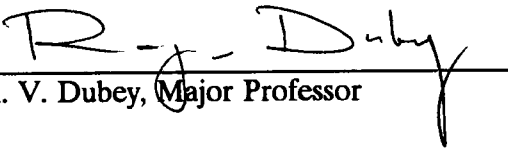


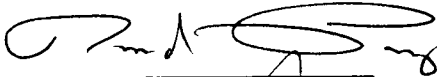
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I am submitting herewith a dissertation written by Brian Damiano entitled "Application of Weighted-Residual Methods to Investigate the Nonlinear Dynamics of BWRs." I have examined the final copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mechanical Engineering.

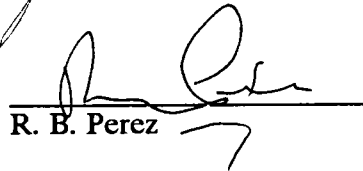

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**APPLICATION OF WEIGHTED-RESIDUAL METHODS TO INVESTIGATE
THE NONLINEAR DYNAMICS OF BOILING WATER REACTORS**

A Dissertation

presented for the

Doctor of Philosophy

Degree

The University of Tennessee, Knoxville

Brian Damiano

December 1992

ACKNOWLEDGMENTS

I have received considerable help in completing the research described in this dissertation and, in completing the dissertation itself, from a number of organizations and individuals. First, I must acknowledge the support of the Office of Nuclear Reactor Regulation, U.S. Nuclear Regulatory Commission, under interagency agreement with the U.S. Department of Energy with Martin Marietta Energy Systems, Inc., who sponsored the majority of my doctoral research. I must also thank the Instrumentation and Controls Division of Oak Ridge National Laboratory for providing the services of the editorial and word-processing staff.

It would have been nearly impossible for me to have completed this work without the assistance and support of James Euler and José March-Leuba. In addition, I need to recognize the value of the many discussions I had with Victor Paré and Richard Wood. I was privileged to have Jackie Miller edit the dissertation and to have Pam Bolden, Judy Pollock, and Lisa Sellers perform the word processing and integration of the equations, tables, and figures into the text. I also need to recognize Kathy McIntyre, who did the initial typing of the majority of the equations in the dissertation.

This work is dedicated to my mother and father, Mary Helen and Litano Carl Damiano, and also to James Euler.

ABSTRACT

This dissertation describes the application of weighted-residual methods to investigate the nonlinear dynamics of boiling water reactors (BWRs). Under proper conditions, BWRs can become linearly unstable and experience steady-state limit-cycle oscillations. Weighted-residual methods were used to obtain approximate solutions describing overall BWR dynamic behavior during these oscillations.

Two BWR dynamic models were used in this investigation. The first model qualitatively describes the reactor dynamics and is useful for examining the effect of various reactor parameters on the limit cycle. The second model uses the more accurate thermal hydraulics model contained in the BWR stability code LAPUR, allowing this model to quantitatively describe the behavior of specific reactors during low-amplitude, steady-state, limit-cycle oscillations. Both dynamic models use the point kinetics equations to describe the neutronics.

High- and low-order approximate solutions describing steady-state limit-cycle oscillations of the overall weighted neutron population were obtained by applying Galerkin's method. The approximate solutions were in the form of Fourier series and can describe either regional or corewide oscillation modes. The approximate solutions were found to converge to the exact solution as more harmonics were included in the approximation. The low-order approximate solutions were used to examine BWR dynamic behavior near the stability boundary, to develop stability conditions for both BWR dynamic models, and to examine the effect of model parameters (and therefore the related reactor parameters) on reactor stability.

Bifurcations of the limit cycle were also investigated. Equations describing the dynamic behavior of perturbations about a candidate limit cycle of the unbifurcated form were derived. Bifurcation tests based on the asymptotic stability of these perturbations were developed by applying both weighted-residual methods and Floquet's theory. If the candidate limit cycle was unstable, numerical results showed that the perturbations established their own limit cycle. An approximate solution for this limit cycle was obtained by applying Galerkin's method. Adding the approximate solution describing the limit cycle of the perturbations to the unbifurcated candidate limit cycle gave an approximate solution for the bifurcated limit cycle.

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1. INTRODUCTION

This dissertation describes research that primarily involves applying the method of weighted residuals to examine the nonlinear dynamics of boiling water reactors (BWR). Although BWR dynamics have been studied for nearly 40 years, this is to the best of my knowledge the first application of weighted-residual methods to this topic.

BWRs are direct-cycle steam generating systems that have been developed and marketed by General Electric Company and separately by ABB-Atom Corporation. Heat produced by nuclear fission in the

core produces wet steam which passes through a series of separators and dryers before leaving the reactor vessel (Figure 1.1). The steam then passes through a turbine-generator unit; through a condenser; and as condensate, through a series of demineralizers and feedwater heaters before being pumped back to the reactor vessel. The total mass flow through the core

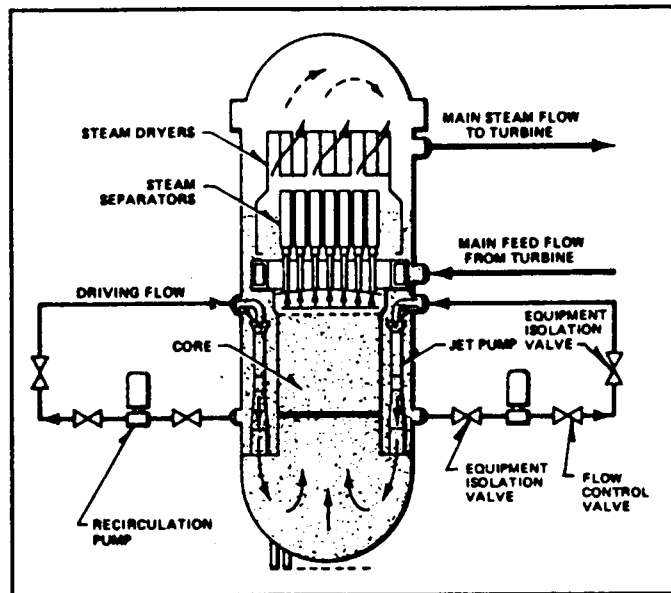


Figure 1.1. Steam and recirculation water flow paths in the boiling water reactor vessel.

Source: Reproduced with permission from Hugh Hexamer, *General Description of a Boiling Water Reactor*, rev., Figure 1-2, p. 1-4, General Electric Company, San Jose, Calif., September 1980.

consists of the amount of evaporated coolant plus the recirculation flow.

The recirculation flow is used primarily for reactor power control. This flow is a mixture of feedwater, the liquid component of the wet steam returned from the steam

separators, and the liquid portion of the coolant flow exiting the reactor core. The recirculation loop injects water at high pressure through jet pumps located in the downcomer between the core and the reactor vessel. The surrounding coolant is entrained by the jet pump flow, providing forced circulation from the upper plenum through the downcomer, through the lower plenum, and finally through the core. The recirculation flow is controlled by either a throttle valve or by changing the recirculation pump speed. Because the water serves the dual role of both moderator and coolant in BWRs, controlling the recirculation flow (which controls the reactor water level and void fraction) controls the degree of neutron moderation and, thus, the reactor power. Increasing the recirculation flow rate decreases the void fraction and increases the reactor water level and power. Conversely, decreasing the recirculation flow rate increases the void fraction and decreases the reactor water level and power.

The coupling between the neutronics and the thermal hydraulics provides a closed-loop system with negative feedback (Figure 1.2). BWRs are designed so that increasing power produces changes in the system that counteract a further power increase. This system characteristic simplifies reactor control and prevents BWRs

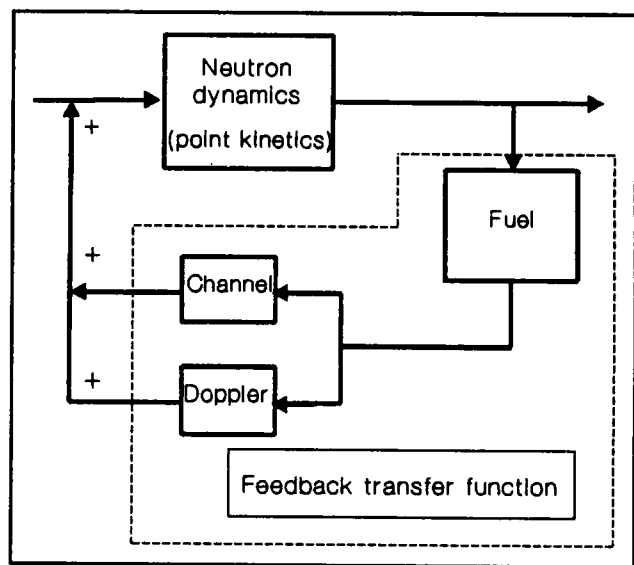


Figure 1.2. Block diagram of boiling water reactor dynamics.

reaching extremely high power levels). The main feedbacks involve the effect of fuel

temperature (through the nuclear Doppler effect) and the moderator density/void fraction (through the fuel channel thermal hydraulics) on reactivity.

The negative fuel temperature and moderator density feedbacks in BWRs make these reactors self-regulating to a large extent. Although the system can become unstable if the void reactivity feedback gain becomes too great, this feedback alone normally is not the cause of the BWR instability. The main mechanism responsible for the observed BWR instabilities is the nuclear-coupled density wave effect, which is a combination of the void-reactivity feedback and a hydrodynamic instability.

The density wave effect occurs because of the feedback between flow rate, rate of void generation, and single- and two-phase pressure drop. An inlet flow perturbation causes enthalpy and pressure perturbations in the single-phase region. The enthalpy perturbation in turn causes fluctuations in the boiling boundary location and produces void fraction perturbations. The void fraction perturbations cause pressure drop perturbations in the two-phase region. Because the total channel pressure drop is constant, the two-phase pressure perturbations produce perturbations of opposite sign in the single-phase region. If the delayed effect is too great (i.e., if the two-phase pressure drop is large compared to the single-phase pressure drop), the channel becomes linearly unstable. The nuclear-coupled density wave effect depends on this same basic mechanism except the heat generation rate is a function of the coolant properties.

Once past the linear stability boundary, BWRs experience exponentially increasing power oscillations. Fortunately, as the oscillation amplitude increases, the nonlinearities in the system act to oppose the oscillation growth. The reactor eventually reaches a steady-state limit cycle in which all the reactor parameters oscillate with a fixed amplitude and frequency.

1.1 Review of Previous Work

A review of the literature on BWR stability was performed to ensure that this doctoral research is original and unique and to place this research into perspective. The interested reader is referred to references 1 through 3 for more detailed reviews of research performed on the dynamics of BWRs.

Instabilities in the form of power and flow oscillations were seen in some of the earliest experimental BWRs. Dietrich and Layman reported power oscillations in the BORAX-I reactor during tests in 1953 (reference 4). Similar instabilities were seen in the BORAX-II and BORAX-III (reference 5), BORAX-IV (reference 6), SPERT-I (reference 7), and the EBWR^{8,9} reactors. Early experimental results indicated that the following factors influenced the onset of oscillations: pressure, average power level, height of water above the core, fuel element heat transfer rate, and magnitude of the void coefficient. Many researchers have recognized that the reactors were linearly unstable during the oscillations and that the oscillation amplitude was bounded by the system nonlinearities.^{10,11}

Dynamic models of two-phase flow through heated channels were developed to gain insight into the observed instabilities. These models included fuel heat transfer; steam formation in the channel; hydraulic circulation; and in some models, neutron kinetics and reactivity feedbacks from temperature and voids. The models consisted of the conservation equations for the coolant mass, momentum, and energy; equations for the fuel heat conduction and heat transfer to the coolant; the neutronics equations (if applicable); empirical relations for single- and two-phase pressure drop; and relationships between coolant properties, power generation rates, steam quality, and void fraction. Most of these early models were linearized for small perturbations about the steady-state

conditions, and the resulting linear equations for the perturbations were transformed into the frequency domain. The techniques of linear control theory were applied, resulting in stability measures based on the Nyquist criterion, root locus methods, or phase and gain margins from Bode plots. Primarily responsible for the early modeling and experimental efforts were Argonne National Laboratory,^{4-6, 8-19} Space Technology Laboratory,²⁰⁻²⁷ Westinghouse Electric Company,²⁸⁻³² and General Electric Company.³³⁻⁴¹

The results of these and other investigations⁴²⁻⁴⁶ led to a good understanding of the basic mechanism responsible for the BWR instability (i.e., the density wave effect) as well as the effect various design parameters such as vessel pressure, inlet subcooling, single- and two-phase pressure drop, power level, and coolant flow rate have on reactor stability. This knowledge allowed reactor designers to develop stable reactors such as the General Electric BWR/2, BWR/3, and BWR/4 product lines introduced in 1955, 1963, and 1965 respectively.⁴⁷ Methods of measuring reactor stability were also developed and used to verify the stability of these reactor types.^{6,11,41,48-52} The methods involved measuring the reactor power output during the application of a known reactivity input and calculating either the closed-loop reactivity-to-power transfer function or the sharpness of resonance of the transfer function peak.

The main differences between the early dynamic models involved the degree of detail and sophistication used to represent coolant flow and heat transfer. Most of these models used the point kinetics equations to describe the neutronics and assumed one dimensional flow. Multichannel models were developed to account for radially varying quantities such as heat generation rate, channel geometry, or flow distribution. Axial nodalization was performed to account for axially varying quantities such as enthalpy, temperature, heat generation rate, flow velocity, and pressure drop. Nodalized multichannel models such as

the General Electric frequency-domain code FABLE^{38,39} or the time-domain code RAMONA^{53,54} were able to satisfactorily estimate the power density or flow at which instability occurred.^{1,2,55,56} The FABLE code was used by General Electric to estimate the stability of its BWR/4, BWR/5, and BWR/6 product lines during their initial design.^{57,58}

Computer codes that could quantitatively predict BWR stability continued to be developed during the 1970s and 1980s. Again, these codes could be classified as linear frequency-domain or nonlinear time-domain codes. The linear frequency-domain codes such as the Rensselaer Polytechnic code NUFREQ,^{59,60} the Oak Ridge National Laboratory code LAPUR,^{61,62} and the General Electric code ODYSY⁶³ were similar in many respects to the FABLE code. Improvements included more accurate modeling of the subcooled and two-phase regions; adding the effects of the steam separators and dryers, the downcomer, and the recirculation loop; and adding multidimensional neutronics.

The nonlinear time-domain codes include the Electric Power Research Institute code RETRAN,⁶⁴⁻⁶⁷ the General Electric code TRAC-G,⁶⁸ the Brookhaven National Laboratory code HIPA,⁶⁹ the RAMONA-3B code,⁷⁰ and the COBRA code.⁷¹ The RETRAN, TRAC-G, and HIPA codes model the complete primary system and include various safety systems, making these codes especially useful for plant accident, safety, and control system analysis. In addition to these functions, the codes can model BWR dynamic behavior in both the linear and nonlinear regions, allowing the effects on the limit cycle of components outside the reactor vessel to be analyzed. The main drawback to these large time-domain codes is the considerable computer time and expense incurred in their use.

In addition to code development, a great number of significant stability tests were performed and improved methods of measuring reactor stability were developed between 1970 and 1985. Stability measurements were performed at the Peach Bottom Atomic Power Station in 1978 (reference 72) and 1980 (reference 73) and at the Vermont Yankee reactor⁷⁴ and the ASEA-ATOM Barseback reactor⁷⁵ in 1980. The Peach Bottom results showed that the newer General Electric reactor designs were less stable than previously thought and the Vermont Yankee and Barseback tests at relatively low flow and high power resulted in limit-cycle oscillations. Subsequent tests at other reactors showed that not only could reactors become unstable but that the instability could appear in one of two spatial modes.⁷⁶⁻⁸⁰ In the first mode, termed the *corewide* or *in-phase* mode, the reactor power oscillates in phase throughout the core. In the second mode, termed the *regional* or *out-of-phase* mode, the power in one half of the core increases while the power in the other half decreases. The test data have also been used extensively to benchmark stability codes.^{63,81-90}

Improved methods of measuring reactor stability were based on the application of autoregressive (AR) modeling techniques.⁹¹⁻⁹⁷ An AR model is formed by fitting model parameters to neutron noise data. Reactor stability is estimated by calculating either the impulse response of the AR model or from the model's most unstable poles. These techniques are closely related to the stability-estimating techniques using the autocorrelation function developed at Argonne National Laboratory in the 1950s (references 6,11,13).

Because the primary goal was to avoid instability, BWR or heated channel dynamics in the nonlinear region was scarcely studied before 1984. Although some early analyses included system nonlinearities,^{14,15,30,32,45,98} these studies addressed the effect the

nonlinearities had on the stability boundary. Two studies involving the application of perturbation methods to examine limit-cycle amplitude and frequency are of particular interest here. Akcasu and Shotkin developed an expression for the critical power level of a reactor with linear feedback and related it and the feedback transfer function to the limit-cycle amplitude and frequency.⁹⁹ Similar results were obtained by Friedly and Krishnan for a boiling channel.¹⁰⁰ Their results indicated that the nonlinear effects were strong even when very close to the stability boundary. Experimental investigation of a Freon TM-113™ (trade name of E. I. du Pont de Nemours & Company) loop by Yadigaroglu and Bergles showed the existence of “higher mode” limit-cycle oscillations.¹⁰¹ This may be the first example of a limit-cycle bifurcation in two-phase flow, although it was not recognized as such at the time.

The first substantial study of nonlinear BWR dynamics was performed by March-Leuba et al.^{94,102-106} A reduced-order phenomenological model based on fits to the LAPUR-calculated closed-loop reactivity-to-power transfer function was used to study BWR nonlinear dynamics. Numerical integration of the model equations indicated that the system experienced a supercritical Hopf bifurcation at the linear stability boundary, leading to steady-state limit-cycle oscillations (Figure 1.3). As the feedback gain was increased, making the system more unstable, the limit-cycle amplitude grew and the power spikes became sharper (Figure 1.4). Increasing the feedback gain still further produced a series of period-doubling pitchfork bifurcations (Figure 1.5), eventually leading to aperiodic or chaotic behavior (Figure 1.6). The transition from periodic to chaotic behavior by a cascade of period-doubling bifurcations exhibited the universal behavior predicted by Feigenbaum.¹⁰⁷

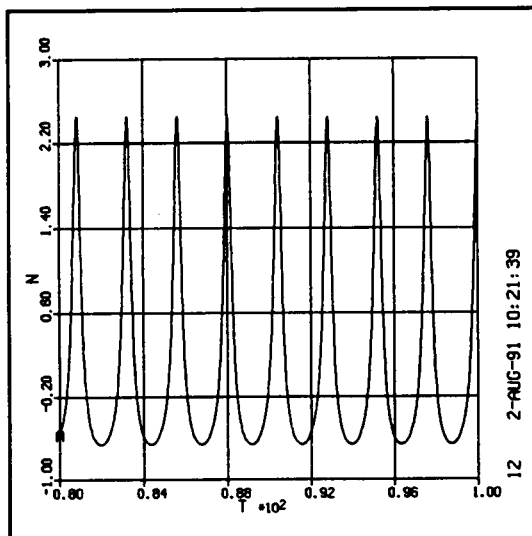


Figure 1.3. Normalized neutron density during steady-state unbifurcated limit-cycle oscillations.

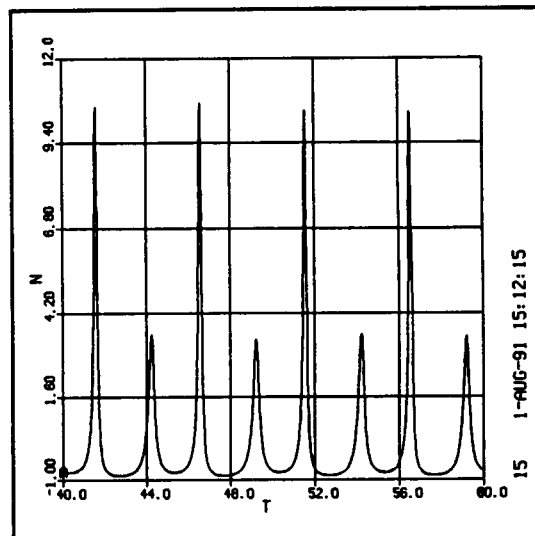


Figure 1.4. Normalized neutron density during steady-state limit-cycle oscillations. Solution between the first and second bifurcation points.

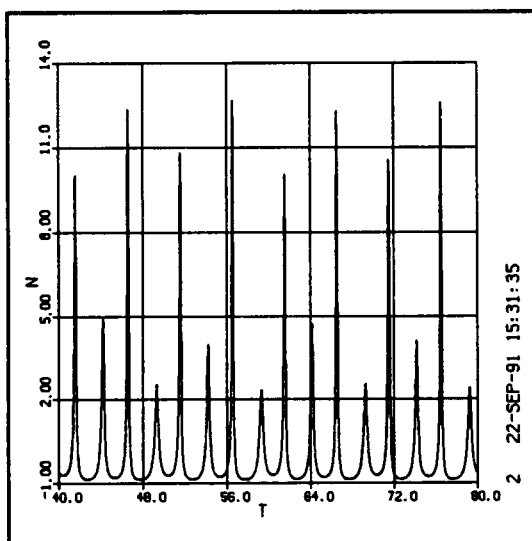


Figure 1.5. Normalized neutron density during steady-state limit-cycle oscillations. Solution between second and third bifurcation points.

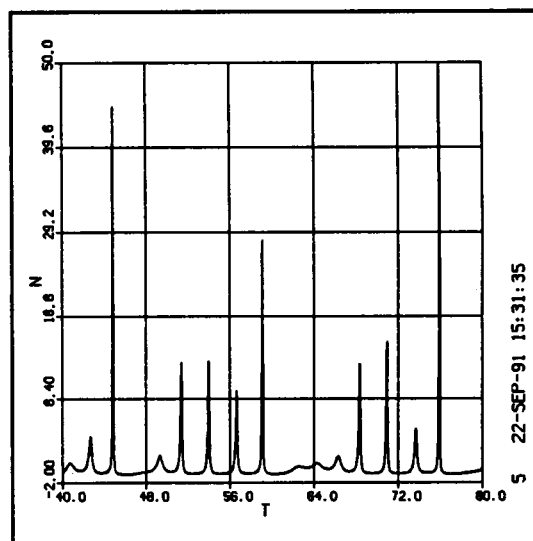


Figure 1.6. Normalized neutron density during steady-state limit-cycle oscillations. Solution is in the chaotic oscillation region.

Additional work in the area of nonlinear BWR dynamics has been performed since 1984. Lahey describes using higher order perturbation techniques to show the existence of both subcritical and supercritical bifurcations in boiling channels.¹⁰⁸ Another application of a perturbation technique by Rizwan-uddin and Dorning showed that a supercritical Hopf bifurcation occurs when the linear stability boundary is crossed.^{109,110} A more practical result from March-Leuba was a simple expression for the average power increase during limit-cycle oscillations.¹¹¹ Blakeman and March-Leuba also developed a model describing the regional oscillation mode.^{112,113} This model explains the origin of these oscillations as an excitement of the first subcritical neutronics mode by the core thermal hydraulics.

Interest in calculating the response of BWRs undergoing limit-cycle oscillations increased markedly after the instability event at the LaSalle-2 power station in March 1988 (reference 114). In this event, a valving error caused the reactor to trip the recirculation pumps, causing it to experience limit-cycle oscillations that gradually grew in amplitude until an automatic scram occurred when the reactor power exceeded the scram set point. The reasons for concern were that

1. the reactor was predicted to be stable at the conditions that caused the limit cycle, and
2. new calculations showed reactor damage may have resulted even if the automatic scram system had worked as designed if the oscillation mode had been regional.

The possibility of reactor damage depends on the oscillation mode and on the steady-state value of the limit-cycle amplitude. Thus, computer calculations were performed by using the RETRAN, HIPA, and TRAC codes to predict the reactor behavior for a number of postulated events.¹¹⁵⁻¹²⁶ In addition to the calculations, a number of studies were performed to investigate the effect of modeling details and numerical methods on code

results.^{88,124,127,128} Most of the post-LaSalle event work on BWR stability performed in the United States has been done under the direction of the BWR Owners Group in close cooperation with the U.S. Nuclear Regulatory Commission (NRC). The goal of this work is to identify long-term BWR stability-problem solutions that are acceptable to both utilities and NRC.^{129,130}

1.2 Description of Research and Original Contributions

The main steps in this dissertation research are described in this section. The research primarily involves applying weighted-residual methods to examine the dynamic behavior of BWRs during limit-cycle oscillations. Unlike the perturbation methods used in similar studies,^{99,100,108,109} weighted-residual methods are not restricted to quasi-linear systems (i.e., to conditions very near the stability boundary). Thus, the full range of nonlinear BWR behavior is potentially open to investigation using these methods.

1.2.1 Application of Galerkin's Method to Calculate the Steady-State Unbifurcated Limit Cycle During Corewide Oscillations Using the Phenomenological Model

The phenomenological model of March-Leuba was used in this research step.⁹⁴ Phenomenological model parameters resulting in a limit cycle were obtained from reference 94. Galerkin's method (a weighted-residual method) was applied to obtain both high- and low-order approximate solutions to the phenomenological model equations. The convergence of the approximate solution to the numerical solution (which plays the role of the "exact" solution here) were examined. Expressions for the limit-cycle frequency and amplitude near the stability boundary were obtained and used to determine the stability condition of the corewide oscillation mode by using a low-order approximate solution.

1.2.2 Application of Galerkin's Method to Calculate the Steady-State Unbifurcated Limit Cycle During Corewide Oscillations Using the Point Kinetics/LAPUR Feedback Transfer Function Model

This work is similar to that described in the preceding step except that a more accurate BWR dynamics model was used. The BWR dynamic model consists of the point kinetics equations as the forward path and the LAPUR-calculated power-to-reactivity feedback transfer function, which describes the thermal hydraulics, as the feedback loop. This model is limited to small-amplitude limit-cycle oscillations (i.e., such that the oscillations in coolant flow remain below ~10% of the steady-state flow rate) because under these conditions the thermal hydraulic nonlinearities will have little effect on the system dynamics. The assumption of linear thermal hydraulics is based on work performed at Space Technology Laboratory which indicated that the thermal hydraulics of boiling channels respond essentially linearly to oscillating flow inputs with oscillation amplitudes up to ~10% of the steady-state flow rate.²³⁻²⁶ This BWR dynamic model can describe BWR dynamics more accurately than the phenomenological model used in the preceding step because it contains a detailed thermal hydraulic description.

The application of Galerkin's method to obtain an approximate solution to the equations resulting from the BWR model comprising the point kinetics equations and the LAPUR-calculated feedback transfer function results in a practical tool that can be used to predict limit cycles in BWRs. The high-order approximate solution can be used to calculate limit-cycle parameters such as oscillation frequency and amplitude, quantities that until now have been calculated by using expensive time-domain codes. The low-order approximate solution can be used to derive a stability condition and to develop expressions for the limit-cycle frequency and amplitude in terms of the parameters in the point kinetics equations and feedback transfer function values.

1.2.3 Application of Galerkin's Method to Calculate the Steady-State Bifurcated Limit Cycle During Corewide Oscillations

This step uses the phenomenological model because the point kinetics/LAPUR feedback transfer function model is valid for only low-amplitude oscillations. For parameter values above the bifurcation point, two mathematical solutions to the phenomenological model equations apparently exist. One of these, describing a bifurcated form of the solution, is stable; the other, describing an unbifurcated form of the solution, is unstable. Application of Galerkin's method results in an unbifurcated form of the solution being calculated after crossing the bifurcation point because the initial guess will typically be obtained from a previously calculated unbifurcated solution. Once past the bifurcation point, the unbifurcated form of the solution should be unstable.

A set of nonlinear equations describing the time-dependent behavior of perturbations about an unbifurcated candidate limit-cycle solution was developed. The behavior of the perturbations can be used to determine whether the candidate limit-cycle solution is stable; if the perturbations return to zero, the candidate solution is stable; if the perturbations diverge or establish their own oscillatory solution, the candidate solution is unstable and presumably the limit cycle has bifurcated.

Bifurcation tests were developed by using a version of the equations for the perturbations about the unbifurcated candidate limit-cycle solution linearized about the equilibrium point. The first test was based on an approximate solution to the equations obtained from the application of weighted-residual methods and the second test was obtained from the application of Floquet's theory.

Weighted-residual methods were applied to the nonlinear form of the equations to obtain an approximate solution describing the perturbations about the unbifurcated

candidate limit-cycle solution. An estimate of the bifurcated limit cycle was obtained by adding the approximate solution for the perturbations to the candidate limit-cycle solution.

If the initial solution is unstable, the nonlinear phenomenological equations for deviations from the approximate solution are used to calculate an approximate bifurcated steady-state limit-cycle solution. This approximate solution for the deviations was added to the previously calculated unstable, unbifurcated solution to obtain a stable bifurcated solution.

1.2.4 Application of Galerkin's Method to Calculate the Steady-State Unbifurcated Limit Cycle During Regional Oscillations Using the Phenomenological Model

This application is similar to the calculation of the corewide oscillations except the phenomenological model and the point kinetics/LAPUR feedback transfer function model were modified to predict regional oscillations.^{112,113} High- and low-order approximate solutions were obtained by applying Galerkin's method. The high-order solution to the phenomenological model is useful for examining the effect of model parameters on the characteristics of the limit cycle. The high-order solution to the point kinetics/LAPUR feedback transfer function model is useful for calculating the limit cycle of specific reactor models. Stability conditions were obtained for both models by using low-order approximate solutions. The stability conditions were used to examine the effect of model parameters on stability. Comparison with the corewide mode stability condition shows that the feedback transfer function and the value of subcritical reactivity are the most significant factors in determining the oscillation mode. This work is the first in-depth investigation of the regional limit cycle.

1.2.5 Original Contributions

The original contributions of the research described in this dissertation are

1. establishment of weighted-residual methods as an alternative to numerical methods for calculating BWR dynamic behavior during limit-cycle oscillations;
2. development of an expression for the stability boundary in terms of parameters in the phenomenological model;
3. application of weighted-residual methods for calculating limit-cycle frequency and amplitude by using the point kinetics equations and the LAPUR-calculated feedback transfer function, thus providing a practical and inexpensive technique for studying the BWR limit cycle;
4. development of bifurcation tests using results from the weighted-residual method calculations and Floquet's theory, thus showing that multiple solutions to the phenomenological model equations exist but that only one solution is stable;
5. development of a technique to apply weighted-residual methods to calculate oscillations during the bifurcated BWR limit cycle;
6. modification of the phenomenological model to predict regional oscillations in BWRs; this model predicts that the regional oscillation mode behaves essentially the same as the corewide oscillation mode; the similarities between oscillation modes include increasing limit-cycle amplitude and decreasing oscillation frequency as the system becomes more unstable as well as the existence of period-doubling bifurcations for both oscillation modes; and
7. development of an expression for the stability boundary of the regional model, by which comparison with the corewide stability boundary shows that the feedback transfer function and the value of subcritical reactivity are the most significant parameters in determining which oscillation mode occurs.

1.3 Outline of Text

This dissertation contains seven chapters. Chapter 2 presents background information that may be unfamiliar to some readers. This information is necessary to fully understand the research described in Chapters 3 through 7 and is included in the dissertation for completeness. Chapter 3 describes the application of weighted-residual methods to the phenomenological model equations to calculate steady-state, corewide, limit-cycle

oscillations. Chapter 4 describes the application of weighted-residual methods to the BWR model consisting of the point kinetics equations and the LAPUR-calculated feedback transfer function to calculate steady-state, corewide, limit-cycle oscillations. Chapter 5 describes the development of the bifurcation tests and the calculation of steady-state, bifurcated, limit-cycle oscillations. Chapter 6 describes the development of the regional BWR dynamic model and the application of weighted-residual methods to calculate BWR behavior during steady-state regional oscillations. Finally, Chapter 7 summarizes the research, presents conclusions, and makes recommendations for future work.

2. PRECURSORY INFORMATION

This chapter presents background material that forms the basis for much of the research described in this dissertation. This presentation is intended as a convenience to readers who are unfamiliar with this material. Three main subject areas are presented: (1) A brief description of some aspects of nuclear reactor theory pertinent to reactor kinetics and control. Discussions are included of nuclear reactions, fission, criticality, the one-speed diffusion equation, the point kinetics equations, and nuclear feedback processes. This material is useful for understanding the BWR dynamic models used in this research. (2) A brief description of the LAPUR computer code. This code, developed at Oak Ridge National Laboratory for BWR stability analysis, was used extensively during this research. (3) A general description of weighted-residual methods. The general principles on which weighted-residual methods are based are presented, and some of the major established techniques are described.

2.1 Selected Topics in Nuclear Reactor Theory

Aspects of nuclear reactor theory necessary to understand the dynamic models of BWRs used in this research are presented in this section. The discussion of each topic is necessarily brief; the interested reader is referred to references 131 through 133 for a thorough treatment of these topics.

2.1.1 Nuclear Reactions

Nuclear reactions can be divided into two categories: (1) spontaneous disintegration of a nucleus, with the emission of particles or energy; and (2) collision reactions involving

nuclei and/or nuclear particles. The spontaneous decay of nuclei, called *radioactive decay*, has been observed to obey the following law: the time rate of change in the number of nuclei is proportional to the number of nuclei present. If λ is the radioactive decay constant and N is the number of nuclei present, this law can be written as

$$\frac{dN(t)}{dt} = -\lambda N(t) . \quad (2.1)$$

The products of the disintegration are (1) a new nucleus, which is new in the sense that the number or type of nuclear particles composing the nucleus is changed; and (2) one or more of the following: an alpha particle (a helium nucleus), a beta particle (an electron), a gamma ray (or photon), a neutrino, or a neutron.

The discussion of collision reactions will be limited here to the case of incident neutrons and target nuclei. As an incident neutron approaches a target nucleus, the particle may scatter elastically off the nuclear potential of the nucleus; or the neutron may be absorbed, forming a compound nucleus. The first type of reaction, called *potential scattering*, is elastic; the total momentum of the neutron and the nucleus is conserved.

If a compound nucleus is formed, several nuclear reactions are possible. If, after the absorption of the incident neutron, the compound nucleus emits a neutron such that the total kinetic energy of the two particles is conserved, this reaction is called *elastic scattering*. The target nucleus is left in an unexcited state. If the compound nucleus emits a neutron but the total kinetic energy of the two particles is not conserved, this reaction is called *inelastic scattering*. The target nucleus is often left in an excited state, and the emitted neutron will usually have less kinetic energy than the incident neutron. The excited nucleus will eventually emit one or more photons to return to the unexcited state. Clearly, high-energy neutrons can lose kinetic energy and be *slowed down* by elastic or

inelastic scattering reactions. The process in which high-energy neutrons slow down to lower energies is called *neutron moderation*, and materials that perform this function in reactors are called *neutron moderators*. Although both types of scattering collisions can slow high-energy neutrons, inelastic scattering is the more effective slowing-down mechanism.

If the compound nucleus emits only photons, the reaction is called *radiative capture*. Another possible reaction occurs when the compound nucleus emits a charged particle such as a proton, an electron, or an alpha particle. These types of reactions remove neutrons from the reactor. Finally, if the compound nucleus *fissions*, the nucleus splits, forming fission fragments in addition to emitting several high-energy neutrons. The fission fragments may be unstable and eventually experience radioactive decay, producing alpha particles, beta particles, photons, or neutrons. The radioactive decay occurs after a time delay relative to the formation of the fission fragment. The *delayed neutrons*, produced by the decay of the fission fragments, are very important in the control of nuclear reactors, as will be described later. Radiative capture and fission are examples of *neutron absorption*.

The type of nuclear reaction that occurs after the formation of a compound nucleus can be characterized by a quantity known as a *nuclear cross section*. This quantity can be described by the following example. Suppose a beam of monoenergetic neutrons of intensity I neutrons/(s-unit area) interacts with a single layer of atoms with a *number density* of N_A nuclei/unit volume. The rate of any specific nuclear reaction is given by

$$rate = \sigma I N_A , \quad (22)$$

where the constant of proportionality σ with units of area, is the *microscopic cross section*. One can think of σ as the effective cross section each nucleus presents to the neutron

beam for a specific reaction. Another way to consider cross sections is as a measure of the probability that a neutron will interact with a nucleus to produce a specific nuclear reaction.

A quantity related to the microscopic cross section is the *macroscopic cross section* Σ . Whereas the microscopic cross section characterizes the probability of a neutron reacting with a single nucleus, the macroscopic cross section characterizes the probability of neutron interaction in a large group of nuclei. The macroscopic cross section is given by

$$\Sigma = N_A \sigma \quad (23)$$

and has units of reciprocal length. The macroscopic cross section can be interpreted as the probability per unit path length that a neutron will undergo a reaction with a target nucleus.

Up to this point, the incident neutrons have been assumed to be monoenergetic. In BWRs, fission neutrons are formed with high kinetic energies and slow down after a series of scattering collisions until they reach thermal equilibrium with their surroundings (i.e., become *thermal neutrons*). Thus, a continuous neutron energy spectrum exists in a reactor. This condition is significant because the probability of nuclear reactions and the corresponding value of the associated cross sections depend heavily on the kinetic energy of the incident neutron calculated from the relative velocities of the incident neutron and the target nucleus (i.e., the *relative kinetic energy*).

The energy dependence of cross sections for reactions that involve the formation of a compound nucleus is due to the quantum nature of nuclei. A nucleus can exist in only certain energy states; for a compound nucleus to form, the sum of the relative neutron kinetic energy and the nuclear binding energy must be such that the compound nucleus is left in one of its excited states after formation. This restriction means that only neutrons

with very specific relative kinetic energies can form a compound nucleus and subsequently cause a nuclear reaction. When microscopic cross-section values are plotted as a function of relative incident neutron kinetic energy, a series of narrow *resonances* is seen to exist.

2.1.2 Fission

In BWRs and all thermal reactors, energy is produced by fission of a specific isotope of uranium, ^{235}U . Examination of the microscopic fission cross section σ_f for ^{235}U shows that the probability of fission is over two orders of magnitude larger at thermal energies than at high energies (>1 keV). Thus, thermal neutrons are more likely to cause ^{235}U fission than are high-energy (*fast*) neutrons. This likelihood accounts for the development of *thermal reactors* (i.e., reactors in which most of the fission neutrons reach thermal energies).

Fission produces fission fragments; several neutrons; a variety of beta particles, photons, and neutrinos; and approximately 200 MeV of energy. Most of the fission energy appears as kinetic energy of the fission fragments and is deposited near the fission site because of collisions between the fission fragments and the fuel atoms. The number of neutrons produced by a fission reaction depends on the target nucleus and the energy of the incident neutron. For any given material composition, the average *number of neutrons per fission* ν can be calculated. The vast majority of the neutrons appear essentially at the instant of fission. Thus, most of the neutrons released during fission are called *prompt neutrons*; prompt neutrons have an average energy of approximately 2 MeV.

As mentioned previously, some neutrons are produced by the decay of the fission fragments. These delayed neutrons can be produced by >20 fission fragment types known as *delayed neutron precursors*. The time delay in the emission of the neutron depends on

the decay constant of each precursor. Although it is possible to treat each precursor individually, the yield of many of the precursors is so small that the behavior of the delayed-neutron precursors can be described accurately by using six precursor groups. The behavior of each precursor group is described by the quantities λ_i , the decay constant of the i th precursor group, and β_i , the fraction of all neutrons emitted that appear because of decay of precursors in the i th precursor group. The total fraction of delayed neutrons is $\beta = \sum_{i=1}^6 \beta_i$. The delayed neutrons appear with an average energy less than that of the prompt neutrons.

2.1.3 Criticality

Reactors operating under steady-state conditions maintain a very delicate balance between the rate of neutron production and the rate of neutron removal. A reactor in which the neutron production rate equals the removal rate is said to be *critical*; the power level and the number of neutrons in the core remain in equilibrium. If the production rate exceeds the removal rate, the reactor is *supercritical*; the power level and the number of neutrons in the core increase with time. If the removal rate exceeds the production rate, the reactor is *subcritical*; the power level and the number of neutrons in the core decrease with time.

Reactor power is changed by adjusting the equilibrium point of the neutron flux. Perturbing the reactor conditions, for example by adding or removing a quantity of absorber material, results in a change in the power level and neutron flux necessary to reestablish equilibrium. After the introduction of the perturbation, the reactor experiences a transient period, during which reactor parameters (e.g., power, neutron

population, and temperature) move from their initial state to a new equilibrium state.

The transient behavior of the reactor is characterized by the *effective time constant*.

A reactor can be critical on the sum of prompt and delayed neutrons, or it can be critical on prompt neutrons alone. If the delayed neutrons are required for criticality, the reactor is *delayed critical*, and the effective time constant of the reactor depends on the decay constants of the precursor groups. In this case, the reactor time constant is on the order of tenths of a second, well within the ability of a control system to respond to power changes. If the reactor is critical on prompt neutrons alone, the reactor is *prompt critical*, and the reactor time constant depends on only the average time between the emission of a neutron and another fission caused by its absorption. This time scale is on the order of 10^{-4} s for thermal reactors; the reactor power under prompt critical conditions can increase to undesirably high levels before the control system can respond. Clearly, the prompt critical condition must be avoided. Comparing the reactor time constants under prompt and delayed critical conditions shows that the delayed neutrons make the practical application and control of nuclear power possible.

The degree of criticality is measured by the *multiplication factor* k or by a related quantity known as the *reactivity* ρ . The multiplication factor is the ratio of the number of neutrons in the present generation to the number of neutrons in the previous generation. For a critical reactor, $k = 1$; $k > 1$ in a supercritical reactor; and $k < 1$ in a subcritical reactor. Two common forms of the multiplication factor exist. The *infinite multiplication factor* k_{∞} is calculated for a specific material composition. The calculation assumes an infinite reactor that experiences no neutron leakage. The *effective multiplication factor* k_{eff} is calculated for a specific material composition and reactor geometry. Leakage of

neutrons from the reactor is taken into account. The effective multiplication factor characterizes the overall dynamic behavior of a reactor.

The reactivity is based on k_{eff} and is given by

$$\rho = \frac{k_{eff} - 1}{k_{eff}}. \quad (2.4)$$

For a critical reactor, $\rho = 0$; $\rho > 0$ in a supercritical reactor; and $\rho < 0$ in a subcritical reactor. Reactivity is the ratio of the difference between neutron production and removal rates to the rate of neutron production. Positive values of reactivity lead to power increases, and negative values of reactivity lead to power decreases. The unit of reactivity used in this work is the *dollar* (\$). The amount of reactivity that will cause a reactor to become prompt critical is \$1. It can be shown that the amount of reactivity corresponding to \$1 equals β , the delayed neutron fraction. Because the material composition of reactors changes with time because of, for example, fuel depletion, control rod motion, or moderator/coolant temperature/phase changes, the multiplication factor and reactivity are time dependent.

2.1.4 The One-Speed Diffusion Equation

A practical and widely used method for describing the distribution of neutrons in both time and space is based on diffusion theory. In diffusion theory, neutrons are assumed to diffuse from regions of high concentration to regions of low concentration. This assumption is valid throughout most of the reactor except very near neutron sources, strong neutron absorbers, or physical boundaries. Even in these regions, diffusion theory can be supplemented with results from the theory of neutron transport to yield accurate results.

In this discussion, we will assume that an average neutron energy is used to determine all energy-dependent quantities. This *one-speed diffusion model* eliminates consideration of the energy dependence of cross sections and other related quantities. More sophisticated *multigroup* or *multispeed diffusion models*, which use a number of energy groups, are analyzed in a similar manner but introduce unnecessary complication, given the purpose of this presentation. Before the development of the one-speed diffusion equation is discussed, several quantities must be defined. The *neutron flux* $\phi(r, t)$ is the product of neutron density $N(r, t)$ and *neutron speed* v :

$$\phi(r, t) = vN(r, t) , \quad (2.5)$$

where r is a vector-defining position. Neutron flux is a time-, energy-, and position-dependent scalar quantity with units of neutrons/unit area-s. Again, in the one-speed diffusion model, the energy dependence is taken into account by using an average neutron energy. The rate of neutron-nucleus interactions in a unit volume [i.e., the reaction rate density $F(r, t)$] will depend on the probability per unit path length of a neutron interaction, the neutron velocity, and the neutron density. Recalling that the probability per unit path length for a specific reaction is given by the macroscopic cross section, the reaction rate density is given by

$$F(r, t) = v \Sigma(r, t)N(r, t) = \phi(r, t) \Sigma(r, t) . \quad (2.6)$$

The expression for the reaction rate density can be used to give the rate at which various reactions such as absorption or fission occur at any location at any given time.

The time rate of change of neutrons in a differential volume dV depends on the rate of neutron production in dV , the rate of neutron absorption in dV , and the net rate

of neutron leakage from dV . Applying this concept to an arbitrary, finite volume V yields

$$\frac{d}{dt} \int_V N(\mathbf{r}, t) d^3r = \frac{d}{dt} \int_V \frac{1}{v} \phi(\mathbf{r}, t) d^3r = \int_V \frac{1}{v} \frac{\partial \phi(\mathbf{r}, t)}{\partial t} d^3r \quad (2.7)$$

= production rate in V - absorption rate in V - net rate of leakage from V ,

where $dV = d^3r$ and the integration is performed over V . Development of expressions for the terms on the right side of the equation follows.

A neutron source density $S(\mathbf{r}, t)$ can be defined such that

$$\text{production rate in } V = \int_V S(\mathbf{r}, t) d^3r . \quad (2.8)$$

An expression for the neutron absorption rate is based on Eq. (2.6) and is given by the product of the neutron flux and the one-speed macroscopic absorption cross section Σ_a :

$$\text{absorption rate in } V = \int_V \Sigma_a \phi(\mathbf{r}, t) d^3r . \quad (2.9)$$

The neutron current density $J(\mathbf{r}, t)$ is used to calculate the net rate of leakage from V .

$J(\mathbf{r}, t)$ is the net rate at which neutrons pass through differential surface area dA . Note that the neutron current density is a vector quantity that is analogous to a conventional flux such as a heat or mass flux. From this definition, the net rate of neutrons leaving V is

$$\text{net rate of leakage from } V = \int_V dA \cdot J(\mathbf{r}, t) , \quad (2.10)$$

where dA is a differential surface element on the surface S enclosing V , and $\mathbf{r}$, is a vector defining locations on S . The surface integral can be converted into a volume integral by using Gauss's theorem:

$$\int_S d\mathbf{S} \cdot \mathbf{J}(\mathbf{r}, t) = \int_V \nabla \cdot \mathbf{J}(\mathbf{r}, t) d^3r, \quad (2.11)$$

where ∇ is the gradient operator. The diffusion assumption is used to write the neutron current density in terms of the neutron flux. Applying Fick's Law, the neutron current density can be approximated by

$$\mathbf{J}(\mathbf{r}, t) = -D(\mathbf{r})\nabla\phi(\mathbf{r}, t), \quad (2.12)$$

where $D(\mathbf{r})$ is the one-speed *diffusion coefficient*, which depends on the material composition and, thus, is position dependent. Note that in an energy-dependent formulation, the diffusion coefficient would also be energy dependent. Substitution of Eqs. (2.8), (2.9), (2.10) and (2.12) into Eq. (2.7) yields

$$\int_V \left[\frac{1}{v} \frac{\partial \phi(\mathbf{r}, t)}{\partial t} - S(\mathbf{r}, t) + \Sigma_a \phi(\mathbf{r}, t) - \nabla \cdot D(\mathbf{r})\nabla\phi(\mathbf{r}, t) \right] d^3r = 0. \quad (2.13)$$

Because the volume used in developing Eq. (2.13) is arbitrary, the integrand must be identically zero. Therefore, the one-speed, time-dependent diffusion equation is

$$\frac{1}{v} \frac{\partial \phi(\mathbf{r}, t)}{\partial t} - S(\mathbf{r}, t) + \Sigma_a \phi(\mathbf{r}, t) - \nabla \cdot D(\mathbf{r})\nabla\phi(\mathbf{r}, t) = 0. \quad (2.14)$$

Equation (2.14) can be simplified by using the following assumptions. First, if the reactor composition is homogeneous, the diffusion coefficient and the macroscopic absorption cross section will be independent of position. Second, if the neutron source term is due to only fission, then $S(\mathbf{r}, t)$ will depend on the one-speed macroscopic fission cross section Σ_f , the average number of neutrons/fission ν , and the neutron flux $\phi(\mathbf{r}, t)$:

$$S(\mathbf{r}, t) = \nu \Sigma_f \phi(\mathbf{r}, t). \quad (2.15)$$

With these simplifications, the time-dependent diffusion equation becomes

$$\frac{1}{v} \frac{\partial \phi(r, t)}{\partial t} = [D \nabla^2 - \Sigma_a + v \Sigma_f] \phi(r, t) . \quad (2.16)$$

The steady-state flux can be calculated by setting the right side of Eq. (2.16) equal to zero:

$$D \nabla^2 \phi(r) + (v \Sigma_f - \Sigma_a) \phi(r) = 0 . \quad (2.17)$$

Equation (2.17) is an eigenvalue problem that can be solved for eigenvalues and the associated eigenvectors. The exact form of the eigenvalue problem will depend on the form of ∇^2 , which in turn depends on the coordinate system. Some general features of the solution to Eq. (2.17) are:

1. The eigenvalues are all real and ordered; no multiple eigenvalues exist.
2. Expressions for the eigenvalues are given in terms of the physical parameters D , v , Σ_a , and Σ_f . Thus, the expressions for the eigenvalues define the relationships these quantities must have to establish criticality.
3. The eigenvectors define the shape of the neutron flux. Each eigenvector corresponds to a spatial mode of the neutron flux. The amplitude of the flux depends on the reactor power and cannot be determined from the eigenvalue problem.
4. The eigenvectors are orthogonal.

If the fundamental spatial mode satisfies its criticality condition, the remaining modes must be subcritical because the eigenvalues are real and ordered. Thus, only the fundamental spatial mode will persist over time.

2.1.5 The Point Kinetics Equations

Solution of the diffusion equation in its energy-, space-, and time-dependent form or even in the form of the one-speed, space- and time-dependent form is a difficult task. Solution for the time-dependent behavior of the neutron flux would be greatly simplified if equations could be developed for flux that depends on time alone. The point kinetics equations are just such equations. The spatial form of the solution is assumed for small perturbations from the steady state, and a set of time-dependent equations is formed by integrating over the reactor volume. The point kinetics equations will be developed starting from the one-speed diffusion equation in this presentation. A similar development is possible by using the multispeed diffusion equation but, again, would involve more detail than is necessary here.

For delayed critical reactors, the behavior of the delayed neutron precursors strongly affects the behavior of the neutron flux. The time dependence of the concentration of one of the precursor groups, $c'_i(r, t)$, depends on the rate the precursor is created by fission and the rate of its decay:

$$\frac{dc'_i(r, t)}{dt} = \beta_i v \Sigma_f \phi(r, t) - \lambda_i c'_i(r, t) \quad (i = 1 \text{ to } 6) . \quad (2.18)$$

The first term on the right gives the rate of precursor production, and the second term gives the rate of precursor removal.

The time-dependent, one-speed diffusion equation must be modified to take into account the delayed neutrons. Because the total fraction of delayed neutrons equals β , the rate of production of prompt neutrons is given by

$$\text{prompt neutron production rate} = (1 - \beta) v \Sigma_f \phi(r, t) . \quad (2.19)$$

The production rate of delayed neutrons depends on the decay rate of the six precursor groups, so

$$\text{delayed neutron production rate} = \sum_{i=1}^6 \lambda_i c'_i(r, t) . \quad (2.20)$$

Replacing the simple neutron production terms due to fission in Eq. (2.16) with Eqs. (2.19) and (2.20) gives a form of the one-speed, time-dependent diffusion equation in which both prompt and delayed neutron production are taken into account:

$$\frac{1}{v} \frac{\partial \phi(r, t)}{\partial t} = \left[D \nabla^2 - \Sigma_a + (1 - \beta) v \Sigma_f \right] \phi(r, t) + \sum_{i=1}^6 \lambda_i c'_i(r, t) . \quad (2.21)$$

Equations (2.18) and (2.21) provide the starting point for the development of the point kinetics equations.

The spatial dependence is taken into account in Eqs. (2.18) and (2.21) by multiplying each equation by a weight function and integrating over the reactor volume. The resulting integrals are weighted sums of the total number of neutrons; these integrals depend only on time. To simplify the development here, a one-dimensional slab geometry consisting of a homogeneous medium will be used because these selections remove the y and z dependence from the neutron flux. The geometry is shown in Fig. 2.1 and the boundary conditions on the neutron flux are

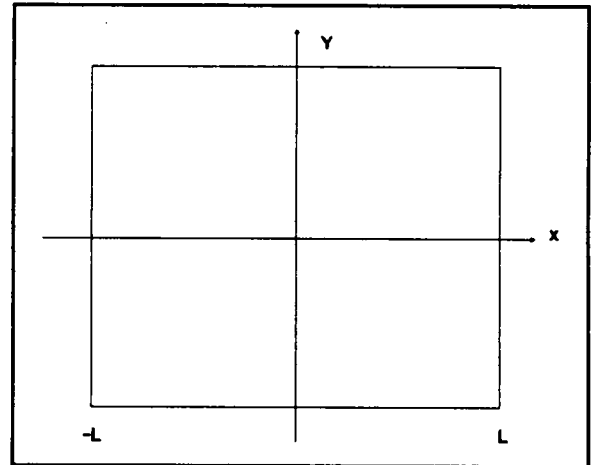


Figure 2.1. One-dimensional slab geometry. The slab is infinite in the Y direction.

$\phi(\pm L, t) = 0$. If the reactor is assumed to be initially critical, the introduction of a perturbation in the absorption cross section $\delta \Sigma_a$ into Eq. (2.21) results in

$$\begin{aligned} \frac{1}{v} \frac{\partial \phi(x, t)}{\partial t} = D \frac{\partial^2 \phi(x, t)}{\partial x^2} + (1 - \beta) v \Sigma_f \phi(x, t) \\ - \Sigma_a \phi(x, t) - \delta \Sigma_a \phi(x, t) + \sum_{i=1}^6 \lambda_i c_i'(x, t) . \end{aligned} \quad (2.22)$$

Equations (2.18) and (2.22) are next multiplied by a weight function and integrated over the reactor volume, which in the one-dimensional case is equivalent to integrating from $-L$ to L . For the one-dimensional case, the best choice for the weight function is the solution for the unperturbed, time-independent neutron flux $\Phi(x) = \sum \Phi_n(x)$ [i.e., the sum of the modes obtained from the one-dimensional form of Eq. (2.17)]. Performing these steps on Eqs. (2.22) and (2.18) gives

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\partial}{\partial t} \int_{-L}^L \frac{\Phi_n(x) \phi(x, t)}{v} dx = D \sum_{n=1}^{\infty} \int_{-L}^L \Phi_n(x) \frac{\partial^2 \phi(x, t)}{\partial x^2} dx \\ + \sum_{n=1}^{\infty} \int_{-L}^L [v \Sigma_f - \Sigma_a] \Phi_n(x) \phi(x, t) dx - \sum_{n=1}^{\infty} \int_{-L}^L [v \beta \Sigma_f + \delta \Sigma_a] \Phi_n(x) \phi(x, t) dx \\ + \sum_{i=1}^6 \lambda_i \left\{ \sum_{n=1}^{\infty} \int_{-L}^L c_i'(x, t) \Phi_n(x) dx \right\} \end{aligned} \quad (2.23)$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\partial}{\partial t} \int_{-L}^L \Phi_n(x) c_i'(x, t) dx = v \beta_i v \Sigma_f \sum_{n=1}^{\infty} \int_{-L}^L \frac{\Phi_n(x) \phi(x, t)}{v} dx \\ - \sum_{n=1}^{\infty} \lambda_i \int_{-L}^L \Phi_n(x) c_i'(x, t) dx \quad (i = 1 \text{ to } 6) . \end{aligned} \quad (2.24)$$

It can be seen that

$$\frac{\partial}{\partial x} \left[\Phi_n(x) D \frac{\partial \phi(x, t)}{\partial x} \right] = \frac{d\Phi_n(x)}{dx} \frac{\partial \phi(x, t)}{\partial x} D + \Phi_n(x) \frac{D \partial^2 \phi(x, t)}{\partial x^2}, \quad (2.25)$$

so the integrand in the first term on the right side in Eq. (2.23) can be expressed as

$$D \Phi_n(x) \frac{\partial^2 \phi(x, t)}{\partial x^2} = \frac{\partial}{\partial x} \left[\Phi_n(x) D \frac{\partial \phi(x, t)}{\partial x} \right] - D \left(\frac{d\Phi_n(x)}{dx} \right) \left(\frac{\partial \phi(x, t)}{\partial x} \right). \quad (2.26)$$

As a result of Eq. (2.26), the first term in Eq. (2.23) becomes

$$\begin{aligned} & \int_{-L}^L \frac{\partial}{\partial x} \left[\Phi_n(x) D \frac{\partial \phi(x, t)}{\partial x} \right] dx - D \int_{-L}^L \left(\frac{d\Phi_n(x)}{dx} \right) \left(\frac{\partial \phi(x, t)}{\partial x} \right) dx \\ &= D \Phi_n(x) \frac{\partial \phi(x, t)}{\partial x} \Big|_{-L}^L - D \int_{-L}^L \left(\frac{d\Phi_n(x)}{dx} \right) \left(\frac{\partial \phi(x, t)}{\partial x} \right) dx \\ &= 0 - D \int_{-L}^L \left(\frac{d\Phi_n(x)}{dx} \right) \left(\frac{\partial \phi(x, t)}{\partial x} \right) dx \end{aligned} \quad (2.27)$$

because $\Phi_n(x)$ satisfies the same boundary conditions as $\phi(x, t)$. The remaining term can be integrated by parts, yielding

$$-D \int_{-L}^L \left(\frac{d\Phi_n(x)}{dx} \right) \left(\frac{\partial \phi(x, t)}{\partial x} \right) dx = D \int_{-L}^L \phi(x, t) \left(\frac{d^2 \Phi_n(x)}{dx^2} \right) dx. \quad (2.28)$$

Substituting this result into Eq. (2.23) gives

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\partial}{\partial t} \int_{-L}^L \frac{\Phi_n(x) \phi(x, t)}{v} dx &= \sum_{n=1}^{\infty} \int_{-L}^L \phi(x, t) \left\{ D \frac{d^2 \Phi_n(x)}{dx^2} + \Phi_n(x) [v \Sigma_f - \Sigma_a] \right\} dx \\ &- \sum_{n=1}^{\infty} \int_{-L}^L \frac{\Phi_n(x) \phi(x, t)}{v} [v v \beta \Sigma_f + v \delta \Sigma_a] dx + \sum_{i=1}^6 \lambda_i \left\{ \sum_{n=1}^{\infty} \int_{-L}^L \Phi_n(x) c_i'(x, t) dx \right\}. \end{aligned} \quad (2.29)$$

The bracketed term on the right side of Eq. (2.29) is identically zero because $\Phi_n(x)$ is an eigenvector of Eq. (2.17), so

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{\partial}{\partial t} \int_{-L}^L \frac{\Phi_n(x) \phi(x, t)}{v} dx \\ &= - \sum_{n=1}^{\infty} \int_{-L}^L \frac{\Phi_n(x) \phi(x, t)}{v} \left[v \nu \beta \Sigma_f + v \delta \Sigma_a \right] dx + \sum_{i=1}^6 \lambda_i \left\{ \sum_{n=1}^{\infty} \int_{-L}^L \Phi_n(x) c_i'(x, t) dx \right\} \end{aligned} \quad (2.30)$$

and

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{\partial}{\partial t} \int_{-L}^L \Phi_n(x) c_i'(x, t) dx \\ &= \nu \beta_i \nu \Sigma_f \sum_{n=1}^{\infty} \int_{-L}^L \frac{\Phi_n(x) \phi(x, t)}{v} dx - \lambda_i \sum_{n=1}^{\infty} \int_{-L}^L \Phi_n(x) c_i'(x, t) dx \quad (i = 1 \text{ to } 6) . \end{aligned} \quad (2.31)$$

At this point, the integrals cannot be evaluated, because the functions $\phi(x, t)$ and $c_i'(x, t)$ are unknown (indeed, if these functions were known, this entire derivation would be pointless). For small perturbations about the steady-state conditions, it will be assumed that the neutron flux and the precursor concentrations can be written as the product of a shape function that depends on only position and an amplitude function that depends on only time. Furthermore, the small perturbation assumption allows the shape function to be approximated by the modes of the time-independent diffusion equation. With this assumption, the approximate time-dependent expressions for the neutron flux and the precursor concentrations after a small perturbation from steady state are

$$\phi(x, t) = \sum_{k=1}^{\infty} n_k(t) \Phi_k(x) \quad (2.32)$$

and

$$c_i'(x, t) = \sum_{n=1}^{\infty} \frac{c_{ik}(t)}{v} \Phi_k(x) \quad (i = 1 \text{ to } 6), \quad (2.33)$$

where $\lambda(t)$ = the amplitude function of the neutron population corresponding to mode k and

$c_{ik}(t)$ = the amplitude function of the concentration of the i th precursor corresponding to mode k .

The assumption of the spatial shape of the flux and precursor concentrations is the main assumption inherent in the point kinetics equations.

Substituting Eqs. (2.32) and (2.33) into Eqs (2.30) and (2.31) gives

$$\begin{aligned} \frac{\partial}{\partial t} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left[\lambda_k(t) \int_{-L}^L \frac{\Phi_n(x) \Phi_k(x)}{v} dx \right] &= -v \left[v \beta \Sigma_f + \delta \Sigma_a \right] \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left[\lambda_k(t) \int_{-L}^L \frac{\Phi_n(x) \Phi_k(x)}{v} dx \right] \\ &+ \sum_{i=1}^6 \lambda_i \left\{ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} c_{ik}(t) \int_{-L}^L \frac{\Phi_n(x) \Phi_k(x)}{v} dx \right\} \end{aligned} \quad (2.34)$$

and

$$\begin{aligned} \frac{\partial}{\partial t} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left[c_{ik}(t) \int_{-L}^L \frac{\Phi_n(x) \Phi_k(x)}{v} dx \right] &= v \beta_i v \Sigma_f \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left[\lambda_k(t) \int_{-L}^L \frac{\Phi_n(x) \Phi_k(x)}{v} dx \right] \\ &- \lambda_i \left[\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} c_{ik}(t) \int_{-L}^L \frac{\Phi_n(x) \Phi_k(x)}{v} dx \right] \quad (i = 1 \text{ to } 6). \end{aligned} \quad (2.35)$$

But because of the orthogonality of the eigenvectors,

$$\int_{-L}^L \frac{\Phi_n(x) \Phi_k(x)}{v} dx = 0 \text{ if } n \neq k. \quad (2.36)$$

Thus, Eqs. (2.34) and (2.35) become

$$\begin{aligned} \frac{d\bar{n}_k(t)}{dt} \int_{-L}^L \frac{\Phi_k^2(x)}{v} dx = & -v \left[v \beta \Sigma_f + \delta \Sigma_a \right] \bar{n}_k(t) \int_{-L}^L \frac{\Phi_k^2(x)}{v} dx \\ & + \sum_{i=1}^6 \lambda_i c_{ik}(t) \int_{-L}^L \frac{\Phi_k^2(x)}{v} dx, \quad (k = 1 \text{ to } \infty) \end{aligned} \quad (2.37)$$

and

$$\begin{aligned} \frac{dc_{ik}(t)}{dt} \int_{-L}^L \frac{\Phi_k^2(x)}{v} dx = & v \beta_i \Sigma_f v \int_{-L}^L \frac{\Phi_k^2(x)}{v} dx \\ & - \lambda_i c_{ik}(t) \int_{-L}^L \frac{\Phi_k^2(x)}{v} dx, \quad (k = 1 \text{ to } \infty, i = 1 \text{ to } 6). \end{aligned} \quad (2.38)$$

Dividing by $\int_{-L}^L \frac{\Phi_k^2(x)}{v} dx$ yields

$$\frac{d\bar{n}_k(t)}{dt} = -v \left[v \beta \Sigma_f + \delta \Sigma_a \right] \bar{n}_k(t) + \sum_{i=1}^6 \lambda_i c_{ik}(t) \quad (k = 1 \text{ to } \infty) \quad (2.39)$$

and

$$\frac{dc_{ik}(t)}{dt} = v \beta_i \Sigma_f \bar{n}_k(t) - \lambda_i c_{ik}(t) \quad (k = 1 \text{ to } \infty, i = 1 \text{ to } 6). \quad (2.40)$$

Recall that reactivity is the ratio of the difference between the neutron production and removal rates to the rate of neutron production:

$$\rho(t) = \frac{\text{production rate} - \text{removal rate}}{\text{production rate}}. \quad (2.41)$$

For the simplified reactor description used in this derivation,

$$\rho(t) = \sum_{k=1}^{\infty} \rho_k(t) = \sum_{k=1}^{\infty} \frac{(v \Sigma_f v + D \nabla^2 - \Sigma_a - \delta \Sigma_a) \bar{n}_k(t) \Phi_k(x)}{v \Sigma_f v \bar{n}_k(t) \Phi_k(x)}. \quad (2.42)$$

But,

$$\frac{(\nu \Sigma_f v + D \nabla^2 - \Sigma_a) \Phi_k(x)}{\nu \Sigma_f \Phi_k(x)} = \rho_{0k} , \quad (2.43)$$

where ρ_{0k} is the subcritical reactivity associated with mode k . Substituting this result into Eq. (2.42),

$$\rho_k(t) = \frac{\rho_{0k} - \delta \Sigma_a}{\nu \Sigma_f} \quad \text{or} \quad \delta \Sigma_a = [\rho_{0k} - \rho_k(t)] \nu \Sigma_f . \quad (2.44)$$

Next, define the *mean generation time* Λ (i.e., the time between the formation of a neutron and its subsequent absorption causing another fission) as

$$\Lambda = \frac{1}{\nu \Sigma_f v} . \quad (2.45)$$

Equations (2.44) and (2.45) can be substituted into Eqs. (2.39) and (2.40) to yield the *modal point kinetics equations* in their standard form:

$$\frac{d n_k(t)}{dt} = \left(\frac{\rho(t) - \rho_{0k} - \beta}{\Lambda} \right) n_k(t) + \sum_{i=1}^6 \lambda_i c_{ik}(t), \quad (n = 1 \text{ to } \infty) \quad (2.46)$$

and

$$\frac{d c_{ik}(t)}{dt} = \frac{\beta_i n_k(t)}{\Lambda} - \lambda_i c_{ik}(t) \quad (n = 1 \text{ to } \infty, i = 1 \text{ to } 6) . \quad (2.47)$$

Equations (2.46) and (2.47) result in a set of seven coupled nonlinear differential equations that describe the time dependence of the weighted neutron and precursor populations associated with spatial mode k . The *one-mode point kinetics equations* are obtained by setting $k = 1$ and $\rho_{0k} = 0$ in Eqs. (2.46) and (2.47):

$$\frac{d n(t)}{dt} = \left(\frac{\rho(t) - \beta}{\Lambda} \right) n(t) + \sum_{i=1}^6 \lambda_i c_i(t) \quad (2.48)$$

and

$$\frac{dc_i(t)}{dt} = \frac{\beta_i \lambda(t)}{\Lambda} - \lambda_i c_i(t) \quad (i = 1 \text{ to } 6) . \quad (2.49)$$

The one-mode point kinetics equations are the most common form of the point kinetics equations because they describe the overall time behavior of a reactor's fundamental spatial mode.

Although these equations have been derived here beginning with the one-speed, one-dimensional diffusion equation, a similar derivation could be performed beginning with the multidimensional, energy-dependent diffusion equation. The main differences are that the integration would be performed over energy as well as space, and the weight function would become the steady-state adjoint flux (the solution to the related set of adjoint equations) rather than the steady-state neutron flux. In each case, the form of the point kinetics equations is the same. The point kinetics equations are used to describe the neutronics in the BWR dynamic models used in the research described in the following chapters.

2.1.6 Inherent Reactivity Feedback Effects

Inherent feedback effects on reactivity, which must be included in dynamic models of nuclear reactors, are described in this section. These effects are caused by changes in neutron moderation and absorption, which occur with temperature or density changes in the fuel, moderator, or coolant. The reaction rate density for absorption or scattering changes because of the temperature dependence of either the microscopic cross sections or the target nuclei number density. A convenient way of expressing the effect of various parameters on reactivity is by defining a *reactivity coefficient* α_i :

$$\alpha_i = \frac{\partial \rho}{\partial \xi_i} , \quad (2.50)$$

where ξ_i is a reactor parameter such as fuel temperature or coolant void fraction. The fact that reactivity coefficients are generally temperature dependent is usually neglected when modeling reactor conditions that remain near the steady-state conditions. In such cases, the reactivity perturbation $\delta \rho_i(t)$ due to a parameter change $\delta \xi_i(t)$ is given by

$$\delta \rho_i(t) = \left(\frac{\partial \rho}{\partial \xi_i} \right) \delta \xi_i(t) . \quad (2.51)$$

Because the reactivity is affected by a number of parameters, the total inherent reactivity feedback is given by an equation of the form

$$\delta \rho(t) = \sum_{i=1}^j \left(\frac{\partial \rho}{\partial \xi_i} \right) \delta \xi_i(t) , \quad (2.52)$$

where ξ_i is a specific parameter and j is the number of parameters affecting reactivity. The two parameters that have the most significant feedback effects in BWRs during limit-cycle oscillations are (1) the coolant void fraction, expressed by using the *void reactivity coefficient* α_v ; and (2) the fuel temperature, expressed by using the *Doppler coefficient* α_T .

A power increase results in greater heating of the coolant/moderator. In BWRs, this heating causes an increase in the average void fraction (because more steam bubbles are created) and a corresponding decrease in coolant density (assuming the flow rate remains constant). As the coolant density decreases and the void fraction increases, the moderating effect of the coolant is decreased because of the decrease in the number density of coolant nuclei. This effect causes neutrons to travel farther between scattering

collisions, increasing the net distance traveled from the point of fission to the point where the neutron reaches thermal energies. Because neutrons must travel farther before they are thermalized, a greater number of neutrons leak from the core without causing fission, thus lowering the reactivity. This effect is greater in small cores. Reduced moderation also raises the average energy level of the neutron spectrum, with the end result that more neutrons have energies corresponding to resonances in the microscopic absorption cross section. This effect increases resonance absorption and lowers reactivity because fewer neutrons are able to reach thermal energies where they are most likely to cause fission. Thus, the effect of coolant density or void fraction changes on neutron moderation produces a negative feedback between reactor power and reactivity (i.e., α_v is negative).

The process in which parasitic absorption increases with fuel temperature is called the *nuclear Doppler effect* because of its similarity to the more familiar Doppler effect seen in the interaction of sound or light waves with moving receptors or sources. As the temperature of a fuel nucleus is increased, the amplitude and velocity of its thermal motion increases. This effect causes the range of neutron velocities at which the relative neutron kinetic energy falls into one of these energy resonances to increase, in effect broadening the energy resonance. The net result is greater nonproductive neutron absorption and correspondingly fewer thermal neutrons available to cause fission. Thus, increasing reactor temperature results in a decrease in reactivity and in a reduction in reactor power. Because increasing reactor power (and thus fuel temperature) results in a reduction in reactivity, the nuclear Doppler effect produces a negative feedback between reactor power and reactivity (i.e., α_T is negative).

2.1.7 Neutronic Models with Inherent Feedback

Equations describing reactivity feedback effects can be formed and combined with the point kinetics equations to form a simple dynamic model of a BWR. These equations can be formed from first principles, but a more heuristic approach is to assume the form of the equations and then fit equation parameters so the equations describe the results of detailed computer calculations or experimental measurements. The resulting models remain relatively simple and can be more useful than detailed computer models for obtaining insight into the physical processes that occur in a BWR. These models retain much of the accuracy of detailed models, particularly so near conditions at which the equation parameter fitting was performed.

Because the main feedback effects during limit-cycle oscillations in BWRs depend on void fraction and fuel temperature, equations must be developed to calculate these quantities. Starting with the equation for fuel temperature, a simple heat balance equation can be written stating that the rate of temperature change depends on the rate of heat production in the fuel and the rate of heat transfer from the fuel. The rate of heat production will depend primarily on the neutron flux $\phi(r, t)$, and the heat transfer rate will depend primarily on the fuel temperature $T'(r, t)$, such that

$$\frac{\partial T'(r, t)}{\partial t} = a_2 T'(r, t) + a_0 \phi(r, t) , \quad (2.53)$$

where a_0 and a_2 are parameters whose values are obtained from fitting to calculated or measured results.

To make Eq. (2.53) compatible with the point kinetics equations, the space dependence must be "removed." This step is accomplished by multiplying Eq. (2.53) by a weight function and then integrating over volume (and energy, in the energy-dependent

case). Next, returning to the slab geometry of Fig. 2.1 and assuming small perturbations from the steady state allow the time- and space-dependent temperature distribution to be approximated by a summation of products of the eigenvectors of Eq. (2.17) and a time-dependent amplitude function $\hat{T}_n$:

$$T'(r, t) = T'(x, t) = \sum_{n=1}^{\infty} \hat{T}_n(t) \Phi_n(x) . \quad (2.54)$$

Substituting Eqs. (2.32) and (2.54) into Eq. (2.53), multiplying by a weight function

$\sum_{k=1}^{\infty} \Phi_k(x)$, and integrating from $-L$ to L gives

$$\begin{aligned} \frac{\partial}{\partial t} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \hat{T}_n(t) \int_{-L}^L \Phi_n(x) \Phi_k(x) dx &= a_2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \hat{T}_n(t) \int_{-L}^L \Phi_n(x) \Phi_k(x) dx \\ &+ a_0 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \hat{n}_n(t) \int_{-L}^L \Phi_n(x) \Phi_k(x) dx . \end{aligned} \quad (2.55)$$

But, because of the orthogonality of the eigenvectors of Eq. (2.17), Eq. (2.55) becomes

$$\frac{\partial}{\partial t} \hat{T}_k(t) \int_{-L}^L \Phi_k^2(x) dx = a_2 \hat{T}_k(t) \int_{-L}^L \Phi_k^2(x) dx + a_0 \hat{n}_k(t) \int_{-L}^L \Phi_k^2(x) dx . \quad (2.56)$$

Dividing by $\int_{-L}^L \Phi_k^2(x) dx$ and setting $k = 1$ (i.e., assume that the fuel temperature is affected only by the fundamental mode of the neutronics) gives

$$\frac{d\hat{T}(t)}{dt} = a_2 \hat{T}(t) + a_0 \hat{n}(t) . \quad (2.57)$$

Equation (2.57) can be used to calculate a weighted fuel temperature over the reactor volume. March-Leuba shows that for this simple fuel temperature model, a_0 is proportional to the energy released per fission and is inversely proportional to both the

fuel density and the heat capacity.⁹⁴ Similarly, a_2 is shown to be proportional to the effective fuel-to-coolant heat-transfer coefficient and inversely proportional to the fuel density, heat capacity, and fuel pellet outer radius.⁹⁴

Selecting a form for the equation describing the effect of void fraction on reactivity is more difficult because no simple expression such as a heat balance is applicable. Fortunately, examination of calculated results from detailed computer codes, such as the LAPUR code, can be used to select the form of the equation. For typical BWR conditions, it has been shown that the average void reactivity $\hat{\rho}_v(t)$, which is the product of the void reactivity coefficient and the excess void fraction, is given by

$$\frac{d^2 \hat{\rho}_v(t)}{dt^2} + a_3 \frac{d \hat{\rho}_v(t)}{dt} + a_4 \hat{\rho}_v(t) = K \hat{T}(t) , \quad (2.58)$$

where a_3 , a_4 , and K are constants whose values are determined by fitting to measured or calculated results.⁹⁴ It has been shown that a_3 and a_4 are proportional to coolant mass flux and inversely proportional to void fraction.⁹⁴ K is proportional to the coolant mass flux and to the magnitude of the void reactivity feedback and is inversely proportional to the channel transit time.⁹⁴

Finally, the total effect on reactivity is given by a sum of the void reactivity and the effect of fuel temperature:

$$\rho(t) = \hat{\rho}_v(t) + \alpha_T \hat{T}(t) . \quad (2.59)$$

Equations (2.48), (2.49), (2.57), (2.58), and (2.59) form a simple dynamic model of a BWR. This particular model has been shown to qualitatively predict BWR dynamic behavior.⁹⁴ The equations can be linearized about a set of operating conditions to examine the asymptotic stability or the nonlinear model can be solved, either numerically

or using weighted-residual methods, to examine the model's behavior in the nonlinear range.

2.2 Description of the LAPUR Computer Code

The LAPUR computer code was developed at Oak Ridge National Laboratory to calculate the linear dynamics of large BWR cores. LAPUR is a linear code; the dynamic equations used in the code have been linearized for small perturbations about the steady-state conditions. The linearization process, with the assumption of small perturbations and the resulting neglect of second-order nonlinear terms, limits the applicability of LAPUR to the region of linear stability. The degree of stability calculated by LAPUR is given in terms of the decay ratio and by gain and phase margins of the closed-loop transfer function. Additional output includes transfer functions giving the frequency-dependent relationships between various reactor parameters. The code is divided into two autonomous modules, LAPURX, which calculates the steady-state core conditions, and LAPURW, which calculates the core dynamics. This section presents a brief description of the LAPUR code. The calculations performed by this code are described thoroughly in reference 61.

2.2.1 Program Input

The required input for the steady-state calculation performed by module LAPURX is listed below.

System operating status

1. State parameters such as reactor pressure, total core power, total core flow, and inlet coolant temperature.

2. The radial and axial power shape. These quantities define both the distribution of total core power between channel types and the axial distribution of power along each channel type.
3. The fraction of power that is transferred to the coolant by neutron moderation or by direct heating effects such as absorption of photons.

System design parameters

1. Mechanical parameters such as the dimensions of the fuel channel boxes or the fuel rods.
2. Physical parameters such as hydraulic diameters, friction multipliers, fuel and cladding densities, and fuel-clad gap conductances.

User options

1. Error criteria for iterative calculations.
2. Adjustable parameters for two-phase correlations.
3. Output options.

The required input for the dynamic calculation performed by module LAPURW is listed below.

System parameters

The recirculation loop pressure to flow-rate gain and time constant.

Neutronics parameters

1. An effective neutron lifetime table.
2. A steady-state core reactivity table.
3. The delayed neutron fractions and their decay constants.
4. The Doppler reactivity coefficient.
5. A table of density (void) reactivity coefficients.

Output options

1. The frequencies of interest for the dynamic analysis.
2. Editing and plotting options.

3. Stability calculation options.

2.2.2 The Steady-State Module LAPURX

The steady-state conditions of the core must be calculated before the core dynamics can be calculated. The core is modeled as a collection of parallel channels with each channel being divided into a series of axial nodes. A maximum of seven channel types and 100 nodes per channel type are available. All channel types have the same number of axial nodes. The channels connect upper and lower plenum regions, which provide hydrodynamic coupling between the channels. Three flow regions are modeled: nonboiling, subcooled boiling, and bulk boiling. The location of the boundary between flow regions is calculated by LAPUR. The steady-state coolant conditions must be calculated before the steady-state fuel conditions to determine the appropriate heat transfer relations for each channel type.

The steady-state coolant calculation determines the mass flow rate of coolant entering each channel and the axial distribution of density, temperature, velocity, and pressure in each channel type. The steps used to calculate the steady-state coolant distribution are:

1. Estimate the inlet flow rate for each channel type.
2. Calculate the axial enthalpy distribution for each channel type by using the coolant inlet conditions, the axial and radial power distribution information, and the coolant thermodynamic properties.
3. Use the enthalpy distributions to determine the location of the boiling boundary regions for each channel type.
4. Solve the coolant mass and energy balance equations along each channel type.
5. Solve the momentum equation to obtain the axial pressure distributions.
6. Verify that the outlet pressures are equal for all the channel types. If the pressures are unequal, the channel inlet flow rates are adjusted and these steps are repeated until this condition is satisfied.

The code includes the effects of direct coolant heating due to photon absorption and neutron moderation. The fuel steady-state conditions can be calculated once the coolant conditions are known.

The fuel temperature and heat transfer calculations use four radial regions to model the fuel: the UO_2 fuel pellet, the clad-pellet gap, the cladding, and the coolant film adjacent to the cladding. The fuel pellet is divided into ten annular rings to account for temperature differences in the fuel heat transfer coefficient and for the effect of nuclear self-shielding. Because of the large temperature differences across the fuel pellet, the axial heat transfer is negligible compared to that of the radial heat transfer. Thus, the axial heat transfer is assumed to be zero in these calculations.

An iterative procedure is used to calculate the fuel temperature because of the temperature dependence of the fuel conduction heat transfer coefficient. Equations for the thermal resistances of the fuel pellet rings, the pellet-clad gap, the cladding, and the coolant film are developed for each axial node. Knowing the axial coolant temperature distribution allows a finite-difference scheme to be used to solve for the temperatures at the center of the cladding, the center of the pellet-clad gap, and the center of each fuel ring.

The LAPURX output consists of the flow rate, pressure drop, and exit quality of each channel type; the coolant density, void fraction, enthalpy, quality, velocities, and friction components at each node; and a set of coefficients for the dynamic calculations. This information is stored in three separate files that are used as input to the module LAPURW.

2.2.3 The Dynamic Module LAPURW

The calculations performed by the dynamic module LAPURW determine

1. the response of the fuel heat flux and average temperature to perturbations in power generation, coolant temperature, and coolant flow at each axial node in every channel type;
2. the response of the coolant parameters to perturbations in power generation, coolant flow rate, and coolant inlet temperature;
3. the open-loop transfer function for the channel hydrodynamics;
4. the reactivity feedback at each axial node due to perturbations in parameters such as coolant voids and fuel temperature;
5. feedback reactivity transfer functions for perturbations in power, coolant inlet temperature, and core flow rate;
6. the core natural frequency and decay ratio; and
7. the closed-loop reactivity-to-power transfer function.

Calculation of the fluid dynamics requires conservation equations for mass, energy, and momentum as well as equations relating steam velocity to slip ratio, heat flux, subcooled boiling, and relating frictional forces to the physical parameters of the coolant channels. The equations for each axial node are linearized about the steady-state conditions and Laplace-transformed for solution in the frequency domain. The transformed equations are used to form transfer functions between perturbations in coolant parameters and fluctuating state variables. Once the coolant dynamics have been calculated, the void reactivity coefficients can be used to calculate transfer functions relating perturbations in coolant parameters to perturbations in reactivity.

A similar linearization is performed by using the fuel temperature and heat transfer equations. The resulting transformed equations relate perturbations in heat generation rate to perturbations in fuel temperature and in the rate of convective heat transfer to the coolant. The fuel dynamics can be linked to the fluid dynamics by using conservation of

energy, and the fuel dynamics can be linked to the neutronics by using the Doppler reactivity feedback relationship. This action allows transfer functions between fuel temperature and reactor power or between fuel temperature and various coolant parameters to be formed.

The neutronics model used in recent versions of LAPUR is the point kinetics model. The point kinetics equations are linearized for small perturbations about the steady-state conditions and Laplace transformed into the frequency domain. The transformed equations can be combined with the coolant dynamics equations, the fuel dynamics equations, the table of void reactivity coefficients, and the Doppler coefficient to form a complete dynamic model of the BWR in the frequency domain. Because of the detailed description of the reactor conditions made possible by the use of different channel types and axial nodes, this model has the potential to describe BWR dynamics very accurately.

The main output data of the LAPUR code are transfer functions relating perturbations in BWR parameters. The open-loop, power-to-reactivity transfer function and the closed-loop, reactivity-to-power transfer functions are of particular interest because they contain information that can be used to determine reactor stability. The Nyquist criteria can be used to determine system stability from the open-loop transfer function, or the stability can be determined from the gain and phase margins obtained from the Bode plot of the closed-loop transfer function. The closed-loop transfer function is used also to calculate the decay ratio and the oscillation frequency. These quantities are calculated from the most unstable pole of the closed-loop transfer function (i.e., the pole with the most positive real part).

2.3 Weighted-Residual Methods

Weighted-residual methods can be used to solve both linear and nonlinear differential equations. The applicability of this method to nonlinear equations makes it useful for investigating the nonlinear dynamics of BWRs. In general, weighted-residual methods will not yield exact solutions; an approximate solution is obtained whose accuracy depends on the number of terms in the approximation. Weighted-residual methods are described thoroughly in references 134 through 137.

Consider a general differential equation:

$$L(u) = f \text{ in } \mathfrak{R}, B_1(u) = u' \text{ on } S_1, B_2 = u'' \text{ on } S_2, \quad (2.60)$$

where L = a differential operator, not necessarily linear;
 u = the dependent variable;
 f = a specified forcing function in the domain $\mathfrak{R}$;
 B_1 and B_2 = boundary operators associated with the operator L ;
 u' and u'' = specified functions on the portions S_1 and S_2 of the boundary S of the domain $\mathfrak{R}$.

In the method of weighted residuals, a solution, u_n of order n , is sought in the form

$$u_n = \sum_{i=1}^n g_i \psi_i + \psi_0, \quad (2.61)$$

where ψ_i ($i = 0$ to n) is the set of known trial functions and the g_i is the corresponding set of undetermined constants.

The trial functions cannot be selected arbitrarily but must satisfy the following conditions:

1. ψ_0 should satisfy all of the specified boundary conditions,
2. the ψ_i ($i = 1$ to n) should satisfy the homogeneous form of all specified boundary conditions,
3. the ψ_i ($i = 0$ to n) should be continuous, and
4. the ψ_i ($i = 0$ to n) should be linearly independent and complete.

In general, u_n will not satisfy the differential equation exactly. A measure of the error in the approximate solution is given by the equation residual, R_e :

$$R_e(g_i, \psi_i, f) = L(u_n) - f = L\left(\sum_{i=1}^n g_i \psi_i + \psi_0\right) - f. \quad (2.62)$$

Note that it is assumed that a small value of the equation residual corresponds to an approximate solution that closely matches the exact solution. Systems for which this assumption is justified are called *well-behaved* or *well-conditioned* systems.

Weighted-residual methods cannot be successfully applied to systems of equations that are not well behaved. Thus, the discussion in this section pertains to only well-behaved systems. In an assumed well-behaved system of equations, R_e would be identically zero if u_n were the exact solution. If u_n is not an exact solution, values of the g_i can be found by giving the best approximate solution of the assumed form. Equations for the g_i are formed by requiring the equation residual to be orthogonal to a linearly independent set of n weight functions W_i in the domain $\mathcal{R}$:

$$\int_{\mathcal{R}} W_i R_e(g_i, \psi_i, f) d\mathcal{R} = 0. \quad (2.63)$$

This condition results in n algebraic equations for the n values of the g_i . If the differential operator L is linear, the algebraic equations for the g_i will be linear. Nonlinear algebraic equations for the g_i occur when L is nonlinear. For a given complete set of trial functions, the magnitude of the error in the approximate solution depends on the number of terms in the approximation. As more terms are included, the approximate solution converges to the exact solution.

The particular criteria satisfied by the g_i depend on the choice of the weight functions. Except for the requirement of linear independence, no restrictions are on the selection of the weight functions. Four methods, differing only in the choice of the weight functions, are often applied: the collocation method, the subdomain method, the least-squares method, and Galerkin's method. Approximate solutions calculated by different methods will eventually converge to the exact solution. The methods differ in their computational complexity and in the rate of convergence of the approximate solution as additional terms are included.

In the collocation method, the weight functions are given by

$$W_i = \delta(x - x_i), \quad x_i \text{ in } \mathfrak{R}, \quad i = 1 \text{ to } n, \quad (2.64)$$

where the x_i represents the n collocation points and δ is the Dirac delta function. The collocation points are arbitrary but are usually selected to cover $\mathfrak{R}$ more or less uniformly by a simple pattern. The integral of the weighted residual becomes

$$\int_{\mathfrak{R}} W_i R_e(g_i, \psi_i, f) d\mathfrak{R} = R_e(g_i, \psi_i, f) \Big|_{x_i} = 0. \quad (2.65)$$

The collocation method requires that the equation residual be zero at the n collocation points. As the number of terms in the approximate solution and the number of collocation points increase, the equation residual presumably approaches zero everywhere in $\mathfrak{R}$, and the approximate solution approaches the exact solution. The collocation method is the easiest weighted-residual method to apply because it avoids integration over the domain and works directly with the expression for the equation residual.

The subdomain method divides the domain $\mathfrak{R}$ into n subdomains $\mathfrak{R}_i$ and uses the weight functions

$$W_i = \begin{cases} 1 & x \text{ in } \mathfrak{R}_i \\ 0 & x \text{ not in } \mathfrak{R}_i \end{cases} . \quad (2.66)$$

The average of the equation residual is zero in each subdomain. The subdomains are arbitrary but must cover $\mathfrak{R}$, usually according to some simple pattern. As the number of terms in the approximate solution and the number of subdomains increase, the differential equation is satisfied on the average over smaller and smaller subdomains, eventually approaching the exact solution.

The least-squares method is the oldest weighted-residual method and has the most rapid rate of convergence to the exact solution as the number of terms is increased. For a given number of terms in the approximate solution, the values of the g_i calculated by using the least-squares method result in the most accurate approximate solution. The weight functions used in the least-squares method are:

$$W_i = \left(\frac{\partial \mathbf{R}_e(g_i, \psi_i, f)}{\partial g_i} \right) , \quad (2.67)$$

so the equations for the g_i are given by

$$\frac{\partial}{\partial g_i} \int_{\mathfrak{R}} \mathbf{R}_e^2(g_i, \psi_i, f) d\mathfrak{R} = 0 . \quad (2.68)$$

Application of the least-squares method tends to result in very cumbersome equations for the g_i ; thus, it is often rejected in favor of one of the other weighted-residual methods.

Galerkin's method is arguably the best-known and most-often-used weighted-residual method. The weight functions are the same as the trial functions:

$$W_i = \psi_i . \quad (2.69)$$

The equations for the g_i are obtained from

$$\int_{\mathcal{R}} \psi_j \mathbf{R}_\epsilon(c_i, \psi_i, f) d\mathcal{R} = 0, (i, j = 1 \text{ to } n) . \quad (2.70)$$

Because the weight functions are members of a complete set, Galerkin's method forces the residual to be zero by requiring it to be orthogonal to a complete set of functions over the domain $\mathcal{R}$. It can be shown that Galerkin's method is closely related to variational methods.¹³⁷

Any weighted-residual method can be used to construct an approximate solution that will converge to the exact solution as terms are added to the approximation. Selection of a particular method depends on the desired rate of convergence and on the ease of implementation. In this dissertation research, one main objective was to obtain expressions describing the behavior of BWR state variables during limit-cycle oscillations in closed form; this allows the BWR limit cycle to be examined in a way that is complementary to previous numerical studies. This objective requires the equations for the undetermined constants in the approximate solution to be easily evaluated.

Galerkin's method was selected to construct most of the approximate solutions in this dissertation research. With Galerkin's method, the expressions for the undetermined constants contain integrals of products of trigonometric functions for which closed-form expressions can be obtained. Thus, the expressions for the undetermined constants can be evaluated without resorting to numerical integration. Furthermore, Galerkin's method converges relatively rapidly to the exact solution and avoids the selection of arbitrary quantities such as collocation points or subdomain boundaries. The combination of (1) closed-form expressions for the undetermined constants, (2) lack of the need to select arbitrary quantities to form the expressions for the undetermined constants, and (3) a

reasonable rate of convergence lead to the selection of Galerkin's method as the weighted-residual method most often used to construct the approximate solutions in this dissertation research.

3. APPROXIMATE SOLUTION OF THE PHENOMENOLOGICAL MODEL EQUATIONS

This chapter describes the application of Galerkin's method to obtain an approximate solution to the phenomenological equations developed by March-Leuba.⁹⁴ These equations qualitatively describe BWR dynamics in both the linearly stable and the linearly unstable regions. The approximate solution is developed for only the linearly unstable region; a linearized version of the equations, which can be solved directly, describes the dynamic behavior in the region of linear stability.

The phenomenological model equations will be used to develop a set of equations describing the behavior of perturbations from the steady state. These equations will be normalized, and the linear stability of the solution about the normal operating point will be examined. The numerically calculated time-dependent behavior of the normalized perturbations for both linearly stable and linearly unstable cases will be shown. The application of Galerkin's method to obtain an approximate solution for the dynamic behavior in the linearly unstable region will be described in detail. This description will be followed by examination of high-order and low-order approximate solutions.

3.1 The Phenomenological Model Equations

The phenomenological model equations were discussed in Subsections 2.1.5 and 2.1.7. These equations are

$$\frac{d\mathcal{N}(t)}{dt} = \left(\frac{\rho(t) - \beta}{\Lambda} \right) \mathcal{N}(t) + \lambda \mathcal{C}(t) , \quad (3.1)$$

$$\frac{d\mathcal{C}(t)}{dt} = \frac{\beta \mathcal{N}(t)}{\Lambda} - \lambda \mathcal{C}(t) , \quad (3.2)$$

$$\frac{d\hat{T}(t)}{dt} = a_2 \hat{T}(t) + a_0 \mathcal{N}(t) , \quad (3.3)$$

$$\frac{d^2 \hat{\rho}_*(t)}{dt^2} + a_3 \frac{d \hat{\rho}_*(t)}{dt} + a_4 \hat{\rho}_*(t) = K \hat{T}(t) , \quad (3.4)$$

and

$$\rho(t) = \hat{\rho}_*(t) + \alpha_T \hat{T}(t) . \quad (3.5)$$

Equations (3.1) and (3.2) are the one-mode point kinetics equations where, for simplicity, only one precursor group has been included. Equations (3.3), (3.4), and (3.5) describe the behavior of fuel temperature, void reactivity, and total reactivity. These equations form a simple model that retains the most significant aspects of BWR dynamics.

3.1.1 Normalization of the Phenomenological Model Equations

Letting the steady-state weighted neutron population be N_0 , the perturbation about N_0 be $\mathcal{N}(t)$, the steady-state weighted precursor population be C_0 , and the perturbation about C_0 be $\mathcal{C}(t)$ results in

$$\mathcal{N}(t) = N_0 + \mathcal{N}(t) \quad (3.6)$$

and

$$\mathcal{C}(t) = C_0 + \mathcal{C}(t) . \quad (3.7)$$

Substituting Eqs. (3.6) and (3.7) into Eq. (3.1) yields

$$\frac{d\mathcal{N}(t)}{dt} = \frac{-N_0 \beta}{\Lambda} + \frac{N_0 \rho(t)}{\Lambda} + C_0 \lambda + \mathcal{C}(t) \lambda + \frac{\mathcal{N}(t)}{\Lambda} [\rho(t) - \beta] . \quad (3.8)$$

For steady-state conditions, the time rate of change of both $n(t)$ and $\rho(t)$ are zero, so

Eq. (3.1) can be solved for the value of C_0 :

$$0 = -\frac{N_0 \beta}{\Lambda} + C_0 \lambda \quad \text{or} \quad C_0 = \frac{\beta N_0}{\lambda \Lambda} . \quad (3.9)$$

Substituting this result into Eq. (3.8) and normalizing by the steady-state neutron population yields

$$\frac{dn(t)}{dt} = \frac{n(t)}{\Lambda} [\rho(t) - \beta] + \frac{\rho(t)}{\Lambda} + \lambda c(t) , \quad (3.10)$$

where $n(t) = \frac{\bar{n}(t)}{N_0}$ and $c(t) = \frac{\bar{c}(t)}{N_0}$.

Substituting Eqs. (3.6) and (3.7) into Eq. (3.2) and performing a similar normalization yields

$$\frac{dc(t)}{dt} = -\lambda c(t) + \frac{\beta n(t)}{\Lambda} . \quad (3.11)$$

Equations (3.10) and (3.11) describe the time rates of change for perturbations about the steady-state values of neutron and precursor populations, both normalized to the steady-state neutron population. The nonlinearity appears as the product of normalized neutron population and the reactivity in Eq. (3.10).

A similar procedure is used to obtain equations for the perturbations about the steady-state values for the void reactivity and fuel temperature. Let

$$\hat{T}(t) = T_0 + T(t) \quad (3.12)$$

and

$$\hat{\rho}_s(t) = \rho_{s0} + \rho_s(t) , \quad (3.13)$$

where T_0 = the steady-state weighted fuel temperature,
 $T(t)$ = the fuel temperature perturbation,
 ρ_{s0} = the steady-state void reactivity,
 $\rho_s(t)$ = the void reactivity perturbation.

Substituting Eqs. (3.6) and (3.12) into Eq. (3.3) yields

$$\frac{dT(t)}{dt} = a_0 N_0 + a_2 T_0 + a_0 \rho(t) + a_2 T(t) . \quad (3.14)$$

Equation (3.3) shows that for steady-state conditions, $a_0 N_0 + a_2 T_0 = 0$. Letting

$a_1 = N_0 a_0$ allows Eq. (3.14) to be rewritten

$$\frac{dT(t)}{dt} = a_1 \rho(t) + a_2 T(t) . \quad (3.15)$$

Equation (3.15) describes the dependence of perturbations about the steady-state fuel temperature on perturbations in the normalized neutron density.

An equation describing the void reactivity perturbations is obtained by substituting Eq. (3.12) and (3.13) into Eq. (3.4):

$$\frac{d^2 \rho_s(t)}{dt^2} + a_3 \frac{d \rho_s(t)}{dt} + a_4 \rho_s(t) = K T(t) + K T_0 - a_4 \rho_{s0} . \quad (3.16)$$

Equation (3.4) shows that for steady-state conditions, $K T_0 - a_4 \rho_{s0} = 0$; thus,

$$\frac{d^2 \rho_s(t)}{dt^2} + a_3 \frac{d \rho_s(t)}{dt} + a_4 \rho_s(t) = K T(t) . \quad (3.17)$$

Equation (3.17) describes the dependence of perturbations about the steady-state void reactivity on perturbations in the void reactivity and fuel temperature.

Substituting Eqs. (3.12) and (3.13) into Eq. (3.5) gives

$$\rho(t) = \rho_{a0} + \alpha_T T_0 + \rho_a(t) + \alpha_T T(t) . \quad (3.18)$$

Recognizing from Eq. (3.5) that $\rho_{a0} + \alpha_T T_0 = 0$ leaves

$$\rho(t) = \rho_a(t) + \alpha_T T(t) . \quad (3.19)$$

Equation (3.19) describes the dependence of total reactivity on perturbations in void reactivity and fuel temperature. The asymptotic stability of Eqs. (3.10), (3.11), (3.15), (3.17), and (3.19) will be examined, and an approximate solution to these equations in the linearly unstable region will be obtained by using Galerkin's method.

3.1.2 Equilibrium Points and Linear Stability

All time rates of change of state variables are zero at the equilibrium points. The values of $n(t)$, $c(t)$, $T(t)$, and $\rho_a(t)$ at the equilibrium points are calculated by setting the time derivatives in Eqs. (3.10), (3.11), (3.15), and (3.17) to zero and solving for the equilibrium values of $n(t)$, $c(t)$, $T(t)$, and $\rho_a(t)$. If the initial perturbation in normalized weighted neutron population is Δ , it can be shown that two equilibrium points exist and are given by

$$P_{e1}(n, c, T, \rho_a) = (-\Delta, -\beta\Delta/\lambda\Lambda, 0, 0)$$

and

$$P_{e2}(n, c, T, \rho_a) = (-1, -\beta/\lambda\Lambda, -a_1/a_2, (\Delta-1)a_1, K/a_2 a_4).$$

P_{e1} corresponds to normal operation, and P_{e2} corresponds to shutdown.

The asymptotic stability during normal operation is examined by linearizing Eqs. (3.10), (3.11), (3.15), and (3.17) about equilibrium point P_{e1} . Letting

$$n(t) = -\Delta + \delta n(t) , \quad (3.20)$$

$$c(t) = -\frac{\beta \Delta}{\lambda \Lambda} + \delta c(t) , \quad (3.21)$$

$$T(t) = 0 + \delta T(t) , \quad (3.22)$$

and

$$\rho_*(t) = 0 + \delta \rho_*(t) , \quad (3.23)$$

where $\delta n(t)$, $\delta c(t)$, $\delta T(t)$, and $\delta \rho_*(t)$ are perturbations about P_{e1} , the equations for the time rate of change of perturbations about equilibrium point P_{e1} become

$$\begin{aligned} \frac{d \delta n(t)}{dt} = & \frac{(1 - \Delta)}{\Lambda} \delta \rho_*(t) - \frac{\beta}{\Lambda} \delta n(t) + \lambda \delta c(t) + \frac{\delta n(t) \delta \rho_*(t)}{\Lambda} \\ & + \frac{\alpha_T \delta T(t)}{\Lambda} + \frac{\alpha_T \delta n(t) \delta T(t)}{\Lambda} - \frac{\alpha_T \delta T(t) \Delta}{\Lambda} , \end{aligned} \quad (3.24)$$

$$\frac{d \delta c(t)}{dt} = \frac{\beta}{\Lambda} \delta n(t) - \lambda \delta c(t) , \quad (3.25)$$

$$\frac{d \delta T(t)}{dt} = -a_1 \Delta + a_1 \delta n(t) - a_2 \delta T(t) , \quad (3.26)$$

and

$$\frac{d^2 \delta \rho_*(t)}{dt^2} + a_3 \frac{d \delta \rho_*(t)}{dt} + a_4 \delta \rho_*(t) = K \delta T(t) . \quad (3.27)$$

Letting

$$\delta Y_*(t) = \frac{d \delta \rho_*(t)}{dt} \quad (3.28)$$

allows Eq. (3.25) to be written as

$$\frac{d \delta Y_*(t)}{dt} = -a_3 \delta Y_*(t) - a_4 \delta \rho_*(t) + K \delta T(t) . \quad (3.29)$$

Thus, Eq. (3.25) is transformed from a second-order differential equation into the two first-order differential equations, Eqs. (3.26) and (3.27). For small perturbations, the terms $\frac{\delta n(t) \delta \rho_a(t)}{\Lambda}$ and $\frac{\delta n(t) \delta T(t)}{\Lambda}$ are negligible compared to other terms in Eq. (3.22); thus, these terms can be neglected. This results in a set of equations linearized about operating point P_{e1} . These equations can be written in matrix form:

$$\begin{bmatrix} -\frac{\beta}{\Lambda} & \lambda & \frac{\alpha_T}{\Lambda} & \frac{1}{\Lambda} & 0 \\ \frac{\beta}{\Lambda} & -\lambda & 0 & 0 & 0 \\ a_1 & 0 & -a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & K & -a_4 & -a_3 \end{bmatrix} \begin{bmatrix} \delta n(t) \\ \delta c(t) \\ \delta T(t) \\ \delta \rho_a(t) \\ \delta Y_a(t) \end{bmatrix} = \begin{bmatrix} \frac{d \delta n(t)}{dt} \\ \frac{d \delta c(t)}{dt} \\ \frac{d \delta T(t)}{dt} \\ \frac{d \delta \rho_a(t)}{dt} \\ \frac{d \delta Y_a(t)}{dt} \end{bmatrix}. \quad (3.30)$$

The stability of the system of equations can be determined from the eigenvalues of the matrix in Eq. (3.30). If any eigenvalues have positive real parts, the equilibrium point P_{e1} is unstable.

Values for the phenomenological model equation parameters were obtained from Ref. 94. The parameter values were calculated by fitting the model's closed-loop transfer function to the LAPUR-calculated closed-loop transfer function. The particular conditions used in the LAPUR calculation were selected because they resulted in limit-cycle oscillations during stability tests at the Vermont Yankee reactor in 1980.⁹⁴ The parameter values are shown in Table 3.1.

The parameter K is proportional to the coolant mass flux, the magnitude of the void reactivity feedback, the fuel heat transfer coefficient and is inversely proportional to the

Table 3.1. Phenomenological model equation parameter values

Parameter	Value
a_1	$25.04 \text{ K} \cdot \text{s}^{-1}$
a_2	0.23 s^{-1}
a_3	2.25 s^{-1}
a_4	6.82 s^{-2}
α_T	$-2.52 \times 10^{-5} \text{ K}^{-1}$
β	0.0056
Λ	$4.00 \times 10^{-5} \text{ s}^{-1}$

channel transit time. K controls the gain of the feedback loop; thus, the stability of the equations is highly dependent on the value of K . The eigenvalues were calculated with K ranging from -0.009 to 0 . The real part of the eigenvalue with the most positive real part is shown in

Figure 3.1. This figure shows that the system becomes linearly unstable for values of K less than -0.0037 . The value of K at the stability boundary (where the eigenvalue with the most positive real part equals zero) is designated K_0 .

3.1.3 Time-Dependent Behavior of the Phenomenological Model Equations

The time-dependent behavior of the normalized neutron population was examined for the linearly stable and the linearly unstable regions by using the Advanced Continuous Simulation Language (ACSL) computer code to numerically integrate Eqs. (3.10), (3.11), (3.15), (3.17), and (3.19).¹³⁸ Eight cases were examined: all cases used the equation parameter values given in Table 3.1 and a delta function in fuel temperature at time zero as the initial perturbation. The values of K used were -0.0030 , -0.00369 , -0.00371 , -0.0038 , -0.0044 , -0.0050 , -0.0056 , -0.0057 , and -0.0060 .

Figure 3.2 shows the time behavior of the normalized neutron population for $K = -0.0030$. This case is in the linearly stable region. Figure 3.2 shows that the initial perturbation quickly damps out, indicating that the normalized neutron population rapidly returns to its steady-state value. Figure 3.3 shows the time behavior of the normalized neutron population for $K = -0.00369$. This value of K puts the system of

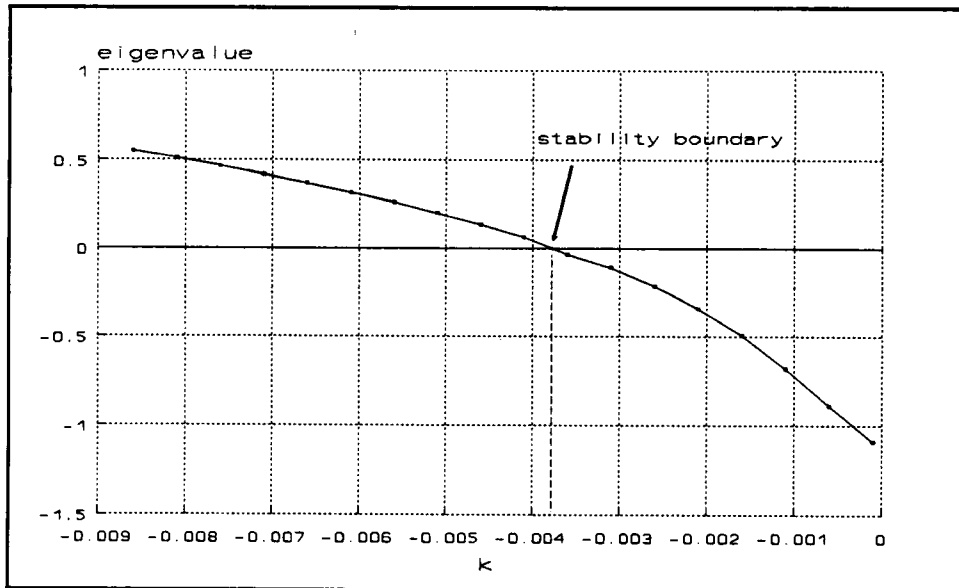


Figure 3.1. Effect of K on the real part of the eigenvalue with the most positive real part.

equations on the verge of instability. Although the equations are linearly stable, comparison with Figure 3.2 shows that the perturbation in normalized neutron population persists for a longer time period (the transient response “rings” for a relatively long time after the initial perturbation).

Figure 3.4 shows the time behavior of the normalized neutron population for slightly unstable conditions, $K = -0.00371$. The initial perturbation grows with time, eventually reaching a steady-state limit cycle. The final amplitude and frequency of the limit cycle are independent of the initial perturbation. Similar behavior is seen in Figures 3.5, 3.6, and 3.7 for $K = -0.0038$, -0.0044 , and -0.0050 . Comparison of these figures shows that as K becomes larger in magnitude (as the system of equations becomes more unstable), the limit-cycle amplitude increases and the limit-cycle frequency decreases.

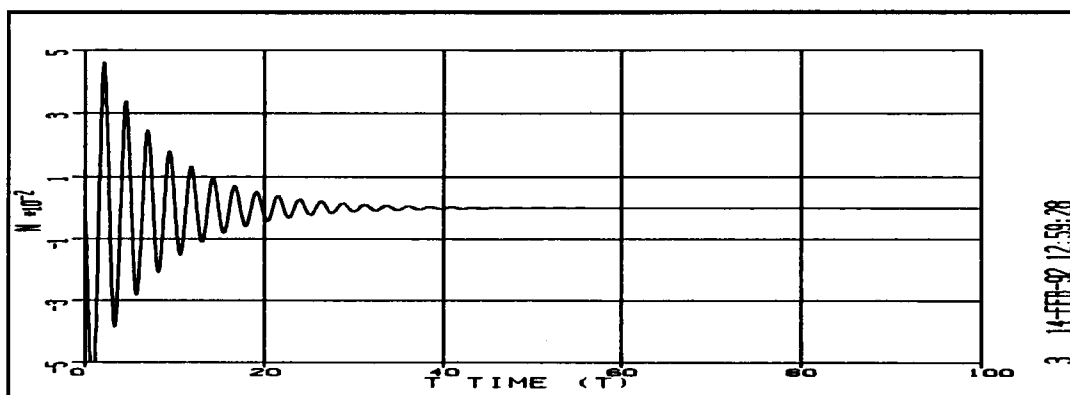


Figure 3.2. Numerical solution for $n(t)$, $K = -0.00300$.

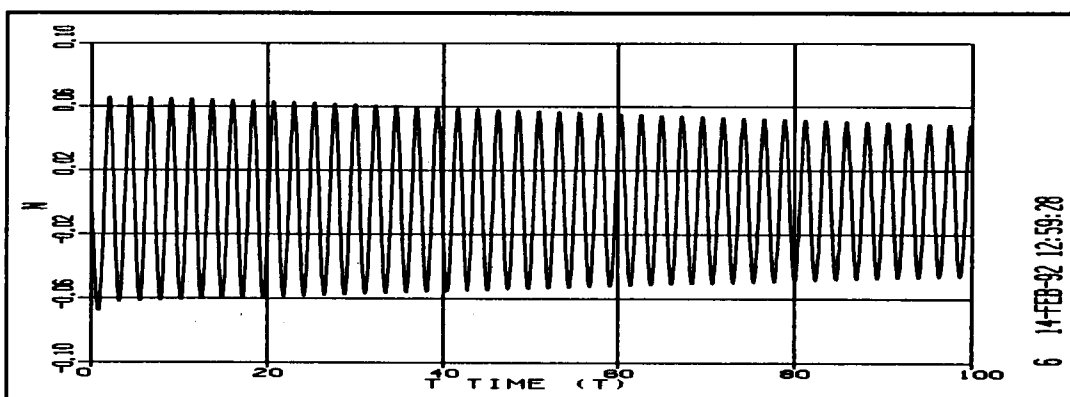


Figure 3.3. Numerical solution for $n(t)$, $K = -0.00369$.

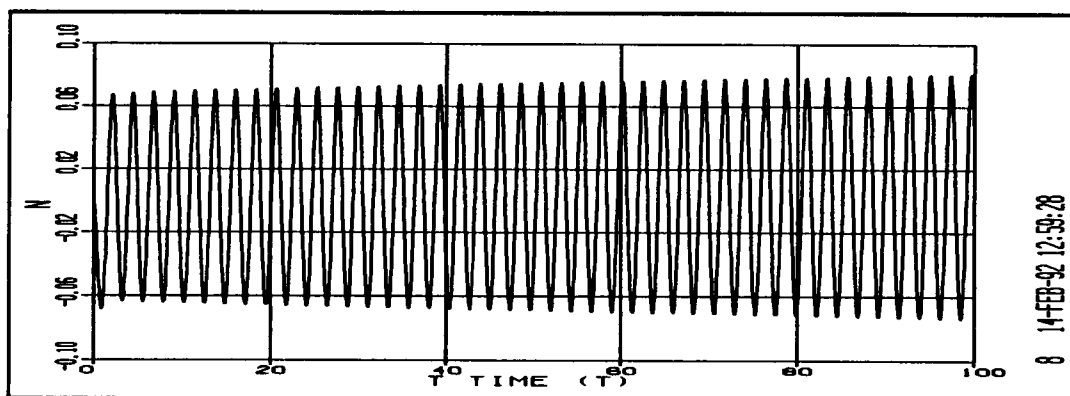


Figure 3.4. Numerical solution for $n(t)$, $K = -0.00372$.

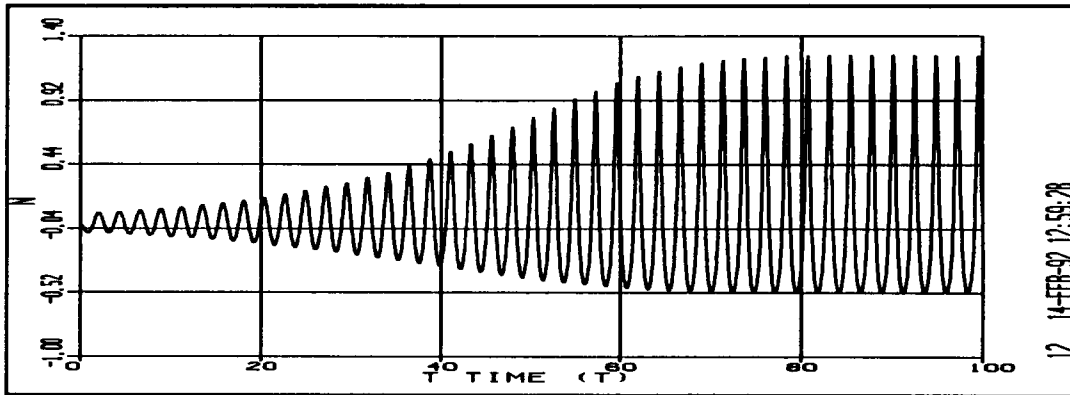


Figure 3.5. Numerical solution for $n(t)$, $K = -0.00400$.

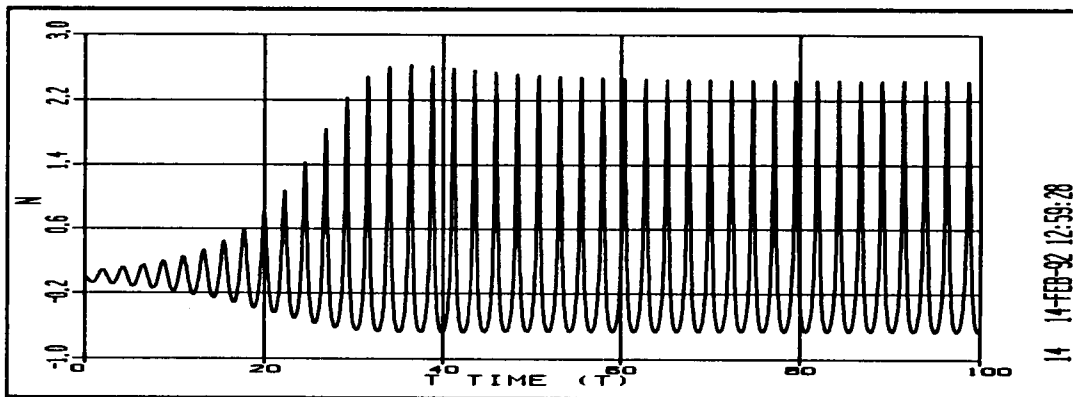


Figure 3.6. Numerical solution for $n(t)$, $K = -0.00440$.

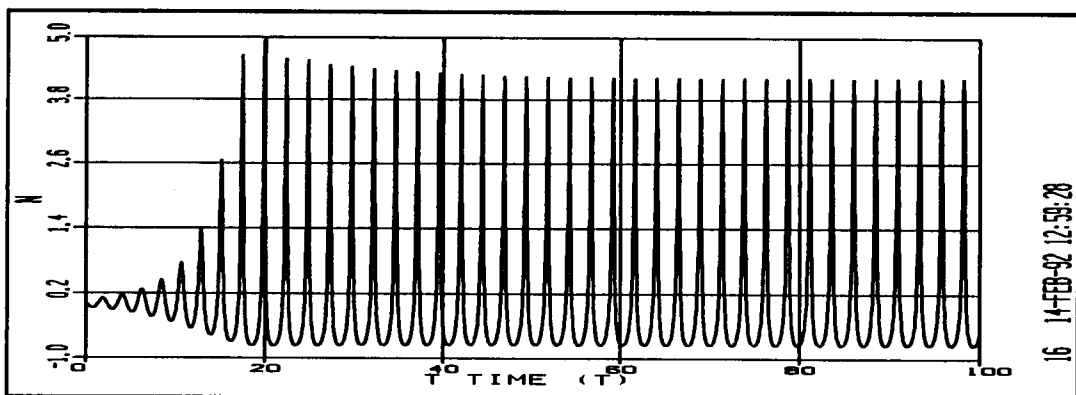


Figure 3.7. Numerical solution for $n(t)$, $K = -0.00500$.

As the magnitude of K is increased further, the limit cycle itself becomes unstable. Figure 3.8 shows that as the normalized neutron population approaches the steady-state limit cycle, a definite ringing of the limit-cycle amplitude about its steady-state value occurs. If the equations are made still more unstable, a supercritical Hopf bifurcation (a period-doubling pitchfork bifurcation) occurs. The value of K at the first bifurcation point is designated K_{b1} . Note in Figure 3.9, with $K = -0.0057 < K_{b1}$, that the period for a complete cycle doubles and that the maximum amplitude continues to increase with K as compared to the steady-state amplitude of Figure 3.8. This behavior can be considered an instability in the original limit cycle.

If K is increased still further, a series of period-doubling bifurcations occur; the K values at which the bifurcations occur are designated K_{b2} , K_{b3} , ... Figure 3.10 shows the time behavior for the normalized neutron population between the second and third bifurcation points. The oscillation period in Figure 3.10 is approximately four times the oscillation period shown in Figure 3.8, and the maximum limit-cycle amplitude has increased. It has been shown that the bifurcation points occur for smaller increments of K as K increases, eventually converging to an accumulation point.⁹⁴ At this point, the system has experienced an infinite number of bifurcations and has become aperiodic (chaotic). The system continues to oscillate within bounds fixed by a strange attractor with the oscillations never exactly repeating. The phenomenological model equations have been shown to follow the universal behavior predicted by Feigenbaum.¹⁰⁷ The limit-cycle amplitudes and the values of K at the bifurcation points follow a predictable pattern, which is characteristic of the classic transition to chaos through a series of period-doubling bifurcations.

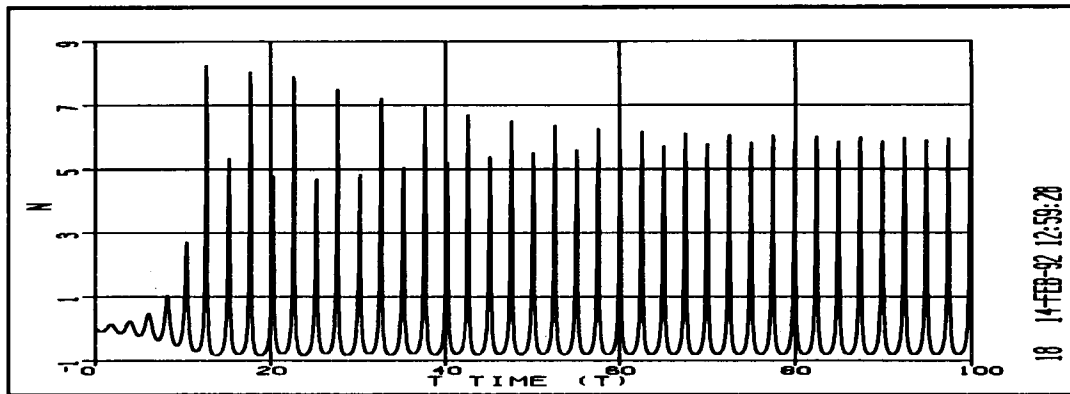


Figure 3.8. Numerical solution for $n(t)$, $K = -0.00560$.

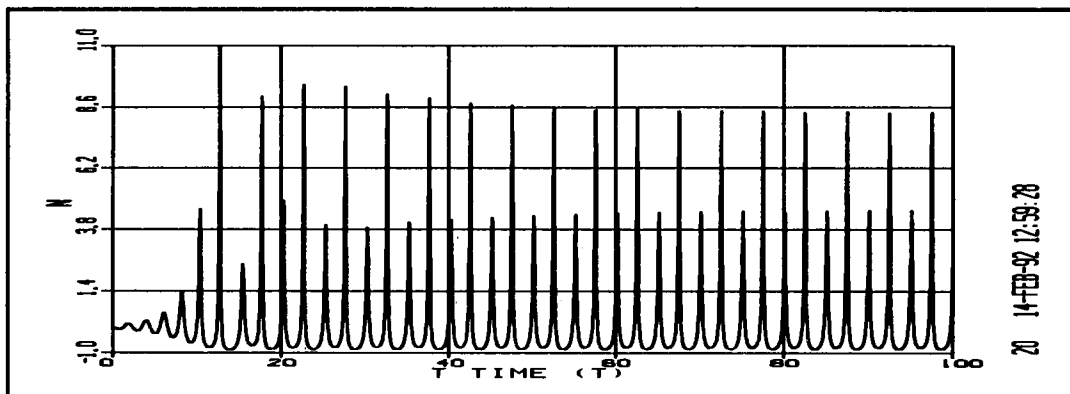


Figure 3.9. Numerical solution for $n(t)$, $K = -0.00580$.

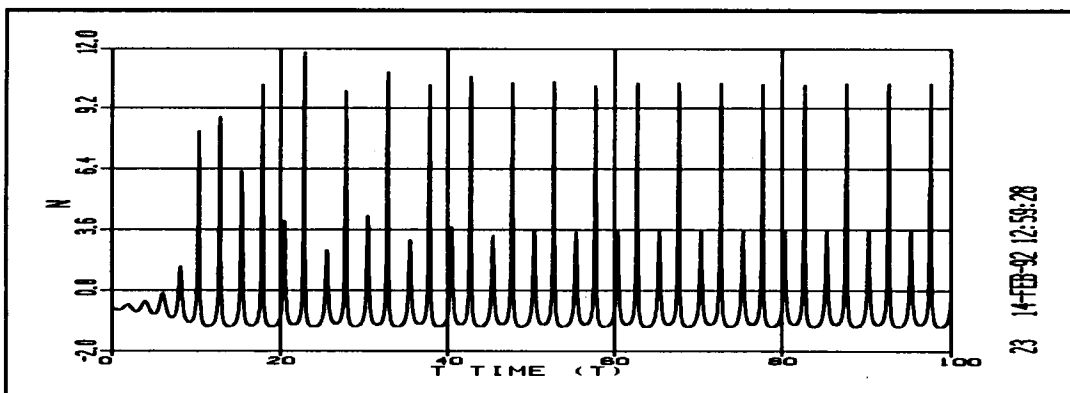


Figure 3.10. Numerical solution for $n(t)$, $K = -0.00600$.

3.2 Development of the Approximate Solution

The character of solutions to the phenomenological model equations have been shown for both the linearly stable and the linearly unstable regions. In the linearly unstable region, any initial perturbation in the state variables will either cause the solution to approach a steady-state limit cycle or will result in chaotic oscillations. The amplitude and period of these oscillations depends on the nature of the phenomenological model equations and is independent of the initial perturbation.

An approximate solution to the phenomenological model equations is developed in this section. This solution is limited to predicting steady-state, limit-cycle oscillations. The development of the approximate solution involves the application of Galerkin's method, which was described in Section 2.3.

3.2.1 Selection of Trial Functions and Development of an Expression for the Equation Residual

Figures 3.4 through 3.8 show that during the steady-state limit-cycle, the solution to the phenomenological equations is an oscillation of fixed frequency and amplitude. Such an oscillation can be described by using a Fourier series. Thus, the normalized neutron population during steady-state, limit-cycle oscillations, $n_{ss}(t)$, can be written as

$$n_{ss}(t) = R + \sum_{n=1}^{\infty} \{P_n \sin(n \omega t) + Q_n \cos(n \omega t)\}, \quad (3.31)$$

where R = the steady-state component of the solution,
 P_n = the amplitude of the sine component at a frequency of $n\omega$,
 Q_n = the amplitude of the cosine component at a frequency of $n\omega$,
 ω = the fundamental oscillation frequency,
 n = an index integer.

Equation (3.31) was selected as the form of the approximate solution for the steady-state, limit-cycle oscillations of the normalized neutron population. The trial functions are the trigonometric functions; the P_n , Q_n , R , and ω are the undetermined constants. This choice of trial functions meets the requirements that these functions satisfy the boundary conditions and be continuous, linearly independent, and complete.

The total reactivity $\rho(t)$ can be eliminated from the phenomenological model equations, reducing by one the number of state variables and equations. Substituting Eq. (3.19) into Eq. (3.10) allows Eq. (3.10) to be rewritten as

$$\Lambda \frac{dn(t)}{dt} = \rho_{\bullet}(t)n(t) + \rho_{\bullet}(t) + \alpha_T n(t)T(t) - \beta n(t) + \lambda \Lambda c(t) . \quad (3.32)$$

This step reduces the phenomenological model to Eqs. (3.11), (3.15), (3.17), and (3.32).

Equations (3.11), (3.15), and (3.17) are linear differential equations whose steady-state solutions can be obtained if the steady-state expression for $n_{ss}(t)$ is known. Using Eq. (3.31) for $n_{ss}(t)$ and solving Eqs. (3.11), (3.15), and (3.17) for the normalized weighted precursor population $c_{ss}(t)$, the weighted fuel temperature $T_{ss}(t)$, and the void reactivity $\rho_{ss\bullet}(t)$ during steady-state oscillations gives

$$c_{ss}(t) = W + \sum_{n=1}^{\infty} \left\{ \sigma_{n1} S_n + \sigma_{n2} C_n \right\} , \quad (3.33)$$

$$T_{ss}(t) = U + \sum_{n=1}^{\infty} \left\{ \tau_{n1} S_n + \tau_{n2} C_n \right\} , \quad (3.34)$$

and

$$\rho_{ss\bullet}(t) = V + \sum_{n=1}^{\infty} \left\{ R_{n1} S_n + R_{n2} C_n \right\} , \quad (3.35)$$

where $S_n = \sin(n\omega t)$ and $C_n = \cos(n\omega t)$ and the values of W , σ_{n1} , σ_{n2} , U , τ_{n1} , τ_{n2} , V , R_{n1} , and R_{n2} are given in Appendix A. The values of these constants depend on R , ω , P_n , and Q_n .

If an infinite number of terms could be included in the expression for $n_{\text{ex}}(t)$, the solution could be represented exactly. Inclusion of only j terms produces an error, called the equation residual R_e , which is a function of only the undetermined constants in the approximate solution. The equation residual, formed by substituting Eqs. (3.31), (3.33), (3.34), and (3.35) into Eq. (3.32), is given by

$$\begin{aligned}
 R_e = & \sum_{n=1}^j \sum_{i=1}^j \left\{ \left[(F_{5n} + \alpha_T F_{3n}) Q_n + (F_{6n} + \alpha_T F_{4n}) P_n \right] S_n \right. \\
 & + \left[(F_{7n} + \alpha_T F_{4n}) Q_n + (F_{8n} - \alpha_T F_{3n}) P_n \right] C_n \left. \right\} (P_i S_i + Q_i C_i) \\
 & + \sum_{n=1}^j \left\{ \left(R F_{5n} + \alpha_T R F_{3n} + F_{5n} + \alpha_T F_{3n} + \lambda \Lambda F_{1n} + \Lambda n \omega \right) Q_n \right. \\
 & + \left[R F_{6n} + \alpha_T R F_{4n} + F_{6n} + \alpha_T F_{4n} - \beta + \lambda \Lambda F_{2n} + \left(\frac{K}{a_4} + \alpha_T \right) \frac{a_1 R}{a_2} \right] P_n \left. \right\} S_n \\
 & + \left\{ \left[R F_{7n} + \alpha_T R F_{4n} + F_{7n} + \alpha_T F_{4n} - \beta + \lambda \Lambda F_{2n} + \left(\frac{K}{a_4} + \alpha_T \right) \frac{a_1 R}{a_2} \right] Q_n \right. \\
 & + \left. \left(R F_{8n} - \alpha_T R F_{3n} + F_{8n} - \alpha_T F_{3n} - \lambda \Lambda F_{1n} - \Lambda n \omega \right) P_n \right\} C_n \left. \right\} + \left(\frac{K}{a_4} + \alpha_T \right) \frac{a_1}{a_2} R(R+1),
 \end{aligned} \tag{3.36}$$

where expressions for F_{1n} through F_{8n} are given in Appendix A. The double summation is due to the product of two time-dependent terms [i.e., due to the nonlinearity in Eq. (3.32)]. Combining terms and simplifying yields

$$\begin{aligned}
R_e = & \sum_{n=1}^j \sum_{i=1}^j \left\{ (G_{1n} Q_n + G_{2n} P_n) P_i S_n S_i + [(G_{3n} Q_n + G_{4n} P_n) P_i \right. \\
& + (G_{1i} Q_i + G_{2i} P_i) Q_n] C_n S_i + (G_{3n} Q_n + G_{4n} P_n) Q_i C_n C_i \Big\} \\
& + \sum_{n=1}^j \left\{ (R G_{1n} + G_{5n}) Q_n + [R (G_{2n} + G_{9n}) + G_{6n}] P_n S_n \right\} \\
& + \left\{ [R (G_{3n} + G_{9n}) + G_{7n}] Q_n + (R G_{4n} + G_{8n}) P_n \right\} C_n \\
& + \left(\frac{K}{a_4} + \alpha_T \right) \frac{a_1}{a_2} R (R + 1) , \tag{3.37}
\end{aligned}$$

where expressions for G_{1n} through G_{9n} are given in Appendix A. An equivalent expression for the equation residual can also be obtained in terms of the constants in Eqs. (3.33), (3.34), and (3.35):

$$\begin{aligned}
R_e = & (V + \alpha_T U)(R + 1) + \lambda \Lambda W - \beta R \\
& + \sum_{n=1}^j \left\{ S_n \left[P_n (V + \alpha_T U - \beta) + (R_{n1} + \alpha_T \tau_{n1})(R + 1) + n \Lambda \omega_k Q + \sum_{m=1}^M \lambda_m \Lambda \sigma_{mn1} \right] \right. \\
& + C_n \left[Q_n (V + \alpha_T U - \beta) + (R_{n2} + \alpha_T \tau_{n2})(R + 1) - n \Lambda \omega P_n + \lambda \Lambda \sigma_{n2} \right] \Big\} \\
& + \sum_{n=1}^j \sum_{l=1}^j \left[(R_{n1} + \alpha_T \tau_{n1})(P_l S_l S_n + Q_l S_n C_l) + (R_{n2} + \alpha_T \tau_{n2})(P_l S_l C_n + Q_l C_l C_n) \right] \tag{3.38}
\end{aligned}$$

Equations (3.37) and (3.38) give the equation residual in terms of parameters that depend on the constants in the approximate solution. Thus, the solution of the set of nonlinear differential equations, Eqs. (3.11), (3.15), (3.17), and (3.32), has effectively been reduced to solving for the undetermined constants in the approximate solution. Galerkin's method is used to develop equations for these constants.

3.2.2 Equations for the Undetermined Constants

Equations for the undetermined constants are developed by requiring the weighted integral of the equation residual, evaluated over a suitable interval or region, to equal zero. This requirement results in values of the constants that force the approximate solution to satisfy the equations in an average sense over the interval. For the case of limit-cycle oscillations, the integration interval is a time interval corresponding to one complete cycle (i.e., from $t = 0$ to $t = 2\pi/\omega$).

For this application, Galerkin's method was selected to form the equations for the undetermined constants. In Galerkin's method, the weighting functions are the same as the trial functions. Thus, the equations for the undetermined constants are obtained from

$$\int_{t=0}^{\frac{2\pi}{\omega}} \mathbf{R}_e S_n dt = 0 \quad (n = 1 \text{ to } j) , \quad (3.38)$$

$$\int_{t=0}^{\frac{2\pi}{\omega}} \mathbf{R}_e C_n dt = 0 \quad (n = 1 \text{ to } j) , \quad (3.39)$$

and

$$\int_{t=0}^{\frac{2\pi}{\omega}} \mathbf{R}_e R dt = \int_{t=0}^{\frac{2\pi}{\omega}} \mathbf{R}_e = 0 . \quad (3.40)$$

Substituting the expression for $\mathbf{R}_e$ from Eq. (3.37) into Eq. (3.39) and neglecting integrals that are identically zero gives

$$\begin{aligned}
\int_0^{\frac{2\pi}{\omega}} \mathbf{R}_e S_k dt = \sum_n^j \sum_i^j \left\{ \left[(G_{3n} Q_n + G_{4n} P_n) P_i + (G_{1i} Q_i + G_{2i} P_i) Q_n \right] \int_0^{\frac{2\pi}{\omega}} S_i S_k C_n dt \right\} \\
+ \left\{ (R G_{1k} + G_{5k}) Q_k + \left[R (G_{2k} + G_{9k}) + G_{6k} \right] P_k \right\} \left(\frac{\pi}{\omega} \right) = 0 . \quad (3.42)
\end{aligned}$$

Similarly, substituting the expression for $\mathbf{R}_e$ from Eq. (3.37) into Eq. (3.40) and neglecting integrals that are identically zero gives

$$\begin{aligned}
\int_0^{\frac{2\pi}{\omega}} \mathbf{R}_e C_n dt = \sum_k^j \sum_i^j \left[(G_{1k} Q_k + G_{2k} P_k) P_i \int_0^{\frac{2\pi}{\omega}} S_i S_k C_n dt + (G_{3k} Q_k + G_{4k} P_k) Q_i \int_0^{\frac{2\pi}{\omega}} C_i C_k C_n dt \right] \\
+ \left\{ \left[R (G_{3n} + G_{9n}) + G_{7n} \right] Q_n + (R G_{4n} + G_{8n}) P_n \right\} \left(\frac{\pi}{\omega} \right) = 0 . \quad (3.43)
\end{aligned}$$

Finally, substituting the expression for $\mathbf{R}_e$ from Eq. (3.37) into Eq. (3.41) and neglecting integrals that are identically zero gives

$$\begin{aligned}
\int_0^{\frac{2\pi}{\omega}} \mathbf{R}_e R dt = \sum_{n=1}^j \left[(G_{1n} Q_n + G_{2n} P_n) P_n + (G_{3n} Q_n + G_{4n} P_n) Q_n \right] \\
+ 2 \left(\frac{K}{a_4} + \alpha_T \right) \frac{a_1 R}{a_2} (R + 1) = 0 . \quad (3.44)
\end{aligned}$$

Appendix B contains a table of definite integrals of products of trigonometric functions, evaluated over one cycle, that are used to evaluate Eqs. (3.42) and (3.43). Equations (3.42), (3.43), and (3.44) are used to determine the values of the undetermined constants that force the weighted integral of the equation residual over one cycle to equal zero.

Indexing k from 1 to j yields j equations from Eq. (3.42), indexing n from 1 to j yields j equations from Eq. (3.43), and one equation is obtained from Eq. (3.44), giving a total of $2j + 1$ equations. Because the absolute phase of the steady-state oscillations described by the approximate solution is arbitrary, the value of Q_j can be assigned a value of zero without affecting the magnitude or the relative phases of the harmonics in the approximate solution. This assignment leaves $2j + 1$ unknowns, which can be solved for by using the $2j + 1$ equations obtained from Eqs. (3.42), (3.43), and (3.44).

Equation (3.44) forces the average value of the equation residual over one cycle to be zero. At any point during the cycle, however, the residual may be relatively large, especially so for approximate solutions containing few harmonics. Equations formed from Eqs. (3.42) and (3.43) are used to calculate values of P_n and Q_n , which minimize the magnitude of the residual. Thus, by satisfying Eqs. (3.42), (3.43), and (3.44), an approximate solution is obtained for which the average value of the residual over one cycle is zero and for which the magnitude of the residual during the cycle is minimized.

After the values of the constants are calculated, Eq. (3.31) can be used to approximate the steady-state time behavior of the normalized neutron population. The steady-state time behavior of the precursor population, the fuel temperature, and the void reactivity can be calculated by using the undetermined constants; Eqs. (3.33), (3.34), and (3.35); and the expressions given in Appendix A.

3.2.3 Solution of the Equations for the Undetermined Constants

A FORTRAN computer code was written to form and solve the equations for the undetermined constants. The code input consists of the phenomenological model parameters, the number of harmonics in the approximation, and initial estimates of the

undetermined constants. The code output consists of the calculated values of the constants.

The code forms and solves the system of equations consisting of Eq. (3.44) and those equations generated by indexing k from 1 to j in Eq. (3.42) and indexing n from 1 to j in Eq. (3.43). Except for very low order approximations, this system of $2j + 1$ equations must be solved numerically. This numerical solution is performed by the Numerical Algorithms Group (NAG) routine C05NBF.¹³⁹ This routine finds a zero of a system on N nonlinear functions of N variables. The routine requires the user to supply an initial estimate for the values of the N unknowns. This requirement determines the strategy used to calculate high-order approximations.

Normally, routine C05NBF converges rapidly to a low-order solution. Thus, the initial estimates for a low-order approximation need not be very accurate. This is not true for high-order approximations (i.e., approximations containing four or more harmonics). Experience with the code has shown it to be nearly impossible to select the initial estimates for a high-order solution without some guidance; NAG routine C05NBF is unlikely to converge to a solution when the initial estimates are selected on the basis of only user judgement.

Two strategies have been successfully employed to calculate high-order approximate solutions. Each strategy uses previously calculated results as the initial estimate for the current calculation. The first strategy involves holding all equation parameters constant while performing a series of calculations in which one additional harmonic is added to the approximate solution in each calculation. The solution to the previous calculation is used as the initial estimate for the current calculation.

In the second strategy, the high-order approximate solution is calculated directly by using a previously calculated solution of equal order for the initial estimates. If a high-order approximate solution has been calculated for some set of equation parameters, this approximation can be used as the initial estimate for the calculation of an equally high-order approximate solution for a slightly different set of equation parameters. These two strategies are complementary and have been used effectively to calculate high-order approximate solutions.

3.3 The High-Order Approximate Solution

The approximate solution to the phenomenological model equations for steady-state, limit-cycle oscillations was developed in Section 3.2. A solution for the normalized neutron population in the form of Eq. 3.31 was obtained with the constants P_n , Q_n ($n = 1, 2, 3, \dots, j$), R , and ω given by Eqs. (3.42), (3.43), and (3.44). In this section, the characteristics of high-order approximate solutions are examined. The results were calculated by using the computer code described in Subsection 3.2.3.

3.3.1 The Nine-Harmonic Approximate Solution

Nine-harmonic approximate solutions were calculated for the phenomenological model equations by using the parameter values shown in Table 3.1 with K ranging from -0.00375 to -0.0052 . The results are tabulated in Table 3.2, and the normalized neutron population calculated for $K = -0.0040$, -0.0044 , and -0.0050 are plotted in Figures 3.11, 3.12, and 3.13. As the value of K is decreased (as the system of equations gets further into the region of linear instability), the magnitude of each harmonic in the approximate solution increases. This result is consistent with the numerical solutions

Table 3.2. Approximate solution constants for two- to eight-harmonic solutions

Constant	Number of harmonics						
	2	3	4	5	6	7	8
ω (Hz)	0.4160	0.4250	0.4192	0.4182	0.4172	0.4171	0.4170
R	0.0416	0.0425	0.0505	0.0546	0.0561	0.0562	0.0564
P_1	0.5785	0.9560	1.0400	1.0760	1.0905	1.0913	1.0927
Q_1	0.7423	-0.1630	-0.0505	-0.1257	-0.1043	-0.1144	-0.1142
P_2	0.3620	-0.1450	-0.0913	-0.1749	-0.1606	-0.1714	-0.1721
Q_2		-0.4207	-0.5120	-0.5508	-0.5686	-0.5684	-0.5698
P_3		-0.1560	-0.2098	-0.2680	-0.2841	-0.2842	-0.2853
Q_3			0.0734	0.1430	0.1469	0.1555	0.1576
P_4			0.0375	0.0810	0.1089	0.1151	0.1193
Q_4				0.1211	0.1301	0.1328	0.1333
P_5				0.0464	0.0475	0.0563	0.0558
Q_5					-0.0679	-0.0728	-0.0805
P_6					-0.0318	-0.0355	-0.0496
Q_6						-0.0213	-0.0191
P_7						-0.0072	-0.0040
Q_7							0.0274
P_8							0.0121

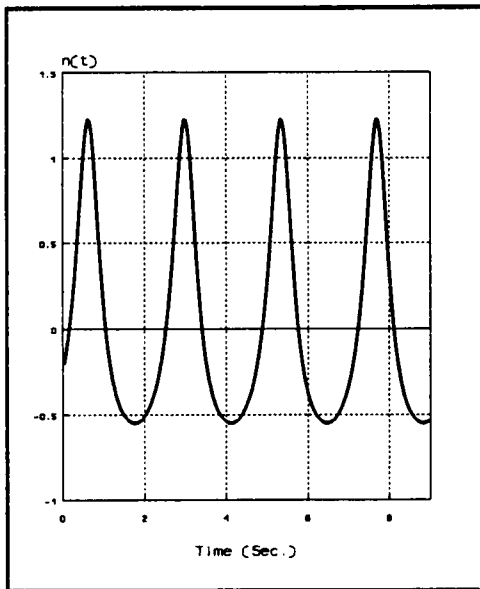


Figure 3.11. Steady-state limit cycle for $K = -0.0040$.

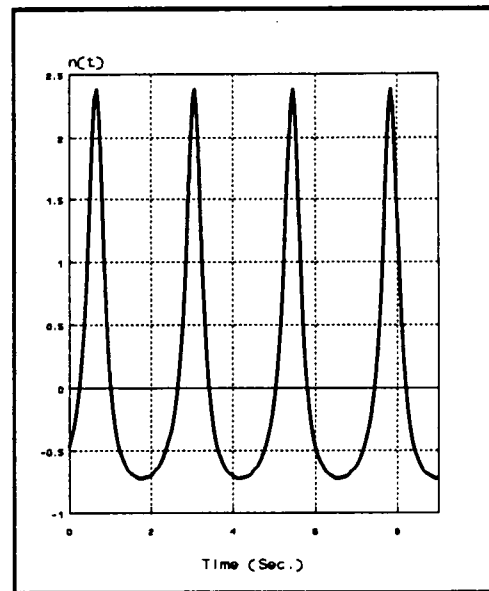


Figure 3.12. Steady-state limit cycle for $K = -0.0044$.

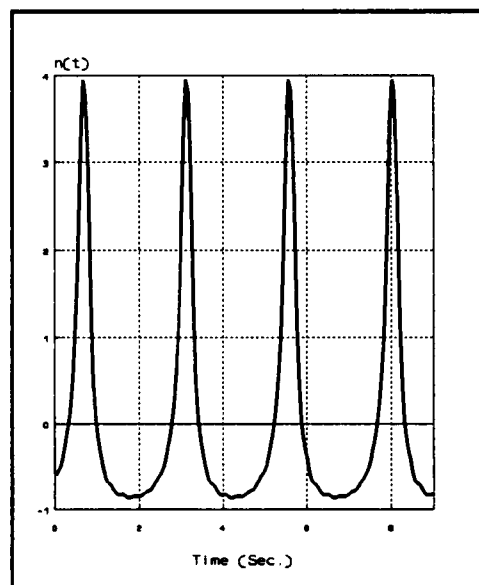


Figure 3.13. Steady-state limit cycle for $K = -0.0050$.

shown in Figures 3.2 through 3.8; in these figures, both the limit-cycle amplitude and period increase as the system of equations becomes more unstable. An important consequence of this anticipated result is that more harmonics are needed to accurately approximate the limit cycle as the system of equations becomes more unstable. Near the stability boundary, however, only one harmonic is needed to approximate the solution. This finding is used in the next section, which describes how the low-order approximate solution can be used to examine the stability boundary.

Comparing Figure 3.3 to Figure 3.11, Figure 3.4 to Figure 3.12, and Figure 3.5 to Figure 3.13 shows overall excellent agreement between the numerical results and the nine-harmonic approximate solution. The greatest disagreement appears for $K = -0.0050$. This disagreement is not surprising, because the number of harmonics needed to accurately represent the solution increases as the system becomes more unstable. Thus, for a comparison of approximate solutions with the same number of harmonics, the poorest agreement between the approximate and exact solutions will occur for the most unstable case. In the comparison between Figures 3.5 and 3.13, the inaccuracy in the approximate solution appears as a failure of the approximate solution to obtain the same peak amplitude as the numerical solution and in the appearance of undulations in the trough between peaks.

A clearer picture of the effect of the degree of stability on the magnitude of the harmonics is shown in Figure 3.14. The magnitudes of the first eight harmonics of the nine-harmonic approximate solution are shown as a function of the value of K . As K decreases, all the harmonics increase in magnitude, with the magnitude of the higher harmonics increasing at a greater rate than that of the lower harmonics. Thus, for

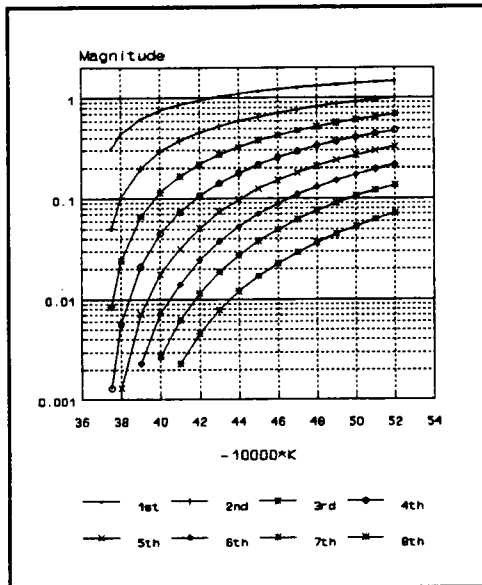


Figure 3.14. Growth of harmonics in the approximate solution with K .

more negative values of K , higher harmonics form a significant portion of the solution and cannot be neglected.

The effects of the degree of nonlinearity on the steady-state component and the frequency of the normalized neutron population and on the limit-cycle amplitude are shown in Figures 3.15 and 3.16. Figure 3.15 shows that the steady-state component, which

is proportional to an increase in the average power, increases with system instability and the oscillation frequency decreases. This result agrees with that of March-Leuba,¹¹¹ who has shown that the average power must increase during the limit cycle to provide a negative reactivity bias to counter the positive reactivity introduced by the oscillation. Figure 3.16 shows the rapid increase in limit-cycle amplitude with decreasing values of K (i.e., as the nonlinear effects become more important). Likely, the relationship between limit-cycle amplitude and the value of K is more linear than shown in Figure 3.16. The failure of the nine-harmonic approximate solution to accurately represent the limit cycle as the system of equations becomes more unstable results in increasing underestimation of the limit-cycle amplitude as the value of K is decreased.

The results from the nine-harmonic approximate solution have shown that for the phenomenological model,

1. more harmonics are needed to adequately represent the solution for the normalized neutron population as the system becomes more unstable;

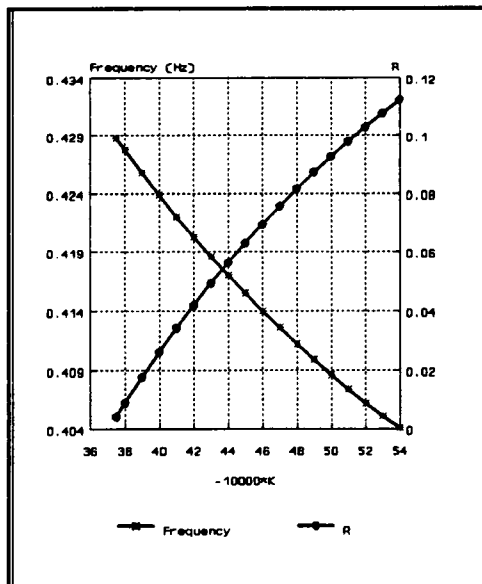


Figure 3.15. Effect of K on the frequency and the steady-state component of $n(t)$ during limit-cycle oscillations.

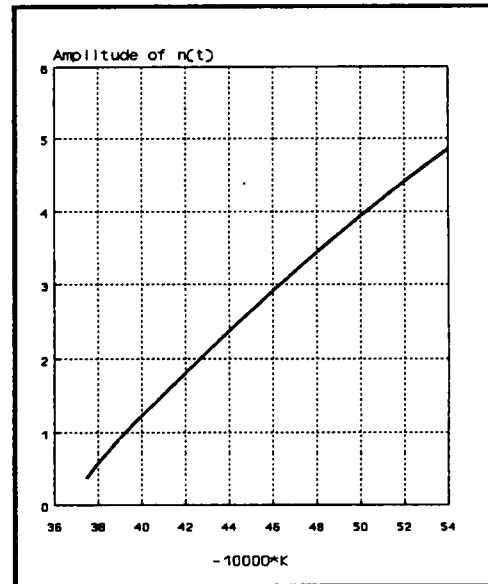


Figure 3.16. Effect of K on the steady-state limit-cycle amplitude.

2. the limit-cycle oscillation frequency decreases and the amplitude increases nearly linearly with system instability;
3. nonlinear effects, shown by the rapid growth of the higher harmonics with decreasing values of K , rapidly become important as the system of equations becomes more unstable;
4. the steady-state component of the limit cycle increases as the system becomes more unstable, further confirming the work of March-Leuba,¹¹¹ showing that an average power increase must accompany limit-cycle oscillations;
5. a single harmonic adequately represents the solution for the normalized neutron density at and near the stability boundary; and
6. the limit-cycle amplitude approaches zero as the stability boundary is approached.

3.3.2 Convergence of the Approximate Solution

The effect of adding terms to the approximate solution is to increase the accuracy of the approximation. Figure 3.17 shows approximate solutions for normalized neutron population containing 2, 4, 6, and 8 harmonics respectively. The approximate solutions were calculated by using the phenomenological model equation parameter values listed in Table 3.1 with $K = -0.0044$. Comparison of the approximate solutions to the numerical solution shown in Figure 3.6 (which substitutes here for the exact solution) shows the approximate solution converging to the exact solution as the number of harmonics is increased.

The integral of the absolute value of the equation residual is a quantitative measure of the accuracy of the approximate solution. The effect of the number of approximate solution harmonics on this integral is shown in Figure 3.18. Again, the phenomenological model equation parameter values shown in Table 3.1 with $K = -0.0044$ were used in these calculations. The integral of the absolute value of the equation residual approaches zero as the number of harmonics increases, indicating that the approximate solution converges to the exact solution.

3.3.3 The Approximate Solution Near the First Bifurcation Point

As the system of equations becomes more unstable, more harmonics are needed to accurately represent the solution. The numerical solution shown in Figures 3.9 and 3.10 show that if the system is made sufficiently unstable, a bifurcation will occur. Before reaching the bifurcation point, the limit cycle is stable; after passing the bifurcation point, the original limit cycle is unstable.

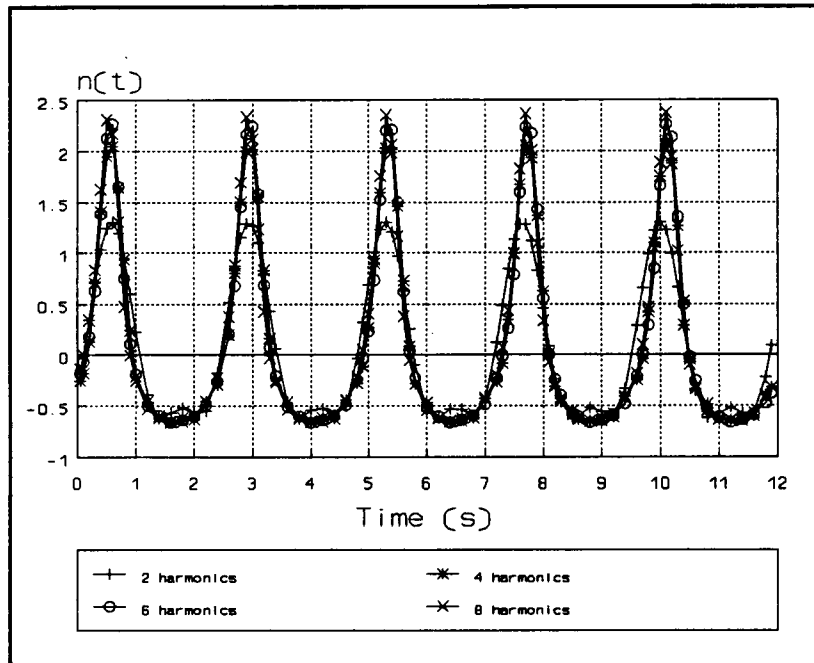


Figure 3.17. Comparison of 2-, 4-, 6-, and 8-harmonic approximate solutions.

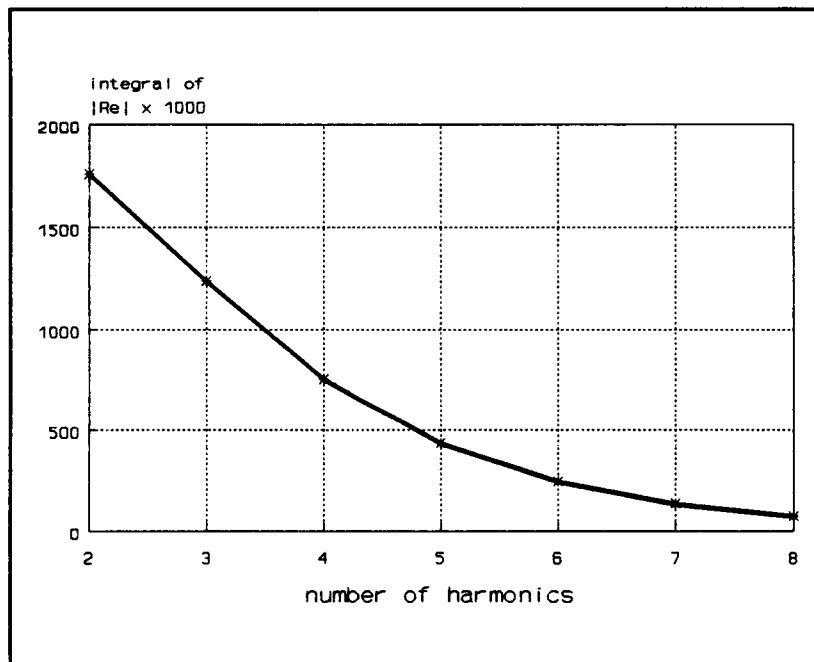


Figure 3.18. Effect of the number of harmonics in the approximate solution on the integral over one cycle of the absolute value of the equation residual.

Figures 3.19 and 3.20 show ten-harmonic approximate solutions calculated for $K = -0.0056$ and $K = -0.0060$. Comparing Figures 3.8 and 3.19 shows that for $K = -0.0056$, the approximate solution gives a reasonably good approximation of the numerical solution. Presumably, the accuracy of the approximation could be improved by including more harmonics. Comparing Figures 3.10 and 3.20 shows that the approximate solution does not represent the bifurcated solution.

Although the approximate solution shown in Figure 3.20 satisfies all the requirements imposed by Galerkin's method for an approximation solution, it fails to accurately represent the corresponding numerical solution. One explanation for this discrepancy is that after passing the bifurcation point, two mathematical solutions exist—one stable and the other unstable. The stable solution corresponds to the numerical solution shown in Figure 3.10, and the unstable solution corresponds to the approximate solution shown in Figure 3.20. Recalling the two strategies described in Subsection 3.2.3 for calculating high-order approximate solutions and assuming the existence of an unstable solution of the unbifurcated form suggest an explanation concerning why the unbifurcated form of the solution is calculated after passing the bifurcation point.

The initial estimates used in the calculation of the approximate solution describe an unbifurcated form of the solution. If this form of the solution satisfies the requirements imposed by Galerkin's method, the solution will converge to this solution form. The solution stability has no effect on the steady-state approximate solution calculated by using Galerkin's method. If initial estimates describing the bifurcated form of the solution were available, Galerkin's method should calculate the bifurcated form of the solution. Thus, the form of the solution to which Galerkin's method

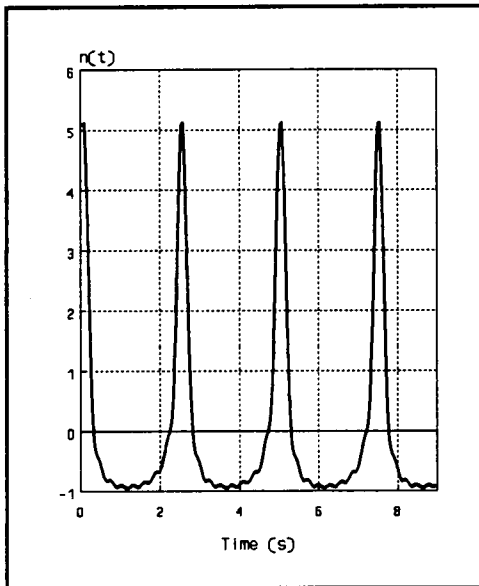


Figure 3.19. Steady-state limit cycle for $K = -0.0056$.

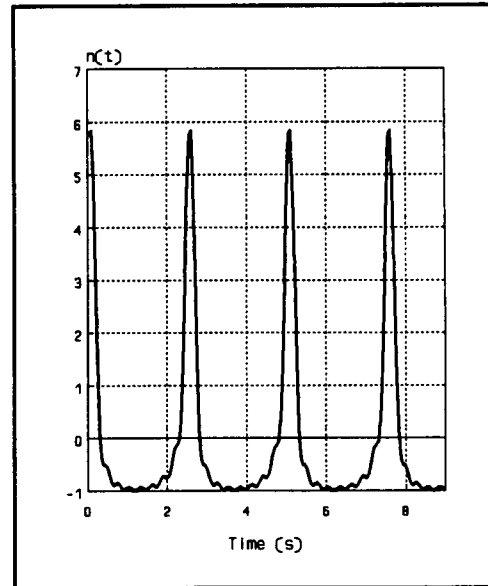


Figure 3.20. Steady-state limit cycle for $K = -0.0060$.

converges depends on the form of the initial estimate. Approximate solution stability and calculation of the bifurcated form of the solution are discussed in Chapter 5.

3.4 The Low-Order Approximate Solution

Figure 3.14 shows that an approximate solution containing a single harmonic will be accurate at or very near the stability boundary. This simple form of the solution allows the equations for the undetermined constants to be solved in closed form. The expressions for the undetermined constants can be used to examine the sensitivity of the stability boundary to changes in the parameters in the phenomenological model equations.

3.4.1 The One-Harmonic Solution

The form of the one-harmonic solution for the normalized neutron population is

$$n_{ss}(t) = R + P_1 \sin(\omega t) . \quad (3.45)$$

Substituting Eq. (3.45) into Eq. (3.38), applying Galerkin's method, and using the integrals listed in Appendix B results in the following equations for the constants R , P_1 , and ω :

$$0 = 2[(V + \alpha_T U)(R + 1) + \lambda \Lambda W - \beta R] + (R_{11} + \alpha_T \tau_{11})P_1 , \quad (3.46)$$

$$0 = P_1(V + \alpha_T U - \beta) + (R_{11} + \alpha_T \tau_{11})(R + 1) + \lambda \Lambda \sigma_{11} , \quad (3.47)$$

and

$$0 = (R_{12} + \alpha_T \tau_{12})(R + 1) - \omega \Lambda P_1 + \lambda \Lambda \sigma_{12} . \quad (3.48)$$

Equation (3.48) can be used to determine the oscillation frequency near the stability boundary. Examination of the equations for R_{11} , R_{12} , τ_{11} , and τ_{12} in Appendix A shows that, for $n_{ss}(t)$ given by Eq. (3.45),

$$R_{11} + \alpha_T \tau_{11} = A_1(\omega)P_1 = \frac{\{a_1 K[a_2 a_4 - (a_2 + a_3)\omega^2] + \alpha_T a_1 a_2[(a_2 - \omega^2)^2 + (a_3 \omega)^2]\}P_1}{(a_2^2 + \omega^2)[(a_4 - \omega^2)^2 + (a_3 \omega)^2]} \quad (3.49)$$

and

$$R_{12} + \alpha_T \tau_{12} = A_2(\omega_k)P_1 = \frac{-\{a_1 K(a_2 a_3 + a_4 - \omega^2)\omega + a_1 \omega[(a_2 - \omega^2)^2 + (a_3 \omega)^2]\}P_1}{(a_2^2 + \omega^2)[(a_4 - \omega^2)^2 + (a_3 \omega)^2]} . \quad (3.50)$$

Substituting Eqs. (3.50), using the expressions in Appendix A, and dividing by P_1 , yields

$$\left[\frac{\Lambda + \lambda \beta}{(\lambda^2 + \omega^2)} \right] \omega = \frac{-\{a_1 K(a_2 a_3 + a_4 - \omega^2)\omega + a_1 \omega[(a_2 - \omega^2)^2 + (a_3 \omega)^2]\}(R + 1)}{(a_2^2 + \omega^2)[(a_4 - \omega^2)^2 + (a_3 \omega)^2]} . \quad (3.51)$$

An equation for R is obtained from Eq. (3.47). Substituting Eqs. (3.49) into Eq. (3.47), using the expressions in Appendix A, and dividing the resulting equation by P_1 yields

$$R = \frac{\left[\beta - A_1(\omega_k) - \frac{\lambda^2 \beta}{(\lambda^2 + \omega^2)} \right] a_2 a_4}{K a_1 + \alpha_T a_1 a_4 + a_2 a_4 A_1(\omega)}, \quad (3.52)$$

where $A_1(\omega)$ is defined in Eq. (3.49). Equations (3.51) and (3.52) can be solved simultaneously for ω and R near the stability boundary.

An equation for P_1 as a function of R and ω is obtained from Eq. (3.46). Substituting Eqs. (3.49) into Eq. (3.46), using the expressions in Appendix A, and solving for P_1 yields

$$P_1 = \sqrt{2 \frac{-(K a_1 + \alpha_T a_1 a_4) R (1 + R)}{a_2 a_4 A_1(\omega)}}, \quad (3.53)$$

where $A_1(\omega)$ is defined in Eq. (3.49). Thus, if R and ω are known, Eq. (3.53) can be solved for the oscillation amplitude near the stability boundary.

3.4.2 Stability Condition

An equation for the oscillation frequency at the stability boundary can be obtained from Eq. (3.51). As the stability boundary is approached, R approaches zero. Thus, at the stability boundary,

$$\left[\Lambda + \frac{\lambda \beta}{(\lambda^2 + \omega_0^2)} \right] \omega_0 = \frac{-\{a_1 K (a_2 a_3 + a_4 - \omega_0^2) \omega_0 + a_1 \omega_0 [(a_2 - \omega_0^2)^2 + (a_3 \omega_0)^2]\}}{(a_2^2 + \omega_0^2) [(a_4 - \omega_0^2)^2 + (a_3 \omega_0)^2]}. \quad (3.54)$$

The only unknown in Eq. (3.54) is ω_0 , the oscillation frequency at the stability boundary. Because R equals zero at the stability boundary, the stability condition for the phenomenological model is obtained by setting the numerator of Eq. (3.52) equal to zero:

$$\beta - \frac{\lambda^2 \beta}{(\lambda^2 + \omega_0^2)} = \frac{a_1 K [a_2 a_4 - (a_2 + a_3) \omega_0^2] + \alpha_T a_1 a_2 [(a_2 - \omega_0^2)^2 + (a_3 \omega_0)^2]}{(a_2^2 + \omega_0^2) [(a_4 - \omega_0^2)^2 + (a_3 \omega_0)^2]} \quad (3.55)$$

Equation (3.55) can be solved for K_0 , the value of K at the stability boundary:

$$K_0 = \frac{\left\{ (a_2^2 + \omega_0^2) \left[\beta - \frac{\lambda^2 \beta}{(\lambda^2 + \omega_0^2)} \right] - \alpha_T a_1 a_2 \left[(a_4 - \omega_0^2)^2 + (a_3 \omega_0)^2 \right] \right\}}{a_1 [a_2 a_4 - (a_2 + a_3) \omega_0^2]} \quad (3.56)$$

If $K > K_0$, the phenomenological model equations are stable; if $K < K_0$, the phenomenological model equations are unstable.

Equation (3.56) can be used to determine the influence of perturbations in the phenomenological model parameters on the value of K_0 and, therefore, on the stability boundary. If a positive perturbation causes K_0 to decrease, the perturbation is stabilizing. Conversely, if a positive perturbation causes K_0 to increase, the perturbation is destabilizing. A simple way to examine the effect of various perturbations is by evaluating partial derivatives of Eq. (3.56). The Macsyma computer code was used to form the expressions for the partial derivatives of Eq. (3.56).¹⁴⁰ The phenomenological model parameters listed in Table 3.1, along with an approximate oscillation frequency of 0.4 Hz, were used to evaluate the expressions for the partial derivatives. The results of these calculations are listed in Table 3.3.

The first line in Table 3.3 shows that increasing a_2 , a_3 , a_4 , or β is stabilizing and that increasing a_1 or α_T is destabilizing. Recalling that a_3 and a_4 have been shown to be

proportional to coolant mass flux and inversely proportional to void fraction or, equivalently, inversely proportional to the channel transit time, the results in Table 3.3 indicate that increasing the coolant flow or decreasing the average void fraction is stabilizing. Similarly, because a_1 is inversely proportional to fuel density and heat capacity (i.e., to fuel thermal inertia), the results in Table 3.3 indicate that reducing either quantity is destabilizing.

The magnitude of the partial derivative shown in Table 3.3 give a somewhat misleading measure of the relative importance of the phenomenological model parameter values on the value of K_0 because of the wide range in the magnitude of the phenomenological model parameters. A more representative measure of the relative importance of each equation parameter is given by the product of the partial derivative and the absolute value of the parameter used to form the partial derivative. This “weighted partial derivative” takes the relative magnitude of each parameter into account; the weighted partial derivative indicates the effect on K_0 of an equal fractional change in the equation parameters. The weighted partial derivatives are shown in the second line in Table 3.3. The weighted partial derivatives show that changing a_2 , a_3 , a_4 , and β have approximately equal effects on K_0 ; changes in a_1 and α_T have much less effect on K_0 .

Table 3.3. Values of some partial derivatives of Eq.(3.56) evaluated by using the phenomenological model parameters listed in Table 3.1

	$\partial K_0 / \partial a_1$	$\partial K_0 / \partial a_2$	$\partial K_0 / \partial a_3$	$\partial K_0 / \partial a_4$	$\partial K_0 / \partial \alpha_T$	$\partial K_0 / \partial \beta$
Partial derivative	1.26 $\times 10^{-4}$	-4.08 $\times 10^{-4}$	-1.39 $\times 10^{-3}$	-1.57 $\times 10^{-4}$	5.26 $\times 10^{-1}$	-5.71 $\times 10^{-1}$
Weighted partial derivative	3.20 $\times 10^{-3}$	-9.38 $\times 10^{-5}$	-3.13 $\times 10^{-3}$	-1.07 $\times 10^{-3}$	1.33 $\times 10^{-5}$	-3.19 $\times 10^{-3}$

3.5 Chapter Summary

In this chapter, the application of Galerkin's method to the phenomenological model equations was described. The normalized phenomenological model equations were shown to have two equilibrium points, one corresponding to the normal operating point and the other corresponding to shutdown. Examining the linear stability about the normal operating point by using the eigenvalues of the linearized system of equations showed that the system became linearly unstable as the value of K was decreased. Numerically calculated results showed that in the linearly unstable region, the solution to the phenomenological model equations describes a limit cycle whose amplitude and period increase as the system becomes more unstable. A cascade of period-doubling bifurcations occur if the system stability is decreased further, ultimately resulting in chaotic oscillations.

Galerkin's method was applied to the phenomenological model equations, producing an approximate solution in the form of a Fourier series. It was shown that the number of harmonics required to accurately approximate the solution increases as the system of equations becomes more unstable. An important consequence is that near the stability boundary, a single harmonic accurately represents the solution. The limit-cycle amplitude was shown to increase nearly linearly with the degree of instability and the frequency was shown to decrease. The limit-cycle amplitude approaches zero as the stability boundary is approached. The accuracy of the approximate solution was shown to improve as more terms were included in the approximation.

The one-harmonic approximate solution was used to show that the parameters in the phenomenological model equations influence the system stability in the following way: Increasing a_1 , which corresponds to decreasing the fuel thermal inertia, is a destabilizing effect. Increasing a_3 , or a_4 , which corresponds to decreasing the average void fraction or

decreasing the channel transit time, is a stabilizing effect. Increasing the magnitude of either α_T or β is stabilizing. The sensitivity of the stability boundary to changes in α_T is over an order of magnitude less than the sensitivity of the stability boundary to changes in a_1 , a_3 , a_4 , and β .

4. APPROXIMATE SOLUTION OF THE EQUATIONS OBTAINED FROM A BWR DYNAMIC MODEL CONSISTING OF THE POINT KINETICS EQUATIONS AND THE LAPUR-CALCULATED FEEDBACK TRANSFER FUNCTION

The application of Galerkin's method to obtain an approximation of overall BWR dynamic behavior during limit-cycle oscillations by using a BWR dynamic model consisting of the point kinetics equations and the LAPUR-calculated feedback transfer function is described in this chapter. Unlike the phenomenological model equations described in Chapter 3, this dynamic model can be used to obtain quantitative information about BWR dynamic behavior. The model enhancement responsible for the improved accuracy is the relatively detailed thermal hydraulics description used in the LAPUR code. Because the thermal hydraulics model is linearized about the steady-state conditions, the dynamic model is accurate in only the region of linear stability or for small-amplitude limit-cycle oscillations.

Investigation of BWR dynamics has been performed primarily by using either linear frequency-domain or nonlinear time-domain computer codes. Frequency-domain codes such as the LAPUR code are limited to predicting BWR dynamics in the linearly stable region because these codes typically use versions of the equations describing reactor dynamics that have been linearized about the steady-state conditions.⁶¹ Thus, these codes cannot estimate reactor response during limit-cycle oscillations and, in general, lack the detailed descriptions of reactor conditions found in many nonlinear time-domain codes. These codes can predict the onset of instability and quantify the degree of stability for stable systems. Frequency-domain codes are useful for examining the effect of reactor parameters such as flow, power, and power shape on stability and for predicting the

stability boundary.⁹⁴ Frequency-domain codes are usually less computation intensive than nonlinear time-domain codes.

Nonlinear time-domain codes are typically large codes that model BWR behavior by using a multinodal representation in both space and time. These codes predict BWR dynamic behavior in the linear and nonlinear regions and can calculate reactor transient response. The main disadvantage of the nonlinear, multinodal, time-domain codes is the large amount of computer resources they require. Other disadvantages include difficulty and expense in debugging and checking models and the possible effect of nodalization on code results.¹²⁷

The calculation of the approximate solution, being based on results from the LAPUR code, retains most of the advantages and disadvantages inherent in frequency-domain codes while providing additional information that has previously been available from only the results of time domain-code calculations. Thus, this application of Galerkin's method to the BWR dynamic model consisting of the point kinetics equations and the LAPUR-calculated feedback transfer function should be a practical and useful tool for examining overall BWR dynamic behavior during small-amplitude limit-cycle oscillations.

The BWR dynamic model and the formation of the equation residual are described in the next section, followed by a description of the application of Galerkin's method to obtain an approximate solution for the normalized neutron population during low-amplitude limit-cycle oscillations. This description will be followed by a comparison of the approximate solution to a numerical (exact) solution and then by a comparison of the approximate solution calculated for the instability event at the LaSalle-2 power station to measured data. The chapter concludes with a description of the application of the

low-order approximate solution to determine the stability condition and to predict how changes in feedback transfer function values affect limit-cycle amplitude.

4.1 The BWR Dynamic Model and the Equation Residual

A block diagram of the BWR dynamics is shown in Figure 1.2. The neutron dynamics forms the forward-transfer function relating reactivity to power. The feedback loop describes the effect of power on reactivity and consists of fuel heat generation and its transfer to the coolant, the channel hydraulics and its effect on reactivity, and the effect of fuel temperature on reactivity through the nuclear Doppler effect. The forward-transfer function is modeled by using the point-kinetics equations, and the feedback loop is modeled by using the power-to-reactivity transfer function.

The normalized point-kinetics equations are given by Eqs. (3.10) and (3.11).

Modifying those equations to include M precursor groups results in

$$\frac{dn(t)}{dt} = \frac{n(t)}{\Lambda} [\rho(t) - \beta] + \frac{\rho(t)}{\Lambda} + \sum_{m=1}^M c_m(t) \lambda_m \quad (4.1)$$

and

$$\frac{dc_m(t)}{dt} = -\lambda_m c_m(t) + \frac{\beta_m n(t)}{\Lambda}, \quad (m = 1, 2, 3, \dots, M). \quad (4.2)$$

Because the normalized neutron population during limit-cycle oscillations is periodic, it can be written as

$$n_{ss}(t) = R + \sum_{n=1}^{\infty} \left\{ P_n \sin(n\omega t) + Q_n \cos(n\omega t) \right\}, \quad (4.3)$$

where, as in Eq. (3.31),

- ω = the fundamental limit-cycle oscillation frequency,
- R = the steady-state component of the solution,
- P_n = the amplitude of the sine component at a frequency of $n\omega$,

Q_n = the amplitude of the cosine component at a frequency of $n\omega$,
 n = an index integer.

If steady-state limit-cycle oscillations are assumed, the expression for $n_{ss}(t)$ given by Eq. (4.3) can be substituted for $n(t)$ in Eq. (4.2). Solving for the normalized precursor population of the m th precursor during steady-state limit-cycle oscillations $c_{mss}(t)$ yields

$$c_{mss}(t) = \frac{\beta_m R}{\Lambda \lambda_m} + \sum_{n=1}^{\infty} \left[\frac{\beta_m (n\omega Q_n + \lambda_m P_n)}{\Lambda (\lambda_m^2 + n^2 \omega^2)} \sin(n\omega t) + \frac{\beta_m (\lambda_m Q_n - n\omega P_n)}{\Lambda (\lambda_m^2 + n^2 \omega^2)} \cos(n\omega t) \right]. \quad (4.4)$$

The power-to-reactivity feedback transfer function $H(\omega)$, which describes the BWR thermal hydraulics is used with Eq. (4.3) to form an expression for the reactivity during steady-state limit-cycle oscillations $\rho_{ss}(t)$ in terms of $n_{ss}(t)$. If the neutron population undergoes a simple harmonic oscillation in time, $n_{ss}(t) = P \sin(\omega t)$, the reactivity can be written as the product of $n_{ss}(t)$ and the transfer function $H(\omega)$,

$$\rho_{ss}(t) = |H(\omega)| P \sin[\omega t + \phi(\omega)], \quad (4.5)$$

where $|H(\omega)|$ is the magnitude of the transfer function and $\phi(\omega)$ is the phase of the transfer function at a frequency of ω rad/s.

A more useful form of Eq. (4.5) is obtained by letting the harmonically oscillating normalized neutron population be expressed as the sum of sine and cosine terms,

$$n_{ss}(t) = P \sin(\omega t) + Q \cos(\omega t), \quad (4.6)$$

and letting the transfer function be expressed as a complex number,

$$H(\omega) = A(\omega) + iB(\omega), \quad (4.7)$$

where $i = \sqrt{-1}$.

The product of $H(\omega)$ and $n_{ss}(t)$ is given by

$$\rho_{ss}(t) = [A(\omega)Q + B(\omega)P]\cos(\omega t) + [A(\omega)P - B(\omega)Q]\sin(\omega t) . \quad (4.8)$$

Extending this result to the normalized neutron population given by Eq. (4.3) yields the following expression for the reactivity during limit-cycle oscillations:

$$\begin{aligned} \rho_{ss}(t) = A_0 R + \sum_{n=1}^{\infty} \left\{ [A(n\omega)Q_n + B(n\omega)P_n]\cos(n\omega t) \right. \\ \left. + [A(n\omega)P_n - B(n\omega)Q_n]\sin(n\omega t) \right\}, \end{aligned} \quad (4.9)$$

where A_0 is the real part of $H(\omega)$ for $\omega = 0$.

Substituting Eqs. (4.3), (4.4), and (4.9) into Eq. (4.1) results in a single equation where the unknowns are the constants in the approximate solution for $n_{ss}(t)$. Thus, the solution of the original system of nonlinear differential equations is reduced to determining values for these constants.

If the assumed solution contains a finite number of harmonics, it cannot satisfy Eq. (4.1) exactly; the resulting error is the equation residual $\mathbf{R}_e$. Thus, for an approximate solution of j harmonics, the result of substituting Eqs. (4.3), (4.4), and (4.9) into Eq. (4.1) is the following expression for the equation residual:

$$\begin{aligned}
\mathbf{R}_e = & \sum_{n=1}^j \sum_{l=1}^j \left\{ P_l (A_n P_n - B_n Q_n) S_l S_n + \left[B_n P_l P_n + (A_l + A_n) P_l Q_n - B_l Q_l Q_n \right] S_l C_n \right. \\
& + Q_l (B_n P_n + A_n Q_n) C_l C_n \left. \right\} + \sum_{n=1}^{\infty} \left[\left(n\omega \left[\Lambda + \sum_{m=1}^M \left(\frac{\lambda_m \beta_m}{\lambda_m^2 + n^2 \omega^2} \right) \right] \right. \right. \\
& - B_n (R+1) \left. \right) Q_n + \left[R(A_0 + A_n) - n^2 \omega^2 \sum_{m=1}^M \left(\frac{\beta_m}{\lambda_m^2 + n^2 \omega^2} \right) + A_n \right] P_n \left. \right) S_n \\
& + \left(\left[R(A_0 + A_n) - n^2 \omega^2 \sum_{m=1}^M \left(\frac{\beta_m}{\lambda_m^2 + n^2 \omega^2} \right) + A_n \right] Q_n \right. \\
& - \left. \left(n\omega \left[\Lambda + \sum_{m=1}^M \left(\frac{\lambda_m \beta_m}{\lambda_m^2 + n^2 \omega^2} \right) \right] - B_n (R+1) \right) P_n \right) C_n \right] + A_0 R(R+1), \tag{4.10}
\end{aligned}$$

where

$$\begin{aligned}
C_l &= \cos(l\omega t), \\
S_l &= \sin(l\omega t), \\
C_n &= \cos(n\omega t), \\
S_n &= \sin(n\omega t), \\
P_n &= P(n\omega), \\
Q_n &= Q(n\omega), \\
A_n &= A(n\omega), \\
B_n &= B(n\omega).
\end{aligned}$$

Galerkin's method is used with Eq. (4.10) to obtain equations for the undetermined constants in the approximate solution.

4.2 Application of Galerkin's Method

Selection of Galerkin's method for this application results in relatively simple algebraic equations for the constants in the approximate solution while still retaining a reasonable rate of solution convergence. The equations for the undetermined constants are formed by requiring the integral of the product of the equation residual and a weighting function, integrated over one oscillation period, to be zero. In Galerkin's

method, the weighting functions are the same as the trial functions; thus, the equations for the undetermined constants are given by

$$\int_0^{\frac{2\pi}{\omega}} \mathbf{R}_e \sin(k\omega t) dt = 0, \quad (k = 1, 2, 3, \dots j), \quad (4.11)$$

$$\int_0^{\frac{2\pi}{\omega}} \mathbf{R}_e \cos(k\omega t) dt = 0, \quad (k = 1, 2, 3, \dots j), \quad (4.12)$$

and

$$\int_0^{\frac{2\pi}{\omega}} \mathbf{R}_e R dt = \int_0^{\frac{2\pi}{\omega}} \mathbf{R}_e dt = 0. \quad (4.13)$$

Substituting Eqs. (4.10) into Eqs. (4.11), (4.12), and (4.13) and performing the integrations results in the following nonlinear algebraic equations:

$$\begin{aligned} 0 = & \sum_{l=1}^j \sum_{n=1}^j \left\{ [B_n P_l P_n + (A_l + A_n) P_l Q_n - B_l Q_l Q_n] \int_0^{\frac{2\pi}{\omega}} S_l S_k C_n dt \right\} \\ & + \left\{ k\omega \left[\Lambda + \sum_{m=1}^M \left(\frac{\lambda_m \beta_m}{\lambda_m^2 + k^2 \omega^2} \right) \right] - B_k (R + 1) \right\} Q_k \\ & + \left[R(A_0 + A_k) - k^2 \omega^2 \sum_{m=1}^M \left(\frac{\beta_m}{\lambda_m^2 + k^2 \omega^2} \right) + A_k \right] P_k \right\} \frac{\pi}{\omega}, \end{aligned} \quad (4.14)$$

$$\begin{aligned} 0 = & \sum_{l=1}^j \sum_{k=1}^j \left[P_l (A_k P_k - B_k Q_k) \int_0^{\frac{2\pi}{\omega}} S_l S_k C_n dt + Q_l (B_k P_k + A_k Q_k) \int_0^{\frac{2\pi}{\omega}} C_l C_k C_n dt \right] \\ & + \left[[R(A_0 + A_n) - n^2 \omega^2 \sum_{m=1}^M \left(\frac{\beta_m}{\lambda_m^2 + n^2 \omega^2} \right) + A_n] Q_n \right. \\ & \left. - \left\{ n\omega \left[\Lambda + \sum_{m=1}^M \left(\frac{\lambda_m \beta_m}{\lambda_m^2 + n^2 \omega^2} \right) \right] - B_n (R + 1) \right\} P_n \right] \frac{\pi}{\omega}, \end{aligned} \quad (4.15)$$

and

$$0 = \sum_{n=1}^j [(P_n^2 + Q_n^2)A_n] + 2A_0R(R + 1) . \quad (4.16)$$

Equations (4.14), (4.15), and (4.16) result in $2j + 1$ equations for R , ω , P_n , and Q_n ($n = 1$ to j). Because the absolute phase of the steady-state oscillations described by the approximate solution is arbitrary, the value of Q_j can be assigned a value of zero without affecting the magnitude or the relative phases of the harmonics in the approximate solution. This assignment leaves $2j + 1$ unknowns, which can be solved for by using the $2j + 1$ equations obtained from Eqs. (4.14), (4.15), and (4.16).

Equation (4.16) forces the average value of the equation residual over one cycle to be zero. At any point during the cycle, however, the equation residual may be relatively large, especially so for approximate solutions containing few harmonics. Equations generated from Eqs. (4.14) and (4.15) calculate the P_n and Q_n values that minimize the magnitude of the equation residual. Thus, by satisfying Eq. (4.14), (4.15), and (4.16), an approximate solution is obtained for which the average value of the equation residual over one cycle is zero and for which the equation residual during the cycle is minimized.

A FORTRAN computer code was written to calculate the approximate solution. The code forms the simultaneous nonlinear algebraic equations given by Eq. (4.14), (4.15), and (4.16) and solves for the undetermined constants in the approximate solution. The code input consists of the point kinetics parameters, the feedback transfer function, and an initial estimate of the values of the undetermined constants. The initial estimate is used by the NAG routine CO5NBF to solve the set of nonlinear algebraic equations generated by the program.¹³⁹ Program output consists of the calculated values of the constants in the approximate solution. Similar experience to that described in Subsection 3.2.3

regarding both solution convergence and strategies for obtaining high-order approximate solutions has been obtained by using this code.

4.3 Comparison of Approximate and Numerical Solutions

A comparison of an approximate solution and a numerical solution was used to examine the accuracy and rate of convergence of the approximate solution. The phenomenological model equations described in Chapter 3 were used in the comparison. A feedback transfer function relating power to reactivity was formed by using Eqs. (3.15), (3.17), and (3.19). The feedback power-to-reactivity transfer function of the phenomenological model is given by

$$H(s) = \frac{a_1(-\alpha_T \omega^2 + i\alpha_T a_3 \omega + \alpha_T a_4 + K)}{(i\omega + a_2)(-\omega^2 + ia_3 \omega + a_4)} . \quad (4.17)$$

Table 3.1 lists parameter values that cause the solution of the phenomenological model equations to be described by a limit cycle; these values were used in the calculations performed for this comparison.

The values of constants in approximate solutions containing from two to eight harmonics are shown in Table 4.1. The results show the values of the constants converging as the number of harmonics increases. Figure 4.1 shows $N(t)/N_0$ for approximate solutions consisting of two, four, six, and eight harmonics over five cycles. The convergence of the solution is apparent.

Equations (3.10), (3.11), (3.15), (3.17) and (3.19) were also solved numerically by using the ACSL program.¹³⁸ The steady-state results of this calculation and the eight-harmonic approximate solution are shown in Figure 4.2. Comparing the results shows excellent agreement between the two solutions. The convergence of the

approximate and numerical solutions and the excellent agreement between the eight-harmonic approximate solution and the numerical solution imply that the error in the approximate solution can be reduced to an acceptable level if a sufficient number of harmonics are included in the approximate solution. This result agrees with those of Chapter 3, which were obtained by using the phenomenological model equations.

Table 4.1. Approximate solution constants for two- to eight-harmonic solutions

Constant	Number of harmonics						
	2	3	4	5	6	7	8
ω (Hz)	0.4240	0.4210	0.4181	0.4174	0.4171	0.4170	0.4170
R	0.0386	0.0497	0.0542	0.0557	0.0562	0.0564	0.0564
P_1	0.6376	0.4836	0.3580	0.2663	0.1994	0.1474	0.1067
Q_1	0.6639	0.9176	1.0180	1.0600	1.0788	1.0884	1.0934
P_2	0.3623	0.4424	0.3898	0.3195	0.2579	0.2063	0.1643
Q_2		0.2615	0.4149	0.4933	0.5343	0.5579	0.5721
P_3		0.2158	0.2690	0.2512	0.2172	0.1824	0.1517
Q_3			0.1120	0.1917	0.2396	0.2695	0.2885
P_4			0.1249	0.1557	0.1500	0.1336	0.1156
Q_4				0.0498	0.0899	0.1178	0.1365
P_5				0.0706	0.0880	0.0863	0.0786
Q_5					0.0225	0.0431	0.0584
P_6					0.0393	0.0490	0.0487
Q_6						0.0106	0.0210
P_7						0.0218	0.0271
Q_7							0.0050
P_8							0.0120

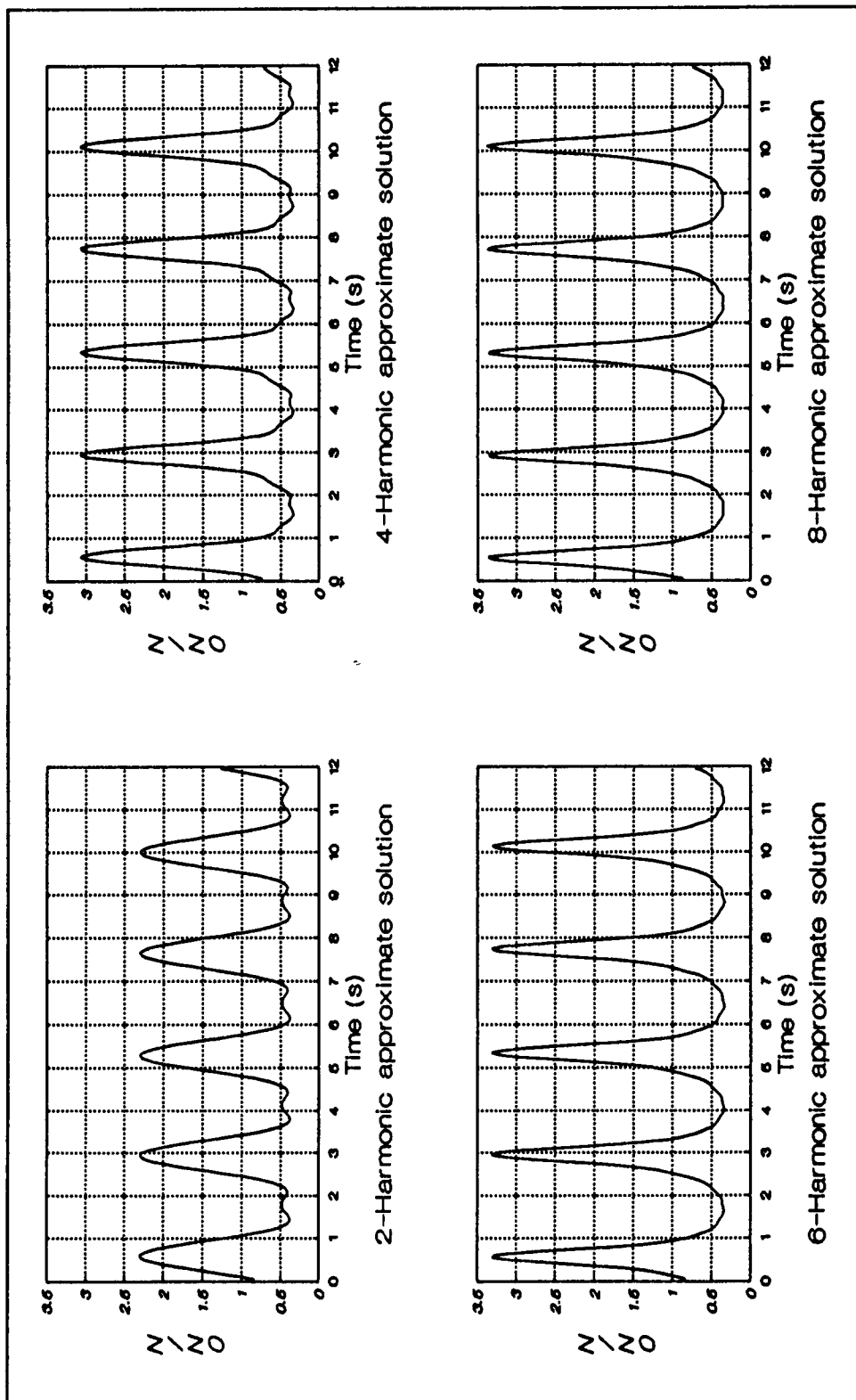


Figure 4.1 . Approximate solutions for two, four, six, and eight harmonics (N/N_0 = neutron population normalized to the steady-state neutron population).

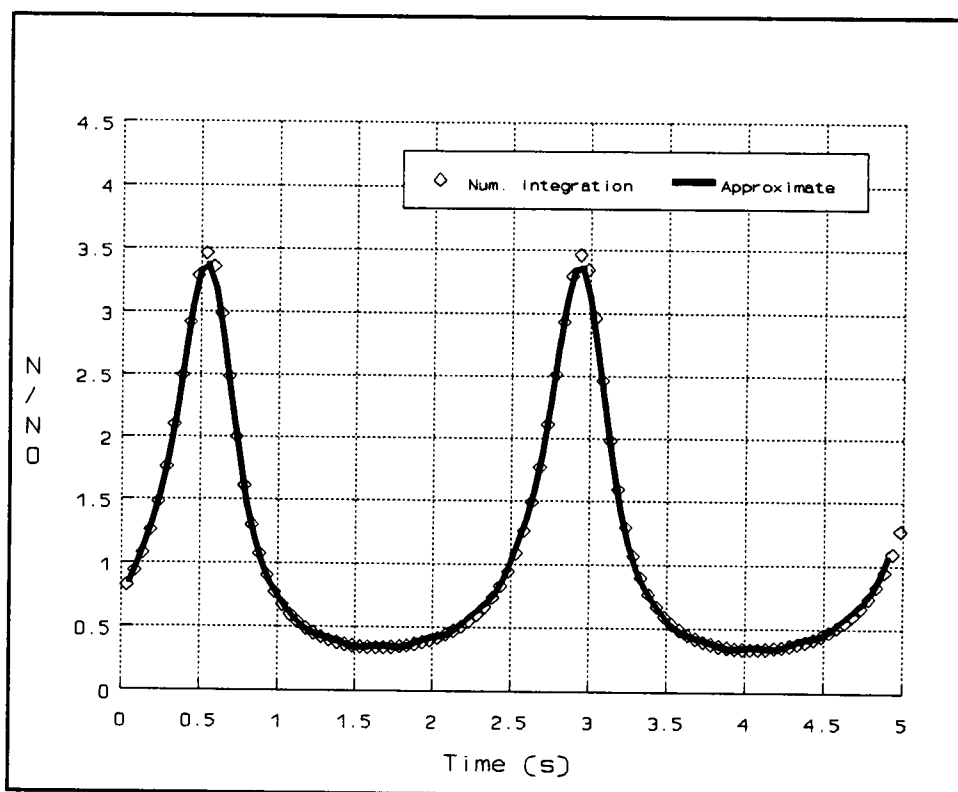


Figure 4.2. Comparison of numerical and eight-harmonic approximate solutions (N/N_0 = neutron population normalized to the steady-state neutron population).

4.4 Comparison of the Approximate Solution to Measured Limit-Cycle Data

The approximate solution corresponding to the instability event at the LaSalle-2 power station is compared to average power-range-monitor data recorded during the event in this section.¹¹⁴ The LaSalle Unit 2 reactor experienced core-wide limit-cycle oscillations on March 9, 1988. The reactor had been operating at ~83% power and 75% core flow when an error by a technician caused both recirculation pumps to trip. Core flow then coasted down to natural circulation conditions, ~28% of rated core flow, and the reactor power decreased to ~43%. The reduction in steam flow caused a reduction in the amount of extraction steam available for feedwater heating, resulting in the automatic isolation of

several feedwater preheaters. Approximately five minutes after the feedwater pumps had tripped, the reduction in feedwater temperature caused by the isolation of the feedwater preheaters resulted in an increase in reactor power to ~45%.⁸⁷ At this point, oscillations were observed on the average- and local-power-range monitors. The oscillations continued for two minutes until the reactor exceeded the high-neutron-flux trip point and automatically scrammed. The oscillation frequency was ~0.45 Hz, and the average power range monitor (APRM) peak-to-peak oscillation amplitude varied between 20 and 100%. During the event, a malfunction in the operation of the feedwater control valve caused core flow to fluctuate between ~26 and 28% of rated core flow with an oscillation period of ~35 s. Figure 4.3 shows an APRM signal recorded during the event. The variation in the amplitude of the power oscillations is due primarily to the changing core flow; the maximum power oscillation amplitude coincided with the minimum core flow. Figure 4.4 shows reactor power, core flow, steam flow, and water level recorded during the event.

The LAPUR frequency-domain code was used to calculate the power-to-reactivity feedback transfer function used in the approximate-solution calculation.⁶¹ The parameters used in the point kinetics equations during this calculation are listed in Table 4.2.

Approximate solutions for $n_{\infty}(t)$ were calculated by using the LAPUR-calculated feedback transfer function calculated for 45% power and core flows ranging from 25 to 29%. These power/flow combinations were selected because they bracket the conditions observed during the LaSalle event. The results are tabulated in Table 4.3, and the effect of core flow on limit-cycle amplitude is shown in Figure 4.5.

Comparison with the APRM data shown in Figure 4.3 shows that the limit-cycle amplitudes of the normalized neutron population calculated for 26% flow and 28% flow, 2.38 and 1.39, closely bracket the range of normalized-power limit-cycle amplitude

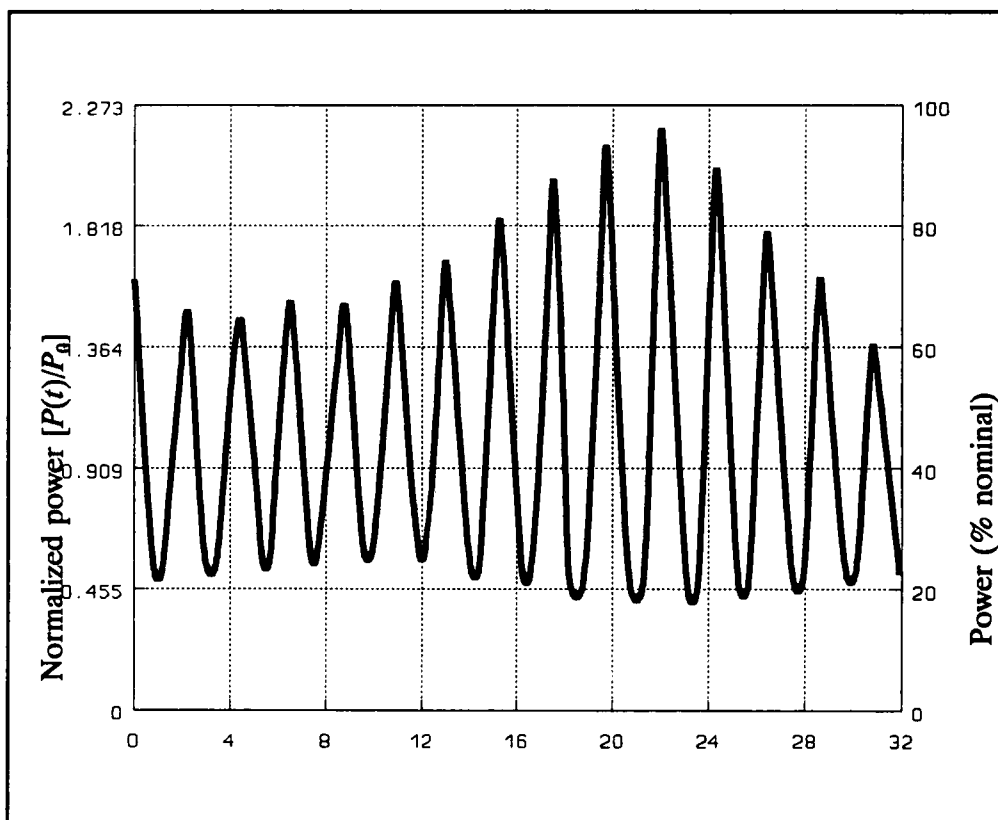


Figure. 4.3. Average power range monitor data recorded during the LaSalle event. Initiation of the event does not correspond to $t = 0$. *Source:* J. C. Shaug, J. G. M. Anderson, and J. K. Garret, "TRAC-G Analysis of BWR Plant Stability Data," p. 354 in proceedings of the International Workshop on Boiling Water Reactor Stability, held in Holtsville, N.Y., October 17-19, 1990, CSNI Report 178, OECD Nuclear Energy Agency, Paris, 1990.

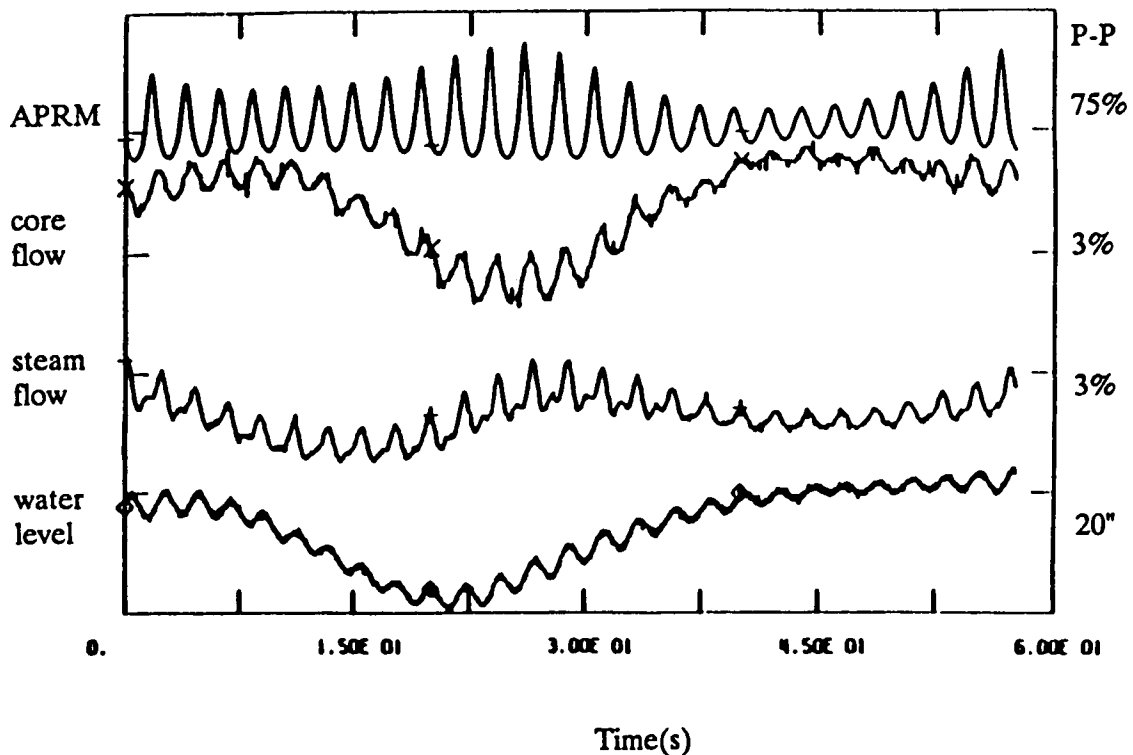


Figure 4.4. LaSalle-2 limit-cycle oscillation characteristics. Initiation of the event does not correspond to $t = 0$ (APRM = average power-range monitor, P-P = peak-to-peak).
Source: Adapted with permission from Thomas Rausch, Commonwealth Edison Company, Chicago, handout at the Boiling Water Reactor owners' group presentation to the Nuclear Regulatory Commission, November 8, 1989, Rockville, Maryland.

Table 4.2. Point kinetics parameters used to calculate the approximate solution for conditions corresponding to the LaSalle event

i	β_i	$\lambda_i(\text{s}^{-1})$
1	4.00×10^{-5}	4.00×10^{-5}
2	1.18×10^{-3}	3.05×10^{-5}
3	1.06×10^{-3}	1.11×10^{-1}
4	2.13×10^{-3}	3.01×10^{-1}
5	6.24×10^{-4}	1.13
6	2.27×10^{-4}	3.00
$\Lambda = 4.00 \times 10^{-5} \text{ s}$		

observed during the event. Poorer agreement was obtained between the measured and calculated oscillation frequencies. The low predicted frequency is consistent with previous results from the LAPUR code. Benchmark calculation results have shown that the LAPUR code consistently underestimates the oscillation frequency by 10 to 25%.¹⁴¹ Thus, the underpredicted oscillation frequency in the approximate solution

should be expected and can be traced to the LAPUR-calculated feedback transfer function.

Once an approximate solution for $n_{ss}(t)$ has been obtained, the behavior of additional reactor parameters can be calculated by using the appropriate LAPUR-calculated transfer functions. If an approximate solution containing j harmonics has been obtained for $n_{ss}(t)$, a multiplication similar to that shown in Eq. (4.8), using a transfer function relating the neutron population to another parameter, results in an approximate solution of similar form for the new parameter. Approximate solutions for the normalized reactivity and inlet flow were obtained by multiplying $n_{ss}(t)$ by the appropriate LAPUR-calculated transfer functions. Table 4.4 lists the values of the LAPUR-calculated reactivity-to-power and power-to-inlet flow transfer functions calculated for 45% power and 26% core flow. Figure 4.6 shows ten-harmonic approximate solutions for $n_{ss}(t)$, $\rho_{ss}(t)$, and the normalized

Table 4.3. Limit-cycle frequency and amplitude for normalized neutron population calculated for the LaSalle event (45% power and core flows ranging from 25 to 29%)

Core flow (%)	Oscillation frequency (Hz)	Limit-cycle amplitude
25.00	0.383	3.321
25.25	0.385	2.992
25.50	0.387	2.757
25.75	0.388	2.582
26.00	0.390	2.382
26.25	0.391	2.224
26.50	0.393	2.084
26.75	0.394	1.951
27.00	0.395	1.846
27.25	0.396	1.723
27.50	0.398	1.610
27.75	0.399	1.500
28.00	0.400	1.390
28.25	0.402	1.112
28.50	Stable	Stable
28.75	Stable	Stable
29.00	Stable	Stable

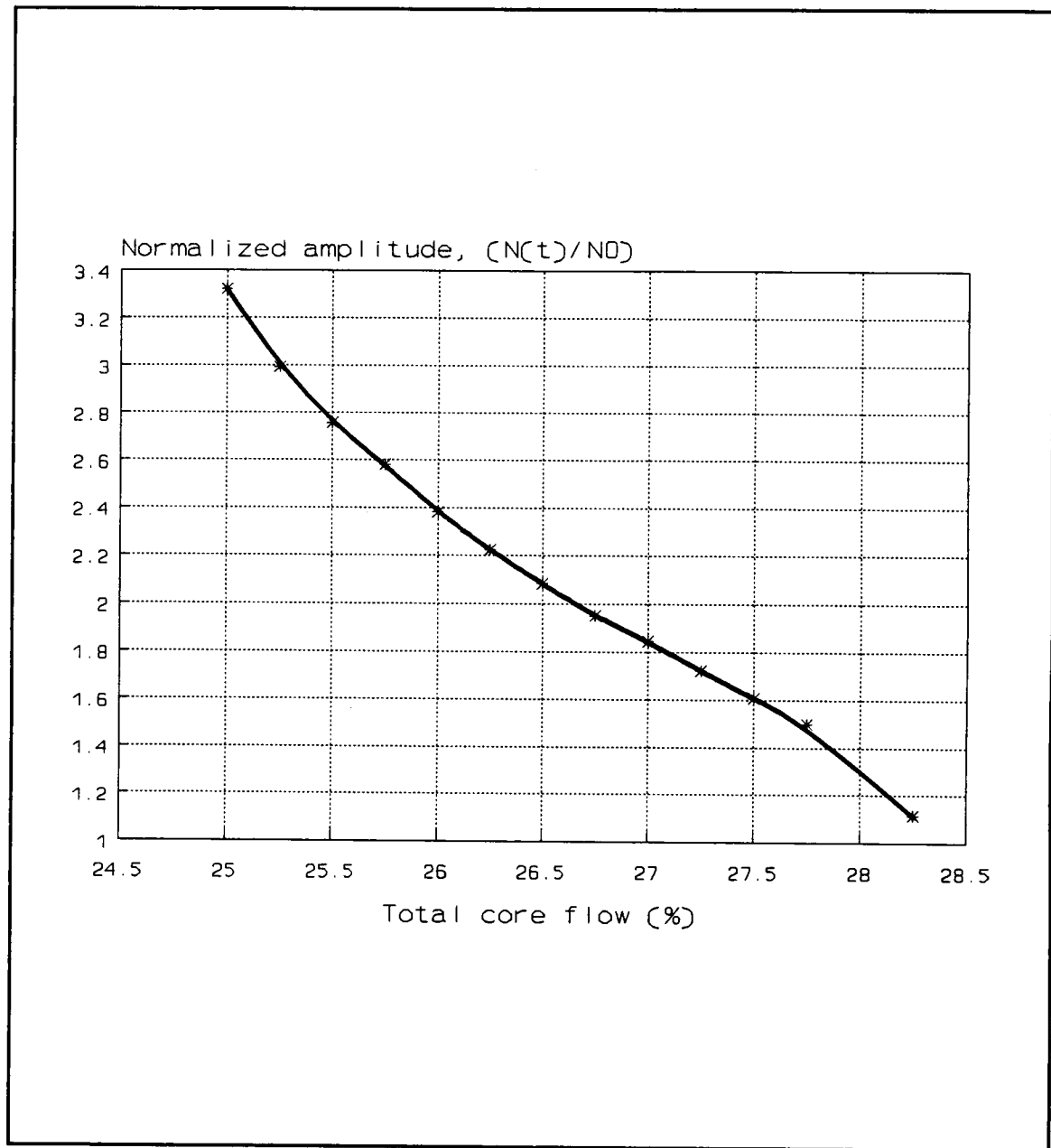


Figure 4.5. Normalized limit-cycle amplitude $[N(t)/N_0]$ for 45% power and total core flow ranging from 25 to 28.25%.

**Table 4.4. Transfer function values for the LaSalle event
(45% power, 25% flow)**

Frequency (Hz)	Power-Reactivity		Power-to-Flow	
	Real	Imaginary	Real	Imaginary
0.001	-3.9899	0.1121	-0.0824	0.0001
0.200	0.2291	1.0133	-0.0835	0.0850
0.250	0.4108	0.7660	-0.0496	0.1017
0.300	0.5013	0.5390	-0.0152	0.1001
0.330	0.5397	0.4219	0.0016	0.0934
0.360	0.5881	0.2795	0.0168	0.0842
0.400	0.5252	0.0261	0.0315	0.0631
0.430	0.3925	-0.0666	0.0321	0.0472
0.460	0.2848	-0.0883	0.0287	0.0364
0.500	0.1917	-0.0795	0.0233	0.0277
0.550	0.1267	-0.0558	0.0178	0.0221
0.600	0.0929	-0.0347	0.0142	0.0192
0.750	0.0608	-0.0069	0.0103	0.0150
1.000	0.0355	-0.0128	0.0082	0.0087
2.000	0.0071	-0.0036	0.0031	0.0024

inlet flow, calculated by using these transfer functions, over four cycles. Note the similarity in shape of the power oscillations shown in Figures 4.3 and 4.4 and the normalized neutron population oscillations shown in Figure 4.6. The effects of the higher harmonics, which are prevalent in the solution for $n_{ss}(t)$, are reduced in magnitude in the solutions for $\rho_{ss}(t)$ and flow, making them appear more sinusoidal. This reduction is expected because the magnitudes of the power-to-reactivity and power-to-inlet flow transfer functions decrease rapidly with increasing frequency.

A true comparison of the measured data and the approximate solution is somewhat difficult because the actual limit cycle never reached a true steady state. Relatively good agreement was found between the calculated and measured values of the limit-cycle amplitude over the range of conditions observed during the LaSalle event and between the shapes of the measured power and calculated neutron population oscillations. This agreement supports the assertion that the application of Galerkin's method to the BWR model consisting of the point kinetics equations and a power-to-reactivity feedback transfer function can be used to extend results obtained from a frequency-domain code such as LAPUR to calculate reactor parameters such as total power, reactivity, or inlet flow during low-amplitude limit-cycle oscillations.

4.5 The Low-Harmonic Approximate Solution at the Stability Boundary

The results in Chapter 3 show that the normalized neutron population near the stability boundary during limit-cycle oscillations can be approximated by

$$n_{ss}(t) = R + P_1 \sin(\omega t) . \quad (4.18)$$

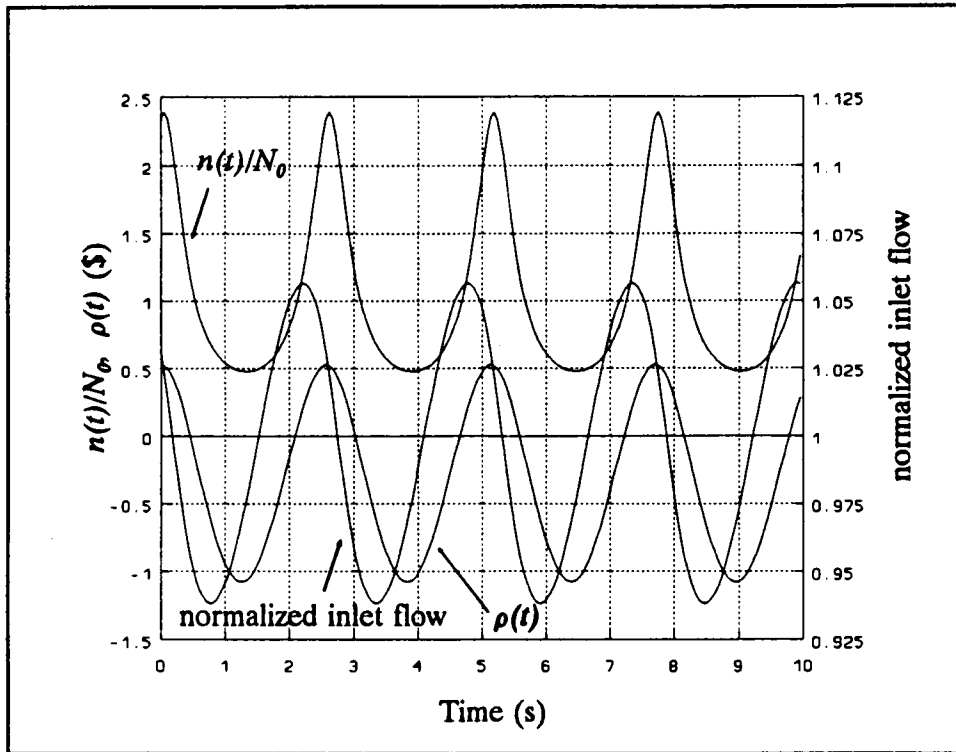


Figure 4.6. Ten-harmonic approximate solution for the LaSalle event (45% power, 26% flow).

Evaluating Eqs. (4.14), (4.15), and (4.16) for $j = 1$ results in three equations for the three unknowns P_1 , R , and ω :

$$R[A_0 + A(\omega)] - \omega^2 \sum_{m=1}^M \frac{\beta_m}{(\lambda_m^2 + \omega^2)} + A(\omega) = 0, \quad (4.19)$$

$$P_1^2 A(\omega) + 2A_0 R(R+1) = 0, \quad (4.20)$$

and

$$B(\omega)(R+1) - \omega \left[\Lambda + \sum_{m=1}^M \left(\frac{\beta_m \lambda_m}{\lambda_m^2 + \omega^2} \right) \right] = 0, \quad (4.21)$$

where A_0 is the real part of the feedback transfer function at 0 Hz.

Results from Chapter 3 show that $\lambda_m \ll \omega$ and $R \ll 1$ near the stability boundary; this result allows Eqs. (4.19), (4.20), and (4.21) to be simplified by omitting negligibly small terms, yielding

$$[A_0 + A(\omega)]R + A(\omega) - \beta = 0, \quad (4.22)$$

$$P_1^2 A(\omega) + 2A_0 R = 0, \quad (4.23)$$

and

$$\omega \Lambda + \sum_{m=1}^M \frac{\lambda_m \beta_m}{\omega} - B(\omega) = 0, \quad (4.24)$$

where $\beta = \sum_{m=1}^M \beta_m$. Solving for R and P_1^2 ,

$$R = \frac{\beta - A(\omega)}{A_0 + A(\omega)} \quad (4.25)$$

and

$$P_1^2 = \frac{-2A_0 R}{A(\omega)} = \frac{-2A_0 [\beta - A(\omega)]}{A(\omega) [A_0 + A(\omega)]}. \quad (4.26)$$

Equation (4.24) can be solved for ω ; $A(\omega)$ can then be determined (because the feedback transfer function is a known quantity), and R and P_1 can be calculated by using Eqs. (4.25) and (4.26).

4.5.1 Stability Condition

For the limit-cycle solution to exist, the values of R and P_1 calculated by using Eqs. (4.25) and (4.26), with a value of ω satisfying Eq. (4.23), must be real. Thus,

$\frac{-2A_0[\beta - A(\omega)]}{A(\omega)[A_0 + A(\omega)]}$ must be positive for the solution to exist. For transfer functions typical of normal BWR operation,⁶¹ $A_0 < 0$, $A(\omega) > 0$, and $|A_0| > |A(\omega)|$, so $A_0 + A(\omega) < 0$; therefore, $\beta - A(\omega) < 0$ if $P_1^2 > 0$. Thus, the stability condition is

$$A(\omega) > \beta. \quad (4.27)$$

The sign of R can be determined from the requirement that P_1^2 be positive. Because A_0 is negative and $A(\omega)$ is positive, R must be positive. This result indicates that the average reactor power must increase during limit-cycle oscillations, a fact that has been observed in previous numerical and analytical studies.¹¹¹ Thus, the one-harmonic solution has been used to show that for the limit-cycle solution to exist, the nonoscillatory part of the solution for $n_{\infty}(t)$ must be positive, $A(\omega)$ must be greater than β , and the average reactor power must increase during limit-cycle oscillations.

The feedback transfer function values can be converted from real and imaginary parts to magnitude and phase by using

$$A(\omega) = M(\omega) \cos[\phi(\omega)] \quad (4.28)$$

and

$$B(\omega) = M(\omega) \sin[\phi(\omega)], \quad (4.29)$$

where $M(\omega)$ is the transfer function magnitude at a frequency of ω , and $\phi(\omega)$ is the transfer function phase at a frequency of ω . The expressions for ω , P_1^2 , and R obtained from the transfer function given in terms of magnitude and phase are

$$\omega^2 \Lambda + \sum_{m=1}^M \lambda_m \beta_m - \omega M(\omega) \sin[\phi(\omega)] = 0, \quad (4.30)$$

$$P_1^2 = \frac{-2M_0 \{ \beta - M(\omega) \cos[\phi(\omega)] \}}{M(\omega) \{ M_0 + M(\omega) \cos[\phi(\omega)] \} \cos[\phi(\omega)]}, \quad (4.31)$$

and

$$R = \frac{\beta - M(\omega) \cos[\phi(\omega)]}{-M_0 + M(\omega) \cos[\phi(\omega)]}, \quad (4.32)$$

where M_0 is the transfer function magnitude at 0 Hz.

Equations (4.24), (4.27), (4.28), and (4.29) can be used to give the stability condition in terms of $M(\omega)$ and $\phi(\omega)$ or in terms of $\phi(\omega)$ alone:

$$\beta < M(\omega) \cos[\phi(\omega)] \quad (4.33)$$

or

$$\tan[\phi(\omega)] < \frac{\omega^2 \Lambda + \sum_{m=1}^M \lambda_m \beta_m}{\omega \beta}. \quad (4.34)$$

4.5.2 Dependence of Limit-Cycle Amplitude on Feedback Transfer Function Values

The one-harmonic solution was used to qualitatively determine the effect of the feedback transfer function values on the limit-cycle amplitude. The limit-cycle amplitude near the stability boundary is determined primarily by P_1 , the magnitude of the first harmonic in the approximate solution. The variation of P_1^2 with $A(\omega)$ is obtained by taking the derivative of Eq. (4.26) with respect to $A(\omega)$,

$$\frac{\partial(P_1^2)}{\partial A(\omega)} = \frac{2A_0 \{ \beta [A_0 + 2A(\omega)] - A^2(\omega) \}}{A^2(\omega) [A_0 + A(\omega)]^2}. \quad (4.35)$$

Setting this derivative equal to zero and solving for $A(\omega)$ yields an equation for the value of $A(\omega)$ at which P_1^2 , is an extreme,

$$A_{mx}(\omega) = \beta \pm \sqrt{\beta(\beta + A_0)}. \quad (4.36)$$

For $A_{mx}(\omega)$ to be real, $\beta + A_0$ must be greater than zero. Thus, a local maximum or minimum in limit-cycle amplitude will exist if $\beta > -A_0$. For transfer functions characteristic of normal BWR operation,⁶¹ $\frac{\partial(P_1^2)}{\partial A(\omega)} > 0$ and $\beta < -A_0$; thus, no maximum exists and P_1 (and, consequently, the limit cycle amplitude) must increase with $A(\omega)$.

In terms of feedback transfer function magnitude and phase, the limit-cycle amplitude will increase as $M(\omega) \cos[\phi(\omega)]$ becomes greater. If $\phi(\omega)$ is constant, increasing $M(\omega)$ will increase the limit-cycle amplitude. If $M(\omega)$ is constant, the limit-cycle amplitude will either increase or decrease with $\phi(\omega)$ depending on the quadrant of $\phi(\omega)$. Equation (4.29) shows that $\sin[\phi(\omega)]$ must be positive because all other quantities in the equation are positive. Thus, $\phi(\omega)$ must be in the first or second quadrant. Similarly, Eq. (4.33) shows that $\cos[\phi(\omega)]$ must be positive, thus $\phi(\omega)$ must be in the first quadrant. If the system is near the stability boundary,

$$\tan[\phi(\omega)] = \frac{\omega^2 \Lambda + \sum_{m=1}^M \lambda_m \beta_m}{\omega \beta}. \quad (4.37)$$

Substituting parameter values describing a typical 3300-MW(th) BWR, ($\Lambda = 4 \times 10^{-5} \text{ s}$, $\omega = 0.4 \times 2\pi \text{ s}^{-1}$, $\sum \lambda_m \beta_m = 2 \times 10^{-3} \text{ s}^{-1}$, and $\beta = 0.0056$) results in $\phi(\omega) = 9.5^\circ$. Thus, it appears that $\phi(\omega)$ will be in the first quadrant, should be near zero, and decreasing $\phi(\omega)$ will increase the limit-cycle amplitude if the transfer function magnitude remains constant.

4.5.3 Calculated Results Near the Stability Boundary

Calculations whose results illustrate the relationships presented in Subsections 4.5.1 and 4.5.2 have been performed by using Eqs. (4.24), (4.25), and (4.26) and a feedback transfer function obtained from the phenomenological model. The model parameters used in this work are given in Table 3.1 with $K = -0.0044$. Equation (4.17) gives the model's feedback transfer function $H(\omega)$.

To examine the effect of the feedback transfer function on the limit cycle, the calculated feedback transfer function values were modified by changing the time delay in the feedback loop by using a phase lag. The phase lag $\Delta\phi$ for a particular time delay Δt is given by

$$\Delta\phi = 2\pi\omega\Delta t. \quad (4.38)$$

The effect of a pure time delay is destabilizing because it reduces both the gain and phase margins. Table 4.5 shows values of the feedback transfer function at the limit-cycle oscillation frequency calculated for various time delays, as well as the corresponding values of R and P_1 calculated by using Eqs. (4.25) and (4.26). Note that a negative time delay corresponds to reducing the phase lag of the system. The relationship between $A(\omega)$ and the limit-cycle amplitude and frequency is shown in Figure 4.7.

The calculated results show the following.

1. As predicted, $\phi(\omega)$ remains small in magnitude and positive. Thus, the increase in $M(\omega)$ is primarily due to an increase in $A(\omega)$. The limit-cycle amplitude increases with $A(\omega)$ as predicted because $\beta < -A_0$.
2. The value of $A(\omega)$ calculated at the stability boundary is 0.0056, [i.e., $A(\omega) = \beta$].
3. The oscillation frequency decreases and the limit-cycle amplitude increases as the time lag increases (i.e., as the system becomes more linearly unstable).

**Table 4.5. Effect of a time delay on limit-cycle parameters calculated
by using the one-harmonic approximate solution**

Δt	$A(\omega)$	ω (Hz)	$M(\omega)$	$\phi(\omega)$ (degrees)	R	P_1	$R + P_1$
0.058	0.005600	0.4609	0.005607	2.7654	0.00001	0.0174	0.0174
0.050	0.005747	0.4564	0.005753	2.6993	0.0021	0.2353	0.2375
0.040	0.005940	0.4506	0.005946	2.6168	0.0050	0.3530	0.3581
0.030	0.006118	0.4453	0.006124	2.5461	0.0077	0.4298	0.4376
0.020	0.006302	0.4399	0.006308	2.4773	0.0105	0.4938	0.5043
0.010	0.006468	0.4350	0.006474	2.4190	0.0130	0.5427	0.5557
0.000	0.006643	0.4300	0.006649	2.3608	0.0157	0.5879	0.6036
-0.010	0.006800	0.4255	0.006806	2.3117	0.0181	0.6239	0.6421
-0.020	0.006964	0.4208	0.006971	2.2625	0.0206	0.6584	0.6791
-0.040	0.007271	0.4122	0.007276	2.1784	0.0254	0.7144	0.7398
-0.080	0.007838	0.3963	0.007843	2.0424	0.0343	0.7998	0.8342
-0.120	0.008357	0.3818	0.008362	1.9375	0.0426	0.8632	0.9059
-0.160	0.008793	0.3687	0.008798	1.8632	0.0497	0.9087	0.9585
-0.200	0.009232	0.3560	0.009237	1.7976	0.0569	0.9491	1.0061
-0.250	0.009733	0.3413	0.009737	1.7344	0.0653	0.9899	1.0553
-0.300	0.010144	0.3277	0.010148	1.6936	0.0723	1.0200	1.0924
-0.350	0.010556	0.3147	0.010560	1.6586	0.0793	1.0477	1.1271
-0.400	0.010947	0.3025	0.010951	1.6312	0.0861	1.0720	1.1582
-0.450	0.011284	0.2910	0.011288	1.6151	0.0921	1.0916	1.1837
-0.500	0.011667	0.2800	0.011672	1.5958	0.0989	1.1126	1.2116
-0.550	0.011961	0.2698	0.011966	1.5902	0.1042	1.1279	1.2322
-0.600	0.012333	0.2599	0.012338	1.5777	0.1110	1.1462	1.2573
-0.680	0.012821	0.2455	0.012826	1.5733	0.1200	1.1690	1.2890
-0.760	0.013333	0.2323	0.013338	1.5695	0.1296	1.1913	1.3210
-0.850	0.013958	0.2186	0.013963	1.5639	0.1416	1.2169	1.3585
-0.900	0.014245	0.2119	0.014250	1.5671	0.1471	1.2281	1.3752
-1.000	0.014955	0.1990	0.014961	1.5638	0.1612	1.2544	1.4156

Δt = time delay in seconds.

$A(\omega)$ = real part of feedback transfer function at a frequency of ω (no units).

ω = oscillation frequency.

$M(\omega)$ = magnitude of feedback transfer function at a frequency of ω .

$\phi(\omega)$ = phase of feedback transfer function at a frequency of ω .

R = steady-state component of one-harmonic approximate solution.

P_1 = magnitude of first harmonic in approximate solution.

$R + P_1$ = amplitude of approximate solution.

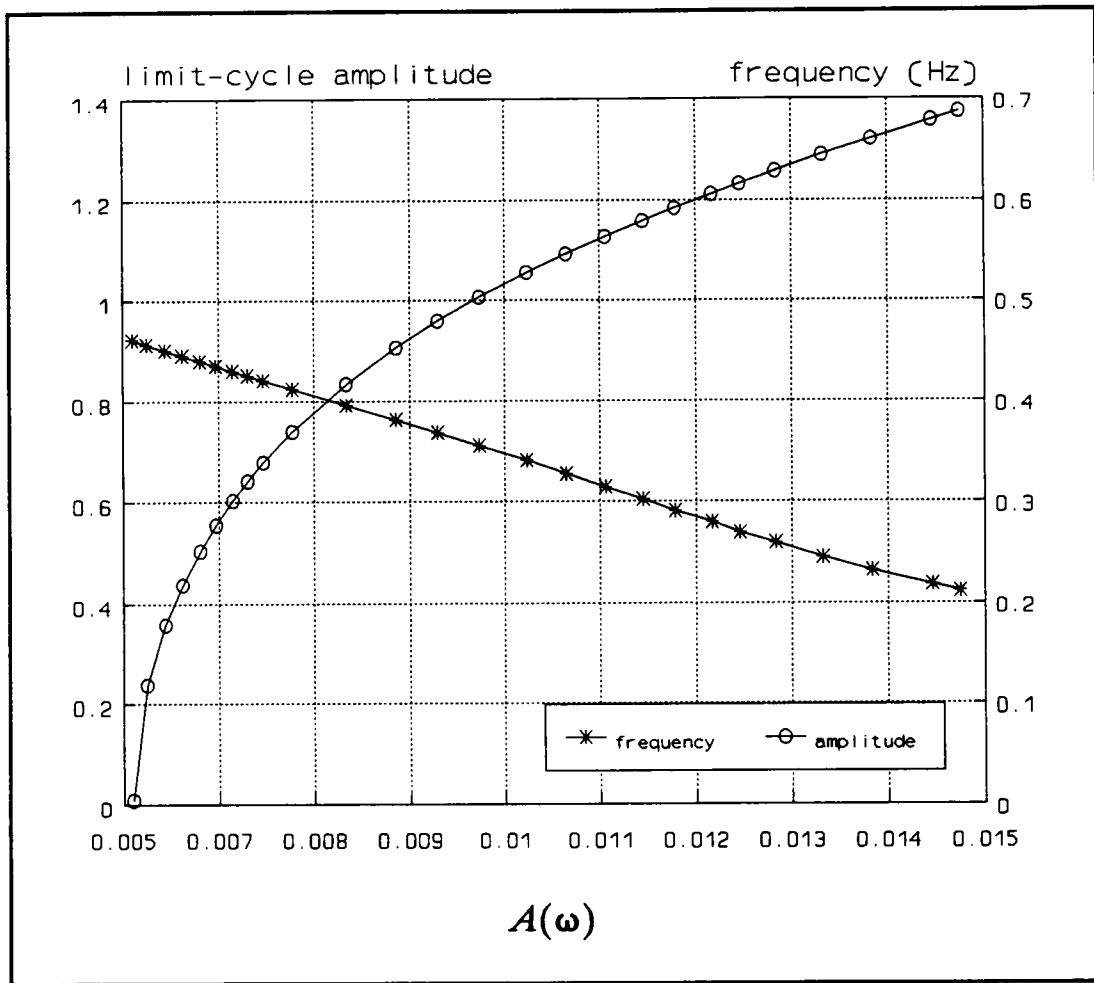


Figure 4.7. Effect of $A(\omega)$ on limit-cycle oscillation frequency and amplitude.

4.6 Chapter Summary

It has been shown that the combination of the LAPUR-calculated feedback transfer function and the point kinetics equations can be used with Galerkin's method to approximate overall BWR dynamic behavior during small-amplitude limit-cycle oscillations. It was shown how the feedback transfer function can be used to express reactivity as a function of the normalized neutron population and how this expression was substituted into the point kinetics equations, forming the equation residual. The application of

Galerkin's method to obtain an approximate solution for the normalized neutron population was then described.

Comparison of the approximate solution with a numerical solution showed that the approximate solution converges to the numerical (exact) solution as more harmonics are included in the approximation. Thus, it was concluded that Galerkin's method can be used to determine an accurate solution to the system of nonlinear differential equations formed from the BWR dynamic model consisting of the point kinetics equations and the power-to-reactivity feedback transfer function.

The approximate solution for the LaSalle event was calculated and used to determine neutron population, reactivity, and inlet flow oscillations during the event. Comparison of the approximate solution to APRM data recorded during the event show good agreement between the amplitudes of the limit-cycle oscillations over the range of observed reactor conditions. The general shape of the measured power oscillations and the calculated normalized neutron population oscillations also showed good agreement. Underprediction of the oscillation frequency in the approximate solution was attributed to the LAPUR-calculated feedback transfer function because the LAPUR code has consistently underpredicted BWR oscillation frequency by 10 to 25% in benchmark tests. It was shown that if transfer functions relating reactor power to other reactor parameters are available, the behavior of these parameters during steady-state limit-cycle oscillations can also be calculated. Based on the comparisons between approximate and numerical solutions and between the approximate solution and APRM data collected during the LaSalle event, it was concluded that the application of Galerkin's method to the equations obtained from a BWR dynamic model consisting of the point kinetics equations and a power-to-reactivity

feedback transfer function can extend the results from a frequency-domain code to calculate overall reactor parameters during low-amplitude limit-cycle oscillations.

The low-order approximate solution obtained by using this technique has been used to determine a stability condition in terms of feedback transfer function values and to qualitatively predict how changing the feedback transfer function affects limit-cycle amplitude. This technique provides an additional indication of reactor stability and qualitatively confirms results from numerical simulations of BWR dynamic behavior.

Specific results of the work with the low-order approximate solution are as follows.

1. For ω given by Eq. (4.23), the reactor will be unstable if $A(\omega)$ is greater than β .
2. During limit-cycle oscillations, the nonoscillatory part of the solution for $n(t)$ must be positive. Thus, the average reactor power must increase during limit-cycle oscillations.
3. A peak value for the limit-cycle amplitude will exist if $\beta + A_0 > 0$. If $\beta + A_0 < 0$ (which is always expected to be the case), the limit-cycle amplitude will always increase as $A(\omega)$ increases.
4. $\phi(\omega)$ will be positive and of small magnitude.
5. Increasing $M(\omega)\cos[\phi(\omega)]$ will increase the limit-cycle amplitude. Thus, increasing either $M(\omega)$ while holding $\phi(\omega)$ constant or decreasing $\phi(\omega)$ while holding $M(\omega)$ constant will increase the limit cycle amplitude.

5. APPROXIMATE SOLUTION FOR THE BIFURCATED LIMIT CYCLE

The approximate steady-state solution for the bifurcated limit cycle is described in this chapter. In Chapters 3 and 4, the application of Galerkin's method to calculate an approximate solution describing the normalized neutron population during limit-cycle oscillations was described. Numerical solutions presented in these chapters showed that the limit cycle is initially unbifurcated, but if the system is made sufficiently unstable, the limit cycle will experience a period-doubling bifurcation. It was shown in Chapter 3 that when calculating high-order approximate solutions, the strategies for selecting the initial estimates for the undetermined constants in the approximate solution result in the calculation of approximate solutions of the unbifurcated form. Thus, two issues that must be examined are (1) **when** the limit cycle has bifurcated and (2) **how** the bifurcated form of the approximate solution can be obtained.

Section 5.1 details the derivation of equations describing the time-dependent behavior of perturbations about a candidate unbifurcated limit-cycle solution. If the perturbations return to zero, indicating that the candidate solution is stable, the limit cycle is of the unbifurcated form. If the perturbations grow with time, eventually reaching their own limit cycle, the limit cycle must be of the bifurcated form.

Section 5.2 describes the application of the subdomain method, a weighted-residual method, and Floquet's theory to develop a test that can be applied to determine whether a candidate approximate solution is stable. Sections 5.3 and 5.4 describe the application of Galerkin's method and a general weighted-residual method to obtain an approximate solution for the bifurcated limit cycle. This application consists of calculating an approximate solution for the perturbations about the candidate limit-cycle solution and

then adding this solution to the candidate limit-cycle solution to obtain an approximate solution of the bifurcated form. The approximate solution is compared to a numerical solutions to examine the accuracy and rate of convergence of the approximate solutions. The chapter concludes with a summary section.

5.1 Equations Describing the Perturbations about the Unbifurcated Approximate Solution

Equations describing the time-dependent perturbations about a candidate steady-state limit-cycle solution are developed in this section. The phenomenological model equations, written as a set of five nonlinear first-order differential equations, are

$$\Lambda \frac{dn(t)}{dt} = [\rho_a(t) + \alpha_T T(t)] [n(t) + 1] - \beta n(t) + \lambda c(t) , \quad (5.1)$$

$$\frac{dc(t)}{dt} = \frac{\beta}{\Lambda} n(t) - \lambda c(t) , \quad (5.2)$$

$$\frac{dT(t)}{dt} = a_1 n(t) - a_2 T(t) , \quad (5.3)$$

$$\frac{d\rho_a(t)}{dt} = Y(t) , \quad (5.4)$$

and

$$\frac{dY(t)}{dt} = -a_3 Y(t) - a_4 \rho_a(t) + KT(t) . \quad (5.5)$$

Let perturbations from this steady-state solution be given by

$$n(t) - n_{ss}(t) = \epsilon_n(t) \quad \text{or} \quad n(t) = n_{ss}(t) + \epsilon_n(t) , \quad (5.6)$$

$$c(t) - c_{ss}(t) = \epsilon_c(t) \quad \text{or} \quad c(t) = c_{ss}(t) + \epsilon_c(t) , \quad (5.7)$$

$$T(t) - T_{ss}(t) = \epsilon_T(t) \quad \text{or} \quad T(t) = T_{ss}(t) + \epsilon_T(t) , \quad (5.8)$$

$$\rho_a(t) - \rho_{ssa}(t) = \epsilon_\rho(t) \quad \text{or} \quad \rho_a(t) = \rho_{ssa}(t) + \epsilon_\rho(t) , \quad (5.9)$$

and

$$Y_{\alpha}(t) - Y_{ss\alpha}(t) = \varepsilon_y(t) \text{ or } Y_{\alpha}(t) = Y_{ss\alpha}(t) + \varepsilon_y(t) , \quad (5.10)$$

where

- $\varepsilon_n(t)$ = perturbation about normalized neutron population during limit cycle,
- $\varepsilon_c(t)$ = perturbation about precursor population during limit cycle,
- $\varepsilon_T(t)$ = perturbation about fuel temperature during limit cycle,
- $\varepsilon_p(t)$ = perturbation about void reactivity during limit cycle,
- $\varepsilon_y(t)$ = perturbation about first derivative of void reactivity during limit cycle.

Substituting Eqs. (5.6) through (5.10) into Eqs. (5.1) through (5.5) and solving for $\dot{\varepsilon}_n(t)$, $\dot{\varepsilon}_c(t)$, $\dot{\varepsilon}_T(t)$, $\dot{\varepsilon}_p(t)$, and $\dot{\varepsilon}_y(t)$ results in the following set of first-order differential equations for the perturbations from the steady-state limit-cycle solution.

$$\begin{aligned} \Lambda \dot{\varepsilon}_n(t) = & R_e + [\rho_{ss} + \alpha_T T_{ss} - \beta] \varepsilon_n(t) + (n_{ss} + 1) \varepsilon_p(t) \\ & + \alpha_T (n_{ss} + 1) \varepsilon_T(t) + \lambda \Lambda \varepsilon_c(t) + \varepsilon_p(t) \varepsilon_n(t) + \alpha_T \varepsilon_n(t) \varepsilon_T(t) , \end{aligned} \quad (5.11)$$

$$\dot{\varepsilon}_c(t) = \frac{\beta \varepsilon_n(t)}{\Lambda} - \lambda \varepsilon_c(t) , \quad (5.12)$$

$$\dot{\varepsilon}_T(t) = a_1 \varepsilon_n(t) - a_2 \varepsilon_T(t) , \quad (5.13)$$

$$\dot{\varepsilon}_p(t) = \varepsilon_y(t) , \quad (5.14)$$

and

$$\dot{\varepsilon}_y(t) = -a_3 \varepsilon_y(t) - a_4 \varepsilon_p(t) + K \varepsilon_T(t) . \quad (5.15)$$

The equilibrium point corresponding to the steady-state solution is $(\varepsilon_n, \varepsilon_c, \varepsilon_T, \varepsilon_p, \varepsilon_y) = (0, 0, 0, 0, 0)$. The asymptotic stability of these equations about this equilibrium point indicates whether the approximate steady-state limit-cycle solution is stable. If this system of equations is asymptotically stable, the unbifurcated approximate limit-cycle solution is stable. If the system is not asymptotically stable, the unbifurcated approximate limit-cycle solution is unstable and a bifurcated solution must exist.

Before examining the stability of Eqs. (5.11) through (5.15), a series of ACSL calculations were performed to examine the behavior of a linearized version of the equations.¹³⁸ After neglecting products of perturbations from the steady-state solutions and also neglecting $\mathbf{R}_e$ (which can be made negligibly small if a sufficient number of harmonics are included in the candidate solution), Eq. (5.11) becomes

$$\Lambda \dot{\mathbf{e}}_n(t) = [\rho_{ss} + \alpha_T T_{ss} - \beta] \mathbf{e}_n(t) + (n_{ss} + 1) \mathbf{e}_\rho(t) + \alpha_T (n_{ss} + 1) \mathbf{e}_T(t) + \lambda \Lambda \mathbf{e}_c(t). \quad (5.16)$$

ACSL calculations were performed by using Eqs. (5.12) through (5.16) with the phenomenological model constants listed in Table 3.1. Two cases were calculated:

$K = -0.0056 > K_{b1}$ (unbifurcated solution) and $K = -0.0060 < K_{b1}$ (bifurcated solution).

For each case, the approximate steady-state solution was of the unbifurcated form and the initial perturbation used was letting $\mathbf{e}_n(t)$ equal 1. ACSL results are shown in Figures 5.1 and 5.2. As expected, the results for $K > K_{b1}$ decay rapidly from the initial value to a small amplitude oscillation and the results for $K < K_{b1}$ grow exponentially in amplitude.

These results indicate the following.

1. For $K > K_{b1}$, the equilibrium point $(\mathbf{e}_n, \mathbf{e}_c, \mathbf{e}_T, \mathbf{e}_\rho, \mathbf{e}_y) = (0, 0, 0, 0, 0)$ is stable and, consequently, the unbifurcated approximate solution describes the stable solution of Eqs. (5.1) through (5.5).
2. For $K < K_{b1}$, the equilibrium point $(\mathbf{e}_n, \mathbf{e}_c, \mathbf{e}_T, \mathbf{e}_\rho, \mathbf{e}_y) = (0, 0, 0, 0, 0)$ is unstable and, consequently, the unbifurcated approximate solution does not describe the stable solution of Eqs. (5.1) through (5.5). Presumably, a stable bifurcated solution exists.

The failure of the ACSL results calculated by using $K = -0.0056$ to approach zero is probably due to errors in the steady-state approximate solution. Nevertheless, the basic character of the results indicates that the stability of the approximate steady-state limit-cycle solution to Eqs. (5.1) through (5.5) can be determined from the behavior of the solutions to Eqs. (5.12) through (5.16).

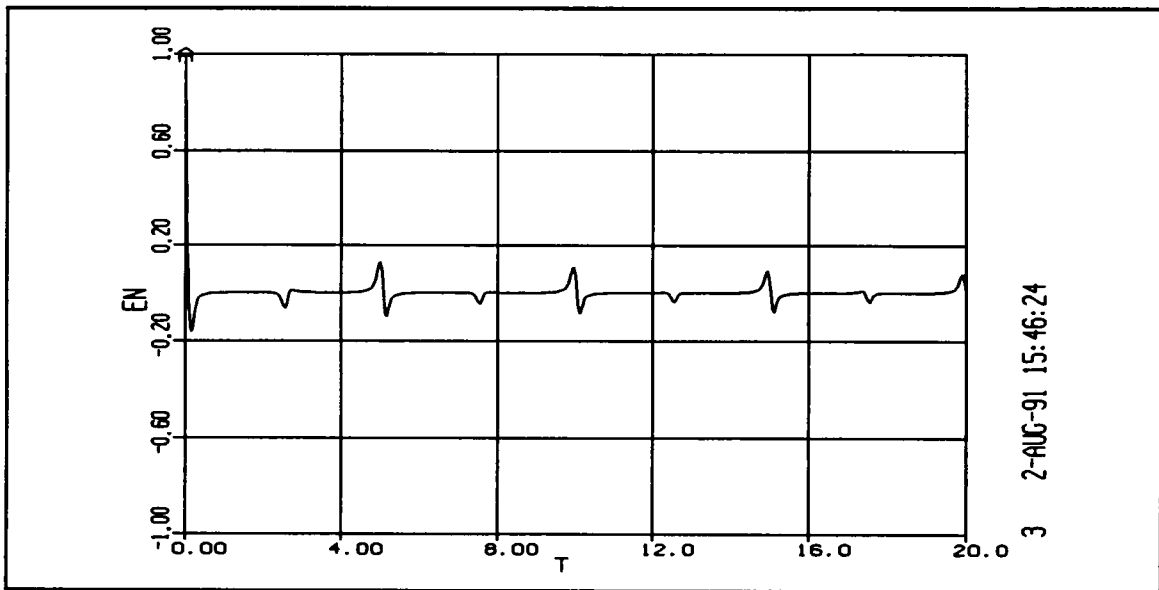


Figure 5.1. $\epsilon_n(t)$ calculated for $K = -0.0056$.

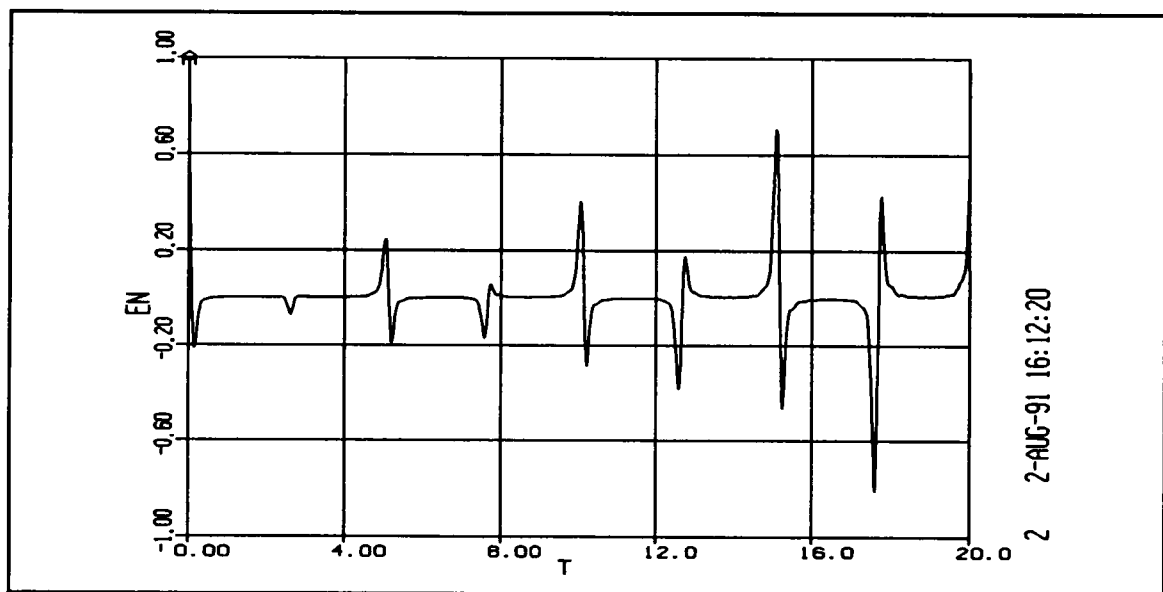


Figure 5.2. $\epsilon_n(t)$ for $K = -0.0060$.

Equations (5.12) through (5.16) are a set of linear equations; Eqs. (5.12) through (5.15) have constant coefficients, and Eq. (5.16) has harmonic coefficients (or, more accurately, a sum of harmonic coefficients). Thus, determining the asymptotic stability of this equation set is considerably more difficult than determining the asymptotic stability of a linearized form of Eqs. (5.1) through (5.5), all of which have constant coefficients. Two approaches were considered: (1) using a weighted-residual technique to develop an approximate solution and (2) applying Floquet's theory.

5.2 Development of a Bifurcation Test

This section describes the application of a weighted-residual method and Floquet's theory to develop a test that indicates whether an approximate steady-state limit-cycle solution is stable.

5.2.1 Bifurcation Test Based on Weighted-Residual Methods

The first and most critical step in applying any weighted-residual method is to choose the form of the approximate solution. The approximate solution normally consists of a sum of trial functions, which are functions of time in this case, each multiplied by a undetermined constant. The trial functions cannot be completely arbitrary but must satisfy the following conditions:

1. The trial functions must be linearly independent.
2. The trial functions must satisfy the initial or boundary conditions regardless of the values of the undetermined constants.

It is also beneficial if the trial functions form a complete set of functions, but this is not an absolute requirement. Examination of Figures 5.1 and 5.2 gives considerable insight into the form of the approximate solution. First, each case shows an exponential decay from

its initial value followed by an oscillation whose amplitude either grows indefinitely or reaches a small steady-state value. Next, the oscillation frequency is half the fundamental frequency of the candidate steady-state solution for the unbifurcated case. Finally, the phase of the oscillations is completely determined by the initial condition of $\epsilon_n(t)$; successive oscillation "peaks" occur at twice the period of the steady-state approximate solution, with the initial "peak" occurring at $t = 0$. An approximate solution possessing these features is given by

$$\epsilon_n(t) = \epsilon_{n0} \left[e^{\alpha_1 t} + A t e^{\alpha_2 t} \right] \cos(\Omega t), \quad (5.17)$$

where

α_1, α_2 , and A = undetermined constants;

$\Omega = \omega/2$;

ϵ_{n0} = initial value of $\epsilon_n(t)$;

trial functions = $e^{\alpha_1 t} \cos(\Omega t)$ and $t e^{\alpha_2 t} \cos(\Omega t)$.

Note that the trial functions are linearly independent and that the approximate solution satisfies the initial condition that $\epsilon_n(t = 0) = \epsilon_{n0}$ for all values of α_1, α_2 , and A . It is intended that the initial decay be approximated by the $e^{\alpha_1 t} \cos(\Omega t)$ term and the eventual amplitude growth, if any, be approximated by the $A t e^{\alpha_2 t} \cos(\Omega t)$ term.

Once the form of $\epsilon_n(t)$ has been selected, Eqs. (5.12) through (5.15) can be solved analytically, yielding expressions for $\epsilon_c(t)$, $\epsilon_T(t)$, and $\epsilon_\rho(t)$:

$$\begin{aligned} \epsilon_c(t) = & C_e e^{-\lambda t} + C_1 e^{\alpha_1 t} \cos(\Omega t) + C_2 e^{\alpha_1 t} \sin(\Omega t) \\ & + C_3 t e^{\alpha_2 t} \cos(\Omega t) + C_4 t e^{\alpha_2 t} \sin(\Omega t) \\ & + C_5 e^{\alpha_2 t} \cos(\Omega t) + C_6 e^{\alpha_2 t} \sin(\Omega t), \end{aligned} \quad (5.18)$$

$$\begin{aligned} \epsilon_T(t) = & T_e e^{-\alpha_2 t} + T_1 e^{\alpha_1 t} \cos(\Omega t) + T_2 e^{\alpha_1 t} \sin(\Omega t) \\ & + T_3 t e^{\alpha_2 t} \cos(\Omega t) + T_4 t e^{\alpha_2 t} \sin(\Omega t) \\ & + T_5 e^{\alpha_2 t} \cos(\Omega t) + T_6 e^{\alpha_2 t} \sin(\Omega t), \end{aligned} \quad (5.19)$$

and

$$\begin{aligned}
\varepsilon_p(t) = & \dot{c}_1 e^{-\frac{a_3 t}{2}} \cos(gt) + \dot{c}_2 e^{-\frac{a_3 t}{2}} \sin(gt) \\
& + C_c e^{-\lambda t} + R_2 e^{a_1 t} \cos(\Omega t) + R_3 e^{a_1 t} \sin(\Omega t) \\
& + R_4 e^{a_2 t} \cos(\Omega t) + R_5 e^{a_2 t} \sin(\Omega t) \\
& + R_6 t e^{a_2 t} \cos(\Omega t) + R_7 t e^{a_2 t} \sin(\Omega t) ,
\end{aligned} \tag{5.20}$$

where expressions for the constants C_1 through C_6 , T_1 through T_6 , R_1 through R_7 , and C_c , T_p , $\dot{c}_1$, $\dot{c}_2$, and g are given in Appendix C. These expressions can be substituted, along with Eq. (5.17) and its derivative, into Eq. (5.16). If Eq. (5.17) were an exact solution for $\varepsilon_n(t)$, Eq. (5.16) would be satisfied identically. Because this is an approximate solution, a discrepancy will exist between the left and right sides of Eq. (5.16) after making these substitutions. This error is the perturbation residual $\mathbf{R}_p$; it represents the error in the solution of the equations for perturbations about the candidate steady-state solution. The perturbation residual is given by

$$\begin{aligned}
\mathbf{R}_p = & z_1 e^{a_1 t} \cos(\Omega t) + z_2 e^{a_2 t} \sin(\Omega t) + z_3 e^{a_2 t} \cos(\Omega t) + z_4 e^{a_2 t} \sin(\Omega t) \\
& + z_5 t e^{a_2 t} \cos(\Omega t) + z_6 t e^{a_2 t} \sin(\Omega t) + z_7 e^{-\frac{a_3 t}{2}} \cos(gt) + z_8 e^{-\frac{a_3 t}{2}} \sin(gt) \\
& + z_9 e^{-a_3 t} + \lambda \Lambda C_c e^{-\lambda t} \\
& + \sum_{n=1}^j \{ G_{1n} e^{a_1 t} \cos(\Omega t) \sin(n\omega t) + G_{2n} e^{a_1 t} \cos(\Omega t) \cos(n\omega t) \\
& + G_{3n} e^{a_1 t} \sin(\Omega t) \sin(n\omega t) + G_{4n} e^{a_1 t} \sin(\Omega t) \cos(n\omega t) \\
& + G_{5n} e^{a_2 t} \cos(\Omega t) \sin(n\omega t) + G_{6n} e^{a_2 t} \cos(\Omega t) \cos(n\omega t) \\
& + G_{7n} e^{a_2 t} \sin(\Omega t) \sin(n\omega t) + G_{8n} e^{a_2 t} \sin(\Omega t) \cos(n\omega t) \\
& + G_{9n} t e^{a_2 t} \cos(\Omega t) \sin(n\omega t) + G_{10n} t e^{a_2 t} \cos(\Omega t) \cos(n\omega t) \\
& + G_{11n} t e^{a_2 t} \sin(\Omega t) \sin(n\omega t) + G_{12n} t e^{a_2 t} \sin(\Omega t) \cos(n\omega t) \\
& + G_{13n} e^{-\frac{a_3 t}{2}} \cos(gt) \sin(n\omega t) + G_{14n} e^{-\frac{a_3 t}{2}} \cos(gt) \cos(n\omega t) \\
& + G_{15n} e^{-\frac{a_3 t}{2}} \sin(gt) \sin(n\omega t) + G_{16n} e^{-\frac{a_3 t}{2}} \sin(gt) \cos(n\omega t) \\
& + G_{17n} e^{-a_3 t} \sin(n\omega t) + G_{18n} e^{-a_3 t} \cos(n\omega t) \} .
\end{aligned} \tag{5.21}$$

Expressions for z_1 through z_9 and G_{1n} through G_{18n} are given in Appendix C. The values of the undetermined constants are selected to minimize $\mathbf{R}_p$ in some way. The choice of a specific weighted-residual method determines how the error is minimized.

The subdomain method was selected to calculate the values of the undetermined constants. In this method, the approximate solutions satisfy Eqs. (5.12) through (5.16) in an average sense; the integral of $\mathbf{R}_p$ over each subinterval is set equal to zero. The number of subintervals equals the number of undetermined constants in the approximate solution. Thus, a set of simultaneous equations is formed which can be solved for values of the undetermined constants that cause the integral of $\mathbf{R}_p$ to equal zero in each subinterval.

A particular advantage of the subdomain method for this application is the ability to choose the boundaries of the subdomains. By selecting the first subdomain to be very near $t = 0$ (i.e., from $t = 0$ to $t = 0.02$), e^{*t} can be forced to approximate the decay from the initial condition because the remaining trial function te^{*t} will be negligibly small in this time range. The eventual growth in the solution must then be described by te^{*t} . Three subdomains are required to form the equations for the three unknowns in the approximate solution. If T_2 specifies the upper time boundary, the time intervals specifying the three subdomains are $(0, 0.02)$, $[0.02, (T_2 + 0.02)/2]$, and $[(T_2 + 0.02)/2, T_2]$. The value of T_2 was selected so each integration was performed over one or more complete cycles. Thus, T_2 equals an even-integer number of periods. The nonlinear algebraic equations for the undetermined constants are given by

$$\int_{t=0}^{0.02} \mathbf{R}_p dt = 0, \quad (5.22)$$

$$\frac{(T_2 + 0.02)}{2} \int_{0.02}^{T_2} \mathbf{R}_p dt = 0 , \quad (5.23)$$

and

$$\int_{\frac{(T_2 + 0.02)}{2}}^{T_2} \mathbf{R}_p dt = 0 . \quad (5.24)$$

The integrals in Eq. (5.21) through (5.23) were evaluated by using the MACSYMA computer code.¹⁴⁰ These expressions were incorporated into a FORTRAN computer code which used the NAG routine CO5NBF to determine values for α_1 , α_2 , and A .¹³⁹

Results from these calculations for $K = -0.0056$ (unbifurcated), $K = -0.0060$ (bifurcated), and $T_2 = 20$ s (4 cycles) are given in Table 5.1. These results show that α_1 is relatively large in magnitude and

Table 5.1. Constants in the approximate solution for $\epsilon_n(t)$

k	A	α_1	α_2
-0.0056	0.0322	-24.96	-0.0683
-0.0060	0.0336	-22.65	-0.0144

negative, as expected. These characteristics are necessary if the first term in the approximate solution is to describe the initial decay. Comparisons of the numerical and approximate solutions for each case are given in Figures 5.3 and 5.4. These particular results show good qualitative agreement between the numerical and the approximate solutions. Specifically, the approximate solution grows with time for the bifurcated case and appears to reach a steady-state solution for the unbifurcated case.

Although these results appear promising, further calculations have shown the existence of multiple solutions that presumably satisfy the conditions imposed on the approximate solution by the subdomain method. The results shown in Figures 5.3 and 5.4 indicate that the approximate solution can give a reasonable representation of the

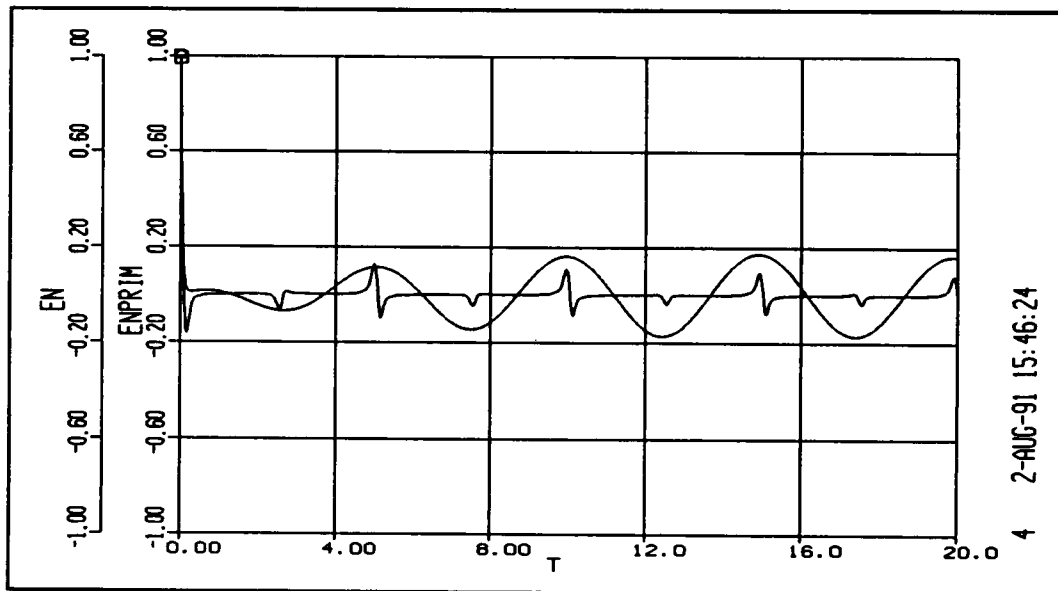


Figure 5.3. Comparison of numerical and approximate solutions for $\varepsilon_n(t)$, $K = -0.0056$.

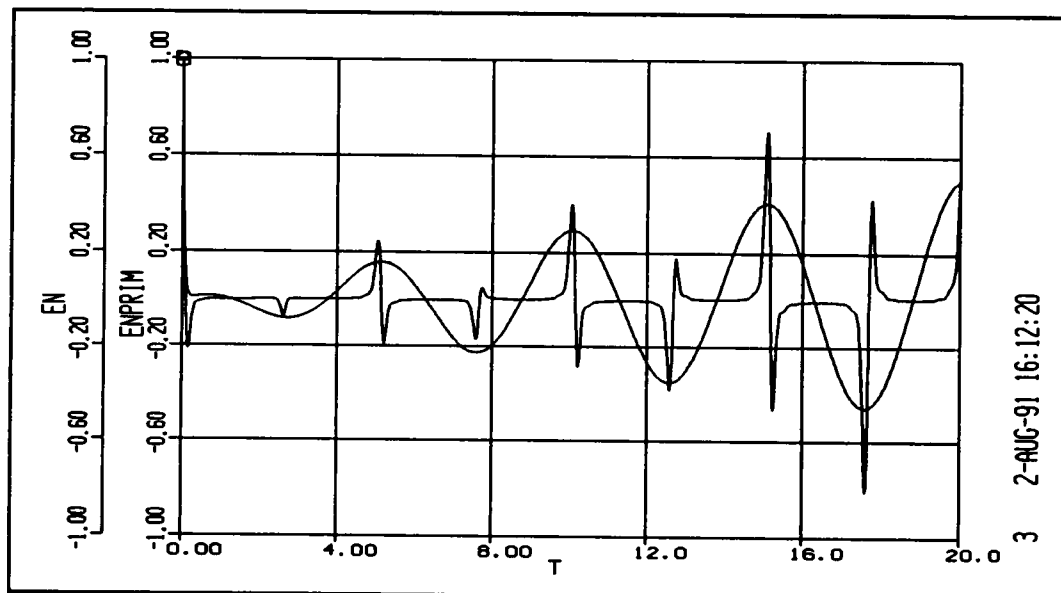


Figure 5.4. Comparison of numerical and approximate solutions for $\varepsilon_n(t)$, $K = -0.0060$.

solution; however, without knowing the exact solution, selecting the appropriate approximation from the number of approximate solutions that have been found to exist would be impossible.

An investigation of the condition of the system of equations was performed by using the collocation method. This weighted-residual method was selected for this investigation because it is the easiest weighted-residual method to apply. Two cases were investigated; the first case used an approximate solution calculated by using the condition that the residual at the collocation points equal zero. The second case used an approximate solution calculated by using the condition that the value of the approximate solution at each collocation point equals the ACSL-calculated values. The values of the approximate solution, the numerical solution, and the residual were calculated at the collocation points. Each case used a candidate 10-harmonic approximate solution calculated for $K = -0.00565$ (an unbifurcated case) and used $t = 0.01$ s, $t = 1$ period, and $t = 2$ periods as the collocation points. This selection of integer periods for the collocation points corresponds to local maximums in the numerically calculated values of $\epsilon_n(t)$.

The results from the first case showed that the residual at the collocation points was zero (not surprising, because this condition was used to determine the constants in the approximate solution) but the values of the numerical and approximate solutions differed by more than an order of magnitude at the collocation points. The results from the second case showed that the numerical and approximate solutions were identical at the collocation points (again not surprising, because this condition was used to determine the constants in the approximate solution) but the values of the residual at the collocation points was not zero and, indeed, was relatively large at the collocation points. Thus, these results show that for this system of equations, a small value of the residual does not

necessarily indicate that the approximate solution accurately represents the exact solution. On the basis of the discussion in Section 2.3, a system of equations with this characteristic is classified as a *not-well-behaved* system; the method of weighted residuals is not applicable to such systems. Thus, weighted-residual methods cannot be applied to the set of equations describing the perturbations from a candidate steady-state limit cycle. A successful bifurcation test based on Floquet's theory is described in the next subsection.

5.2.2 Bifurcation Test Based on Floquet's Theory

The development of an alternate bifurcation test based on Floquet's theory is described in this subsection. Floquet's theory is applicable to linear systems of differential equations with harmonic coefficients. This type of system can be written as

$$\dot{x} = A(t)x, \quad (5.25)$$

where $\dot{x}$ and x are the vectors of state variables and the elements of $A_{ij}(t)$ are continuous periodic functions of time with period T , that is

$$A(t+T) = A(t). \quad (5.26)$$

Floquet's theory deals with fundamental matrices. A fundamental matrix is one whose columns are composed of linearly independent solutions of a homogeneous differential equation such as Eq. (5.25) (the coefficients do not have to be functions of time for this definition). Floquet's theory¹⁴² states the following.

"If $\varphi(t)$ is a fundamental matrix of the system described by [Eqs. (5.25) and (5.26)], then $\varphi(t+T)$ is also a fundamental matrix. Moreover, for every such $\varphi(t)$ there exist a periodic nonsingular matrix $Q(t)$ with period T and a constant matrix R such that

$$\varphi(t) = Q(t)e^{R}, \quad Q(t) = Q(t+T). \quad (5.27)$$

Thus, a fundamental matrix for a system of linear equations with periodic coefficients can be written as the product of a periodic matrix of period T and a solution matrix for some system with constant coefficients. A proof of this theorem is given in reference 142. This proof shows that because $\varphi(t)$ and $\varphi(t + T)$ are fundamental matrices, they are related by the monodromy matrix C :

$$\varphi(t+T) = C \varphi(t) . \quad (5.28)$$

It is also shown that a matrix R can be found such that

$$C = e^{TR} . \quad (5.29)$$

The eigenvalues of the monodromy matrix C , denoted by λ_i , are the characteristic multipliers associated with the periodic matrix $A(t)$. The eigenvalues of R , denoted by ρ_i , are the characteristic exponents associated with the periodic matrix $A(t)$. The characteristic multipliers and characteristic exponents are related by

$$\lambda_i = e^{T \rho_i} . \quad (5.30)$$

The real part of the characteristic exponent is given by

$$\rho_i = \frac{1}{T} \log |\lambda_i| . \quad (5.31)$$

When all the characteristic exponents are distinct, the following can be shown.

1. If all the characteristic exponents have negative real parts, all solutions of Eq. (5.27) are asymptotically stable.
2. If any of the characteristic exponents have a positive real part, the system of equations given by Eq. (5.25) is unstable.
3. A purely periodic solution will exist if one of the characteristic exponents is identically zero.

Thus, the stability of the system is determined by the real parts of the characteristic exponents. The characteristic exponents can be found if either matrix R or C is available.

Unfortunately, no general method is available to obtain these matrices, so Floquet's theory cannot be used to obtain an analytical solution. However, a semianalytical method can be used to generate the monodromy matrix C , after which the characteristic exponents can be calculated and the stability of the system determined.

Note that from Eq. (5.28) the monodromy matrix relates two fundamental matrices, one at time t and the other at time $t + T$. Postmultiplying by the inverse of $\varphi(t)$,

$$\varphi(t+T) \varphi(t)^{-1} = C . \quad (5.32)$$

Because this is an initial value problem, any initial conditions that are convenient can be selected. The initial conditions will, by definition, be a solution to the equations at $t = 0$. Thus, the fundamental matrix at $t = 0$ can be the identity matrix I . The fundamental matrix at $t + T$ can be obtained by numerically integrating Eq. (5.25) for initial conditions corresponding to the columns of I over the time period $t = 0$ to $t = T$. The result of each numerical integration becomes a column in the matrix $\varphi(t + T)$. If $\varphi(t) = I$,

$$C(t) = \varphi(t+T) , \quad (5.33)$$

so the monodromy matrix is equal to the principle matrix of Eq. (5.25) evaluated at the end of one time period (the principle matrix at time t is the fundamental matrix at time t calculated by using an identity matrix as the fundamental matrix at $t = 0$).

To apply Floquet's theory, Eqs. (5.12) through (5.16) were integrated numerically by using the ACSL program. Because this is a set of five equations, five integrations were performed to generate the column vectors in the monodromy matrix. The initial conditions for integration i consisted of setting all initial values equal to 0 except for the value of variable i , which was set equal to 1. Once the monodromy matrix was obtained,

NAG routine F02AAF was used to obtain the characteristic multipliers and Eq. (5.31) was used to obtain the characteristic exponents.¹³⁹

The results for the phenomenological model with the parameter values given in Table 3.1 are shown in Figure 5.5. The phenomenological model with these parameter values is known from numerical integration to undergo its first bifurcation for a value of $K_{b1} = -0.0057$. The value of K at which one of the characteristic exponents becomes positive is approximately -0.0057 . This agreement between the calculated bifurcation points is excellent; thus, it is concluded that the application of Floquet's theory results in an accurate, though cumbersome, test indicating whether an approximate limit-cycle solution is stable.

5.3 Approximate Solution to the Perturbed Equations

The development of an approximate steady-state solution to Eqs. (5.11) through (5.15) is described in this section. The approximate solution is obtained by applying Galerkin's method, and the development proceeds along lines similar to those described in Chapters 3 and 4. The development begins with the set of four coupled, nonlinear differential equations with harmonic coefficients comprising Eqs. (5.11), (5.12), (5.13), and a second-order differential equation obtained by combining Eqs. (5.14) and (5.15):

$$\ddot{\epsilon}_p(t) + a_3 \dot{\epsilon}_p(t) + a_4 \epsilon_p(t) = K \epsilon_T(t) . \quad (5.34)$$

This set of equations describes the perturbations about the candidate steady-state solution given by $n_{ss}(t)$, $c_{ss}(t)$, $T_{ss}(t)$, and $\rho_{ss}(t)$.

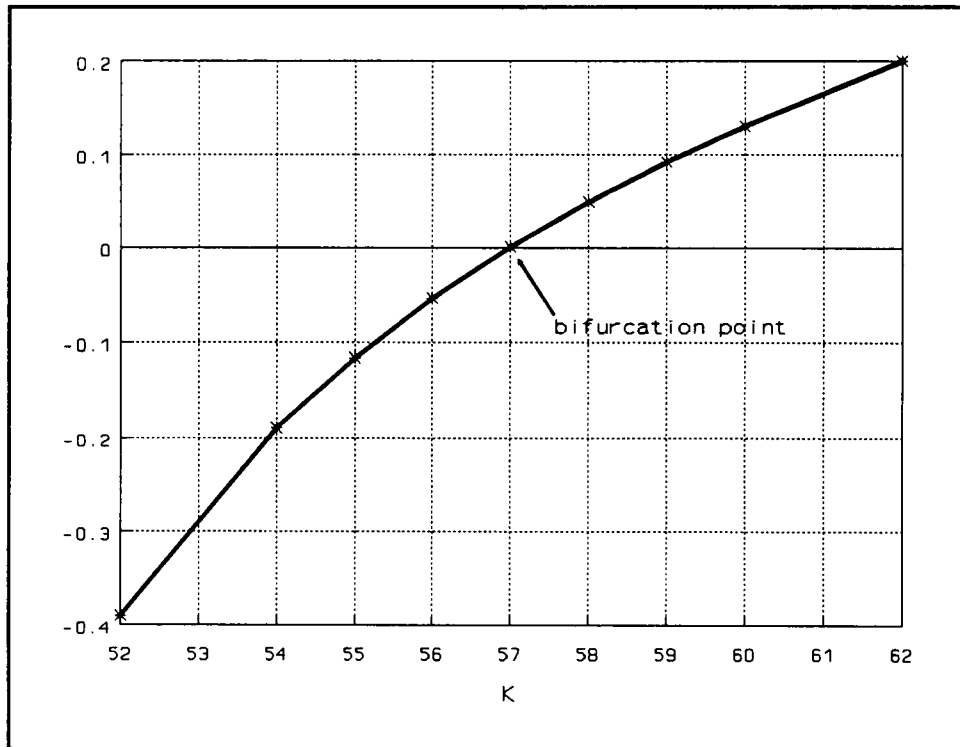


Figure 5.5. Characteristic exponent as a function of K . A bifurcation occurs when the characteristic exponent becomes positive.

5.3.1 Numerical Solution of the Equations for the Perturbations about the Limit Cycle

A numerical solution of Eqs. (5.11), (5.12), (5.13), and (5.34) was performed to examine the form of the approximate solution. The time behavior of $\epsilon_n(t)$ is shown in Figure 5.6. The calculation was performed by using the ACSL computer code with values of the phenomenological equation parameters given in Table 3.1 with $K = -0.0060$. As expected, the numerical results show the perturbation not returning to zero (because $K < K_{b1}$). The perturbations eventually establish a limit cycle about the candidate steady-state solution. Figure 5.7 shows $n_{ss}(t)$, $\epsilon_n(t)$, and $n_{ss}(t) + \epsilon_n(t)$ calculated by using

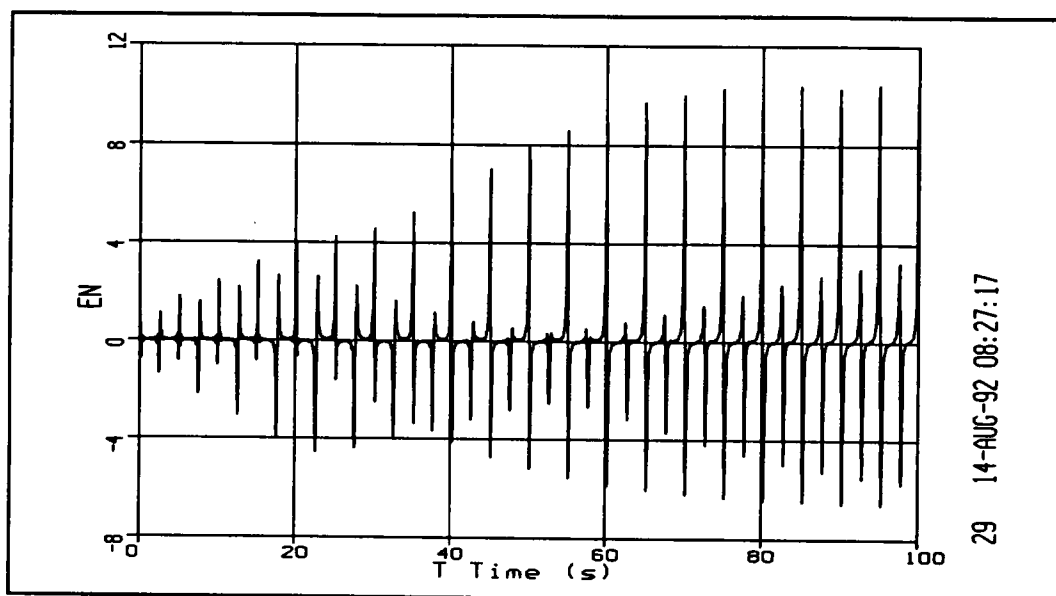


Figure 5.6. Advanced Continuous Simulation Language-calculated results showing $e_n(t)$ approaching a steady-state limit cycle, $K = -0.0060$.

the ACSL computer code. These results show that the sum of $n_{ss}(t)$ and $e_n(t)$ is a bifurcated limit cycle. Also, the frequency of the limit cycle is half that of the candidate solution. Thus, it has been shown that for $K < K_{b1}$:

1. The steady-state solution to Eqs. (5.11), (5.12), (5.13), and (5.34) is a limit cycle.
2. This limit cycle can be summed with the candidate unbifurcated approximate solution to obtain the approximate bifurcated solution to Eqs. (5.1) through (5.5).
3. The frequency of the perturbation limit cycle is half that of the candidate steady-state solution.

5.3.2 Selection of Trial Functions and Development of an Expression for the Perturbation Residual

It has been shown that the steady-state solution to Eqs. (5.11), (5.12), (5.13), and (5.34) is a limit cycle. Thus, the approximate solution for $e_n(t)$ can be expressed by using a Fourier series:

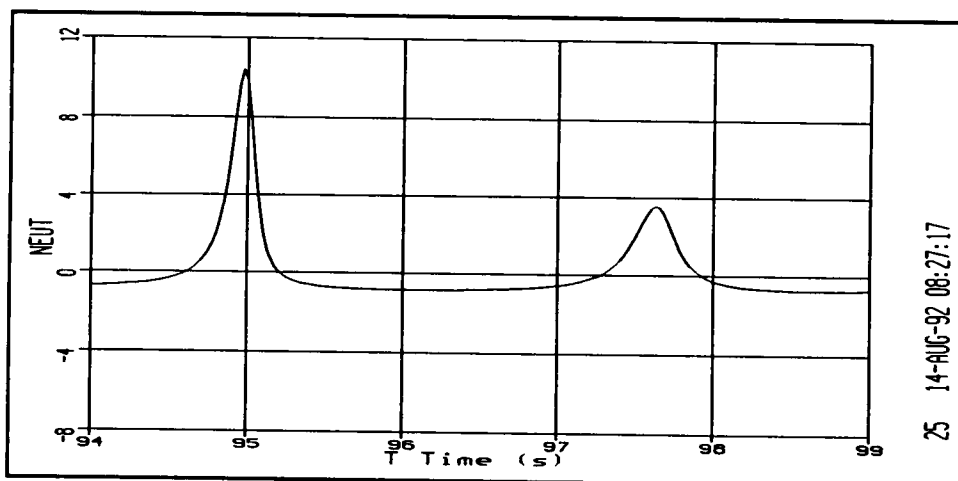
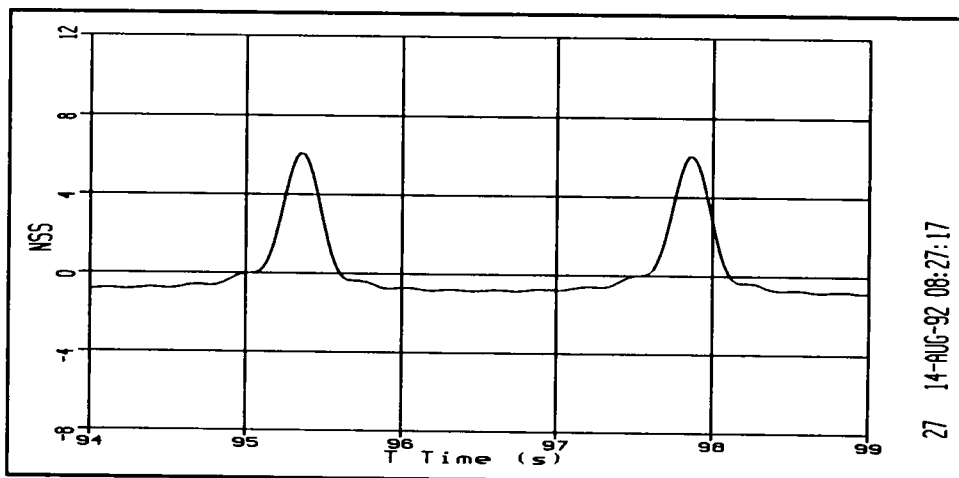
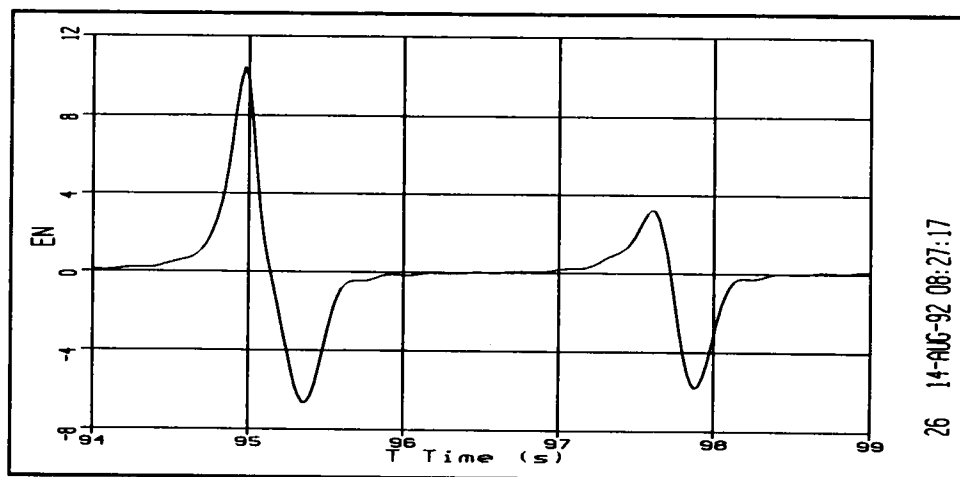


Figure 5.7. Advanced Continuous Simulation Language calculations showing $\epsilon_n(t)$, $n_n(t)$, and $\epsilon_n(t) + n_n(t)$ over one cycle, $K = -0.0060$.

$$\epsilon_n(t) = F + \sum_{n=1}^{\infty} \left\{ G_n \sin(n \omega_e t) + H_n \cos(n \omega_e t) \right\}, \quad (5.35)$$

where

F = steady-state component of the solution,
 G_n = amplitude of the sine component at a frequency of $n\omega_e$,
 H_n = amplitude of the cosine component at a frequency of $n\omega_e$,
 ω_e = fundamental oscillation frequency,
 n = an index integer.

The trial functions are the sine and cosine functions, and G_n , H_n , and F are the undetermined constants.

Equations (5.12), (5.13), and (5.34) are linear differential equations with constant coefficients. Using Eq. (3.35) for $\epsilon_n(t)$ allows these equations to be solved for $\epsilon_c(t)$, $\epsilon_T(t)$, and $\epsilon_p(t)$ during the limit cycle:

$$\begin{aligned} \epsilon_c(t) = V + \sum_{n=1}^j \left\{ (\alpha_n G_n - \delta_n H_n) \sin(n \omega_e t) \right. \\ \left. + (\delta_n G_n + \alpha_n H_n) \cos(n \omega_e t) \right\}, \end{aligned} \quad (5.36)$$

$$\begin{aligned} \epsilon_T(t) = X + \sum_{n=1}^j \left\{ (\sigma_n G_n - \tau_n H_n) \sin(n \omega_e t) \right. \\ \left. + (\tau_n G_n + \sigma_n H_n) \cos(n \omega_e t) \right\}, \end{aligned} \quad (5.37)$$

and

$$\begin{aligned} \epsilon_p(t) = W + \sum_{n=1}^j \left\{ (\zeta_n G_n + \phi_n H_n) \sin(n \omega_e t) \right. \\ \left. + (-\phi_n G_n + \zeta_n H_n) \cos(n \omega_e t) \right\}, \end{aligned} \quad (5.38)$$

where the constants α_n , δ_n , σ_n , τ_n , ζ_n , ϕ_n , V , X , and W are given in Appendix C.

An equation for the perturbation equation residual R_p is formed by substituting Eqs. (5.36), (5.37), and (5.38) as well as expressions for $n_{ss}(t)$, $c_{ss}(t)$, $T_{ss}(t)$, $\rho_{ss}(t)$, and R_e ,

written as functions of ω_e , into Eq. (5.11). Using the observation that $\omega_e = \omega/2$ allows the following trigonometric relationships to be used to write the equations for $n_{ss}(t)$, $c_{ss}(t)$, $T_{ss}(t)$, $\rho_{ss}(t)$, and R_e in terms of ω_e :

$$\sin(2\omega) = 2\sin(\omega)\cos(\omega) \quad (5.39)$$

and

$$\cos(2\omega) = \cos^2(\omega) - \sin^2(\omega) . \quad (5.40)$$

Expressions for $n_{ss}(t)$, $c_{ss}(t)$, $T_{ss}(t)$, $\rho_{ss}(t)$, and R_e , written in terms of ω_e , are

$$\begin{aligned} n_{ss}(t) = R + \sum_{n=1}^{j_{ss}} \left\{ 2P_n \sin(n\omega_e t) \cos(n\omega_e t) \right. \\ \left. - Q_n \cos^2(n\omega_e t) - Q_n \sin^2(n\omega_e t) \right\} , \end{aligned} \quad (5.41)$$

$$\begin{aligned} c_{ss}(t) = \frac{\beta R}{\lambda \Lambda} + \sum_{n=1}^{j_{ss}} \left\{ 2(F_{1n}Q_n + F_{2n}P_n) \sin(n\omega_e t) \cos(n\omega_e t) \right. \\ \left. + (F_{2n}Q_n - F_{1n}P_n) \cos^2(n\omega_e t) \right. \\ \left. - (F_{2n}Q_n - F_{1n}P_n) \sin^2(n\omega_e t) \right\} , \end{aligned} \quad (5.42)$$

$$\begin{aligned} T_{ss}(t) = \frac{a_1 R}{a_2} + \sum_{n=1}^{j_{ss}} \left\{ 2(F_{3n}Q_n + F_{4n}P_n) \sin(n\omega_e t) \cos(n\omega_e t) \right. \\ \left. + (F_{4n}Q_n - F_{3n}P_n) \cos^2(n\omega_e t) \right. \\ \left. - (F_{4n}Q_n - F_{3n}P_n) \sin^2(n\omega_e t) \right\} , \end{aligned} \quad (5.43)$$

$$\begin{aligned} \rho_{ss}(t) = \frac{Ka_1 R}{a_2 a_4} + \sum_{n=1}^{j_{ss}} \left\{ 2(F_{5n}Q_n + F_{6n}P_n) \sin(n\omega_e t) \cos(n\omega_e t) \right. \\ \left. + (F_{7n}Q_n + F_{8n}P_n) \cos^2(n\omega_e t) \right. \\ \left. - (F_{7n}Q_n + F_{8n}P_n) \sin^2(n\omega_e t) \right\} , \end{aligned} \quad (5.44)$$

and

$$\begin{aligned}
\mathbf{R}_e(t) = & \sum_{n=1}^{j_m} \sum_{k=1}^{j_m} \left\{ 4(G_{1n}Q_n + G_{2n}P_n)P_k \dot{S}_n \dot{S}_k \dot{C}_n \dot{C}_k \right. \\
& + \left[(G_{3n}Q_n + G_{4n}P_n)P_k - (G_{1k}Q_k + G_{2k}P_k)Q_n \right] (\dot{S}_k \dot{C}_n^2 \dot{C}_k - \dot{S}_n^2 \dot{S}_k \dot{C}_k) \\
& + (G_{3n}Q_n + G_{4n}P_n)Q_k \left[\dot{C}_n^2 \dot{C}_k^2 - \dot{S}_n^2 \dot{C}_k^2 - \dot{S}_k^2 \dot{C}_n^2 + \dot{S}_n^2 \dot{S}_k^2 \right] \Big\} \\
& - \sum_{k=1}^{j_m} \left[2 \left\{ (RG_{1k} + G_{5k})Q_k + \left[R(G_{2k} + G_{4k}) + G_{6k} \right] P_k \right\} \dot{S}_k \dot{C}_k \right. \\
& + \left. \left\{ [R(G_{3k} + G_{4k}) + G_{7k}]Q_k + [RG_{4k} + G_{8k}]P_k \right\} \left[\dot{C}_n^2 - \dot{S}_n^2 \right] \right] \\
& + \left[\frac{K}{a_4} + \alpha_T \right] \frac{a_1 R(R+1)}{a_2}, \tag{5.45}
\end{aligned}$$

where

$$\begin{aligned}
\dot{S}_n &= \sin(n\omega_e t), \\
\dot{C}_n &= \cos(n\omega_e t), \\
\dot{S}_k &= \sin(k\omega_e t), \\
\dot{C}_k &= \cos(k\omega_e t).
\end{aligned}$$

Substituting Eqs. (5.35) through (5.38) and (5.41) through (5.45) into Eq. (5.11) and limiting the summation to a finite number of terms yields an expression for the perturbation residual:

$$\begin{aligned}
\mathbf{R}_p = & \sum_{n=1}^{j_s} \sum_{k=1}^{j_s} \left\{ 4(G_{1n}Q_n + G_{2n}P_n)P_k \dot{S}_n \dot{S}_k \dot{C}_n \dot{C}_k + 2[(G_{3n}Q_n + G_{4n}P_n)P_k \right. \\
& + (G_{1k}Q_k + G_{2k}P_k)Q_n] \left(\dot{S}_k \dot{C}_n^2 \dot{C}_k - \dot{S}_n^2 \dot{S}_k^2 \dot{C}_k \right) \\
& + (G_{3n}Q_n + G_{4n}P_n)Q_k \left[\dot{C}_n^2 \dot{C}_k^2 - \dot{S}_n^2 \dot{C}_k^2 - \dot{S}_k^2 \dot{C}_n^2 + \dot{S}_n^2 \dot{S}_k^2 \right] \left. \right\} \\
& + (1+F+R) \left(\frac{Ka_1R}{a_2a_4} + \frac{\alpha_T a_1 R}{a_2} + W + \alpha_T X \right) - \beta F + \lambda \Lambda V + \sum_{k=1}^{j_s} \left\{ 2[Q_k(RG_{1k} + G_{5k} + FG_{1k}) \right. \\
& + P_k(R(G_{2k} + G_{9k}) + G_{6k} + W + \alpha_T X + FG_{2k})] \dot{S}_k \dot{C}_k + [Q_k(R(G_{3k} + G_{9k}) \\
& + G_{7k} + W + \alpha_T X + FG_{3k}) + P_k(RG_{4k} + G_{8k} + FG_{4k})] \left(\dot{C}_k^2 - \dot{S}_k^2 \right) \left. \right\} \quad (5.46) \\
& + \sum_{n=1}^j \left\{ \left[\frac{Ka_1R}{a_2a_4} + \frac{\alpha_T a_1 R}{a_2} - \beta + \lambda \Lambda \alpha_n + W + \alpha_T X + (1+F+R)(\alpha_T \sigma_n + \zeta_n) \right] [G_n \dot{S}_n + H_n \dot{C}_n] \right. \\
& + [n\Lambda \omega_n - \lambda \Lambda \delta_n + (1+F+R)(\phi_n - \alpha_T \tau_n)] [H_n \dot{S}_n - G_n \dot{C}_n] \left. \right\} \\
& + \sum_{n=1}^j \sum_{k=1}^{j_s} \left\{ 2[P_k(\alpha_T \sigma_n + \zeta_n + G_{2k}) + Q_k G_{1k}] (G_n \dot{S}_k \dot{S}_n \dot{C}_k + H_n \dot{S}_k \dot{C}_n \dot{C}_k) \right. \\
& + [P_k G_{4k} + (\alpha_T \sigma_n + \zeta_n + G_{3k}) Q_k] (G_n \dot{S}_n \dot{C}_k \dot{C}_k - H_n \dot{S}_k \dot{S}_n \dot{C}_k) \\
& + 2P_k(\phi_n - \alpha_T \tau_n) (H_n \dot{S}_k \dot{C}_k \dot{S}_n - G_n \dot{S}_k \dot{C}_k \dot{C}_n) \\
& + Q_k(\phi_n - \alpha_T \tau_n) (H_n \dot{C}_k \dot{C}_n \dot{S}_k + G_n \dot{C}_n \dot{S}_k \dot{S}_k - G_n \dot{C}_k \dot{C}_k \dot{C}_n - H_n \dot{S}_n \dot{S}_k \dot{S}_k) \left. \right\} .
\end{aligned}$$

Equation (5.46) gives the perturbation residual in terms of the undetermined constants in Eq. (5.35). Thus, the solution of the set of equations for the perturbations about the candidate solution has been reduced to solving for these constants, which are calculated by applying Galerkin's method.

5.3.3 Application of Galerkin's Method

Galerkin's method was selected to form the equations for the constants in the approximate solution. As in Chapters 3 and 4, Galerkin's method results in relatively simple equations for the constants because many of the terms initially appearing in the equations are identically zero when integrated over one cycle. The equations for the undetermined constants are formed by requiring that the integral of the product of the perturbation residual and a weighting function equal zero when integrated over one oscillation period. In Galerkin's method, the weighting functions are the same as the trial functions, so the equations for the undetermined constants are given by

$$\int_0^{\frac{2\pi}{\omega_e}} \mathbf{R}_p(t) dt = 0 , \quad (5.47)$$

$$\int_0^{\frac{2\pi}{\omega_e}} \mathbf{R}_p(t) \sin(n \omega_e t) dt = 0 , \quad (n = 1 \text{ to } j) , \quad (5.48)$$

and

$$\int_0^{\frac{2\pi}{\omega_e}} \mathbf{R}_p(t) \cos(n \omega_e t) dt = 0 , \quad (n = 1 \text{ to } j) . \quad (5.49)$$

Substituting Eq. (5.46) into Eqs. (5.47), (5.48), and (5.49) results in the following nonlinear algebraic equations:

$$\begin{aligned}
& \frac{2\pi}{\omega_i} \left[F \left(\frac{Ka_1 R}{a_2 a_4} + \frac{\alpha_T a_1 R}{a_2} - \beta + W + \alpha_T X \right) + \lambda \Lambda V + (W + \alpha_T X)(1+R) \right] \\
& + \sum_{n=1}^j \sum_{k=1}^{j_n} \left\{ 2 \left\{ [P_k(\alpha_T \sigma_n + \zeta_n + G_{2k}) + Q_k G_{1k}] G_n + (\phi_n - \alpha_T \tau_n) P_k H_n \right\} \int_0^{\frac{2\pi}{\omega_i}} \dot{S}_k \dot{S}_n \dot{C}_k dt \right. \\
& \quad + \left\{ H_n [P_k G_{4k} + Q_k(\alpha_T \sigma_n + \zeta_n + G_{3k})] - (\phi_n - \alpha_T \tau_n) Q_k G_n \right\} \int_0^{\frac{2\pi}{\omega_i}} \dot{C}_k^2 \dot{C}_n dt \\
& \quad \left. + \left\{ (\phi_n - \alpha_T \tau_n) Q_k G_n - H_n [P_k G_{4k} + Q_k(\alpha_T \sigma_n + \zeta_n + G_{3k})] \right\} \int_0^{\frac{2\pi}{\omega_i}} \dot{S}_k^2 \dot{C}_n dt \right\} \\
& \quad + \frac{\pi}{\omega_i} \sum_{n=1}^j \left[(\alpha_T \sigma_n + \zeta_n)(G_n^2 + H_n^2) \right] = 0 ; \tag{5.50}
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^{\frac{2\pi}{\omega_i}} \mathbf{R}_p \sin(i\omega_i t) dt = \sum_{n=1}^{j_n} \sum_{k=1}^{j_n} \left\{ 2 \left[(G_{3n} Q_n + G_{4n} P_n) P_k + (G_{1k} Q_k + G_{2k} P_k) Q_n \right] \right. \\
& \quad \times \left[\int_0^{\frac{2\pi}{\omega_i}} \dot{S}_k \dot{S}_i \dot{C}_n \dot{C}_n \dot{C}_k dt - \int_0^{\frac{2\pi}{\omega_i}} \dot{S}_n \dot{S}_n \dot{S}_k \dot{S}_i \dot{C}_k dt \right] \left. + \sum_{k=1}^{j_n} \left[2 \left\{ Q_k (R G_{1k} + G_{3k} + F G_{1k}) \right. \right. \right. \\
& \quad \left. \left. + P_k [R (G_{2k} + G_{9k}) + G_{6k} + W + \alpha_T X + F G_{2k}] \right\} \int_0^{\frac{2\pi}{\omega_i}} \dot{S}_k \dot{S}_i \dot{C}_k dt \right] \\
& \quad + \left\{ \left[\frac{Ka_1 R}{a_2 a_4} + \frac{\alpha_T a_1 R}{a_2} - \beta + \lambda \Lambda \alpha_i + W + \alpha_T X + (1+F+R)(\alpha_T \sigma_i + \zeta_i) \right] G_i \right.
\end{aligned}$$

$$\begin{aligned}
& + \left[i\Lambda \omega_e - \lambda \Lambda \zeta_i + (1 + F + R)(\phi_i - \alpha_T \tau_i) \right] H_i \left\{ \left(\frac{\pi}{\omega_e} \right) \right. \\
& + \sum_{n=1}^j \sum_{k=1}^{j_n} \left\{ 2 \left\{ \left[P_k (\alpha_T \sigma_n + \zeta_n + G_{2k}) + Q_k G_{1k} \right] H_n - P_k (\phi_n - \alpha_T \tau_n) G_n \right\} \int_0^{\frac{2\pi}{\omega_e}} \dot{S}_k \dot{S}_i \dot{C}_n \dot{C}_k dt \right. \\
& + \left\{ \left[P_k G_{4k} + Q_k (G_{3k} + \zeta_n + \alpha_T \sigma_n) \right] G_n + Q_k (\phi_n - \alpha_T \tau_n) H_n \right\} \left[\int_0^{\frac{2\pi}{\omega_e}} \dot{C}_k \dot{C}_k \dot{S}_n \dot{S}_i dt \right. \\
& \left. \left. - \int_0^{\frac{2\pi}{\omega_e}} \dot{S}_n \dot{S}_k \dot{S}_k \dot{S}_i dt \right] \right\} + \sum_{n=1}^j \sum_{k=1}^j \left\{ \left[(\zeta_k + \zeta_n + \alpha_T [\sigma_k + \sigma_n]) G_k H_n \right. \right. \\
& \left. \left. + (\phi_k - \alpha_T \tau_k)(H_n H_k - G_k G_n) - (\phi_k \tau_k) H_k H_n \right] \int_0^{\frac{2\pi}{\omega_e}} \dot{S}_k \dot{C}_n \dot{S}_i dt \right\} = 0 ; \quad (5.51)
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^{\frac{2\pi}{\omega_e}} \mathbf{R}_p \cos(i \omega_e t) dt = 0 \\
& = \sum_{n=1}^{j_n} \sum_{k=1}^{j_n} \left\{ 4 (G_{1n} Q_n + G_{2n} P_n) P_k \int_0^{\frac{2\pi}{\omega_e}} \dot{S}_n \dot{S}_k \dot{C}_n \dot{C}_k \dot{C}_i dt \right. \\
& \quad + (G_{3n} Q_n + G_{4n} P_n) Q_k \left[\int_0^{\frac{2\pi}{\omega_e}} \dot{C}_n \dot{C}_n \dot{C}_k \dot{C}_k \dot{C}_i dt \right. \\
& \quad \left. \left. - \int_0^{\frac{2\pi}{\omega_e}} \dot{S}_k \dot{S}_k \dot{C}_n \dot{C}_n \dot{C}_i dt - \int_0^{\frac{2\pi}{\omega_e}} \dot{S}_n \dot{S}_n \dot{C}_k \dot{C}_k \dot{C}_i dt + \int_0^{\frac{2\pi}{\omega_e}} \dot{S}_n \dot{S}_n \dot{S}_k \dot{S}_k \dot{C}_i dt \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{k=1}^{j_{\infty}} \left[\left\{ Q_k [R(G_{3k} + G_{9k}) + G_{7k} + W + \alpha_T X + FG_{3k}] \right. \right. \\
& \quad \left. \left. + P_k (RG_{4k} + G_{8k} + FG_{4k}) \right\} \left(\int_0^{\frac{2\pi}{\omega_s}} \dot{C}_k \dot{C}_k \dot{C}_i dt - \int_0^{\frac{2\pi}{\omega_s}} \dot{S}_k \dot{S}_k \dot{C}_i dt \right) \right] \\
& \left(\left[\frac{Ka_1 R}{a_2 a_4} + \frac{\alpha_T a_1 R}{a_2} - \beta + \lambda \Lambda \alpha_i + W + \alpha_T X + (1 + F + R)(\alpha_T \sigma_i + \zeta_i) \right] H_i \right. \\
& \quad \left. - \left[i \Lambda \omega_s - \lambda \Lambda \delta_i + (1 + F + R)(\phi_i - \alpha_T \tau_i) \right] G_i \right) \frac{\pi}{\omega_s} \\
& + \sum_{n=1}^j \sum_{k=1}^{j_{\infty}} \left\{ 2 \left[P_k (\alpha_T \sigma_n + \zeta_n + G_{2k}) + Q_k G_{1k} \right] G_n \right. \\
& \quad \left. + 2 P_k (\phi_n - \alpha_T \tau_n) H_n \right\} \int_0^{\frac{2\pi}{\omega_s}} \dot{S}_k \dot{S}_n \dot{C}_k \dot{C}_i dt + \left\{ Q_k (\phi_n - \alpha_T \tau_n) G_n \right. \\
& \quad \left. - \left[P_k G_{4k} + (\alpha_T \sigma_n + \zeta_n + G_{3k}) Q_k \right] H_n \right\} \int_0^{\frac{2\pi}{\omega_s}} \dot{S}_k \dot{S}_k \dot{C}_n \dot{C}_i dt \\
& \quad \left. + \left\{ \left[P_k G_{4k} + (\alpha_T \sigma_n + \zeta_n + G_{3k}) Q_k \right] - Q_k (\phi_n - \alpha_T \tau_n) G_n \right\} \int_0^{\frac{2\pi}{\omega_s}} \dot{C}_k \dot{C}_k \dot{C}_n \dot{C}_i dt \right\} \\
& + \sum_{n=1}^j \sum_{k=1}^j \left\{ \left[(\alpha_T \sigma_k + \zeta_k) G_k + (\phi_k - \alpha_T \tau_k) H_k \right] G_n \int_0^{\frac{2\pi}{\omega_s}} \dot{S}_k \dot{S}_n \dot{C}_i dt \right. \\
& \quad \left. + \left[(\alpha_T \sigma_k + \zeta_k) H_k - (\phi_k - \alpha_T \tau_k) G_k \right] H_n \int_0^{\frac{2\pi}{\omega_s}} \dot{C}_k \dot{C}_n \dot{C}_i dt \right\}. \tag{5.52}
\end{aligned}$$

Equations (5.50), (5.51), and (5.52) result in $2j + 1$ equations for the $2j + 1$ unknowns F , G_n , and H_n ($n = 1$ to j).

A FORTRAN computer code was written to form the simultaneous equations given by Eqs. (5.50), (5.51), and (5.52) and to solve for the constants in the approximate solution. The code input consists of the constants in the phenomenological model equations and an initial estimate of the values of the constants in the approximate solution. The code output consists of the calculated values of the constants in the approximate solution.

Approximate solutions containing 3, 5, 7, and 9 harmonics are shown in Figure 5.8, and the solution constants are listed in Table 5.2. These solutions were calculated by using Eqs. (5.50), (5.51), and (5.52) with the phenomenological model parameter values listed in Table 3.1 and by using $K = -0.00575$. This set of parameter values places the system of equations just beyond the first bifurcation point. A numerical solution using the same set of parameter values is shown in Figure 5.9. Comparison of Figures 5.8 and 5.9 shows that the approximate solution appears to be converging toward the numerical solution as more harmonics are included in the approximation but that this convergence is relatively slow. The approximation remains poor, mainly because of an insufficient number of harmonics in the approximate solution. This deficiency in the number of harmonics is also evident in the constant values listed in Table 5.2; unlike the results shown in Table 3.2, in which the higher harmonics are considerably smaller in magnitude than the lower harmonics, all the harmonics in Table 5.2 are comparable in magnitude. This behavior is characteristic of approximate solutions that do not contain enough terms to accurately represent a solution.

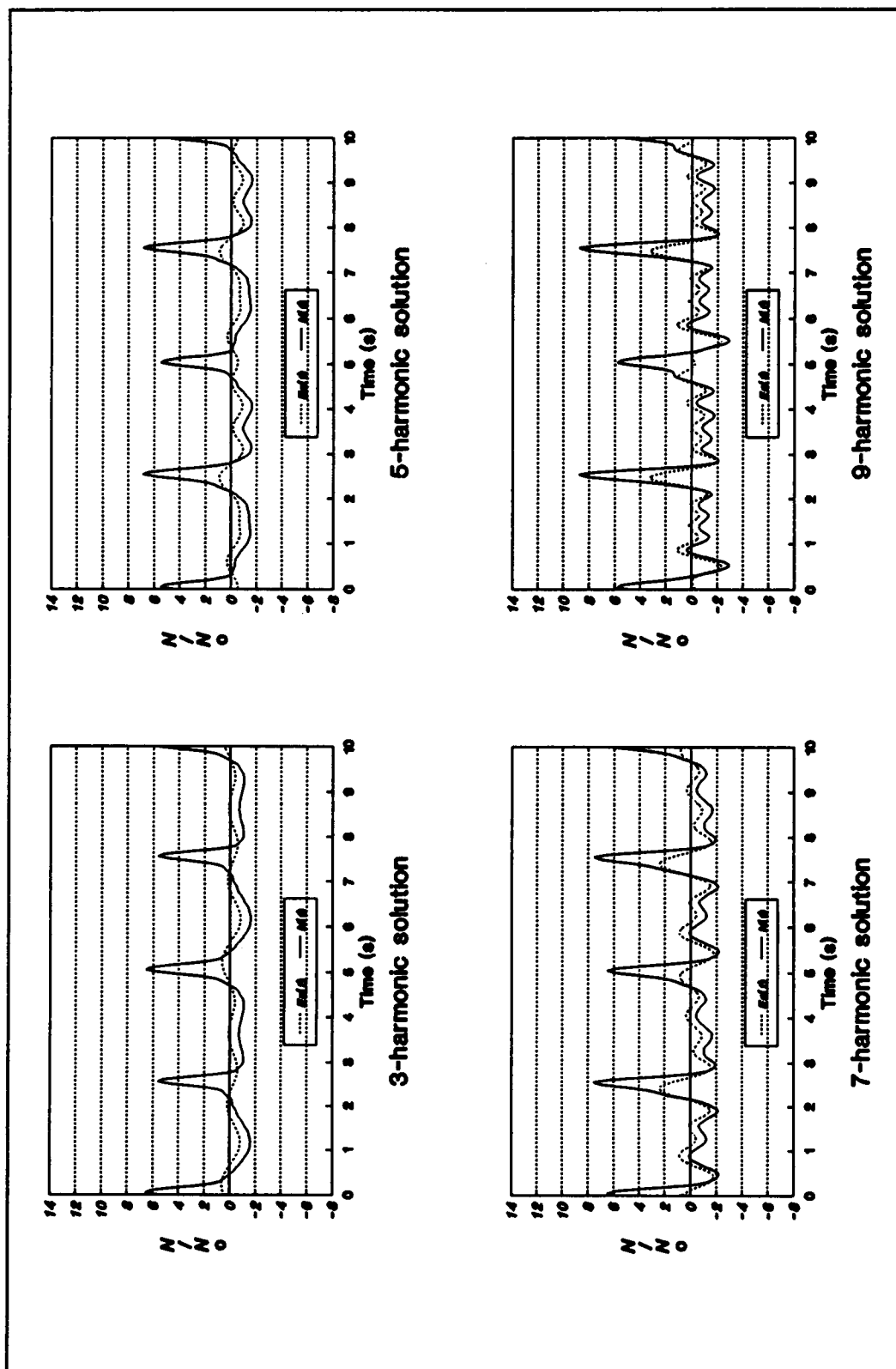


Figure 5.8. Low-order approximate solutions calculated for $K = -0.00575$.

Table 5.2 Approximate solution constants for three-, five-, seven-, and nine-harmonic approximate solutions

Constant	3	5	7	9
ω (Hz)	0.20006	0.20006	0.20006	0.20006
F	-0.13255	-0.27541	-0.19155	-0.19482
G_1	-0.05326	0.08948	0.09107	0.09048
H_1	0.18349	-0.09908	-0.08842	-0.19787
G_2	-0.01351	0.07096	-0.24234	-0.25717
H_2	0.21662	0.36131	0.21332	0.23758
G_3	0.38063	0.16962	-0.09362	-0.19083
H_3	0.21841	-0.29144	-0.47111	-0.28186
G_4		-0.18126	-0.41781	-0.37645
H_4		0.11030	0.62226	0.58272
G_5		-0.09980	0.10653	-0.04117
H_5		-0.32260	-0.01751	-0.12698
G_6			-0.40682	-0.07900
H_6			0.71913	0.71407
G_7			0.21768	0.13790
H_7			-0.07021	-0.35597
G_8				0.02792
H_8				0.14910
G_9				0.07999
H_9				-0.75772

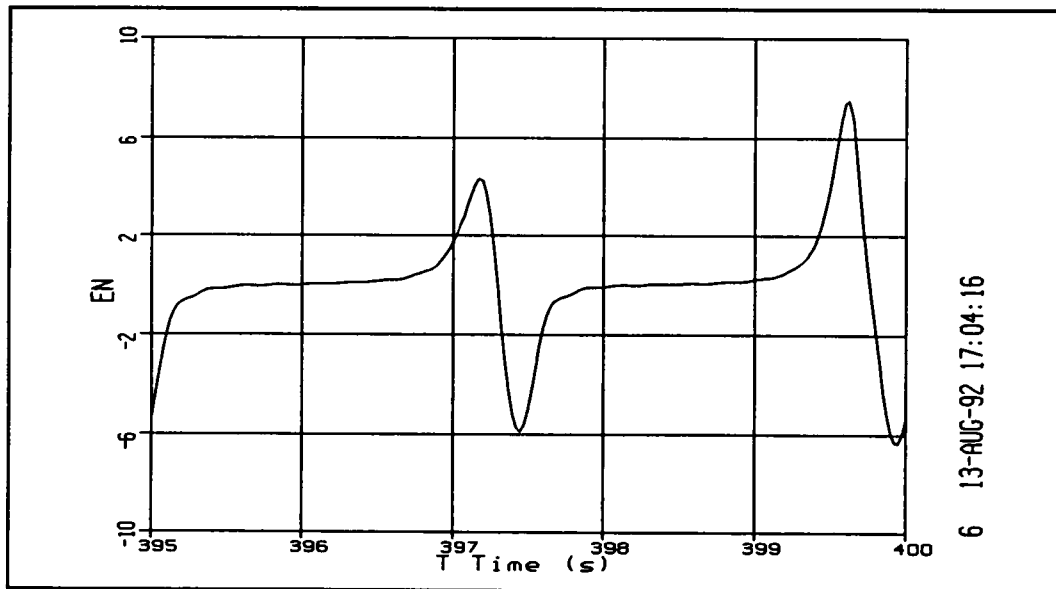


Figure 5.9. Advanced Continuous Simulation Language-calculated solution for $e_n(t)$ over approximately 1 cycle, $K = -0.00575$.

Unfortunately, it was not possible to obtain an approximate solution with more than 9 harmonics. The computer code that calculates the undetermined constants either failed to converge or converged to a solution wildly different from previously calculated solutions. These “rogue” solutions also did not approximate the numerical solution well and, in fact, were poorer approximations than was the 9-harmonic solution shown in Figure 5.8. Two explanations are likely for this result; first, because high-order solutions require more accurate initial estimates of the actual solutions for convergence, possibly an appropriate initial estimate was not used. A second explanation is that the calculations do not contain sufficient precision to calculate an approximate solution containing more than 9 harmonics. Two results that tend to support the second explanation are that when approximate solutions calculations are attempted by using only odd harmonics, rapid convergence was obtained up to a 17-harmonic solution, and an 18-harmonic solution was

obtained when the approximate solution was limited to only cosine terms. In each of these cases, the maximum number of simultaneous equations and the number of undetermined constants was 19. Thus, the number of equations that can be solved simultaneously by this particular computer code apparently are limited. A likely cause for this apparent limit is the precision limitations of the computer used to perform the calculations. In any event, this initial approach to the approximate solution can be considered only partially successful.

5.4 Application of a General Weighted-Residual Technique

A second attempt at obtaining an approximate solution using weighted-residual methods was performed by using a different choice of trial functions combined with a general weighted-residual technique. Comparison of the numerical solutions for $\epsilon_n(t)$ (Figure 5.9) and the time derivative of the candidate approximate solution $\dot{n}_{ss}(t)$ (Figure 5.10) suggests that the solution for $\epsilon_n(t)$ can be approximated by the product of $\dot{n}_{ss}(t)$ and a low-frequency oscillation with a period twice that of the candidate solution. Thus, the form of the approximate solution based on $\dot{n}_{ss}(t)$ is given by

$$\epsilon_n(t) = \sum_{k=1}^j \{AA_k \dot{S}_k + BB_k \dot{C}_k\} \dot{n}_{ss}(t) + F. \quad (5.55)$$

Substituting the first time derivative of the candidate solution for $\dot{n}_{ss}(t)$ and using trigonometric identities yields

$$\begin{aligned} \epsilon_n(t) = F + \sum_{n=1}^{j_{ss}} \sum_{k=1}^j \{n \omega_e [(AA_k P_n - BB_k Q_n) \dot{S}_{2n+k} - (AA_k P_n - BB_k Q_n) \dot{S}_{2n-k} \\ + (BB_k P_n - AA_k Q_n) \dot{C}_{2n-k} + (AA_k Q_n + BB_k P_n) \dot{C}_{2n+k}]\}. \end{aligned} \quad (5.56)$$

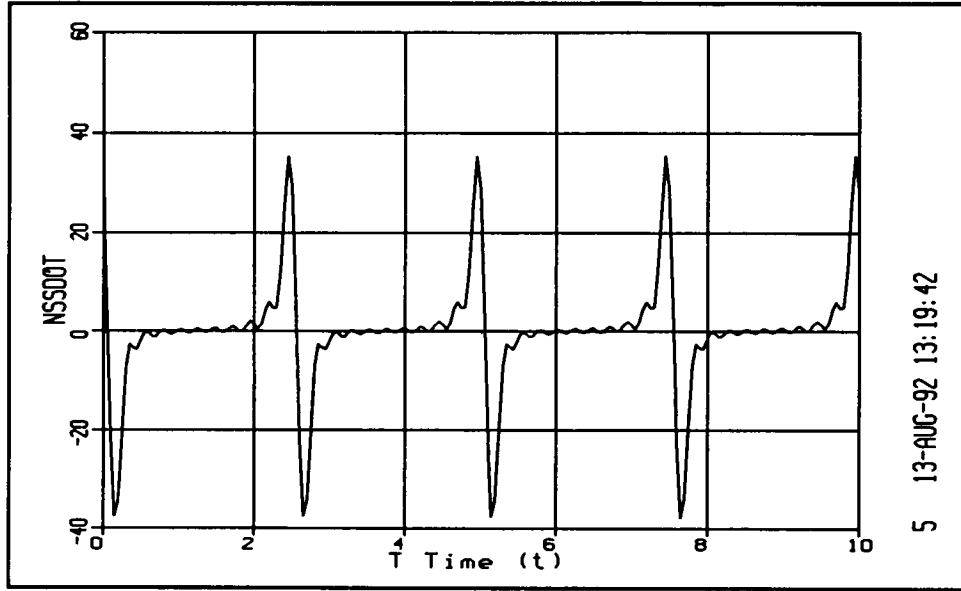


Figure 5.10. 10-harmonic approximate solution for $n_m(t)$ over approximately 4 cycles, $K = -0.00575$.

Further manipulation of Eq. (5.55) allows the removal of the double summation, yielding

$$\varepsilon_n(t) = F + \sum_{l=1}^{2j_m+j} \{G_l \dot{S}_l + H_l \dot{C}_l\}, \quad (5.57)$$

where

$$G_l = \omega_s \sum_{k_1=1}^j \sum_{n_1=1}^{j_m} \{n_1(AA_{k_1}P_{n_1} - BB_{k_1}Q_{n_1})\} - \omega_s \sum_{k_2=1}^j \sum_{n_2=1}^{j_m} \{n_2(AA_{k_2}P_{n_2} + BB_{k_2}Q_{n_2})\}, \quad (5.58)$$

$$H_l = \omega_s \sum_{k_1=1}^j \sum_{n_1=1}^{j_m} \{n_1(AA_{k_1}Q_{n_1} + BB_{k_1}P_{n_1})\} - \omega_s \sum_{k_2=1}^j \sum_{n_2=1}^{j_m} \{n_2(BB_{k_2}P_{n_2} - AA_{k_2}Q_{n_2})\}, \quad (5.59)$$

and where

$$\begin{aligned} 2n_1 + k_1 &= l, \\ 2n_2 - k_2 &= l. \end{aligned} \quad (5.60)$$

Equations (5.57) through (5.60) result in an approximate solution of the same form as that shown in Eq. (5.35) but with additional conditions placed on the constants G_i and H_i .

Thus, when Eq. (5.56) is used as the approximate solution for $\epsilon_n(t)$, expressions for $\epsilon_c(t)$, $\epsilon_T(t)$, and $\epsilon_p(t)$ are still given by Eqs. (5.36), (5.37), (5.38), and the equations in Appendix C.

Equation (5.56) describes an approximate solution of $2j_{ss} + j$ harmonics with only j unknowns. This solution has the advantage of including a large number of harmonics while requiring the calculation of only a relatively few undetermined constants. For example, a 10-harmonic candidate approximate solution with j equal to 3 results in a 23-harmonic approximate solution obtained after a calculation of only 7 constants.

Because the number of harmonics and the number of unknowns in Eq. (5.56) are not equal, a direct application of Galerkin's method cannot be performed. Instead, a general weighted-residual method, using sines and cosines as the weighting functions, was used to form the equations for the undetermined constants:

$$\int_0^{2\pi} R_p \sin((n-1)\omega_s t) dt, n = 2 \text{ to } 2j + 1 \quad (5.61)$$

and

$$\int_0^{2\pi} R_p \cos((n-1)\omega_s t) dt, n = 1 \text{ to } 2j + 1. \quad (5.62)$$

Unlike previous applications of weighted-residual methods described in this dissertation, algebraic expressions corresponding to Eq. (5.61) and (5.62) were not formed, primarily because their complexity precludes obtaining any additional information that would not be available from a numerical solution. A computer code written to form the equations for

the undetermined constants used the NAG routine D01ARF to perform the integrations shown in Eqs. (5.61) and (5.62).¹³⁹

An approximate solution for $j = 3$, calculated for $K = -0.00575$ and using the phenomenological model constants listed in Table 3.1 is shown in Figure 5.11.

Comparison of the approximate solution for $\epsilon_n(t)$ with the numerical solution of

Figure 5.9 shows that

1. the overall shape of the approximate solution is correct in that it is a series of alternating major and minor peaks separated by relatively broad troughs;
2. the difference between the peak amplitudes of the major and minor peaks is approximately the same for the approximate and the numerical solutions;
3. the major and minor peaks for the approximate solution for $\epsilon_n(t)$ are more narrow and have smaller amplitudes compared with the corresponding peaks in the approximate solution;
4. the major and minor peak amplitudes in $n(t)$ are larger for the approximate solution; and
5. the amplitude of the negative peaks vary more in the approximate solution than in the numerical solution.

These results are not surprising considering the comparison of $\epsilon_n(t)$ and $n_{ss}(t)$. The narrowness of the peaks in the approximate solution, as well as the alternating values of the negative peaks, are directly related to $n_{ss}(t)$.

Figure 5.11 shows approximate solutions calculated for $K = -0.00575$, -0.00580 , -0.00590 , and -0.00600 . Comparing these results shows that the approximate solution predicts the major peak in $\epsilon_n(t)$ to grow as K decreases while the minor peak remains at the same amplitude. This behavior causes the major peak amplitude in the approximate solution for $n(t)$ to increase faster than the minor peak amplitude as K decreases.

Clearly, the approximate solution obtained by using weighted-residual methods with an approximate solution based on $n_{ss}(t)$ cannot reproduce the bifurcated solution

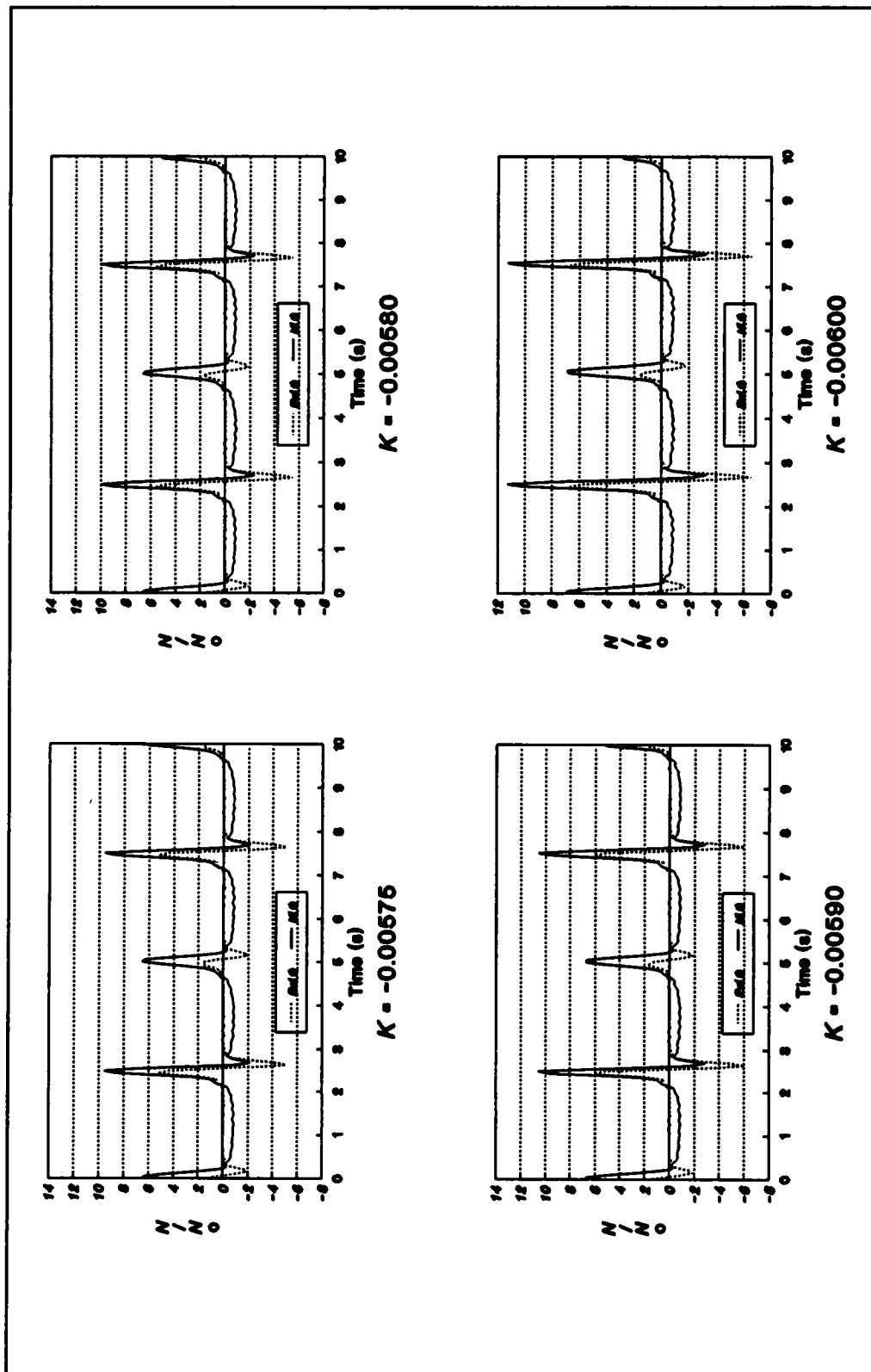


Figure 5.11. Approximate solutions for $e_a(t)$ and $n(t)$ calculated by using values of K equal to -0.00575 , -0.00580 , -0.00590 , and -0.00600 .

nearly so well as the approximations shown in Chapters 3 and 4 were able to reproduce the unbifurcated solution. The results presented here do suggest that it should be possible to form an accurate approximate solution by using a sum of known harmonic functions multiplied by an oscillating function containing relatively few undetermined constants. This was the procedure used in forming the approximation based on $n_{ss}(t)$; presumably, a solution based on the sum of $n_{ss}(t)$ and other harmonic functions (constructed so this sum has the appearance of the actual solution) would result in more accurate approximate solutions than were obtained here. To formulate such solutions, the form of the exact solution must be known.

Although it is reasonable to expect success from such a procedure, the usefulness and advantages of an approximate solution of this type are somewhat in question. Even if perfect agreement were obtained from such a solution, no additional information would be available than could be obtained from a numerical solution. Unlike the applications of weighted-residual methods described in Chapters 3 and 4, in which the low-order approximate solutions could be used to obtain information about the stability boundary or which allowed additional information to be calculated, the approximate solutions described in this chapter cannot be readily used to provide similar information, because even the low-order solutions are too complicated to yield additional insight into the nature of the solution. Thus, we can conclude that although, apparently, an accurate approximate solution containing a relatively small number of undetermined parameters can be constructed, such a solution would be more difficult to obtain and would yield no information that could not be obtained from a direct application of numerical methods.

5.5 Chapter Summary

Significant results of the work described in this chapter are as follows:

1. That the approximate steady-state solution can be either stable or unstable has been shown.
2. The stability boundary for the steady-state solution corresponds to a bifurcation point. A second, stable, bifurcated solution exists if the initial, unbifurcated, steady-state solution is unstable.
3. Linearizing the equations describing BWR dynamic behavior about the approximate steady-state solution results in a set of linear equations with harmonic coefficients. The solution of this set of equations gives the behavior of perturbations about the steady-state solution and, thus, gives the asymptotic stability of the approximate steady-state solution.
4. The linearized set of equations are not well behaved, in the sense that a small residual does not necessarily mean that the approximate solution accurately represents the exact solution. Thus, weighted-residual methods are not applicable to obtaining an approximate solution to this set of equations.
5. Floquet's theory can be applied to the linearized set of equations, resulting in an accurate but tedious semianalytical test to determine whether a candidate approximate solution is stable. A stable solution corresponds to an exact solution of the unbifurcated form, and an unstable solution corresponds to an exact solution of the bifurcated form.
6. An approximate solution to the nonlinear form of the equations for the perturbations about a candidate approximate solution can be obtained by using Galerkin's method. The method used to calculate the undetermined constants in the approximate solution failed to obtain approximate solutions with more than 9 harmonics. Low-order solutions remained too complicated to allow additional insight to be obtained that is not available from a direct application of numerical methods.
7. An approximate solution based on a trial solution consisting of the product of the first time derivative on the candidate solution for the normalized neutron population and a harmonic function produced approximate solutions that agreed reasonably well with the exact solution. The constants in the approximate solution were obtained by an application of a general weighted-residual method. This solution required the calculation of relatively few terms. Again, little advantage could be seen in obtaining approximate solutions by using weighted-residual methods because these solutions did not yield any additional information compared to a direct application of numerical methods.

6. APPROXIMATE SOLUTION DESCRIBING REGIONAL LIMIT-CYCLE OSCILLATIONS

This chapter describes the application of weighted-residual methods to investigate regional BWR stability. Regional stability is of particular concern because regional power oscillations may not be detected by the reactor safety system (which relies on signals from average-power-range monitors) until relatively large power oscillation amplitudes are reached.

It has been established that regional instabilities are caused by the first subcritical neutronics mode combined with a parallel-channel thermal hydraulic instability.^{112,113} During such instabilities, the core power and flow oscillate, with increasing power and flow in one half core and decreasing power and flow in the other half core. The spatial distribution of the power depends on the shape of the first subcritical mode, and the oscillation frequency depends on the transit time of the reactor coolant through the core. During regional power and flow oscillations, the inlet flow remains constant, unlike during corewide power and flow oscillations in which the inlet flow oscillates, with the oscillation characteristics being determined primarily by the recirculation-loop dynamics.

Two BWR dynamic models are used in this chapter; each model uses the modal point kinetics equations to include the effects of subcritical neutronics modes. The first model is a multimodal phenomenological model that is an extension of the phenomenological model used in Chapter 3. The values of the constants in this model are determined by performing a nonlinear fit to the LAPUR-calculated feedback transfer function. The second dynamic model consists of the modal point kinetics equations, which form the forward transfer function, and the LAPUR-calculated feedback transfer function, which

forms the feedback loop. This model is an extension of the dynamic model used in Chapter 4.

Section 6.1 describes the development of the multimodal phenomenological model and shows numerically calculated time-dependent behavior of this model. The approximate solution of the model equations, calculated by using Galerkin's method, is described in Section 6.2; the low- and high-order solutions are shown and described in Sections 6.3 and 6.4. The development of the second BWR dynamic model is described in Section 6.5. The low- and high-order approximate solutions are shown and described in Sections 6.6 and 6.7. Section 6.8 summarizes the significant results of this chapter.

6.1 The Multimodal Phenomenological Model

The multimodal phenomenological model is based on the modal point kinetics equations and on equations describing fuel temperature, void reactivity, and total reactivity that are identical in form to Eqs. (3.3), (3.4), and (3.5).

6.1.1 Development of the Multimodal Phenomenological Model Equations

A BWR dynamic model similar to the phenomenological model used in Chapter 3, but with the first-mode point kinetics equations replaced by the modal point kinetics equations, is given by

$$\frac{d\lambda_k(t)}{dt} = \frac{[\rho_k(t) - \rho_{0k} - \beta]}{\Lambda} \lambda_k(t) + \sum_{m=1}^M \lambda_m c_{mk}(t) , \quad (6.1)$$

$$\frac{dc_{mk}(t)}{dt} = \frac{\beta_m \lambda_k(t)}{\Lambda} - \lambda_m c_{mk}(t) , (m = 1 \text{ to } M) , \quad (6.2)$$

$$\frac{dT(t)}{dt} = a_2 T(t) + a_0 \lambda(t) , \quad (6.3)$$

$$\frac{d^2 \hat{\rho}_*(t)}{dt^2} + a_3 \frac{d \hat{\rho}_*(t)}{dt} + a_4 \hat{\rho}_*(t) = K \hat{T}(t) , \quad (6.4)$$

and

$$\rho(t) = \hat{\rho}_*(t) + \alpha_T \hat{T}(t) . \quad (6.5)$$

where $n_k(t)$ = the weighted neutron population corresponding to neutronic mode k ,
 $c_{mk}(t)$ = the precursor concentration of the m th precursor group corresponding to neutronic mode k ,
 $\hat{\rho}_k(t)$ = the total reactivity associated with neutronic mode k ,
 ρ_{0k} = the subcritical reactivity of neutronic mode k .

The remaining variables have the same definitions used in Chapter 3. Equations (6.1) and (6.2) are the modal point kinetics equations that describe the weighted neutron population and the precursor concentrations of the k th neutronic mode. Equation (6.3) describes the time-dependent fuel temperature, Eq. (6.4) describes the time-dependent void reactivity, and Eq. (6.5) gives the relationship between the total reactivity and the void reactivity and fuel temperature. The nonlinearity appears in Eq. (6.1) in the product of the terms $\rho_k(t)$ and $n_k(t)$.

The weighted neutron population can be expanded as a sum over the neutronic modes; thus, Eq. (6.3) can be rewritten

$$\frac{d \hat{T}(t)}{dt} - a_2 \hat{T}(t) = a_0 n(t) = \sum_{k=1}^{\infty} a_0 n_k(t) . \quad (6.6)$$

Because Eq. (6.6) is linear, the sum of the solutions obtained by applying the $n_k(t)$ individually equals the solution obtained by applying $n(t)$. If these partial solutions are denoted by $\hat{T}_k(t)$, then

$$\hat{T}(t) = \sum_{k=1}^{\infty} \hat{T}_k(t) . \quad (6.7)$$

The linearity of Eq. (6.6) allows equations for the $\hat{T}_k(t)$ to be formed:

$$\frac{d\hat{T}_k(t)}{dt} = a_2\hat{T}_k(t) + a_0n_k(t) . \quad (6.8)$$

A similar procedure applied to Eq. (6.4) results in the equation for the void reactivity $\hat{\rho}_{vk}$ associated with neutronic mode k ,

$$\frac{d^2\hat{\rho}_{vk}(t)}{dt^2} + a_3\frac{d\hat{\rho}_{vk}(t)}{dt} + a_4\hat{\rho}_{vk}(t) = K\hat{T}_k(t) . \quad (6.9)$$

Finally, a similar procedure applied to Eq. (6.5) results in the equation for the total reactivity ρ_{0k} of mode k ,

$$\rho_k(t) = \hat{\rho}_{vk}(t) + \alpha_T\hat{T}_k(t) . \quad (6.10)$$

Thus, a phenomenological model describing the dynamic behavior of the weighted neutron population, precursor concentrations, fuel temperature, void reactivity, and total reactivity associated with each mode is given by Eqs. (6.1), (6.2), (6.8), (6.9), and (6.10).

Equations for the perturbations from the steady-state values of $n_k(t)$, $c_{mk}(t)$, $T_k(t)$, and $\hat{\rho}_{vk}(t)$ can be found by letting

$$n_k(t) = N_0 + \tilde{n}_k(t) , \quad (6.11)$$

$$c_{mk}(t) = C_{m0} + \tilde{c}_{mk}(t) , \quad (6.12)$$

$$T_k(t) = T_0 + \tilde{T}_k(t) , \quad (6.13)$$

and

$$\hat{\rho}_{vk}(t) = \rho_{v0} + \tilde{\rho}_{vk}(t) , \quad (6.14)$$

where N_0 = steady-state value of $n(t)$,
 C_{m0} = steady-state value of $c_m(t)$,
 T_0 = steady-state value of $T(t)$,
 ρ_{v0} = steady-state value of $\hat{\rho}_{vk}(t)$,

$\tilde{n}_k(t)$ = perturbation about N_0 due to neutronic mode k ,
 $\tilde{c}_{mk}(t)$ = perturbation about C_{m0} due to neutronic mode k ,
 $T_k(t)$ = perturbation about T_0 due to neutronic mode k ,
 $\rho_{sk}(t)$ = perturbation about ρ_{s0} due to neutronic mode k .

Substituting Eq. (6.11) and (6.12) into (6.1) gives

$$\begin{aligned}
 \frac{d\tilde{n}_k(t)}{dt} = & \frac{N_0[\rho_k(t) - \rho_{0k} - \beta]}{\Lambda} + \frac{\tilde{n}_k(t)[\rho_k(t) - \rho_{0k} - \beta]}{\Lambda} \\
 & + \sum_{m=1}^M \lambda_m C_{m0} + \sum_{m=1}^M \lambda_m \tilde{c}_{mk}(t) .
 \end{aligned} \tag{6.15}$$

Equation (6.1) can be used to show that

$$0 = \frac{-(\rho_{0k} + \beta)N_0}{\Lambda} + \sum_{m=1}^M \lambda_m C_{m0} . \tag{6.16}$$

This result allows Eq. (6.15) to be simplified, leaving

$$\frac{d\tilde{n}_k(t)}{dt} = \frac{\rho_k(t)N_0}{\Lambda} + \frac{[\rho_k(t) - \rho_{0k} - \beta]\tilde{n}_k(t)}{\Lambda} + \sum_{m=1}^M \lambda_m \tilde{c}_{mk}(t) . \tag{6.17}$$

Normalizing by N_0 results in

$$\frac{dn_k(t)}{dt} = \frac{[\rho_k(t) - (\beta + \rho_{0k})]n_k(t)}{\Lambda} + \frac{\rho_k(t)}{\Lambda} + \sum_{m=1}^M \lambda_m c_{mk}(t) , \tag{6.18}$$

where

$$n_k(t) = \frac{\tilde{n}_k(t)}{N_0} \tag{6.19}$$

and

$$c_{mk}(t) = \frac{\tilde{c}_{mk}(t)}{N_0} . \tag{6.20}$$

An equation for the precursor concentration perturbations about C_{m0} is obtained by substituting Eqs. (6.11) and (6.12) into Eq. (6.2):

$$\frac{d\epsilon_{mk}(t)}{dt} = \frac{\beta_m N_0}{\Lambda} + \frac{\beta_m \tilde{n}_k(t)}{\Lambda} - \lambda_m C_{m0} - \lambda_m \epsilon_{mk}(t) . \quad (6.21)$$

Equation (6.2) can be used to show that $\frac{\beta N_0}{\Lambda} - \lambda_m C_{m0} = 0$. This result allows Eq. (6.21) to be simplified, leaving

$$\frac{d\epsilon_{mk}(t)}{dt} = \frac{\beta_m \tilde{n}_k(t)}{\Lambda} - \lambda_m \epsilon_{mk}(t) . \quad (6.22)$$

Normalizing by N_0 results in

$$\frac{dc_{mk}(t)}{dt} = \frac{\beta_m n_k(t)}{\Lambda} - \lambda_m c_{mk}(t) . \quad (6.23)$$

An equation for the fuel temperature perturbations about T_0 is obtained by substituting Eqs. (6.11) and (6.13) into Eq. (6.8):

$$\frac{dT_k(t)}{dt} = a_2 T_0 + a_2 T_k(t) + a_0 N_0 + a_0 \tilde{n}_k(t) . \quad (6.24)$$

Equation (6.8) can be used to show that $a_2 T_0 + a_0 N_0 = 0$. This result allows Eq. (6.24) to be simplified, leaving

$$\frac{dT_k(t)}{dt} = a_1 n_k(t) + a_2 T_k(t) , \quad (6.25)$$

where $a_1 = N_0 a_0$. Similarly, an equation for the void reactivity perturbations about ρ_{s0} is obtained by substituting Eqs. (6.13) and (6.14) into Eq. (6.9):

$$\frac{d^2 \rho_{sk}(t)}{dt^2} + a_3 \frac{d\rho_{sk}(t)}{dt} + a_4 \rho_{s0} + a_4 \rho_{sk}(t) = K T_0 + K T_k(t) . \quad (6.26)$$

Equation (6.9) can be used to show that $a_4 \rho_{s0} = KT_0$. This result allows Eq. (6.26) to be simplified, leaving

$$\frac{d^2 \rho_{sk}(t)}{dt^2} + a_3 \frac{d \rho_{sk}(t)}{dt} + a_4 \rho_{sk}(t) = KT_k(t) . \quad (6.27)$$

Finally, an equation for the perturbations in total reactivity is obtained by substituting Eqs. (6.13) and (6.14) into Eq. (6.10):

$$\rho_k(t) = \rho_{s0} + \rho_{sk}(t) + \alpha_T T_0 + \alpha_T T_k(t) . \quad (6.28)$$

Equation (6.10) can be used to show that $\rho_{s0} + \alpha_T T_0 = 0$. This result allows Eq. (6.28) to be simplified, leaving

$$\rho_k(t) = \rho_{sk}(t) + \alpha_T T_k(t) . \quad (6.29)$$

The total reactivity can be eliminated from the model equations by substituting Eq. (6.29) into Eq. (6.18):

$$\begin{aligned} \frac{dn_k(t)}{dt} = & \frac{\left[\rho_{sk}(t) + \alpha_T T_k(t) - (\beta + \rho_{0k}) \right]}{\Lambda} n_k(t) \\ & + \frac{\rho_{sk}(t) + \alpha_T T_k(t)}{\Lambda} + \sum_{m=1}^M \lambda_m c_{mk}(t) . \end{aligned} \quad (6.30)$$

Thus, the multimodal phenomenological model equations describing the time dependence of perturbations of the normalized neutron population, the precursor concentrations, the fuel temperature, and the void reactivity, all associated with mode k , about their steady-state values consist of Eqs. (6.23), (6.25), (6.27), and (6.30). These equations are of the same form as the phenomenological model equations described in Chapter 3, with the exception that β in the original model is replaced by $\beta + \rho_{0k}$ in the current model. The

multimodal phenomenological model reduces to the original model if $\rho_{0k} = 0$ and $k = 1$; thus, this model is an extension of the model described in Chapter 3.

6.1.2 Time-Dependent Behavior of the Multimodal Phenomenological Model Equations

The ACSL program was used to numerically integrate Eqs. (6.23), (6.25), (6.27), and (6.30) to investigate the time behavior of the normalized neutron population.¹³⁸ The model constants were calculated by using the LAPUR-calculated transfer function corresponding to a model of the Ringhals-1 BWR power station. This particular power station model was selected for these calculations because regional oscillations were observed at this plant under the modeled conditions.⁸⁰ The multimodal phenomenological model constants, calculated for both the inlet flow determined by the recirculation-loop dynamics (i.e., for inlet flow conditions satisfied by the corewide instability mode) and for constant inlet flow (i.e., for inlet flow conditions satisfied by the regional instability mode), are listed in Table 6.1. Note that all values of the fitted constants change because of the change in inlet flow conditions. ACSL-calculated numerical results for each set of constants, calculated by using a value of \$1.0 for the subcritical reactivity of the first subcritical mode, are shown in Figures 6.1 and 6.2. As expected, these results show that the corewide mode is stable and the regional mode is unstable. Decreasing the value of K without changing the remaining equation constants allows the behavior of the system to be examined as the equations become more unstable. ACSL-calculated values of $n_k(t)$ for the regional oscillation mode of the Ringhals-1 power station model calculated by using values of K equal to -0.04494 , -0.050 , -0.060 , and -0.063 are shown in Figures 6.3 through 6.6. Figure 6.3 shows that $n_k(t)$ is nearly sinusoidal for very low amplitude limit cycles. Figures 6.3 through 6.5 show that the limit-cycle amplitude increases as the system of

**Table 6.1. Multimodal phenomenological model constants for the
Ringhals-1 power station calculated for conditions
corresponding to regional oscillations.**

Constant	Corewide	Regional
a_1 ($K \cdot s^{-1}$)	0.5917	2.0690
a_2 (s^{-1})	0.2496	0.3266
a_3 (s^{-1})	2.4880	1.0180
a_4 (s^{-2})	8.3970	8.8700
K	-1.332×10^{-1}	-4.739×10^{-2}
Λ (s^{-1})	3.290×10^{-5} ,	
α_T (K^{-1})	-2.640×10^{-5} ,	
β_1, λ_1 (s^{-1})	1.950×10^{-4} , 0.0127	
β_2, λ_2 (s^{-1})	1.100×10^{-3} , 0.0317	
β_3, λ_3 (s^{-1})	9.670×10^{-4} , 0.1150	
β_4, λ_4 (s^{-1})	2.090×10^{-3} , 0.3310	
β_5, λ_5 (s^{-1})	6.580×10^{-4} , 1.4000	
β_6, λ_6 (s^{-1})	1.340×10^{-4} , 3.8700	

equations becomes more unstable. Figure 6.6 shows that the regional instability mode will undergo a period-doubling bifurcation if K is sufficiently negative. Thus, the numerical results indicate that regional limit-cycle oscillations, modeled by using the multimodal phenomenological model equations, behave essentially the same as the corewide limit-cycle oscillations described in Chapter 3.

6.2 The Approximate Solution to the Multimodal Phenomenological Model Equations

An approximate solution to the multimodal phenomenological model equations is described in this section. The approximate solution is limited to predicting steady-state, limit-cycle oscillations.

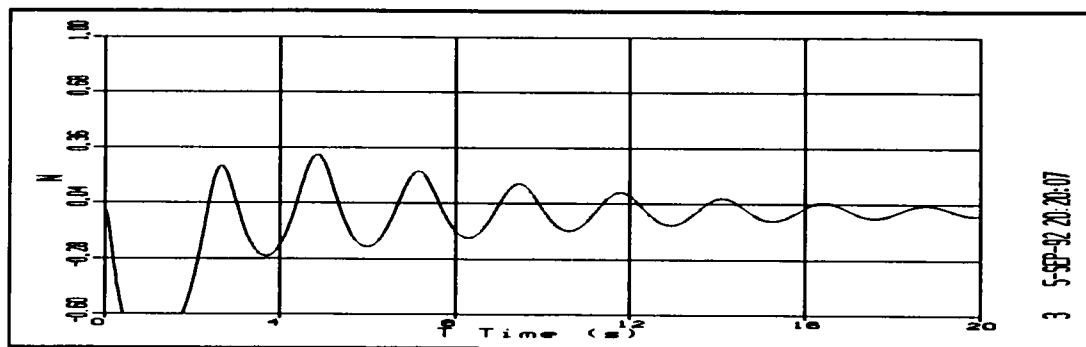


Figure 6.1. Advanced Continuous Simulation Language-calculated behavior of $n(t)$ for the Ringhals-1 corewide mode, $K = -1.332 \times 10^{-1}$.

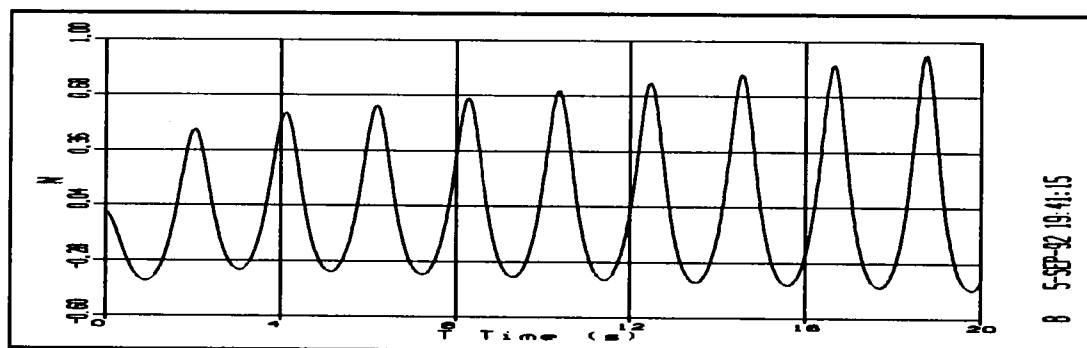


Figure 6.2. Advanced Continuous Simulation Language-calculated behavior of $n(t)$ for the Ringhals-1 regional mode, $K = -4.739 \times 10^{-2}$.

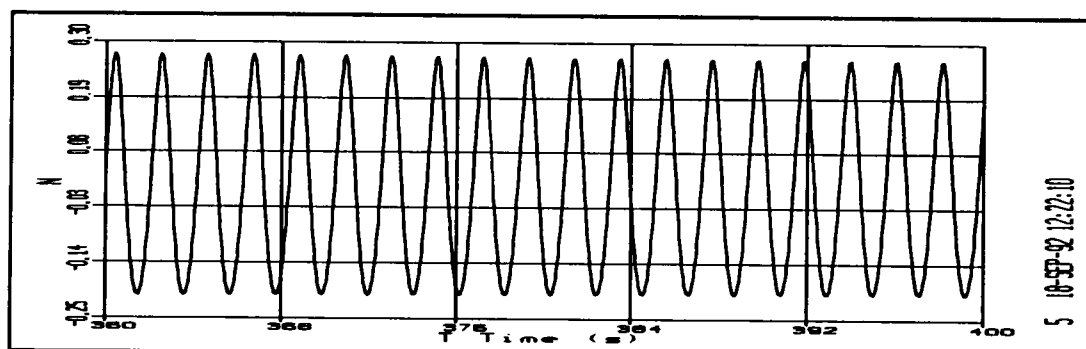


Figure 6.3. Advanced Continuous Simulation Language-calculated behavior of $n(t)$ for the Ringhals-1 regional mode showing a nearly sinusoidal oscillation, $K = -4.494 \times 10^{-2}$.

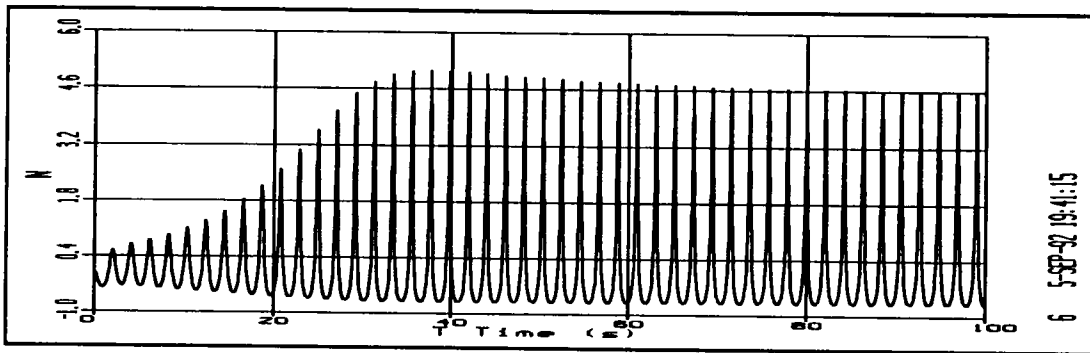


Figure 6.4. Advanced Continuous Simulation Language-calculated behavior of $n(t)$ for the Ringhals-1 regional mode, $K = -5.000 \times 10^{-2}$.

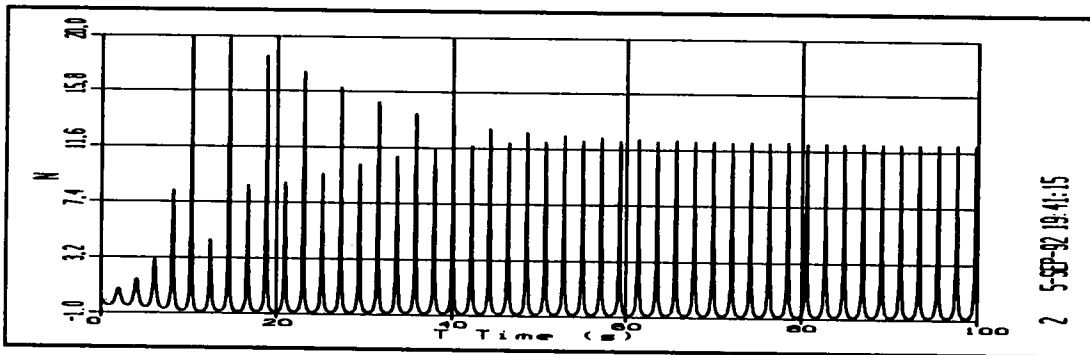


Figure 6.5. Advanced Continuous Simulation Language-calculated behavior of $n(t)$ for the Ringhals-1 regional mode, $K = -6.000 \times 10^{-2}$.

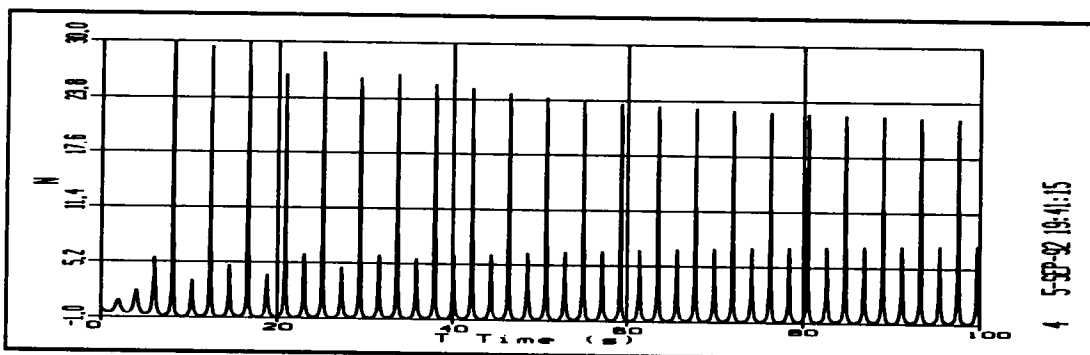


Figure 6.6. Advanced Continuous Simulation Language-calculated behavior of $n(t)$ for the Ringhals-1 regional mode showing a bifurcated limit cycle, $K = -6.300 \times 10^{-2}$.

6.2.1 Development of an Expression for the Equation Residual

Because the modal phenomenological model equations are of the same form as the phenomenological model equations used in Chapter 3, an identical procedure can be used to obtain an approximate solution. During limit-cycle oscillations, let the normalized neutron population $n_{ssk}(t)$, with precursor concentration $c_{ssmk}(t)$, fuel temperature $T_{ssk}(t)$, and void reactivity $\rho_{ssak}(t)$ be expanded in Fourier series:

$$n_{ssk}(t) = R_k + \sum_{n=1}^j \{P_{nk} S_{nk} + Q_{nk} C_{nk}\}, \quad (6.31)$$

$$c_{ssmk}(t) = W_{mk} + \sum_{n=1}^j \{\sigma_{mn1k} S_{nk} + \sigma_{mn2k} C_{nk}\}, \quad (6.32)$$

$$T_{ssk}(t) = U_k + \sum_{n=1}^j \{\tau_{n1k} S_{nk} + \tau_{n2k} C_{nk}\}, \quad (6.33)$$

and

$$\rho_{ssak}(t) = V_k + \sum_{n=1}^j \{R_{n1k} S_{nk} + R_{n2k} C_{nk}\}, \quad (6.34)$$

where $S_{nk} = \sin(n\omega_k t)$,
 $C_{nk} = \cos(n\omega_k t)$,
 ω_k = oscillation frequency of mode k ,
 k = an integer designating the mode number,
 m = an integer designating the precursor,
 j = the number of harmonics used in the expansion,
 $R_k, W_{mk}, U_k, V_k, P_{nk}, Q_{nk}, \sigma_{mn1k}, \sigma_{mn2k}, \tau_{n1k}, \tau_{n2k}, R_{n1k}, R_{n2k}$ = constants.

Equations for $W_{mk}, U_k, V_k, \sigma_{mn1k}, \sigma_{mn2k}, \tau_{n1k}, \tau_{n2k}, R_{n1k}$, and R_{n2k} in terms of the R_k, P_{nk} , and Q_{nk} are obtained by solving Eqs. (6.23), (6.25), (6.27), and (6.30) after substituting Eq. (6.31) for $n_k(t)$. These equations are given in Appendix D. Thus, the approximate solution will be completely determined if values are obtained for the R_k, P_{nk} , and Q_{nk} .

Substituting Eqs. (6.31) through (6.34) into Eq. (6.30) and solving for the equation residual R_{ek} for mode k ,

$$\begin{aligned}
R_{ek} &= (V_k + \alpha_T U_k)(R_k + 1) + \sum_{m=1}^M \lambda_m \Lambda W_{mk} - (\beta + \rho_{0k})R_k \\
&+ \sum_{n=1}^j \left\{ S_{nk} \left\{ P_{nk} [V_k + \alpha_T U_k - (\beta + \rho_{0k})] + (R_{n1k} + \alpha_T \tau_{n1k})(R_k + 1) \right. \right. \\
&\quad \left. \left. + n \Lambda \omega_k Q_k + \sum_{m=1}^M \lambda_m \Lambda \sigma_{mn1k} \right\} \right. \\
&+ C_{nk} \left\{ Q_{nk} [V_k + \alpha_T U_k - (\beta + \rho_{0k})] + (R_{n2k} + \alpha_T \tau_{n2k})(R_k + 1) \right. \\
&\quad \left. \left. - n \Lambda \omega_k P_{nk} + \sum_{m=1}^M \lambda_m \Lambda \sigma_{mn2k} \right\} \right\} \\
&+ \sum_{n=1}^j \sum_{l=1}^j \left[(R_{n1k} + \alpha_T \tau_{n1k})(P_{lk} S_{lk} S_{nk} + Q_{lk} S_{nk} C_{lk}) \right. \\
&\quad \left. + (R_{n2k} + \alpha_T \tau_{n2k})(P_{lk} S_{lk} C_{nk} + Q_{lk} C_{lk} C_{nk}) \right].
\end{aligned} \tag{6.35}$$

Equations for the R_k , ω_k , P_{nk} , and Q_{nk} are obtained by applying Galerkin's method.

6.2.2 Equations for the Undetermined Constants

Equations for the undetermined constants are developed by requiring the weighted integral of the equation residual, evaluated over a suitable interval, to equal zero. This requirement results in values of the constants that force the approximate solution to satisfy the model equations in an average sense over the interval. For this case, the interval is one cycle (i.e., from $t = 0$ to $t = 2\pi/\omega_k$).

Galerkin's method was selected to form the equations for the undetermined constants because its application results in relatively compact equations for the undetermined constants and because it was successfully applied to similar problems in both Chapters 3 and 4. In Galerkin's method, the weight functions are the same as the trial functions; thus, the equations for the undetermined constants are given by

$$\frac{2\pi}{\omega_k} \int_0^{\frac{2\pi}{\omega_k}} \mathbf{R}_{ek} dt = 0, \quad (6.36)$$

$$\frac{2\pi}{\omega_k} \int_0^{\frac{2\pi}{\omega_k}} \mathbf{R}_{ek} S_{ik} dt = 0, \quad (i = 1 \text{ to } j), \quad (6.37)$$

and

$$\frac{2\pi}{\omega_k} \int_0^{\frac{2\pi}{\omega_k}} \mathbf{R}_{ek} C_{ik} dt = 0, \quad (i = 1 \text{ to } j). \quad (6.38)$$

Equations (6.36), (6.37), and (6.38) will generate $2j + 1$ equations as i is incremented from 1 to j . By setting $Q_{jk} = 0$, fixing the absolute phase of the approximate solution, the number of unknowns is reduced to $2j + 1$. Substituting Eq. (6.35) into Eqs. (6.36), (6.37), and (6.38) results in the following equations that can be solved simultaneously for the values of the R_k , P_{nk} , and Q_{nk} :

$$\begin{aligned} \frac{2\pi}{\omega_k} \int_0^{\frac{2\pi}{\omega_k}} \mathbf{R}_{ek} dt = 0 = & 2 \left[(V_k + \alpha_T U_k)(R_k + 1) + \sum_{m=1}^M \lambda_m \Lambda W_{mk} - (\beta + \rho_{0k}) R_k \right] \\ & + \sum_{n=1}^j \left[(R_{n1k} + \alpha_T \tau_{n1k}) P_{nk} + (R_{n2k} + \alpha_T \tau_{n2k}) Q_{nk} \right], \end{aligned} \quad (6.39)$$

$$\begin{aligned} \frac{2\pi}{\omega_k} \int_0^{\frac{2\pi}{\omega_k}} \mathbf{R}_{ek} S_{ik} dt = 0 = & \left\{ P_{ik} \left[V_k + \alpha_T U_k - (\beta + \rho_{0k}) \right] + (R_{i1k} + \alpha_T \tau_{i1k})(R_k + 1) + i \Lambda \omega_k Q_{ik} + \sum_{m=1}^M \lambda_m \Lambda \sigma_{mi1k} \right\} \left(\frac{\pi}{\omega_k} \right) \\ & + \sum_{n=1}^j \sum_{l=1}^j \left[(R_{n1k} + \alpha_T \tau_{n1k}) Q_{lk} \int_0^{\frac{2\pi}{\omega_k}} S_{nk} S_{ik} C_{lk} dt + (R_{n2k} + \alpha_T \tau_{n2k}) P_{lk} \int_0^{\frac{2\pi}{\omega_k}} S_{lk} S_{ik} C_{nk} dt \right], \end{aligned} \quad (6.40)$$

and

$$\begin{aligned}
& \int_0^{\frac{2\pi}{\omega_k}} \mathbf{R}_{ek} C_{ik} dt = 0 \\
& = \left\{ \mathcal{Q}_{ik} \left[V_k + \alpha_T U_k - (\beta + \rho_{0k}) \right] + (R_{i2k} + \alpha_T \tau_{i2k})(R_k + 1) - i\Lambda \omega_k P_{ik} + \sum_{m=1}^M \lambda_m \Lambda \sigma_{mi1k} \right\} \left(\frac{\pi}{\omega_k} \right) \\
& + \sum_{n=1}^j \sum_{l=1}^j \left[(R_{n1k} + \alpha_T \tau_{n1k}) P_{lk} \int_0^{\frac{2\pi}{\omega_k}} S_{nk} S_{lk} C_{ik} dt + (R_{n2k} + \alpha_T \tau_{n2k}) \mathcal{Q}_{lk} \int_0^{\frac{2\pi}{\omega_k}} C_{lk} C_{nk} C_{ik} dt \right].
\end{aligned} \tag{6.41}$$

6.3 The Low-Order Approximate Solution of the Multimodal Phenomenological Model Equations

Investigation of the solution of the multimodal phenomenological model equations near the stability boundary by using the low-order approximate solution is described in this section. This investigation includes the determination of a stability condition as well as the development of expressions for ω_k , R_{1k} , and P_{1k} near the stability boundary.

Figure 6.3 shows that the regional oscillations near the stability boundary will be essentially sinusoidal and can be accurately approximated by a single harmonic. Thus, the approximate solution for $n_{ssk}(t)$ near the stability boundary is given by

$$n_{ssk}(t) = R_k + P_k \sin(\omega_k t). \tag{6.42}$$

Setting $k = 1$ in Eqs. (6.39), (6.40), and (6.41) and using the integrals listed in Appendix B results in the following equations for the constants R_k , P_{1k} , and ω_k :

$$0 = 2 \left[(V_k + \alpha_T U_k)(R_k + 1) + \sum_{m=1}^M \lambda_m \Lambda W_{mk} - (\beta + \rho_{0k}) R_k \right] + (R_{11k} + \alpha_T \tau_{11k}) P_{1k}, \tag{6.43}$$

$$0 = P_{1k} [V_k + \alpha_T U_k - (\beta + \rho_{0k})] + (R_{11k} + \alpha_T \tau_{11k})(R_k + 1) + \sum_{m=1}^M \lambda_m \Lambda \sigma_{m1k}, \quad (6.44)$$

and

$$0 = (R_{12k} + \alpha_T \tau_{12k})(R_k + 1) - \omega_k \Lambda P_{1k} + \sum_{m=1}^M \lambda_m \Lambda \sigma_{m12k}. \quad (6.45)$$

Equation (6.45) can be used to determine the oscillation frequency near the stability boundary. Examination of the equations for R_{11k} , R_{12k} , τ_{11k} , and τ_{12k} in Appendix D shows that, for $n_{sk}(t)$ given by Eq. (6.42),

$$\begin{aligned} R_{11k} + \alpha_T \tau_{11k} &= A_1(\omega_k) P_{1k} \\ &= \frac{\left\{ a_1 K [a_2 a_4 - (a_2 + a_3) \omega_k^2] + \alpha_T a_1 a_2 [(a_2 - \omega_k^2)^2 + (a_3 \omega_k)^2] \right\} P_{1k}}{(a_2^2 + \omega_k^2) [(a_4 - \omega_k^2)^2 + (a_3 \omega_k)^2]}, \end{aligned} \quad (6.46)$$

and

$$\begin{aligned} R_{12k} + \alpha_T \tau_{12k} &= A_2(\omega_k) P_{1k} \\ &= \frac{-\left\{ a_1 K (a_2 a_3 + a_4 - \omega_k^2) \omega_k + a_1 \omega_k [(a_2 - \omega_k^2)^2 + (a_3 \omega_k)^2] \right\} P_{1k}}{(a_2^2 + \omega_k^2) [(a_4 - \omega_k^2)^2 + (a_3 \omega_k)^2]}. \end{aligned} \quad (6.47)$$

Substituting Eq. (6.47) into Eq. (6.45), using the expressions in Appendix D, and dividing by P_{1k} yields

$$\begin{aligned} &\left[\Lambda + \sum_{m=1}^M \frac{\lambda_m \beta_m}{(\lambda_m^2 + \omega_k^2)} \right] \omega_k \\ &= \frac{-\left\{ a_1 K (a_2 a_3 + a_4 - \omega_k^2) \omega_k + a_1 \omega_k [(a_2 - \omega_k^2)^2 + (a_3 \omega_k)^2] \right\} (R_k + 1)}{(a_2^2 + \omega_k^2) [(a_4 - \omega_k^2)^2 + (a_3 \omega_k)^2]}. \end{aligned} \quad (6.48)$$

An equation for R_k is obtained from Eq. (6.44). Substituting Eq. (6.46) into Eq. (6.44), using the expressions in Appendix D, and dividing the resulting equation by P_{1k} yields

$$R_k = \frac{\left[(\beta + \rho_{0k}) - A_1(\omega_k) - \sum_{m=1}^M \frac{\lambda_m^2 \beta_m}{(\lambda_m^2 + \omega_k^2)} \right] a_2 a_4}{K a_1 + \alpha_T a_1 a_4 + a_2 a_4 A_1(\omega_k)}, \quad (6.49)$$

where $A_1(\omega_k)$ is defined in Eq. (6.46). Equations (6.48) and (6.49) can be solved simultaneously for ω_k and R_k near the stability boundary.

An equation for P_{1k} as a function of R_k and ω_k is obtained from Eq. (6.43).

Substituting Eq. (6.46) into Eq. (6.43), using the expressions in Appendix D, and solving for P_{1k} yields

$$P_{1k} = \sqrt{2 \frac{(a_2 a_4 \rho_{0k} - K a_1 - \alpha_T a_1 a_4) R_k - (K a_1 + \alpha_T a_1 a_4) R_k^2}{a_2 a_4 A_1(\omega_k)}}. \quad (6.50)$$

where again, $A_1(\omega_k)$ is defined in Eq. (6.46). Thus, if R_k and ω_k are known, Eq. (6.50) can be solved for the oscillation amplitude near the stability boundary.

6.3.1 Stability Condition

An equation for the oscillation frequency at the stability boundary can be obtained from Eq. (6.48). As the stability boundary is approached, R_k approaches zero. Thus, at the stability boundary,

$$\begin{aligned} & \left[\Lambda + \sum_{m=1}^M \frac{\lambda_m \beta_m}{(\lambda_m^2 + \omega_{k0}^2)} \right] \omega_{k0} \\ &= \frac{- \left\{ a_1 K (a_2 a_3 + a_4 - \omega_{k0}^2) \omega_{k0} + a_1 \omega_{k0} [(a_2 - \omega_{k0}^2)^2 + (a_3 \omega_{k0})^2] \right\}}{(a_2^2 + \omega_{k0}^2) [(a_4 - \omega_{k0}^2)^2 + (a_3 \omega_{k0})^2]}. \end{aligned} \quad (6.51)$$

The only unknown in Eq. (6.51) is ω_{k0} , the oscillation frequency at the stability boundary. Because R_k equals zero at the stability boundary, the stability condition for the multimodal phenomenological model is obtained by setting the numerator of Eq. (6.49) equal to zero

$$\begin{aligned} & \beta + \rho_{0k} - \sum_{m=1}^M \frac{\lambda_m^2 \beta_m}{(\lambda_m^2 + \omega_{k0}^2)} \\ &= \frac{a_1 K [a_2 a_4 - (a_2 + a_3) \omega_{k0}^2] + \alpha_T a_1 a_2 [(a_2 - \omega_{k0}^2)^2 + (a_3 \omega_{k0})^2]}{(a_2^2 + \omega_{k0}^2) [(a_4 - \omega_{k0}^2)^2 + (a_3 \omega_{k0})^2]} \end{aligned} \quad (6.52)$$

Equation (6.52) can be solved for K_0 , the value of K at the stability boundary:

$$K_0 = \frac{\left\{ (a_2^2 + \omega_{k0}^2) \left[\beta + \rho_{0k} - \sum_{m=1}^M \frac{\lambda_m^2 \beta_m}{(\lambda_m^2 + \omega_{k0}^2)} \right] - \alpha_T a_1 a_2 \right\} [(a_4 - \omega_{k0}^2)^2 + (a_3 \omega_{k0})^2]}{a_1 [a_2 a_4 - (a_2 + a_3) \omega_{k0}^2]} \quad (6.53)$$

If $K > K_0$, the multimodal phenomenological model equations are stable; if $K < K_0$, the multimodal phenomenological model equations are unstable.

6.3.2 Calculated Results near the Stability Boundary

Equations (6.51) and (6.52) were used to calculate the value of K_0 for four cases: the first two cases model the LaSalle-2 power station for conditions corresponding to corewide limit-cycle oscillations, with one case modeling the corewide stability mode and the other case modeling the regional stability mode. The second two cases model the Ringhals-1 power station for conditions corresponding to regional limit-cycle oscillations, again with one case modeling the corewide stability mode and the other case modeling the regional stability mode. Table 6.2 lists the values of the multimodal phenomenological model constants calculated for the LaSalle corewide and regional stability modes. Table 6.3 lists the LAPUR-calculated values of decay ratio and oscillation frequency, as well as values

**Table 6.2. Multimodal phenomenological model constants for the
LaSalle-2 power station calculated for conditions
corresponding to corewide oscillations**

Constant	Corewide	Regional
a_1 ($K \cdot s^{-1}$)	63.380	204.90
a_2 (s^{-1})	0.2620	0.3520
a_3 (s^{-1})	1.5290	0.8702
a_4 (s^{-2})	5.6600	6.6560
K	-7.895×10^{-4}	-3.202×10^{-4}
Λ (s^{-1})	4.000×10^{-5} ,	
α_T (K^{-1})	-2.520×10^{-5} ,	
β_1, λ_1 (s^{-1})	1.780×10^{-4} , 0.0124	
β_2, λ_2 (s^{-1})	1.180×10^{-3} , 0.0305	
β_3, λ_3 (s^{-1})	1.060×10^{-3} , 0.1110	
β_4, λ_4 (s^{-1})	2.130×10^{-3} , 0.3010	
β_5, λ_5 (s^{-1})	6.240×10^{-4} , 1.1300	
β_6, λ_6 (s^{-1})	2.270×10^{-4} , 3.000	

**Table 6.3. Comparison of stability parameters for the LaSalle-2
and the Ringhals-1 power station models**

Constant	LaSalle-2		Ringhals-1	
	Corewide mode	Regional mode	Corewide mode	Regional mode
Decay ratio	1.02	0.76	0.76	1.07
ω (Hz)	0.40	0.41	0.47	0.49
K_0	-7.544×10^{-4}	-4.259×10^{-4}	-1.675×10^{-1}	-4.4900×10^{-2}
ω_{k0} (Hz)	0.3889	0.4261	0.4504	0.4774
K/K_0	1.047	0.752	0.795	1.055

of K_0 , ω_{k0} , and K/K_0 for the four cases calculated, by using Eqs. (6.51) and (6.52). The values of subcritical reactivity used in these calculations were \$1.5 for the LaSalle-2 regional mode and \$1 for the Ringhals-1 regional mode. These results show that for the unstable cases (the LaSalle-2 corewide and Ringhals-1 regional oscillation modes), $K < K_0$ or, equivalently, $K/K_0 > 1$. The value of K_0 is less than K for both of the stable cases. These results indicate that the stability condition obtained from the low-order solution accurately predicts the model stability.

The effect of subcritical reactivity on the value of K_0 is shown in Figure 6.7. The value of K_0 for the LaSalle-2 and Ringhals-1 regional modes are shown for subcritical reactivities ranging from \$0 to \$2. The value of K/K_0 decreases as the amount of subcritical reactivity increases; thus, increasing the subcritical reactivity is stabilizing.

Figure 6.8 shows the effect on K_0 as each of the model constants a_1 , a_2 , a_3 , and a_4 are varied between 50% and 150% of the values listed in Tables 6.1 and 6.2. Figure 6.8 shows that increasing a_1 or decreasing a_3 or a_4 is destabilizing. The constant a_2 is shown to have little affect on the value of K/K_0 . Recalling that a_1 is inversely proportional to the product of fuel density and specific heat, the results shown in Figure 6.8 indicate that decreasing this product is destabilizing.⁹⁴ The value of a_3 is directly proportional to the product of the coolant mass flux, and quality and is inversely proportional to average void fraction.⁹⁴ The value of a_4 is directly proportional to the square of the product of coolant mass flux, and quality and is inversely proportional to the square of the average void fraction. Thus, the results shown in Figure 6.8 indicate that decreasing the coolant mass flux or increasing the average void fraction is destabilizing. These results agree with those shown in Chapter 3 and also with the findings of earlier investigations.¹⁻³

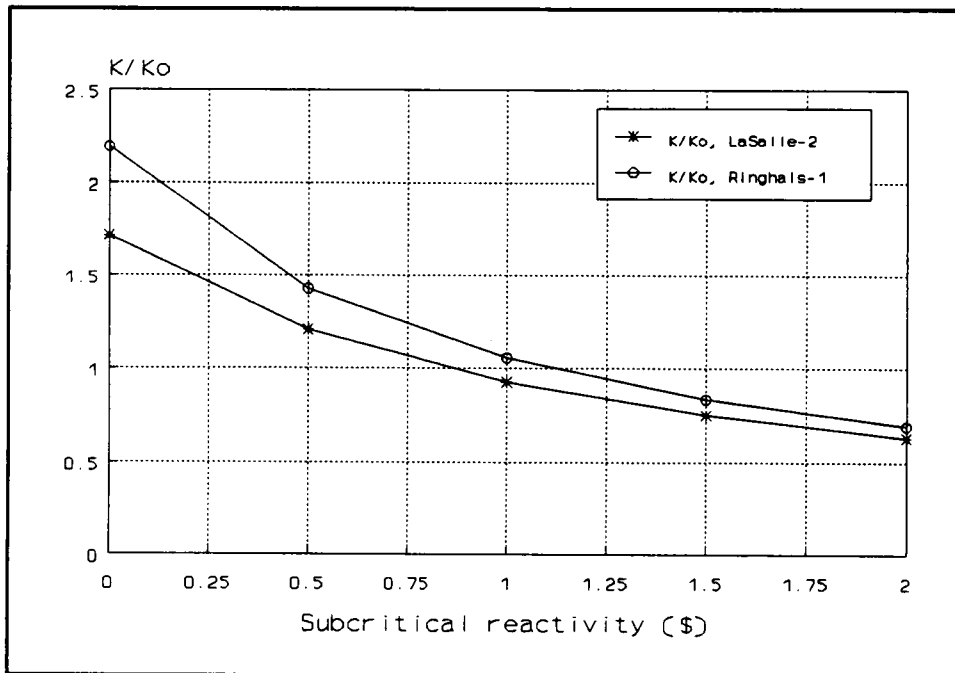


Figure 6.7. Effect of subcritical reactivity on the stability boundary, as measured by K/K_0 .

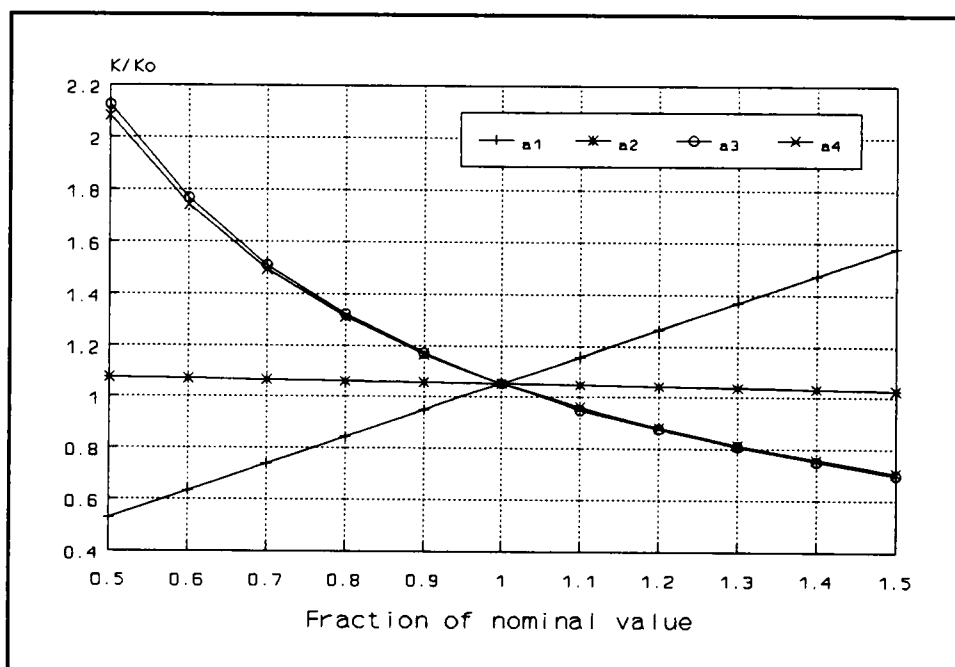


Figure 6.8. Effect of a_1 , a_2 , a_3 , and a_4 on the stability boundary, as measured by K/K_0 .

The effects of a_1 , a_2 , a_3 , a_4 , and ρ_{0k} on ω_{k0} are shown in Figure 6.9. These results were calculated by using Eqs. (6.48) and (6.49). In general, Figure 6.9 shows that increasing any of the equation constants increases ω_{k0} . The equation constant having the greatest affect on ω_{k0} is a_4 . Note that the change in ω_{k0} due to changes in a_4 is displayed on the right border of Figure 6.8; the sensitivity of ω_{k0} to changes in a_4 is over an order of magnitude greater than the sensitivity of ω_{k0} to any of the remaining equation constants. The sensitivity of ω_{k0} to the value of a_4 is caused by the dependence of a_4 on the square of the coolant mass flux. Increasing the coolant mass flux increases a_4 , decreases the channel transit time, and increases ω_{k0} .

The effect of K on the limit-cycle amplitude is shown in Figure 6.10. These results were calculated for both the corewide and regional stability modes of the LaSalle-2 and Ringhals-1 power station models by using Eqs. (6.48), (6.49), and (6.50). The results show that the limit-cycle amplitude grows more rapidly with increasing K_0/K for the regional oscillations than for the corewide oscillations for both power station models. The increase in the regional mode over the corewide mode is ~25% for the LaSalle-2 model and ~34% for the Ringhals-1 model. These results suggest that for similar excursions beyond the stability boundary, a regional oscillation will have a greater oscillation amplitude with correspondingly greater power, temperature, and flow oscillations, than will a corewide oscillation.

The effect of ρ_{0k} on the limit-cycle amplitude, calculated for the LaSalle-2 power station model, is shown in Figure 6.11. The amplitude was calculated for values of K/K_0 ranging from 1 to 1.1 and for subcritical reactivity values of \$0.0, \$1.0, and \$2.0. The results show that the rate of growth in limit-cycle amplitude as K/K_0 increases becomes greater as the subcritical reactivity becomes greater. This increase in the rate of

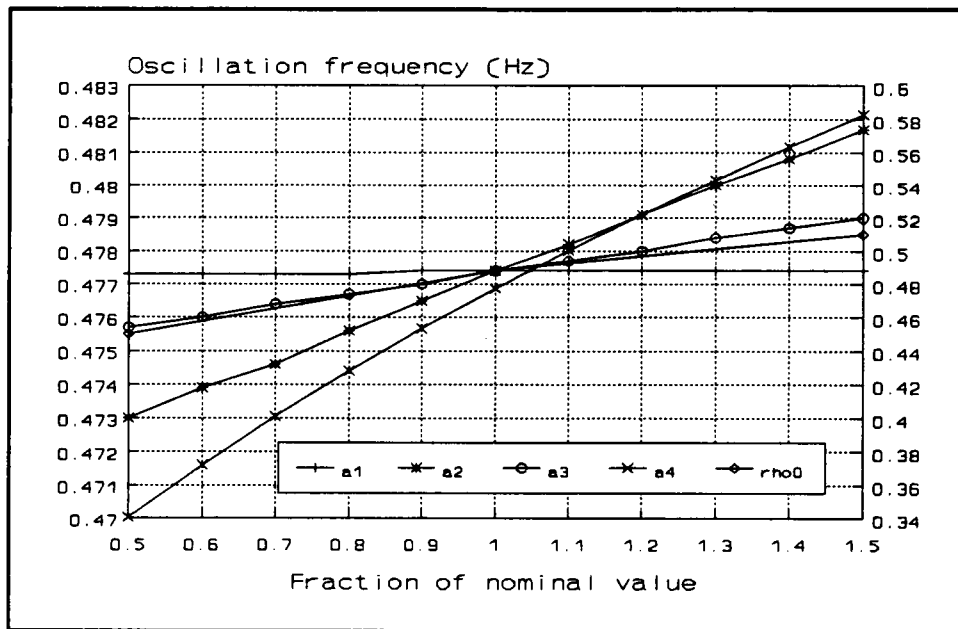


Figure 6.9. Effect of a_1 , a_2 , a_3 , and a_4 on the oscillation frequency at the stability boundary. Note that the curve for a_4 uses the scale on the right border.

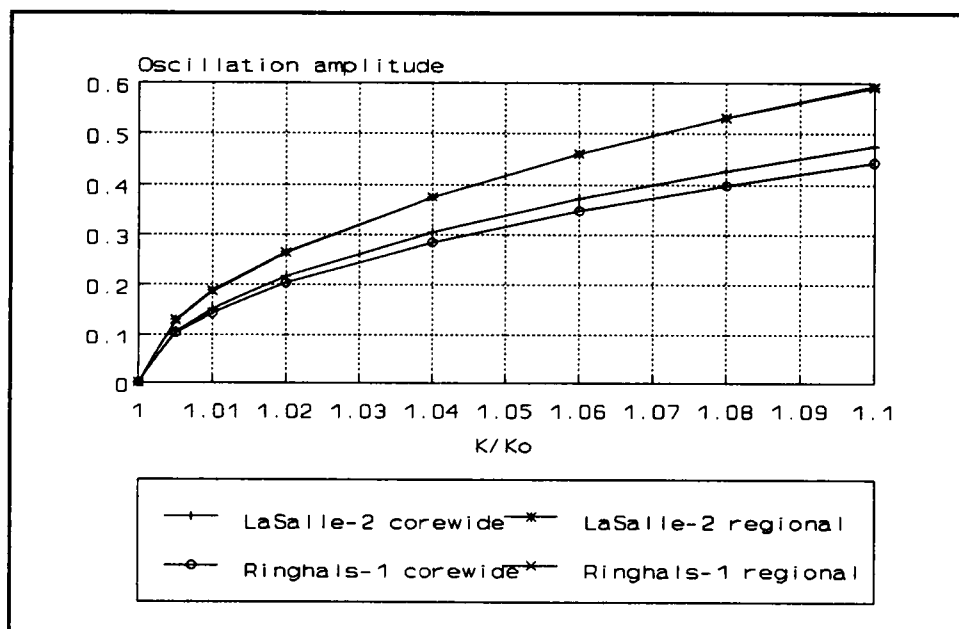


Figure 6.10. Effect of K on the limit-cycle amplitude for the corewide and regional modes of the LaSalle-2 and Ringhals-1 power station models.

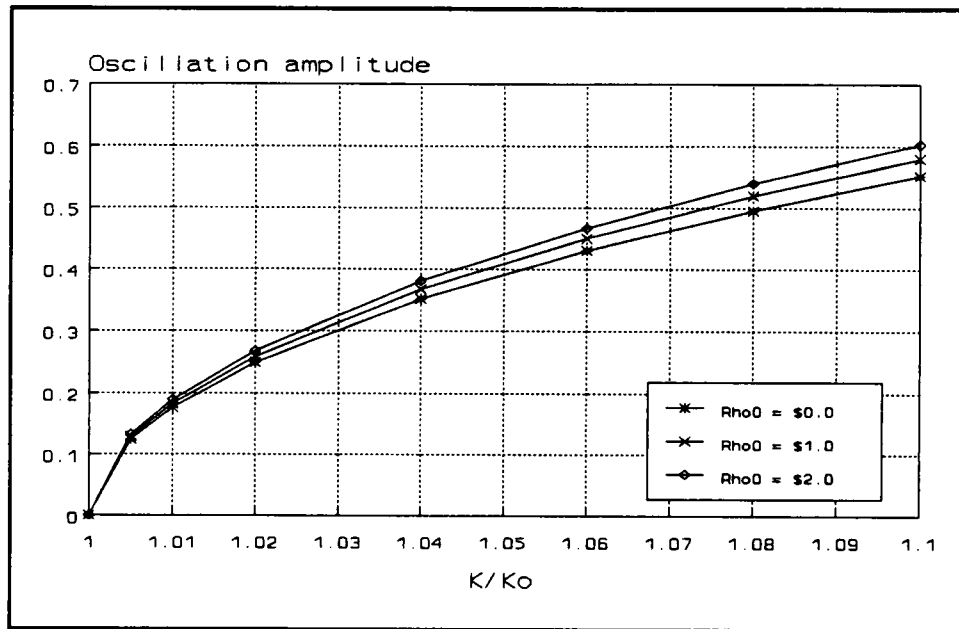


Figure 6.11. Effect of subcritical reactivity on regional limit-cycle amplitude calculated for the LaSalle-2 plant model.

limit-cycle amplitude growth with subcritical reactivity occurs because ρ_{0k} appears in the numerator of Eq. (6.50), the expression for P_{1k} .

6.4 High-Order Solution to the Multimodal Phenomenological Model Equations

This section presents the results from calculations of the high-order approximate solution of the multimodal phenomenological model equations. A FORTRAN computer code was written to form and solve the equations obtained from Eqs. (6.39), (6.40), and (6.41) for the constants in Eq. (6.31). The code uses NAG routine C05NBF to solve for the undetermined constants and requires an initial estimate of the solution.¹³⁹ The strategies for selecting the initial estimates described in Subsection 3.2.3 were found to be applicable when this code is used to calculate high-order solutions.

Table 6.4 lists the constants for two- to eight-harmonic approximate solutions calculated by using the phenomenological model constants listed in Table 3.1. The value of subcritical reactivity was zero because this code input corresponds to a corewide oscillation mode. The results show the solution converging as the number of harmonics increases. Comparison of the calculated results in listed Tables 3.2 and 6.4 shows that the multimodal phenomenological model reproduces the results obtained from the phenomenological model used in Chapter 3. Thus, it has been shown that the solution converges as more terms are included in the approximation and that the phenomenological model results shown in Chapter 3 can be duplicated by the multimodal phenomenological model.

Figure 6.12 shows the limit-cycle amplitude, calculated from a 7-harmonic approximate solution, for values of K/K_0 ranging from 1 to 1.4. The results are calculated for the corewide and regional modes of the LaSalle-2 power station model. These results agree with those shown in Figure 6.10; the regional oscillation mode has an oscillation amplitude greater than that of the corewide mode for the same value of K/K_0 . The oscillation amplitude shows a greater rate of increase with K/K_0 in Figure 6.12 than it does in Figure 6.10. The 7-harmonic approximate solution used to produce the results shown in Figure 6.12 is a better representation of the exact solution than is the 1-harmonic approximate solution used to produce the results shown in Figure 6.10. The curves shown in Figure 6.12 would be more linear if the exact solution were used to produce the results because the approximate solution underpredicts limit-cycle amplitude if a sufficient number of harmonics are not included in the approximation.

The effect of K on the limit-cycle amplitude of the corewide and regional stability modes is shown in Figure 6.13 for the LaSalle-2 power station model and in Figure 6.14

Table 6.4. Approximate solution constants for two- to eight-harmonic solutions of the multimodal phenomenological equations calculated by using the constants listed in Table 3.1

Constant	2	3	4	5	6	7	8
ω (Hz)	0.4240	0.4200	0.4181	0.4174	0.4171	0.4170	0.4170
R	0.0386	0.0497	0.0542	0.0557	0.0562	0.0564	0.0564
P_1	0.6376	0.4836	0.3580	0.2663	0.1988	0.1462	0.1067
Q_1	0.6639	0.9176	1.0180	1.0600	1.0789	1.0885	1.0930
P_2	0.3623	0.4424	0.3898	0.3195	0.2573	0.2051	0.1643
Q_2		0.2615	0.4149	0.4933	0.5346	0.5584	0.5721
P_3		0.2158	0.2689	0.2512	0.2168	0.1815	0.1517
Q_3			0.1120	0.1917	0.2399	0.2701	0.2885
P_4			0.1249	0.1557	0.1498	0.1331	0.1156
Q_4				0.0498	0.0902	0.1183	0.1365
P_5				0.0706	0.0879	0.0860	0.0786
Q_5					0.0227	0.0436	0.0584
P_6					0.0393	0.0490	0.0487
Q_6						0.0109	0.0210
P_7						0.0218	0.0271
Q_7							0.0050
P_8							0.0120

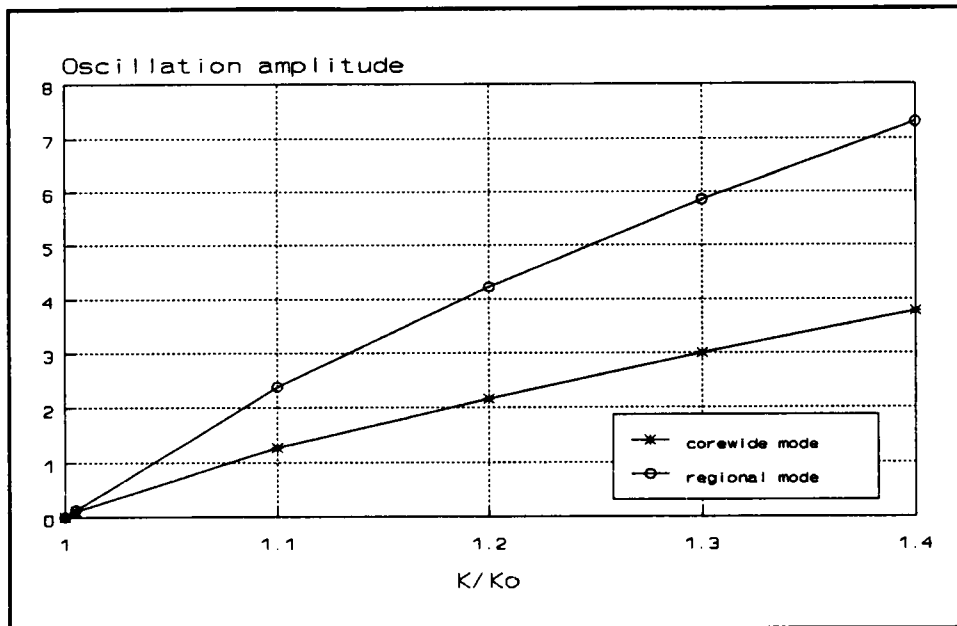


Figure 6.12. Effect of K on the limit-cycle amplitude for the corewide and regional modes of the LaSalle-2 power station model.

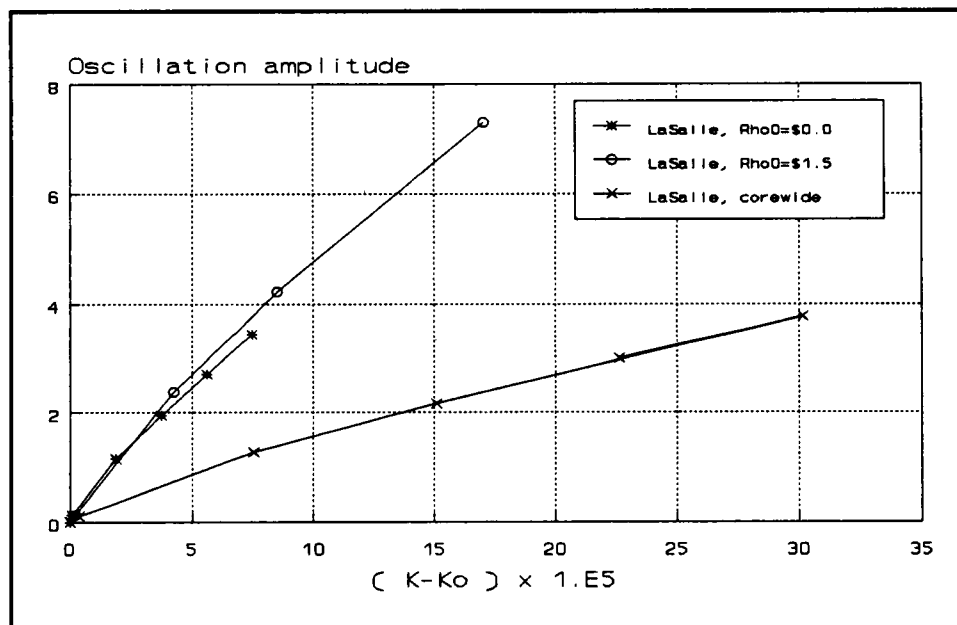


Figure 6.13. Effect of K on the limit-cycle amplitude of corewide and regional modes calculated for the LaSalle-2 power station model.

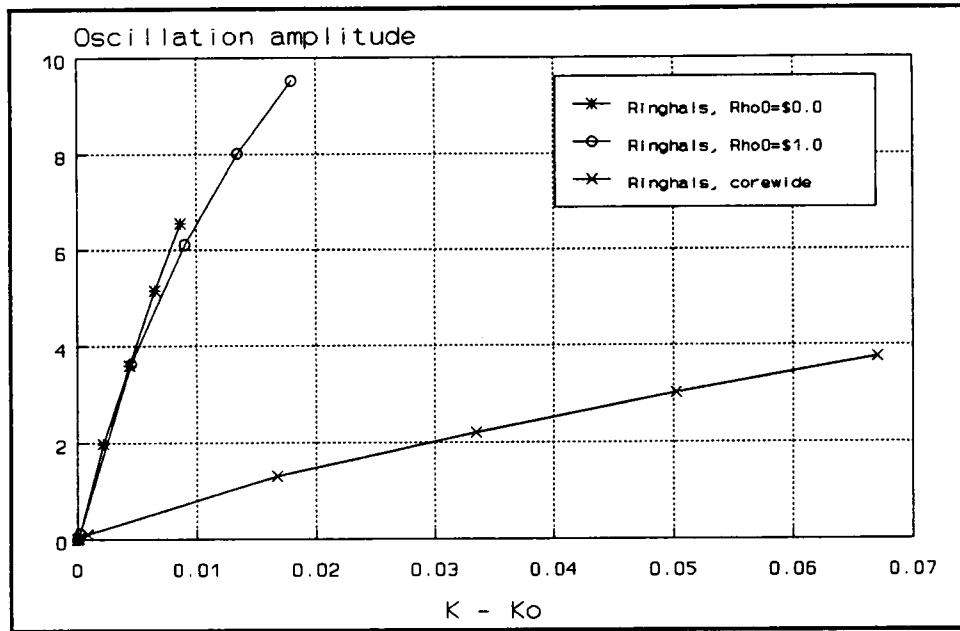


Figure 6.14. Effect of K on the limit-cycle amplitude of corewide and regional modes calculated for the Ringhals-1 power station model.

for the Ringhals-1 power station model. These figures show results obtained from 7-harmonic approximate solutions. Subcritical reactivity values of \$0.0 and \$1.5 were used to calculate regional modes of the LaSalle-2 power station model, and subcritical reactivity values of \$0.0 and \$1.5 were used to calculate regional modes of the Ringhals-1 power station model. These results clearly show that the regional mode is more sensitive than the corewide mode to changes in K . Because K depends heavily on the void coefficient of reactivity, it can be concluded that the limit-cycle amplitude of the regional mode is more sensitive to changes in void coefficient than is the limit-cycle amplitude of the corewide mode. Comparison of the results calculated for the regional modes by using different values of subcritical reactivity shows the effect of subcritical reactivity on limit-cycle amplitude to be relatively minor.

Figure 6.15 shows the effect of increasing K/K_0 on the limit-cycle oscillation frequency for the corewide and regional modes of the LaSalle-2 and the Ringhals-1 power station models. The oscillation frequency decreases with increasing K/K_0 for the corewide modes calculated for both the LaSalle-2 and the Ringhals-1 power station models. A similar decrease in oscillation frequency was calculated for the Ringhals-1 regional modes; a slight increase in oscillation frequency with increasing K/K_0 was calculated for the LaSalle-2 regional modes. In all cases, the change in oscillation frequency was small, indicating that K has little effect on oscillation frequency; it was shown in Section 6.3 that the oscillation frequency is determined primarily by the value of a_4 .

Figure 6.16 shows the effect of K/K_0 and the subcritical reactivity on R_k , the steady-state component of $n_{ssk}(t)$, obtained from the 7-harmonic approximate solution calculated for both the corewide and regional oscillation modes of the LaSalle-2 and Ringhals-1 power station models. R_k is proportional to the power increase that accompanies limit-cycle oscillations. The results show that both regional modes have greater values of R_k than the corresponding corewide modes and that the value of R_k increases with increasing subcritical reactivity. Because any subcritical mode in a commercial power reactor will have a significant value of subcritical reactivity (i.e., on the order of \$1.0 or more), it can be concluded from these results that the steady-state component of the neutron population during limit-cycle oscillations will increase more rapidly with K/K_0 for a regional oscillation than for a corewide oscillation.

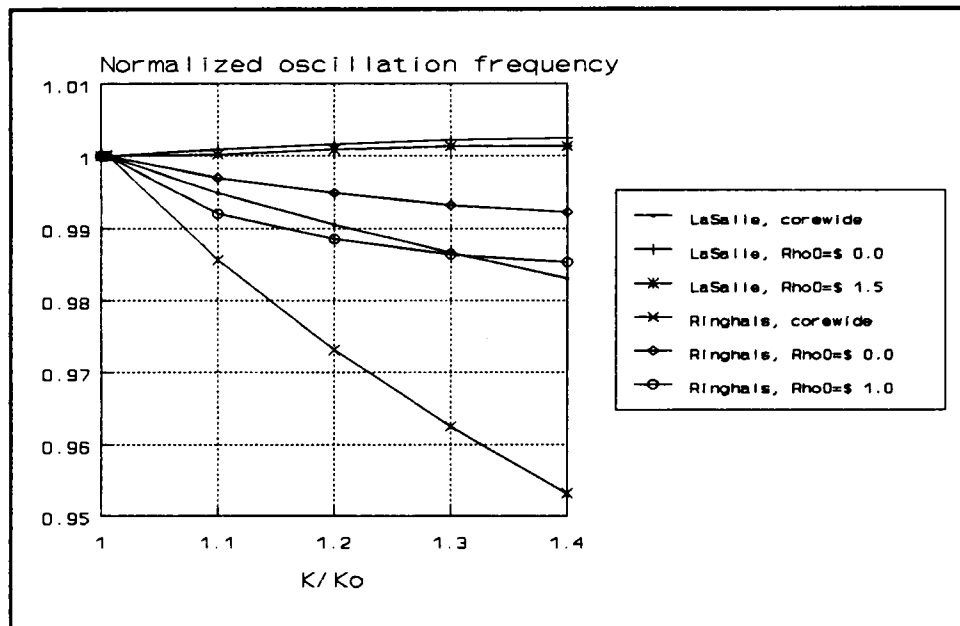


Figure 6.15. Effect of K and subcritical reactivity on the oscillation frequency for the corewide and regional modes of the LaSalle-2 and Ringhals-1 power station models.

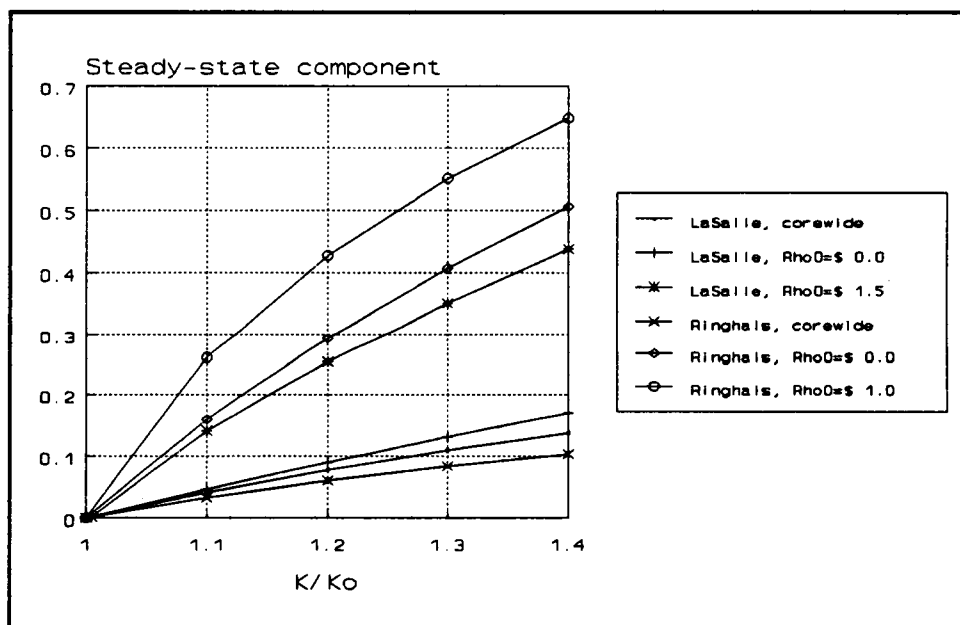


Figure 6.16. Effect of K and subcritical reactivity on steady-state power increase calculated for the LaSalle-2 and Ringhals-1 power station models.

6.5 Approximate Solution for a Multimodal BWR Dynamic Model Consisting of the Modal Point Kinetics Equations and the LAPUR-Calculated Feedback Transfer Function

The development of an approximate solution to a BWR dynamic model consisting of the modal point kinetics equations and the LAPUR-calculated feedback transfer function is presented in this section. This model is an extension of the BWR dynamic model used in Chapter 4. The modal point kinetics equations form the forward transfer function relating total reactivity to neutron population and the feedback transfer function forms the feedback loop relating neutron population to total reactivity.

Because this model uses the detailed thermal hydraulic description from the LAPUR code, it can be used to calculate quantitative information about BWR dynamic behavior. Like the BWR dynamic model used in Chapter 4, this dynamic model is accurate for only small-amplitude limit-cycle oscillations because the thermal hydraulic model is linearized about the steady-state conditions.

6.5.1 The BWR Dynamic Model and the Equation Residual

The forward transfer function relating the normalized neutron population to total reactivity is formed by the normalized modal point kinetics equations given by Eqs. (6.18) and (6.23). Assuming a periodic solution for $n_{ssk}(t)$ in the form of Eq. (6.31) results in a solution for $c_{ssmk}(t)$ given by Eq. (6.32) with W_{mk} , σ_{mn1k} , and σ_{mn2k} given by the equations in Appendix D. If the feedback transfer function is given by

$$H(\omega) = A(\omega) + iB(\omega) , \quad (6.54)$$

where $A(\omega)$ = the real part of the feedback transfer function at a frequency of ω ,
 $B(\omega)$ = the imaginary part of the feedback transfer function at a frequency of ω ,

then the expression for the reactivity during limit-cycle oscillations written in terms of the normalized neutron population given by Eq. (6.31) is

$$\begin{aligned} \rho_{ssk}(t) = & A_0 R_k + \sum_{n=1}^j \left\{ \left[A(n\omega_k) Q_{nk} + B(n\omega_k) P_{nk} \right] \cos(n\omega_k t) \right. \\ & \left. + \left[A(n\omega_k) P_{nk} - B(n\omega_k) Q_{nk} \right] \sin(n\omega_k t) \right\}, \end{aligned} \quad (6.55)$$

where A_0 is the real part of $H(\omega_k)$ for $\omega_k = 0$. Thus, the BWR dynamic model is given by Eqs. (6.18), (6.31), (6.32), and (6.55). Substituting Eqs. (6.32) and (6.55) into Eq. (6.18) results in an expression for the equation residual $\mathbf{R}_{ek}$ for mode k :

$$\begin{aligned} \mathbf{R}_{ek} = & (A_0 - \rho_{0k}) R_k + A_0 R_k^2 + \sum_{n=1}^j \sum_{l=1}^j \left\{ \left[A(n\omega_k) Q_{nk} + B(n\omega_k) P_{nk} \right] (P_{lk} C_{nk} S_{lk} + Q_{lk} C_{nk} C_{lk}) \right. \\ & \left. + \left[A(n\omega_k) P_{nk} - B(n\omega_k) Q_{nk} \right] (P_{lk} S_{nk} S_{lk} + Q_{lk} S_{nk} C_{lk}) \right\} \\ & + \sum_{n=1}^j \left\{ \left[A(n\omega_k) (R_k + 1) - \rho_{0k} + A_0 R_k - n^2 \omega_k^2 \sum_{m=1}^M \left\{ \frac{\beta_m}{\lambda_m^2 + n^2 \omega_k^2} \right\} \right] (P_{nk} S_{nk} + Q_{nk} C_{nk}) \right. \\ & \left. + \left[\Lambda n \omega_k - B(n\omega_k) (R_k + 1) + \sum_{m=1}^M \left\{ \frac{n \lambda_m \beta_m \omega_k}{\lambda_m^2 + n^2 \omega_k^2} \right\} \right] (Q_{nk} S_{nk} - P_{nk} C_{nk}) \right\}. \end{aligned} \quad (6.56)$$

6.5.2 Application of Galerkin's Method

Equations for the undetermined constants in Eq. (6.31) are again obtained by applying Galerkin's method. This application results in the following equations:

$$\int_0^{\frac{2\pi}{\omega_k}} \mathbf{R}_{ek} dt = 2 \left[(A_0 - \rho_{0k}) R_k + A_0 R_k^2 \right] + \sum_{n=1}^j \left\{ A(n\omega_k) (P_{nk}^2 + Q_{nk}^2) \right\} = 0, \quad (6.57)$$

$$\begin{aligned}
\int_0^{\frac{2\pi}{\omega_k}} \mathbf{R}_{ek} S_{ik} dt = & \left[P_{ik} \left(A(i\omega_k)(R_k + 1) - \rho_{0k} + A_0 R_k - i^2 \omega_k^2 \sum_{m=1}^M \left\{ \frac{\beta_m}{\lambda_m^2 + i^2 \omega_k^2} \right\} \right) \right. \\
& + Q_{ik} \left(\Lambda i \omega_k - B(i\omega_k)(R_k + 1) + \sum_{m=1}^M \left\{ \frac{i \lambda_m \beta_m \omega_k}{\lambda_m^2 + i^2 \omega_k^2} \right\} \right) \left. \right] \left(\frac{\pi}{\omega_k} \right) \\
& + \sum_{n=1}^j \sum_{l=1}^j \left\{ P_{lk} \left[A(n\omega_k) Q_{nk} + B(n\omega_k) P_{nk} \right] \int_0^{\frac{2\pi}{\omega_k}} S_{lk} S_{ik} C_{nk} dt \right. \\
& \left. + Q_{lk} \left[A(n\omega_k) P_{nk} - B(n\omega_k) Q_{nk} \right] \int_0^{\frac{2\pi}{\omega_k}} S_{nk} S_{ik} C_{lk} dt \right\} = 0,
\end{aligned} \tag{6.58}$$

and

$$\begin{aligned}
\int_0^{\frac{2\pi}{\omega_k}} \mathbf{R}_{ek} C_{ik} dt = & \left[Q_{ik} \left(A(i\omega_k)(R_k + 1) - \rho_{0k} + A_0 R_k - i^2 \omega_k^2 \sum_{m=1}^M \left\{ \frac{\beta_m}{\lambda_m^2 + i^2 \omega_k^2} \right\} \right) \right. \\
& - P_{ik} \left(\Lambda i \omega_k - B(i\omega_k)(R_k + 1) + \sum_{m=1}^M \left\{ \frac{i \lambda_m \beta_m \omega_k}{\lambda_m^2 + i^2 \omega_k^2} \right\} \right) \left. \right] \left(\frac{\pi}{\omega_k} \right) \\
& + \sum_{n=1}^j \sum_{l=1}^j \left\{ Q_{lk} \left[A(n\omega_k) Q_{nk} + B(n\omega_k) P_{nk} \right] \int_0^{\frac{2\pi}{\omega_k}} C_{nk} C_{lk} C_{ik} dt \right. \\
& \left. + P_{lk} \left[A(n\omega_k) P_{nk} - B(n\omega_k) Q_{nk} \right] \int_0^{\frac{2\pi}{\omega_k}} S_{nk} S_{lk} C_{ik} dt \right\} = 0.
\end{aligned} \tag{6.59}$$

Equations (6.57), (6.58), and (6.59) result in $2j + 1$ equations for the unknowns in Eq. (6.31) as i is indexed from 1 to j . By assigning $Q_{jk} = 0$, fixing the absolute phase of the approximate solution, the number of unknowns is reduced to $2j + 1$. This assignment allows Eqs. (6.57), (6.58), and (6.59) to be solved for the remaining constants.

6.6 The Low-Order Approximate Solution

The low-order approximate solution of the BWR dynamic model consisting of the modal point kinetics equations and the LAPUR-calculated feedback transfer function is examined in this section. This solution is accurate only near the stability boundary; nevertheless, it can be used to develop a stability condition and to examine the effects of subcritical reactivity and transfer function values on the limit-cycle amplitude.

6.6.1 Development of the Low-Order Approximate Solution

Let the solution for $n_{ssk}(t)$ near the stability boundary be given by Eq. (6.42). The following equations for R_k , P_{1k} , and ω_k are formed by setting j in Eqs. (6.57), (6.58), and (6.59) equal to 1:

$$2[A_0 R_k (R_k + 1) - \rho_{0k} R_k] + A(\omega_k) P_{1k}^2 = 0, \quad (6.60)$$

$$A(\omega_k)(R_k + 1) - \rho_{0k} + A_0 R_k - \omega_k^2 \sum_{m=1}^M \left\{ \frac{\beta_m}{\lambda_m^2 + \omega_k^2} \right\} = 0, \quad (6.61)$$

and

$$B(\omega_k)(R_k + 1) - \omega_k \left[\Lambda + \sum_{m=1}^M \left\{ \frac{\lambda_m \beta_m}{\lambda_m^2 + \omega_k^2} \right\} \right] = 0. \quad (6.62)$$

Solving Eq. (6.62) for R_k ,

$$R_k = \frac{\omega_k}{B(\omega_k)} \left[\Lambda + \sum_{m=1}^M \left\{ \frac{\lambda_m \beta_m}{\lambda_m^2 + \omega_k^2} \right\} - \frac{B(\omega_k)}{\omega_k} \right]. \quad (6.63)$$

Substituting Eq. (6.63) into Eq. (6.60) results in an equation for P_{1k} :

$$P_{1k} = \frac{-2[A_0 - \rho_{0k}]\omega_k}{A(\omega_k)B(\omega_k)} \left[\Lambda + \sum_{m=1}^M \left\{ \frac{\lambda_m \beta_m}{\lambda_m^2 + \omega_k^2} \right\} - \frac{B(\omega_k)}{\omega_k} \right]. \quad (6.64)$$

Equation (6.63) can be substituted into Eq. (6.61), resulting in an equation in which the only unknown is ω_k :

$$\frac{\omega_k[A_0 + A(\omega_k)]}{B(\omega_k)} \left[\Lambda - \frac{B(\omega_k)}{\omega_k} + \sum_{m=1}^M \left\{ \frac{\lambda_m \beta_m}{\lambda_m^2 + \omega_k^2} \right\} \right] + A(\omega_k) - \rho_{0k} - \omega_k^2 \sum_{m=1}^M \left\{ \frac{\beta_m}{\lambda_m^2 + \omega_k^2} \right\} = 0. \quad (6.65)$$

Equation (6.65) can be solved for ω_k ; $A(\omega_k)$, and $B(\omega_k)$ can then be determined from the LAPUR-calculated transfer function. Equations (6.63) and (6.64) can then be evaluated for R_k and P_{1k} .

6.6.2 Stability Condition

At the stability boundary $R_k = 0$; evaluating Eq. (6.61) at the stability boundary yields the following stability condition:

$$A(\omega_{k0}) - \rho_{0k} - \omega_{k0}^2 \sum_{m=1}^M \left\{ \frac{\beta_m}{\lambda_m^2 + \omega_{k0}^2} \right\} = 0. \quad (6.66)$$

Equation (6.66) reduces to the stability condition given by Eq. (4.27) if $\rho_{0k} = 0$ and if $\lambda_m^2 + \omega_{k0}^2$ is approximated by ω_{k0}^2 . Equation (6.66) can be evaluated for ω_{k0} ; the LAPUR-calculated feedback transfer function can then be used to determine $A(\omega_{k0})$ and $B(\omega_{k0})$. The system of equations will be unstable if $A(\omega_k) > A(\omega_{k0})$, that is, if

$$A(\omega_k) > \rho_{0k} + \omega_{k0}^2 \sum_{m=1}^M \left\{ \frac{\beta_m}{\lambda_m^2 + \omega_{k0}^2} \right\} = A(\omega_{k0}). \quad (6.67)$$

Equation (6.67) shows that $A(\omega_{k0})$ increases with subcritical reactivity, indicating that increasing subcritical reactivity is stabilizing and that increasing the real part of the feedback transfer function at the oscillation frequency (which essentially equals the magnitude of the feedback transfer function) is destabilizing.

6.6.3 Calculated Results near the Stability Boundary

Equations (6.63), (6.64), and (6.65) were used to investigate the effect of subcritical reactivity, feedback gain, and feedback time delay on the limit-cycle amplitude. All results presented in this subsection were calculated by using the Ringhals-1 power station model. The LAPUR-calculated feedback transfer function for this power station model is shown in Figure 6.17. The regional mode has a greater feedback transfer function magnitude than that of the corewide mode over the frequency range displayed, and the phase of each transfer function crosses zero at approximately 0.49 Hz (recall from Chapter 4 that the limit-cycle oscillation frequency is near the frequency at which the phase of the feedback transfer function is zero). The sharp spike in the regional transfer function magnitude near 0.49 Hz is responsible for the instability of the regional mode of this power station model.

Figure 6.18 shows the effect of subcritical reactivity on limit-cycle amplitude. These results were calculated by holding all equation parameters constant except ρ_{0k} . As the subcritical reactivity decreases, the limit-cycle amplitude increases; the system of equations becomes stable for values of ρ_{0k} greater than approximately \$1.61. The increase in limit-cycle amplitude with decreasing subcritical reactivity occurs because the system of equations is pushed further into the unstable region. The quantity $\Delta A = A(\omega_k) - A(\omega_{k0})$ can be used as a measure of the degree of instability; this quantity is zero at the stability boundary and increases as the system of equations becomes more unstable. Figure 6.19

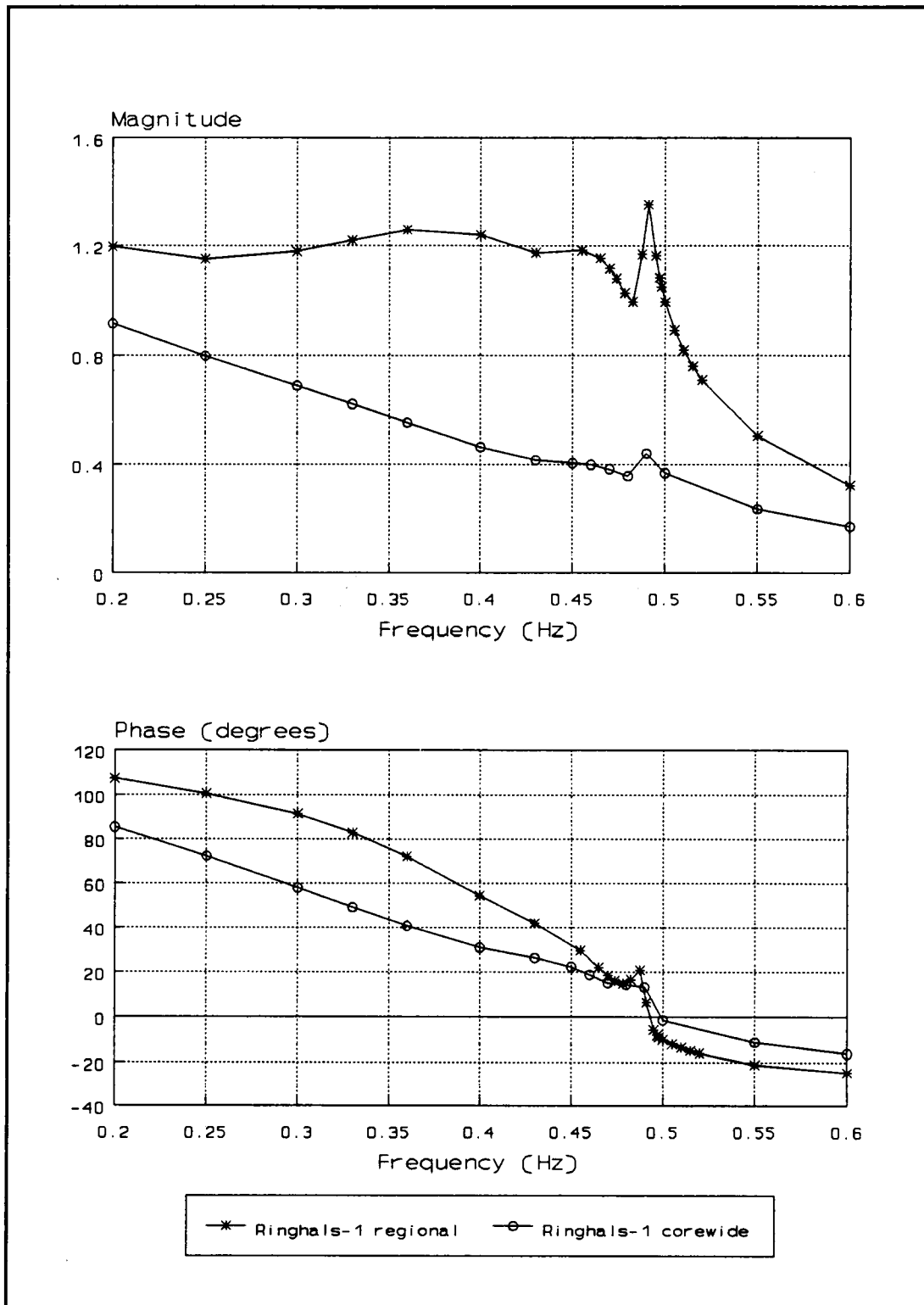


Figure 6.17. Magnitude and phase of the LAPUR-calculated power-to-reactivity feedback transfer functions of the Ringhals-1 and LaSalle-2 power station models.

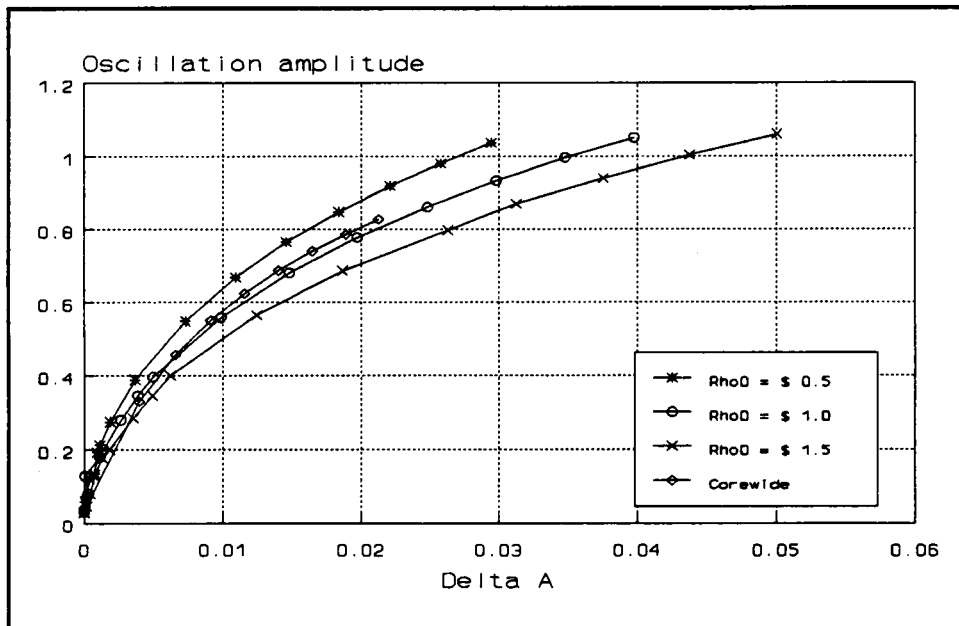


Figure 6.18. Sensitivity of the regional and corewide limit-cycle amplitude to subcritical reactivity calculated for the Ringhals-1 power station model. The amplitude was obtained from the one-harmonic solution.

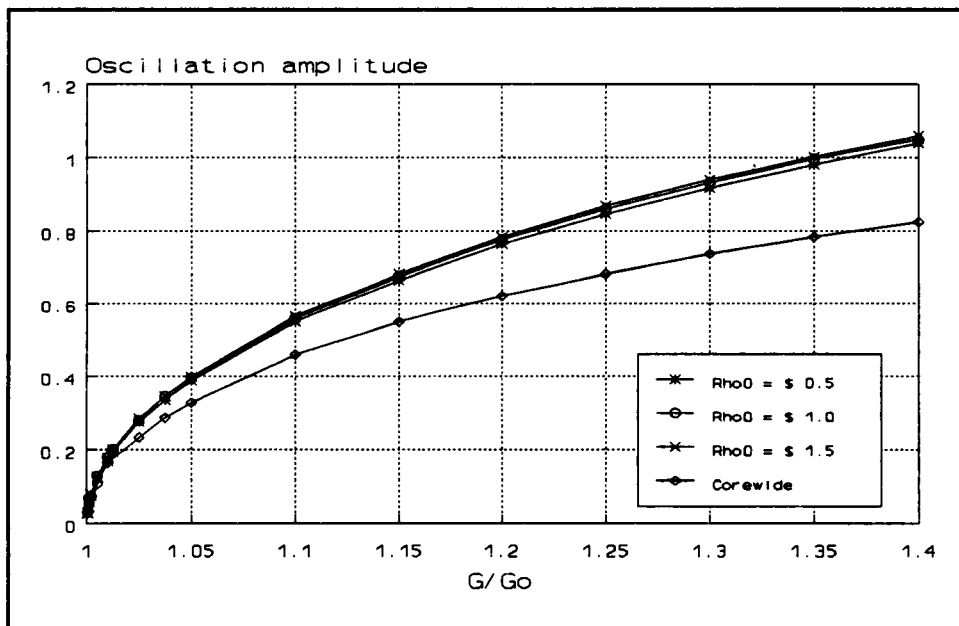


Figure 6.19. Effect of feedback gain on the limit-cycle amplitude.

shows how ΔA increases as ρ_{0k} decreases, again, indicating that decreasing subcritical reactivity is destabilizing, resulting in an increase in limit-cycle amplitude.

The sensitivity of limit-cycle amplitude to subcritical reactivity is shown in Figure 6.20. A comparison of limit-cycle amplitudes calculated for the same value of ΔA shows that increasing the subcritical reactivity causes the limit-cycle amplitude to decrease. Figure 6.20 also shows that for this particular model, the corewide and regional modes are equally sensitive to changes in ΔA .

Figures 6.21 and 6.22 show the effect of feedback gain on the limit-cycle amplitude and frequency. The value of feedback gain putting the system of equations at the stability boundary G_0 was found for each curve shown in Figure 6.21. The limit-cycle amplitude was calculated for feedback gains ranging from G_0 to $1.4 G_0$ (i.e., from $G/G_0 = 1$ to $G/G_0 = 1.4$). Because the parameter K in the multimodal phenomenological model equations is proportional to the feedback gain, increasing the value of G/G_0 from 1 to 1.4 is equivalent to increasing the value of K/K_0 from 1 to 1.4. Comparing Figures 6.11 and 6.21 shows similar results; the limit-cycle amplitude increases with G/G_0 and K/K_0 with a slightly greater amplitude being calculated for greater values of subcritical reactivity. The regional limit-cycle amplitude increases faster than the corewide limit-cycle amplitude with increasing feedback gain. Little effect on the oscillation frequency is seen because the frequency at which the transfer function phase equals zero is not affected by a gain change.

Figures 6.23 and 6.24 show the effect of a time delay on the limit-cycle oscillation frequency and amplitude. A time delay results in the value of zero phase being shifted upward in the phase plot of Figure 6.17 (note that this upward shift is linearly dependent on frequency). Because the zero-phase crossing point of the transfer function is directly

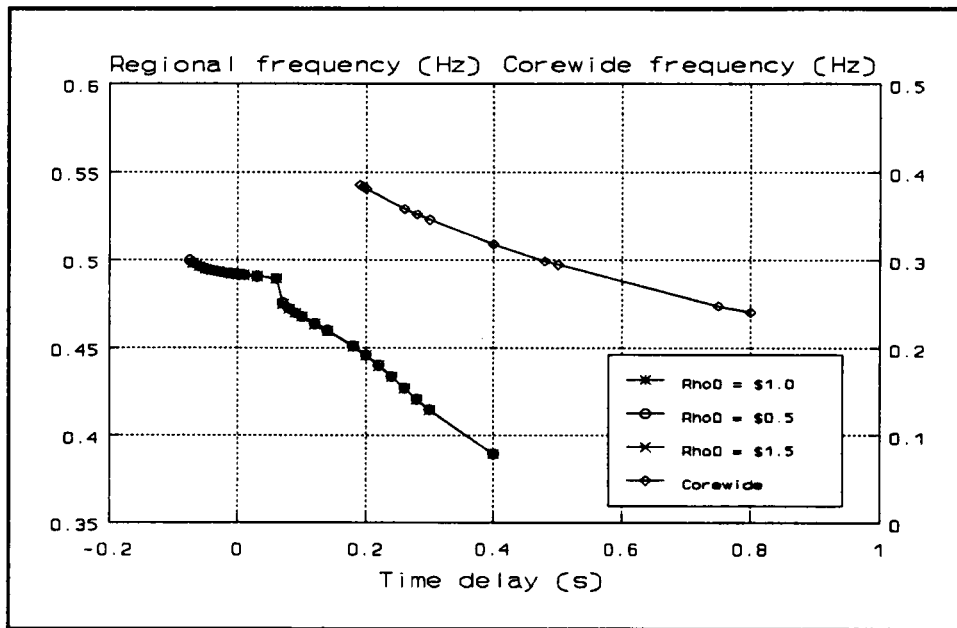


Figure 6.20. Effect of time delay on limit-cycle oscillation frequency calculated for the Ringhals-1 power station model. Note that the oscillation frequency of the corewide mode is plotted on the right vertical axis.

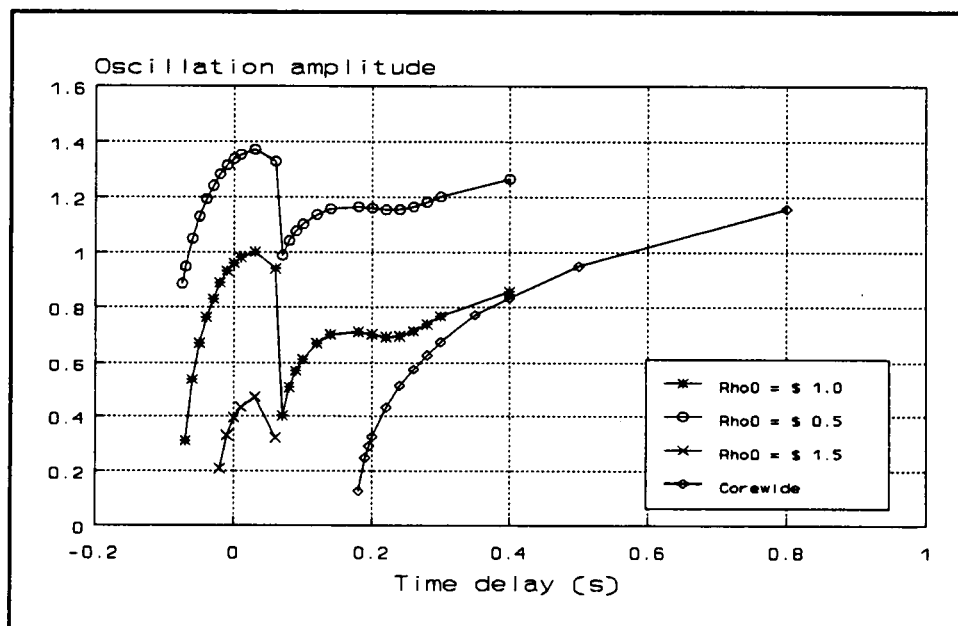


Figure 6.21. Effect of time delay on limit-cycle amplitude calculated for the Ringhals-1 power station model. The amplitude was obtained from the one-harmonic solution.

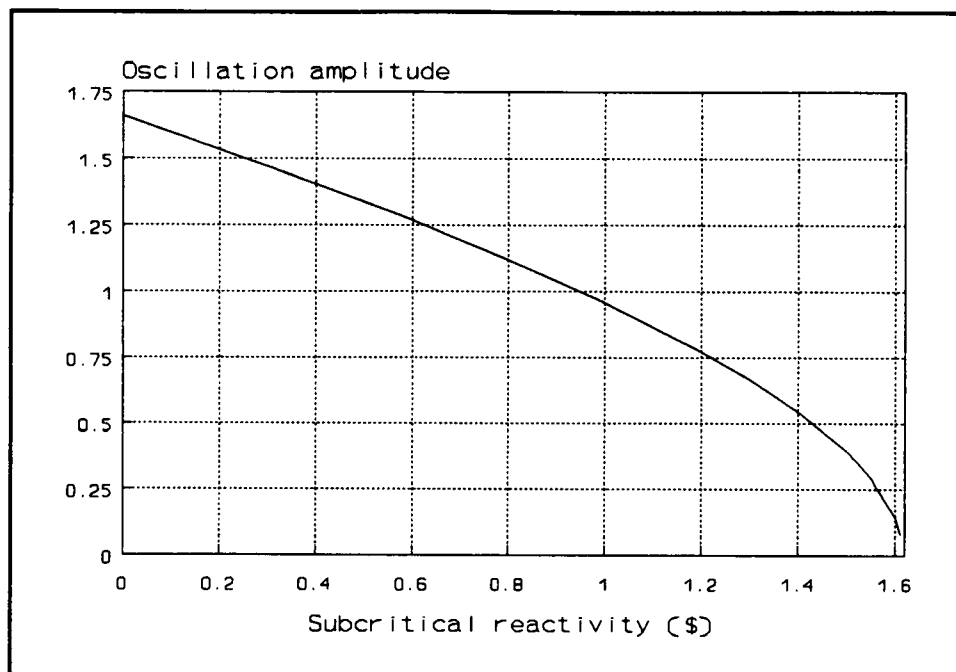


Figure 6.22. Effect of subcritical reactivity on limit-cycle amplitude calculated for the regional mode of the Ringhals-1 power station model. The amplitude was obtained from the one-harmonic solution.

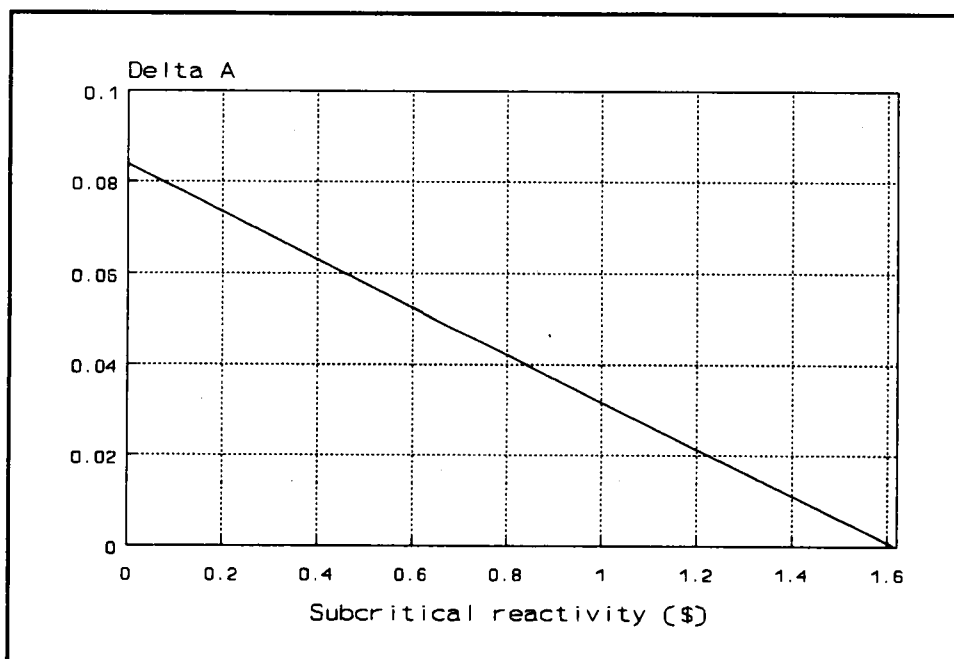


Figure 6.23. Effect of subcritical reactivity on ΔA .

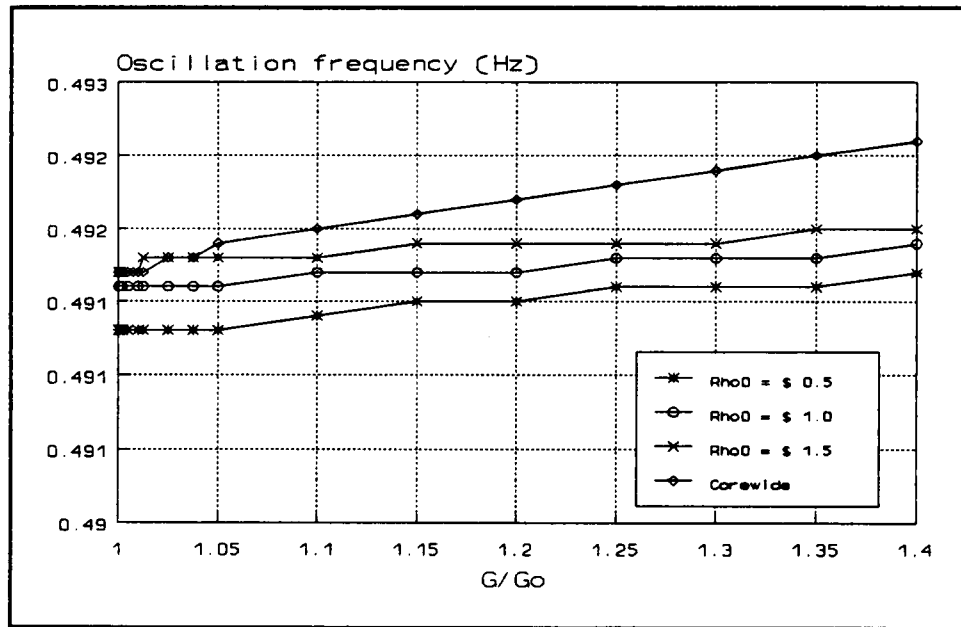


Figure 6.24. Effect of feedback gain on limit-cycle oscillation frequency calculated for the Ringhals-1 power station model. The oscillation frequencies were obtained from the one-harmonic solution.

affected, a decrease in oscillation frequency occurs (Figure 6.23). This frequency decrease causes a large change in the magnitude of the feedback transfer function at the oscillation frequency and, therefore, a large change in the limit-cycle amplitude. Figure 6.24 shows the limit-cycle amplitude for the regional and corewide modes of the Ringhals-1 power station calculated for a range of time delays. Because the regional modes calculated for subcritical reactivities of \$1.5, \$1.0, and \$0.5 are unstable for no time delay, a negative time delay (i.e., a time advance) was used to bring the equation conditions to the stability boundary; a time delay of just under 0.2 s was needed to make the corewide mode unstable. The results show that for a given time delay, the limit-cycle amplitude increases with decreasing subcritical reactivity. This increase occurs because the subcritical reactivity

decrease causes the system of equations to be more unstable (i.e., the value of ΔA is greater at a given time delay for smaller values of subcritical reactivity). The spike in the limit-cycle amplitude for the regional mode with increasing time delay is caused by the spike near 0.49 Hz in the magnitude of the feedback transfer function. Figure 6.25 shows the same limit-cycle amplitudes used to form Figure 6.24 plotted against ΔA . The dependence of limit-cycle amplitude on ΔA confirms that the spikes in limit-cycle amplitude shown in Figure 6.24 occur because of increases in ΔA . The results for a time delay show that the oscillation frequency decreases significantly with increasing time delay, with the decrease depending on the transfer function phase, and that the limit-cycle amplitude for a given value of subcritical reactivity depends indirectly on the magnitude of the feedback transfer function at the oscillation frequency.

6.7 The High-Order Approximate Solution to the Equations Obtained from the BWR Dynamic Model Consisting of the Modal Point Kinetics Equations and the LAPUR-Calculated Feedback Transfer Function

This section presents the results from calculations of the high-order approximate solution of the equations obtained from the BWR dynamic model consisting of the modal point kinetics equations and the LAPUR-calculated feedback transfer function. A FORTRAN computer code was written to form and solve the equations obtained from Eqs. (6.57), (6.58), and (6.59) for the constants in the approximate solution for $n_{\infty}(t)$, which is given by Eq. (6.31). The code was used to calculate the approximate solution for two cases. The first case corresponds to a corewide oscillation with the feedback transfer function given by Eq. (4.17). The transfer function was calculated for the phenomenological model constants listed in Table 3.1. The second case is a regional oscillation of the Ringhals-1 power station model with a subcritical reactivity value of \$1.5.

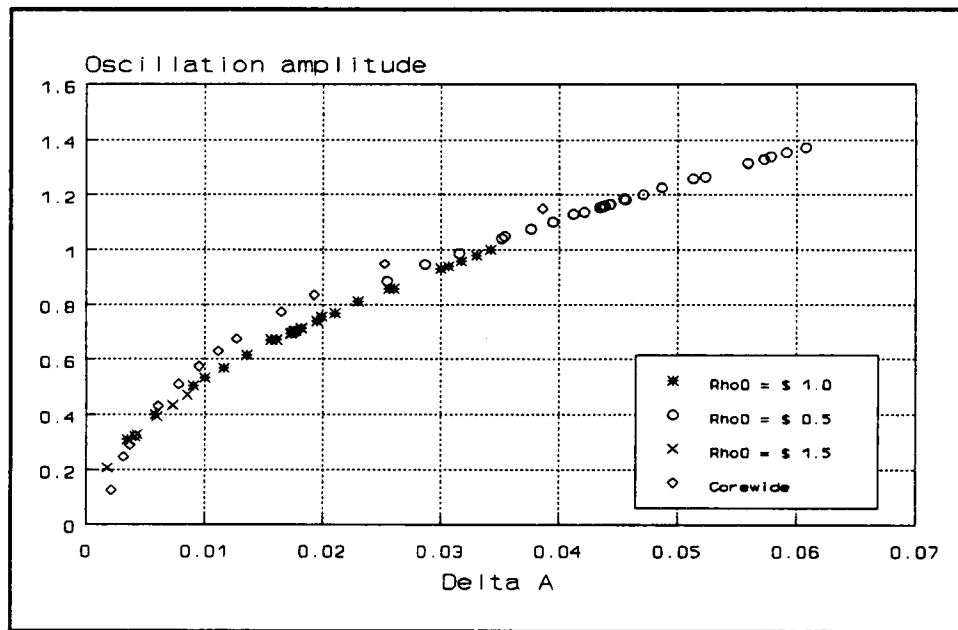


Figure 6.25. Effect of ΔA on limit-cycle amplitude. These limit-cycle amplitudes were also used to form Figure 6.24, in which they were plotted against time delay.

Table 6.5 lists the constants for two- to eight-harmonic approximate solutions calculated by using the phenomenological model constants listed in Table 3.1 and by using Eq. (4.17) to calculate the feedback transfer function. The value of subcritical reactivity was zero because this input corresponds to a corewide oscillation mode. The results show the solution converging as the number of harmonics is increased. Comparison of the calculated results listed in Tables 4.1 and 6.5 shows that the BWR dynamic model used in Chapter 6 can reproduce the results obtained from the BWR dynamic model used in Chapter 4. Thus, it has been shown that the solution converges as the number of harmonics is increased and that the BWR dynamic model using the modal point kinetics equations can reproduce the results obtained from the BWR dynamic model using the

Table 6.5. Approximate solution constants for two- to eight-harmonic solutions of the modal point kinetics-LAPUR feedback transfer function dynamic model equations calculated by using the constants listed in Table 3.1

Constant	Number of harmonics						
	2	3	4	5	6	7	8
ω (Hz)	0.4240	0.4210	0.4181	0.4174	0.4171	0.4170	0.4170
R	0.0386	0.0497	0.0542	0.0557	0.0562	0.0564	0.0564
P_1	0.6376	0.4836	0.3580	0.2663	0.1994	0.1474	0.1067
Q_1	0.6639	0.9176	1.0180	1.0600	1.0788	1.0884	1.0934
P_2	0.3623	0.4424	0.3898	0.3195	0.2579	0.2063	0.1643
Q_2		0.2615	0.4149	0.4933	0.5343	0.5579	0.5721
P_3		0.2158	0.2690	0.2512	0.2172	0.1824	0.1517
Q_3			0.1120	0.1917	0.2396	0.2695	0.2885
P_4			0.1249	0.1557	0.1500	0.1336	0.1156
Q_4				0.0498	0.0899	0.1178	0.1365
P_5				0.0706	0.0880	0.0863	0.0786
Q_5					0.0225	0.0431	0.0584
P_6					0.0393	0.0490	0.0487
Q_6						0.0106	0.0210
P_7						0.0218	0.0271
Q_7							0.0050
P_8							0.0120

single-mode point kinetics equations. This second result indicates that the BWR dynamic model using the modal point kinetics equations is an extension of the BWR dynamic model using the one-mode point kinetics equations.

The seven-harmonic approximate solution for $n_{ss}(t)$, $\rho_{ss}(t)$, and normalized inlet flow, calculated for the regional mode of the Ringhals-1 power station model for a value of subcritical reactivity equal to β_{eff} is shown in Figure 6.26. Note that the inlet flow oscillation amplitude is practically zero because of the constant pressure drop boundary condition imposed by the LAPUR code during the calculation of the transfer functions.

6.8 Chapter Summary

This chapter described the application of Galerkin's method to obtain approximate solutions to sets of nonlinear differential equations describing BWR dynamics during limit-cycle oscillations. Two BWR dynamic models were used. A multimodal phenomenological model was formed by substituting the modal point kinetics equations for the single-model point kinetics equations in the phenomenological model used in Chapter 3. A second BWR dynamic model was formed by using the modal point kinetics equations as the forward transfer function and the LAPUR-calculated feedback transfer function as the feedback loop. Each of these models allowed the effect of subcritical reactivity on limit-cycle behavior to be examined.

Approximate solutions calculated for each of these BWR dynamic models show the following significant results:

1. Increasing the subcritical reactivity is stabilizing. For a given set of conditions, increasing the subcritical reactivity will either reduce the amplitude or totally eliminate limit-cycle oscillations.

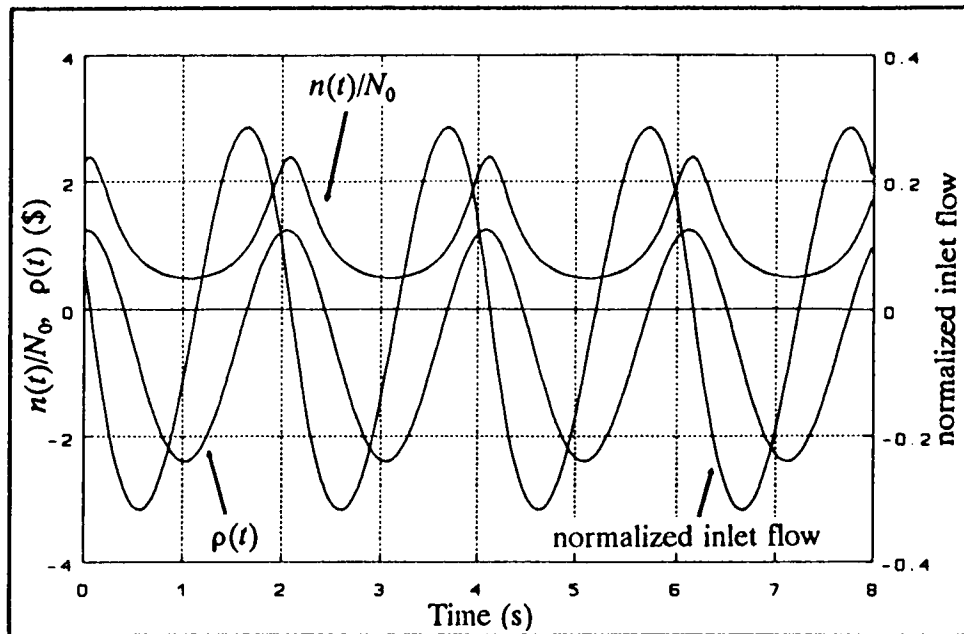


Figure 6.26. Ten-harmonic approximate solution for the regional model of the Ringhals-1 power station model calculated for a subcritical reactivity of \$1.5.

2. Both the corewide and regional limit-cycle amplitude and steady-state power increase as the magnitude of the feedback gain increases. The regional oscillation mode is more sensitive to changes in the feedback gain than is the corewide mode.
3. The limit-cycle oscillation frequency is relatively insensitive to the feedback gain. The oscillation frequency is very sensitive to the value of a_4 in the multimodal phenomenological model, indicating that the oscillation frequency is sensitive to the coolant velocity.
4. The sensitivity of the regional oscillation mode to feedback gain is not greatly affected by the subcritical reactivity.
5. A time delay in the feedback transfer function causes a significant reduction in the limit-cycle oscillation frequency.
6. A change in the limit-cycle amplitude is caused by a time delay in the feedback transfer function. The magnitude of this amplitude change depends on the magnitude of the feedback transfer function at the oscillation frequency.
7. Stability conditions were obtained for both BWR dynamic models. The stability conditions are given by Eqs. (6.52) and (6.67).

7. SUMMARY AND CONCLUSIONS

This chapter summarizes significant results obtained from the research described in Chapters 3, 4, 5, and 6. In addition to this summary, a number of general conclusions drawn regarding the applicability of weighted-residual methods to the examination of BWR stability are presented in Section 7.2. The chapter concludes with suggestions in Section 7.3 for future work that are related to the research described in this dissertation.

7.1 Summary of Significant Results

The significant results obtained from Chapters 3, 4, 5, and 6 are listed in Subsections 7.1.1 through 7.1.4.

7.1.1 Significant Results Obtained from the Research Described in Chapter 3

1. Galerkin's method can be applied successfully to calculate approximate solutions to the phenomenological model equations.
2. The number of harmonics required to accurately represent the solution increases as the system of equations becomes more unstable. The waveform of the normalized neutron population is characterized by narrow peaks separated by relatively broad troughs. The peak amplitude increases and the peaks become more narrow as the system of equations becomes more unstable. Conversely, near the stability boundary, the normalized neutron population can be accurately represented by a simple sinusoidal oscillation.
3. The limit-cycle amplitude and oscillation period increase as the system of equations becomes more unstable. Conversely, near the stability boundary, the limit-cycle amplitude approaches zero.
4. Near the stability boundary, a single harmonic accurately represents the solution. This result means that a low-order solution can be used to examine the solution properties near the stability boundary.
5. The degree of stability and the limit-cycle amplitude increase with the magnitude of the feedback gain as measured by the parameter K in the phenomenological model equations.

6. An expression was developed for the value of K at the stability boundary in terms of the remaining parameters in the phenomenological model equations by using the low-order approximate solution (Eq. 3.51).
7. Increasing a_1 , which corresponds to decreasing the fuel thermal inertia, is destabilizing.
8. Increasing a_3 and a_4 , which corresponds to decreasing the average void fraction or decreasing the channel transit time, is stabilizing.
9. Increasing either α_T or β is stabilizing.

7.1.2 Significant Results Obtained from the Research Described in Chapter 4

1. An approximate solution to the equations obtained from the BWR dynamic model consisting of the one-mode point kinetics equations and the LAPUR-calculated feedback transfer function can be obtained by applying Galerkin's method.
2. The approximate solution was shown to converge to the exact solution as additional terms were included in the approximation.
3. The approximate solution for the normalized neutron population was calculated for the instability event at the LaSalle-2 power station. Comparison with measured neutron detector data recorded during the event showed good agreement. This agreement indicates that the application of Galerkin's method to the nonlinear differential equations obtained from the BWR dynamic model consisting of the one-mode point kinetics equations and the LAPUR-calculated feedback transfer function can be used to accurately approximate the overall dynamic behavior of BWRs during steady-state limit-cycle oscillations.
4. In addition to the normalized neutron population, the behavior of other reactor parameters that can be related to the normalized neutron population by a product of LAPUR-calculated transfer functions can be calculated during limit-cycle oscillations.
5. A stability condition written in terms of the feedback transfer function values and the point kinetics parameters was formed (Eq. 4.27).
6. Equations for the limit-cycle amplitude and frequency near the stability boundary were formed by using the low-order approximate solution (Eqs. 4.24, 4.25, and 4.26).
7. Equation 4.26 was used to show that the steady-state part of the solution for the normalized neutron population during limit-cycle oscillations must be positive. This result indicates that the average power must increase during limit-cycle oscillations.

8. The limit-cycle amplitude must increase as the real part of the transfer function increases.
9. The phase of the feedback transfer function at the oscillation frequency will be of small magnitude and positive. This result means that the limit-cycle oscillation frequency will be near the zero crossing point of the feedback transfer function phase plot.

7.1.3 Significant Results Obtained from the Research Described in Chapter 5

1. A set of equations describing the time-dependent behavior of perturbations from a candidate approximate solution for limit-cycle oscillations was derived. Examination of the asymptotic stability of a linearized version of these equations showed that the candidate approximate solution can be either stable or unstable. A stable solution indicates that the form of the candidate solution (i.e., either an unbifurcated or bifurcated form) is stable. An unstable solution indicates that the form of the stable solution is different from that of the candidate approximate solution.
2. The stability boundary of the perturbations from a candidate limit-cycle solution corresponds to a bifurcation point in the limit cycle.
3. The linearized version of the equations for the perturbations about a candidate solution is not well behaved (i.e., a small equation residual does not necessarily mean the approximate solution closely approximates the exact solution). Thus, weighted-residual methods cannot be applied to obtain an approximate solution to the linearized version of these equations.
4. Because the linearized version of the equations for the perturbations about a candidate solution contains harmonic coefficients, a stability test based on Floquets theory can be developed to determine the asymptotic stability of these equations. This stability test can be used to determine whether a candidate limit-cycle solution is stable (i.e., the test can be used to determine when a bifurcation occurs in the limit cycle).
5. An approximate solution to the nonlinear form of the equations describing the perturbations about the candidate steady-state solution was obtained by using Galerkin's method. The approximate solution for the perturbation in normalized neutron population is in the form of a Fourier series. A maximum of nine harmonics was calculable—far too few to accurately represent the solution. The limitation to only nine harmonics was most probably due to precision limitations in the numerical calculation of the constants in the approximate solution. Low-order solutions were too complex to allow any additional insight into the nature of the bifurcated solution to be obtained that is not available from direct numerical integration.

6. A second attempt at an approximate solution to the nonlinear equations describing the perturbations about the candidate steady-state solution was made by using the product of a low-order harmonic oscillation and the first time derivative of the normalized neutron population as the approximate solution. This particular solution form was selected because the numerical solution for the perturbation in normalized neutron population closely resembles the first derivative of the normalized neutron population. This approximate solution required the calculation of relatively few terms and agreed reasonably well with the numerical solution for the bifurcated limit cycle. Again, the approximate solution showed no particular advantage when compared to calculating the bifurcated solution by using numerical integration.

7.1.4 Significant Results Obtained from the Research Described in Chapter 6

1. The BWR dynamic models of Chapters 3 and 4 were modified to include the effects of subcritical reactivity. This modification allowed examination of the regional limit cycle.
2. The regional limit cycle behaves essentially the same as the corewide limit cycle. The similarities include both increasing amplitude and decreasing frequency as the system of equations becomes more unstable as well as the existence of bifurcations in the regional limit cycle.
3. Stability conditions were obtained for both regional BWR dynamic models (Eqs. 6.52 and 6.66).
4. Increasing subcritical reactivity was found to be stabilizing. An increase in subcritical reactivity will either reduce limit-cycle amplitude or eliminate the oscillations altogether.
5. The regional oscillation mode is more sensitive than the corewide oscillation mode to a gain change in the feedback loop. The greater sensitivity of the regional mode appears to be due to the difference in the feedback transfer functions of the two oscillation modes.
6. The sensitivity of the regional oscillation mode to feedback gain is not greatly affected by changes in the subcritical reactivity.

7.2 Conclusions

A number of general conclusions can be drawn concerning the application of weighted-residual methods to examine BWR dynamic behavior during limit-cycle oscillations. First, the work described in this dissertation shows that weighted-residual

methods can be used to obtain a considerable amount of information about the BWR limit cycle. Much of the most significant information, such as the sensitivity of the stability boundary to various parameters or the effect of parameters on the steady-state limit-cycle amplitude or frequency, was able to be determined by using the low-order approximate solutions obtained from the application of weighted-residual methods. That many of the results obtained from the application of weighted-residual methods had been found previously by numerical methods does not alter the fact that weighted-residual methods are a reasonable alternative method for examining BWR stability.

The main shortcoming of the work presented in this dissertation is that the approximate solutions apply to only steady-state oscillations; transient solutions cannot be calculated by any of the approximate solutions presented in this dissertation. Thus, weighted-residual methods cannot be used in simulations of plant events and cannot replace or even augment the use of numerical methods for simulation of plant transients during limit-cycle oscillations.

The use of the low-order approximate solution to obtain information about the BWR limit cycle near the stability boundary is the main advantage of the application of weighted-residual methods to examine BWR stability. Although it should be possible to obtain similar information about the BWR stability boundary and its sensitivity to the various model parameters from examination of the conditions for asymptotic stability of a linearized version of the model equations about the appropriate equilibrium point, it is unlikely that the stability condition could be obtained without considerably more work than was required here. The stability conditions obtained from the low-order approximate solutions have been used to show the effect of various model parameters on the stability boundary, allowing changes in these parameters to be classified as either stabilizing or

destabilizing. In some cases, in which the model parameters can be related to physical parameters such as void reactivity coefficient or channel transit time, the effect of these physical parameters on the stability boundary has been determined. It has been a relatively simple matter to examine the effect of model parameters on stability by using the stability condition because of the relatively simple form of the stability conditions. To obtain similar information by using numerical integration would have required a large parametric study involving considerable computer time and expense. Thus, the use of weighted-residual methods to examine BWR dynamic behavior near the stability boundary appears to have considerable advantages compared to numerical studies.

It was shown that given a sufficient number of harmonics, the weighted-residual solution will closely approximate numerical solutions describing limit-cycle oscillations. The work described in Chapter 5 indicates that the number of harmonics that can be included in an approximate solution may ultimately be limited by the precision of the computer used to perform the calculations. This limitation has been a severe problem when attempts were made to calculate the bifurcated limit cycle, which requires on the order of 20 or more harmonics to result in an accurate approximate solution. This limitation is not expected to be significant when calculating unbifurcated limit cycles, the type of limit cycle most frequently encountered during the majority of BWR instability events. Comparison of the calculation of the steady-state limit cycle by using weighted-residual techniques and numerical integration would indicate that the approximate solution calculated by using weighted-residual techniques can be obtained more rapidly than the solution obtained by numerical integration, but some error will still be associated with the approximate solution. An indication of the general magnitude of this error can be obtained from the magnitude of the equation residual.

It was shown in Chapter 4 that the high-order approximate solution to the BWR dynamic model consisting of the one-mode point kinetics equations and the LAPUR-calculated feedback transfer function can be used to accurately approximate the amplitude of steady-state limit-cycle oscillations. This result means that the combination of weighted-residual methods with the BWR dynamic model used in Chapter 4 yields a tool that can be used to extend the results of a frequency-domain code such as the LAPUR code into the time domain to examine the limit-cycle behavior of actual BWRs. This tool can be used to obtain information on the BWR limit cycle that until now has been available from only time-consuming and expensive BWR transient analysis codes.

The results in Chapter 5 indicate some of the limitations of the application of weighted-residual methods to obtaining approximate solutions. As discussed above, it appears that accurate solutions obtained by using weighted-residual methods are limited to solutions that do not require a very large number of harmonics. The number of harmonics that can be calculated for any given solution will be determined by the extensiveness of the calculations required to form the equations for the undetermined constants in the approximate solution. Another limitation is that weighted-residual methods are limited to systems that are well behaved. The weighted-residual methods failed to produce a bifurcation test because the linearized version of the equations describing the perturbations from the candidate steady-state solutions was not well behaved. No known test exists to determine whether a system of equations is well behaved before attempting to apply weighted-residual methods. Thus, successful application of these methods is not guaranteed when attempting a weighted-residual solution.

7.3 Suggestions for Future Work

It appears that the further application of weighted-residual methods in the area of BWR stability would have to involve the two applications that did not yield particularly favorable results in the work described in this dissertation; the development of a bifurcation test and the calculation of the bifurcated limit cycle. All of the steady-state solutions described in this dissertation have used trigonometric functions as the trial functions in the approximate solution. Using other complete sets of functions may be useful to form the approximate solution for the nonlinear set of equations describing the perturbations from a candidate steady-state solution. A particularly appealing choice for the trial functions for this application may be wavelets. These functions have orthogonal properties that should make the equations for the undetermined constants in the approximate solution reasonably compact, and wavelets can approximate complex waveforms with relatively few terms.

Some interesting work that could be attempted with either wavelets or some other set of complete orthogonal functions are listed below.

1. Form an approximate solution for the perturbations from the candidate solution of the normalized neutron population by using a set of complete orthogonal functions (other than the trigonometric functions) as the trial functions. The approximate solution will be much like those used in the work described in this dissertation, with the approximate solution comprising a summation of trial functions each multiplied by an undetermined constant.
2. Form an expression for the equation residual. The equations for the undetermined constants are obtained by applying a weighted-residual method. Note that the candidate approximate solution for the normalized neutron population may need to be written in terms of the trial functions used in the approximate solution for the perturbation about the candidate solution to take full advantage of the orthogonal properties of the trial functions.
3. Examine the properties of the approximate solution. It would be most advantageous if the approximate solution behaved essentially the same as those described in Chapters 3, 4, and 6, in that a low-order solution gives accurate

results near the stability boundary and that a reasonable number of terms would result in an accurate approximate solution.

4. If an accurate approximate solution can be found, then an accurate approximate solution for the bifurcated limit cycle can be obtained by adding the candidate approximate solution for the normalized neutron population to the approximate solution for the perturbations about the normalized neutron population. Furthermore, whether the approximate solution obtained by adding these two solutions can be used as the initial estimate for the calculation of normalized neutron population should be investigated. If such be possible, the bifurcated limit cycle could be calculated directly by using the approximate solution described in Chapter 3.
5. If a low-order approximate solution described the behavior of the approximate solution for the perturbations about the normalized neutron population near the stability boundary, then the low-order solution may be able to be used to yield a stability condition similar to those developed in Chapters 3, 4, and 6. Such a stability condition would essentially serve as a bifurcation test and could be used to determine the bifurcation points.

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APPENDIXES

APPENDIX A.

EQUATIONS FOR CONSTANTS USED IN THE FORMATION OF THE APPROXIMATE SOLUTION OF THE PHENOMENOLOGICAL MODEL EQUATIONS

This appendix lists equations for constants used in the development of the approximate solution to the phenomenological model equations.

$$W = \frac{\beta R}{\Lambda \lambda} \quad (\text{A.1})$$

where β = the delayed-neutron precursor fraction,
 R = the steady-state component of the approximate solution for neutron population,
 Λ = the mean generation time,
 λ = the precursor decay constant.

$$\sigma_{n1} = \frac{\beta (n \omega Q_n + \lambda P_n)}{\Lambda (\lambda^2 + n^2 \omega^2)} \quad (\text{A.2})$$

where ω = the fundamental oscillation frequency of the approximate solution for neutron population,
 Q_n = the multiplier of the n th cosine term in the approximate solution for neutron population,
 P_n = the multiplier of the n th sine term in the approximate solution for neutron population,

$$\sigma_{n2} = \frac{\beta (\lambda Q_n - n \omega P_n)}{\Lambda (\lambda^2 + n^2 \omega^2)} \quad (\text{A.3})$$

$$U = \frac{a_1 R}{a_2} \quad (\text{A.4})$$

where a_1 and a_2 = fitted constants in the phenomenological model equations.

$$\tau_{n1} = \frac{a_1(n\omega Q_n + a_2 P_n)}{a_2^2 + n^2\omega^2} \quad (\text{A.5})$$

$$\tau_{n2} = \frac{a_1(a_2 Q_n - n\omega P_n)}{a_2^2 + n^2\omega^2} \quad (\text{A.6})$$

$$V = \frac{K a_1 R}{a_2 a_4} \quad (\text{A.7})$$

where K and a_4 = fitted constants in the phenomenological model equations.

$$R_{n1} = \frac{a_1 K \left\{ P_n [a_2 a_4 - (a_2 + a_3) n^2 \omega^2] + Q_n (a_2 a_3 + a_4 - n^2 \omega^2) n \omega \right\}}{(a_2^2 + n^2 \omega^2) [(a_4 - n^2 \omega^2)^2 + (a_3 n \omega)^2]} \quad (\text{A.8})$$

where a_3 = a fitted constant in the phenomenological model equations.

$$R_{n2} = \frac{a_1 K \left\{ Q_n [a_2 a_4 - (a_2 + a_3) n^2 \omega^2] - P_n (a_2 a_3 + a_4 - n^2 \omega^2) n \omega \right\}}{(a_2^2 + n^2 \omega^2) [(a_4 - n^2 \omega^2)^2 + (a_3 n \omega)^2]} \quad (\text{A.9})$$

$$F_{1n} = \frac{\beta n \omega}{\Lambda (\lambda^2 + n^2 \omega^2)} \quad (\text{A.10})$$

$$F_{2n} = \frac{\lambda \beta}{\Lambda (\lambda^2 + n^2 \omega^2)} \quad (\text{A.11})$$

$$F_{3n} = \frac{n \omega a_1}{a_2^2 + n^2 \omega^2} \quad (\text{A.12})$$

$$F_{4n} = \frac{a_1 a_2}{(a_2^2 + n^2 \omega^2)} \quad (\text{A.13})$$

$$F_{5n} = \frac{K \left[a_3 n \omega F_{4n} + F_{3n} (a_4 - n^2 \omega^2) \right]}{\left[(a_4 - n^2 \omega^2)^2 + (a_3 n \omega)^2 \right]} \quad (\text{A.14})$$

$$F_{6n} = \frac{K \left[F_{4n} (a_4 - n^2 \omega^2) - a_3 n \omega F_{3n} \right]}{\left[(a_4 - n^2 \omega^2)^2 + (a_3 n \omega)^2 \right]} \quad (\text{A.15})$$

$$F_{7n} = \frac{F_{5n} (a_4 - n^2 \omega^2) - K F_{3n}}{a_3 n \omega} \quad (\text{A.16})$$

$$F_{8n} = \frac{F_{6n} (a_4 - n^2 \omega^2) - K F_{4n}}{a_3 n \omega} \quad (\text{A.17})$$

$$G_{1n} = F_{5n} + \alpha_T F_{3n} \quad (\text{A.18})$$

where α_T = the Doppler coefficient of reactivity.

$$G_{2n} = F_{6n} + \alpha_T F_{4n} \quad (\text{A.19})$$

$$G_{3n} = F_{7n} + \alpha_T F_{4n} \quad (\text{A.20})$$

$$G_{4n} = F_{8n} - \alpha_T F_{3n} \quad (\text{A.21})$$

$$G_{5n} = F_{5n} + \alpha_T F_{3n} + \lambda \Lambda F_{1n} + \Lambda n \omega \quad (\text{A.22})$$

$$G_{6n} = F_{6n} + \alpha_T F_{4n} - \beta + \lambda \Lambda F_{2n} \quad (\text{A.23})$$

$$G_{7n} = F_{7n} + \alpha_T F_{4n} + \lambda \Lambda F_{2n} - \beta \quad (\text{A.24})$$

$$G_{8n} = F_{8n} - \alpha_T F_{3n} - \lambda \Lambda F_{1n} - \Lambda n \omega \quad (\text{A.25})$$

$$G_{9n} = \left(\frac{K}{a_4} + \alpha_T \right) \frac{a_1}{a_2} \quad (\text{A.26})$$

APPENDIX B.

INTEGRALS OVER ONE CYCLE FOR PRODUCTS OF SINES AND COSINES

This appendix lists integrals of products of trigonometric functions evaluated over one cycle. The frequency of each trigonometric function is an integer multiple of the fundamental frequency ω .

$$\int_0^{\frac{2\pi}{\omega}} \sin(n\omega t) \sin(k\omega t) dt = \begin{cases} 0 & \text{if } n \neq k, \\ \frac{\pi}{\omega} & \text{if } n = k, \end{cases} \quad (\text{B.1})$$

where n and k = integers,
 t = time.

$$\int_0^{\frac{2\pi}{\omega}} \sin(n\omega t) \cos(k\omega t) dt = 0 \quad \text{for all } n \text{ and } k. \quad (\text{B.2})$$

$$\int_0^{\frac{2\pi}{\omega}} \cos(n\omega t) \cos(k\omega t) dt = \begin{cases} 0 & \text{if } n \neq k, \\ \frac{\pi}{\omega} & \text{if } n = k. \end{cases} \quad (\text{B.3})$$

$$\int_0^{\frac{2\pi}{\omega}} \sin(i\omega t) \sin(j\omega t) \sin(k\omega t) dt = 0 \quad \text{for all } i, j, \text{ and } k. \quad (\text{B.4})$$

$$\int_0^{\frac{2\pi}{\omega}} \sin(i\omega t) \sin(k\omega t) \cos(n\omega t) dt = \begin{cases} -\pi/2\omega & \text{if } i+k=n, \\ \pi/2\omega & \text{if } i+n=k, \\ \pi/2\omega & \text{if } k+n=i, \\ 0 & \text{otherwise.} \end{cases} \quad (\text{B.5})$$

$$\int_0^{\frac{2\pi}{\omega}} \sin(i\omega t) \cos(j\omega t) \cos(k\omega t) dt = 0 \text{ for all } i, j, \text{ and } k. \quad (\text{B.6})$$

$$\int_0^{\frac{2\pi}{\omega}} \cos(i\omega t) \cos(k\omega t) \cos(n\omega t) dt = \begin{cases} \pi/2\omega & \text{if } i+k=n, \\ \pi/2\omega & \text{if } i+n=k, \\ \pi/2\omega & \text{if } k+n=i, \\ 0 & \text{otherwise.} \end{cases} \quad (\text{B.7})$$

$$\int_0^{\frac{2\pi}{\omega}} \sin(n\omega t) \sin(k\omega t) \sin(l\omega t) \sin(i\omega t) dt = \begin{cases} \pi/4\omega & \text{if } n+k=l+i, \\ \pi/4\omega & \text{if } k+l=n+i, \\ \pi/4\omega & \text{if } l+n=k+i, \\ -\pi/4\omega & \text{if } n+k+i=l, \\ -\pi/4\omega & \text{if } k+l+i=n, \\ -\pi/4\omega & \text{if } l+n+i=k, \\ -\pi/4\omega & \text{if } n+k+l=i, \\ 0 & \text{otherwise.} \end{cases} \quad (\text{B.8})$$

where $l = \text{an integer.}$

$$\int_0^{\frac{2\pi}{\omega}} \sin(n\omega t) \sin(k\omega t) \sin(i\omega t) \cos(l\omega t) dt = 0 \text{ for all } i, n, k, \text{ and } l. \quad (\text{B.9})$$

$$\frac{2\pi}{\omega} \int_0^{\omega} \sin(n\omega t) \sin(k\omega t) \cos(l\omega t) \cos(i\omega t) dt = \begin{cases} \pi/4\omega & \text{if } k+l=n+i, \\ \pi/4\omega & \text{if } n+l=k+i, \\ \pi/4\omega & \text{if } k+l+i=n, \\ \pi/4\omega & \text{if } l+n+i=k, \\ -\pi/4\omega & \text{if } n+k=i+l, \\ -\pi/4\omega & \text{if } l+n+k=i, \\ -\pi/4\omega & \text{if } n+k+i=l, \\ 0 & \text{otherwise .} \end{cases} \quad (\text{B.10})$$

$$\frac{2\pi}{\omega} \int_0^{\omega} \sin(l\omega t) \cos(n\omega t) \cos(k\omega t) \cos(i\omega t) dt = 0 \text{ for all } i, n, k, \text{ and } l. \quad (\text{B.11})$$

$$\frac{2\pi}{\omega} \int_0^{\omega} \cos(n\omega t) \cos(k\omega t) \cos(l\omega t) \cos(i\omega t) dt = \begin{cases} \pi/4\omega & \text{if } n+k=i+l, \\ \pi/4\omega & \text{if } k+l=n+i, \\ \pi/4\omega & \text{if } l+n=k+i, \\ \pi/4\omega & \text{if } n+k+i=l, \\ \pi/4\omega & \text{if } k+l+i=n, \\ \pi/4\omega & \text{if } l+n+i=k, \\ \pi/4\omega & \text{if } n+k+l=i, \\ 0 & \text{otherwise .} \end{cases} \quad (\text{B.12})$$

$$\frac{2\pi}{\omega} \int_0^{\omega} \sin(i\omega t) \sin(j\omega t) \sin(k\omega t) \sin(l\omega t) \sin(n\omega t) dt = 0 \text{ for all } i, j, k, l, \text{ and } n. \quad (\text{B.13})$$

$$\begin{aligned}
& \frac{2\pi}{\omega} \int_0^{\omega} \sin(i\omega t) \sin(j\omega t) \sin(k\omega t) \sin(l\omega t) \cos(n\omega t) dt \\
& = \begin{cases} -\pi/8\omega & \text{if } i-j+k+l-n=0, \\ -\pi/8\omega & \text{if } i-j-k-l+n=0, \\ \pi/8\omega & \text{if } i-j+l+n-k=0, \\ \pi/8\omega & \text{if } i-j-l-n+k=0, \\ \pi/8\omega & \text{if } i-j+n+k-l=0, \\ \pi/8\omega & \text{if } i-j-n-k+l=0, \\ -\pi/8\omega & \text{if } i-j+n+k+l=0, \\ -\pi/8\omega & \text{if } i-j-n-k-l=0, \\ \pi/8\omega & \text{if } i+j+k+l-n=0, \\ \pi/8\omega & \text{if } i+j-k-l+n=0, \\ -\pi/8\omega & \text{if } i+j+l+n-k=0, \\ -\pi/8\omega & \text{if } i+j-l-n+k=0, \\ -\pi/8\omega & \text{if } i+j+n+k-l=0, \\ -\pi/8\omega & \text{if } i+j-n-k+l=0, \\ \pi/8\omega & \text{if } i+j-n-k-l=0, \\ 0 & \text{otherwise .} \end{cases} \quad (\text{B.14})
\end{aligned}$$

$$\frac{2\pi}{\omega} \int_0^{\omega} \sin(i\omega t) \sin(j\omega t) \sin(k\omega t) \cos(l\omega t) \cos(n\omega t) dt = 0 \text{ for all } i, j, k, l, \text{ and } n. \quad (\text{B.15})$$

$$\begin{aligned}
& \frac{2\pi}{\omega} \int_0^{\omega} \sin(i\omega t) \sin(j\omega t) \cos(k\omega t) \cos(l\omega t) \cos(n\omega t) dt \\
& = \begin{cases} \pi/8\omega & \text{if } i-j+k+l-n=0, \\ \pi/8\omega & \text{if } i-j-k-l+n=0, \\ \pi/8\omega & \text{if } i-j+l+n-k=0, \\ \pi/8\omega & \text{if } i-j-l-n+k=0, \\ \pi/8\omega & \text{if } i-j+n+k-l=0, \\ \pi/8\omega & \text{if } i-j-n-k+l=0, \\ \pi/8\omega & \text{if } i-j+n+k+l=0, \\ \pi/8\omega & \text{if } i-j-n-k-l=0, \\ -\pi/8\omega & \text{if } i+j+k+l-n=0, \\ -\pi/8\omega & \text{if } i+j-k-l+n=0, \\ -\pi/8\omega & \text{if } i+j+l+n-k=0, \\ -\pi/8\omega & \text{if } i+j-l-n+k=0, \\ -\pi/8\omega & \text{if } i+j+n+k-l=0, \\ -\pi/8\omega & \text{if } i+j-n-k+l=0, \\ -\pi/8\omega & \text{if } i+j-n-k-l=0, \\ 0 & \text{otherwise .} \end{cases} \quad (\text{B.16})
\end{aligned}$$

$$\frac{2\pi}{\omega} \int_0^{\omega} \sin(i\omega t) \cos(j\omega t) \cos(k\omega t) \cos(l\omega t) \cos(n\omega t) dt = 0 \text{ for all } i, j, k, l, \text{ and } n. \quad (\text{B.17})$$

$$\begin{aligned}
& \frac{2\pi}{\omega} \int_0^{\omega} \cos(i\omega t) \cos(j\omega t) \cos(k\omega t) \cos(l\omega t) \cos(n\omega t) dt \\
& = \begin{cases} \pi/8\omega & \text{if } i+j+k+l-n=0, \\ \pi/8\omega & \text{if } i+j-k-l-n=0, \\ \pi/8\omega & \text{if } i+j+l+n-k=0, \\ \pi/8\omega & \text{if } i+j-l-n+k=0, \\ \pi/8\omega & \text{if } i+j+n+k-l=0, \\ \pi/8\omega & \text{if } i+j-n-k+l=0, \\ \pi/8\omega & \text{if } i+j-n-k-l=0, \\ \pi/8\omega & \text{if } i-j+k+l-n=0, \\ \pi/8\omega & \text{if } i-j-k-l+n=0, \\ \pi/8\omega & \text{if } i-j+l+n-k=0, \\ \pi/8\omega & \text{if } i-j-l-n+k=0, \\ \pi/8\omega & \text{if } i-j+n+k-l=0, \\ \pi/8\omega & \text{if } i-j-n-k+l=0, \\ \pi/8\omega & \text{if } i-j+n+k+l=0, \\ \pi/8\omega & \text{if } i-j-n-k-l=0, \\ 0 & \text{otherwise .} \end{cases} \tag{B.18}
\end{aligned}$$

APPENDIX C.

**EQUATIONS FOR CONSTANTS USED IN THE FORMATION OF
APPROXIMATE SOLUTIONS OF THE EQUATIONS
DESCRIBING THE BIFURCATED LIMIT CYCLE**

This appendix lists equations for constants used in the derivation of approximate solutions for the linearized and nonlinear versions of the equations describing perturbations about a candidate unbifurcated limit cycle.

$$C_1 = \frac{\beta \epsilon_{n0} (\lambda + \alpha_1)}{\Lambda [\Omega^2 + (\lambda + \alpha_1)^2]} \quad (C.1)$$

where β = the delayed-neutron precursor fraction,
 ϵ_{n0} = initial value of the approximate solution to the linearized equations describing the perturbations about the neutron population,
 λ = the precursor decay constant,
 α_1 = an undetermined constant in the approximate solution to the linearized equations describing the perturbations about the neutron population,
 Λ = the mean generation time,
 Ω = the oscillation frequency of the approximate solution to the linearized equations.

$$C_2 = \frac{\beta \epsilon_{n0} \Omega}{\Lambda [\Omega^2 + (\lambda + \alpha_1)^2]} \quad (C.2)$$

where α_2 and A = undetermined constants in the approximate solution to the linearized equations describing the perturbations about the neutron population.

$$C_3 = \frac{A \beta \epsilon_0 (\lambda + \alpha_2)}{\Lambda [\Omega^2 + (\lambda + \alpha_2)^2]} \quad (C.3)$$

$$C_4 = \frac{A \beta \epsilon_{n0} \Omega}{\Lambda [\Omega^2 + (\lambda + \alpha_2)^2]} \quad (C.4)$$

$$C_5 = \frac{A \beta \epsilon_{n0} [\Omega^2 - (\lambda + \alpha_2)^2]}{\Lambda [\Omega^2 + (\lambda + \alpha_2)^2]^2} \quad (C.5)$$

$$C_6 = \frac{-2A \beta \epsilon_{n0} \Omega (\lambda + \alpha_2)}{\Lambda [\Omega^2 + (\lambda + \alpha_2)^2]^2} \quad (C.6)$$

$$C_c = -(C_1 + C_5) \quad (C.7)$$

$$T_1 = \frac{a_1 \epsilon_{n0} (a_2 + \alpha_1)}{\Omega^2 + (a_2 + \alpha_1)^2} \quad (C.8)$$

where a_1 and a_2 = fitted constants in the phenomenological model equations.

$$T_2 = \frac{a_1 \epsilon_{n0} \Omega}{\Omega^2 + (a_2 + \alpha_1)^2} \quad (C.9)$$

$$T_3 = \frac{A a_1 \epsilon_{n0} (a_2 + \alpha_2)}{\Omega^2 + (a_2 + \alpha_2)^2} \quad (C.10)$$

$$T_4 = \frac{A a_1 \epsilon_{n0} \Omega}{\Omega^2 + (a_2 + \alpha_2)^2} \quad (C.11)$$

$$T_5 = \frac{A a_1 \epsilon_{n0} [\Omega^2 - (a_2 + \alpha_2)^2]}{[\Omega^2 + (a_2 + \alpha_2)^2]^2} \quad (C.12)$$

$$T_6 = \frac{-2A a_1 \epsilon_{n0} \Omega (a_2 + \alpha_2)}{[\Omega^2 + (a_2 + \alpha_2)^2]^2} \quad (C.13)$$

$$T_i = -(T_1 + T_5) \quad (C.14)$$

$$R_1 = \frac{KT_i}{a_2^2 - a_2 a_3 + a_4} \quad (C.15)$$

where K , a_3 , and a_4 = fitted constants in the phenomenological model equations.

$$R_2 = \frac{-K[\Omega(2\alpha_1 + a_3)T_2 + (\Omega^2 - \alpha_1^2 - a_3\alpha_1 - a_4)T_1]}{[\Omega^4 + (2\alpha_1^2 + 2a_3\alpha_1 - 2a_4 + a_3^2)\Omega^2 + \alpha_1^4 + 2a_3\alpha_1^3 + (2a_4 + a_3^2)\alpha_1^2 + 2a_3a_4\alpha_1 + a_4^2]} \quad (C.16)$$

$$R_3 = \frac{K[\Omega(2\alpha_1 + a_3)T_1 + (-\Omega^2 + \alpha_1^2 + a_3\alpha_1 + a_4)T_2]}{[\Omega^4 + (2\alpha_1^2 + 2a_3\alpha_1 - 2a_4 + a_3^2)\Omega^2 + \alpha_1^4 + 2a_3\alpha_1^3 + (2a_4 + a_3^2)\alpha_1^2 + 2a_3a_4\alpha_1 + a_4^2]} \quad (C.17)$$

$$R_4 = - \left\{ K(2\alpha_2 + a_3)\Omega T_6 + (\alpha_2^2 + a_3\alpha_2 + a_4)KT_5 + (2\alpha_2^2 + 2a_3\alpha_2 - 2a_4 + a_3^2)\Omega R_7 \right. \\ \left. + [(-4\alpha_2 - 2a_3)\Omega^2 - 2\alpha_2^3 - 3a_3\alpha_2^2 + (-2a_4 - a_3^2)\alpha_2 - a_3a_4]R_6 \right\} / \\ [(4\alpha_2^2 + 4a_3\alpha_2 + a_3^2)\Omega^2 + \alpha_2^4 + 2a_3\alpha_2^3 + (2a_4 + a_3^2)\alpha_2^2 + 2a_3a_4\alpha_2 + a_4^2] \quad (C.18)$$

$$R_5 = \left\{ K(2\alpha_2 + a_3)\Omega T_5 + (\alpha_2^2 + a_3\alpha_2 + a_4)KT_6 - (2\alpha_2^2 + 2a_3\alpha_2 - 2a_4 + a_3^2)\Omega R_6 \right. \\ \left. + [(-4\alpha_2 - 2a_3)\Omega^2 - 2\alpha_2^3 - 3a_3\alpha_2^2 + (-2a_4 - a_3^2)\alpha_2 - a_3a_4]R_7 \right\} / \\ [(4\alpha_2^2 + 4a_3\alpha_2 + a_3^2)\Omega^2 + \alpha_2^4 + 2a_3\alpha_2^3 + (2a_4 + a_3^2)\alpha_2^2 + 2a_3a_4\alpha_2 + a_4^2] \quad (C.19)$$

$$R_6 = \frac{-K[\Omega(2\alpha_2 + a_3)T_4 + (\Omega^2 - \alpha_2^2 - a_3\alpha_2 - a_4)T_3]}{[\Omega^4 + (2\alpha_2^2 + 2a_3\alpha_2 - 2a_4 + a_3^2)\Omega^2 + \alpha_2^4 + 2a_3\alpha_2^3 + (2a_4 + a_3^2)\alpha_2^2 + 2a_3a_4\alpha_2 + a_4^2]} \quad (C.20)$$

$$R_7 = \frac{K[\Omega(2\alpha_2 + a_3)T_3 + (-\Omega^2 + \alpha_2^2 + a_3\alpha_2 + a_4)T_4]}{[\Omega^4 + (2\alpha_2^2 + 2a_3\alpha_2 - 2a_4 + a_3^2)\Omega^2 + \alpha_2^4 + 2a_3\alpha_2^3 + (2a_4 + a_3^2)\alpha_2^2 + 2a_3a_4\alpha_2 + a_4^2]} \quad (C.21)$$

$$g = \sqrt{\left| \frac{a_3^2}{4} - a_4 \right|} \quad (C.22)$$

$$S_g = \sin(gt), \quad C_g = \cos(gt) \quad (C.23)$$

$$\dot{c}_1 = -(R_1 + R_2 + R_4) \quad (C.24)$$

$$c_2 = \frac{[(a_3/2)\dot{c}_1 + a_2R_1 - \alpha_1R_2 - \Omega R_3 - \alpha_2R_4 - \Omega R_5 - R_6]}{g} \quad (C.25)$$

$$F_1 = -\Lambda \epsilon_{n0} \alpha_1 - \beta \epsilon_0 + R_2 + \alpha_T T_1 + \lambda \Lambda C_1 \quad (C.26)$$

where α_T = the nuclear Doppler coefficient.

$$F_2 = R_2 + \alpha_T T_1 \quad (C.27)$$

$$F_3 = \Lambda \varepsilon_{n0} \Omega + R_3 + \alpha_T T_2 + \lambda \Lambda C_2 \quad (C.28)$$

$$F_4 = R_3 + \alpha_T T_2 \quad (C.29)$$

$$F_5 = -\Lambda A \varepsilon_{n0} + R_4 + \alpha_T T_5 + \lambda \Lambda C_5 \quad (C.30)$$

$$F_6 = R_4 + \alpha_T T_5 \quad (C.31)$$

$$F_7 = R_5 + \alpha_T T_6 + \lambda \Lambda C_6 \quad (C.32)$$

$$F_8 = R_5 + \alpha_T T_6 \quad (C.33)$$

$$F_9 = -\Lambda \alpha_2 A \varepsilon_{n0} - \beta A \varepsilon_{n0} + R_6 + \alpha_T T_3 + \lambda \Lambda C_3 \quad (C.34)$$

$$F_{10} = R_6 + \alpha_T T_3 \quad (C.35)$$

$$F_{11} = \Lambda A \varepsilon_{n0} \Omega + R_9 + \alpha_T T_4 + \lambda \Lambda C_4 \quad (C.36)$$

$$F_{12} = R_7 + \alpha_T T_4 \quad (C.37)$$

$$F_{13} = R_1 + \alpha_T T_1 \quad (C.38)$$

$$z_1 = F_1 + F_2 R + \varepsilon_{n0} (\alpha_T U + V) \quad (C.39)$$

where R = the steady-state component of the approximate solution for normalized neutron population,
 U = the steady-state component of the approximate solution for fuel temperature,
 V = the steady-state component of the approximate solution for reactivity.

$$z_2 = F_3 + F_4 R \quad (\text{C.40})$$

$$z_3 = F_5 + F_6 R \quad (\text{C.41})$$

$$z_4 = F_7 + F_8 R \quad (\text{C.42})$$

$$z_5 = F_9 + F_{10} R + A \epsilon_{n0} (\alpha_T U + V) \quad (\text{C.43})$$

$$z_6 = F_{11} + F_{12} R \quad (\text{C.44})$$

$$z_7 = \dot{c}_1 (R + 1) \quad (\text{C.45})$$

$$z_8 = \dot{c}_2 (R + 1) \quad (\text{C.46})$$

$$z_9 = F_{13} (R + 1) \quad (\text{C.47})$$

$$G_{1n} = F_2 P_n + \epsilon_{n0} (R_{n1} + \alpha_T \tau_{n1}) \quad (\text{C.48})$$

where R_{n1} = multiplier of the n th sine term in the approximate solution for reactivity,
 τ_{n1} = multiplier of the n th sine term in the approximate solution for fuel temperature,
 Q_n = multiplier of the n th sine term in the approximate solution for neutron population.

$$G_{2n} = F_2 Q_n + \epsilon_0 (R_{n2} + \alpha_T \tau_{n2}) \quad (C.49)$$

$$G_{3n} = F_4 P_n \quad (C.50)$$

$$G_{4n} = F_4 Q_n \quad (C.51)$$

$$G_{5n} = F_6 P_n \quad (C.52)$$

$$G_{6n} = F_6 Q_n \quad (C.53)$$

$$G_{7n} = F_8 P_n \quad (C.54)$$

$$G_{8n} = F_8 Q_n \quad (C.55)$$

$$G_{9n} = F_{10} P_n + A \epsilon_{n0} (R_{n1} + \alpha_T \tau_{n1}) \quad (C.56)$$

$$G_{10n} = F_{10} Q_n + A \epsilon_{n0} (R_{n2} + \alpha_T \tau_{n2}) \quad (C.57)$$

$$G_{11n} = F_{12} P_n \quad (C.58)$$

$$G_{12n} = F_{12} Q_n \quad (\text{C.59})$$

$$G_{13n} = c_1^* P_n \quad (\text{C.60})$$

$$G_{14n} = c_1^* Q_n \quad (\text{C.61})$$

$$G_{15n} = c_2^* P_n \quad (\text{C.62})$$

$$G_{16n} = c_2^* Q_n \quad (\text{C.63})$$

$$G_{17n} = F_{13} P_n \quad (\text{C.64})$$

$$G_{18n} = F_{13} Q_n \quad (\text{C.65})$$

$$V = \frac{\beta F}{\lambda \Lambda} \quad (\text{C.66})$$

where F = the steady-state component of the approximate solution to the nonlinear equations describing the perturbation about the neutron population.

$$X = \frac{a_1 F}{a_2} \quad (\text{C.67})$$

$$W = \frac{K a_1 F}{a_2 a_4} \quad (\text{C.68})$$

$$\alpha_n = \frac{\beta \lambda}{\Lambda(n^2 \omega_*^2 + \lambda^2)} \quad (\text{C.69})$$

where ω_* = the oscillation frequency of the approximate solution to the nonlinear version of the equations for the perturbations about a candidate unbifurcated limit-cycle solution.

$$\delta_n = \frac{-\beta n \omega_*}{\Lambda(n^2 \omega_*^2 + \lambda^2)} \quad (\text{C.70})$$

$$\sigma_n = \frac{a_1 a_2}{n^2 \omega_*^2 + a_2^2} \quad (\text{C.71})$$

$$\tau_n = \frac{-a_1 n \omega_*}{n^2 \omega_*^2 + a_2^2} \quad (\text{C.72})$$

$$\zeta_n = \frac{\left[(a_4 - n^2 \omega_*^2) a_2 - a_3 n^2 \omega_*^2 \right]}{(a_2^2 + n^2 \omega_*^2) \left[(a_3 n \omega_*)^2 + (a_4 - n^2 \omega_*^2)^2 \right]} \quad (\text{C.73})$$

$$\phi_n = \frac{\left[n \omega_z (a_4 - n^2 \omega_z^2 + a_2 a_3) \right]}{(a_2^2 + n^2 \omega_z^2) \left[(a_3 n \omega_z)^2 + (a_4 - n^2 \omega_z^2) \right]} \quad (\text{C.74})$$

APPENDIX D.

EQUATIONS FOR CONSTANTS USED IN THE FORMATION OF THE APPROXIMATE SOLUTION OF THE MULTIMODAL PHENOMENOLOGICAL MODEL EQUATIONS

This appendix lists equations used in the derivation of the approximate solution to the multimodal phenomenological equations.

$$W_{mk} = \frac{\beta_m R_k}{\Lambda \lambda_m} \quad (\text{D.1})$$

where β_m = the fraction of delayed-neutron precursors from the m th precursor group,
 R_k = the steady-state component of the approximate solution for mode k ,
 Λ = the mean generation time,
 λ_m = the decay constant of the m th precursor group.

$$\sigma_{mn1k} = \frac{\beta_m (n \omega_k Q_{nk} + \lambda_m P_{nk})}{\Lambda (\lambda_m^2 + n^2 \omega_k^2)} \quad (\text{D.2})$$

where ω_k = the fundamental oscillation frequency of mode k ,
 Q_{nk} = the multiplier of the n th cosine term in the approximate solution for mode k ,
 P_{nk} = the multiplier of the n th sine term in the approximate solution for mode k .

$$\sigma_{mn2k} = \frac{\beta_m (\lambda_m Q_{nk} - n \omega_k P_{nk})}{\Lambda (\lambda_m^2 + n^2 \omega_k^2)} \quad (\text{D.3})$$

$$U_k = \frac{a_1 R_k}{a_2} \quad (\text{D.4})$$

where a_1 and a_2 = fitted constants in the phenomenological model equations.

$$\tau_{n1k} = \frac{a_1(n\omega_k Q_{nk} + a_2 P_{nk})}{a_2^2 + n^2 \omega_k^2} \quad (\text{D.5})$$

$$\tau_{n2k} = \frac{a_1(a_2 Q_{nk} - n\omega_k P_{nk})}{a_2^2 + n^2 \omega_k^2} \quad (\text{D.6})$$

$$V_k = \frac{K a_1 R_k}{a_2 a_4} \quad (\text{D.7})$$

where K and a_4 = fitted constants in the phenomenological model equations.

$$R_{n1k} = \frac{a_1 K \left\{ P_{nk} [a_2 a_4 - (a_2 + a_3) n^2 \omega_k^2] + Q_{nk} (a_2 a_3 + a_4 - n^2 \omega_k^2) n \omega_k \right\}}{(a_2^2 + n^2 \omega_k^2) \left[(a_4 - n^2 \omega_k^2)^2 + (a_3 n \omega_k)^2 \right]} \quad (\text{D.8})$$

where a_3 = a fitted constant in the phenomenological model equations.

$$R_{n2k} = \frac{a_1 K \left\{ Q_{nk} [a_2 a_4 - (a_2 + a_3) n^2 \omega_k^2] - P_{nk} (a_2 a_3 + a_4 - n^2 \omega_k^2) n \omega_k \right\}}{(a_2^2 + n^2 \omega_k^2) \left[(a_4 - n^2 \omega_k^2)^2 + (a_3 n \omega_k)^2 \right]} \quad (\text{D.9})$$

VITA

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