To the Graduate Council:

I am submitting herewith a dissertation written by David F. Snyder entitled "Partially Acyclic Manifold Decompositions Yielding Generalized Manifolds". I have examined the final copy of this dissertation for form and content, and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Robert J. Daverman, Major Professor

We have read this dissertation
and recommend its acceptance:

Accepted for the Council:

Vice Provost
and Dean of The Graduate School

PARTIALLY ACYCLIC MANIFOLD DECOMPOSITIONS YIELDING GENERALIZED MANIFOLDS

A Dissertation

Presented for the

Doctor of Philosophy

Degree

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David F. Snyder

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DEDICATION

To my wife, Danielle,

and sons, Jyota and Ian.

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ABSTRACT

Certain classes of upper semicontinuous decompositions (uscd's) are investigated. The methods established in the literature of manifold decompositions are compared with methods arising from the theory of sheaves. A wide range of results from the theories of sheaf cohomology, generalized manifolds, and ANRs are discussed in order to develop proofs of the theorems below. Examples and applications are given.

DEFINITION: An upper semicontinuous decomposition G of a $(n+k)$–manifold is called a *codimension–k manifold decomposition* if each element of G is [a continuum having the shape of] a closed orientable n–manifold. We say G is *j–acyclic* if the reduced Čech homology of each element of G is trivial up through dimension j.

THEOREM. *Every non–degenerate $(k-2)$–acyclic codimension–k manifold decomposition G of an orientable $(n+k)$–manifold M $(3 \leq k \leq n+1)$ yields a k–manifold B as its decomposition space, if B is finite dimensional and if the set E (which consists of those points of B at which the Leray sheaf in dimension n of the decomposition map is not locally constant) does not locally separate B. Moreover, the set E is actually a locally finite subset of B.*

Addendum: *In the case $k = n+1$, the assumptions of non–degeneracy of G and of orientability of M are not necessary.*

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INTRODUCTION

Can a given $(n+k)$-manifold M be decomposed upper semicontinuously into closed submanifolds of dimension n? If such a decomposition exists, what are the topological properties of the decomposition space? What can be said about the structure of M? One expects that restriction of the decomposition elements to a class of manifolds with certain structure allows one to draw some rather strong conclusions about the decomposition space, as well as the decomposition map.

In particular, we mainly consider here decompositions whose elements are closed orientable n-manifolds with

$$\tilde{H}_i(g) = 0, \quad i \in \{0, 1, \ldots, k-2\},$$

where $2 < k < n+2$, extending Theorem 3.1 in [**DW4**], as well as weakly extending the main result in [**CD2**].

An *upper semicontinuous decomposition* (uscd) G of a metric space X is a collection of pairwise disjoint compact subsets which cover X so that for each neighborhood U of an element $g \in G$ the set $\cup\{g' |\ g' \in G,\ g' \subset U\}$ (called the *saturate* of U by G) is an open subset of X. The *decomposition space*, denoted X/G, is the quotient space of X under the relation $\sim$, where $x_1 \sim x_2$ if, and only if, there is $g \in G$ with $x_1, x_2 \in g$. The quotient map is called the *decomposition map*. This map is closed.

As usual, $\mathbf{Z}$ denotes the integers, $\mathbf{Z}_p$ denotes the integers modulo p. For the functors of Algebraic Topology we use the notation of [**S**] and [**Br1**], where the coefficient module is taken to be $\mathbf{Z}$ unless indicated otherwise. For a geometric intuition of the algebra [**F**] and [**BRS**] are highly recommended.

A *codimension-k manifold decomposition* will signify a uscd of an $(n+k)$-manifold M into subcontinua having the shape of closed orientable n-manifolds. If for $i < n$ we have $\tilde{H}_j(g) = 0$ for all $j \leq i$ we call G a *i-acyclic manifold decomposition.*

Note that a codimension-0 manifold decomposition is cell-like; the literature on this subject is vast. The interested reader is refered to [**L2**]. In all that follows the reader can think of the decomposition elements as being actual submanifolds without losing any understanding of the proofs.

To put the results and techniques of this paper into proper context, the literature pertaining to codimension-2 manifold decompositions, then higher codimensional decompositions, is reviewed. The reader is also pointed to the survey article by R.J. Daverman [**D4**] as an excellent information source.

In the codimension-2 case, I think it is important to point out certain classical results of the 1930's which use very specific circle decompositions of 3-manifolds to display the topological structure of either the decomposition space or the source (i.e. decomposed) space. In [**Ho**], H. Hopf presented certain maps from the 3-sphere to the 2-sphere, including the famous Hopf fibration. This latter map represents a generator of $\Pi_3(S^2)$. Soon after, H. Seifert [**Se**] developed his elegant theory of Seifert fibred spaces. Each of these fiberings is a uscd of a 3-manifold into circles, though for a decomposition theorist these are atypically rigid decompositions. However, because of this rigidity, these spaces are useful aids in the analysis of geometric structures on 2-manifolds (see [**Sc**] and [**BoSb**]), and have interesting geometric structures themselves as well [**Th**].

In the 1970's D.S. Coram and P.F. Duvall investigated decompositions of the 3-sphere into compacta having the shape of the circle; for example, they prove [**CD3**]

that the decomposition map is approximable by either the Hopf fibration or a Seifert fiber map (R.J. Daverman and J.J. Walsh proved the decomposition space to be the 2-sphere [**DW2**]). The techniques that Coram and Duvall applied are fundamental for following results in this area.

In particular, winding functions are primary tools of that investigation. Suppose G to be a uscd of a manifold M into closed orientable submanifolds of dimension n. Pick an element g of G. Since g is an ANR there is a neighborhood U of g in M which retracts by a map r to g.

For a given element $g' \subset U$ of G we define the *n-winding of g' about g* by $\mid (r|g')_*(1) \mid$, where $(r|g')_*$ is the map on the nth homology induced by r, which is an endomorphism of $\mathbf{Z}$. Let b, b', W denote the images under the decomposition map of g, g', and the saturate of U by G, respectively. Note that W is open in M/G. We define an *n-winding function at b* on W, denoted α_b; its value at b' is the n-winding of g' about g. This function is lower semicontinuous on W. A Baire category argument can then be used to show that it is continuous on a dense open subset of W.

Using winding functions, amongst other things, Daverman and Walsh proceeded to get more general results. First they broadened the scope of the problem to include codimension-2 decompositions into spheres [**DW2**]. Further extending the scope of their investigation to *any* codimension-two decomposition of an orientable manifold into closed orientable submanifolds, they prove (in [**DW3**]), for example, that the decomposition space is a 2-manifold. The decomposition map is shown to be an approximate fibration if a locally finite set of points is removed from the resulting 2-manifold. Daverman extended the result to include the case where G

consists of non–orientable closed 1-manifolds.

One would expect that manifold decompositions of codimension higher than 2 are not going to behave as nicely as those of codimension-2 and this is the case. Daverman and Walsh give an example of a 5-manifold decomposed into orientable surfaces so that the decomposition space is not a generalized manifold [**DW4**]. One of the elements is a torus, the others are spheres. So some restriction on the homology of the elements is usually dictated if one expects the decomposition space to be a generalized manifold.

Yet some theorems may be obtained without restricting the decomposition elements in such a way. Daverman established that codimension-3 manifold decompositions (under the modest restriction of orientability of the elements and source space) have finite-dimensional decomposition spaces [**D2**].

Of crucial importance to this paper is the result of J. Dydak and J. Segal [**DyS2**], that shows the decomposition space of a codimension-k manifold decomposition of a manifold is LC^1 (this is discussed in more detail in section 1.5). In conjunction with Dydak's extension of the Hurewicz theorem [**Hr**], this gives a simple way of determining if the decomposition space is an ANR; the problem is reduced to determining the local cohomological connectedness of the space.

Daverman and L.S. Husch establish [**DH**] the strong result that there is an open dense subset of the decomposition space over which the elements are all shape equivalent.

In [**CD2**] and [**CD4**], Coram and Duvall look at Lacher's k-sphere mappings (i.e. mappings from the $(2k+1)$-sphere onto a manifold whose point inverses have the shape of the k-sphere [**L1**]), showing them to be approximate fibrations off of a

finite subset of the image. The paper is very similar in technique to the material in chapter 2, which can be thought of as a generalization of their result. Part of the strategy is the same: show a certain subset of the image space to be locally finite. However, Coram and Duvall draw conclusions about the map assuming manifold structure of the image, whereas here the structure on the image is established by studying the map and the local structure on a dense open subset of the image.

Recent work of Daverman and Walsh uses sheaf theory and spectral sequences to determine certain restrictions of the cohomological dimension of the decomposition space. In particular,

THEOREM (DAVERMAN–WALSH) [**DW4**, Theorem 1.2].

Suppose M is a complete metrizable space and let $f : M \to B$ be a proper map such that each set $f^{-1}(y)$ has the shape of a closed connected orientable n–manifold. Then

$$dim_{\mathbf{Z}} B \leq dim_{\mathbf{Z}} M - n.$$

In the same paper, they also establish, using essentially the Vietoris–Begle mapping theorem and winding functions, that if the source space is an orientable $(n+k)$-manifold, $n \geq k \geq 3$, decomposed into continua having the shape of orientable closed n-manifolds with trivial reduced homology in dimensions less than k such that the decomposition space is finite dimensional, then the decomposition space is a generalized k-manifold. This result is generalized herein.

My investigation followed a route suggested by lacunary phenomena of spectral sequences, exploiting its naturality with respect to inclusion maps, along with duality properties of the source manifold.

The main results are now stated.

THEOREM 1. *Let M^{2n+1} be an manifold, $n > 1$, and G a uscd of M into compacta having the shape of homology n-spheres such that dim $M/G < \infty$. Suppose the discontinuity set $E = D \cup K$ (see sections 1.2 and 1.4) does not locally separate B. Then M/G is a generalized $(n+1)$-manifold. Moreover, E is a locally finite subset of B.*

This Theorem generalizes the main theorem in [**CD2**] about k-sphere mappings, where M, the decomposition elements, and the decomposition space are assumed to be spheres. Their conclusion that the map is an approximate fibration off of the locally finite subset if the elements have the shape of the n-sphere still follows without assuming the image to be a sphere.

COROLLARY 1. *In addition to the hypotheses of Theorem 1, assume all the elements have the shape of the n–sphere. Then the decomposition map is an approximate fibration off of the set E.*

COROLLARY 2. *There can be no uscd of euclidean $(2n+1)$–space, E^{2n+1}, into continua having the shape of homology n-spheres such that the set E does not locally separate the decomposition space.*

COROLLARY 3. *If G is a uscd of S^{2n+1} into homology n-spheres with exceptional set not locally separating the decomposition space, then there is at most one decomposition element g having the property that for every neighborhood U of g which retracts to g there is a neighborhood V which deforms to g in U such that every element $g' \in G - \{g\}$ which lies in V maps degree 0 into g under the retraction of U.*

Example: Given $n > 1$, let Σ_1, Σ_2 be homology n-spheres. Let $M = \Sigma_1 \star \Sigma_2$

denote the external join [**RS**, p.23] of Σ_1 and Σ_2. Then M is a $(2n+1)$-sphere ([**E**], [**C**]). Let G_1 denote the decomposition of M which has all elements, but one, being the copies of Σ_1 parallel to the obvious end of the join, the remaining one being the end of the join corresponding to Σ_2. Similarly, G_2 will be the decomposition of M all of whose elements are parallel to the Σ_2 end of the join, but for one corresponding to the Σ_1 end. The space M/G_i is homeomorphic to the suspension of Σ_i, $i = 1, 2$, a generalized manifold but not a manifold at the suspension points if Σ_i is not a sphere. The distinguished element of the decomposition G_i represents such an element as described in Corollary 3.

Lest the reader think that the non-manifold points are always so sparse, recall the example of Daverman and Walsh [**DW1**] of a "ghastly" generalized n-manifold X^n $(n > 2)$ which is not locally Euclidean at *any* point, yet crossed with the real line is a manifold. Then $M = X^n \times \Sigma$, where Σ is any homology $(n-1)$-sphere, is a manifold with a natural decomposition into homology $(n-1)$-spheres yielding X^n as the decomposition space. The decomposition map is a trivial bundle over X. The point is that the structure of the decomposition space can be highly non-Euclidean if the fibers of the map are not nicely embedded with respect to the manifold structure on M.

THEOREM 2. *Let M^{n+k} be an orientable manifold, $n \geq k \geq 3$. Let G be a uscd of M into compacta having the shape of closed orientable n-manifolds with $\check{H}_i(g) \cong 0$ for $i < k-1$. If $\mathcal{H}^n[p]$ [see section 1.1] is Hausdorff and D does not locally separate B, and dim $M/G < \infty$, then M/G is a generalized k-manifold.*

This Theorem generalizes the result of Daverman and Walsh mentioned above; their hypotheses are identical except "$i < k-1$" is replaced by "$i < k$", and here

G is assumed to be non–degenerate.

COROLLARY 4. *There cannot be a non-degenerate uscd G of $(n+k)$–dimensional Euclidean space, E^{n+k}, into compacta having the shape of n–manifolds, each of which is a product of spheres of dimension at least $k-1$, with the exceptional set E not locally separating the decomposition space.*

Chapter 1

Tools and Methods

This chapter contains the bulk of observations needed to understand the more arcane aspects of the proofs in chapters 2 and 3.

Section 1 discusses presheaves, sheaves, and the Leray sheaf of a closed map. The second section is a discussion of winding functions, which we define using cohomology. Section 3 deals with the relationship of winding functions to the Leray sheaf, establishing two useful equivalences. Cohomology with coefficients in the Leray sheaf is discussed in section 4 and salient points of the theory of sheaf cohomology are stated. In particular, the Leray spectral sequence of a closed map is described axiomatically. Section 5 defines the concept of generalized manifold as used in this paper, briefly describing properties of generalized manifolds and ANR's established in the literature. The last section contains various algebraic lemmas that are used in chapter 2.

§1.1 Sheaves and Presheaves.

This section defines sheaf, presheaf, sections, and the Leray sheaf. Several examples of sheafs illustrating crucial points are discussed. For a rigorous and more general treatment of sheaves the reader is referred to the books by Swan [**Sw**] and Bredon [**Br1**].

Definitions A *sheaf* on a topological space X is a pair $(\mathcal{S}, \pi)$ where $\mathcal{S}$ is a topological space, and π: $\mathcal{S} \to X$ is a continuous map satisfying the properties:

(1) $\mathcal{S}_x = \pi^{-1}(x)$,the *stalk* of $\mathcal{S}$ at x, is a $\mathbf{Z}$-module with the discrete topology for all $x \in X$.

(2) For all $s \in \mathcal{S}$ there exists a neighborhood U of s in $\mathcal{S}$ such that $\pi|_U$ is a homeomorphism onto its image.

(3) Given $x \in X$, $s, t \in \mathcal{S}_x$, and a neighborhood U of $s+t$, there is a neighborhood V of s and neighborhood W of t such that, for all $y \in \pi(U) \cap \pi(V) \cap \pi(W)$, if $s' \in V \cap \mathcal{S}_y$ and $t' \in W \cap \mathcal{S}_y$, then $s' + t' \in U$.

Given $Y \subset X$ a *section* of $\mathcal{S}$ over Y is a (continuous) map $\sigma : Y \to \mathcal{S}$ such that $\pi \circ \sigma$ is the identity. Let $\check{H}^0(Y; \mathcal{S})$ denote the set of sections over Y; by (2) and (3) this set is an $\mathbf{Z}$-module. The *restriction of* $\mathcal{S}$ *to* Y, denoted $\mathcal{S}|_Y$, is $(\pi^{-1}(Y), \pi|_{\pi^{-1}(Y)})$.

$\mathcal{S}$ is *constant* if $\mathcal{S} = X \times G$ (G a $\mathbf{Z}$–module) and π is projection onto the first factor. $\mathcal{S}$ is *locally constant* at $x \in X$ if there exists a neighborhood $U \subset X$ of x such that $\mathcal{S}|_U$ is constant on U.

Note that, if X is Hausdorff, a sheaf $\mathcal{S}$ on X is Hausdorff if, and only if, given any stalk $\mathcal{S}_x$, any two points in the stalk can be separated by disjoint open neighborhoods in $\mathcal{S}$. Moreover, if two sections of $\mathcal{S}$ agree at a point, then they agree on an open and closed set.

Given a section σ over $Y \subset X$ into $\mathcal{S}$, we define the *support of* σ to be the closure in Y of the set

$$\{y \mid \sigma(y) \neq 0 \in \pi^{-1}(y)\}.$$

A family of supports ϕ *on* X denotes a collection of closed subsets of X with the following properties:

(1) A closed subset of an element of ϕ is an element of ϕ.

(2) ϕ is closed under finite unions.

We say ϕ is *paracompactifying* if, in addition, it satisfies:

(3) Each element of ϕ is paracompact.

(4) Each element of ϕ has a closed neighborhood in ϕ.

We concern ourselves only with three families of supports: the family of closed subsets of X, the family of compact subsets of X, or the family consisting of a singleton and the null set. Since our space X will always be locally compact metric, the first two of these families are paracompactifying. The last one is not.

Many times we must consider sections with restricted support. Let $H^0_\phi(Y;\mathcal{S})$ denote the set of sections of $\mathcal{S}$ over Y with support being an element of the family of supports ϕ on Y.

A *presheaf* on X is a contravariant functor from the category of inclusions of open subsets of X to the category of homomorphisms of $\mathbf{Z}$–modules. A presheaf S on a space X induces a sheaf $\mathcal{S}$ on X (see [**Br**,pp.2–3]) with the following key properties:

(1) $\mathcal{S}_x = \varinjlim S(U)$, for all $x \in X$, where the direct limit is taken over all neighborhoods of x in X

(2) the topology of $\mathcal{S}$ is generated by sets of the form $\{\alpha_x \mid x \in U\}$ where U is any open subset of X, $\alpha \in S(U)$, and α_x is the image of α in $\mathcal{S}_x$ under the direct limit.

Given a closed proper map $p : Y \to X$, the *cohomology presheaf on X in dimension $n \geq 0$ induced by p* is given by the functor $U \mapsto \check{H}^n(p^{-1}(U);\mathbf{Z})$. The sheaf induced by this presheaf, denoted $\mathcal{H}^n[p]$, is called the *Leray sheaf* in dimension n induced by p. When there is no confusion as to which map p is being discussed, we write briefly $\mathcal{H}^n$.

We now conclude this section with some pertinent examples.

EXAMPLE 1.1.1: Let $Y' = (S^1 \times [-1,1]) \setminus (S^1 \setminus \{*\}) \times \{0\}$, where $* \in S^1$; let Y be the wedge of Y' with S^1 at $(*,0)$. Let $X = [-1,1]$ and $p : Y \to X$ given by

$$p(s,t) = \begin{cases} t, & \text{for } (s,t) \in Y' \\ 0, & \text{for } s \text{ in the } S^1 \text{ factor of the wedge.} \end{cases}$$

Then $\mathcal{H}^1[p]$ is a sheaf on X with stalks isomorphic to $\mathbf{Z}$, and is constant on $[-1,0) \cup (0,1]$. Note that the group of sections $\mathcal{H}^1[p](X)$ is generated by the following section:

$$\sigma(t) = \begin{cases} 0, & \text{for } t < 0 \\ 0, & \text{for } t > 0 \\ 1, & \text{for } t = 0. \end{cases}$$

The group of sections $\mathcal{H}^1[p](X - \{0\})$ is generated by

$$\sigma_1(t) = \begin{cases} 0, & \text{for } t > 0 \\ 1, & \text{for } t < 0, \end{cases}$$

$$\sigma_2(t) = \begin{cases} 0, & \text{for } t < 0 \\ 1, & \text{for } t > 0. \end{cases}$$

Furthermore, the sheaf $\mathcal{H}^1[p]$ is *not* Hausdorff.

EXAMPLE 1.1.2: Consider the Möbius band as $S^1 \times_{\mathbf{Z}_2} [-1,1]$, the twisted product over $\mathbf{Z}_2$, where $\mathbf{Z}_2$ acts on S^1 by rotating through π radians and acts on $[-1,1]$ by reflection through 0. Let $p_0 : S^1 \times [-1,1] \to M$ be the natural (two-to-one) projection.There is a natural decomposition G of M into circles, to wit,

$$\{p_0(S^1 \times \{t\}) | t \in [-1,1]\}.$$

The decomposition space B is $[0,1]$. The sheaf $\mathcal{H}^1[p]$ of the decomposition map p in dimension one is constant on $(0,1]$. Select a generator of the stalk over 0; a typical neighborhood of this generator in the sheaf is homeomorphic to $[0,t), t < 1$,

and, for $0 < s < t$, meets the stalk over s at twice a generator of that stalk. After appropriate choice of stalk generators, $\mathcal{H}^1[p](B)$ is generated by

$$\sigma(t) = \begin{cases} 1, & \text{for } t = 0 \\ 2, & \text{for } t > 0. \end{cases}$$

This sheaf is Hausdorff.

Another important sheaf is the *orientation sheaf* of an n-manifold X. This sheaf is described briefly in section 5 of this chapter; see [Gr] for a lucid account.

EXAMPLE 1.1.3: The orientation sheaf on the open Möbius band is *locally* constant, but *not* constant. More generally, the orientation sheaf for *any* manifold is locally constant, but a manifold is orientable if,and only if, its orientation sheaf is constant.

§1.2 Winding Functions.

This section briefly discusses the definition and elementary properties of winding functions. The definition is not exactly the definition used by Coram-Duvall and Daverman-Walsh only in that the cohomology groups of decomposition elements are used rather than the homology groups. All pertinent properties of these functions are retained. Examples of winding functions for several maps, including those of section 1, close the section.

DEFINITION:

Let G denote an upper semicontinuous decomposition of a $(n+k)$- manifold into [compacta having the shape of] closed orientable n-manifolds, with $n, k \geq 0$. Select a point b in the decomposition space B, letting $\tilde{b}$ be the element of G corresponding to b. Let U be a neighborhood of b whose preimage under the decomposition map, $\tilde{U}$, [shape] retracts to $\tilde{b}$ via a (fixed) map $r_b : \tilde{U} \to \tilde{b}$. The restriction of the

retraction to any decomposition element contained in $\tilde{U}$ induces a homomorphism $h : \check{H}^n(\tilde{b}) \mapsto \check{H}^n(\tilde{y})$ for all $y \in U$. The absolute value of the degree of this map between copies of $\mathbf{Z}$ is denoted $\alpha_b(y)$. The function α_b is called *the n-winding function of G at b*. Note that $\alpha_b(b) = 1$.

The proofs of the following list of lemmas may be found in the work of Coram and Duvall [**CD2**].

LEMMA 1.2.1 (CORAM–DUVALL). *Given $x_0, x_1, x_2 \in B$ such that x_1, x_2 are in the domain α_{x_0} and x_2 is in the domain of α_{x_1}. Then*

$$\alpha_{x_0}(x_2) = \alpha_{x_0}(x_1) \cdot \alpha_{x_1}(x_2).$$

LEMMA 1.2.2 (CORAM–DUVALL). *Each winding function α_b is lower semicontinuous on its domain.*

LEMMA 1.2.3 (CORAM–DUVALL). *Each winding function α_b is continuous on a dense open subset C_b of U.*

Remark: The proof of this lemma uses a Baire category argument, and lemma 1.2.1.

We now study the winding functions in particular cases, including the examples of section 1.1.

EXAMPLE 1.2.1: Consider Example 1.1.1. For each $t \neq 0$, the domain of α_t may be taken to be equal to the component of $[-1,1] \setminus \{0\}$ containing t, and obviously $\alpha_t \equiv 1$. On the other hand, the domain for α_0 is, without loss of generality, $[-1,1]$ and $\alpha_0(t) = 0$ for all $t \neq 0$.

EXAMPLE 1.2.2: Consider Example 1.1.2. For $t > 0$, the domain of α_t may be taken to be $(0,1]$, and $\alpha_t \equiv 1$. On the other hand, the domain for α_0 is , without loss of generality, $[0,1]$, and $\alpha_0(t) = 2$ for all $t > 0$.

EXAMPLE 1.2.3: Let $M = S^1 \times D^2$ and

$$G = \{S^1 \times \{0\}, \{*\} \times rS^1 \mid * \in S^1, 0 < r \leq 1\}.$$

Then $B = M/G = D^2$. Let $b_0 = p(S^1 \times \{0\}) = 0$. Then, without loss of generality, the domain of α_0 is B and $\alpha_0 \equiv 0$ on $D^2 \setminus \{0\}$; for $b \in D^2 \setminus \{0\}$ the domain of α_b may be taken to be the open disk D_b of radius $|b|$ centered at b and $\alpha_b \equiv 1$ on D_b.

EXAMPLE 1.2.4: Let $M = S^1 \times D^2 = ((S^1 \times S^1) \times (0,1]) \cup S^1 \times \{0\}$; let p, q be coprime integers and G_0 the decomposition of $S^1 \times S^1$ into (p,q)-curves (each travels p times meridionally and q times longitudinally). Let

$$G_1 = \{g \times \{t\} \mid g \in G_0, t \in (0,1]\}$$

decompose $M \setminus (S^1 \times \{0\})$. Finally, let $G = \{S^1 \times \{0\}, g \mid g \in G_1\}$ decompose M. Then $B = M/G = D^2$. The domains of the 1-winding functions may be taken as in the previous example. Then $\alpha_b \equiv 1$ on D_b for $b \neq 0$ and $\alpha_0 \equiv |q|$ on $D^2 \setminus \{0\}$.

§1.3 Relations between winding functions and the Leray sheaf.

Inspecting definitions suggests some relationship between the winding functions and the Leray sheaf. Having ascertained these, we can have the luxury of using either tool at certain points in the analyses in the following chapters.

In this section, we establish equivalency of a given condition on the sheaf (for example, local constancy at b) with a condition upon the winding function(s) (continuity of α_b at b). The examples of sections 1 and 2 hint at these results. The

section closes with a useful corollary, which also follows from more general work of J. Dydak and J. J. Walsh [**DyW**], as well as being a consequence of results in [**DH**].

In general, the Leray sheaf is not a Hausdorff space. As noted in Section 1, if two sections into a Hausdorff sheaf agree at a point, then they agree on the component containing that point. So if the base space of a Hausdorff sheaf is locally connected we expect the module of sections to be particularly amenable to analysis. Thus the importance of the following proposition.

PROPOSITION 1.3.1.

Let M,G be as in §1.2, with p denoting the decomposition map. The sheaf $\mathcal{H}^n$ is Hausdorff if, and only if, each n-winding function α_b is non-zero on some neighborhood of b.

PROOF: Note that each stalk of $\mathcal{H}^n$ is isomorphic to $\mathbf{Z}$; for $b \in B$ and $\zeta \in \mathbf{Z}$ let ζ_b denote the element of the stalk over b corresponding to ζ under a given isomorphism: ζ_b is well–defined modulo the non-trivial automorphism of $\mathbf{Z}$.

($\Rightarrow$) Suppose that $\mathcal{H}^n$ is Hausdorff. Let U be any neighborhood of b. Pick a neighborhood V of b in U so that $p^{-1}(V)$ shape deforms in $p^{-1}(U)$ to $p^{-1}(\{b\})$.

Suppose there exists a sequence of points $\{x_i\}_{i=1}^{\infty}$ converging to b with the property that $\alpha_b(x_i) = 0$. Thus, x_i lies in V for sufficiently large i. This implies that $0_{x_i} \in \mathcal{H}^n_{x_i}$ is a sequence of points in $\mathcal{H}^n$, which converge to $1_b \in \mathcal{H}^n_b$, since $\alpha_b(b) = 1$ and the set $\{(y, \alpha_b(y)) | y \in V\}$ is a neighborhood of 1_b in $\mathcal{H}^n$. It is clear then that 0_b and 1_b cannot be separated by disjoint neighborhoods, since in particular every basic neighborhood of 1_b, and of 0_b, contains infinitely many of the 0_{x_i}'s.

($\Leftarrow$) Suppose that $\mathcal{H}^n$ is not Hausdorff. Then there are two points in $\mathcal{H}^n$ which cannot be separated by disjoint open sets; in particular, since B is Hausdorff, any

two such points must lie in the same stalk, say $\mathcal{H}_b^n$. Let $\gamma_b, \beta_b \in \mathcal{H}_b^n$ be two such points. Find a connected neighborhood U of b in B so that U is contained in the domain of α_b and such that there are sections $s_\gamma, s_\beta : U \to \mathcal{H}^n$ with $s_\gamma(b) = \gamma_b, s_\beta(b) = \beta_b$. Note that $U_\gamma = s_\gamma(U)$ and $U_\beta = s_\beta(U)$ intersect over arbitrarily small neighborhoods of b. Let $s_\delta \equiv s_\gamma - s_\beta$, $\delta_b = \gamma_b - \beta_b$, and let s_0 be the restriction of the zero section to U. Now the previous remark gives that s_δ and s_0 agree on a set of points having b as a limit point. Let $\{x_i\}_{i=1}^\infty$ be a subset of such points converging to b. We then obtain that

$$|\delta_b| \cdot \alpha_b(x_i) = \pm s_\delta(x_i) = \pm s_0(x_i) = 0 \in \mathcal{H}_{x_i}^n.$$

This contradicts the non-degeneracy of α_b. ∎

Sheaf cohomology with coefficients in a constant sheaf can be calculated as cohomology with coefficients in the module which is the stalk of the sheaf. Since we are interested in determining local properties of B, the following characterization is useful.

PROPOSITION 1.3.2.

The Leray sheaf in dimension n is locally constant at $b \in B$ if, and only if, the n-winding function α_b is continuous at b.

PROOF: ($\Rightarrow$) Suppose that $\mathcal{H}^n = U \times \mathbf{Z}$, where U is a neighborhood of b. By choosing a smaller neighborhood of b if necessary, we may assume that U is connected and contained in the domain of α_b (i.e. the retraction r_b is defined on $p^{-1}(U)$). Now $\alpha_b(y) \cdot 1_y$ is in a neighborhood of 1_b for y sufficiently close to b ; in particular, $\alpha_b(y) = 1$ for all y sufficiently close to b. Hence α_b is continuous on a neighborhood of b.

($\Leftarrow$) Conversely, suppose that α_b is continuous on a connected neighborhood U of b. Then $\alpha_b(y) = 1$ for all y in U. Let γ_b be the selected generator of $\check{H}^n(p^{-1}(b))$. Let γ_y denote the element of $\check{H}^n(p^{-1}(y))$ which is the image of γ_b under the induced homomorphism $(r_b|p^{-1}(y))^*$. Obviously γ_y generates $\check{H}^n(p^{-1}(y))$ (since γ_y is $\alpha_b(y)$ times one of the generators of this stalk) and, for all y in a small enough neighborhood V of b (for example a neighborhood whose preimage over p shape deforms into $p^{-1}(b)$ over U), the set $N = \{\gamma_y \mid y \in V\}$ forms a basic neighborhood of γ_b in $\mathcal{H}^n$. Define a map $h : \mathcal{H}^n|_{\pi(N)} \to V \times \mathbf{Z}$ as follows: $i \cdot \gamma_y \mapsto (y, i)$. Then h is a well defined continuous map which is clearly bijective with continuous inverse. Thus $\mathcal{H}^n|V$ is constant. ∎

As a corollary of theorem 1.3.2 and lemma 1.2.3, we obtain

COROLLARY 1.3.3. *The Leray sheaf in dimension n is locally constant over a dense open subset of B.*

§1.4 Sheaf Cohomology and the Leray Spectral Sequence of a Map.

In this section, we give an extremely brief outline of the obtuse technical tools used in this work. There is no attempt to give detailed definitions of sheaf cohomology or spectral sequences, just the bare minimum required to orient the reader in the proofs of Theorems 1 and 2. For an easy transition into aspects of the theory, the reader is advised to read [**A**] and [**DyW**]; for all the gory details one should read [**Br1**], [**Sw**], and [**S**].

For a family of supports ϕ and sheaf $\mathcal{S}$ on the space X, the sheaf cohomology of X with coefficients in $\mathcal{S}$ and supports in ϕ, denoted $H^*_\phi(X;\mathcal{S})$, is defined ([**Br1**]). If $A \subset X$, then we write $H^*_\phi(A;\mathcal{S})$ for $H^*_{\phi|A}(A;\mathcal{S}|_A)$. Sheaf cohomology of the pair

(X, A) can also be defined. For constant sheaves, the functor so defined agrees with the Alexander–Spanier theory in general and, in addition, agrees with the singular theory when X is an ANR. Also, ϕ is elided from the expression if, and only if, ϕ is the family of closed supports on X.

For the pair (X, A) we have a long exact sequence

$$\cdots \to H^q_\phi(X, A; \mathcal{S}) \to H^q_\phi(X; \mathcal{S}) \to H^q_\phi(A; \mathcal{S}) \to H^{q+1}_\phi(X, A; \mathcal{S}) \to \cdots .$$

A property of sheaf cohomology which will be of some importance to us is that [**Br1**, p.80] if ϕ is the family of closed supports and $x \in X$ such that $\{x\}$ is a closed subset of X, then there is a natural isomorphism

$$H^*_x(X; \mathcal{S}) \cong H^*_\phi(X, X - x; \mathcal{S}).$$

Since all our spaces (excepting sheaves) are locally compact metric spaces, we use this property without further ado.

We need a corollary of the following Theorem; our statement of it is not the most general, but is sufficient for our needs. The proof is found on page 59 of [**Br1**].

THEOREM 1.4.1.

If A is an open subset of X and ϕ is the family of closed supports on X, then there is a natural isomorphism

$$H^*_\phi(X, A; \mathcal{S}) \cong H^*_{\phi|X-A}(X; \mathcal{S}).$$

LEMMA 1.4.2.

Let g be a compact subset of X. Then there is a natural isomorphism

$$H^*_\phi(X, X - g; \mathcal{S}) \cong H^*_{\phi|g}(X; \mathcal{S}).$$

PROOF:

Let $A = X - g$ and apply the previous theorem. ∎

The following theorem is the foundation of our approach to Theorems 1 and 2; its proof is found in [**Br1**, pp. 140–141].

THEOREM 1.4.3 (LERAY–GROTHENDIECK).

For a closed map $f : X \to Y$, there is a first quadrant spectral sequence

$$E_2^{p,q} = H_\phi^p(Y; \mathcal{H}^q[p]) \Rightarrow H_{\phi(\psi)}^{p+q}(X).$$

Remarks:

(1) For our purposes we need only consider the family ψ of closed supports on X, and ϕ either the family of closed supports on Y or $\phi = \{x\}$. In the former case $\phi(\psi) = \psi$, and in the latter $\phi(\psi) = \phi|\tilde{x}$, the closed subsets of $\tilde{x}$.

(2) $E_{r+1}^{p,q} = \ker(d_r)/\mathrm{im}(d_r)$, where the differential

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

has bidegree $(r, 1-r)$.

(3)

$$E_r^{p,q} = E_{r+1}^{p,q} = E_{r+2}^{p,q} = \cdots = E_\infty^{p,q}$$

for $r \geq p + q + 1$.

(4) There is a filtration $0 \subset J_0 \subset J_1 \subset \cdots \subset J_p = H^p(X)$, where $J_0 = E_\infty^{p,0}$ and $J_i/J_{i-1} = E_\infty^{p-i,i}$, $i \leq p$.

(5) We also may use Čech sheaf cohomology for ϕ being closed supports [**Sw**].

Note for the choice of supports in the latter of (1) above that, by lemma 1.4.1,

$$H_{\phi(\psi)}^i(X) = H^i(X, X - f^{-1}(x)).$$

We now give a relatively easy application of the Theorem 1.4.3.

LEMMA 1.4.4.

Let G be a decomposition of the complete metrizable space M into subcontinua having the shape of closed n–manifolds. Let A be an arc in $B = M/G$.

Then there is a natural isomorphism $H^1(A; \mathcal{H}^n) \cong H^{n+1}(\tilde{A})$.

Addendum: *Suppose that G is 1–acyclic. Then there is a natural isomorphism $H^0(A; \mathcal{H}^n) \cong H^n(\tilde{A})$.*

PROOF: We consider the map $f = \pi|_{\tilde{A}} : \tilde{A} \to A$, where q denotes the decomposition map. Staring at the Leray–Grothendieck spectral sequence for f brings one to the observation that $E_2^{p,q} = 0$ for $q > n$; this is also true for $p > 1$, since sheaf cohomology with paracompactifying supports respects dimension of a space [**Sw**, pp. 109–111].

In particular, $d_i \equiv 0$ for all $i \geq 2$. Thus,

$$E_2^{p,q} = E_\infty^{p,q}$$

for *all* p and q. Considering the filtration of remark (4) above, and observing that

$$J_{n+1}/J_n = E^{n+1,0} = 0,$$

it is clear that

$$H^{n+1}(\tilde{A}) = J_{n+1} = J_n = E_\infty^{1,n+1} = E_2^{1,n+1} = H^1(A; \mathcal{H}^n).$$

In the case of the addendum, the extra hypothesis ensures that

$$E_2^{p,n-1} = E_\infty^{p,n-1} = 0.$$

Using remark (4) again, the desired conclusion follows. ∎

We now apply 1.4.4 to the case of M being a manifold, in order to get two important lemmata.

Let G be a codimension–k manifold decomposition; let B be the decomposition space, p the decomposition map. Let

(1) $K = \cap\{K' \subset B \mid \mathcal{H}^n|_{B-K'}$ is Hausdorff $\}$.

(2) $D = \{x \in B - K \mid \mathcal{H}^n$ is not locally constant at $x\}$.

(3) $C = \{x \in B \mid \mathcal{H}^n$ is locally constant at $x\}$.

Note that K, D, and C form a partition of B; that K is closed, and C is open, in B; and that $E \equiv K \cup D$ is closed in B.

LEMMA 1.4.5.

Assume that G and M are as in the previous paragraph. Let A be an arc in B such that

(a) $\partial A = \{d_1, d_2\} \subset D$,

(b) $A - \partial A \subset C$,

(c) $\alpha_{d_i}(c_i) = p_i$ *for some* $c_i \in A \cap C$ *in the domain of* α_{d_i} $(i = 1, 2)$.

Then $H^{n+1}(\tilde{A}) \cong \mathbf{Z}_d$, where d is $gcd(p_1, p_2)$ (the greatest common divisor of p_1, p_2).

Addendum: *$H^n(\tilde{A}) \cong \mathbf{Z}$, if G is 1–acyclic.*

PROOF: By lemma 1.4.4, it is enough to calculate this for $H^1(A; \mathcal{H}^n)$. By the addendum to lemma 1.4.4, $H^n(\tilde{A}) = H^0(A; \mathcal{H}^n) = \mathbf{Z}$. Note that by 1.3.1 $p_1, p_2 \neq 0$. We do not prove here the case that at least one of the p_i is equal to one, for, though these are simple to prove, we do not need the result. There are then two remaining cases to analyze.

Case 1: $d = p_1 = p_2$

We define a sheaf map from the constant sheaf $A \times \mathbf{Z}$ to $\mathcal{H}^n|_A$ Now, $\mathcal{H}^n|_{A-\{d_1,d_2\}}$ is constant, so for $c \in A - \{d_1, d_2\}$ pick elements γ_c of the stalk $\mathcal{H}^n_c$ so that γ_c corresponds to p under the map; we can extend the map to $A \times \mathbf{Z}$ by picking the generator γ_{d_i} of $\mathcal{H}^n_{d_i}$ that corresponds under the defining retraction to $p_i \cdot \gamma_c$. Obviously this map is an injection; let $\mathcal{Q}$ denote the quotient sheaf. We thus have a short exact sequence of sheaves:

$$0 \to A \times \mathbf{Z} \to \mathcal{H}^n|_A \to \mathcal{Q} \to 0.$$

As a consequence we get a long exact cohomology sequence

$$\cdots \to H^i(A) \to H^i(A; \mathcal{H}^n) \to H^i(A; \mathcal{Q}) \to \ldots.$$

Since $H^1(A) = 0 = H^2(A)$, to calculate $H^1(A; \mathcal{H}^n)$ we need only calculate $H^1(A; \mathcal{Q})$.

Since $p_1 = p_2$, we have

$$\mathcal{Q}_b = \begin{cases} 0, & \text{for } b = d_1, d_2 \\ \mathbf{Z}_d, & \text{otherwise.} \end{cases}$$

Moreover this sheaf is constant over the interior of A. In fact the topology of the sheaf is the same as taking the constant sheaf $A \times \mathbf{Z}_d$, restricting it to the interior of the arc, and then extending the sheaf so it is trivial over the endpoints of the arc. So $H^1(A; \mathcal{Q}) = H^1(A, \{d_1, d_2\}; \mathbf{Z}_d) = \mathbf{Z}_d$.

Case 2: $p_1 \neq p_2$

Note in the argument of the first case that if, say, the endpoint d_2 of A were also not in $D \cup K$, then $H^{n+1}(\tilde{A}) = 0$ and $H^n(\tilde{A}) = \mathbf{Z}$. We use this observation in

conjunction with the Mayer-Vietoris sequence to calculate the cohomology in the second case.

Let l be the least common multiple of p_1 and p_2. Pick $c \in A - \partial A$. Let A_i denote the closure in A of the component of $A - \{c\}$ containing d_i. Applying the cohomology version of the Mayer-Vietoris sequence for the pair (A_1, A_2) we obtain

$$0 \to H^n(\tilde{A}) \to \sum_{i=1}^{2} H^n(\tilde{A}_i) \to H^n(\tilde{c}) \to H^{n+1}(\tilde{A}) \to 0.$$

So applying the above note to A_i we have

$$0 \to \mathbf{Z} \to \mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z} \to H^{n+1}(\tilde{A}) \to 0.$$

The first map is determined (up to signs) by $1 \mapsto (q_1, q_2)$, where $q_i = l/p_i$ (to see this consider the appropriate sections). Since l is the least common multiple of the p_i, we have q_1, q_2 relatively prime. Let r_1, r_2 be the coprime pair so that (q_1, q_2) and (r_1, r_2) generate $\mathbf{Z} \oplus \mathbf{Z}$. By comparing generating sections it is apparent that $(1,0) \mapsto p_1$ and $(0,1) \mapsto p_2$; hence $(r_1, r_2) \mapsto r_1 p_1 + r_2 p_2$.

Let $m = r_1 p_1 + r_2 p_2$. It is elementary number theory to show that $m = gcd(p_1, p_2)$.

The proof of the addendum is trivial upon observing that $H^n(\tilde{A}) \cong H^0(A; \mathcal{H}^n)$. The above discussion should make it clear that the latter group is $\mathbf{Z}$. ∎

LEMMA 1.4.6. *Assume that G and M are as in 1.4.5. Let A be an arc in B such that*

(a) $\partial A = \{k_1, k_2\} \subset K$,

(b) $A - \partial A \subset C$,

(c) $\alpha_{k_i}(c_i) = 0$ *for some* $c_i \in A \cap C$ *in the domain of* α_{k_i} $(i = 1, 2)$.

Then $H^{n+1}(\tilde{A}) \cong \mathbf{Z}$.

PROOF: Using 1.4.5 as in the proof above, we need only calculate $H^1(A; \mathcal{H}^n)$.

Choose $c \in A - \partial A$. Let A_i denote the closure of the component of $A - \{c\}$ containing k_i $(i = 1, 2)$. Apply the Mayer–Vietoris sequence (coefficients in $\mathcal{H}^n$) to the triple (A, A_1, A_2), to see that

$$0 \to \mathbf{Z} \to \check{H}^1(A; \mathcal{H}^n) \to \check{H}^1(A_1; \mathcal{H}^n) \oplus \check{H}^1(A_2; \mathcal{H}^n) \to 0.$$

Thus it suffices to show that $\check{H}^1(A_1; \mathcal{H}^n)$ is trivial. To prove this choose a nested collection of subarcs $\varpi_1, \varpi_2, \ldots$ of $A_1 = \varpi_0$ whose intersection is k_1. The inclusion–induced homomorphism

$$\check{H}^1(A_1; \mathcal{H}^n) \to \check{H}^1(\varpi_j; \mathcal{H}^n)$$

is obviously an isomorphism for all j. Thus, using continuity of sheaf cohomology [**Br1**, p.71],

$$0 = \check{H}^1(k_1; \mathcal{H}^n) = \check{H}^1(\varprojlim \varpi_j; \mathcal{H}^n) = \varinjlim \check{H}^1(\varpi_j; \mathcal{H}^n) \cong \check{H}^1(A_1; \mathcal{H}^n).$$

Hence, by lemma 1.4.4,

$$\mathbf{Z} \cong \check{H}^1(A; \mathcal{H}^n) \cong \check{H}^{n+1}(\tilde{A}). \quad \blacksquare$$

§1.5 Properties of Generalized Manifolds and ANRs.

In this section we define and discuss crucial properties of generalized manifolds and ANRs. Theorem 1.5.5 and Corollary 1.5.13 are especially important for the investigations of Chapters 2 and 3. For details about proofs of the deep results, the

reader is referred to [**Hu**], [**W**], [**Br1**], [**Br2**], [**DyW**], [**A**] and [**D3**]. Throughout this section X is assumed to be a locally compact metrizable space.

DEFINITIONS:

X is a *generalized k-manifold over* $\mathbf{Z}$, abbreviated as k–gm$_{\mathbf{Z}}$ or k-gm, if

(1) X is a finite dimensional ANR and

(2) for all $x \in X$,

$$H_i(X, X - x) = \begin{cases} 0, & \text{for } i \neq k \\ \mathbf{Z}, & i = k. \end{cases}$$

The *orientation sheaf* $\mathcal{O}$ of a k-gm is generated by the presheaf

$$U \mapsto H_k(X, X - U).$$

If $\mathcal{O}$ is (locally) constant, them we say X is *(locally) orientable*; otherwise X is *non-orientable.*

Remark: Item (1) of our definition of k–gm is less general than the local cohomological connectivity and finite cohomological dimension posited in the "classical" definition (see [**Br2**]); many of the results that follow were proved in the more general setting.

The following Theorem is a rather deep result; we use it often. It was first proved by G. Bredon in [**Br2**]; it also follows as a Corollary of a much more general result of J. Dydak and J.J. Walsh [**DyW**].

THEOREM 1.5.1 (BREDON). *Every k-gm is locally orientable.*

The following Corollary is used in the applications.

COROLLARY 1.5.2. *Every simply–connected k-gm is orientable.*

PROOF: Show that $\mathcal{O}$ is constant, using the technique of monodromy. ∎

As another consequence of the previous theorem, we have

COROLLARY 1.5.3. *Every non–orientable k-gm X has an orientable double cover $\tilde{X}$.*

PROOF: Let $\tilde{X} = \{x \mid x \text{ generates } \mathcal{O}_{\pi(x)}\}$. ∎

For orientable k-gm's there is an Alexander Duality theorem (see [**A**] or [**W**]). This will be quite useful to us in the characterizations of sections 2.2 and 3.2.

THEOREM 1.5.4. *Let X be an orientable k-gm. Let $B \subset A$ be closed subsets of X. Then there is a natural isomorphism*

$$D : H_i(X - B, X - A) \cong H_c^{k-i}(A, B).$$

In particular, the following diagram commutes:

$$\begin{array}{ccccccc}
\cdots \to & H_c^{k-i}(X, A) & \to & H_c^{k-i}(X, B) & \to & H_c^{k-i}(A, B) & \to \cdots \\
& \uparrow \cong & & \uparrow \cong & & \uparrow \cong & \\
\cdots \to & H_i(X - A) & \to & H_i(X - B) & \to & H_i(X - B, X - A) & \to \cdots
\end{array}$$

Using the local duality of a k-gm and a Mayer-Vietoris argument, one can show the following

THEOREM 1.5.5. *Let A be an arc in a k-gm X. Then*

$$H_i(X, X - A) = \begin{cases} 0, & \text{for } i \neq k \\ \mathbf{Z}, & i = k. \end{cases}$$

We are now finished with our discussion of k-gm's and turn our attention to properties of ANRs. To start with we need the following definition (for this section only).

DEFINITION: A space X is *locally connected in dimension k*, denoted k–LC, if for every $x \in X$ and neighborhood U of x in X there is a neighborhood V of x in U with

$$\pi_k(V) \to \pi_k(U)$$

being trivial. If X is k–LC for every $k \leq i$ [resp. $k < \infty$], we say X is LC^i [resp. LC$^\infty$]. Moreover, we say X is LC* if X is locally contractible (i.e. for every x and neighborhood U of x there is a neighborhood V of x in U and a homotopy $h_t : V \to U$ such that h_0 is the inclusion of V into U and h_1 is a constant map.

A crucial result about ANRs is the following Theorem.

THEOREM 1.5.6 ([**Hu**, p.168]). *If X is a metric space with dim $X \leq n$, then the following staements are equivalent:*

(a) *X is an ANR.*

(b) *X is LC*.*

(c) *X is LC^n.*

In the second chapter we wish to recognize our decomposition space as an ANR. We now proceed to develop the material needed there.

DEFINITION: Let $\mathcal{S}$ be a sheaf on X. Then we say X is *cohomologically locally connected in dimension i with respect to $\mathcal{S}$*, abbreviated i-$clc_{\mathcal{S}}$, at x if for all neighborhoods U of x, there is a neighborhood $V \subset U$ of X so that the image of the inclusion–induced map

$$\check{H}^i(U;\mathcal{S}) \to \check{H}^i(V;\mathcal{S})$$

is finitely generated as a $\mathbf{Z}$–module. If the above property holds for all $x \in X$ then we say X *is i-$clc_{\mathcal{S}}$*. Furthermore, in the special case that $\mathcal{S}$ is the constant sheaf

$\mathcal{S} = X \times \mathbf{Z}$, then we write i-$clc_{\mathbf{Z}}$.

Finally, X is $clc^k_{\mathcal{S}}$ [resp. $clc^\infty_{\mathcal{S}}$ or $clc_{\mathcal{S}}$] at x if X is i-$clc_{\mathcal{S}}$ at x for all $i \leq k < \infty$ [resp. $i < \infty$]. If this is true for all $x \in X$, then we drop the phrase "at x". A subscript of $\mathbf{Z}$ indicates the trivial sheaf $\mathcal{S} = X \times \mathbf{Z}$.

Remarks:

- ♠ The Alexander–Spanier and the Čech cohomology theories are isomorphic for paracompact Hausdorff spaces [**S**, Corollary **8**, p.334].
- ♡ There are corresponding definitions for homology, denoted $i - hlc$, hlc^k_∞ etc. [**Br1**, p.253]; we will use the Alexander–Spanier homology functor in this paper.

The following lemma is of use to us in proving Corollary 1.5.3.

LEMMA 1.5.7. *Let X be a metric space which is $clc \equiv clc^\infty_{\mathbf{Z}}$. Then X is $hlc \equiv hlc^\infty_{\mathbf{Z}}$.*

PROOF: Given $k \geq 0$, $x \in X$ select neighborhoods $W \subset V \subset U$ of x in X so that

$$\text{image } (i^*_{U,V} : H^{k+1}(U) \to H^{k+1}(V))$$

and

$$\text{image } (i^*_{V,W} : H^j(V) \to H^j(W))$$

are trivial, where i^* is the inclusion induced homomorphism and $j = k, k+1$. The crucial observation here is that the short exact sequence of the Universal Coefficient Theorem for Alexander–Spanier cohomology is natural [**S**, p.244]. So the following

diagram commutes

$$\begin{array}{ccccccc}
0 & \to \mathrm{Ext}(H^{k+1}(U),\mathbf{Z}) & \to H_k(U) & & & & \\
 & \qquad\uparrow{\scriptstyle \beta} & \qquad\uparrow{\scriptstyle i_*^{U,V}} & & & & \\
0 & \to \mathrm{Ext}(H^{k+1}(V),\mathbf{Z}) & \to H_k(V) & \to \mathrm{Hom}_{\mathbf{Z}}(H^k(V),\mathbf{Z}) & \to 0 & & \\
 & \qquad\uparrow{\scriptstyle \alpha} & \qquad\uparrow{\scriptstyle i_*^{V,W}} & \qquad\uparrow{\scriptstyle \gamma} & & & \\
0 & \to \mathrm{Ext}(H^{k+1}(W),\mathbf{Z}) & \to H_k(W) & \to \mathrm{Hom}_{\mathbf{Z}}(H^k(W),\mathbf{Z}) & \to 0 & &
\end{array}$$

Now $\alpha = \mathrm{Ext}(i^*_{V,W})$, $\beta = \mathrm{Ext}(i^*_{U,V})$, and $\gamma = \mathrm{Hom}_{\mathbf{Z}}(i^*_{U,V})$; the usual categorical nonsense establishes that these maps are trivial. One concludes that

$$i_*^{U,W} = i_*^{U,V} \circ i_*^{V,W}$$

is trivial by chasing the diagram.

Thus, X is k-$hlc_{\mathbf{Z}}$, for $k \geq 0$, implying that X is *hlc*. ∎

The following Theorem, an extension of the Hurewicz theorem ([**Hr**]), appears as Theorem 3.2 in [**DyS2**].

PROPOSITION 1.5.8 (DYDAK–SEGAL). *Suppose X is a LC^k ($k \geq 1$) complete metrizable space such that for each $x \in X$ and for each neighborhood U of x there exists a neighborhood V of x in U with the composition of homomorphisms*

$$\pi_{k+1}(V,x) \to H_{k+1}(V) \to \check{H}_{k+1}(V) \to \check{H}_{k+1}(U)$$

being trivial. Then X is LC^{k+1}.

LEMMA 1.5.9. *Suppose X is a complete metrizable space which is LC^1 and clc. Then X is LC^∞.*

PROOF: By lemma 1.5.6, X is *hlc*. The proof is by induction; we merely establish the basis of the induction, the rest of the argument being clarified by this.

Fix $x \in X$; given the neighborhood U of x in X, find a neighborhood V of x in U with $H_2(V) \to H_2(U)$ being trivial. Then consider that the following diagram commutes:

$$\begin{array}{ccccc} \pi_2(V) & \xrightarrow[\mu_1]{} & H_2(V) & \xrightarrow[\mu_2]{} & \check{H}_2(V) \\ & & \downarrow{\scriptstyle \nu_1} & & \downarrow{\scriptstyle \nu_2} \\ & & H_2(U) & \xrightarrow[\omega]{} & \check{H}_2(U) \end{array}$$

Since ν_1 is has trivial image, we have

$$\omega \circ \nu_1 \circ \mu_1 = \nu_2 \circ \mu_2 \circ \mu_1$$

is trivial. By hypothesis X is LC^1. Thus, the hypotheses of Proposition 1.5.8 are being met with $k = 1$, implying that X is LC^2. The rest of the induction is now clear. ∎

The following corollary has been the goal of the previous two pages.

COROLLARY 1.5.10. *If X is a finite dimensional LC^1, clc metric space then X is an ANR.*

The following proposition is found in [**DyS2**]; we use it to establish a useful criterion for determining if the decomposition space of a codimension–k manifold decomposition is an ANR.

THEOREM 1.5.11 (DYDAK–SEGAL). *Let $f : X \to Y$ be a proper surjection of locally compact spaces whose point inverses are FANRs. If X is an ANR and f has the AHLP with respect to all j–cells, $0 \leq j \leq k$, then Y is LC^{k+1}.*

Remarks:

(A) A finite–dimensional compactum is an FANR if it has the shape of an ANR; see [**DyS1**, p.74–75].

(B) A map $f : X \to Y$ between locally compact metrizable spaces has the *approximate homotopy lifting property with respect to the space Z* if, given any open cover $\mathcal{U}$ of Y and commutative diagram

$$\begin{array}{ccc} Z \times \{0\} & \xrightarrow[H_0]{} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ Z \times [0,1] & \xrightarrow[h]{} & Y \end{array}$$

where i denotes the obvious inclusion, there is an extension $H : Z \times [0,1] \to X$ of H_0 so that for any (z,t) there is an element U of $\mathcal{U}$ containing both $(f \circ H)(z,t)$ and $h(z,t)$.

A map is an *approximate fibration* if it has the AHLP with respect to all metrizable spaces; this concept was developed in [**CD1**].

COROLLARY 1.5.12. *Let G be a codimension-k manifold decomposition of the manifold M^{n+k}. Then B is LC^1.*

PROOF: It is an easy exercise to show that p has the AHLP with respect to a point. The result is then a corollary of 1.5.11. ∎

COROLLARY 1.5.13. *Assume the hypotheses of 1.5.12 and, in addition, that dim $B \leq d < \infty$. Then the following statements are equivalent:*

(1) *B is an ANR.*

(2) *B is $clc^{\infty}_{\mathbf{Z}}$.*

(3) *B is $clc^{d}_{\mathbf{Z}}$.*

PROOF: This is just combining Lemmata 1.5.10 and 1.5.12 with Theorem 1.5.1. ∎

§1.6 Potpourri.

We include in this section various algebraic and sheaf theoretic results needed in the sequel in order to retain clarity. The first is a special result about torsion **Z**–modules used in the proof of Theorem 2 exactly once. The second is a sort of "zig–zag" lemma needed in our discussion of local connectivity properties of the decomposition space. The last one gives some sufficient conditions for which a sheaf that is locally constant off a closed set is actually constant off that set.

The first lemma is used in Chapter 2 to show that the module of extensions of a certain group by **Z** is not finitely generated. The proof is simple, and uses the following elementary proposition.

PROPOSITION [**H**, P.222]. *Let A be a torsion module over a UFD R. Let*

$$A(p) = \{a \in A \mid a \text{ has order of some power of } p\}$$

for any prime $p \in R$. Then $A = \sum A(p)$ with the sum being taken over all primes $p \in R$.

THEOREM 1.6.1. *Suppose A_i are $\mathbf{Z}$-modules for $i = 1, 2, \ldots$, and there exist surjections $\psi_i : A_i \to A_{i+1}$ with kernel isomorphic to $\mathbf{Z}_{p_i}$, where $\{p_i \mid i = 1, 2, 3, \ldots\}$ is a list of distinct positive integers with any three being coprime. If the direct limit of the system $\{(A_i, \psi_i)\}$ is trivial, then A_1 is an infinite direct sum of finitely generated torsion $\mathbf{Z}$-modules.*

PROOF: Let K_i denote the kernel of $\theta_i \equiv \psi_i \circ \psi_{i-1} \circ \ldots \psi_2 \circ \psi_1$. Obviously $K_i \subset K_{i+1}$. By hypothesis K_1 is isomorphic to $\mathbf{Z}_{p_1}$. By induction we see that each K_i is a finitely generated torsion module; moreover, A_1 is the countable nested union of

the K_i since the direct limit is trivial. Therefore A_1, and each A_i, is a torsion $\mathbf{Z}$-module. By the previous proposition it suffices to show that each $A_1(q)$ is finitely generated (as a $\mathbf{Z}$-module) for all primes $q \in \mathbf{Z}$.

Fix a prime q. Let N be the largest integer such that q divides p_N (N exists since any three p_i's are coprime). We show that $A_1(q) = K_N(q)$, which is finitely generated. That $K_N(q) \subset A_1(q)$ is obvious. Let $\gamma \in A_1(q)$; we wish to show that $\theta_N(q) = 0$. Obviously, for each i, $\theta_i(\gamma)$ has order some power of q. There is a j so that $\theta_j(\gamma) = 0$; pick the least such j, say j_0. Then $\theta_{j_0-1}(\gamma) \neq 0$ has order some power of q, so q divides p_{j_0}. Thus, $j_0 \leq N$. Hence, $A_1(q) \subset K_N(q)$. ∎

The following theorem is used heavily. The proof is straightforward and may be found in the references cited.

THEOREM 1.6.2 [**Br2**, P.77], [**DyW**]. *Let the following diagram denote a commutative diagram of* $\mathbf{Z}$*-modules, where each of the vertical maps* ρ, σ *has finitely generated image, and the middle row is exact. Then the image of* L_1 *under the map* $\tau_2 \circ \tau_1$ *is finitely generated.*

$$\begin{array}{ccccc} & & L_1 & \longrightarrow & L_2 \\ & & \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \sigma} \\ M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \tau_2} & & \\ N_1 & \longrightarrow & N_2 & & \end{array}$$

The following lemma is applied at certain crucial instances. Some of the hypotheses may be weakened further, but as this is not necessary for us we leave it to the interested reader.

LEMMA 1.6.3. *Suppose U is a metrizable ANR, dim $U \geq 3$, and V is a connected neighborhood of x in U such that closed loops in V are contractible in U. Suppose that, for $i \leq 2$, $H_i(U, U-y) = 0$, for all $y \in U$. Suppose that $D = \{d_1, d_2, \dots\} \subset U$ is a sequence of points converging to x. Furthermore, assume $\mathcal{S}$ to be a sheaf on U with stalks isomorphic to $\mathbf{Z}$ which is locally constant on $V - (D \cup \{x\})$.*

Then $\mathcal{S}$ is constant on $V - (D \cup \{x\})$.

Addendum: *the above is true if D is a closed 0–dimensional subset of U.*

PROOF: Let $\overline{D}$ denote $D \cup \{x\}$. We exhibit a sheaf isomorphism

$$\psi : (V - \overline{D}) \times \mathbf{Z} \to \mathcal{S}|_{V-\overline{D}}.$$

(1) Definition of the map:

Fix $y_0 \in V - \overline{D}$. Choose an isomorphism

$$\psi_0 : y_0 \times \mathbf{Z} \to \mathcal{S}_{y_0}.$$

For each $y \in V - (\overline{D} \cup y_0)$, choose a simple path, τ_y, in $V - \overline{D}$ from y_0 to y. We shall abuse notation and also consider τ to be a map of the unit interval $[0, 1]$ into U.

Note that, because of the hypothesis concerning local constancy, $\mathcal{S}|_{\tau_y}$ is constant since τ_y is simple. Let $\tilde{\tau}_{y,j}$ be the lift of τ_y into $\mathcal{S}$ starting at $\psi_0(y_0, j)$. We define $\psi(y, j)$ as $\tilde{\tau}_{y,j}(1)$.

(2) ψ is an isomorphism:

We assume for now that ψ is well-defined and show it has the requisite properties.

(2a) ψ is bijective:

If $\psi(y,j) = \psi(y,k)$, then the lifts $\tau_{y,j}$ and $\tau_{y,k}$ end at the same point and, by local constancy are identical over a neighborhood of y; by continuation, the lifts agree everywhere, hence we have $\psi(y_0,k) = \psi(y_0,j)$. So $j = k$.

Onto is obvious, since we may choose a lift $\tilde{\tau}$ starting at the desired point.

(2b) ψ is continuous:

We show continuity at (y_0,j); a slight variation of the argument yields continuity at all other points. Choose a connected neighborhood W of y_0 disjoint from $\overline{D}$ such that $\mathcal{S}|_W$ is constant; continuity is clear if ψ is well-defined, since we may then choose each path τ_y ($y \in W$) to lie in W.

It is clear then that the inverse is also continuous.

(2c) ψ is a homomorphism:

We must show

$$\psi(y,l) + \psi(y,m) = \psi(y,m+l),$$

for all y and all $l, m \in \mathbf{Z}$.

Choose the path τ_y. Recall that $\psi(y,l+m) = \tilde{\tau}_{y,l+m}(1)$, etc. So it suffices to show that

$$\tilde{\tau}_{y,l} + \tilde{\tau}_{y,m} = \tilde{\tau}_{y,l+m},$$

where the addition on the left side is in reference to the addition operations in the stalks of the sheaf. By definition, the paths on each side of this equation have identical initial values; local constancy then guarantees that they agree everywhere.

(3) Well-definedness:

Let τ_y and υ_y be simple paths in $V - \overline{D}$ from y_0 to y. We wish to show that

$$\tilde{\tau}_{y,1}(1) = \tilde{\upsilon}_{y,1}(1).$$

By local constancy, each endpoint represents a generator of $\mathcal{S}_y$, hence differ at most by sign.

Let λ denote the (singular) loop obtained by concatenating τ_y and υ_y; we also let λ denote a map of the unit circle into V realizing this loop. Our hypotheses imply that this map extends to a map of the unit disc into U. The hypotheses concerning local homology allow us to conclude that λ can be extended to a map μ of a disc-with-handles Σ into $U - \overline{D}$. Let T denote the image of Σ under λ.

Note that $\mathcal{S}|_T$ is locally constant.

Let $\mu^*(\mathcal{S}|_T) = \wp$ be the inverse image of $\mathcal{S}|_T$ over μ, i.e. the pull-back of the diagram

$$\begin{array}{ccc} & & \mathcal{S}|_T \\ & & \downarrow \pi \\ \Sigma & \xrightarrow[\mu]{} & T \end{array}$$

Claim: $\wp$ is locally constant.

Pick $y \in \Sigma$; choose a neighborhood $\tilde{W}$ of $\mu(y)$ in T such that $\mathcal{S}|_{\tilde{W}}$ is constant. Choose a connected neighborhood W of y such that $\mu(W) \subset \tilde{W}$. Without loss of generality, W is a disc. The claim is proved by observing that $\wp|_W \cong W \times \mathbf{Z}$.

This last statement is easily verified by considering first that

$$\wp|_W = \{(y, s) \in W \times (\tilde{W} \times \mathbf{Z}) \mid \mu(y) = \pi(s)\}.$$

We then define a map from $\wp$ to $W \times \mathbf{Z}$ as

$$(y, (z, l)) \mapsto (y, l).$$

It is easy to check that the map is a sheaf isomorphism.

Thus the claim is proved.

Let $\pi_\wp$ denote the projection of $\wp$ onto Σ. Let

$$\Omega = \{ \text{ generators } p \text{ of } \pi_\wp^{-1}(s) | s \in \Sigma \}.$$

So Ω contains two elements of each stalk and, by the claim, is an orientable double cover of Σ.

Now if our map ψ is *not* well-defined, we may assume that $\tau_{y_1,1}(1) = -\tau_{y_1,1}(1)$. It follows quite easily that then Ω is in fact a disk–with–handles double covering Σ. Yet this is impossible: considering the Euler characteristic plus the nature of the covering we obtain

$$\chi(\Omega) = 2\chi(\Sigma) = 2(1 - 2g),$$

where g is the genus of Σ. On the other hand, since Ω is a disk–with–handles, $\chi(\Omega)$ is odd.

This contradiction establishes that ψ is well-defined. ∎

The following lemma is useful in chapter 3.

LEMMA 1.6.4.

Let g_1, g_2 denote orientable closed n-manifolds and let $f : g_1 \to g_2$ be a map which is an injection with respect to the global orientation class $\zeta_1 \in H_n(g_1)$, say $f_*(\zeta_1) = k \cdot \zeta_2$. Let $j = min\{i \mid \tilde{H}_i(g_2) \neq 0\}$.

Then $f^* : H^j(g_2) \to H^j(g_1)$ is an injection.

PROOF:

First note that $j > 0$ and, by the Universal Coefficient Theorem, $H^j(g_2)$ is free.

Choose $\xi \in H^j(g_2)$. Now we apply **16** on page 254 of [**S**]. Namely,

$$\begin{aligned} f_*(f^*\xi \frown \zeta_1) &= \xi \frown f_*\zeta_1 \\ &= \xi \frown k \cdot \zeta_2 \\ &= k \cdot (\xi \frown \zeta_2). \end{aligned}$$

Note $H_{n-j}(g_2) \cong H^j(g_2)$ is free; so if $\xi \neq 0$ then $k \cdot (\xi \frown \zeta_2)$ is non-zero. Therefore, f^* is an injection. ∎

CHAPTER 2

HOMOLOGY SPHERE DECOMPOSITIONS IN THE MIDDLE DIMENSION

This chapter contains the proof of Theorem 1 and consequences. The following are the standing hypotheses for this chapter. Let M be a $(n+k)$-manifold,$k > 2$, G an upper semicontinuous decomposition (uscd) of M into homology n-spheres, $n \geq 2$, p the decomposition map, and B the decomposition space. For $A \subseteq B$, let $\tilde{A}$ denote $p^{-1}(A)$. Furthermore, we assume that dim $B < \infty$ and that $E = D \cup K$ does not locally separate B.

In section 2.1, we find there are long exact sequences for p, and use these to determine immediately discernible local properties of B. Under the hypotheses of Theorem 1, it is shown, in section 2.2, that B being a k–gm is equivalent to the discontinuity set $E = D \cup K$ being locally finite; this is used frequently in section 2.3, which contains the proof of Theorem 1.

§2.1 The Long Exact Sequences of a Homology Sphere Decomposition.

The following lemma may be thought of as a type of Vietoris-Begle theorem [**Sp**, p.344].

LEMMA 2.1.1. *For all open sets $U \subseteq B$ there is a long exact sequence*

$$\cdots \to H^i(U;\mathbf{Z}) \to H^i(\tilde{U};\mathbf{Z}) \to H^{i-n}(U;\mathcal{H}^n) \to \cdots$$

which is natural with respect to inclusion maps.

LEMMA 2.1.2. *For all $b \in B$ and sets U containing b, there is a long exact sequence*

$$\cdots \to H^i(U, U-b) \to H^i(\tilde{U}, \tilde{U}-\tilde{b}) \to H_b^{i-n}(U;\mathcal{H}^n) \to \cdots$$

which is natural with respect to inclusion maps of neighborhoods of b.

PROOF OF 2.1.1 AND 2.1.2: Note that

$$\mathcal{H}^q[p]_x = \begin{cases} 0, & q \neq 0, n \\ \mathbf{Z}, & q = 0, n \end{cases}$$

and, therefore, in the Leray spectral sequence for the map p we have $E_2^{p,q} = 0$ for $q \neq 0, n$. It is a well known lacunary property of spectral sequences (see [**McL**, pp. 6–9]), that from this we obtain a long exact sequence

$$\cdots \to E_2^{i,0} \to H^i_{\phi(\psi)}(M) \to E_2^{i-n,n} \to E_2^{i+1,0} \to \cdots$$

Lemma 2.1.1 follows by taking the family of supports ϕ to be the closed subsets of U; taking ϕ to be the family consisting of the singleton $\{x\}$ and applying lemma 1.4.2 proves 2.1.2. ∎

The following is useful in investigating local properties of B.

PROPOSITION 2.1.3. *Given $b \in B$ and $i \in \mathbf{Z}$, the following are equivalent:*

(1) *B is i-clc$_{\mathbf{Z}}$ at b.*

(2) *B is (i-n-1)-clc$_{\mathcal{H}^n}$ at b.*

PROOF: Given $b \in B$ and U a neighborhood of b so that $\tilde{U}$ [shape] retracts to $\tilde{b}$. Let V be neighborhood of b contained in U so that $\tilde{V}$ [shape] deforms in $\tilde{U}$ to $\tilde{b}$. Let W be a neighborhood of b in V so that $\tilde{W}$ [shape] deforms to $\tilde{b}$ in $\tilde{V}$. In particular,

$$\text{image}\{H^j(\tilde{A}_1; \mathbf{Z}) \to H^j(\tilde{A}_2; \mathbf{Z})\} = \begin{cases} 0, & j \neq 0, n \\ \mathbf{Z}, & j = 0, n \end{cases}$$

where (A_1, A_2) is either of the pairs (U, V) or (V, W).

We also have

$$\begin{array}{ccccccc}
H^{i-1}(\tilde{U}) & \xrightarrow{\alpha_1} & H^{i-n-1}(U;\mathcal{H}^n[p]) & \xrightarrow{\alpha_2} & H^i(U) & \xrightarrow{\alpha_3} & H^i(\tilde{U}) \\
\downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle \delta_3} & & \downarrow{\scriptstyle \delta_4} \\
H^{i-1}(\tilde{V}) & \xrightarrow{\beta_1} & H^{i-n-1}(V;\mathcal{H}^n[p]) & \xrightarrow{\beta_2} & H^i(V) & \xrightarrow{\beta_3} & H^i(\tilde{V}) \\
\downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle \eta_3} & & \downarrow{\scriptstyle \eta_4} \\
H^{i-1}(\tilde{W}) & \xrightarrow{\gamma_1} & H^{i-n-1}(W;\mathcal{H}^n[p]) & \xrightarrow{\gamma_2} & H^i(W) & \xrightarrow{\gamma_3} & H^i(\tilde{W})
\end{array}$$

(1)⇒(2)

By choosing V smaller if necessary, we may assume that image δ_3 is finitely generated.

Now we apply lemma 1.6.2 to the following portion of the diagram:

$$\begin{array}{ccccc}
 & & H^{i-n-1}(U;\mathcal{H}^n[p]) & \xrightarrow{\alpha_2} & H^i(U) \\
 & & \downarrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle \delta_3} \\
H^{i-1}(\tilde{V}) & \xrightarrow{\beta_1} & H^{i-n-1}(V;\mathcal{H}^n[p]) & \xrightarrow{\beta_2} & H^i(V) \\
\downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \eta_2} & & \\
H^{i-1}(\tilde{W}) & \xrightarrow{\gamma_1} & H^{i-n-1}(W;\mathcal{H}^n[p]) & &
\end{array}$$

(2)⇒(1) By choosing V smaller if necessary, we may assume that image η_2 is finitely generated.

We apply lemma 1.6.2 to the following portion of the diagram:

$$\begin{array}{ccccc}
 & & H^i(U) & \xrightarrow{\alpha_3} & H^i(\tilde{U}) \\
 & & \downarrow{\scriptstyle \delta_3} & & \downarrow{\scriptstyle \delta_4} \\
H^{i-n-1}(V;\mathcal{H}^n[p]) & \xrightarrow{\beta_2} & H^i(V) & \xrightarrow{\beta_3} & H^i(\tilde{V}) \\
\downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle \eta_3} & & \\
H^{i-n-1}(W;\mathcal{H}^n[p]) & \xrightarrow{\gamma_2} & H^i(W) & &
\end{array}$$

∎

COROLLARY 2.1.4. *B is $clc^n_{\mathbf{Z}}$ at all points.*

PROOF: Since for $i \leq n$ we get $H^{i-n-1}(U; \mathcal{H}^n) = 0$ trivially, the result follows from the previous proposition. ∎

COROLLARY 2.1.5. *If the Leray sheaf in dimension n ($n > 2$) is locally constant at each $b \in B$, then B is a generalized k-manifold.*

PROOF: First we show that B is an ANR, then that B has the correct local relative homology.

Pick $b \in B$. Let U be a neighborhood of b over which $\mathcal{H}^n$ is constant, so $\mathcal{H}^n \cong U \times \mathbf{Z}$. Then for all $A \subseteq U$, $H^i(A; \mathcal{H}^n) \cong H^i(A; \mathbf{Z})$. By Corollary 2.1.4, pick a neighborhood $V_1 \subset U$ of b so that for all $i \leq n$

$$\text{image}\{H^i(U) \to H^i(V_1)\}$$

is trivial. By proposition 2.1.3 we obtain that B is $clc^{2n}_{\mathbf{Z}}$ at b. Continuing inductively, we find a neighborhood $V_i \subset U$ of b to get that B is $clc^{n \cdot 2^i}_{\mathbf{Z}}$ at b. When i is such that $n \cdot 2^i$ exceeds dim B, then we must conclude that B is $clc^{\infty}_{\mathbf{Z}}$ at b. Since b was arbitrarily chosen, B is *clc*. Now apply 1.5.13.

This concludes the first step.

We now show that

$$H^i(B, B - b) = \begin{cases} 0, & i \neq k \\ \mathbf{Z}, & i = k \end{cases}$$

It is an exercise in dimension theory (first see [N, p.206]) that for $i >$ dim$B$, $H^i(B, B - b) = 0$ (consider $B - b$ as a direct limit of complements of neighborhoods of b).

Pick a neighborhood U of b so that $\tilde{U}$ is an orientable submanifold of M (this is possible since the elements have the shape of homology n–spheres and $n \geq 2$).

Then using the Universal Coefficient Theorem and Alexander Duality we obtain

$$H^i(\tilde{U}, \tilde{U} - \tilde{b}) = \begin{cases} 0, & i \neq k, n+k \\ \mathbf{Z}, & i = k, n+k, \end{cases}$$

$$H_i(\tilde{U}, \tilde{U} - \tilde{b}) \cong H^{n+k-i}(\tilde{b}) = \begin{cases} 0, & i \neq k, n+k \\ \mathbf{Z}, & i = k, n+k, \end{cases}$$

By this and the long exact sequence of Proposition 2.1.2 we get

$$H^i(U, U - b) \cong H^{i-n-1}(U, U - b),$$

for $i < k$, $k+1 < i < n+k$,and $n+k+1 < i$. Arguing as in the first step, we may conclude, for $i < k, k < i < n+k$, and $i \geq n+k$, that $H^i(U, U-b) = 0$ (the case $i = n+k$ follows from dimension theory).

For $i = k$ we get (from the same long exact sequence)

$$0 \to H^k(U, U-b) \to \mathbf{Z} \to H^{k-n}(U, U-b) = 0.$$

Thus, $H^k(U, U-b) \cong \mathbf{Z}$.

Thus we have shown B to be a generalized cohomology manifold. Now apply the Universal Coefficient Theorem [**S**, p.248]. ∎

COROLLARY 2.1.6. *The set of points at which B is a generalized k-manifold contains an open dense subset of B.*

PROOF: Let $C \subset B$ be the open dense subset of B on which the sheaf is locally constant. Then $\Omega = \tilde{C}$ is an open submanifold of M. Apply Corollary 2.1.5 to $p|_\Omega : \Omega \to C$. So C is a generalized manifold. ∎

The following is also a corollary of Proposition 3.1 in [**DW3**].

COROLLARY 2.1.7. *If $k < n$, then B is a generalized k-manifold.*

PROOF: Since $\dim B \leq k < n < \infty$ and B is, by 2.1.4, $clc^k_{\mathbf{Z}}$ (hence clc), it follows from 1.5.13 that B is an ANR.

All that remains to be shown is that for all $b \in B$

$$H_j(B, B-b) = \begin{cases} 0, & j \neq k \\ \mathbf{Z}, & j = k. \end{cases}$$

By the proof of the latter half of Corollary 2.1.5, we have $H^i(B, B-b) = 0$ for $i < k$; the dimension of B ensures that the same is true for $i > k$. Finally, examining the long exact sequence of 2.1.2, we see

$$0 \to H^k(B, B-b) \to H^k(M, M-\tilde{b}) = \mathbf{Z} \quad \to H^{k-n}_x(B; \mathcal{H}^n) = 0.$$

Hence $H^k(B, B-b) = \mathbf{Z}$. ∎

Remark: Actually, if $k = n$, the result still holds (see [**DW3**]). The above method can, with additional care, also prove it.

§2.2 Characterization.

This section shows the equivalence of the conclusion of Theorem 1 (that B is a generalized manifold) with a limitation on the structure of the sheaf.

PROPOSITION 2.2.1. *Let G be a usc decomposition of the $(2n+1)$-manifold M into homology n-spheres. Then M/G is a generalized $(n+1)$–manifold if, and only if, the set $D \cup K$ is locally finite in M/G.*

PROOF: ($\Rightarrow$)

Suppose $B = M/G$ is a generalized $(n+1)$-manifold. An important property of such spaces is that they are locally orientable, and support Alexander Duality. For

example, if U is an orientable open set and A an arc in U, then

$$H_{k-1}(U, U - A) \cong H^1(A) = 0.$$

The proof proceeds as follows: we first show D to be countable , then locally finite in $B - K$, and, last, locally finite in B; then we show that K is locally finite.

D is countable

Express D as $D_1 \cup D_2$ where D_1 is countable and D_2 is dense in itself; we prove that D_2 is empty. Suppose not. Then D_2 is a non-empty closed subset of $B - K$ having a dense open continuity set, denoted $C(D_2)$, on which $\mathcal{H}^n$ is locally constant (see 1.2.3, 1.3.3, and [**DyW**]). Pick a point $d_0 \in C(D_2)$ and a neighborhood U of d_0 in B so that $\tilde{U}$ is orientable, U is orientable and connected, U is contained in the domain of α_{d_0}, and $U \cap D_2 \subset C(D_2)$; select a connected neighborhood V of d_0 in U so that $\tilde{V}$ shape deforms in $\tilde{U}$ to $\tilde{d}_0$ and also that $H_i(V) \to H_i(U)$ is trivial (where $i = n, n-1$); find a neighborhood W of d_0 in V with the same property with respect to V. There is a natural number ν for all $d \in U \cap C(D_2)$ and $c \in C \cap U$ such that $\alpha_d(c) = \nu$.

Fix a point c_0 in $W \cap C$. Pick two points d_1', d_2' in $W \cap C(D_2)$ distinct from d_0; find arcs A_i', $i = 1, 2$, in $W - (D_1 \cup \{d_0\})$ so that $\partial A_i' = \{c_0, d_i'\}$. Let A_i'' be the path in A_i' from c_0 to the first point in D that it meets, say d_i. Let A be a path in $A_1'' \cup A_2''$ from d_1 to d_2. Note that $A - \partial A \subset C$. By using lemma 1.4.5 one verifies that $H^n(\tilde{A}) = \mathbf{Z}$ and $H^{n+1}(\tilde{A}) = \mathbf{Z}_\nu$.

Now, since $\tilde{U}$ is orientable, we may apply Alexander Duality to conclude that

$H_n(\tilde{U}, \tilde{U} - \tilde{A}) \cong \mathbf{Z}_\nu$. We now consider the diagram

$$\begin{array}{ccccccc}
H_n(\tilde{W}) & \xrightarrow[\alpha_1]{} & H_n(\tilde{W}, \tilde{W} - \tilde{A}) & \xrightarrow[\beta_1]{} & H_{n-1}(\tilde{W} - \tilde{A}) & \longrightarrow & H_{n-1}(\tilde{W}) \\
\downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} & & \downarrow{\scriptstyle \tau_3} & & \downarrow{\scriptstyle \tau_4} \\
H_n(\tilde{V}) & \xrightarrow[\alpha_2]{} & H_n(\tilde{V}, \tilde{V} - \tilde{A}) & \xrightarrow[\beta_2]{} & H_{n-1}(\tilde{V} - \tilde{A}) & \xrightarrow[\gamma_2]{} & H_{n-1}(\tilde{V}) \\
\downarrow{\scriptstyle \chi_1} & & \downarrow{\scriptstyle \chi_2} & & \downarrow{\scriptstyle \chi_3} & & \downarrow{\scriptstyle \chi_4} \\
H_n(\tilde{U}) & \xrightarrow[\alpha_3]{} & H_n(\tilde{U}, \tilde{U} - \tilde{A}) & \xrightarrow[\beta_3]{} & H_{n-1}(\tilde{U} - \tilde{A}) & \longrightarrow & H_{n-1}(\tilde{U})
\end{array}$$

Note that χ_2, τ_2 are excision isomorphisms and that χ_4,τ_4 are zero maps. The latter fact implies that the image of τ_3 lies in the image of β_2. It can also be seen that, since A misses d_0 and $\tilde{d}_0$ carries the image of τ_1, χ_1, that the maps preceding α_1, α_2 in the long exact sequence are surjections and, hence, the maps β_1, β_2 are injective; the image of each, recall, is $\mathbf{Z}_\nu$. Note that $\tau_3|_{\mathrm{im}\beta_1}$ is injective. Thus, the image of τ_3 is non–trivial and contains $\mathbf{Z}_\nu$. But $\mathrm{im}\tau_3 \subset \ker\gamma_2 = \mathrm{im}\beta_2 = \mathbf{Z}_\nu$. Thus, we must have the image of τ_3 isomorphic to $\mathbf{Z}_\nu$.

We now use the Vietoris-Begle theorem to observe that the image of

$$H_{n-1}(W - A) \to H_{n-1}(V - A)$$

is isomorphic to $\mathbf{Z}_\nu$. This allows us to conclude that $H_n(V, V - A) \cong \mathbf{Z}_\nu$. But this is impossible, by Theorem 1.5.5, since B is a generalized $(n+1)$–manifold. Thus D is countable.

D is discrete

We now show that D is locally finite in $B - K$. Suppose not. Then there is a $d_0 \in D$ with points $\{d_i \in D \mid i > 0\}$ that converge to d_0. Pick a neighborhood U as before; we assume without loss of generality that each d_i lies in U. Say

$\alpha_{d_0}(c) = n_0 > 1$ for all c in $C \cap U$. Noting that $\alpha_{d_0}(c) = \alpha_{d_0}(d_i) \cdot \alpha_{d_i}(c)$ for c close enough to d_i, we conclude that $\{\alpha_{d_i}(c) | i \geq 0\}$ is a bounded set of positive integers. So there exists a positive integer $N > 1$ such that $\alpha_{d_i}(c) = N$ for infinitely many of the i . Now build an arc A connecting two such points (using that $n > 1$ and B is a generalized $(n+1)$-manifold), as in the proof of D's countability, in a small enough neighborhood V of d_0 to obtain a contradiction.

Thus D is locally finite in $B - K$.

K is discrete

Let k_0 be an element of K; we show the existence of a neighborhood of k_0 in B which meets $D \cup K$ in k_0 only. Let U, V, W be neighborhoods of k_0 with the usual properties. Suppose there are infinitely many points of K in W. Find an arc A in $W - \{k_0\}$ so that $\partial A = A \cap (K \cup D) = A \cap K = \{k_1, k_2\}$. By lemma 1.4.6, $H^{n+1}(\tilde{A}) \cong \mathbf{Z}$. Alexander Duality yields

$$H_n(\tilde{N}, \tilde{N} - \tilde{A}) \cong \mathbf{Z},$$

where N is U, V, or W. We now consider the diagram

$$\begin{array}{ccccccc}
H_n(\tilde{W}) & \xrightarrow[\alpha_1]{} & H_n(\tilde{W}, \tilde{W} - \tilde{A}) & \xrightarrow[\beta_1]{} & H_{n-1}(\tilde{W} - \tilde{A}) & \longrightarrow & H_{n-1}(\tilde{W}) \\
\downarrow \tau_1 & & \downarrow \tau_2 & & \downarrow \tau_3 & & \downarrow \tau_4 \\
H_n(\tilde{V}) & \xrightarrow[\alpha_2]{} & H_n(\tilde{V}, \tilde{V} - \tilde{A}) & \xrightarrow[\beta_2]{} & H_{n-1}(\tilde{V} - \tilde{A}) & \xrightarrow[\gamma_2]{} & H_{n-1}(\tilde{V}) \\
\downarrow \chi_1 & & \downarrow \chi_2 & & \downarrow \chi_3 & & \downarrow \chi_4 \\
H_n(\tilde{U}) & \xrightarrow[\alpha_3]{} & H_n(\tilde{U}, \tilde{U} - \tilde{A}) & \xrightarrow[\beta_3]{} & H_{n-1}(\tilde{U} - \tilde{A}) & \longrightarrow & H_{n-1}(\tilde{U})
\end{array}$$

Again note that χ_2, τ_2 are excision isomorphisms and that χ_4,τ_4 are zero maps. The latter fact implies that the image of τ_3 lies in the image of β_2. It can also be seen

that, since A misses k_0 and $\tilde{k}_0$ carries the image of τ_1, χ_1, that the maps preceding α_1, α_2 in the long exact sequence are surjections and, hence, the maps β_1, β_2 are injective; the image of each, recall, is $\mathbf{Z}$. Note that $\tau_3|_{im\beta_1}$ is injective. Thus, the image of τ_3 is non–trivial and contains $\mathbf{Z}$. But $\text{im}\tau_3 \subset \ker\gamma_2 = \text{im}\beta_2 = \mathbf{Z}$. Thus, we must have the image of τ_3 isomorphic to $\mathbf{Z}$.

We now use the Vietoris-Begle theorem to observe that the image of

$$H_{n-1}(W - A) \to H_{n-1}(V - A)$$

is isomorphic to $\mathbf{Z}$. This allows us to conclude that $H_n(V, V - A) \cong \mathbf{Z}$. But this is impossible, by Theorem 1.5.5, since B is a generalized $(n+1)$–manifold. Thus K is discrete.

E is discrete

Thus, without loss of generality, $U \cap K = \{k_0\}$. Note that $H^0_{k_0}(U; \mathcal{H}^n) \cong \mathbf{Z}$ is generated by the section that is zero everywhere on U except at k_0 where it takes the value of a generator of the stalk $\mathcal{H}^n_{k_0}$; this is since k_0 being isolated in U means a basic neighborhood of any element of $\mathcal{H}^n_{k_0}$ meets all the other stalks in the elements corresponding to 0. We now have $\dim(K \cup D) \leq 0$. So we may further suppose, by lemma 1.6.3, that $\mathcal{H}^n|_{U-(D\cup K)}$ is constant.

We have

$$0 \to H^0_{k_0}(U; \mathcal{H}^n) \to H^{n+1}(U, U - k_0) \to H^{n+1}(\tilde{U}, \tilde{U} - \tilde{k}_0) \to H^1_{k_0}(U; \mathcal{H}^n) \to 0.$$

Note that the first three terms in the sequence are isomorphic to $\mathbf{Z}$. Therefore, $H^1_{k_0}(U; \mathcal{H}^n) = \mathbf{Z}$. Restricting to V implies the existence of a non-zero section on $U - k_0$ which upon restriction to $V - k_0$ does not extend to V and every section

of $U - k_0$ restricted to $V - k_0$ is an integer multiple of this section's restriction. In particular, if $D \cap V = \{d_1, d_2, \ldots, d_i, \ldots\}$, the collection $\{\alpha_{d_i}(c_i) = p_i \mid i > 0\}$'s ($c_i$ close to d_i) is bounded, which is a contradiction, unless the set is finite.

Thus $D \cup K$ is locally finite.

∎

($\Leftarrow$)

Suppose that $D \cup K$ is locally finite. In order to show that B is a generalized $(n+1)$-manifold it suffices to show that B is LC^{n+1} at each $x \in D \cup K$ and $H_i(B, B - x)$ is the appropriate group for all x and i.

Choose $x \in D \cup K$. Choose a small enough connected neighborhood U of x so that $U \cap (D \cup K) = \{x\}$ and $\mathcal{H}^n|_{U-\{x\}}$ is constant.

Whether x is in D or K we get $H^0(U; \mathcal{H}^n) = \mathbf{Z}$. So by Proposition 2.1.3 we get B is $clc^{\infty}_{\mathbf{Z}}$ at x. Thus, B is an ANR, by lemma 1.5.13.

We now determine $H^{n+1}(B, B - x) = H^{n+1}(U, U - x)$.

Case 1: $x \in D$.

There is a $\nu \in \mathbf{Z}$ so that for *any* section over $U - \{x\}$ (and by lemma 1.6.3 we may assume $\mathcal{H}^n|_{U-\{x\}}$ to be constant, so non-trivial sections exist), the ν multiple of it extends to U. This implies that $H^1_x(U; \mathcal{H}^n)$ is a subgroup of $\mathbf{Z}_\nu$. In any event, the sequence

$$0 = H^0_x(U; \mathcal{H}^n) \to H^{n+1}(U, U - x) \to H^{n+1}(\tilde{U}, \tilde{U} - \tilde{x}) = \mathbf{Z} \to H^1_x(U; \mathcal{H}^n)$$

then reveals that $H^{n+1}(U, U - x) = \mathbf{Z}$.

Case 2: $x \in K$.

We analyzed isolated points of K in the first half of this proof. Using that analysis,

we obtain

$$H^1_x(U;\mathcal{H}^n) = \mathbf{Z} = H^0_x(U;\mathcal{H}^n);$$

and $H^{n+1}(\tilde{U}, \tilde{U} - \tilde{x}) = \mathbf{Z}$ maps onto, hence isomorphically to, $H^1_x(U;\mathcal{H}^n)$ in the sequence. Hence $H^{n+1}(U, U - x) = H^0_x(U;\mathcal{H}^n) = \mathbf{Z}$. ∎

This concludes the proof of the proposition.

§2.3 Proof of Theorem 1.

In this section we prove Theorem 1, by showing $E = D \cup K$ to be locally finite. The strategy is identical with that used in the last section when proving the necessity of the local finiteness of $D \cup K$, except the use there of local duality of a generalized manifold is replaced by a more sophisticated analysis.

The non-degeneracy set D.

Since the sheaf $\mathcal{H}^n|_{B-K}$ is Hausdorff and B is locally connected, we may conclude that $B - K$ is *clc* by lemma 2.1.3. (for given a section on a connected neighborhood of a point of B, any section defined on that same neighborhood agreeing with the given section at a point agrees with it everywhere (see section 2.2)). Thus, $B - K$ is an ANR, by 1.5.13.

Given a point d in D we observe that $H^0_d(U;\mathcal{H}^n) = 0$ for any neighborhood U of d. Any section which is non-zero at d is also non-zero on a small neighborhood of d, thus the Leray spectral sequence yields

$$0 = H^0_d(U;\mathcal{H}^n) \to H^{n+1}(U, U - d) \to H^{n+1}(\tilde{U}, \tilde{U} - \tilde{d}) = \mathbf{Z} \to \ldots.$$

Thus,

$$H^{n+1}(U, U - d) = H^{n+1}(B, B - d)$$

is either trivial or $\mathbf{Z}$. The sequence also yields $H^i(B, B-d) = 0$ for $i < n+1$. In consequence, we need only analyze the former case.

Again we describe D as $D_1 \cup D_2$ where D_1 is countable and D_2 is dense in itself. The first step is to show that D_2 is empty.

The critical observation about points in D is

LEMMA 2.3.1.

Suppose $d_0 \in D$ and $q \in \mathbf{Z}_+$ $(q > 1)$. Choose connected neighborhoods U, V, W of d_0 so that

(1) $W \subset V \subset U$,

(2) $\tilde{U}$ *is orientable,*

(3) $H_i(W) \to H_i(V)$ *has trivial image for* $i = k-2, k-1$,

(4) $\tilde{V}$ *[resp.* $\tilde{W}$*] shape deformation retracts to* $\tilde{d}_0$ *in* $\tilde{U}$ *[resp.* $\tilde{V}$*] and,*

(5) $U \cap K = \emptyset$.

Then the set

$$F(q) = \{x \in (D \cap W) - \{d_0\} \mid \alpha_x|_C = q \text{ and } H^{n+1}(B, B-x) = 0\}$$

consists of at most one point.

PROOF:

If there are distinct points $x_1, x_2 \in F(q)$, using, among other things, the fact that D does not locally separate B to build an arc A in W whose endpoints are x_1, x_2 and whose interior lies in C. By lemma 1.4.5, we obtain

$$\mathbf{Z}_q \cong H^{n+1}(\tilde{A}) \cong H_n(\tilde{U}, \tilde{U} - \tilde{A}),$$

the latter map given by Alexander duality. We then consider the commutative diagram with exact rows

$$\begin{array}{ccccccc}
H_n(\tilde{W}) & \xrightarrow[\alpha_1]{} & H_n(\tilde{W}, \tilde{W}-\tilde{A}) & \xrightarrow[\beta_1]{} & H_{n-1}(\tilde{W}-\tilde{A}) & \longrightarrow & H_{n-1}(\tilde{W}) \\
\downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} & & \downarrow{\scriptstyle \tau_3} & & \downarrow{\scriptstyle \tau_4} \\
H_n(\tilde{V}) & \xrightarrow[\alpha_2]{} & H_n(\tilde{V}, \tilde{V}-\tilde{A}) & \xrightarrow[\beta_2]{} & H_{n-1}(\tilde{V}-\tilde{A}) & \xrightarrow[\gamma_2]{} & H_{n-1}(\tilde{V}) \\
\downarrow{\scriptstyle \chi_1} & & \downarrow{\scriptstyle \chi_2} & & \downarrow{\scriptstyle \chi_3} & & \downarrow{\scriptstyle \chi_4} \\
H_n(\tilde{U}) & \xrightarrow[\alpha_3]{} & H_n(\tilde{U}, \tilde{U}-\tilde{A}) & \xrightarrow[\beta_3]{} & H_{n-1}(\tilde{U}-\tilde{A}) & \longrightarrow & H_{n-1}(\tilde{U})
\end{array}$$

Note that χ_2, τ_2 are excision isomorphisms and that χ_4,τ_4 are zero maps. The latter fact implies that the image of τ_3 lies in the image of β_2. It can also be seen that, since A misses d_0 and $\tilde{d}_0$ carries the image of τ_1, χ_1, that the maps preceding α_1, α_2 in the long exact sequence are surjections and, hence, the maps β_1, β_2 are injective; the image of each, recall, is $\mathbf{Z}_q$. Note that $\tau_3|_{im\beta_1}$ is injective. Thus, the image of τ_3 is non–trivial and contains $\mathbf{Z}_q$. But $\mathrm{im}\tau_3 \subset \ker\gamma_2 = \mathrm{im}\beta_2 = \mathbf{Z}_q$. Thus, we must have the image of τ_3 isomorphic to $\mathbf{Z}_q$.

We now use the Vietoris-Begle theorem to observe that the image of

$$H_{n-1}(W-A) \to H_{n-1}(V-A)$$

is isomorphic to $\mathbf{Z}_q$. This allows us to conclude that $H_n(V, V-A) \cong \mathbf{Z}_q$.

We now obtain a contradiction. Let A_1, A_2 denote subarcs of A whose union is A. Let c denote their common endpoint. Applying the Mayer-Vietoris theorem to the triple $(V, V-A_1, V-A_2)$ we obtain the exact sequence

$$
\begin{array}{ccc}
H_n(V, V-A_1) \oplus H_n(V, V-A_2) & = & H_n(V, V-x_1) \oplus H_n(V, V-x_2) = 0 \\
\downarrow & & \\
H_n(V, V-c) & = & \mathbf{Z} \\
\downarrow & & \\
H_n(V, V-A) & = & \mathbf{Z}_q
\end{array}
$$

To show the horizontal map at the beginning of this diagram is an isomorphism, use the fact that C is a generalized manifold (the technique is in [**D3**, p.192], [**DyW**]).

The exactness of the sequence is patently impossible. ∎

We now prove that D_2 is void. Choose neighborhoods of $d_0 \in D_2$ as in the above lemma so that $\mathcal{H}^n|_{U \cap D_2}$ is constant. By hypothesis, the intersection of C with each connected open set is a connected set. Let $N = \alpha_{d_0}(c)$, where c is any point in C. Then N is a bound such that $F(q) = \emptyset$ for all $q > N$. Hence, as a consequence of lemma 2.3.1, we may assume that $H_k(B, B-x) = \mathbf{Z}$ for all $x \in V - \{d_0\}$. Thus, by proposition 2.2.1, $D \cap (V - \{d_0\})$ is a discrete subset of $V - \{d_0\}$. But this contradicts that D_2 is dense in itself. Therefore, $D_2 = \emptyset$.

Thus, D is countable.

<u>D is locally finite in $B - K$</u>

Suppose $d_0 \in D$ is a limit point of D. Choose neighborhoods of d_0 satisfying the hypotheses of lemma 2.3.1, and so that α_{d_0} is defined on U. Since D is countable we may write

$$(U - d_0) \cap D = \{d_1, d_2, d_3, \dots\}.$$

Then the values of each α_{d_i} on C is bounded by the value of α_{d_0} on C. Hence, we

may assume, by lemma 2.3.1, that

$$H_{n+1}(B, B - d_i) = \mathbf{Z}$$

for all $i > 0$ (in particular, by 2.2.1, each d_i is isolated). So $H_{n+1}(B, B - d_0) \neq \mathbf{Z}$, as another consequence of 2.2.1, and which implies $H_{n+1}(B, B - d_0) = 0$. By lemma 1.6.3, we may assume $\mathcal{H}^n|_{V - \cup\{d_i | i \geq 0\}}$ is constant.

Since $H_{n+1}(B, B - d_0) = 0$, the long exact sequence of the map insures the existence of a (non-zero) section σ on $V - \{d_0\}$ such that no non-zero multiple of σ extends to V. To see this, first consider that

$$\mathbf{Z} \cong H^{n+1}(\tilde{V}, \tilde{V} - \tilde{d}_0) \cong H^1_{d_0}(V; \mathcal{H}^n);$$

also we may assume that V was chosen so that the map

$$H^1(U; \mathcal{H}^n) \to H^1(V; \mathcal{H}^n)$$

has trivial image, by proposition 2.1.3 and the fact that B is $(n+1)$–clc. Now consider the diagram

$$\begin{array}{ccc} H^1_{d_0}(U; \mathcal{H}^n) & \longrightarrow & H^1(U; \mathcal{H}^n) \\ \downarrow \cong & & \downarrow 0 \\ H^1_{d_0}(V; \mathcal{H}^n) & \xrightarrow[\alpha]{} & H^1(V; \mathcal{H}^n) \end{array}$$

The horizontal maps are from the long exact sequence of the pair $(U, U - \{d_0\})$, $(V, V - \{d_0\})$. The vertical maps are induced by inclusion. The first vertical map is an excision isomorphism. Thus, the map α has trivial image. Finally, consider the following portion of the long exact sequence for $(V, V - \{d_0\})$:

$$H^0(V; \mathcal{H}^n) \to H^0(V - d_0; \mathcal{H}^n) \to H^1_{d_0}(V; \mathcal{H}^n) \cong \mathbf{Z} \to 0.$$

The last map, to 0, is of course α. The second map thus being onto implies the existence of an element (i.e. section) of $H^0(V - d_0; \mathcal{H}^n)$ which maps onto the generator of $H^1_{d_0}(V; \mathcal{H}^n)$. It is clear that no integral multiple of this section extends to V for then that multiple would be in the image of the first of these maps, which is identical with the kernel of the second map.

Suppose that, for $c \in C \cap V$, $\alpha_{d_0}(c) = q_0$. Now, if $\sigma(c) = l$ (which, without loss of generality, is positive), then consider the function $\sigma' : V \to \mathcal{H}^n$ defined by

$$\sigma'(x) = \begin{cases} q_0 \cdot \sigma, & x \in V - \{d_0\} \\ l, & x = d_0. \end{cases}$$

We show that σ' is continuous at d_0, i.e. a section, contradicting the non-extendibility of σ.

Consider a sequence of points $x_i \to d_0$. We show $\sigma'(x_i) \to \sigma'(d_0)$.

<u>case 1</u> $\{x_i\} \subset C$.

The function is defined so that this case is clear.

<u>case2</u> $\{x_i\} \subset D$.

Say $x_i = d_{j_i}$ and $\alpha_{d_{j_i}}(c) = q_i$ for $c \in C$ sufficiently close to d_{j_i}. Note that $q_0 = \alpha_{d_0}(d_i) \cdot q_i$.

Then

$$\begin{aligned} \sigma'(x_i) &= q_0 \cdot l/q_i \\ &= l \cdot \alpha_{d_0}(d_i). \end{aligned}$$

These points in the stalk obviously lie in a neighborhood of $\sigma'(d_0)$.

This concludes the proof that D is a discrete subset of $B - K$.

The degeneracy set K

We move on to analyse the situation at the points in K, a decidedly more subtle matter.

To give the argument a more coherent logical structure we proceed as follows:

(1) show that the isolated points of K are not in the closure of D and, consequently, that B is a generalised manifold at such points,

(2) show that K consists entirely of isolated points.

Isolated points of K are isolated points of $D \cup K$

Suppose that k is an isolated point of K which is in the closure of D. By the proof of 2.2.1, it must be that B is not a generalized manifold at that point. Choose neighborhoods U, V of k so that $\tilde{U}$ is orientable and retracts to $\tilde{k}$, $U \cap K = \{k\}$, $V \subset U$ with $\tilde{V}$ deforming in $\tilde{U}$ to $\tilde{k}$, and $U \cap (D - cl_B V)$ has finite cardinality.

It should first be noted that U is an ANR, since B is $clc^{n+1}_{\mathbf{Z}}$ at k as can be checked by noting that the only non-zero sections defined on U (or V) are those which are zero in every stalk except the one over k; then apply 2.1.3 and 1.5.13.

It is clear, by examining sections, that $H^0(U; \mathcal{H}^n) = \mathbf{Z}$. If B is to fail to be a generalized manifold at k, which must be the case here by 2.2.1, we must have $H^{n+1}(U, U-k) = \mathbf{Z} \oplus \mathbf{Z}$ and $H^1_k(U; \mathcal{H}^n)$ must be torsion. In fact, the latter module will be trivial (since otherwise there is a non–zero section on $U - k$ with some multiple extending to U).

Find an arc $A \subset V$ so that $D \cap V = D \cap A$ and $k \in \partial A$.

By using the Universal Coefficient Theorem for singular cohomology and Alexan-

der Duality we get

$$\begin{aligned} H^{n+1}(\tilde{U}, \tilde{U} - \tilde{A}) &= Hom_{\mathbf{Z}}(H_{n+1}(\tilde{U}, \tilde{U} - \tilde{A}), \mathbf{Z}) \oplus Ext(H_n(\tilde{U}, \tilde{U} - \tilde{A}), \mathbf{Z}) \\ &= Hom_{\mathbf{Z}}(H^n(\tilde{A}), \mathbf{Z}) \oplus Ext(H^{n+1}(\tilde{A}), \mathbf{Z}) \\ &= \mathbf{Z} \oplus Ext(H^{n+1}(\tilde{A}), \mathbf{Z}) \\ &= \mathbf{Z} \oplus \Xi \end{aligned}$$

where $\Xi = Ext(H^{n+1}(\tilde{A}, \mathbf{Z})$. Also

$$H^{n+1}(U, U - A) \cong H^{n+1}(U, U - k) = \mathbf{Z} \oplus \mathbf{Z}.$$

By applying lemma 1.6.3, we may assume $\mathcal{H}^n$ is constant on $U - (D \cup \{k\})$; thus, there is a generating section of $U - A$, say σ, with no non-zero multiple of its restriction to $V - A$ extending continuously to V, where V is a small neighborhood of k in U. We also observe that

$$H^0(U - A; \mathcal{H}^n) \to H^1(U, U - A; \mathcal{H}^n) \to H^1(U; \mathcal{H}^n);$$

the last module in this sequence, when restricted to V is trivial. From this we conclude that $H^1(U, U - A; \mathcal{H}^n) = \mathbf{Z}$.

From all the above, and the long exact sequence of the map, we obtain the exact sequence

$$\mathbf{Z} \to \mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z} \oplus \Xi \to \mathbf{Z} \to 0.$$

Note first that Ξ is finitely generated. Also Ξ is non-trivial; otherwise the last of these maps would be an isomorphism, which in turn would imply that the first is onto –a contradiction.

We now analyze Ξ more carefully in order to conclude that it actually is countably generated. In particular we will demonstrate that $H^{n+1}(\tilde{A}, \mathbf{Z}) \cong \sum_{i=1}^{\infty} \mathbf{Z}_{q_i{}^{n_i}}$ where $\{q_i | i = 1, 2, 3, \dots\}$ is a subset of the prime integers and n_i is either 1 or 2. It will then follow from a well-known property of the functor Ext that Ξ is the direct product of the $\mathbf{Z}_{q_i}$'s, which is not finitely generated.

First a short digression. Let $p_i = \alpha_{d_i}(c)$ for $c \in C$ close to d_i. If $gcd(p_i, p_j) = q > 1$, then suppose π is a path in $V - \{k, d_l \mid l \neq i, j\}$ connecting d_i and d_j. Then lemma 1.4.5 applies so we know the cohomology of $\tilde{\pi}$ in dimension $n+1$ is isomorphic to $\mathbf{Z}_q$.

Proceeding with our analysis of E, let $A_0 = A$, c_i a point in C separating d_i and d_{i+1} in A, and A_i the subinterval of A with endpoints c_i and k. A Mayer-Vietoris argument in conjunction with the above digression reveals that

$$H^{n+1}(\tilde{A}_i) \to H^{n+1}(\tilde{A}_{i+1})$$

is onto with kernel isomorphic to $\mathbf{Z}_{p_i}$. Also, $H^{n+1}(\cap \tilde{A}_i) = H^{n+1}(\tilde{k}) = 0$.

Therefore, by Theorem 1.6.1, there exist primes $q_1, q_2, \dots$ with

$$\Gamma = H^{n+1}(\tilde{A}) \cong \sum_{i=1}^{\infty} T_i$$

where

$$T_i = \begin{cases} \mathbf{Z}_{q_i{}^{n_i}}, & n_i \text{ is either 1 or 2} \\ \mathbf{Z}_{q_i} \oplus \mathbf{Z}_{q_i}. \end{cases}$$

There may be repetition in the list of primes, but any given prime appears at most twice.

Thus $E = Ext(\Gamma, \mathbf{Z}) \cong \prod T_i$, by a well-known property of Ext (see [**R**, p.199]). Thus, E cannot possibly be finitely generated.

This contradiction proves the assertion that the isolated points of K are isolated points of $K \cup D$.

K is discrete

Let K' denote the points of K which are not isolated points of K. Note that K' is a closed subset of B. We conclude the proof of the Theorem 1 by showing that K' is void. We define $C(K')$ to be an open dense subset of K' where the sheaf $\mathcal{H}^n|_{K'}$ is locally constant (see 1.2.3, 1.3.3, and [**DyW**]). The strategy is to first show that every point of $C(K')$ is a limit point of $K - K'$ (which is obvious for points of K' isolated in K'); we then establish a contradiction using this fact. Since $C(K')$ is an open dense subset of K', we conclude that $C(K') = K' = \emptyset$.

Let $k \in C(K')$. Let U be a neighborhood of k in B so that $\alpha_k|U \cap K'$ is continuous with $\tilde{U}$ orientable, $V \subset U$ a neighborhood such that $\tilde{V}$ [shape] deforms to $\tilde{k}$ in $\tilde{U}$.

Suppose that k is not isolated in K'. So there are infinitely many points of K' in any neighborhood of k. Furthermore suppose that k is not in the closure of $K - K'$. Then we can assume that U contains no isolated points of K. Note that $H^{n+1}(U, U - \{k'\})$ is either trivial or $\mathbf{Z}$ for every $k' \in U \cap K'$, since $H^0_x(U; \mathcal{H}^n) = 0$.

Claim 1: There cannot be distinct points $k_1, k_2 \in V \cap K'$ with $H^{n+1}(V, V - \{k_i\}) = 0$ for $i = 1, 2$. Else, using the long exact sequence of the map as was done previously, we detect sections

$$\sigma_i : U - \{k_i\} \to \mathcal{H}^n$$

so that no (non-zero) multiple of σ_i extends to U, $i = 1, 2$. Because k is not in the closure of $K - K'$ we have for each i that $\sigma_i|U - K' \equiv 0$. Moreover, since $U \cap K'$ is in the continuity set of K', we have $\sigma_i(x) = m_i > 0$ for all $x \in U \cap K'$. But then we can easily see that $m_2 \cdot \sigma_1$ can be extended continuously to U by comparing it

to $m_1 \cdot \sigma_2$ which *is* defined at k_1. Claim 1 is verified.

Before proceeding, we should observe that U is an ANR. This is suspect only at points $k' \in U \cap K = U \cap C(K')$, a suspicion which the following claim allays.

Claim 2: The image of $h : H^0(U; \mathcal{H}^n) \to H^0(V'; \mathcal{H}^n)$ is finitely generated (where V' is a sufficiently small neighborhood of k'). Pick distinct non-zero elements σ_1, σ_2 of this image, with $\tilde{\sigma}_1, \tilde{\sigma}_2$ respective elements of $h^{-1}(\sigma_1), h^{-1}(\sigma_1)$. Let $s_i = \sigma_i(k'), i = 1, 2$. Note that $s_i \neq 0$ since otherwise σ_i will be the zero section since V' was chosen small, $k' \in C(K')$, and $U \cap (K - K') = \emptyset$. Let t be the greatest common divisor of s_1 and s_2. By taking the appropriate linear sum of σ_1 and σ_2, we can obtain a non-trivial section σ_3 of V' which is the restriction of a section $\tilde{\sigma}_3$ of U such that $\sigma_3(k') = t$. Let t_0 be the minimum t taken over all possible pairs of such sections, let σ be the corresponding section over V'.

Then clearly σ generates the image of h. If τ is a non-zero element of the image of h, then $\gcd(\sigma(k'), \tau(k'))= t_o$, otherwise the minimality of t_0 is contradicted. Thus, the image is either trivial or infinite cyclic, proving claim 2.

Therefore, U is $(n+1)$-clc$_{\mathbf{Z}}$ and, so, by 2.1.3, clc$^{\infty}_{\mathbf{Z}}$. Hence, U is an ANR, by 1.5.13.

It must be that for the given k $H^{n+1}(U, U - \{k\}) = 0$, for otherwise U is a neighborhood of k which is a generalized manifold by claim 1, but has infinitely many sheaf discontinuities, contradicting the Proposition 2.2.1. As a consequence, there are no points in $(U \cap K') - \{k\}$ since they would have such neighborhoods by claim 1. Thus k is isolated in K'. But $k \in K'$, not being in the closure of $K - K'$, must then be isolated in K, contradicting the defining property of points in K'.

So we assume that each $k \in C(K')$ which is not isolated in K' is a limit point of

$K - K'$. This is necessarily true of any isolated point of K'.

Note that $U - K'$ is a generalized manifold (but U itself is not an ANR!). Thus, for any arc A in $U - K'$, $H^n(U, U - A) = 0$ (this can be seen by first excising K' to form the pair $(U - K', U - (K' \cup A))$ and again using the technique appearing in [**D3**, p.192]). The Universal Coefficient Theorem can then be applied to see that $H_n(U, U - A)$ is a torsion module.

Find an arc A' in $V - K'$ with $(A' - \partial A') \subset C$ and $\partial A' = \{k_1, k_2\} \subset K - K'$.

Let $c \in A' - \{k_1, k_2\}$ and let A_i denote the closure of the component of $A' - \{c\}$ containing k_i. Consider the sequence

$$H^0(A_i; \mathcal{H}^n) \to H^0(k_i; \mathcal{H}^n) \to H^1(A_i, k_i; \mathcal{H}^n) \to H^1(A_i; \mathcal{H}^n) \to 0.$$

The first map is a natural isomorphism in this case since the generating sections are equivalent. Thus the third map is an isomorphism.

It can be calculated, using Proposition 12.2 of [**Br1**, p.59] and our explicit knowledge of the structure of $\mathcal{H}^n|_{A_i}$, that $H^1(A_i, k_i; \mathcal{H}^n)$ is isomorphic to $H^1(A_i; A_i \times \mathbf{Z}_{A_i - k_i}) \cong 0$. So $H^1(A_i; \mathcal{H}^n) = 0$.

Now we examine the Mayer-Vietoris sequence

$$\cdots \to H^0(A_1; \mathcal{H}^n) \oplus H^0(A_2; \mathcal{H}^n) \to H^0(c; \mathcal{H}^n) \to H^1(A'; \mathcal{H}^n) \to 0.$$

The first map is trivial because the sections residing over the A_i's are trivial over c. Thus the second map is an isomorphism; $H^0(c; \mathcal{H}^n)$ obviously being infinite cyclic abelian implies that $H^1(A'; \mathcal{H}^n)$ is as well.

Choose $k_3 \in (K \cap V) - \{k_1, k_2\}$ and an arc A_3 from c to k_3 in V so that $A_3 \cap A' = \{c\}$ and $A_3 \cap K = \{k_3\}$. By applying the Mayer-Vietoris argument to the pair

(A', A_3) we find that

$$H^1(A' \cup A_3; \mathcal{H}^n) = \mathbf{Z} \oplus \mathbf{Z}.$$

Let $A = A' \cup A_3$. First we get, by Alexander Duality, that the image J of

$$H_{n-1}(\tilde{V} - \tilde{A}) \to H_{n-1}(\tilde{U} - \tilde{A})$$

is $\mathbf{Z} \oplus \mathbf{Z}$ (since A misses k_0 we in essence have the map $H_n(\tilde{U} - \tilde{A}) \to H_n(\tilde{U})$ is onto, since $\tilde{k}_0 \subset (\tilde{U} - \tilde{A})$ is the essential carrier of the latter group).

Using the Vietoris-Begle Theorem we conclude that

$$H_{n-1}(V - A) \to H_{n-1}(U - A)$$

has image isomorphic to this group.

Finally we consider

$$\begin{array}{cccccc} H_n(V) & \to H_n(V, V-A) & \to H_{n-1}(V-A) & \to H_{n-1}(V) \\ \downarrow & \downarrow & \downarrow & \downarrow \\ H_n(U) & \to H_n(U, U-A) & \to H_{n-1}(U-A) & \to H_{n-1}(U) \end{array}$$

The second vertical arrow is an excision isomorphism of trivial groups, the fourth is a trivial map. Thus we must have $J = 0$.

Thus, $C(K')$ is empty. In consequence, K' is empty, implying K is a discrete subset of B.

This concludes the proof of the theorem.

§2.4 Corollaries, examples, and questions.

This section shows the possible types of sheaf structures that can be attained, and a certain limitation is proved in case the source space is the $(2n+1)$-sphere.

PROPOSITION 2.4.1. *Suppose G to be a decomposition of a $(2n+1)$-manifold M into continua having the shape of homology n-spheres with dim $M/G < \infty$ such that $E = D \cup K$ does not locally separate M/G. Suppose $H^n(M) = 0$. If M/G is orientable (e.g. if it is simply connected), then K consists of at most one point.*

PROOF: The long exact sequence associated with the map yields

$$0 \to \check{H}^0(M/G; \mathcal{H}^n) \to H^{n+1}(M/G) = \mathbf{Z}.$$

Since each point of K corresponds to a generator of $\check{H}^0(M/G; \mathcal{H}^n)$ due to local finiteness of K, the conclusion follows. ∎

The following is an easy consequence of the fact that maps with connected point pre–images induce epimorphisms of the fundamental group.

COROLLARY 2.4.2. *Each decomposition of the $(2n+1)$-sphere into homology n-spheres (with finite dimensional image and exceptional set E not locally separating the decomposition space) has at most one degenerate element.*

The example of the Introduction illustrates K being non-empty.

Example 2.4.1: Let Σ^n denote a homology n-sphere, $n > 2$. Then the $(2n+1)$-sphere is the $(n+1)$st suspension of Σ^n, since Edward's double suspension theorem ([**E**] and [**C**]) says that the second suspension of a homology sphere is a sphere. There is a rather natural (usc) decomposition of the $(2n+1)$-sphere each of whose elements, but for one, is homeomorphic to Σ^n, and that special element is an n-sphere. That sphere is the one degenerate element of the decomposition. The decomposition space is the $(n+1)$-sphere; over the complement of K the map is a homology-sphere bundle over euclidean $(n+1)$-space. Note that in the special case

where the homology sphere is actually a sphere, then the map is the Hopf fibration; also we may perform surgeries on disjoint PL neighborhoods of finitely many of the non-degenerate spheres to get a decomposition of a compact orientable manifold with the cardinality of K arbitrarily high.

Example 2.4.2: Let Σ denote the Poincare homology sphere; there is a 120-to-1 covering of the 3-sphere to Σ. Let $\mathcal{G}$ denote the covering transformations for this cover. This group action extends naturally to E^4, the open cone on S^3. Now let this group act diagonally on the product $S^3 \times E^4$; let the orbit space be M. There is a natural decomposition of the $(2n+1)$-manifold M into n-spheres and an exceptional homology n-sphere; the latter is the image, under the orbit map, of $S^3 \times \{0\}$, whereas the former are the images, under the orbit map, of $\mathcal{G}(x) \times r \cdot S^3$ where $\mathcal{G}(x)$ denotes the orbit of the given point $x \in S^3$ and $r \cdot S^3$ denotes the sphere in E^4 of radius r centered at the origin.

Note that $K = \emptyset$ and D is a singleton whose winding function equals 120 on its continuity set.

For non-degenerate decompositions of S^{2k+1} into copies of S^k (k odd), we refer the reader to [**CD2**]. They also show that for even k no such decompositions exist; this is achieved by computing the Hopf invariant (which is 0 for even k) to be an upper bound on the cardinality of D. A question we have left unanswered: Is there an analogous bound for a non–degenerate homology n-sphere decomposition of S^{2n+1}?

COROLLARY 2.4.3. *There is no uscd of Euclidean $(2n+1)$-space into (homology) n-spheres.*

PROOF: We consider the Leray spectral sequence of the decomposition map p with

the families of compact supports c, ϕ on domain and image. Clearly, $\check{H}^0_\phi(B; \mathcal{H}^n) = 0$ and $H^{n+1}_c(E^{2n+1}) = 0$. Thus, $H^{n+1}_c(B) = 0$.

On the other hand, since B is an orientable $(n+1)$–gm, Duality theory tells us that $H^{n+1}_c(B) \cong H_0(B) \cong \mathbf{Z}$, a contradiction. ∎

Questions:

(1) Is there a non–degenerate homology n–sphere decomposition of S^{2n+1} each of whose elements is a non–trivial homology n–sphere? Are there any such decompositions for even k? (if all the elements are sphere–like and B is a manifold, this is known to be impossible)

(2) Given any non–degenerate homology n–sphere decomposition of S^{2n+1}, is there an invariant which measures the maximum allowable number of points in D? Can this maximum be realized?

(3) For which n and $n > k > 0$ (if any) is there a homology n–sphere decomposition of $S^{2n+1-k} \times S^k$?

CHAPTER 3

A CLASS OF PARTIALLY ACYCLIC MANIFOLD DECOMPOSITIONS

In this chapter, we prove Theorem 2 of the Introduction. The chapter is organized as follows: section 1 is a discussion of pertinent quantities in the Leray spectral sequence and the immediately discernable connectivity properties of B; section 2 shows the equivalence of B being a generalized k-manifold over $\mathbf{Z}$ with local finiteness of the discontinuity set; section 3 demonstrates that the decompositions under consideration must always have locally finite discontinuity set; and section 4 contains some examples and corollaries.

It should be noted that there is a rather slight difference in some of the details here from those of chapter 2. Here the sheaf $\mathcal{H}^{k-1}$ appears to play a role, though for the most part we avoid dealing with it by deepening our understanding of how to calculate cohomology in sufficiently high dimensions with coefficients in $\mathcal{H}^n$.

§3.1 The Leray spectral sequence of the decomposition map.

In this section, we calculate the crucial parts of the Leray spectral sequence of the decomposition map.

Assume that G is an uscd of the orientable $(n+k)$-manifold into subcontinua so that the following properties are satisfied:

(†) For all $g \in G$, $\tilde{H}_i(g) = 0$ for $i \in \{0, 1, \ldots, k-2\}$.

(††) Each $g \in G$ has the shape of a closed orientable n-manifold.

(†††) $\dim M/G < \infty$.

Note that $E_2^{p,q} = 0$: for all $p > k$, by (†††), the fact that $\dim M/G = k$, and the (sheaf theoretic) fact that $H^i(M/G; \mathcal{S}) = 0$ for all $i > k$; for all $q > n$, by (††); and,

if $0 \leq p \leq k$, for $0 < q < k-1, n-k+1 < q < n$, by (†). In particular, when E_∞ is calculated, we obtain the following:

(1) $E_\infty^{p,0} = E_2^{p,0}$ for $0 \leq p < k$,

(2) $E_\infty^{k,0} = E_2^{k,0}/\text{image } d_{k-1}^{0,k-1}$,

(3) $E_\infty^{p,n} = E_2^{p,n}$ for $1 \leq p \leq k$,

(4) $E_\infty^{0,n} = \ker d_{k-1}^{0,n}$,

(5) $E_\infty^{0,k-1} = \ker d_{k-1}^{0,k-1}$,

(6) $E_\infty^{1,k-1} = E_2^{1,k-1}$, and

(7) $E_\infty^{0,k} = \ker d_2^{0,k}$.

Thus, for supports ψ, ϕ on M, B, respectively, we obtain

(a) $H_\phi^i(U) \cong H_{\phi(\psi)}^i(\tilde{U})$, for $i < k-1$,

(b) an exact sequence of homomorphisms

$$0 \to H_\phi^{k-1}(U) \to H_{\phi(\psi)}^{k-1}(\tilde{U}) \to H_\phi^0(U; \mathcal{H}^{k-1}) \to H_\phi^k(U) \to H_{\phi(\psi)}^k(\tilde{U}),$$

(c) and $H_\phi^k(U; \mathcal{H}^n) \cong H_{\phi(\psi)}^{n+k}(\tilde{U})$.

The information in this list is enough for us.

For example, letting ψ, ϕ be the family of closed subsets of M, B, we see, by (a), that B is $clc_{\mathbf{Z}}^{k-2}$ (see the proof of 2.1.3). By (b), it is easy to see that B is in fact cohomologically connected in all dimensions except possibly k.

By using $\phi = \{x\}$ in (a) and (b), we obtain $H^i(B, B-x) = 0$ for all $x \in B$ and $i < k$.

From (b) and (c), we may derive that $C = \{x \in B \mid \mathcal{H}^n \text{ is locally constant at } x\}$ is a generalized k-manifold over $\mathbf{Z}$ (see 2.1.3–2.1.6).

Another consequence of (b), is that B is k-$clc_{\mathbf{Z}}$ at $x \in B$ if, and only if, B is 0-$clc_{\mathcal{H}^{k-1}}$ at x, an analogue of 2.1.3.

§3.2 Characterization.

In this section we prove the analogue of 2.2.1. The hypotheses on M and G are the same as in section 3.1.

PROPOSITION 3.2.1. *Assume G is a uscd of a $(n+k)$-manifold M satisfying the hypothesis of chapter 3.*

Then B is a generalized k-manifold if, and only if, $E = D \cup K$ is locally finite.

PROOF:

($\Rightarrow$)

This proof is nearly identical to that of Proposition 2.2.1, except when calculating the *homology* groups, we replace $n = (2n+1)-(n+1)$ by $k-1 = (n+k)-(n+1)$.

($\Leftarrow$)

B is an ANR

B is $clc_{\mathbf{Z}}^{k-1}$ and dim $B < \infty$. Therefore, it suffices to show that B is k-$clc_{\mathbf{Z}}$.

If $\mathcal{H}^n$ is Hausdorff on a neighborhood of $x \in B$, then

$$H^0(U; \mathcal{H}^{k-1}) \subset H^{k-1}(\tilde{x})$$

(the latter group is finitely generated) for a small enough connected neighborhood U of x. Thus, B is k-$clc_{\mathbf{Z}}$ at x.

If $x \in K$, then choose a connected neighborhood U which misses $E - \{x\}$ and so that $\mathcal{H}^n$ is constant on $U - x$. Then $U - x$ is connected. Clearly,

$$H^0(U; \mathcal{H}^{k-1}) \subset H^0(U - x; \mathcal{H}^{k-1}) \oplus H_x^0(U; \mathcal{H}^{k-1}).$$

Both summands are finitely generated. So B is k-$clc_{\mathbf{Z}}$ at x. Hence, B is $clc^{\infty}_{\mathbf{Z}}$, LC^1, and, consequently, an ANR.

B is a cohomology k-manifold

By the results of §3.1, we have $H^i(B, B-x) = 0$ for all $x \in B$ and $0 \leq i \leq k-1$. Looking at item (4) in the list of section 3.1, we note that

$$\check{H}^k_x(U; \mathcal{H}^n) \cong H^{n+k}(\tilde{U}, \tilde{U} - \tilde{x}) = \mathbf{Z}$$

for small neighborhoods U of x. So it suffices to demonstrate that

$$\check{H}^k_x(U; \mathcal{H}^n) \cong H^k_x(U; \mathbf{Z}),$$

a suitably chosen neighborhood U. This is trivial for $x \notin E$.

So we assume $x \in E$, and U a connected neighborhood of x such that $U \cap E = \{x\}$ and, by 1.6.3, $\mathcal{H}^n|_{U-x}$ is constant. Then consider the following diagram

$$\begin{array}{ccccc}
H^{k-1}(U) & \overset{\cong}{\longleftarrow} & \check{H}^{k-1}(U; \mathcal{H}^n) & & \\
\downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \beta_1} & & \\
H^{k-1}(U-x) & \overset{\cong}{\longleftarrow} & \check{H}^{k-1}(U-x; \mathcal{H}^n) & & \\
\downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \beta_2} & & \\
H^k_x(U) & & \check{H}^k_x(U; \mathcal{H}^n) & \overset{\cong}{\longleftarrow} & \mathbf{Z} \\
\downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \beta_3} & & \\
H^k(U) & \overset{\cong}{\longleftarrow} & \check{H}^k(U; \mathcal{H}^n) & & \\
\downarrow{\scriptstyle \alpha_4} & & \downarrow{\scriptstyle \beta_4} & & \\
H^k(U-x) & \overset{\cong}{\longleftarrow} & \check{H}^k(U-x; \mathcal{H}^n) & &
\end{array}$$

The vertical maps come from the appropriate long exact sequences of the pair $(U, U-x)$. The second and fourth equal signs are obvious, since $\mathcal{H}^n|_{U-x}$ is constant.

We argue that we can find isomorphisms (represented by the first and third equal signs) so that the squares in the diagram commute. Once we have established this a rather routine diagram chase shows that $H^k(U, U - x) \cong \mathbf{Z}$.

The key to understanding the indicated isomorphisms, is applying dimension theory when calculating $\check{H}^i(U; \mathcal{H}^n)$. The reader is referred to [**S**, pp.327–328], as we content ourselves, for now, with the description

$$\check{H}^i(U; \mathcal{H}^n) = \varinjlim\{H^i(\mathcal{U}; \mathcal{H}^n)\},$$

where the limit is being taken over all open covers $\mathcal{U}$ of U (which form an inverse system via refinement). Since U is k-dimensional, for every open cover $\mathcal{U}$ of U there is a refinement $\mathcal{V}$ of order $k + 1$ such that x is not in the intersection of any two elements of $\mathcal{V}$. A glance at the definition of $H^i(\mathcal{V}; \mathcal{H}^n)$ will show that, because $\mathcal{H}^n|_{U-x}$ is constant, we can use the constant sheaf $U \times \mathbf{Z}$ rather than $\mathcal{H}^n$ when $i > 1$. Thus, there is a natural isomorphism, for $i > 1$, between the limits

$$\varinjlim\{H^i(\mathcal{U}; \mathcal{H}^n)\} \cong \varinjlim\{H^i(\mathcal{U}; \mathbf{Z})\}.$$

In particular, since $k - 1 > 1$, we obtain the desired isomorphisms. Thus, B is a k-gm. ∎

§3.3 Proof of Theorem 2.

In this section we prove Theorem 2. In particular, in addition to the hypotheses of the previous sections of this chapter, we assume

(1) $\mathcal{H}^n$ is a Hausdorff space (i.e. $K = \emptyset$), and

(2) D does not locally separate B.

Note that B is an ANR by (1), since $\check{H}^0(U;\mathcal{H}^{k-1}) \subset \check{H}^{k-1}(\tilde{x})$ for any connected open set U and $x \in U$. Moreover, $H^i(B, B-x) = 0$ for any $x \in B$ and $i \neq k$. Given a connected subset U, we have $U \cap C$ connected by (2).

By the proposition of the previous section, it suffices to show D to be locally finite in B. Let

$$D_1 = \{x \in D \mid x \text{ is isolated in } D\}$$

and $D_2 = D - D_1$. Note that D_1 is an open subset of D and is countable; D_2 is closed in B.

We show $D_2 = \emptyset$, implying $D = D_1$ is, of necessity, locally finite.

Before proceeding, we state the following analogue of lemma 2.3.1; the proof here has the same strategy, the only significant difference being in indices on the algebraic topological functors.

LEMMA 3.3.1.

Suppose $d_0 \in D$ and $q \in \mathbf{Z}_+$ $(q > 1)$. Choose connected neighborhoods U, V, W of d_0 so that

(1) $W \subset V \subset U$,

(2) $H_i(W) \to H_i(V)$ *has trivial image for* $i = k, k-1$,

(3) $\tilde{V}$ *[resp.* $\tilde{W}$*] shape deformation retracts to* $\tilde{d}_0$ *in* $\tilde{U}$ *[resp.* $\tilde{V}$*].*

Then the set

$$F(q) = \{x \in (D \cap W) - \{d_0\} \mid \alpha_x|_C = q \text{ and } H^k(B, B-x) = 0\}$$

consists of at most one point.

PROOF: If there are distinct points $x_1, x_2 \in F(q)$, build an arc A in W whose endpoints are x_1, x_2 and whose interior lies in C, using the hypothesis that D does

not locally separate B. By lemma 1.4.5, we obtain

$$\mathbf{Z}_q \cong \check{H}^{n+1}(\tilde{A}) \cong H_{k-1}(\tilde{U}, \tilde{U} - \tilde{A}),$$

the latter map given by Alexander duality. We then consider the commutative diagram with exact rows

$$\begin{array}{ccccccc}
H_{k-1}(\tilde{W}) & \xrightarrow[\alpha_1]{} & H_{k-1}(\tilde{W}, \tilde{W} - \tilde{A}) & \xrightarrow[\beta_1]{} & H_{k-2}(\tilde{W} - \tilde{A}) & \longrightarrow & H_{k-2}(\tilde{W}) \\
\downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} & & \downarrow{\scriptstyle \tau_3} & & \downarrow{\scriptstyle \tau_4} \\
H_{k-1}(\tilde{V}) & \xrightarrow[\alpha_2]{} & H_{k-1}(\tilde{V}, \tilde{V} - \tilde{A}) & \xrightarrow[\beta_2]{} & H_{k-2}(\tilde{V} - \tilde{A}) & \longrightarrow & H_{k-2}(\tilde{V}) \\
\downarrow{\scriptstyle \chi_1} & & \downarrow{\scriptstyle \chi_2} & & \downarrow{\scriptstyle \chi_3} & & \downarrow{\scriptstyle \chi_4} \\
H_{k-1}(\tilde{U}) & \xrightarrow[\alpha_3]{} & H_{k-1}(\tilde{U}, \tilde{U} - \tilde{A}) & \xrightarrow[\beta_3]{} & H_{k-2}(\tilde{U} - \tilde{A}) & \longrightarrow & H_{k-2}(\tilde{U})
\end{array}$$

Note that χ_2, τ_2 are excision isomorphisms and that χ_4,τ_4 are zero maps. The latter fact implies that the image of τ_3 lies in the image of β_2. It can also be seen that, since A misses d_0 and $\tilde{d}_0$ carries the image of χ_1, τ_1, the maps preceding α_1, α_2 in the long exact sequence are surjections and, hence, the maps β_1, β_2 are injective; the image of each, recall, is $\mathbf{Z}_q$. Note that $\tau_3|_{im\beta_1}$ is injective. Thus, the image of τ_3 is non–trivial and contains $\mathbf{Z}_q$. But $\mathrm{im}\tau_3 \subset \ker\gamma_2 = \mathrm{im}\beta_2 = \mathbf{Z}_q$. Thus, we must have the image of τ_3 isomorphic to $\mathbf{Z}_q$.

We now use the Vietoris-Begle theorem to observe that the image of

$$H_{k-2}(W - A) \to H_{k-2}(V - A)$$

is isomorphic to $\mathbf{Z}_q$. This allows us to conclude that $H_{k-1}(V, V - A) \cong \mathbf{Z}_q$.

We now obtain a contradiction. Let A_1, A_2 denote subarcs of A whose union is A. Let c denote their common endpoint. Applying the Mayer-Vietoris theorem to the excisive triple $(V, V - A_1, V - A_2)$ we obtain the exact sequence

$$\begin{array}{ccc} H_k(V, V-A_1) \oplus H_k(V, V-A_2) & = & H_k(V, V-x_1) \oplus H_k(V, V-x_2) = 0 \\ \downarrow & & \\ H_k(V, V-c) & = & \mathbf{Z} \\ \downarrow & & \\ H_{k-1}(V, V-A) & = & \mathbf{Z}_q \end{array}$$

To show the horizontal map at the beginning of this diagram is an isomorphism, use the fact that C is a generalized manifold (for details on the technique see [**D3**, p.192], [**DyW**]).

This is impossible. ∎

Select $d_0 \in C(D_2)$. Find a connected neighborhood U of d_0 such that

$$\mathcal{H}^n|_{U \cap D_2} \cong (U \cap D_2) \times \mathbf{Z}.$$

By hypothesis $C \cap U$ is connected. Let $N = \alpha_{d_0}(c)$, where c is any point in C. Then N is a bound such that $F(q) = \emptyset$ for all $q > N$. By lemma 3.3.1, we may assume U to be chosen such that, for each $d \in (D \cap U) - \{d_0\}$, $H^k(U, U-d) = \mathbf{Z}$; in other words $U - \{d_0\}$ is a generalized k-manifold. We may assume the winding functions have bounded value.

By proposition 3.2.1, $D \cap (V - \{d_0\})$ is a discrete subset of $V - \{d_0\}$. But this contradicts that D_2 is dense in itself.

Thus, $C(D_2) = \emptyset = D_2$. So D is locally finite, 3.2.1 then implying that B is a k-gm$_{\mathbf{Z}}$. ∎

§3.4 Immediate Corollaries, Examples, and Questions.

We start out this section with some consequences of Theorem 2.

THEOREM 3.4.1.

There is no non–degenerate codimension-k manifold decomposition G of S^{n+k} with the following properties:

(1) *For all $g \in G$, $\tilde{H}_i(g) = 0$ for $i \in \{0, 1, 2, \ldots, k-2\}$.*

(2) *dim $M/G < \infty$.*

(3) *$\mathcal{H}^{k-1}$ is constant and the rank of $\mathcal{H}^{k-1}_x$ as a $\mathbf{Z}$–module is at least 2, for some $x \in B$.*

(4) *The exceptional set $E = D$ does not locally separate B.*

Addendum: *Item (3) can be weakened to: (3 ′) there is a sheaf embedding of $B \times F$ into $\mathcal{H}^{k-1}$, where F is a free $\mathbf{Z}$–module of rank at least 2.*

PROOF: Suppose such a G exists. Property 3 means that $H^0(B; \mathcal{H}^{k-1})$ is a free $\mathbf{Z}$–module on at least two generators. On the other hand, from the Leray spectral sequence we get an injection

$$H^0(B; \mathcal{H}^{k-1}) \to H^k(B).$$

We further realize that B is a simply–connected compact k-gm, hence is orientable; so, by Duality, $H^k(B) \cong \mathbf{Z}$. This is a contradiction. ∎

COROLLARY 3.4.2. *There is no approximate fibration defined on S^{n+k} with fiber F such that*

(1) *F has the shape of a closed orientable n–manifold.*

(2) *$\tilde{H}_i(F) = 0$, for $i \in \{0, 1, \ldots, k-2\}$.*

(3) *rank $H_{k-1}(F) \geq 2$.*

PROOF: For an approximate fibration, $\mathcal{H}^{k-1}$ is locally constant and the set E is empty; by simple connectivity of B, $\mathcal{H}^{k-1}$ is globally constant. ∎

PROPOSITION 3.4.3.

There is no uscd G of S^{n+k} satisfying the hypotheses of Theorem 2 such that rank $H_{k-1}(g) = 0$ for all $g \in G$.

Addendum: *There is no uscd G of a closed manifold M^{n+k} $(k \leq n)$ satisfying the hypotheses of Theorem 2 such that rank $H_{k-1}(g) = 0$ for all $g \in G$, if, in addition, $H_1(M) = 0$ and rank $H_n(M) = 0$.*

PROOF: If rank $H_{k-1}(g) = 0$ for all $g \in G$, then $\mathcal{H}^{k-1}$ is trivial with trivial stalks. On the one hand, since B is a compact simply–connected k-gm, we have $H^k(B) \cong Z$; on the other, we have, by item (b) in section 3.1, $H^k(B) \subset H^k(S^{n+k}) = 0$. This is impossible. Proving the addendum is similar: note that B is a compact, orientable k–gm, since $H_1(M) = 0$; and that $H^k(M^{n+k}) = 0$ because of the constraint on the rank of $H_n(M)$. ∎

THEOREM 3.4.4.

There is no uscd G of E^{n+k} satisfying the hypotheses of Theorem 2.

PROOF: Suppose there is such a decomposition; we derive a contradiction. First note that B is an orientable k–gm. Now apply the Leray spectral sequence with the families ϕ, ψ of compact supports on B, M. In particular, there is an injection

$$\alpha : H_c^0(B; \mathcal{H}^{k-1}) \to H_c^k(B).$$

The cokernel of α is imbedded in $H_c^k(E^{n+k}) = 0$; thus, α is an isomorphism. Obviously, $H_c^0(B; \mathcal{H}^{k-1})$ is trivial, since $\mathcal{H}^{k-1}$ is Hausdorff. By duality, $H_c^k(B) \cong H_0(B) \cong \mathbf{Z}$. It is clear that these last three conditions are logically incompatible. ∎

We now give some examples which are not wholly satisfying.

EXAMPLE 3.4.5: Let X be a k–gm ($k \geq 3$) so that $X \times E^{k-1}$ is a manifold. Fix $j > 0$; for each $1 \leq i \leq j$, let $\Sigma_i^{d_i}$ be a homology (d_i)–sphere, where $d_i \geq k-1$. Let $n = d_1 + d_2 + \cdots + d_j$. Let

$$M = X \times (\prod_{i=1}^{j} \Sigma_i);$$

then M is an $(n+k)$–manifold. Let G be the following product decomposition:

$$G = \{x \times (\prod_{i=1}^{j} \Sigma_i) \mid x \in X\}.$$

The decomposition space is obviously X.

EXAMPLE 3.4.6: Let G_0 be any uscd of N^{2k-1} into homology $(k-1)$–spheres ($k \geq 3$); see section 2.4 for explicit examples. Let

$$M^{n+k} = N^{2k-1} \times \Sigma^{n-k+1},$$

where Σ is a homology $(n-k+1)$–sphere and $n > 2k-2 \geq 4$. Let

$$G = \{g \times \Sigma^{n-k+1} \mid g \in G_0\};$$

then G is a uscd of M into homology $S^{k-1} \times S^{n-k+1}$s. Obviously, $M/G \cong N^{2k-1}/G_0$, a k–gm.

Questions: The following questions are natural (for the curious):

(1) Does Theorem 2 remain valid if the hypothesis concerning non–degeneracy is dropped? (perhaps if we assume B to be an ANR).

(2) Is there an approximate fibration defined on S^{n+k} with fiber the shape of $S^{k-1} \times S^{n-k+1}$? More generally, is there a decomposition of S^{n+k}, in addition to satisfying the hypotheses of Theorem 2, such that rank $H_{k-1}(g) = 1$ for

some or all $g \in G$? (This is really the case of interest in light of 3.4.1 and 3.4.3).

(3) If G were a uscd of S^{n+k} satisfying the hypotheses of Theorem 2, except that $K \neq \emptyset$, then can K contain more than one point? (i.e. is there an analogue of 2.4.2 ?)

(4) Under what conditions is $p|_{M-\tilde{E}}$ an approximate fibration? In particular, if each decomposition element has the shape of a simply–connected closed n–manifold and $M = S^{n+k}$, is $p|_{M-\tilde{E}}$ an approximate fibration?

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