability and extend service life. The objective of this chapter is to develop a structural dynamics model for one face-gear meshing with multiple spur pinions by following the previous work in chapter 4. This model considers periodically time-varying mesh load, face-gear body flexibility and pinion parameters (pinion mass, pinion bearing stiffness and damping). The focus is to explore the relationship between the spatial configuration of pinions and the system dynamic instability under various pinion parameter combinations and different tooth counts.

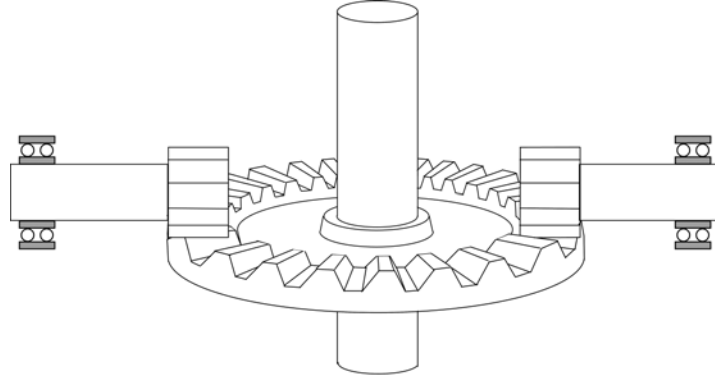
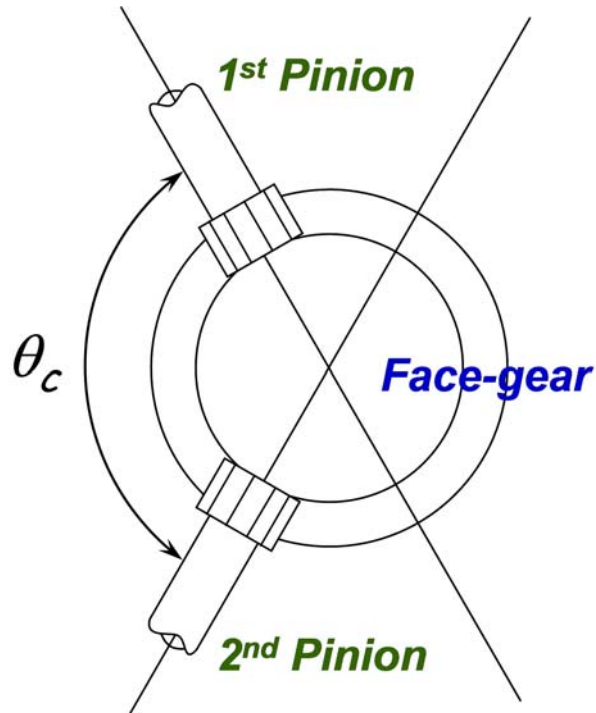


Fig. 5. 1: Sketch of one face-gear meshing with two spur pinions.

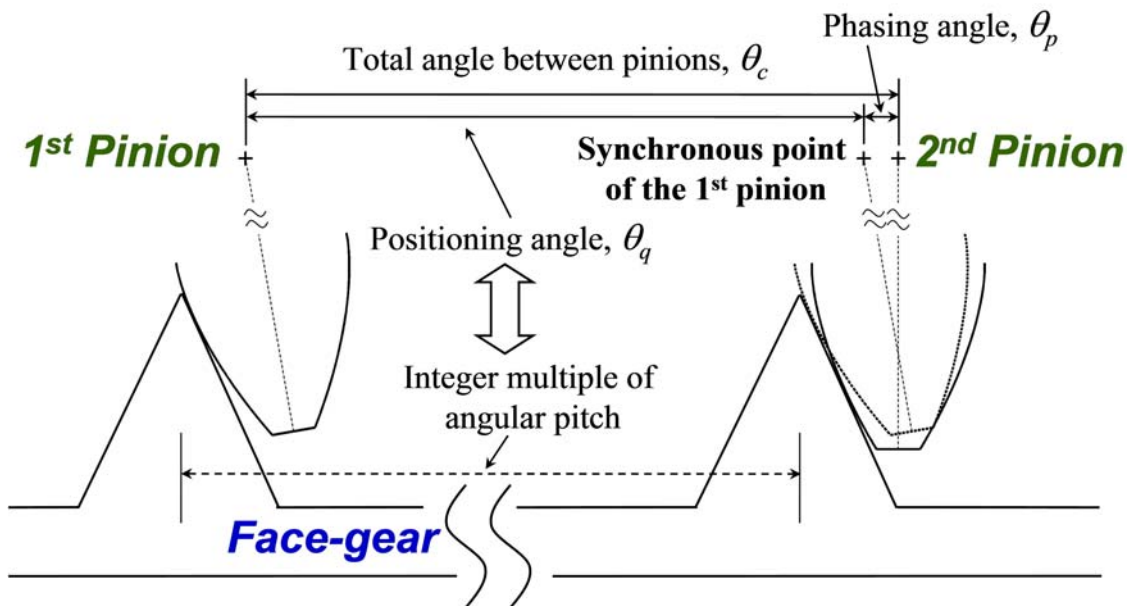
All assumptions for the multiple pinions case are the same as those for the one-pinion case in section 4.2. Furthermore, all pinions meshing with face-gear are identical to the shaper that generates the face-gear in this chapter. Hence the meshing kinematics analysis and the governing equation-of-motion in previous chapters are still applicable for this multiple pinions case.

5.2 Phase Differences between Pinions

Total angle between pinions, θ_c , comprises two parts: positioning angle, θ_q , and phasing angle, θ_p , illustrated in Figure 5.2. The positioning angle is defined as the integral multiple of face-gear angular pitch between pinions, which is equivalent to the amount of face-gear teeth between pinions (from the 1st pinion to the synchronous point of the 1st pinion in Figure 5.2(b)). It reflects the approximate relative circumferential position of pinions. The phasing angle is the difference of total angle subtracted by positioning angle, which corresponds to the micro tuning within one angular pitch (from the synchronous point of the 1st pinion to the 2nd pinion in Figure 5.2(b)).



(a) Top view of spatial configuration of two pinions



(b) Circumferential developing drawing of a face-gear drive with two pinions

Fig. 5. 2: Illustration for total angle, positioning angle and phasing angle.

Positioning angle changes discretely with a grade difference of one angular pitch and its variation contributes nothing to phase but may cause instability; phasing angle varies continuously within one angular pitch, which affects both phase and stability. The ratio of phasing angle to angular pitch defines a dimensionless phase difference between pinions. Consequently, the phase difference between the corresponding meshing tooth pairs at two different pinions is obtained by

$$\begin{cases} r_{2j}(t) = r_{1j}(t + \psi T_0) \\ \theta_{2j}(t) = \theta_{1j}(t + \psi T_0) \end{cases} \quad \text{with} \quad \psi = \frac{\theta_p}{\theta_N} \quad \text{and} \quad \theta_N = \frac{2\pi}{N_2} \quad (5-1)$$

and the positioning angle is given by face-gear angular pitch:

$$\theta_q = \Delta N_2 \theta_N \quad (5-2)$$

where (r_{1j}, θ_{1j}) and (r_{2j}, θ_{2j}) represent the contact point positions of the j^{th} meshing pairs at two different pinions respectively, T_0 is mesh period, ψ is dimensionless phase difference between pinions, θ_N is face-gear angular pitch, N_2 is number of face-gear teeth, and ΔN_2 is the amount of face-gear teeth between pinions. In addition, $\theta_c = \theta_q + \theta_p$, that is to say total angle equals to positioning angle when phasing angle is zero.

The dimensionless phase difference rises from zero to one when the increment of total angle is less than one angular pitch, namely only phasing angle changes; this phase variation occurs over again at each time when this increment exceeds one angular pitch, that is to say positioning angle changes. The relationship of total angle to dimensionless phase difference is displayed in Figure 5.3.

The position of contact centroid for the i^{th} pinion, (r_{ic}, θ_{ic}) , is obtained by adding a subscript “ i ” to Eq. (3-21) as

$$r_{ic}(t) = \frac{\sum_{j=1}^{\text{ceil}(cr)} [r_{ij}(t)l_{ij}(t)]}{\sum_{j=1}^{\text{ceil}(cr)} [l_{ij}(t)]} \quad \theta_{ic}(t) = \frac{\sum_{j=1}^{\text{ceil}(cr)} [\theta_{ij}(t)l_{ij}(t)]}{\sum_{j=1}^{\text{ceil}(cr)} [l_{ij}(t)]} \quad (5-3)$$

where (r_{ij}, θ_{ij}) represents the position of contact point for the j^{th} meshing pair at the i^{th} pinion, and $l_{ij}(t)$ is the corresponding contact line length.

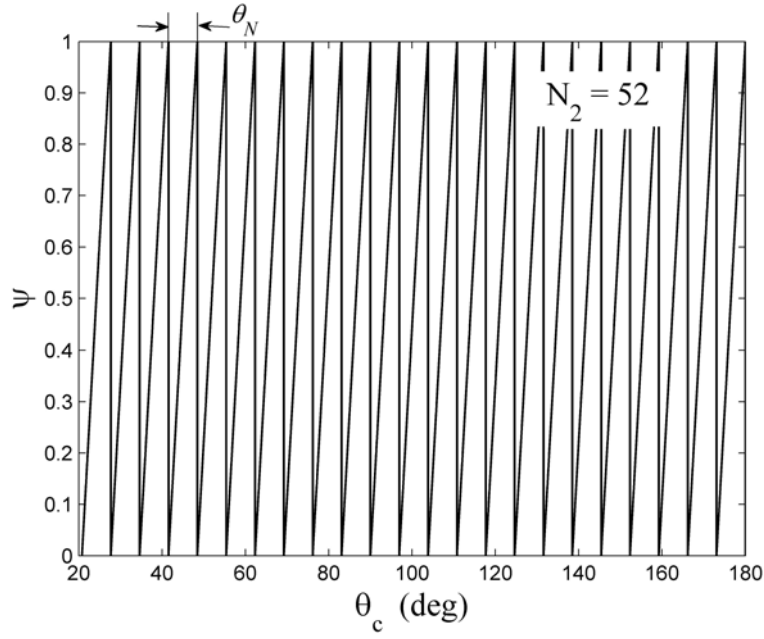


Fig. 5. 3: Relationship of total angle to dimensionless phase difference for $N_2=52$.

Both contact point position and contact line length depend on gear rotation angle, also on time. Their variations have the same period and are in phase. By substituting Eq. (5-1) into Eq. (5-3), the phase difference between contact centroids at two different pinions is given as

$$r_{2c}(t) = \frac{\sum_{j=1}^{\text{ceil}(cr)} [r_{2j}(t)l_{2j}(t)]}{\sum_{j=1}^{\text{ceil}(cr)} [l_{2j}(t)]} = \frac{\sum_{j=1}^{\text{ceil}(cr)} [r_{1j}(t + \psi T_0)l_{1j}(t + \psi T_0)]}{\sum_{j=1}^{\text{ceil}(cr)} [l_{1j}(t + \psi T_0)]} = r_{1c}(t + \psi T_0) \quad (5-4)$$

$$\theta_{2c}(t) = \frac{\sum_{j=1}^{\text{ceil}(cr)} [\theta_{2j}(t)l_{2j}(t)]}{\sum_{j=1}^{\text{ceil}(cr)} [l_{2j}(t)]} = \frac{\sum_{j=1}^{\text{ceil}(cr)} [\theta_{1j}(t + \psi T_0)l_{1j}(t + \psi T_0)]}{\sum_{j=1}^{\text{ceil}(cr)} [l_{1j}(t + \psi T_0)]} = \theta_{1c}(t + \psi T_0)$$

5.3 Structural Dynamics Model of Face-gear Drives with Multiple Spur Pinions

Following the work in chapter 4, the face-gear drive with multiple spur pinions is modeled as a spinning disk with multiple meshing loads. Hereinto, the effect of each pinion is considered an independent concentrated meshing load unit with a relative phase difference. The structural dynamics model is illustrated in Figure 5.4.

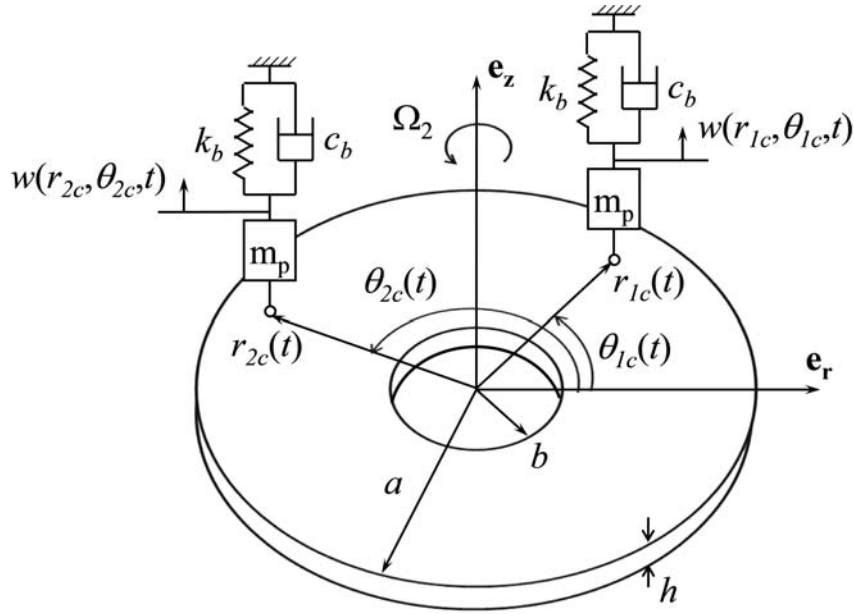


Fig. 5. 4: Face-gear structural dynamics model: spinning elastic disk and multiple meshing load units that move as the unique face-gear meshing kinematics.

The equation-of-motion for the model of face-gear meshing with multiple pinions is obtained by adding multiple meshing load units to Eq. (4-1) as

$$\begin{aligned}
 & \rho h \left(\frac{\partial^2 w}{\partial t^2} + 2\Omega_2 \frac{\partial^2 w}{\partial t \partial \theta} + \Omega_2^2 \frac{\partial^2 w}{\partial \theta^2} \right) + \xi D \nabla^4 \left(\frac{\partial w}{\partial t} + \Omega_2 \frac{\partial w}{\partial \theta} \right) \\
 & + D \nabla^4 w - h \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \sigma_r \frac{\partial w}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\sigma_\theta \frac{\partial w}{\partial \theta} \right) \right] \\
 & + \sum_i \frac{\delta(r - r_{ic}(t)) \delta(\theta - \theta_{ic}(t))}{r} (m_p \ddot{w} + c_b \dot{w} + k_b w) = 0
 \end{aligned} \tag{5-5}$$

where the centrifugal membrane pre-stresses, σ_r and σ_θ , are given in Eq. (4-2) and Eq. (4-3), and the boundary conditions are given in Eq. (4-4).

5.4 System Discretization

By using the same approach in section 4.3, the governing equation can be discretized via substituting the solution form Eq. (4-5) into Eq. (5-5) and applying Galerkin's method. The resulting equation is

$$[\mathbf{M}_0 + \mathbf{M}(t)]\ddot{q}(t) + [\mathbf{C}_0 + \mathbf{C}(t)]\dot{q}(t) + [\mathbf{K}_0 + \mathbf{K}(t)]q(t) = 0 \quad (5-6)$$

where

$$\begin{cases} \mathbf{M}_0 = \text{diag}[\mathbf{I}_\beta] \\ \mathbf{C}_0 = \text{diag}[\xi \mathbf{\beta}_n + \mathbf{i}2n\Omega_2 \mathbf{I}_\beta] \\ \mathbf{K}_0 = \text{diag}[\mathbf{\beta}_n + \mathbf{L}_n + \mathbf{i}\xi n\Omega_2 \mathbf{\beta}_n - n^2 \Omega_2^2 \mathbf{I}_\beta] \\ \mathbf{M}(t) = \sum_i \text{full}[e^{\mathbf{i}(-n+q)\theta_{ic}(t)} \mathbf{H}_{n,q}(r_{ic}(t))] m_p / 2\pi\rho h \\ \mathbf{C}(t) = \sum_i \text{full}[e^{\mathbf{i}(-n+q)\theta_{ic}(t)} \mathbf{H}_{n,q}(r_{ic}(t))] c_b / 2\pi\rho h \\ \mathbf{K}(t) = \sum_i \text{full}[e^{\mathbf{i}(-n+q)\theta_{ic}(t)} \mathbf{H}_{n,q}(r_{ic}(t))] k_b / 2\pi\rho h \end{cases} \quad (5-7)$$

Next, Eq. (5-6) is recast into the standard state-space form and Floquet theory is utilized to determine the system stability by the procedure introduced in section 2.2.1.

5.5 Stability Analysis

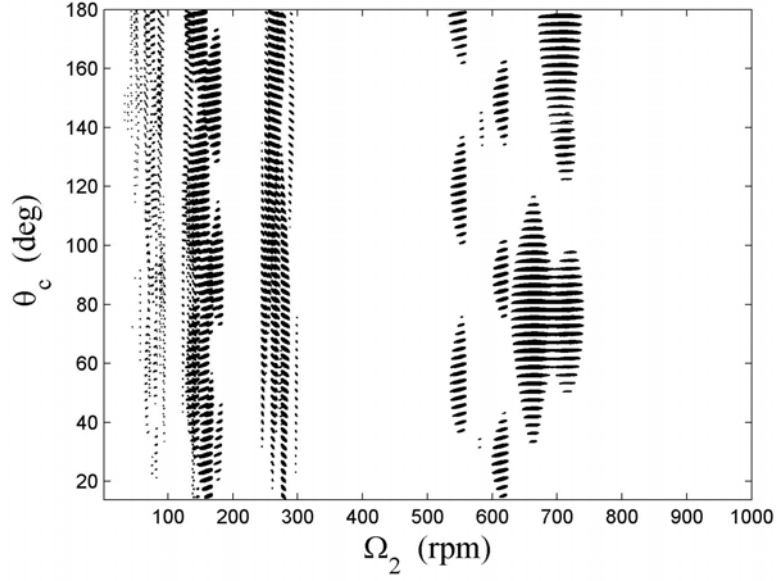
This chapter takes a face-gear meshing with two pinions for example to explore the relationship between system stability and various parameters of face-gear drive, including:

- 1) Total angle between pinions, θ_c ;
- 2) Positioning angle, θ_q ;
- 3) Phasing angle, θ_p ;
- 4) N.D. pinion mass, τ , N.D. pinion bearing stiffness, μ , and proportional pinion bearing damping, ζ .

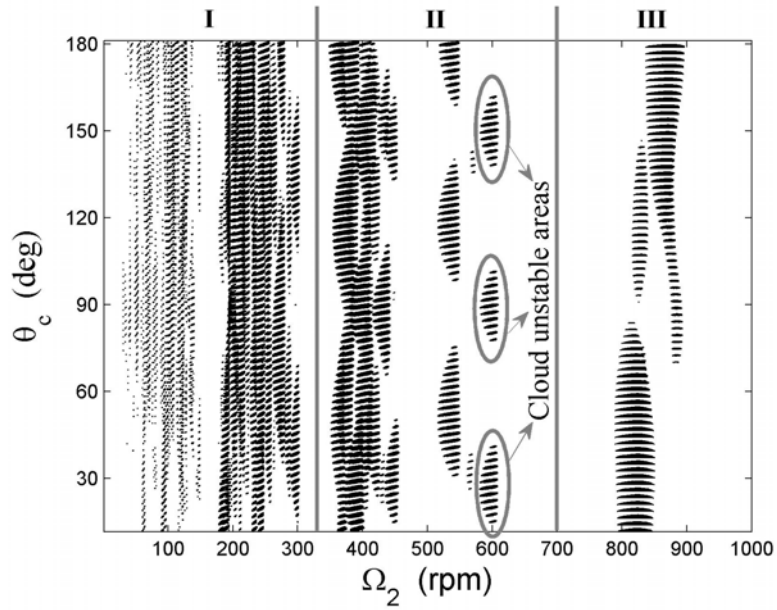
These parameters above are required to be designed for the system safety according to the stability results. At last, the stability analysis is performed for a design based on two space-fixed pinions but with variable number of teeth, constant gear ratio and constant module. All design parameters are listed in Table 4.1-4.3 if no other special assignments.

5.5.1 Stability design based on total angle and rotation speed

The value of total angle between two pinions varies continuously, which affects both positioning and phasing angles. The total angle is chosen as a design variable for a given operating speed range.



(a) *Example A*: $N_1 = 26$, $N_2 = 130$, $h = 0.05a$, $\tau = 0.8$, $\mu = 1.5$, and $\zeta = 1 \times 10^{-6}$ sec



(b) *Example B*: $N_1 = 31$, $N_2 = 155$, $h = 0.07a$, $\tau = 1$, $\mu = 0.25$, and $\zeta = 1 \times 10^{-5}$ sec

Fig. 5. 5: Face-gear/two pinions system parametric instability with respect to total angle and rotation speed.

The system stability of two example face-gear/pinions drives (*Example A* and *Example B*) with respect to total angle variation is plotted first to capture the overall situation, shown in Figure 5.5(a) and (b) respectively. The horizontal axis is face-gear rotation speed; the vertical axis is total angle between pinions. The instabilities can be clearly observed (black areas) and they are cut by stable areas into sawtooth shape from inside. The unstable and stable areas appear alternately as the total angle variations. Since total angle is the summation of positioning angle and phasing angle, their combined actions cause the system instability. The variations of positioning angle and phasing angle represent the coarse tuning and the fine tuning of spatial configuration of pinions respectively.

The *Example B* is chosen for further analysis. For convenience, Figure 5.5(b) is divided into three parts: the part **I** is low speed zone; **II** is medium speed zone; **III** is high speed zone. The instability of *Example B* at low speed zone is obscure and occurs almost through the entire total angle range. This may result from the resonances of single low frequency mode, combinations of multiple low frequency modes, or fractions of high frequency mode. At medium speed zone, the instability situation becomes clear and some cloud form unstable areas at same speed appear intermittently as the increase of total angle, e.g. the areas circled by grey ellipses in Figure 5.5(b). For the high speed zone, the system instability is very clear. The detailed relationships of positioning angle and phasing angle to the system instability are analyzed, based on *Example B*, in the following sections.

5.5.2 Stability design based on positioning angle and rotation speed

As a part of the variation of total angle, the positioning angle variation is possible to excite dynamic instabilities according to the results in Figure 5.5, but it is not clear which unstable area result from the positioning angle variation. Therefore, the effect of the positioning angle on the system stability is explored here for design purpose. Positioning angle is the integer multiple of face-gear angular pitch between pinions, which is equivalent to the space where the face-gear teeth between pinions takes. Its value varies discretely with a grade difference of one angular pitch. The variation of positioning angle reflects a big adjustment (coarse tuning) for the spatial configuration of pinions, but it contributes nothing to the phase difference between pinions.

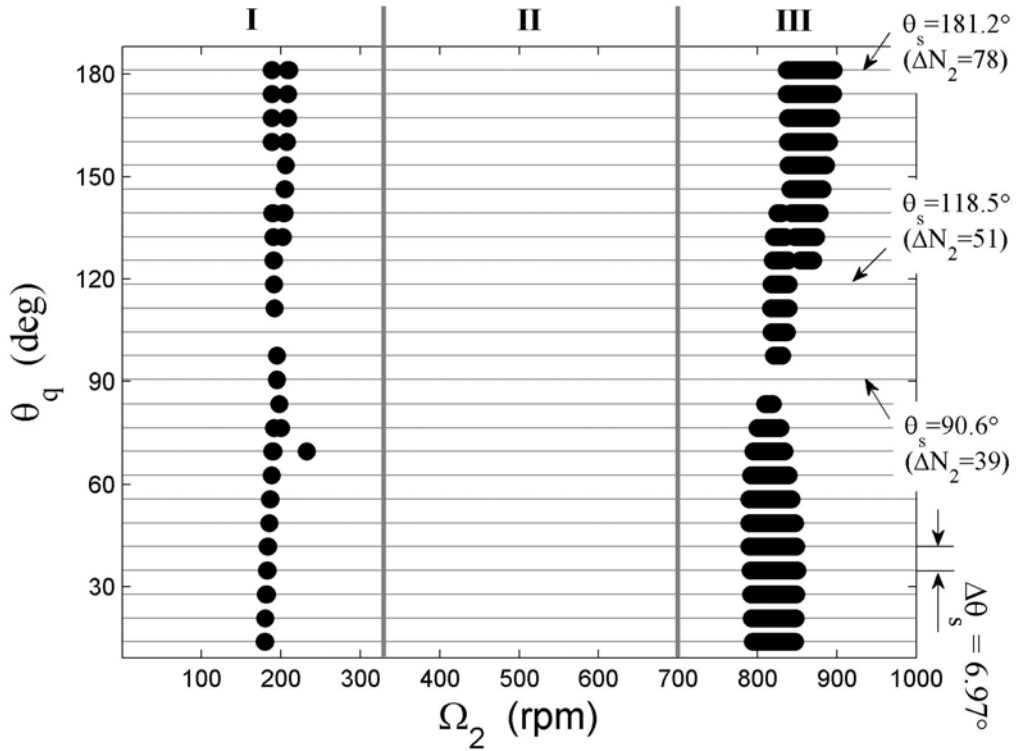


Fig. 5. 6: Face-gear/two pinions system parametric instability with respect to positioning angle and rotation speed for *Example B*.

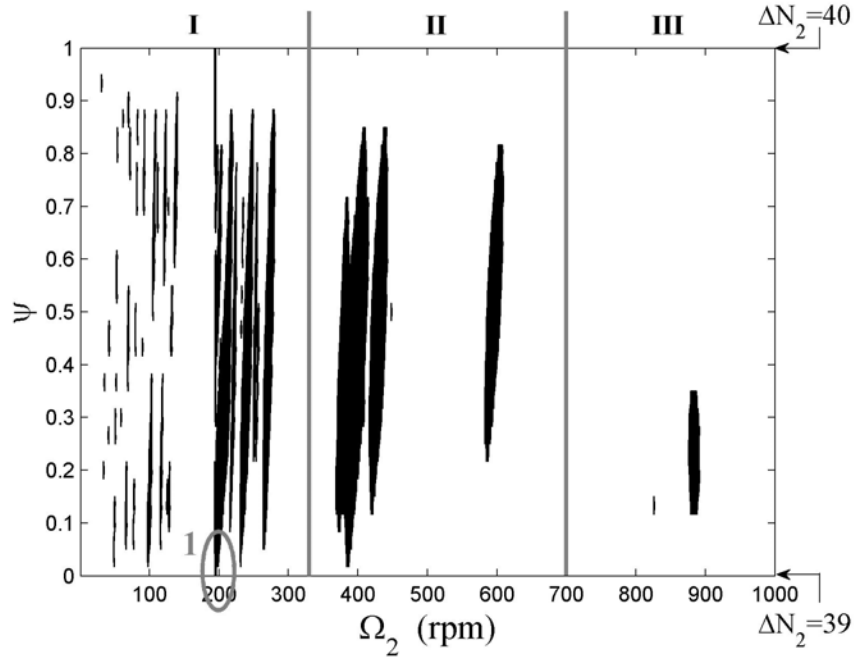
The relationship of positioning angle to system stability for *Example B* is displayed in Figure 5.6. The horizontal axis is face-gear rotation speed; the vertical axis is positioning angle. In order to indicate discrete property, the horizontal grey lines are drawn in Figure 5.6 to represent each value of positioning angle, also an amount of face-gear teeth between pinions. The grade difference between these lines is 6.97° , equivalent to three face-gear teeth. The black areas on these horizontal grey lines indicate instability. This figure shows the positioning angle is responsible for the instabilities at lower speed and high speed zones (I and III). The instability distributions dependent on total angle and positioning angle are highly similar to each other at high speed zone (III).

5.5.3 Stability design based on phasing angle and rotation speed

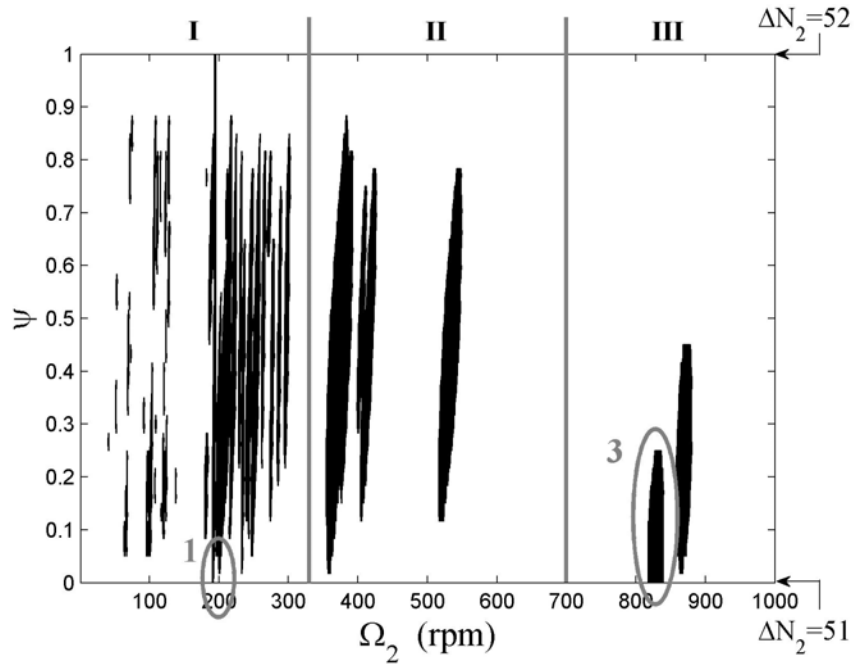
Since the positioning angle variation cannot explain the instability at the medium speed zone (II) in *Example B*, the unstable areas related to the phasing angle must be examined. Also, the sawtooth shape unstable areas in Figure 5.5 indicate the system stability is sensitive to small angle variation. The phasing angle variation is a micro adjustment (fine tuning) to the spatial configuration of pinions, which is very useful to improve the design without much spatial flexibility. The system stability may be affected via changing phasing angle to produce the phase difference between pinions.

Based on *Example B*, the stability plots with respect to phasing angle and rotation speed for three different positioning angles [$\theta_q = 90.6^\circ, 118.5^\circ, 181.2^\circ$] are displayed in Figure 5.7(a)-(c) respectively. The horizontal axis is face-gear rotation speed; the vertical axis is dimensionless phase difference between pinions defined in Eq. (5-1). The dimensionless phase difference is proportional to phasing angle. The positioning angle

equals to total angle when the phasing angle is zero or one, so the stability at $\psi=0$ or $\psi=1$ is identical to the stability along the corresponding grey line in Figure 5.6. As marked in Figure 5.7, the unstable areas appear only at low speed zone (I) for $\theta_q=90.6^\circ$, but at both low speed and high speed zones (I and III) for $\theta_q=118.5^\circ$ and $\theta_q=181.2^\circ$. The grey ellipses 1 and 3 in Figure 5.7 indicate the instabilities at low speed and high speed zones for $\psi=0$ respectively. The instabilities of *Example B* at lower speed zone can not be suppressed completely by adjusting phasing angle, whereas those at high speed zone are successfully stabilized via increasing phasing angle. Moreover, the unstable areas at medium speed zone in Figure 5.7 do not appear in Figure 5.6, which tells the dynamic instability at medium speed zone results mainly from phasing angle for *Example B*.

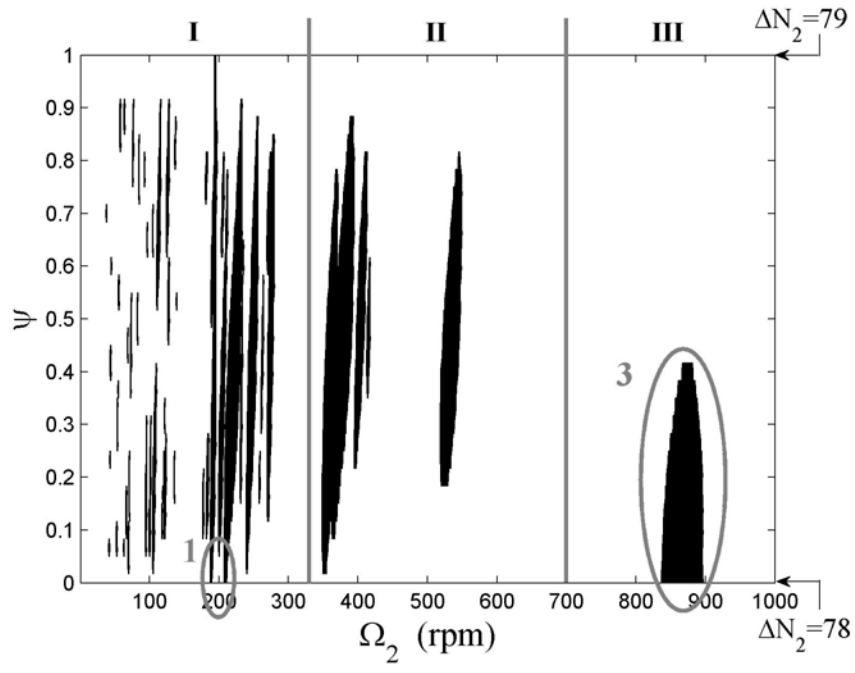


(a) $\theta_q = 90.6^\circ$, $\Delta N_2 = 39$



(b) $\theta_q = 118.5^\circ$, $\Delta N_2 = 51$

Fig. 5. 7: Face-gear/two pinions system parametric instability with respect to phasing angle and rotation speed for *Example B*.



(c) $\theta_q = 181.2^\circ$, $\Delta N_2 = 78$

Fig. 5. 7: Continued

5.5.4 Stability design based on pinion parameters and rotation speed

The pinion mass, pinion bearing stiffness and pinion bearing damping change system mass, stiffness and damping matrices respectively, so the system stability is closely related to the pinion parameters. After the failures of the instability suppression by angle adjustment in some designs, the variation of pinion parameters provides another way to stabilize the system.

The system parametric instability is plotted based on N.D. pinion mass and rotation speed in Figure 5.8; N.D. bearing stiffness and rotation speed in Figure 5.9; proportional bearing damping and rotation speed in Figure 5.10. It can be found from Figure 5.8 - Figure 5.10 that some parameter combinations, for instance $[\tau = 0.4, \mu = 0.25, \zeta = 1 \times 10^{-5} \text{ sec}, \theta_c = 120^\circ]$, stabilize the example face-gear drive over the entire speed range.

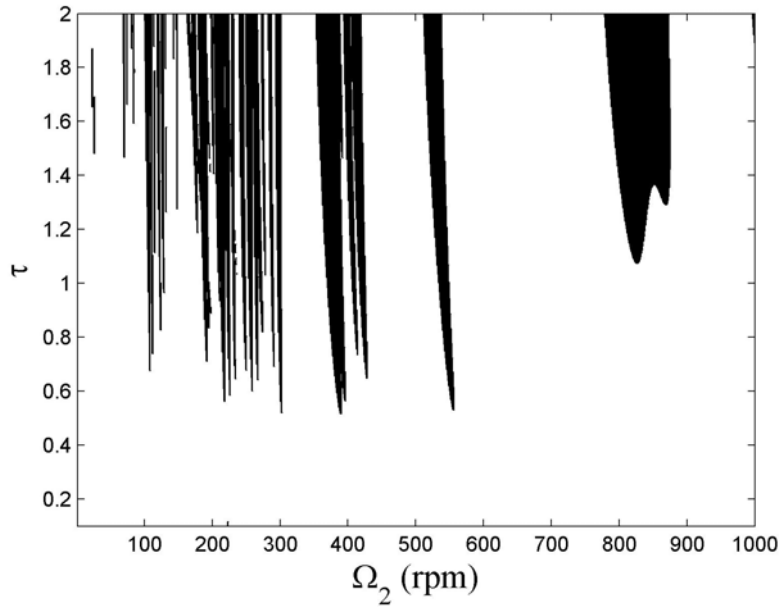
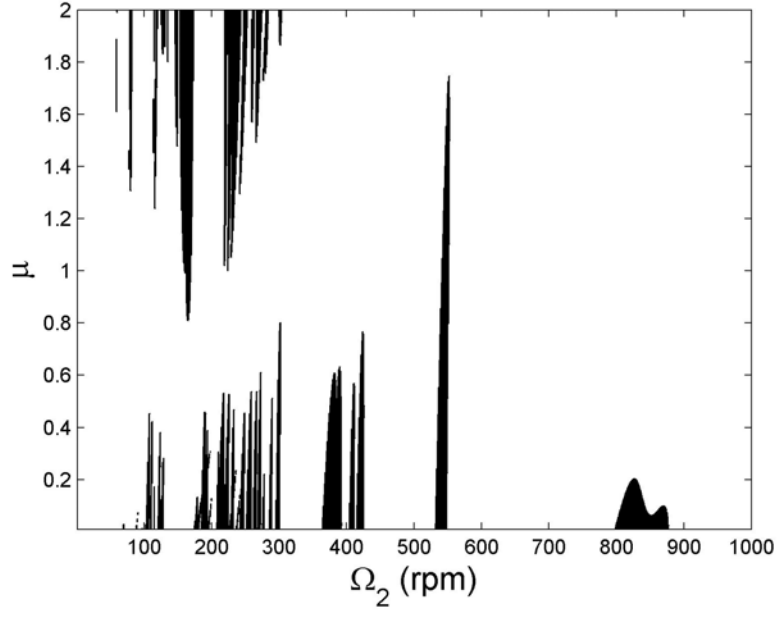
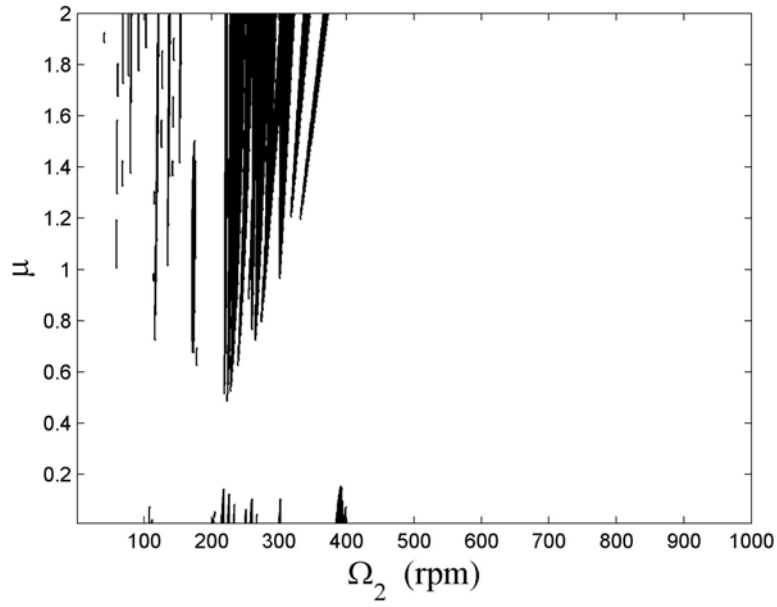


Fig. 5. 8: System parametric instability due to pinion mass, $N_1 = 31, N_2 = 155, h = 0.07a,$
 $\mu = 0.25, \zeta = 1 \times 10^{-5} \text{ sec}, \theta_c = 120^\circ.$



(a) $\tau = 1$



(b) $\tau = 0.4$

Fig. 5. 9: System parametric instability due to pinion bearing stiffness, $N_I = 31$, $N_2 = 155$,

$$h = 0.07a, \zeta = 1 \times 10^{-5} \text{ sec}, \theta_c = 120^\circ.$$

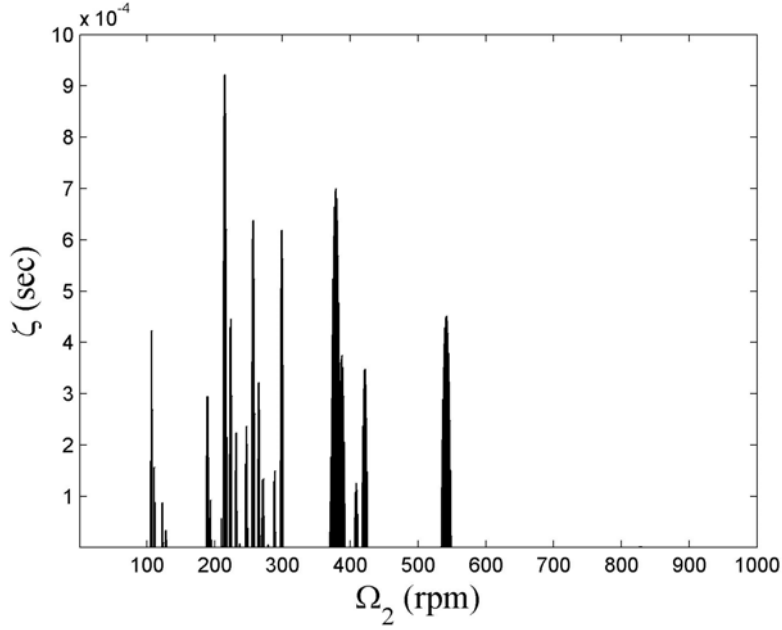


Fig. 5. 10: System parametric instability due to pinion bearing damping, $N_1 = 31$, $N_2 = 155$, $h = 0.07a$, $\tau = 1$, $\mu = 0.25$, $\theta_c = 120^\circ$.

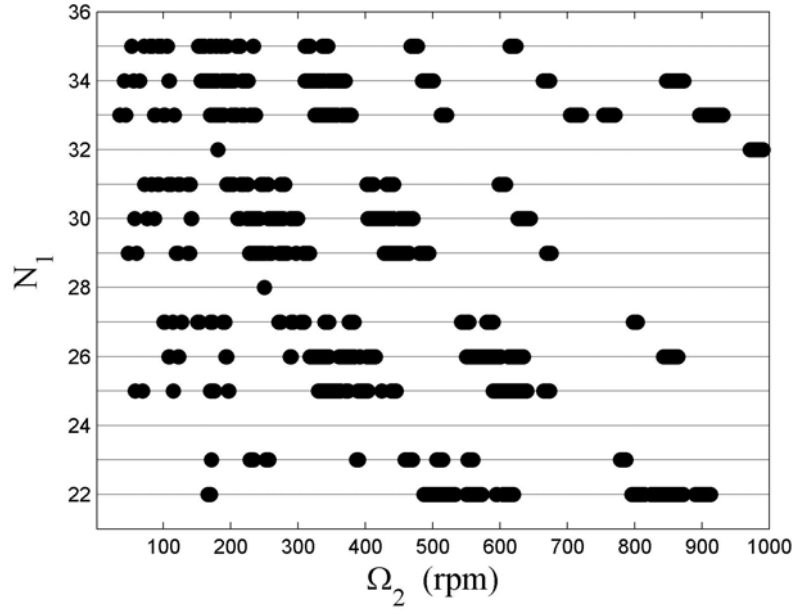
5.5.5 Stability design based on two space-fixed pinions

A possible design based on two space-fixed pinions with constant gear ratio and module is studied in this section. In this case, the total angle is fixed but the number of teeth is variable. Both positioning and phasing angles are changed for different tooth counts so that the system instability may be suppressed. This exploration can help upgrade pre-installed systems without any changes except gears.

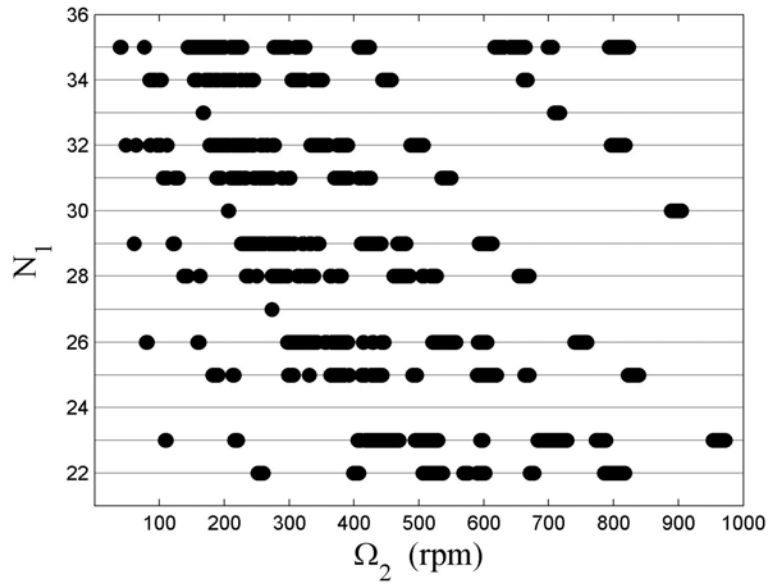
The stability plots for the face-gears meshing with two space-fixed pinions mounted in three different total angles, $[\theta_c = 90^\circ, \theta_c = 120^\circ, \theta_c = 180^\circ]$, are displayed in Figure 5.11(a)-(c) respectively. The horizontal grey line in Figure 5.11 represents each number of pinion teeth. The black areas on these horizontal grey lines indicate instability. It has been shown in Figure 5.11 that the design with 24 pinion teeth is stable over the

entire speed range for three total angles and the design with 22 pinion teeth is also stable for the diametrically opposed pinions ($\theta_c = 180^\circ$).

Moreover, under the same module, the larger number of teeth is chosen, the bigger mass the system possesses. This reduces the system natural frequencies, so the dynamic instability distribution moves towards lower speed zone. This conclusion can be observed in Figure 5.11: the black areas shift along the negative direction of the rotation speed axis as the increase of tooth counts.

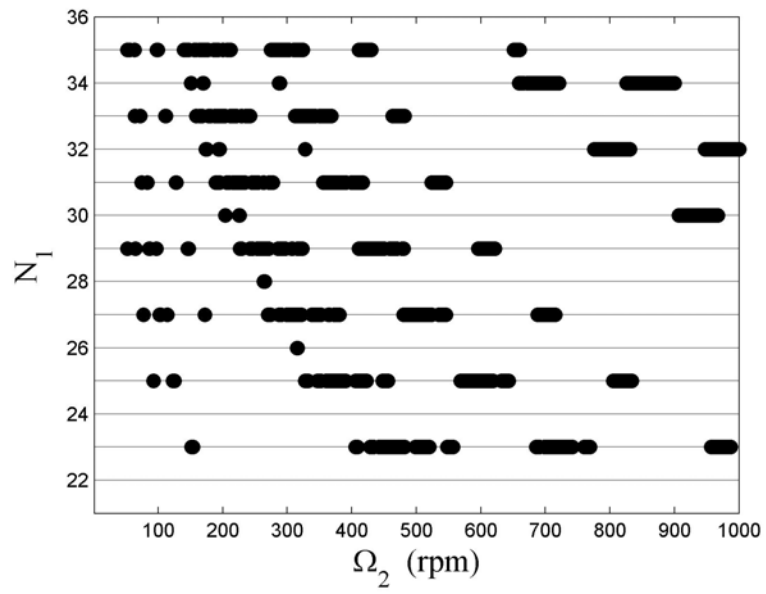


(a) $\theta_c = 90^\circ$



(b) $\theta_c = 120^\circ$

Fig. 5. 11: Stability plots for the face-gears meshing with two space-fixed pinions: constant gear ratio $m_{12} = 5$ and constant module $m_d = 3.175\text{mm}$; $h = 0.07a$, $\tau = 1$, $\mu = 0.25$, $\zeta = 1 \times 10^{-5}$ sec.



(c) $\theta_c = 180^\circ$

Fig. 5. 11: Continued

5.6 Summary

This chapter explores the parametric instability of one face-gear meshing with multiple spur pinions by following the work on one face-gear with one pinion in chapter 4. The phase difference between pinions is detailedly analyzed based on the unique face-gear meshing kinematics and the spatial angle between pinions. The face-gear/multiple pinions system is modeled as a spinning disk with multiple moving loads according to linear plate theory. A helicopter quality face-gear meshing with two spur pinions is taken for example to study the system stability with respect to various parameters: total angle, positioning angle and phasing angle between pinions; pinion mass, pinion bearing stiffness and damping. It has been shown in the results that these parameters can excite dynamic instability.

The spatial configuration of pinions affects the system stability via changing positioning angle (coarse tuning) and phasing angle (fine tuning). The instabilities can be suppressed by adjusting spatial configuration of pinions in some rotation speed zones. When the angle tuning fails to stabilize the system or there is no enough spatial flexibility to change the pinion setting, the adjustment of pinion parameters provides another way to suppress the system instability. For the design practices on two space-fixed pinions with variable number of teeth, constant gear ratio, and constant module, a suitable choice on the number of teeth is possible to make the system stable over the entire speed range by utilizing the variation of tooth counts to produce different phases between pinions. In addition, the system instability moves towards lower speed zone as the increase of tooth counts under the same module.

In summary, the dynamic instability of face-gear/multiple pinions system is closely related to pinion parameters, spatial configuration of pinions and rotation speed. To avoid instability, the system may be designed in a given operation speed range by adjusting total angle between pinions, pinion mass, pinion bearing stiffness, or pinion bearing damping.

Chapter 6

FACE-GEAR STRUCTURAL DYNAMICS MODEL CONSIDERING THE EFFECT OF IN-PLANE DRIVING FORCE

6.1 Overview

The tooth contact surface is neither vertical nor parallel to face-gear body, so the mesh load affects the face-gear dynamics through both out-of-plane and in-plane components. In chapter 3 and 4, the out-of-plane component has been considered by modeling the mesh load as a vertical load unit with prescribed movements, but the in-plane effect is neglected. This chapter is to add this omitted part to the previous model. The in-plane component of the mesh load is represented by a space-fixed in-plane driving force which produces an asymmetric stress field in face-gear body to affect the system stability. The connection between system stability and input power is established via the in-plane driving force. Consequently, the input power becomes a design parameter for the system stability besides serving as a stress constraint in previous designs.

The in-plane stress fields resulting from in-plane edge loads were derived for stationary circular plate in [121-123], and the in-plane free vibrations of annular disk were analyzed by [124-125]. The stationary disk stress field, obtained by Airy stress function, was employed to research the transverse vibration for stationary disk in [126-

127] and also for rotating disk in [128-129]. In addition, Rosen and Libai [130] investigated the transverse vibration of a stationary disk under uniform radial pressure experimentally, and Zajackowski [122] also analyzed the transverse stability for a stationary disk subjected to periodically varying tangential edge loads.

Srinivasan and Ramamurti [131] calculated the steady state in-plane stress response due to an in-plane edge load analytically by transforming the problem of a rotating disk under stationary load to the one of a moving load rotating around stationary disk. However, this transformation ignores the centrifugal initial stress and the Coriolis effect resulting from the rotation of the disk, which makes their conclusion only valid for low speed cases. The symmetric centrifugal stress field is well known and has been given in Eq. (4-2) - Eq. (4-3). The asymmetric stress field due to in-plane edge loads, considering the Coriolis effect, was analytically solved by Chen and Jhu [132-134]. Their research includes natural frequency, critical speed and steady state response. Koh et al. [135] employed a novel moving element method to numerically calculate the in-plane dynamic response of annular disk, and the results were compared with the analytical solutions based on Chen and Jhu's method [134] but using complex Fourier-Hankel series instead of real Fourier-Bessel series.

Radcliffe and Mote [136] conducted an experiment to investigate the effect of in-plane concentrated edge load on the rotating disk transverse stability. Irons [137] analyzed the lateral vibration of a thick disk subjected to symmetric in-plane stress arising from rotational or thermal effects. The transverse instability of a rotating disk was explored based on the asymmetric in-plane stress field of stationary disk in [128,138]. Chen [139-140] applied his in-plane stress results to the spinning disk transverse

vibration by expanding the concentrated in-plane edge force as Fourier series. The cases with uniform in-plane edge loads were investigated in [141-143]. Moreover, Chen [144] also explored the nonlinear parametric resonance for the similar problems.

6.2 In-Plane Stress Field

This section is to introduce Chen and Jhu's work in [132-134] for solving the asymmetric in-plane stress field, $(\sigma_{rr}, \sigma_{r\theta}, \sigma_{\theta\theta})$, in spinning annular disk resulting from a space-fixed concentrated in-plane tangential edge load. The problem is illustrated in Figure 6.1. For the face-gear model, the edge load equals to the driving force on the pinion pitch circle, F_t , but in the opposite direction. F_t is calculated by Eq. (4-21). u_r and u_θ are in-plane radial and circumferential displacements respectively.

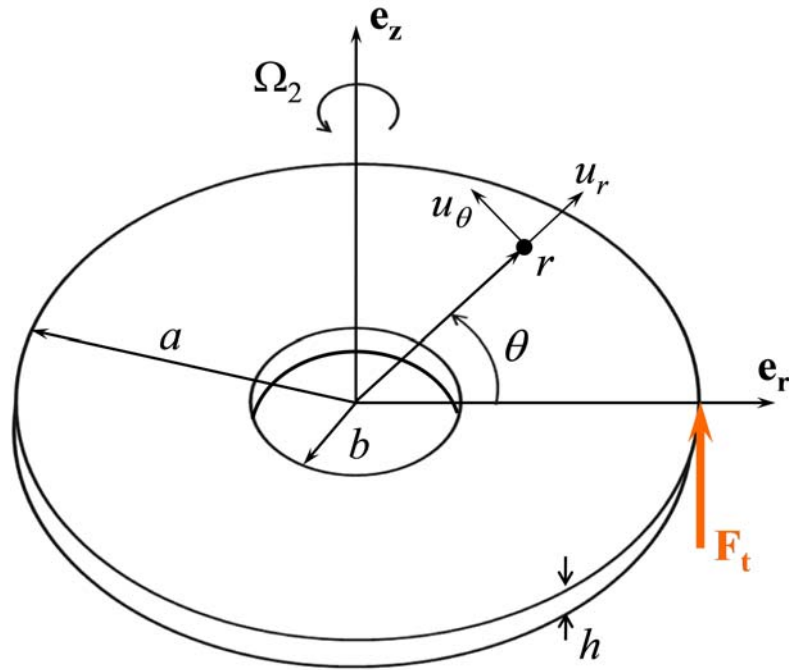


Fig. 6. 1: Spinning annular disk subjected to a space-fixed concentrated in-plane tangential edge load.

First of all, some dimensionless variables are introduced [133-134] and denoted with asterisk,

$$\begin{aligned} r^* &= \frac{r}{a} & r_b^* &= \frac{b}{a} & u_r^* &= \frac{u_r}{a} & u_\theta^* &= \frac{u_\theta}{a} & \Omega_2^* &= \Omega_2 a \sqrt{\frac{\rho}{E}} \\ t^* &= \frac{t}{a} \sqrt{\frac{E}{\rho}} & \sigma_{rr}^* &= \frac{\sigma_{rr}}{E} & \sigma_{r\theta}^* &= \frac{\sigma_{r\theta}}{E} & \sigma_{\theta\theta}^* &= \frac{\sigma_{\theta\theta}}{E} \end{aligned} \quad (6-1)$$

The complete in-plane equations-of-motion are given in the dimensionless space-fixed polar coordinates (r^*, θ^*) as [133-134]

$$\begin{aligned} & \frac{1}{1-\nu^2} \left(\frac{\partial^2 u_r^*}{\partial r^{*2}} + \frac{1}{r^*} \frac{\partial u_r^*}{\partial r^*} - \frac{u_r^*}{r^{*2}} + \frac{1-\nu}{2r^{*2}} \frac{\partial^2 u_r^*}{\partial \theta^2} + \frac{1+\nu}{2} \frac{1}{r^*} \frac{\partial^2 u_\theta^*}{\partial r^* \partial \theta} - \frac{3-\nu}{2} \frac{1}{r^{*2}} \frac{\partial u_\theta^*}{\partial \theta} \right) \\ &= \left(\frac{\partial^2 u_r^*}{\partial t^{*2}} + 2\Omega_2^* \frac{\partial^2 u_r^*}{\partial t^* \partial \theta} + \Omega_2^{*2} \frac{\partial^2 u_r^*}{\partial \theta^2} \right) - 2\Omega_2^* \left(\frac{\partial u_\theta^*}{\partial t^*} + \Omega_2^* \frac{\partial u_\theta^*}{\partial \theta} \right) - \Omega_2^{*2} u_r^* - \Omega_2^{*2} r^* \end{aligned} \quad (6-2)$$

and

$$\begin{aligned} & \frac{1}{1-\nu^2} \left(\frac{1+\nu}{2} \frac{1}{r^*} \frac{\partial^2 u_r^*}{\partial r^* \partial \theta} + \frac{3-\nu}{2} \frac{1}{r^{*2}} \frac{\partial u_r^*}{\partial \theta} + \frac{1-\nu}{2} \frac{\partial^2 u_\theta^*}{\partial r^{*2}} + \frac{1-\nu}{2r^*} \frac{\partial u_\theta^*}{\partial r^*} - \frac{1-\nu}{2r^{*2}} u_\theta^* + \frac{1}{r^{*2}} \frac{\partial^2 u_\theta^*}{\partial \theta^2} \right) \\ &= \left(\frac{\partial^2 u_\theta^*}{\partial t^{*2}} + 2\Omega_2^* \frac{\partial^2 u_\theta^*}{\partial t^* \partial \theta} + \Omega_2^{*2} \frac{\partial^2 u_\theta^*}{\partial \theta^2} \right) + 2\Omega_2^* \left(\frac{\partial u_r^*}{\partial t^*} + \Omega_2^* \frac{\partial u_r^*}{\partial \theta} \right) - \Omega_2^{*2} u_\theta^* \end{aligned} \quad (6-3)$$

The last item in Eq. (6-2), $\Omega_2^{*2} r^*$, results from the centrifugal effect that has been considered in Eq. (4-2), so this item is neglected in the following calculation for the asymmetric stress field due to in-plane tangential load. The dimensionless boundary conditions for in-plane vibration are given [127, 129, 131]: for clamped edge at $r^*=r_b^*$,

$$u_r^* \big|_{r^*=r_b^*} = u_\theta^* \big|_{r^*=r_b^*} = 0 \quad (6-4)$$

and for the free edge at $r^*=1$,

$$\sigma_{rr}^* \big|_{r^*=1} = 0 \quad \sigma_{r\theta}^* \big|_{r^*=1} = \begin{cases} \frac{F_t}{2\theta_f a h E} & -\theta_f < \theta < \theta_f \\ 0 & \text{Otherwise} \end{cases} \quad (6-5)$$

It should be noted that the second equation of Eq. (6-5) is obtained by assuming the concentrated edge load uniformly distributes over a small circumferential area, $(-\theta_f, \theta_f)$ and $\theta_f \rightarrow 0$, at the outer edge. Moreover, $\sigma_{r\theta}^*$ can be further expanded as Fourier series [129],

$$\sigma_{r\theta}^*|_{r^*=1} = \tau_0^* + \sum_{k=1}^{\infty} (\tau_{nc}^* \cos k\theta + \tau_{ns}^* \sin k\theta) \quad (6-6)$$

where

$$\begin{aligned} \tau_0^* &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sigma_{r\theta}^*|_{r=a} d\theta = \frac{1}{2\pi} \int_{-\theta_f}^{\theta_f} \frac{F_t}{2\theta_f ahE} d\theta = \frac{F_t}{2\pi ahE} \\ \tau_{nc}^* &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sigma_{r\theta}^*|_{r=a} \cos k\theta d\theta = \frac{1}{\pi} \int_{-\theta_f}^{\theta_f} \frac{F_t}{2\theta_f ahE} \cos k\theta d\theta \\ &= \frac{\sin k\theta_f}{k\theta_f} \frac{F_t}{\pi ahE} \rightarrow \frac{F_t}{\pi ahE} |_{\theta_f \rightarrow 0} \\ \tau_{ns}^* &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sigma_{r\theta}^*|_{r=a} \sin k\theta d\theta = \frac{1}{\pi} \int_{-\theta_f}^{\theta_f} \frac{F_t}{2\theta_f ahE} \sin k\theta d\theta = 0 \end{aligned} \quad (6-7)$$

By utilizing Lamé's potential ϕ_L and ψ_L [131, 133-134], the coupled equations Eq. (6-2) and Eq. (6-3) are simplified. Next, the steady state solutions are obtained based on all items independent of temporal derivative. Here, Chen and Jhu's work in [133-134] are summarized as following

$$\begin{aligned} u_r^* &= \frac{\partial \phi_L}{\partial r^*} + \frac{1}{r^*} \frac{\partial \psi_L}{\partial \theta} \\ u_\theta^* &= \frac{1}{r^*} \frac{\partial \phi_L}{\partial \theta} - \frac{\partial \psi_L}{\partial r^*} \end{aligned} \quad (6-8)$$

and the resulting equations are

$$\begin{aligned} \frac{\partial G_L}{\partial r^*} + \frac{1}{r^*} \frac{\partial H_L}{\partial \theta} &= 0 \\ \frac{1}{r^*} \frac{\partial G_L}{\partial \theta} - \frac{\partial H_L}{\partial r^*} &= 0 \end{aligned} \quad (6-9)$$

where

$$\begin{aligned} G_L &= \lambda_1^{*2} \nabla^2 \phi_L - \Omega_2^{*2} \frac{\partial^2 \phi_L}{\partial \theta^2} + \Omega_2^{*2} \phi_L - 2\Omega_2^{*2} \frac{\partial \psi_L}{\partial \theta} \\ H_L &= \lambda_2^{*2} \nabla^2 \psi_L - \Omega_2^{*2} \frac{\partial^2 \psi_L}{\partial \theta^2} + \Omega_2^{*2} \psi_L + 2\Omega_2^{*2} \frac{\partial \phi_L}{\partial \theta} \end{aligned} \quad (6-10)$$

and the constants related to Poisson's ratio, ν , are expressed by

$$\begin{aligned} \lambda_1^{*2} &= \frac{1}{1-\nu^2} \\ \lambda_2^{*2} &= \frac{1}{2(1+\nu)} \end{aligned} \quad (6-11)$$

The general solutions of G_L and H_L in Eq. (6-9) are given as,

$$\begin{aligned} G_L &= A_{g0} + \sum_{k=1}^{\infty} \left[(A_{gk} r^{*k} + B_{gk} r^{*-k}) \cos k\theta + (A_{hk} r^{*k} + B_{hk} r^{*-k}) \sin k\theta \right] \\ H_L &= A_{h0} + \sum_{k=1}^{\infty} \left[(A_{hk} r^{*k} - B_{hk} r^{*-k}) \cos k\theta + (-A_{gk} r^{*k} + B_{gk} r^{*-k}) \sin k\theta \right] \end{aligned} \quad (6-12)$$

Here A_{gk} , B_{gk} , A_{hk} , and B_{hk} are constant coefficients. Accordingly, the solutions of ϕ_L and ψ_L are in the similar form as Eq. (6-12),

$$\begin{aligned} \phi_L(r^*, \theta) &= \Phi_{0c}(r^*) + \sum_{k=1}^{\infty} [\Phi_{kc}(r^*) \cos k\theta + \Phi_{ks}(r^*) \sin k\theta] \\ \psi_L(r^*, \theta) &= \Psi_{0c}(r^*) + \sum_{k=1}^{\infty} [\Psi_{kc}(r^*) \cos k\theta + \Psi_{ks}(r^*) \sin k\theta] \end{aligned} \quad (6-13)$$

where the radial functions, Φ_{0c} , Φ_{kc} , Φ_{ks} , Ψ_{0c} , Ψ_{kc} , Ψ_{ks} , are solved by substituting Eq. (6-12) and Eq. (6-13) into Eq. (6-10) and then applying harmonic balance method. For $k=0$, the nontrivial solutions (keep only items that affect in-plane displacements and stresses) are given

$$\begin{aligned} \Phi_{0c}(r^*) &= c_{g0} J_0(\beta_{01}^* r^*) + e_{g0} Y_0(\beta_{01}^* r^*) \\ \Psi_{0c}(r^*) &= -d_{h0} J_0(\beta_{02}^* r^*) - f_{h0} Y_0(\beta_{02}^* r^*) \end{aligned} \quad (6-14)$$

Here, J and Y are Bessel functions; c_{g0} , e_{g0} , d_{h0} , and f_{h0} are constants to be determined by boundary conditions; β_{01}^* and β_{02}^* are:

$$\begin{aligned}\beta_{01}^* &= \frac{\Omega_2^*}{\lambda_1^*} \\ \beta_{02}^* &= \frac{\Omega_2^*}{\lambda_2^*}\end{aligned}\tag{6-15}$$

For $k=l$, the nontrivial solutions are

$$\begin{aligned}\Phi_{1c}(r^*) &= d_{g1}s_{11}J_1(\beta_{12}^*r^*) + f_{g1}s_{11}Y_1(\beta_{12}^*r^*) + A_{g1}\frac{r^*}{2\Omega_2^{*2}} + B_{g1}\frac{r^*\ln r^*}{\lambda_1^{*2} + \lambda_2^{*2}} \\ \Psi_{1s}(r^*) &= d_{g1}J_1(\beta_{12}^*r^*) + f_{g1}Y_1(\beta_{12}^*r^*) + B_{g1}\frac{1}{\lambda_1^{*2} + \lambda_2^{*2}}\left(r^*\ln r^* + \frac{\lambda_1^{*2} - \lambda_2^{*2}}{2r^*\Omega_2^{*2}}\right) \\ \Phi_{1s}(r^*) &= d_{h1}s_{11}J_1(\beta_{12}^*r^*) + f_{h1}s_{11}Y_1(\beta_{12}^*r^*) + A_{h1}\frac{r^*}{2\Omega_2^{*2}} + B_{h1}\frac{r^*\ln r^*}{\lambda_1^{*2} + \lambda_2^{*2}} \\ \Psi_{1c}(r^*) &= -d_{h1}J_1(\beta_{12}^*r^*) - f_{h1}Y_1(\beta_{12}^*r^*) - B_{h1}\frac{1}{\lambda_1^{*2} + \lambda_2^{*2}}\left(r^*\ln r^* + \frac{\lambda_1^{*2} - \lambda_2^{*2}}{2r^*\Omega_2^{*2}}\right)\end{aligned}\tag{6-16}$$

Here, d_{g1} , f_{g1} , d_{h1} , and f_{h1} are constants to be determined by boundary conditions; β_{12}^* and s_{11} are respectively:

$$\begin{aligned}\beta_{12}^* &= \frac{\sqrt{3-\nu}}{\lambda_2^*}\Omega_2^* \\ s_{11} &= -\frac{1-\nu}{2}\end{aligned}\tag{6-17}$$

For $k \geq 2$, the nontrivial solutions are

$$\begin{aligned}\Phi_{kc}(r^*) &= c_{gk}J_k(\beta_{k1}^*r^*) + d_{gk}s_{k1}J_k(\beta_{k2}^*r^*) + e_{gk}Y_k(\beta_{k1}^*r^*) + f_{gk}s_{k1}Y_k(\beta_{k2}^*r^*) \\ \Psi_{ks}(r^*) &= c_{gk}s_{k2}J_k(\beta_{k1}^*r^*) + d_{gk}J_k(\beta_{k2}^*r^*) + e_{gk}s_{k2}Y_k(\beta_{k1}^*r^*) + f_{gk}Y_k(\beta_{k2}^*r^*) \\ \Phi_{ks}(r^*) &= c_{hk}J_k(\beta_{k1}^*r^*) + d_{hk}s_{k1}J_k(\beta_{k2}^*r^*) + e_{hk}Y_k(\beta_{k1}^*r^*) + f_{hk}s_{k1}Y_k(\beta_{k2}^*r^*) \\ \Psi_{1c}(r^*) &= -c_{hk}s_{k2}J_k(\beta_{k1}^*r^*) - d_{hk}J_k(\beta_{k2}^*r^*) - e_{hk}s_{k2}Y_k(\beta_{k1}^*r^*) - f_{hk}Y_k(\beta_{k2}^*r^*)\end{aligned}\tag{6-18}$$

Here, c_{gk} , d_{gk} , e_{gk} , f_{gk} , c_{hk} , d_{hk} , e_{hk} , and f_{hk} , are constants to be determined by boundary conditions; β_{k1}^* and β_{k2}^* are positive real roots of the following equation:

$$\lambda_1^{*2} \lambda_2^{*2} \beta_k^{*4} - (\lambda_1^{*2} + \lambda_2^{*2})(1+k^2)\Omega_2^{*2} \beta_k^{*2} + (1-k^2)^2 \Omega_2^{*4} = 0 \quad (6-19)$$

s_{k1} and s_{k2} are respectively:

$$s_{k1} = \frac{2k\Omega_2^{*2}}{(1+k^2)\Omega_2^{*2} - \lambda_1^{*2} \beta_{k2}^{*2}} \quad (6-20)$$

$$s_{k2} = \frac{2k\Omega_2^{*2}}{(1+k^2)\Omega_2^{*2} - \lambda_2^{*2} \beta_{k1}^{*2}}$$

Finally, the in-plane stress fields are expressed based on Lamé's potential as

$$\begin{aligned} \sigma_{rr}^* &= \frac{\nu}{1-\nu^2} \nabla^2 \phi_L + \frac{1}{1+\nu} \left(\frac{\partial^2 \phi_L}{\partial r^{*2}} - \frac{1}{r^{*2}} \frac{\partial \psi_L}{\partial \theta} + \frac{1}{r^*} \frac{\partial^2 \psi_L}{\partial r^* \partial \theta} \right) \\ \sigma_{r\theta}^* &= \frac{1}{2(1+\nu)} \left(\frac{2}{r^*} \frac{\partial^2 \phi_L}{\partial r^* \partial \theta} - \frac{2}{r^{*2}} \frac{\partial \phi_L}{\partial \theta} - \frac{\partial^2 \psi_L}{\partial r^{*2}} + \frac{1}{r^*} \frac{\partial \psi_L}{\partial r^*} + \frac{1}{r^{*2}} \frac{\partial^2 \psi_L}{\partial \theta^2} \right) \\ \sigma_{\theta\theta}^* &= \frac{\nu}{1-\nu^2} \nabla^2 \phi_L + \frac{1}{1+\nu} \left(\frac{1}{r^*} \frac{\partial \phi_L}{\partial r^*} + \frac{1}{r^{*2}} \frac{\partial^2 \phi_L}{\partial \theta^2} + \frac{1}{r^{*2}} \frac{\partial \psi_L}{\partial \theta} - \frac{1}{r^*} \frac{\partial^2 \psi_L}{\partial r^* \partial \theta} \right) \end{aligned} \quad (6-21)$$

By applying the boundary conditions in Eq. (6-4) - Eq. (6-7) for each harmonic component, all constant coefficients can be determined. It should be noted that the shear stress along the outer edge of the disk contains only cosine components for in-plane tangential edge load. Consequently, the dimensional stress fields are represented in a simplified form as [140]

$$\begin{aligned} \sigma_{rr}(r, \theta) &= \sum_{k=1}^{\infty} \sigma_{rr}^{(k)}(r) \sin k\theta \\ \sigma_{r\theta}(r, \theta) &= \sum_{k=0}^{\infty} \sigma_{r\theta}^{(k)}(r) \cos k\theta \\ \sigma_{\theta\theta}(r, \theta) &= \sum_{k=1}^{\infty} \sigma_{\theta\theta}^{(k)}(r) \sin k\theta \end{aligned} \quad (6-22)$$

where $\sigma_{rr}^{(k)}$, $\sigma_{r\theta}^{(k)}$, $\sigma_{\theta\theta}^{(k)}$ are the k th harmonic component of the asymmetric in-plane stresses. They are actually the Fourier coefficients of Eq. (6-21) multiplied by Young's modulus E .

6.3 Face-gear Model Considering In-Plane Driving Load

The in-plane component of the mesh load is modeled as a space-fixed concentrated in-plane tangential driving force, acting on the outer edge of the face-gear disk. This force is assumed as follower type, which means its direction is always tangential to the circumferential slope of the laterally vibrating disk at the point of application [145-147]. The asymmetric in-plane stress fields due to the in-plane force have been given in Eq. (6-22) as Fourier series and they are incorporated into the disk transverse vibrations by the following operator [128, 140]:

$$\begin{aligned} \ell \equiv & -\frac{h}{r} \left[\frac{\partial}{\partial r} \left(r \sigma_{rr} \frac{\partial}{\partial r} + \sigma_{r\theta} \frac{\partial}{\partial \theta} \right) + \frac{\partial}{\partial \theta} \left(\sigma_{r\theta} \frac{\partial}{\partial r} + \frac{1}{r} \sigma_{\theta\theta} \frac{\partial}{\partial \theta} \right) \right] \\ & = -\frac{h}{r} \left[\frac{\partial}{\partial r} \left(\sigma_{r\theta}^{(0)} \frac{\partial}{\partial \theta} \right) + \frac{\partial}{\partial \theta} \left(\sigma_{r\theta}^{(0)} \frac{\partial}{\partial r} \right) \right] - \sum_{k=1}^{\infty} \frac{h}{r} \left[\frac{\partial}{\partial r} \left(r \sigma_{rr}^{(k)} \sin k\theta \frac{\partial}{\partial r} + \sigma_{r\theta}^{(k)} \cos k\theta \frac{\partial}{\partial \theta} \right) \right. \\ & \quad \left. + \frac{\partial}{\partial \theta} \left(\sigma_{r\theta}^{(k)} \cos k\theta \frac{\partial}{\partial r} + \frac{1}{r} \sigma_{\theta\theta}^{(k)} \sin k\theta \frac{\partial}{\partial \theta} \right) \right] \end{aligned} \quad (6-23)$$

Since the stress field in Eq. (6-22) is derived for the in-plane edge force at the zero circumferential position, $\theta=0$, the initial circumferential position of the corresponding out-of-plane load is also transformed to zero, namely $\theta_c(0)=0$. The face-gear model with both in- and out-of-plane mesh loads is illustrated in Figure 6.2.

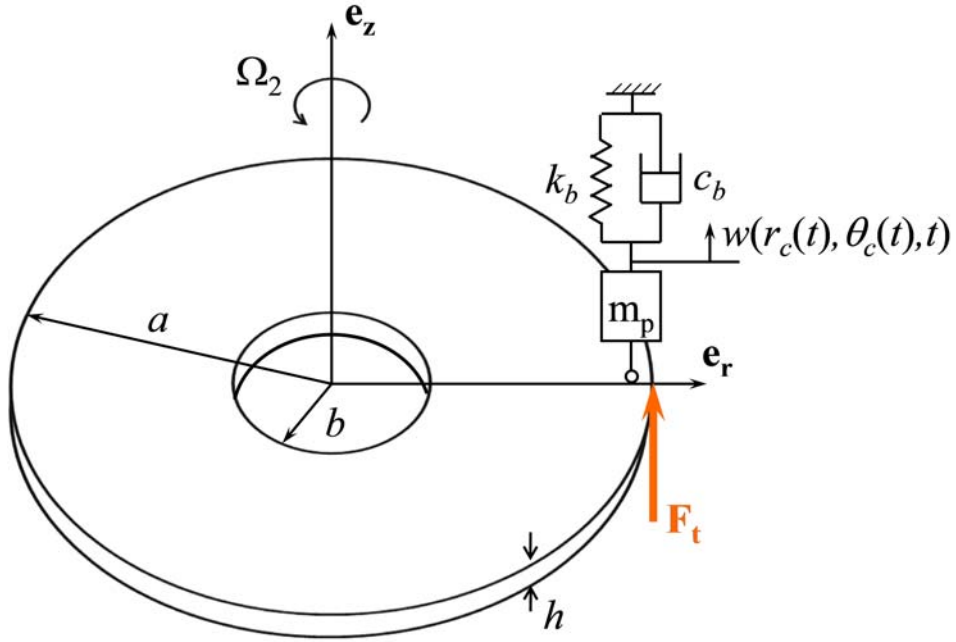


Fig. 6. 2: Face-gear dynamics model: spinning elastic disk with space-fixed in-plane driving force and movable out-of-plane mesh load unit.

By adding Eq. (6-23) to Eq. (4-1), the governing equation-of-motion for the face-gear model considering in- and out-of-plane components of the mesh load is obtained

$$\begin{aligned} \rho h \left(\frac{\partial^2 w}{\partial t^2} + 2\Omega_2 \frac{\partial^2 w}{\partial t \partial \theta} + \Omega_2^2 \frac{\partial^2 w}{\partial \theta^2} \right) + \xi D \nabla^4 \left(\frac{\partial w}{\partial t} + \Omega_2 \frac{\partial w}{\partial \theta} \right) + D \nabla^4 w + \ell w \\ - h \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \sigma_r \frac{\partial w}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\sigma_\theta \frac{\partial w}{\partial \theta} \right) \right] + \frac{\delta(r - r_c(t)) \delta(\theta - \theta_c(t))}{r} (m_p \ddot{w} + c_b \dot{w} + k_b w) = 0 \end{aligned} \quad (6-24)$$

It should be noted that the symmetric stress fields (σ_r , σ_θ) result from centrifugal effect while the asymmetric stress fields (σ_{rr} , $\sigma_{r\theta}$, $\sigma_{\theta\theta}$) are produced by in-plane driving force. The follower type in-plane edge load yields nothing on the boundary conditions of transverse vibrations, so they are the same as Eq. (4-4). Based on the same system discretization procedure in section 4.3, the matrix form equation-of-motion can be

obtained by adding to Eq. (4-10) a stiffness matrix, $\mathbf{K}_f$, resulting from the in-plane asymmetric stress fields,

$$[\mathbf{M}_0 + \mathbf{M}(t)]\ddot{q}(t) + [\mathbf{C}_0 + \mathbf{C}(t)]\dot{q}(t) + [\mathbf{K}_0 + \mathbf{K}_f + \mathbf{K}(t)]q(t) = 0 \quad (6-25)$$

where

$$\mathbf{K}_f = \sum_{k=0}^{\infty} \left(\text{full} \left[\frac{1}{\rho} \mathbf{K}_{n,q}^{(k)} \right] \right) \quad (6-26)$$

with

$$\mathbf{K}_{n,q}^{(k)} = \text{full} [K_{m1,m2}^{(k,n,q)}]$$

$$K_{m1,m2}^{(k,n,q)} = \begin{cases} \begin{pmatrix} -\mathbf{i}q\sigma_{r\theta}^{(0)}(a)R_{m1,n}(a)R_{m2,q}(a) \\ + \mathbf{i} \int_b^a \begin{pmatrix} q\sigma_{r\theta}^{(0)}(r) \frac{\partial R_{m1,n}(r)}{\partial r} R_{m2,q}(r) \\ - n\sigma_{r\theta}^{(0)}(r)R_{m1,n}(r) \frac{\partial R_{m2,q}(r)}{\partial r} \end{pmatrix} dr \end{pmatrix} & \text{for } q-n=0 \\ \begin{pmatrix} -\mathbf{i} \frac{1}{2} q\sigma_{r\theta}^{(k)}(a)R_{m1,n}(a)R_{m2,q}(a) \\ + \mathbf{i} \frac{1}{2} \int_b^a \begin{pmatrix} \frac{q-n}{k} r\sigma_{rr}^{(k)}(r) \frac{\partial R_{m1,n}(r)}{\partial r} \frac{\partial R_{m2,q}(r)}{\partial r} \\ + q\sigma_{r\theta}^{(k)}(r) \frac{\partial R_{m1,n}(r)}{\partial r} R_{m2,q}(r) \\ - n\sigma_{r\theta}^{(k)}(r)R_{m1,n}(r) \frac{\partial R_{m2,q}(r)}{\partial r} \\ + \frac{q-n}{k} \frac{nq}{r} \sigma_{\theta\theta}^{(k)}(r)R_{m1,n}(r)R_{m2,q}(r) \end{pmatrix} dr \end{pmatrix} & \text{for } q-n=\pm k \\ 0 & \text{Otherwise} \end{cases} \quad (6-27)$$

and all other coefficient matrices are presented in Eq. (4-11). Next, Eq. (6-25) is rewritten in state-space form to determine the system stability via Floquet theory:

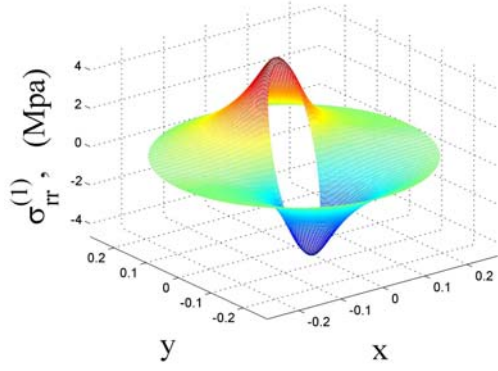
$$\dot{\mathbf{X}} = \mathbf{A}(t)\mathbf{X} \quad \text{with} \quad \mathbf{X} = \begin{bmatrix} q(t)^T & \dot{q}(t)^T \end{bmatrix}^T \quad (6-28)$$

and

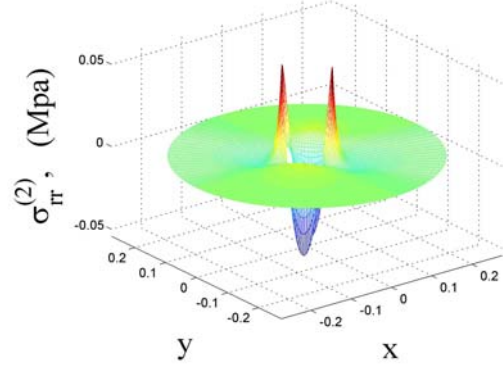
$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -[\mathbf{M}_0 + \mathbf{M}(t)]^{-1}[\mathbf{K}_0 + \mathbf{K}_f + \mathbf{K}(t)] & -[\mathbf{M}_0 + \mathbf{M}(t)]^{-1}[\mathbf{C}_0 + \mathbf{C}(t)] \end{bmatrix} \quad (6-29)$$

6.4 Stability Results for Various Power Levels

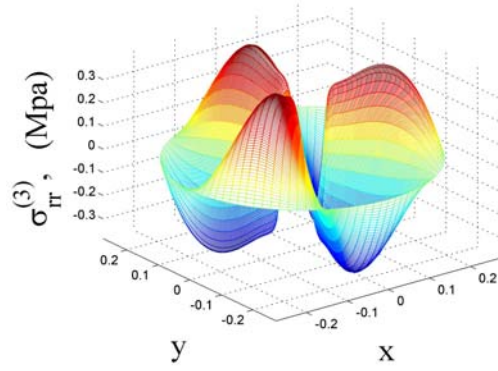
The *Case I* and *II* in section 4.6 are recalculated here to explore the effects of the input power level on the system stability. All parameters are listed in Table 4.1-Table 4.3. The first five (six for $\sigma_{r\theta}$) Fourier components of the asymmetric in-plane stress fields for the *Case II*, (σ_{rr} , $\sigma_{r\theta}$, $\sigma_{\theta\theta}$), are respectively shown in Figure 6.3 - Figure 6.5. The input power level is 800 hp at the input speed 5000 rpm.



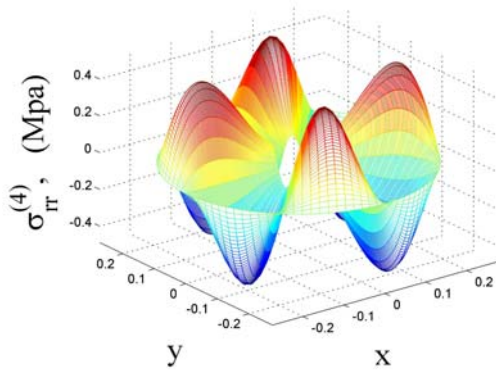
$k=1$



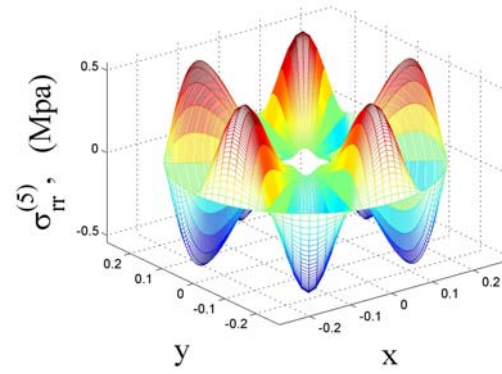
$k=2$



$k=3$



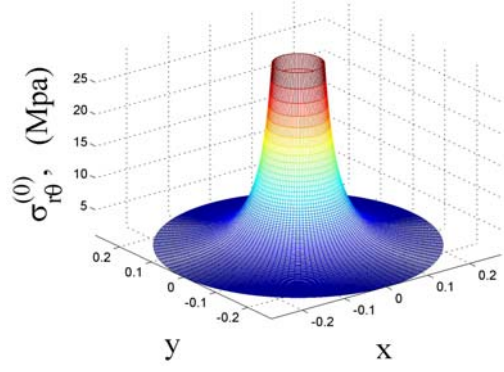
$k=4$



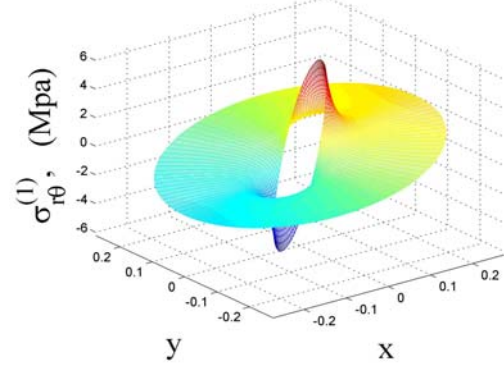
$k=5$

Fig. 6. 3: Asymmetric in-plane radial stress field, σ_{rr} , resulting from in-plane edge load:

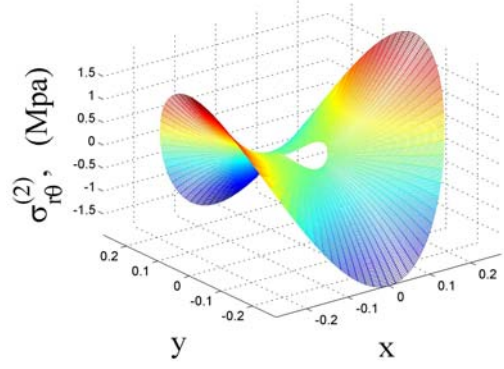
$$N_1=31, N_2=155, a=274.54mm, h=0.05a, P=800hp@\Omega_1=5000rpm.$$



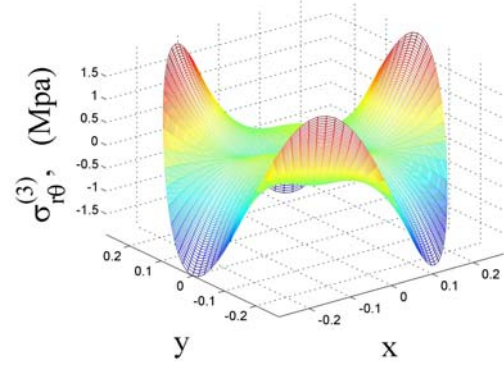
$k=0$



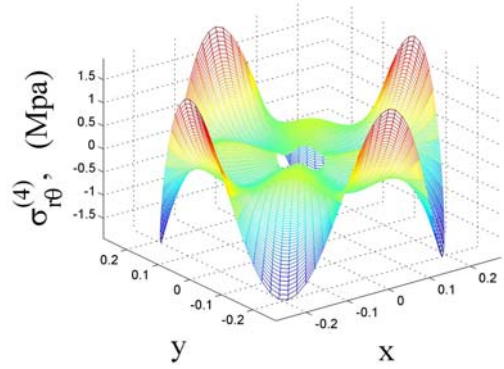
$k=1$



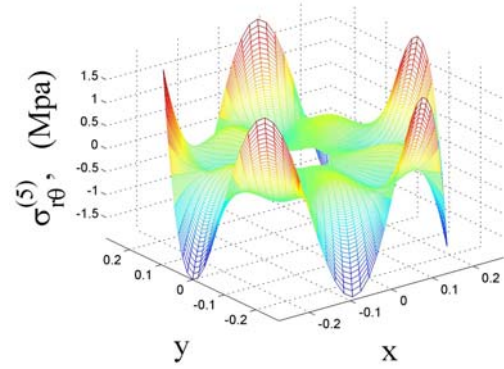
$k=2$



$k=3$



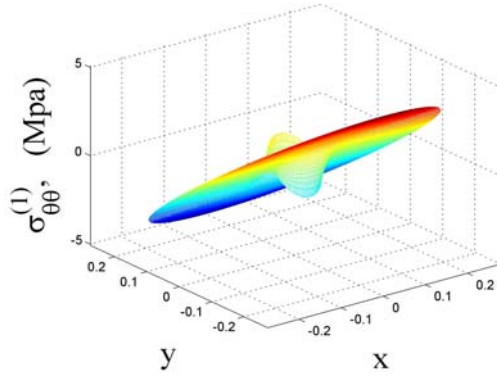
$k=4$



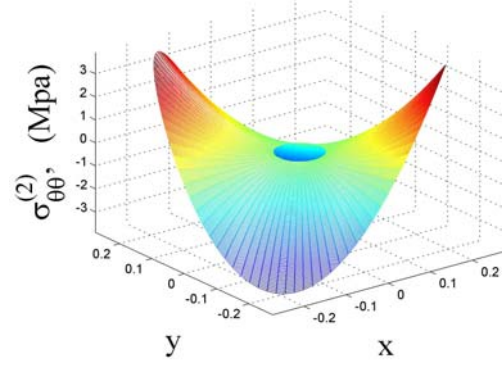
$k=5$

Fig. 6. 4: Asymmetric in-plane shear stress field, $\sigma_{r\theta}$, resulting from in-plane edge load:

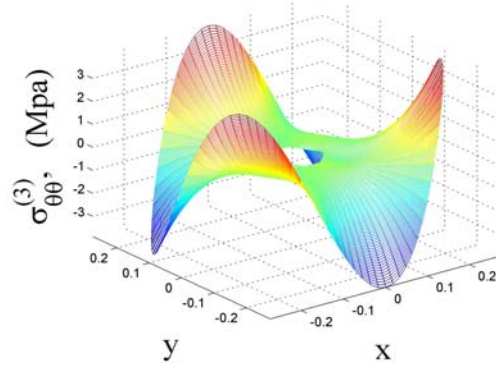
$$N_1=31, N_2=155, a=274.54mm, h=0.05a, P=800hp@\Omega_1=5000rpm.$$



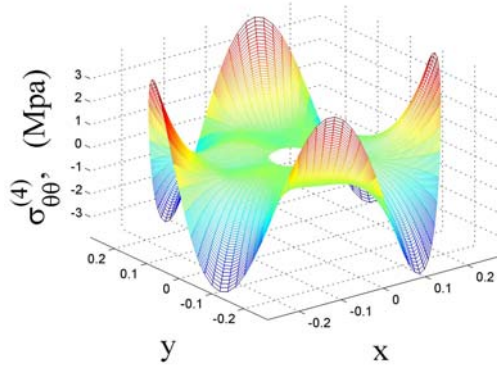
$k=1$



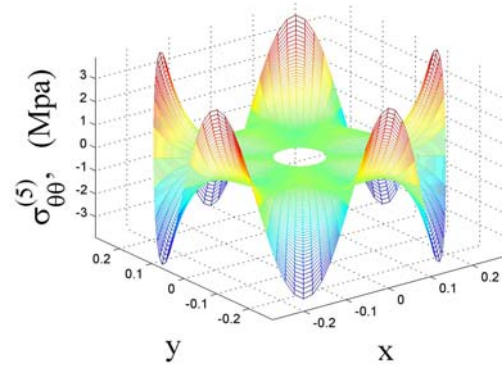
$k=2$



$k=3$



$k=4$

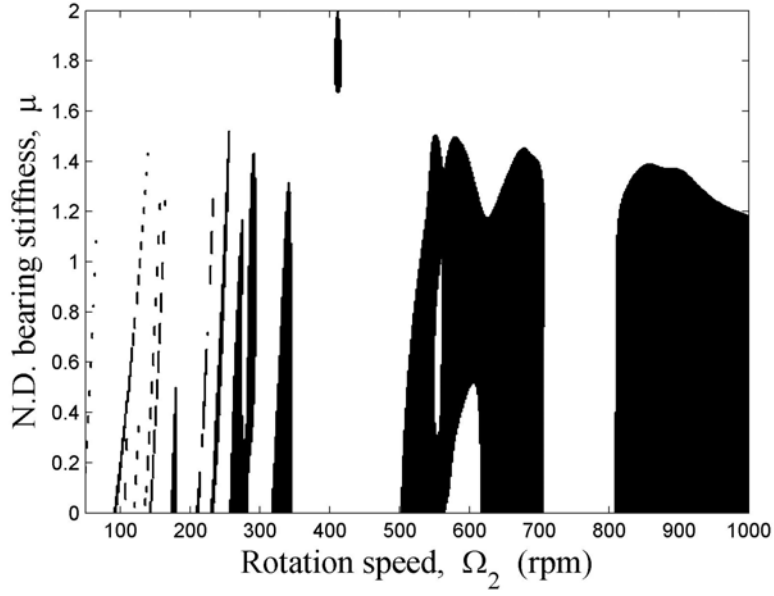


$k=5$

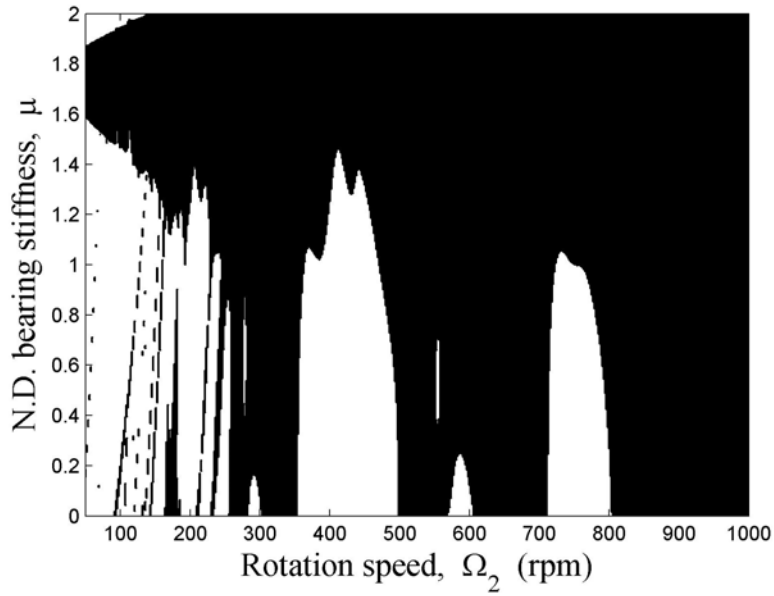
Fig. 6. 5: Asymmetric in-plane circumferential stress field, $\sigma_{\theta\theta}$, resulting from in-plane edge load: $N_1=31$, $N_2=155$, $a=274.54mm$, $h=0.05a$, $P=800hp@\Omega_1=5000rpm$.

Considering the computation efficiency, only first six Fourier components of the asymmetric in-plane stress fields are used in the transverse stability analysis, $k=0\sim5$. The power level is linearly proportional to the rotation speed with a constant slope, that is to say, constant in-plane driving force. The stability results dependent on the rotation speed and N.D. pinion bearing stiffness for the *Case I* and *Case II* are plotted in Figure 6.6 and Figure 6.7 respectively at various power levels.

The maximum power levels are set as [100 hp, 300 hp, 600 hp]@ $\Omega_1=5000$ rpm for *Case I* (Figure 6.6) and [600 hp, 800 hp, 1000 hp]@ $\Omega_1=5000$ rpm for *Case II* (Figure 6.7). Comparing with the stability results without considering the in-plane driving load in Figure 4.4(a) and Figure 4.5(a), it is indicated that high power level enlarges the unstable areas. The results also show that the *Case II* is able to bear larger in-plane load than the *Case I* based on the stability deterioration. This is because: the gear module is the same in both cases, so the pinion with larger tooth count has bigger radius; according to Eq. (4-21), the in-plane force is inversely related to pinion radius, consequently, the *Case I* suffers a higher in-plane force than the *Case II* for the same power level and rotation speed; secondly, the face-gear body thickness in *Case I* is smaller than the one in *Case II*, which yields a higher in-plane boundary stress level in *Case I* under the same in-plane force.

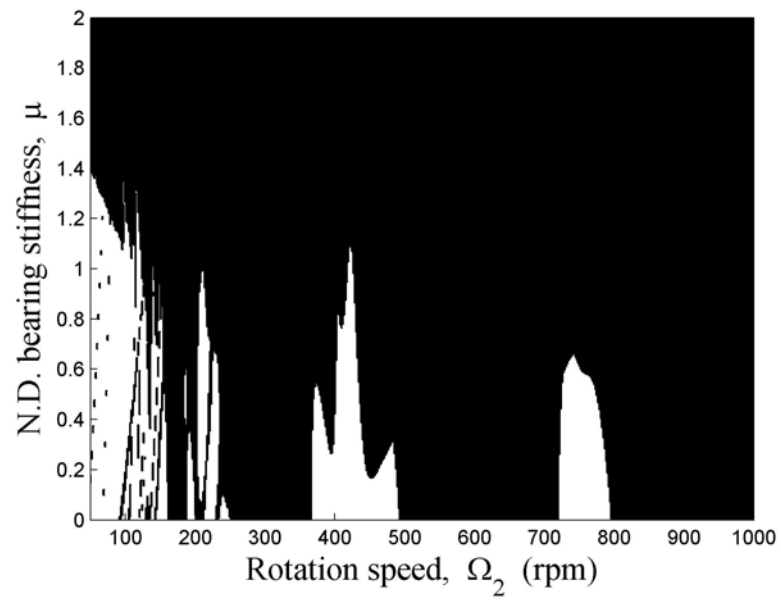


(a) $P=100$ hp@ $\Omega_I=5000$ rpm



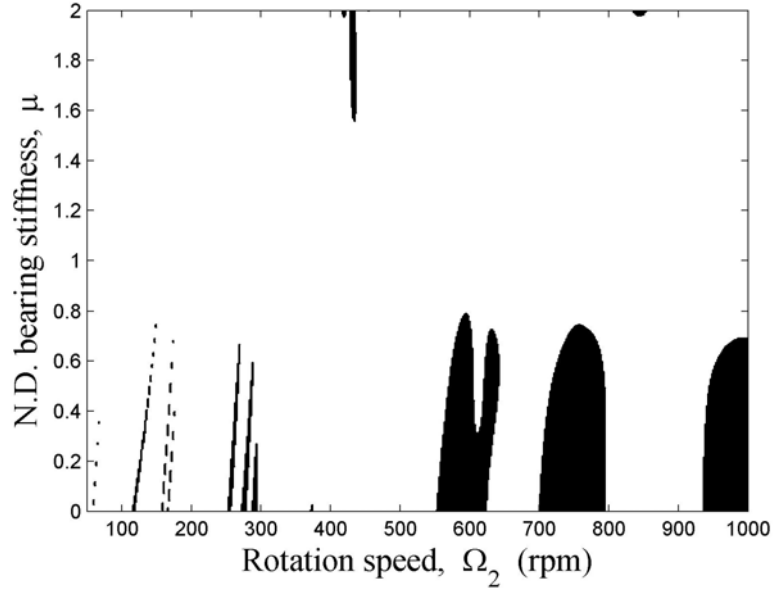
(b) $P=300$ hp@ $\Omega_I=5000$ rpm

Fig. 6. 6: Face-gear/pinion system parametric instability considering in-plane drive load,
Case I: $N_1 = 26$, $N_2 = 130$, $a = 231.94$ mm, $h = 0.03a$, $cr = 1.5532$, $\tau = 1$, $\zeta = 3 \times 10^{-6}$ sec.

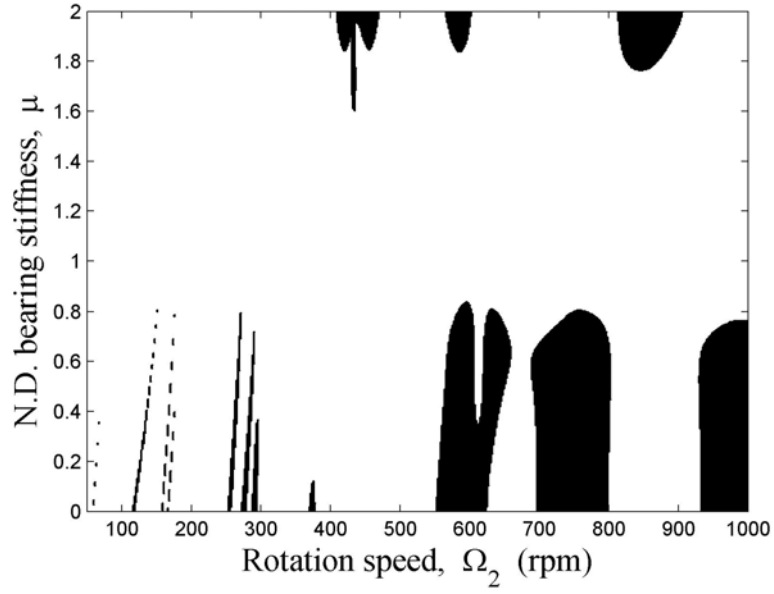


(c) $P=600$ hp@ $\Omega_1=5000$ rpm

Fig. 6. 6: Continued

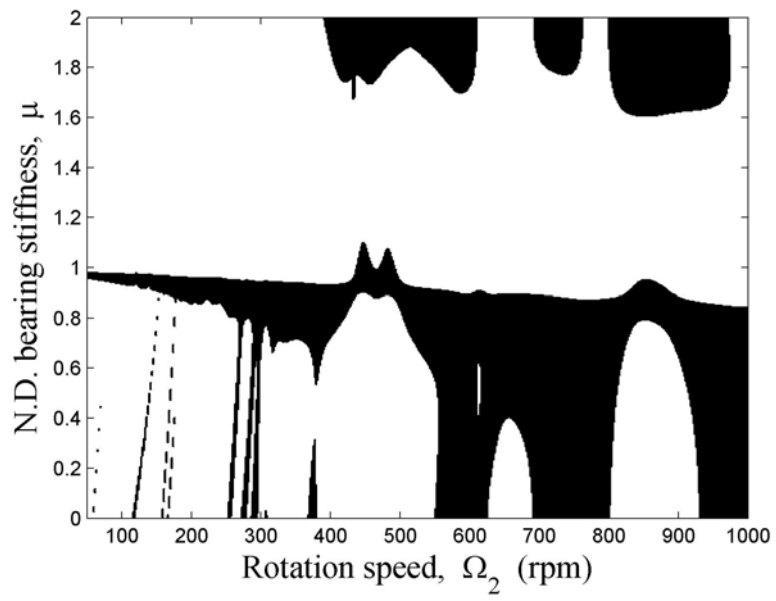


(a) $P=600$ hp@ $\Omega_1=5000$ rpm



(b) $P=800$ hp@ $\Omega_1=5000$ rpm

Fig. 6. 7: Face-gear/pinion system parametric instability considering in-plane drive load,
Case II: $N_1 = 31$, $N_2 = 155$, $a = 274.54$ mm, $h = 0.05a$, $cr = 1.5668$, $\tau = 1$, $\zeta = 1 \times 10^{-6}$ sec.



(c) $P=1000$ hp@ $\Omega_1=5000$ rpm

Fig. 6. 7: Continued

6.5 Summary

The out-of-plane component of the mesh load are modeled as a load unit normal to the face-gear body in chapter 4, but the in-plane component is neglected in the transverse vibration model. However, the in-plane driving force is also able to affect the system transverse stability via the asymmetric in-plane stress fields. This chapter improves the previous face-gear structural dynamics model by considering the in-plane load. The in-plane load is the driving force from the pinion and this load is modeled as an in-plane concentrated tangential edge force acting on the outer edge of the face-gear body. The asymmetric stress field due to in-plane edge force is analytically derived as Fourier series based on Chen and Jhu's work in [133-134]. Next, a stiffness matrix resulting from the in-plane force is added to the system governing equation for stability analyses.

The stability plots show that the in-plane effects enlarge the original unstable areas by comparing with previous results neglecting the in-plane component of the mesh load. The in-plane drive force is proportional to the input power level, namely, the system stability can be affected by the input power through the corresponding in-plane driving force and asymmetric in-plane stress fields.

In a word, it is necessary to take into account input power or in-plane driving force as a stability constraint in stability based designs besides serving as a stress constraint for spur pinion.

Chapter 7

TORSIONAL VIBRATION OF FACE-GEAR DRIVE SYSTEMS

7.1 Overview

Most research on gear torsional vibration focuses on spur gear [48-49, 52] and bevel gear [55-56], but the result for face-gear is scarce. Hereinto, the dynamic instability excited by the fluctuation of mesh stiffness due to a non-unity contact-ratio was observed and modeled for spur gears in [48]. This type of instability mechanism also exists in face-gear drives. The unique meshing kinematics is the other significant source of face-gear instability. In order to improve dynamic performance and reduce noise, it is necessary to investigate the torsional vibration due to the unique face-gear meshing kinematics and non-unity contact ratio kinematics.

This chapter is to establish a structural dynamics model for torsional vibration of gearboxes containing a face-gear drive by taking into account the flexibilities of gear teeth and transmission shafts. The system torsional stability is explored based on this model. The same as in previous transverse vibration models, the face-gear and pinion are cut for line contact mode and the mesh load is assumed as a concentrated one acting on the midpoint of each contact line (*contact point*). The simplification strategy that approximates the movement path of contact point as a skew line for each individual

meshing tooth pair is employed here to analyze the face-gear torsional vibration. This strategy is detailedly described in section 3.7.

7.2 Mesh Stiffness Evaluation

Following the works in [148-149], the tooth stiffness is estimated by considering the tooth as a tip loaded cantilever beam with average tooth dimensions, shown in Figure 7.1. The tooth stiffness, k_t , along the line of action is calculated as

$$k_t = \frac{E cp^3 L}{32(a_g + H)^3 \cos \alpha_c} \quad (7-1)$$

where α_c is pressure angle, E is Young's modulus, L is face width, cp is circular pitch, a_g is tooth addendum, and tooth dedendum is $H=1.25a_g$.

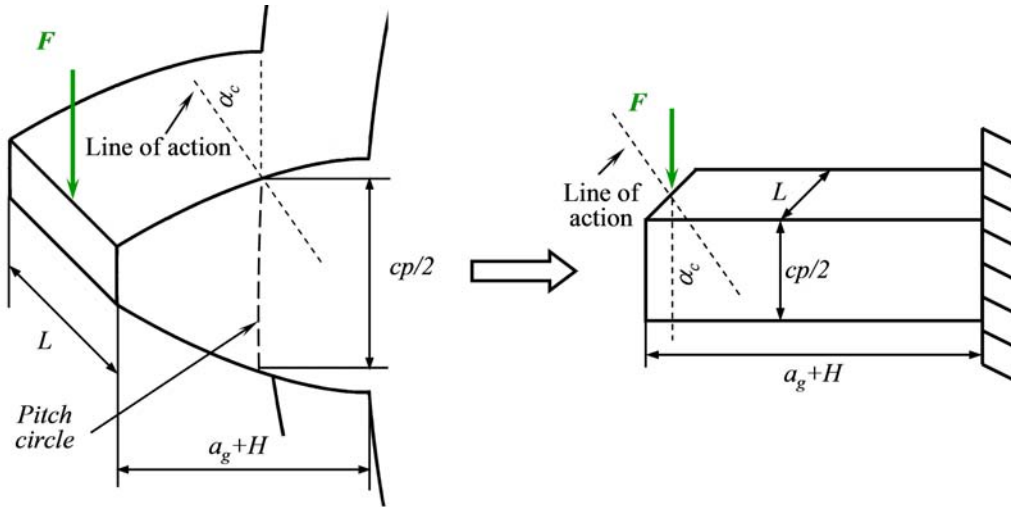


Fig. 7. 1: Tooth stiffness model; a tip loaded cantilever beam with average tooth dimensions.

The mesh stiffness is obtained by connecting two engaging tooth stiffnesses in series and its model is shown in Figure 7.2. Wang and Howard [150] indicated that the

case of one gear having two teeth in mesh simultaneously is possible to produce an up to 45 percent higher value than the actual one by stiffness parallel calculation. Moreover, the total mesh stiffness is further weakened by other cascading stiffnesses, such as gear body stiffness, bending stiffness and bearing stiffness. The average mesh stiffness for each engaging tooth pair, k_m , is given by [149]

$$k_m = cf \frac{1}{\frac{1}{k_t} + \frac{1}{k_i}} \quad (7-2)$$

with a correction factor, cf , to be determined by experiments or further calculations. All mesh stiffnesses must be included for a contact-ratio greater than one when multiple teeth pairs contact at the same time. They are calculated individually, and then superposed with the phase lag of integral mesh periods.

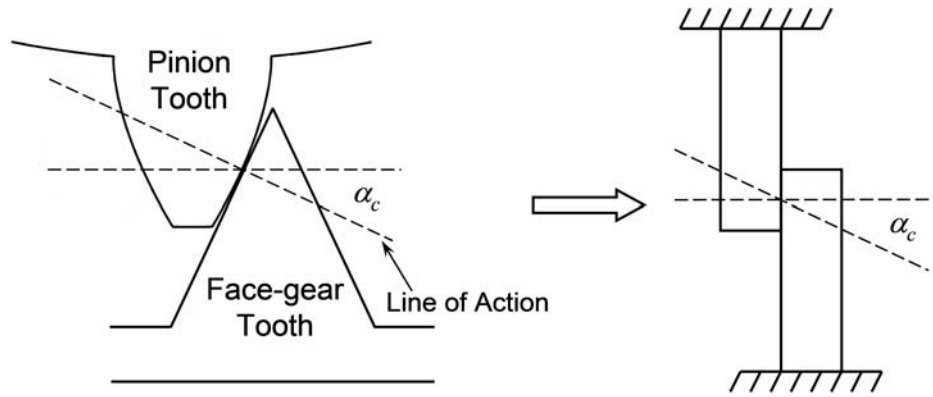


Fig. 7. 2: Mesh stiffness model [149].

In addition to mesh stiffness, the mesh deformation is also measured along the line of action, shown in Figure 7.3. The pinion end of the mesh stiffness moves along the line of action as the pinion rotation (point P in Figure 7.3); the face-gear end of the mesh stiffness is located on the face-gear tooth surface and oscillates with the contact point

along the face-gear radial direction (point Q in Figure 7.3). The radial variations of point Q is easily visualized from the top view of face-gear in Figure 7.4

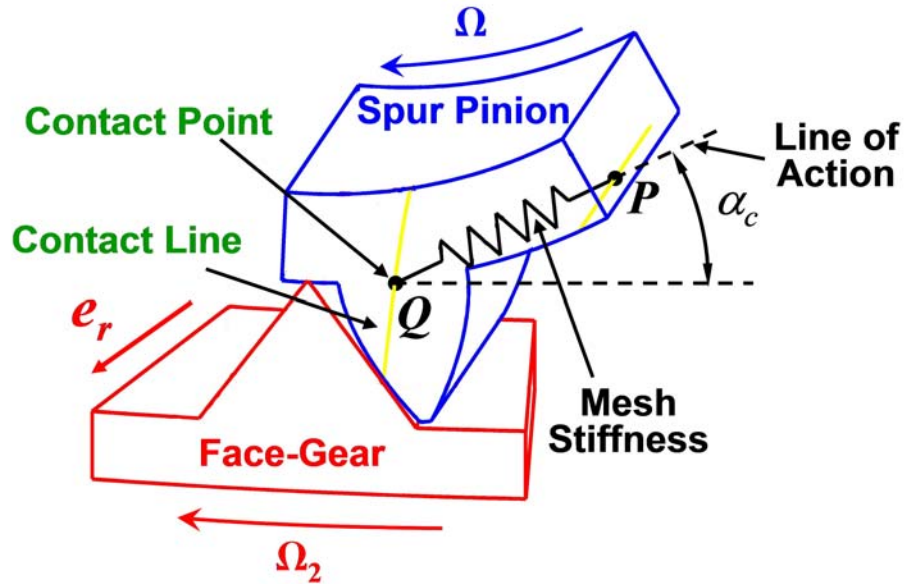


Fig. 7. 3: Mesh stiffness and mesh deformation.

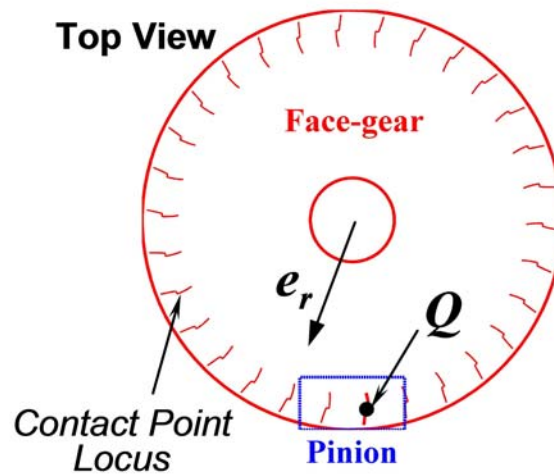


Fig. 7. 4: Top view of the face-gear end of the mesh stiffness.

The single side contact assumption is made here to keep the system linear. This means face-gear and pinion teeth are always compressed and never lose contact. Rigid

body rotation does not produce any mesh deformations since the face-gear is generated by conjugated meshing. The mesh deformation results only from gear vibrations, namely elastic deviation angles. The line of action is tangential to the pinion base circle because the pinion has involute tooth profile. For mesh deformation calculation, point P is extended backwards to the tangency point of the pinion base circle and the line of action, shown in Figure 7.5. The mesh deformation is equivalent to the length variation of PQ line produced by gear vibrations [149],

$$g_i(t_i) = \begin{cases} r_{b1}\psi_1(t_i) - r_k(t_i)\psi_2(t_i)\cos\alpha_c & 0 \leq t_i \leq crT_0 \\ \text{no contact} & crT_0 < t_i \leq T_0\text{ceil}(cr) \end{cases} \quad (7-3)$$

$$t_i = t + (i-1)T_0 \quad \text{for } i = 1, 2, \dots, \text{ceil}(cr)$$

where $g_i(t_i)$ is the deformation of the i^{th} mesh stiffness, r_{b1} is pinion base circle radius, $\psi_1(t_i)$, $\psi_2(t_i)$ are elastic deviation angles of pinion and face-gear from the nominal rigid-body rotation respectively and t is time. $r_k(t_i)$ fluctuates with gear rotation angle as a result of the face-gear meshing kinematics described in Eq. (3-22).

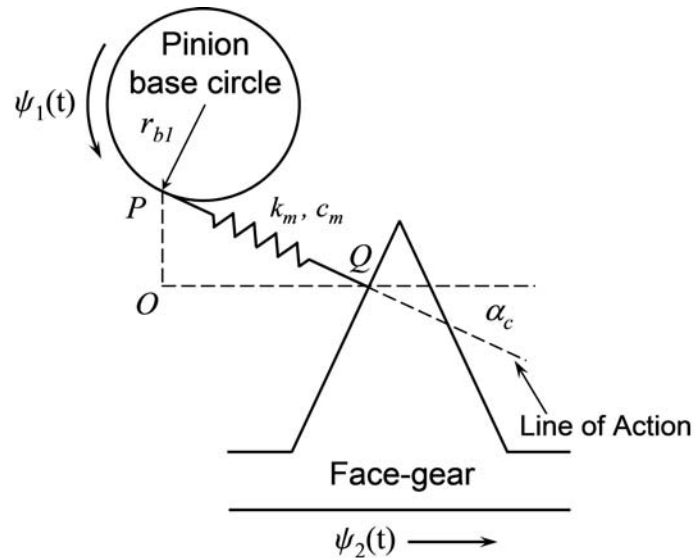


Fig. 7. 5: Mesh deformation due to elastic deviation angles [149].

7.3 Structural Dynamics Model for Torsional Vibration of Face-Gear Drive Systems

This model considers tooth flexibility as meshing stiffness, treats shafts as torsional stiffness and damping, and assumes both face-gear body and pinion body as rigid. The torque is transmitted from input shaft to load via pinion, face-gear and output shaft in sequence. The input speed keeps constant during operations, but the tooth flexibility and elastic twist of shafts make the rotation speed of the face-gear drive unsteady. Figure 7.6 and Figure 7.7 display the schematic diagram and the structural dynamics model of a gearbox containing a face-gear drive respectively.

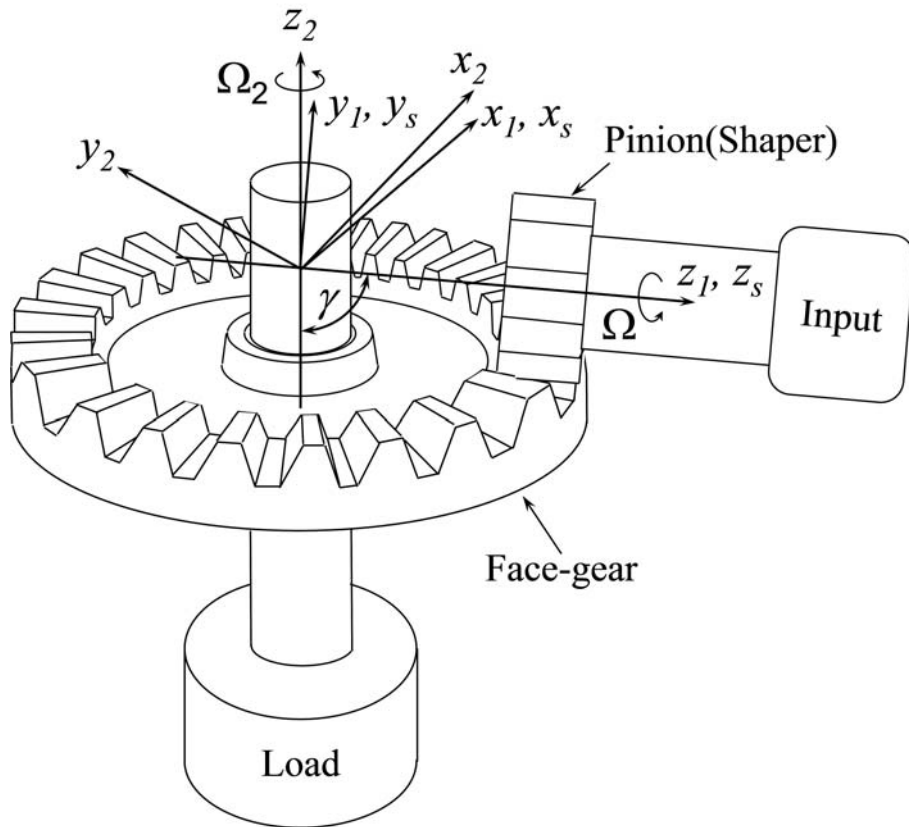


Fig. 7. 6: Schematic diagram of a gearbox containing a face-gear drive [149].

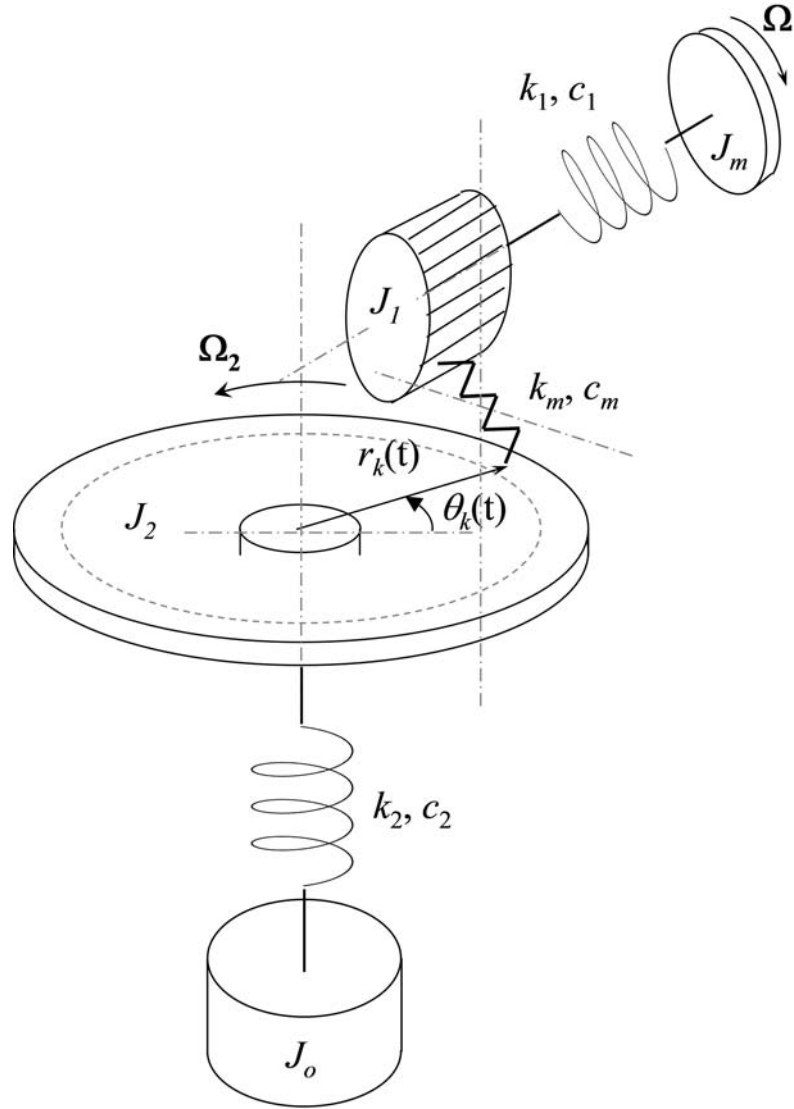


Fig. 7. 7: Structural dynamics model of a face-gear drive system [149].

The total system kinetic energy is [149]

$$T = \frac{1}{2} J_m \dot{\varphi}_m(t)^2 + \frac{1}{2} J_1 \dot{\varphi}_1(t)^2 + \frac{1}{2} J_2 \dot{\varphi}_2(t)^2 + \frac{1}{2} J_o \dot{\varphi}_o(t)^2 \quad (7-4)$$

where J_m, J_1, J_2, J_o are mass moment of inertia of input-side, pinion, face-gear and load respectively, $\varphi_m(t), \varphi_1(t), \varphi_2(t), \varphi_o(t)$ are the corresponding total rotation angles, and “ $\cdot$ ”

represents differentiation with respect to time t . The system strain energy, due to shaft flexibility and mesh deformation, is [149]

$$V = \frac{1}{2}k_1\psi_1(t)^2 + \frac{1}{2}k_2[\psi_2(t) - \psi_o(t)]^2 + \frac{1}{2}k_m \sum_i^{ceil(cr)} g_i(t_i)^2 \quad (7-5)$$

where k_1 and k_2 are torsional stiffnesses of input and output shafts respectively, k_m is mesh stiffness given in Eq. (7-2), and $\psi_1(t)$, $\psi_2(t)$, $\psi_o(t)$ are elastic deviation angles of pinion, face-gear and load from the nominal rigid-body rotation respectively. The Rayleigh dissipation function is [149]

$$R_D = \frac{1}{2}c_1\dot{\psi}_1(t)^2 + \frac{1}{2}c_2[\dot{\psi}_2(t) - \dot{\psi}_o(t)]^2 + \frac{1}{2}c_m \sum_i^{ceil(cr)} \dot{g}_i(t_i)^2 \quad (7-6)$$

where c_1 , c_2 , c_m are respectively damping coefficients of input shaft, output shaft and mesh deformation. The relationships between total angles and the elastic deviation angles are given based on pinion rotation speed, Ω , and gear ratio $m_{21}=N_1/N_2$,

$$\begin{aligned} \varphi_m(t) &= \Omega t \\ \varphi_1(t) &= \Omega t + \psi_1(t) \\ \varphi_2(t) &= m_{21}\Omega t + \psi_2(t) \\ \varphi_o(t) &= m_{21}\Omega t + \psi_o(t) \end{aligned} \quad (7-7)$$

The system equations-of-motion are derived from the energy expressions above via Lagrange's Equations

$$\frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}} \right] - \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} + \frac{\partial R_D}{\partial \dot{q}} = 0 \quad (7-8)$$

with generalized coordinates vector

$$q(t) = [\psi_1(t) \quad \psi_2(t) \quad \psi_o(t)]^T \quad (7-9)$$

The resulting equations are

$$\mathbf{M}\ddot{q}(t) + \mathbf{C}(t)\dot{q}(t) + \mathbf{K}(t)q(t) = 0 \quad (7-10)$$

where the coefficient matrices are [149]

$$\mathbf{M} = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_o \end{bmatrix} \quad (7-11)$$

$$\mathbf{C}(t) = \mathbf{C}_0 + \sum_i^{\text{ceil}(cr)} \mathbf{C}_i(t_i)$$

$$\mathbf{C}_0 = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & -c_2 \\ 0 & -c_2 & c_2 \end{bmatrix} \quad (7-12)$$

$$\mathbf{C}_i(t_i) = c_m \begin{bmatrix} r_{b1}^2 & -r_{b1}r_k(t_i)\cos\alpha_c & 0 \\ -r_{b1}r_k(t_i)\cos\alpha_c & r_k^2(t_i)(\cos\alpha_c)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{K}(t) = \mathbf{K}_0 + \sum_i^{\text{ceil}(cr)} \mathbf{K}_i(t_i)$$

$$\mathbf{K}_0 = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}$$

$$\mathbf{K}_i(t_i) = c_m \begin{bmatrix} 0 & -r_{b1}\dot{r}_k(t_i)\cos\alpha_c & 0 \\ 0 & r_k(t_i)\dot{r}_k(t_i)(\cos\alpha_c)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (7-13)$$

$$+ k_m \begin{bmatrix} r_{b1}^2 & -r_{b1}r_k(t_i)\cos\alpha_c & 0 \\ -r_{b1}r_k(t_i)\cos\alpha_c & r_k^2(t_i)(\cos\alpha_c)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, $t_i = t + (i-1)T_0$ is defined in Eq. (7-3). It should be noted that the mesh stiffness and viscoelastic damping of an engaging tooth pair contribute nothing to the system after this pair goes out of contact. That is to say, $\mathbf{K}_i(t_i)$ and $\mathbf{C}_i(t_i)$ are replaced by zeros when $t_i > crT_0$.

In order to use Floquet theory to determine the stability of the periodically time-varying system described by Eq. (7-10), the system equation is recast into the standard state-space form as:

$$\dot{\mathbf{X}} = \mathbf{A}(t)\mathbf{X} \quad \text{with} \quad \mathbf{X} = \begin{bmatrix} q(t)^T & \dot{q}(t)^T \end{bmatrix}^T \quad (7-14)$$

and

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K}(t) & -\mathbf{M}^{-1}\mathbf{C}(t) \end{bmatrix}_{6 \times 6} \quad (7-15)$$

When the system damping and the time-varying components of the system stiffness are much smaller than the constant component of the system stiffness, the perturbation method is employed to predict the system stability boundary analytically. All time-varying coefficient matrices in Eq. (7-10) are expanded as Fourier series [149],

$$\begin{aligned} \overline{\mathbf{U}}^{(0)} &= \mathbf{K}_0 + \frac{1}{T_0} \int_0^{T_0} \sum_i^{\text{ceil}(cr)} \mathbf{K}_i(t_i) dt \\ \overline{\mathbf{D}}^{(s)} &= \frac{2}{T_0} \int_0^{T_0} \sum_i^{\text{ceil}(cr)} \mathbf{K}_i(t_i) \cos s\omega t dt \\ \overline{\mathbf{E}}^{(s)} &= \frac{2}{T_0} \int_0^{T_0} \sum_i^{\text{ceil}(cr)} \mathbf{K}_i(t_i) \sin s\omega t dt \\ \overline{\mathbf{F}}^{(0)} &= \mathbf{C}_0 + \frac{1}{T_0} \int_0^{T_0} \sum_i^{\text{ceil}(cr)} \mathbf{C}_i(t_i) dt \\ \overline{\mathbf{F}}^{(s)} &= \frac{2}{T_0} \int_0^{T_0} \sum_i^{\text{ceil}(cr)} \mathbf{C}_i(t_i) \cos s\omega t dt \\ \overline{\mathbf{G}}^{(s)} &= \frac{2}{T_0} \int_0^{T_0} \sum_i^{\text{ceil}(cr)} \mathbf{C}_i(t_i) \sin s\omega t dt \end{aligned} \quad (7-16)$$

Next, the constant component of the system stiffness is diagonalized by [149]

$$\begin{aligned}
\mathbf{M}^{-1}\bar{\mathbf{U}}^{(0)}\mathbf{X}_R &= \mathbf{X}_R\mathbf{\Lambda} \\
\mathbf{X}_L^T\mathbf{X}_R &= \mathbf{I} \\
\mathbf{\Lambda} &= \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix}
\end{aligned} \tag{7-17}$$

where $\mathbf{X}_R$ and $\mathbf{X}_L$ are right and left eigenvectors respectively, and $\mathbf{\Lambda}$ is diagonal matrix of eigenvalues. Other coefficient matrices are transformed at the same time as

$$\begin{aligned}
\mathbf{U}^{(0)} &= \mathbf{X}_L^T\mathbf{M}^{-1}\bar{\mathbf{U}}^{(0)}\mathbf{X}_R = \mathbf{\Lambda} & \mathbf{D}^{(s)} &= \frac{1}{\varepsilon}\mathbf{X}_L^T\mathbf{M}^{-1}\bar{\mathbf{D}}^{(s)}\mathbf{X}_R \\
\mathbf{E}^{(s)} &= \frac{1}{\varepsilon}\mathbf{X}_L^T\mathbf{M}^{-1}\bar{\mathbf{E}}^{(s)}\mathbf{X}_R & \mathbf{F}^{(0)} &= \frac{1}{\varepsilon}\mathbf{X}_L^T\mathbf{M}^{-1}\bar{\mathbf{F}}^{(0)}\mathbf{X}_R \\
\mathbf{F}^{(s)} &= \frac{1}{\varepsilon}\mathbf{X}_L^T\mathbf{M}^{-1}\bar{\mathbf{F}}^{(s)}\mathbf{X}_R & \mathbf{G}^{(s)} &= \frac{1}{\varepsilon}\mathbf{X}_L^T\mathbf{M}^{-1}\bar{\mathbf{G}}^{(s)}\mathbf{X}_R
\end{aligned} \tag{7-18}$$

where ε is a small positive parameter. The resulting system equations-of-motion are in the standard form as Eq.(2-15)

$$\begin{aligned}
\ddot{q}(t) + \mathbf{U}^{(0)}q(t) &= -\varepsilon\left[\mathbf{F}^{(0)} + \sum_{s=1}^S\left(\mathbf{F}^{(s)}\cos s\omega t + \mathbf{G}^{(s)}\sin s\omega t\right)\right]\dot{q}(t) \\
&\quad -\varepsilon\left[\sum_{s=1}^S\left(\mathbf{D}^{(s)}\cos s\omega t + \mathbf{E}^{(s)}\sin s\omega t\right)\right]q(t)
\end{aligned} \tag{7-19}$$

The stability criterion is given in Eq. (2-26) when the mesh frequency approaches some combination of k^{th} and j^{th} natural frequencies with a small variation, $\omega = (\omega_k + \omega_j)/s + \varepsilon\lambda$.

Here, λ is a finite real number and $\varepsilon\lambda$ is a small quantity since ε is small.

The diagonal matrix $\mathbf{U}^{(0)}$ results from $\mathbf{K}_0$ and Fourier constant item of $\mathbf{K}_i(t_i)$. It should be noted that $\mathbf{U}^{(0)}$ is a function of rotation speed since the mesh damping related item in $\mathbf{K}_i(t_i)$ depends on the time derivative of $r_k(t_i)$, also on the rotation speed (see Eq.(7-13)). However, the speed effect is very small because the mesh damping is usually a small quantity and the face-gear works at low speed range as the output end of a

deceleration stage. To relate the rotation speed with the natural frequency, a new parameter is defined as the face-gear rotation speed that produces a mesh frequency equal to one of the system natural frequencies, named “*natural speed*” and denoted by Ω_{pn} [149].

$$\Omega_{pn} \equiv \omega_n \times \frac{60}{2\pi N_2} (rpm) \quad \text{for } n = 1, 2, 3 \quad (7-20)$$

7.4 Torsional Stability Results

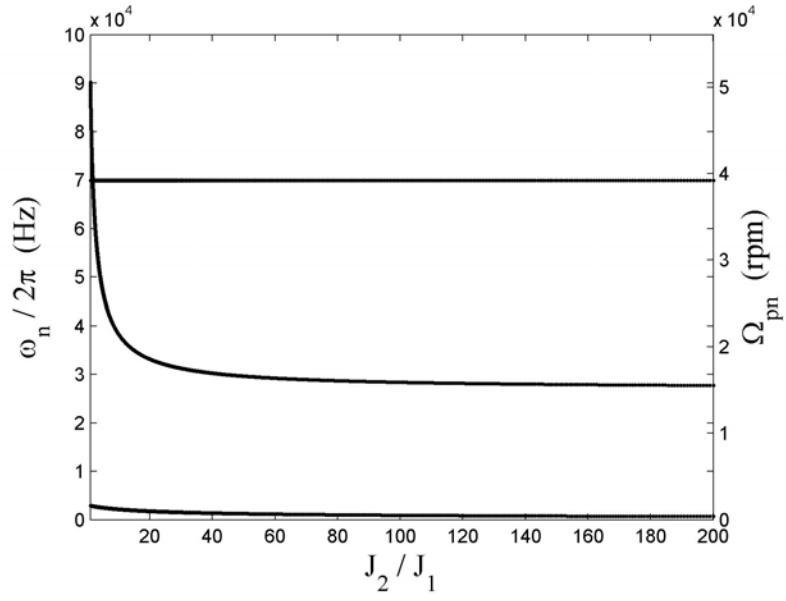
The stability of a sample spur-pinion/face-gear system is explored based on the system inertia distributions and shaft characteristics. The dynamic instability is investigated for various mass moment of inertia distributions among pinion, face-gear and load. The decrease of the face-gear inertia indicates a possible lightweight design. The effects of shaft stiffness and damping on the system stability over the entire operation speed range are calculated by Floquet theory. Subsequently, the perturbation method (Hsu’s method) predicts the stability boundaries approximately for the shaft damping cases and the results are compared with the full numerical analysis by the Floquet method. The parameters in this study are listed in Table 7.1.

Table 7. 1: Requirements and gear parameters [149].

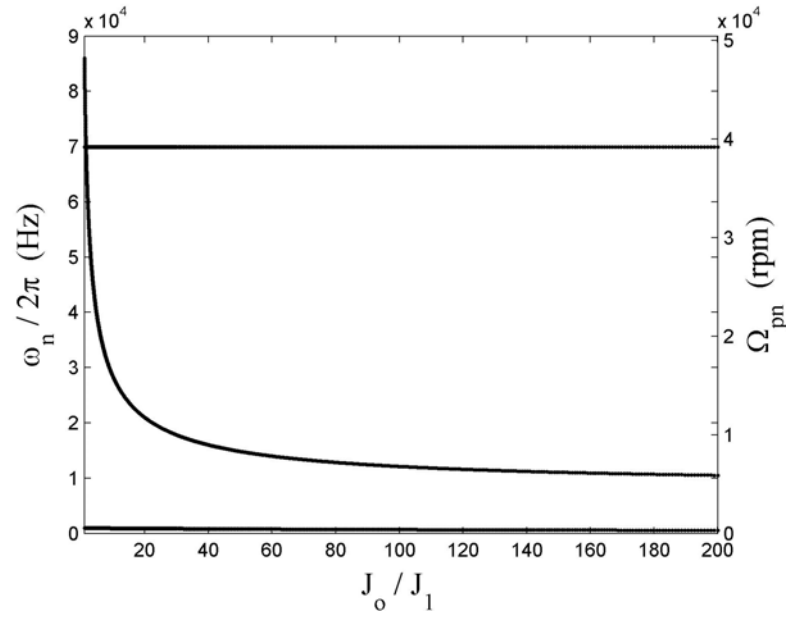
Parameter	Value
Output speed range, Ω_2	0 ~ 1000 (RPM)
Pinion number of teeth, N_1	31
Face-gear number of teeth, N_2	107
Module, m_d	3.175 (mm) [$p_d=8(1/\text{in})$]
Shaft angle, γ	90°
Pressure angle, α_{PA}	25°
Density, ρ	7840 (kg/m ³)
Young's modulus, E	200 (GPa)

7.4.1 Parametric Instability due to Mass Moment of Inertia

The time-varying stiffness resulting from the mesh damping disappears when the face-gear is stationary ($\Omega_2=0$). The stationary system natural frequencies are plotted respectively as the variations of the face-gear mass moment of inertia, J_2 , in Figure 7.8(a) and the load mass moment of inertia, J_o , in Figure 7.8(b). The mass moment of inertia of the pinion, J_I , is calculated by treating it as a solid cylinder. Its radius is the pitch radius and height is the tooth face width.



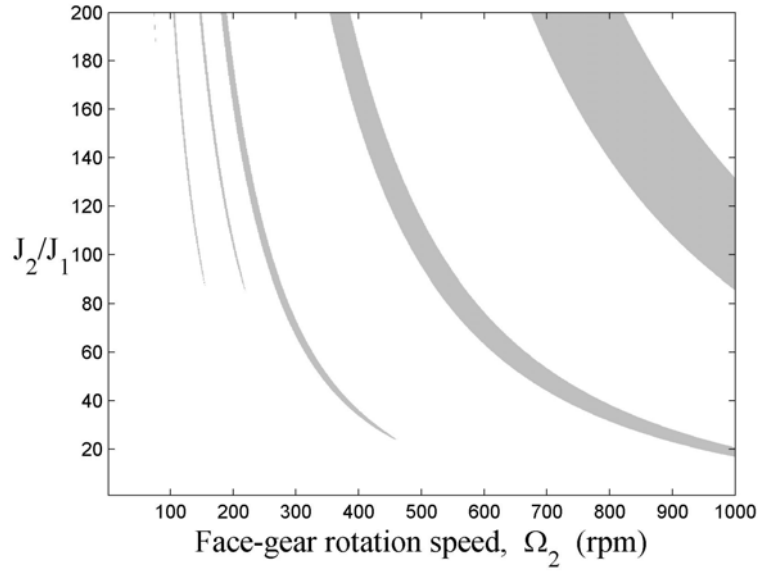
(a) $J_o = 10J_1$



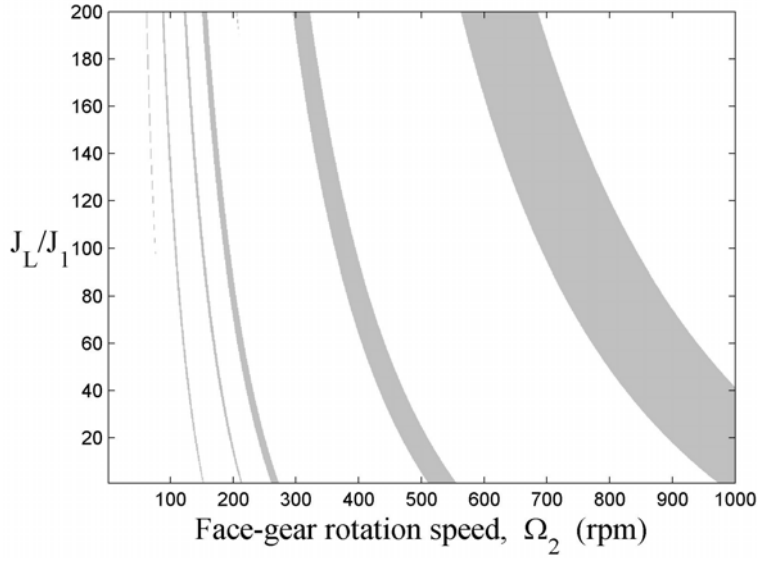
(b) $J_2 = 100J_1$

Fig. 7. 8: System natural frequencies VS face-gear and load mass moment of inertia:

$\Omega_2=0$, $h=0.05R_2$, $cf=0.5$, $J_1=1.6 \times 10^{-3} \text{ kg} \cdot \text{m}^2$, $k_m=1.58 \times 10^8 \text{ N/m}$, $k_l=3.15 \times 10^8 \text{ N} \cdot \text{m/rad}$, $k_2=4.73 \times 10^8 \text{ N} \cdot \text{m/rad}$ [149].



(a) $J_o=10J_l$



(b) $J_2=100J_l$

Fig. 7. 9: Face-gear system torsional instability due to face-gear and load mass moment of inertia and rotation speed: $h=0.05R_2$, $cf=0.5$, $J_l=1.6\times10^{-3}kg\cdot m^2$, $k_m=1.58\times10^8N/m$, $c_m=5\times10^{-6}(sec)k_m$, $k_l=3.15\times10^8N\cdot m/rad$, $c_l=5\times10^{-6}(sec)k_l$, $k_2=4.73\times10^8N\cdot m/rad$, $c_2=5\times10^{-6}(sec)k_2$ [149].

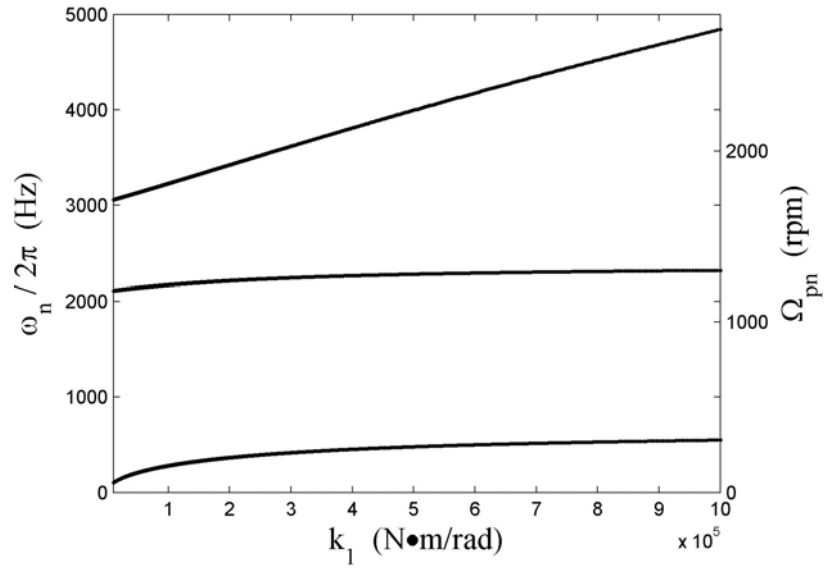
The dynamic instability results with respect to the face-gear and the load mass moment of inertia are displayed in Figures 7.9(a) and (b) respectively over the entire rotation speed range. Grey regions are used to indicate instabilities based on Floquet method in this chapter.

Figure 7.8 reveals that the first and third frequencies decrease slightly; while the second natural frequency decays dramatically when the face-gear or the load increase from 1 to 40 times the pinion mass moment of inertia. The increase in either the face-gear or the load mass moment of inertia shifts the unstable areas towards the low speed zone in Figure 7.9. This movement trend results from the reduced natural frequencies for higher face-gear or load mass moment of inertia.

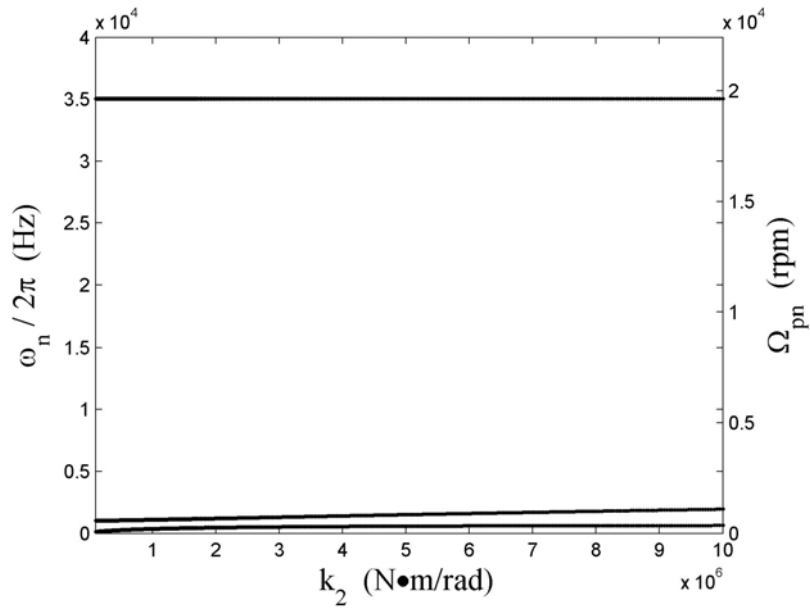
7.4.2 Parametric Instability due to Shaft Stiffness

Next, the effects of input and output shaft stiffnesses on the system stability are explored. The stationary system natural frequencies are shown in Figures 7.10(a) and (b) respectively for various input and output shaft stiffness levels.

Figure 7.11 records the stability results dependent on rotation speed and shaft stiffnesses. The increase of shaft stiffnesses raises the natural frequencies and also boosts the speed range of the instabilities, which is contrary to the inertia effects.



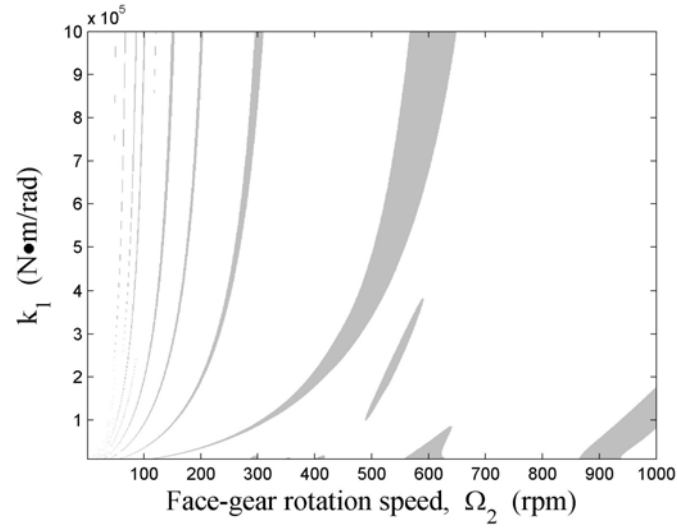
(a) Input shaft stiffness, $k_2=1.58 \times 10^7 \text{ N}\cdot\text{m/rad}$



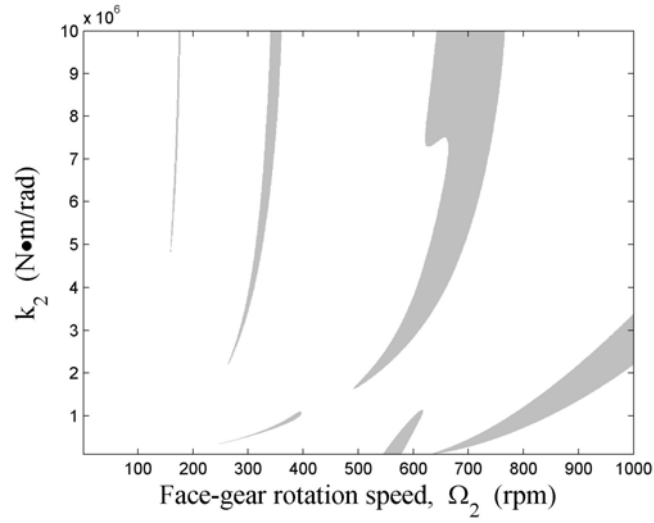
(b) Output shaft stiffness, $k_1=7.88 \times 10^7 \text{ N}\cdot\text{m/rad}$

Fig. 7. 10: System natural frequencies vs. shaft stiffness: $\Omega_2=0$, $h=0.05R_2$, $cf=0.5$,

$$J_1=1.6 \times 10^{-3} \text{ kg}\cdot\text{m}^2, J_2=91.97J_1, J_o=101.17J_1, k_m=1.58 \times 10^8 \text{ N/m} [149].$$



(a) Input shaft stiffness, $c_m=3 \times 10^{-6}(\text{sec})k_m$, $c_l=3 \times 10^{-6}(\text{sec})k_l$, $k_2=1.58 \times 10^7 \text{N} \cdot \text{m}/\text{rad}$, $c_2=3 \times 10^{-6}(\text{sec})k_2$



(b) Output shaft stiffness, $c_m=1 \times 10^{-5}(\text{sec})k_m$, $c_2=1 \times 10^{-5}(\text{sec})k_2$, $k_l=7.88 \times 10^7 \text{N} \cdot \text{m}/\text{rad}$, $c_l=1 \times 10^{-5}(\text{sec})k_l$

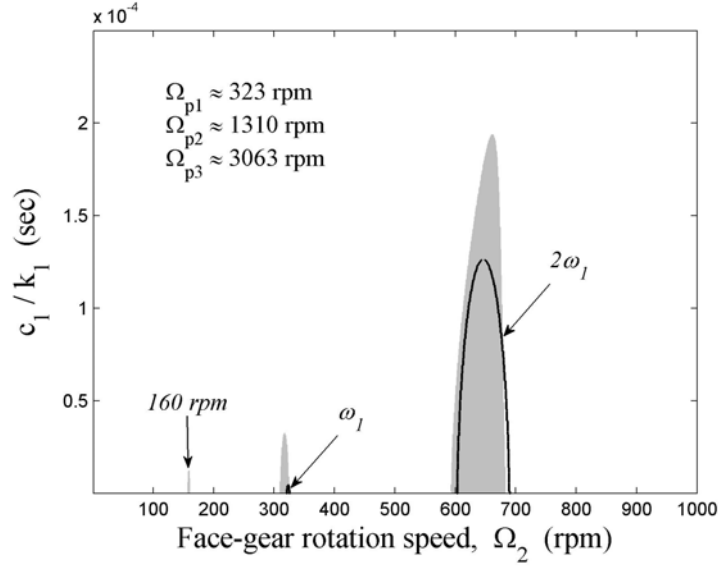
Fig. 7. 11: Face-gear system torsional instability due to shaft stiffnesses and rotation speed: $h=0.05R_2$, $c_f=0.5$, $J_1=1.6 \times 10^{-3} \text{kg} \cdot \text{m}^2$, $J_2=91.97J_1$, $J_o=101.17J_1$, $k_m=1.58 \times 10^8 \text{N}/\text{m}$ [149].

7.4.3 Parametric instability due to shaft damping

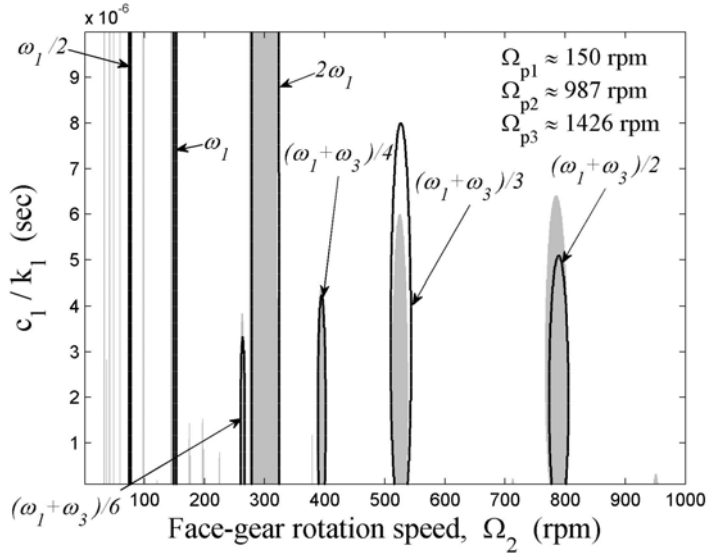
The damping of input and output shafts is proportional to the corresponding shaft stiffness in this chapter. The effects of shaft damping on the system stability are first investigated by Floquet method and the results are shown in Figures 7.12-7.13. Subsequently, the stability boundary is predicted approximately via perturbation method, marked by black lines in Figures 7.12-7.13.

The input shaft damping induced parametric instability is clearly shown in Figure 7.12. The perturbation method predicts the unstable regions of principal resonance (twice natural frequency) around 650 rpm and secondary resonance (natural frequency) near 320 rpm but fails to predict the subharmonic resonance close to 160 rpm in Figure 7.12(a). Furthermore, all predictions overestimate the stability safety. However, the principal, secondary, subharmonic and combination resonances are successfully indicated in Figure 7.12(b) because the later case has smaller mesh stiffness and damping so that the assumption of dominant constant component of stiffness is better satisfied.

The parametric instability resulting from the output shaft damping is displayed in Figure 7.13. The perturbation method finds the unstable regions of principal resonance around 650 rpm and secondary resonance near 320 rpm but fails to find the subharmonic resonance close to 155 rpm and the combination resonance around 570 rpm in Figure 7.13(a), and the principal resonance is estimated well. The same situation as in the input shaft damping case, the principal, secondary, subharmonic and combination resonances are successfully predicted in Figure 7.13(b) even though some results underrate the instability.



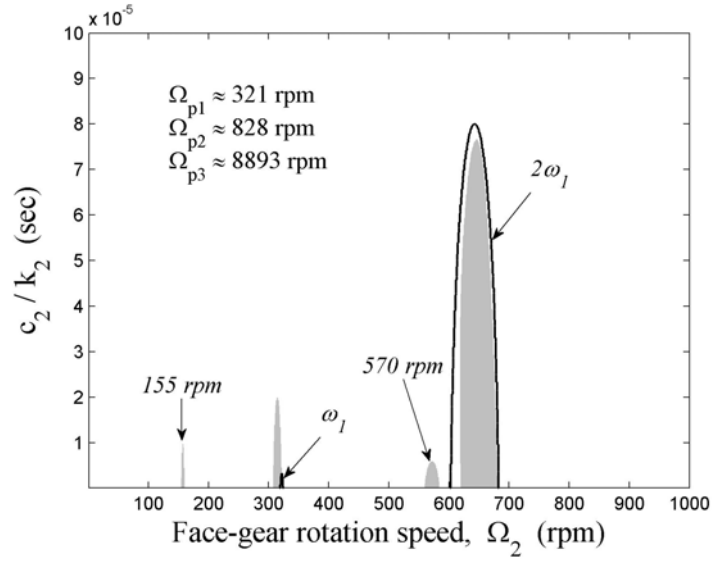
(a) $k_m = 1.58 \times 10^8 \text{ N/m}$, $k_l = 1.42 \times 10^6 \text{ N}\cdot\text{m/rad}$, $k_2 = 1.58 \times 10^7 \text{ N}\cdot\text{m/rad}$, $cf = 0.5$, $c_m = 1 \times 10^{-5}$
 $(\text{sec})k_m$, $c_2 = 1 \times 10^{-5}(\text{sec})k_2$



(b) $k_m = 3.15 \times 10^7 \text{ N/m}$, $k_l = 3.15 \times 10^5 \text{ N}\cdot\text{m/rad}$, $k_2 = 9.46 \times 10^6 \text{ N}\cdot\text{m/rad}$, $cf = 0.1$, $c_m = 2 \times 10^{-6}$
 $(\text{sec})k_m$, $c_2 = 2 \times 10^{-6}(\text{sec})k_2$

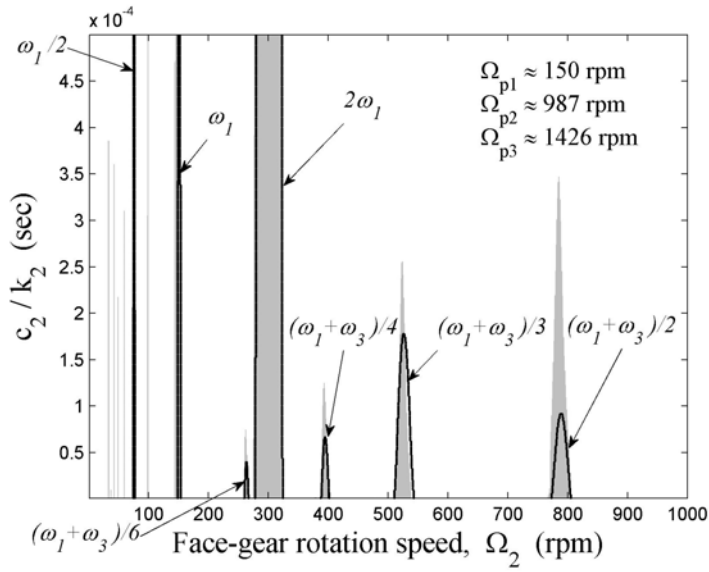
Fig. 7. 12: Face-gear drive-system torsional instability due to input shaft damping and

rotation speed: $J_1 = 1.6 \times 10^{-3} \text{ kg}\cdot\text{m}^2$, $J_2 = 91.97 J_1$, $J_o = 101.17 J_1$, $h = 0.05 R_2$ [149].



(a) $k_m = 1.58 \times 10^8 \text{ N/m}$, $k_l = 1.58 \times 10^7 \text{ N}\cdot\text{m/rad}$, $k_2 = 4.73 \times 10^6 \text{ N}\cdot\text{m/rad}$, $cf = 0.5$, $c_m = 1 \times 10^{-5}$

$(\text{sec})k_m$, $c_l = 1 \times 10^{-5} (\text{sec})k_l$



(b) $k_m = 3.15 \times 10^7 \text{ N/m}$, $k_l = 3.15 \times 10^5 \text{ N}\cdot\text{m/rad}$, $k_2 = 9.46 \times 10^6 \text{ N}\cdot\text{m/rad}$, $cf = 0.1$, $c_m = 2 \times 10^{-6}$

$(\text{sec})k_m$, $c_l = 2 \times 10^{-6} (\text{sec})k_l$

Fig. 7. 13: Face-gear drive-system torsional instability due to output shaft damping and

rotation speed: $J_l = 1.6 \times 10^{-3} \text{ kg}\cdot\text{m}^2$, $J_2 = 91.97 J_l$, $J_o = 101.17 J_l$, $h = 0.05 R_2$ [149].

7.5 Summary

This chapter establishes a structural dynamics model for the torsional vibration of the gearbox containing a face-gear drive. The parametric instability phenomena of the face-gear drive system are investigated by taking into account the unique face-gear meshing kinematics and the non-unity contact-ratio kinematics.

The dynamic instability results with respect to the mass moment of inertia distributions among pinion, face-gear and load are displayed. Passive vibration control can be implemented by changing the face-gear inertia. A lightweight design may be obtained based on a dynamic stability viewpoint if the face-gear inertia can be decreased. Consequently, it is possible to find the optimal design within the stable operation speed ranges.

The effects of input and output shaft characteristics, including stiffness and damping, on the system stability are also explored. This investigation provides necessary information to develop passive and active vibration suppression methods via adjusting the driveshaft stiffness and damping. For design purposes, the perturbation method is also given to calculate the system instability due to the variations of the shaft damping analytically.

Chapter 8

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE WORK

8.1 Summary and Conclusions

This dissertation presents a novel design strategy for face-gear transmission systems based on dynamic stability viewpoint. The structural dynamics models are established respectively for the transverse and torsional vibrations of face-gear drives to determine the system stability. Essentially, the dynamic instability results from the unique face-gear meshing kinematics and fluctuation of mesh load due to non-unity contact-ratio. The parametric study is conducted to disclose the relationships of system stability with geometrical dimensions and physical characteristics. To improve dynamic performance and reduce noise, it is required to sustain the system stability over entire operation speed range. This design constraint of stability may be satisfied by limiting the face-gear dimensions, changing the spatial configuration or adjusting the bearing and driveshaft characteristics.

The meshing kinematics is obtained via the analytical geometry for simulating face-gear generation process. The inner and outer radii of face-gear tooth are limited by nonundercutting condition and pointing condition respectively. Tregold's approximation is employed for face-gear contact-ratio calculation in order to avoid complex numerical computations.

The transverse vibration model considers face-gear body flexibility, pinion mass and pinion bearing characteristics. Face-gear body is modeled as a rotating disk under centrifugal stiffening effects, while pinion and its bearing are treated as a mesh load unit coupling with the out-of-plane dynamics of the face-gear and having prescribed in-plane movements. An equivalent movement of tooth contact area is derived to contain the effects from all mesh load units when the contact-ratio is greater than one. All parameters that are able to affect system stability can be designed to avoid instability and satisfy with pinion tooth stress restriction. The limit of face-gear geometrical dimensions determines the system weight which is an important design parameter. In the stability based design, the requirements for weight and stress conflict with each other under a constant diametral pitch, because the lower stress level designs tend to be less stable and therefore require larger values of face-gear body thickness to sustain stability. Also, this trade off design depends on the desired operating speed range. However, the critical thicknesses for the system stability are almost the same for various tooth counts under a constant pinion pitch radius. So, the key is to find the pinion pitch radius that satisfies with the stress requirement in this design.

In addition, multi-pinion configuration and in-plane driving force induced instabilities are respectively explored for the transverse vibration of face-gear drives. The spatial angle between different pinions is designed to stabilize the system through changing the relative mesh phase difference. In order to upgrade the pre-installed systems without much design flexibility, it is required to redesign other gear parameters instead of pinions' included angle to sustain stability. Besides determining the pinion tooth stress level, the in-plane driving force also affects the system stability via the resulting in-plane

asymmetric stress field in face-gear body. Consequently, the input power and rotation speed are limited by both stability and stress design constraints. The overall diagram for transverse stability based designs is displayed in Figure 8.1. The parameters in round corner rectangles are design variables. The arrow line means that the parameters pointed by the arrow end are affected by the ones at the initial end of the arrow.

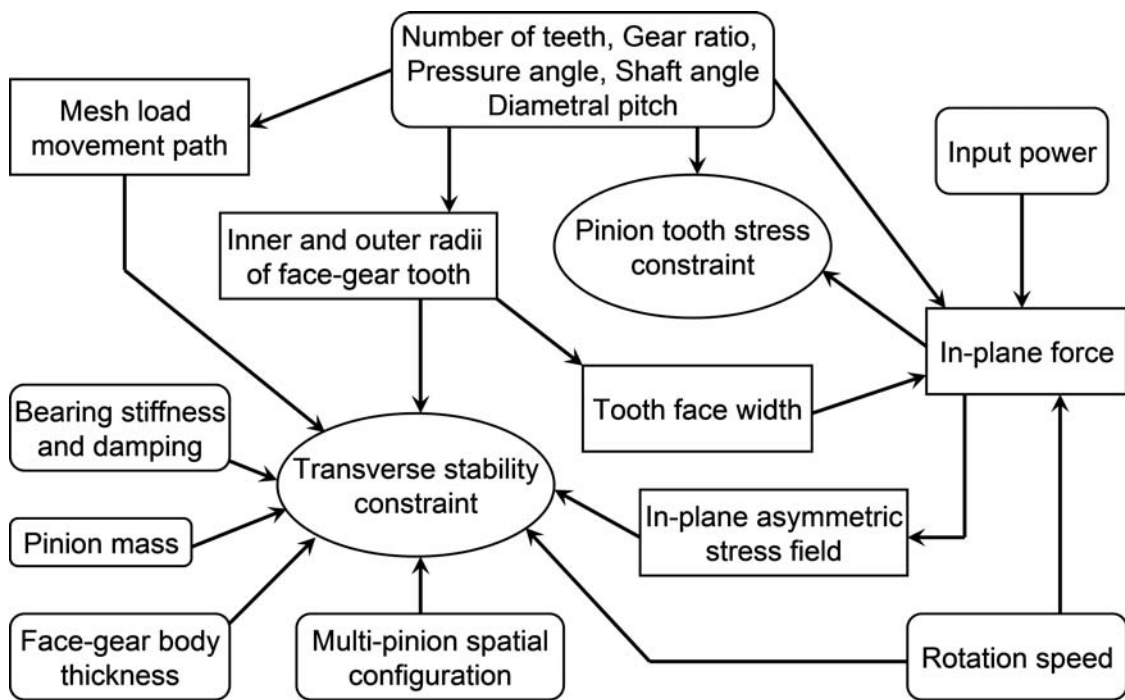


Fig. 8. 1: Transverse stability based design diagram.

The torsional vibration model comprises tooth flexibility, shaft elasticity and rotational inertia. The unique movement path of each mesh load on face-gear is approximated as a linear function to simplify the analysis. For the contact-ratio greater than one, all mesh loads must be included by the linear superposition with relative phase lags. The mass moment of inertia distributions among pinion, face-gear and load is able

to shift the parametric instabilities along the rotation speed axis and can also be designed to suppress vibration passively. The torsional stiffness of transmission shafts has opposite effects to the system stability relative to the system inertia. For design purposes, the instabilities due to the shaft damping are explored by both numerical and analytical methods. The stability results based on the shaft characteristics provide necessary information to design active and passive vibration control strategies via adjusting shaft damping and stiffness. The overall diagram for torsional stability based designs is displayed in Figure 8.2. The gear parameters, shaft characteristics and system inertia configuration are design variables.

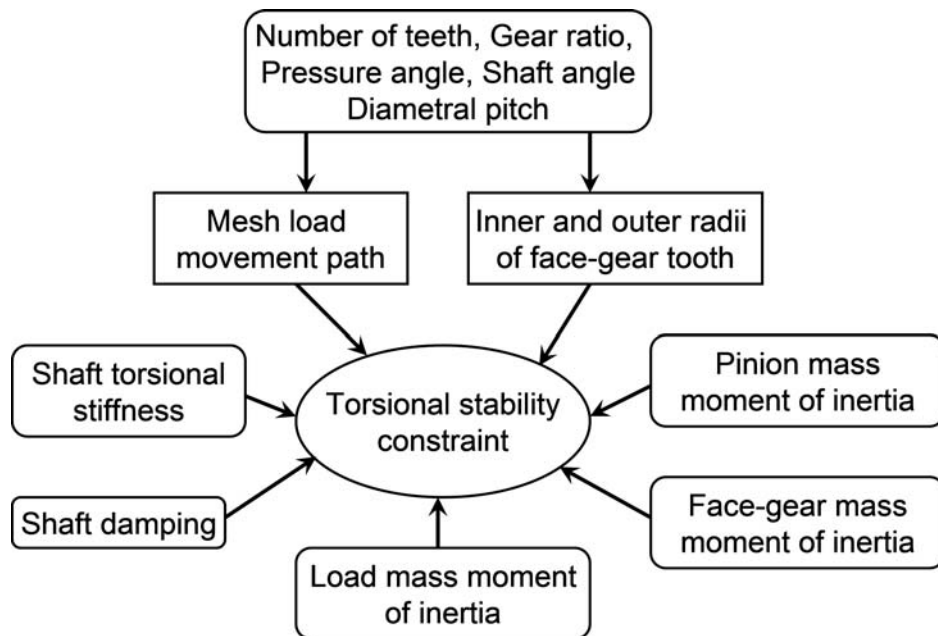


Fig. 8. 2: Torsional stability based design diagram.

The parametric instability studies on transverse and torsional vibrations of face-gear drive systems conducted in this research outline the novel stability based design

methodology for future rotorcraft gear transmission systems. The feasibility has also been demonstrated by this dissertation. This methodology advances the current state of the art and is an important complement to traditional static designs.

8.2 Recommendations for Future Work

One important topic for future investigations is to incorporate tooth stiffness with the existing transverse vibration model. This requires an additional degree-of-freedom to describe the displacement of pinion mass, w_m . The mesh stiffness, pinion mass and bearing stiffness are connected in series as a mesh load oscillator, shown in Figure 8.3. The system stability can be determined by combining the equations-of-motion of the spinning disk and the oscillator. Following this model, the Sikorsky RDS-21 drivetrain system can be analyzed by add two degree-of-freedom since its face-gears mesh with pinions at both sides (See Figure 1.11).

The dynamic load distribution among all meshing teeth is another important topic to help estimate the equivalent mesh position and mesh stiffness better. It is not a simple superposition of individual mesh stiffness for one gear having multiple teeth in mesh simultaneously. This dissertation employs the length of contact line as weight function to calculate the equivalent mesh position. To approach the reality, the experimental or finite element analyses need to be implemented to understand the effects of different meshing teeth on the dynamics of the mesh load oscillator.

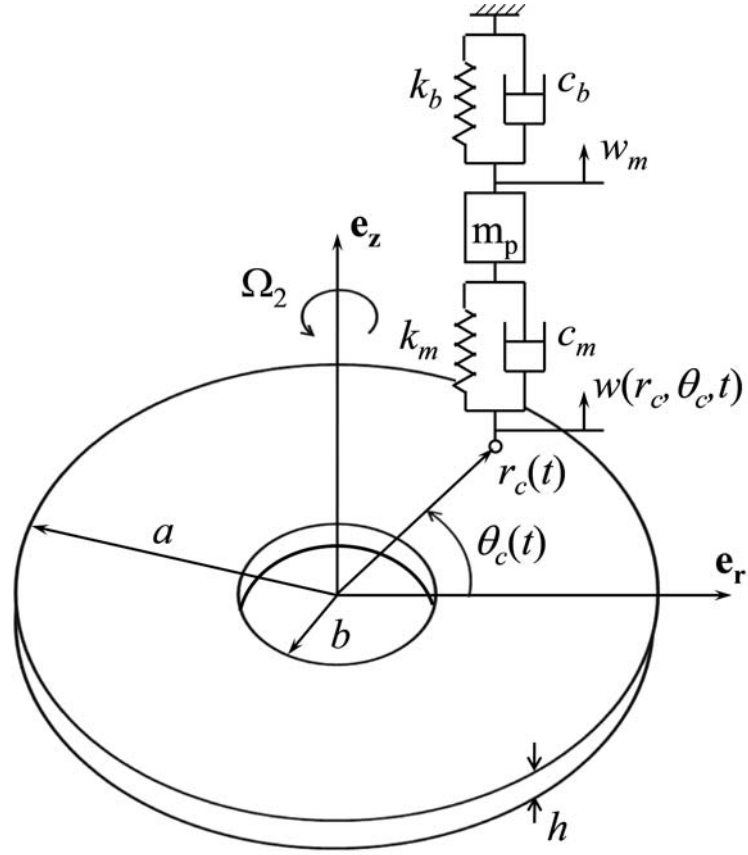


Fig. 8. 3: Face-gear transverse vibration model considering tooth flexibility: spinning disk with a mesh load oscillator.

The in-plane driving force acts on face-gear through its tooth surface, so the force from each meshing tooth actually distributes over the tooth width and tooth root thickness rather than a point. In addition, the system stability also depends on whether the force direction changes with the vibration of the acting position. The accurate stress field due to the in-plane force helps yield better results even though this force is not a major instability source according to previous literatures.

Following this dissertation, the split-torque face-gear system applied in Apache Block III Improved Drive System can be modeled and analyzed (See Figure 1.10). Besides multiple pinions, the system also comprises two face-gears. In this configuration, the relative meshing phase between pinions is zero and only positioning angle affects the system stability.

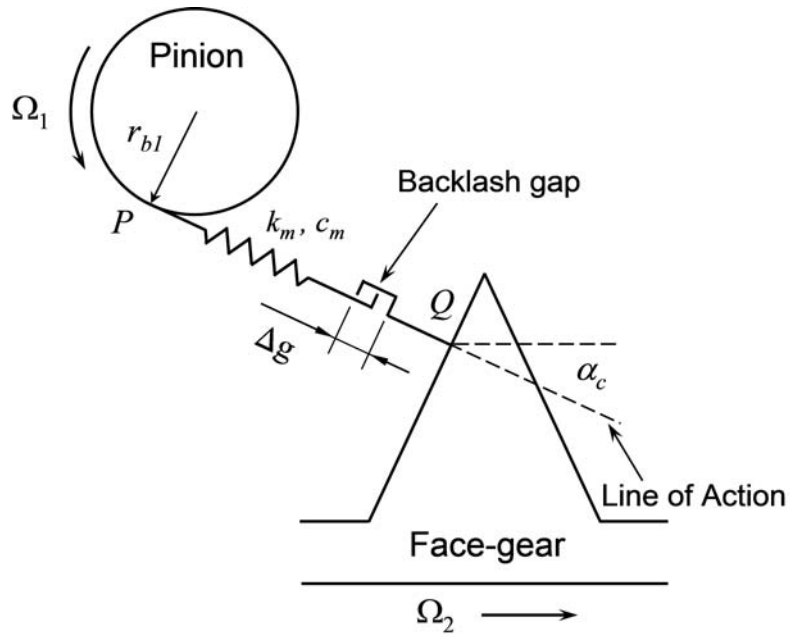


Fig. 8. 4: Nonlinear mesh deformation with backlash gap clearance.

For torsional vibration, the nonlinear backlash effect and shaft mass need to be considered to improve the model. The nonlinear backlash deformation is illustrated in figure 8.4. Double-side tooth contact is investigated in the analysis. The mesh deformation is greater than zero when the pinion tooth runs faster than the face-gear tooth. In this case, the pinion tooth presses the face-gear tooth **1** at the point **A**, as shown in Figure 8.5. The mesh deformation is less than zero when the pinion tooth moves slower

than the face-gear tooth. This means the pinion tooth presses the face-gear tooth **2** at the point **B** in Figure 8.5. It should be noted that the mesh phase of the contact point **A** is not equal to the mesh phase of the contact point **B**. Both backlash gap clearance and backlash mesh phase difference need to be considered in nonlinear analysis.

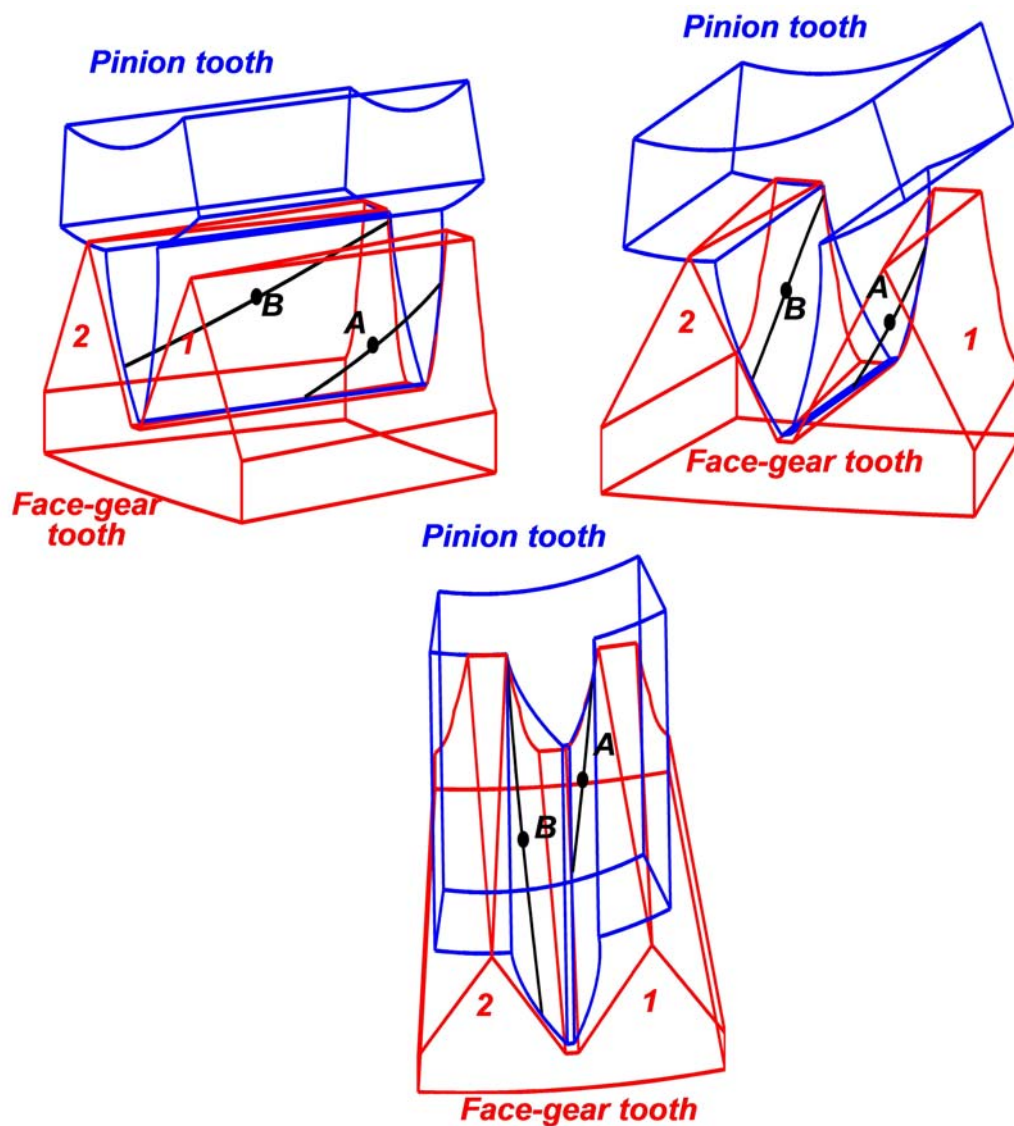


Fig. 8. 5: Double-side tooth contact and backlash effect.

By following the work in [49], the mesh deformation, $G_i(t_i)$, with the backlash effects is given

$$G_i(t_i) = \begin{cases} g_i(t_i + t_f) - \Delta g / 2 & g_i(t_i + t_f) > \Delta g / 2 \\ g_i(t_i - t_b) + \Delta g / 2 & g_i(t_i - t_b) < -\Delta g / 2 \\ 0 & \text{otherwise} \end{cases} \quad (8-1)$$

where Δg is backlash gap clearance, t_f is forward mesh phase offset and t_b is backward mesh phase offset. Based on the Fourier series form of the backlash deformation, the nonlinear system responses can be solved by the numerical arc-length continuation approach method [49].

The multiple degrees-of-freedom system becomes continuous one after including the dynamics of shaft mass, namely, shaft torsional vibration. These explorations provide the baseline to design and apply smart materials and structures.

Since this dissertation develops structural dynamics models of face-gear drives, it is able to obtain the system natural frequency and force response theoretically. This provides the basis for the natural frequency shift or response deviation based structural damage identification methods. The challenge is the time-varying characteristic. The system is periodic, so a nominal hyper-dimensional system matrix that is time invariant can be obtained by transforming the time-varying system matrix through a harmonic balance method [81, 151]. The damage perturbation matrix is inversely deduced from the observed variations of frequency and response via the nominal hyper-dimensional system matrix. Finally the severity and location/distribution of damages are detected and determined based on the damage perturbation matrix and the structural dynamics models.

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